THE UNIVERSITY *of* EDINBURGH

PHILOSOPHY, PSYCHOLOGY AND LANGUAGE SCIENCES

LOGIC, COMPUTABILITY AND INCOMPLETENESS

PHIL10133

Exam Date:	**Tuesday 15th May 2018**	From and To:	**09:30 – 11:30**	Exam Diet: **April/May**

Please read full instructions before commencing writing

Exam paper information

- Answer **ALL** questions from **part A**, exactly **TWO** questions from **part B**, and exactly **TWO** questions from **part C**; NINE questions in total.
- Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each one selected).

Special instructions

- Answer questions in the exam booklet.
- Make sure you write the numbers of the questions answered on the **FRONT** of the booklet.

Special items

- None

Convenor of the Board of Examiners: Dr David Levy

Course Organiser: Dr Paul Schweizer

External: Dr Davide Rizza, University of East Anglia

This examination will be marked anonymously

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Part A

1. Answer all parts of (i), (ii) and (iii) below.

> (i) Consider a 1-place predicate P^1. Construct a sentence in first-order logic with identity which asserts that the predicate is true of *exactly one* object in the domain.
>
> (ii) Consider an interpretation $\Im$ with the 3-element domain $D = \{1,3,7\}$ and the interpreted 2-place relation symbol R^2 such that
> $\Im(R^2) = \{<7,1>, <7,7>, <3,7>, <3,3>, <1,7>, <7,3>\}$.
> Please state whether or not R^2 is irreflexive, asymmetric, and serial, and in each case explain why or why not.
>
> (iii) What are the structural properties that define an equivalence relation?

2. Consider the propositional language ***L*** with denumerably many sentence letters P_1, P_2, ... and the connectives ¬, ∨, ∧,→. Prove by induction on structural complexity that all formulas of ***L*** must have finite length.

3. Establish that the language ***L*** as described above has enumerably many compound formulas.

4. The function **min**(x,y) on the natural numbers is defined by cases as follows:

$\mathbf{min}(x,y) = \{x$, if $x \leq y$
$\quad\quad\quad\quad = \{y$, if $x > y$

Using the primitive recursive functions **diff**(x,y), **sg**(y) and $\underline{\mathbf{sg}}(y)$, along with the appropriate base functions, provide a specification of **min**(x,y) in the standard primitive recursive format for definition by cases, using + and · , as described in the text and lecture slides.

5. Using the derivation system developed in the text and lecture slides, give a proof of the following. Be sure to specify the relevant set Δ, include annotations, and establish that your derivation is a refutation.

$$\exists x \forall y(x = y), Pb \ \vdash \ \neg\exists y \neg Py$$

[Paper continues on next page]

Part B:

1. Suppose $c_1(x,y)$ is the primitive recursive characteristic function of condition C_1 on x and y, and that $c_2(x,y)$ is the primitive recursive characteristic function of condition C_2. Give the primitive recursive characteristic function for each of the following: the negation of C_1, the conjunction of C_1 and C_2, the disjunction of C_1 and C_2, the conditional if C_1 then C_2.

2. Prove *via* diagonalization that the set of 1-place functions from positive integers to positive integers is uncountable. Why does this show that not all such functions are Turing computable?

3. Explain what completeness means in the context of first-order logic. The formal theory of arithmetic Q is closed under first-order logical consequence, yet Gödel's theorems show that Q is incomplete. Explain why these two theoretical facts are not incompatible.

4. Explain what decidability means in the context of first-order logic, and how the halting problem for Turing machines leads to the result that first-order logic is undecidable. How do the completeness and undecidability of first-order logic relate?

Part C:

1. Answer all of parts (i) and (ii) below.

> (i) Consider the Turing machine quadruple $q_2S_1Rq_1$. Adopting the conventions used in the text and lecture slides, including the restriction that the Turing machine in question can read/write only the symbols S_0 and S_1, construct a sentence in first-order logic that formalizes this quadruple.
>
> (ii) Assuming the same conventions as above, give the quadruple instructions for a Turing machine that computes the function $f(x) = x + 2$ on the positive integers. Exhibit the sequence of configurations for computing the value $f(2)$.

2. What does it mean for a set to be *effectively enumerable*? Give an example of a set that is enumerable but not effectively enumerable, and explain why it has these properties. What does it mean for a formal theory to be *axiomatizable*? Is every decidable theory axiomatizable? Explain why or why not.

3. Let Φ be the formula $\exists y R(x, y)$. What is the diagonalization of Φ? As defined in the text and lecture slides, what is the value of the 1-place recursive function **diag**(n)? State the diagonal lemma and give an outline of the strategy whereby the lemma is proved.

4. State Löb's Theorem, and explain its conceptual significance and motivation. Give the formalized version of Löb's Theorem in the language of first-order arithmetic. How does this translate into the axiom G characterizing the modal logic of provability? What is the Henkin Sentence, and what does Löb's Theorem reveal about this sentence?

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