SCHOOL NAME: PHILOSOPHY, PSYCHOLOGY AND LANGUAGE SCIENCES

LOGIC, COMPUTABILITY AND INCOMPLETENESS

PHIL10133

Exam Date:	**Wednesday 21st May, 2014**	From and To:	**9.30am – 11.30am**	Exam Diet:	**April/May**

Please read full instructions before commencing writing

Exam paper information **JUNIOR HONOURS** candidates *must clearly write* **"THREE"** on the cover of the script book Answer **all** questions from part A, exactly **two** questions from part B, and exactly **two** questions from part C. Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each one selected).
Special instructions NONE
Special items NONE

Chairman – Professor Mike Ridge
External Examiners Dr Michael Scott

This examination will be marked anonymously

Answer all questions from part A, exactly two questions from part B, and exactly two questions from part C.

Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each one selected).

Part A

1. Consider the propositional language $\boldsymbol{L}$ with denumerably many sentence letters P_1, P_2, ... and the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$. Prove by induction on structural complexity that all formulas of $\boldsymbol{L}$ must have finite length.

2. Establish that any formal language with a finite alphabet and formulas of finite length must have denumerably many such formulas.

3. Consider the Turing machine quadruple $q_3 S_1 S_0 q_2$. Adopting the conventions used in the lecture slides (and Boolos and Jeffery), including the restriction that the Turing machine in question can read/write only the symbols S_0 and S_1, construct a sentence in first-order logic that formalizes this quadruple.

4. The predecessor of a natural number x, written **pred**(x), is the number immediately preceding it, except that we let the predecessor of 0 be 0. Give the two equations that informally define **pred**(x) as a primitive recursive function, and then provide its official specification in primitive recursive format.

5. Using the derivation system developed in the lecture slides (and Boolos and Jeffery), give a proof of the following. Be sure to specify the relevant set Δ, include annotations, and establish that your derivation is a refutation.

$$\exists x \forall y (x = y), P\mathrm{a} \vdash \forall x P x$$

Part B:

1. Prove *via* diagonalization that the set of 1-place functions from positive integers to positive integers is uncountable. Why does this show that there must be functions which are not Turing computable?

2. What does it mean for a set to be *effectively enumerable*? Give an example of a set that is enumerable but not effectively enumerable, and explain why it has these properties. What does it mean for a formal theory to be *axiomatizable*? Is every decidable theory axiomatizable? Explain why or why not.

3. On the basis of the completeness proof for first-order logic, show that if Γ is a satisfiable set of sentences, then Γ has a model with a domain of cardinality less than or equal to $\aleph_0$. Why might this result be viewed as 'paradoxical'?

4. Show that there can be no 'axiom of finitude', in the sense that if a set of sentences Γ has arbitrarily large finite models, then it has a model whose domain is infinite. Given that a domain is finite iff it is not infinite, why can't the negation of Russell's axiom of infinity be used to characterize finite domains?

Part C:

1. Explain the halting problem for Turing machines, why the halting function is not Turing computable, and how the halting problem leads to the result that first-order logic is undecidable. Why can't a Universal Turing machine solve the halting problem? And why doesn't the completeness of first-order logic supply a method for solving the problem?

2. State the diagonal lemma, and explain in some detail how it is used to construct the Gödel Sentence *S*, and why this sentence leads to the conclusion that *Q* is incomplete. Let *Q** be the theory obtained by adding *S* as an axiom to supplement *Q*. Why does *Q** not thereby escape Gödel's incompleteness result?

3. First-order logic is provably complete, and the formal theory of arithmetic Q is closed under first-order logical consequence. Yet Gödel's theorems show that Q is incomplete. Explain why these two theoretical facts are not incompatible. Why do considerations of brute cardinality reveal that there must be truths of the full theory of natural numbers that are not theorems of formal arithmetic? How does this type of 'incompleteness' compare with Gödel's incompleteness results?

4. State Löb's Theorem, and explain why it is relevant to an 'internal' consistency proof for formal arithmetic. How does Gödel's second incompleteness theorem follow as a direct corollary of Löb's Theorem? What is the Henkin Sentence, and what does Löb's Theorem reveal about this sentence?

[End of Paper]