SCHOOL NAME: PHILOSOPHY, PSYCHOLOGY AND LANGUAGE SCIENCES

LOGIC, COMPUTABILITY AND INCOMPLETENESS

PHIL10133

Exam Date:	**18/05/2016**	From and To:	**2.30pm - 4.30pm**	Exam Diet	**May**

Please read full instructions before commencing writing

Exam paper information

JUNIOR HONOURS candidates *must clearly write* **"THREE"** on the cover of the script book

Answer all questions from part A, exactly two questions from part B, and exactly two questions from part C.

Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each question selected).

Special instructions

NONE

Special items

NONE

Chairman – Dr David Levy
External Examiners Dr Arif Ahmed

This examination will be marked anonymously

Answer all questions from part A, exactly two questions from part B, and exactly two questions from part C.
Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each question selected).

Part A:

1. Prove that the set of all infinite strings of 0's and 1's is not enumerable

2. Please answer all of parts (i) and (ii) below.

(i) The set of formulas $\{\forall x \exists y Rxy, \neg \exists y \forall x Rxy\}$ is *consistent*. Demonstrate this fact by producing a simple interpretation $\Im$ which is a model of this set.

(ii) Consider a 1-place predicate P^1. Construct a sentence in first-order logic which asserts that the predicate is true of *exactly two* objects in the domain.

3. Consider the propositional language $\boldsymbol{L}$ with denumerably many sentence letters S_1,S_2, S_3,... and the two connectives $\neg$, $\vee$. Suppose that the set of sentences $\mathbf{\Gamma}$ is a formal theory in $\boldsymbol{L}$ and that for each sentence letter S_i, either $S_i \in \mathbf{\Gamma}$ or $\neg S_i \in \mathbf{\Gamma}$. Prove by induction on structural complexity that $\mathbf{\Gamma}$ is complete.

4. The arithmetical difference between x and y, written **dif**(x,y) [and abbreviated informally as $x \dot{-} y$] is defined as $x - y$ if $x \geq y$ and 0 otherwise. Assuming that **pred**(x) is the primitive recursive function yielding the predecessor of x, give the two equations that informally capture arithmetical difference as a primitive recursive function, and then provide the formal specification of **dif** in full primitive recursive format.

5. Using the formal derivation system developed in the lecture slides (and Boolos and Jeffrey), prove that the two formulas

$$\neg \forall x \exists y \neg Lyx \text{ and } \exists y \forall x Lxy$$

are logically equivalent.

Part B:

1. Please answer all of parts (i) and (ii) below.

(i) Using the derivation system developed in the lecture slides (and Boolos and Jeffrey), prove that any asymmetric relation is irreflexive.

(ii) Consider the interpretation $\Im$ with the 3-element domain $D = \{a,b,c\}$ and the interpreted 2-place relation symbol R^2 such that $\Im(R^2) = \{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle a,c\rangle, \langle c,a\rangle, \langle c,b\rangle\}$. Please state whether or not R^2 is reflexive, symmetric, asymmetric, and serial, and in each case explain why or why not. Could R^2 be used to define a corresponding 1-place function f^1? Explain why or why not.

2. Give a proof of Cantor's theorem that for any set $\mathbf{\Gamma}$, the power set $\mathcal{P}(\mathbf{\Gamma})$ must have greater cardinality than $\mathbf{\Gamma}$ itself.

3. What is the fundamental relationship between validity and unsatisfiability? Explain in detail how this relationship is utilized in the completeness proof for first-order logic given by Boolos and Jeffrey. Is there an effective positive test for unsatisfiability? Explain why or why not.

4. Church originally proved the undecidability of first-order logic relative to the undecidability of the theory *Q*. Give a detailed exposition of the proof. Does the conclusion rely upon the Church-Turing thesis? Explain why or why not. Are the theorems of first-order logic effectively enumerable? Explain why or why not.

Part C:

1. Please answer all of parts (i) and (ii) below.

(i) Consider the Turing machine quadruple $q_3 S_1 S_0 q_2$. Adopting the conventions used in the lecture slides (and Boolos and Jeffrey), including the restriction that the Turing machine in question can read/write only the symbols S_0 and S_1, construct a sentence in first-order logic that formalizes this quadruple.

(ii) Give the quadruple instructions for a Turing machine ***M*** that, if started in state q_1 scanning the leftmost of an unbroken string of 1's on an otherwise blank tape, will halt in standard configurations scanning a 0 if the number of 1's in the string is *even*, and will halt scanning a 1 if the number of 1's is *odd*. Exhibit the sequence of configurations on the even input string '11'.

2. Show that there is an effective procedure for deciding the validity of prenex sentences in which no function symbols occur, and in which no existential quantifier is to the left of any universal quantifier.

3. State the diagonal lemma, and explain in detail how it is used to construct the Gödel sentence *S*, and why this sentence leads to the conclusion that *Q* is incomplete. Let *Q** be the theory obtained by adding *S* as an axiom to supplement *Q*. Why does *Q** not thereby escape Gödel's incompleteness result?

4. Gödel's second incompleteness theorem is sometimes described as a formalization of his first theorem. Explain why this is the case. State Löb's theorem, and explain why it is relevant to an 'internal' consistency proof for formal arithmetic. How does Gödel's second incompleteness theorem follow as a direct corollary of Löb's theorem?

[END OF PAPER]