PHILOSOPHY, PSYCHOLOGY AND LANGUAGE SCIENCES

Logic, Computability and Incompleteness MSc

PHIL11114

Exam Date:	**Wednesday 18th May 2016**	From and To:	**2.30 – 4.30 PM**	Exam Diet:	**April / May**

Please read full instructions before commencing writing

Exam paper information

- Answer **all questions from part A**, exactly **two questions from part B**, and exactly **two questions from part C**.
- Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each question selected).

Special instructions

- None

Special items

- None

Convener of the Board of Examiners: Dr Mits Ota

This examination will be marked anonymously.

Answer all questions from part A, exactly two questions from part B, and exactly two questions from part C.

Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each question selected).

Part A:

1. Prove that the set of all infinite strings of 0's and 1's is not enumerable.

2. Please answer all of parts (i) and (ii) below.

(i) The set of formulas $\{\forall x\exists yRxy, \neg\exists y\forall xRxy\}$ is *consistent*. Demonstrate this fact by producing a simple interpretation $\mathfrak{I}$ which is a model of this set.

(ii) Consider a 1-place predicate P^1. Construct a sentence in first-order logic which asserts that the predicate is true of *exactly two* objects in the domain.

3. Consider the propositional language $\boldsymbol{L}$ with denumerably many sentence letters S_1, S_2, S_3,... and the two connectives $\neg$, $\vee$. Suppose that the set of sentences $\boldsymbol{\Gamma}$ is a formal theory in $\boldsymbol{L}$ and that for each sentence letter S_i, either $S_i \in \boldsymbol{\Gamma}$ or $\neg S_i \in \boldsymbol{\Gamma}$. Prove by induction on structural complexity that $\boldsymbol{\Gamma}$ is complete.

4. The arithmetical difference between x and y, written **dif**(x,y) [and abbreviated informally as $x \dot{-} y$] is defined as $x - y$ if $x \geq y$ and 0 otherwise. Assuming that **pred**(x) is the primitive recursive function yielding the predecessor of x, give the two equations that informally capture arithmetical difference as a primitive recursive function, and then provide the formal specification of **dif** in full primitive recursive format.

5. Using the formal derivation system developed in the lecture slides (and Boolos and Jeffrey), prove that the two formulas

$$\neg\forall x\exists y\neg\, Lyx \quad \text{and} \quad \exists y\forall x\, Lxy$$

are logically equivalent.

Part B:

1. Define a 1-place function of positive integers, $u(x)$, that cannot be computed by any Turing machine, and establish that it has this property. Explain the connection between $u(x)$ and the halting problem. Is there an effective positive test for halting? What is the relation between the halting problem and the entailment relation for first-order logic?

2. Give a proof of Cantor's theorem that for any set **Γ**, the power set **$\mathcal{P}$(Γ)** must have greater cardinality than **Γ** itself.

3. What is the fundamental relationship between validity and unsatisfiability? Explain in detail how this relationship is utilized in the completeness proof for first-order logic given by Boolos and Jeffrey. Suppose Γ is a set of sentences of first-order logic, ϕ is a single sentence, and $\Gamma \vDash \phi$. Show in detail how the completeness proof yields the result that there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \phi$.

4. Church's theorem on the undecidability of first-order logic can be proved relative to the undecidability of the theory *Q*. Give a detailed exposition of the proof. Does the conclusion rely upon the Church-Turing thesis? Explain why or why not. Are the theorems of first-order logic effectively enumerable? Explain why or why not.

Part C:

1. What does it mean for a formal theory to be *axiomatizable*? Is every decidable theory axiomatizable? Is every axiomatizable theory decidable? Explain why or why not. What does it mean for a formal theory to be *complete*? Is every axiomatizable complete theory decidable? Explain why or why not.

2. On the basis of the completeness proof for first-order logic, show that if Γ is a satisfiable set of sentences, then Γ has a model with a domain of cardinality less than or equal to $\aleph_0$. Why might this result be viewed as 'paradoxical'?

3. Let $\boldsymbol{F}$ be the formula $\forall y R(x, y)$. What is the diagonalization of $\boldsymbol{F}$?
Following the proof of the diagonal lemma, construct the sentence $\boldsymbol{G}$ such that

$$\vdash_Q \boldsymbol{G} \leftrightarrow \forall y\, R(\ulcorner \boldsymbol{G} \urcorner, y)$$

Give an outline of how the diagonal lemma is used to obtain Tarski's result that the set of Gödel numbers of the sentences true in *Q* is not definable in *Q*.

4. State Gödel's second incompleteness theorem as proved in the lecture slides, and give an exposition of the basic conceptual strategy of the proof. What is the relationship between Gödel's first theorem and the second theorem? State Löb's theorem and show how Gödel's second theorem can be derived as a consequence. Given the Gödel-Löb results, is S4 the appropriate modal logic for provability in arithmetic? Explain why or why not.