PHILOSOPHY, PSYCHOLOGY AND LANGUAGE SCIENCES

Logic, Computability and Incompleteness MSc

PHIL11114

Exam Date:	**Wednesday 21st May 2014**	From and To:	**09.30 - 11.30AM**	Exam Diet:	**May**

Please read full instructions before commencing writing

Exam paper information

- Answer **all** questions from Part A, exactly **two** questions from Part B, and exactly **two** questions from Part C.

Special instructions

- None

Special items

- None

Convener of the Board of Examiners: Professor Holly Branigan

This examination will be marked anonymously

Logic, Computability and Incompleteness

Answer **all** questions from part A, exactly **two** questions from part B, and exactly **two** questions from part C.

Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each one selected).

Part A

1. Consider the propositional language $\boldsymbol{L}$ with denumerably many sentence letters P_1, P_2, ... and the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$. Prove by induction on structural complexity that all formulas of $\boldsymbol{L}$ must have finite length.

2. Establish that any formal language with a finite alphabet and formulas of finite length must have denumerably many such formulas.

3. Consider the Turing machine quadruple $q_3 S_1 S_0 q_2$. Adopting the conventions used in the lecture slides (and Boolos and Jeffery), including the restriction that the Turing machine in question can read/write only the symbols S_0 and S_1, construct a sentence in first-order logic that formalizes this quadruple.

4. The predecessor of a natural number x, written $\mathbf{pred}(x)$, is the number immediately preceding it, except that we let the predecessor of 0 be 0. Give the two equations that informally define $\mathbf{pred}(x)$ as a primitive recursive function, and then provide its official specification in primitive recursive format.

5. Using the derivation system developed in the lecture slides (and Boolos and Jeffery), give a proof of the following. Be sure to specify the relevant set Δ, include annotations, and establish that your derivation is a refutation.

$$\exists x \forall y (x = y),\ P\mathrm{a} \vdash \forall x P x$$

Part B

1. Define a 1-place function of positive integers, $u(x)$, that cannot be computed by any Turing machine, and prove that it has this property. Explain the connection between $u(x)$ and the halting problem. Is there an effective positive test for halting? Why can't a Universal Turing machine compute the halting function?

2. What does it mean for a set to be *effectively enumerable*? Give an example of a set that is enumerable but not effectively enumerable, and explain why it has these properties. What does it mean for a formal theory to be *axiomatizable*? Is every decidable theory axiomatizable? Explain why or why not.

3. On the basis of the completeness proof for first-order logic, show that if Γ is a satisfiable set of sentences, then Γ has a model with a domain of cardinality less than or equal to $\aleph_0$. Why might this result be viewed as 'paradoxical'?

4. Suppose Γ is set of sentences of first-order logic, ϕ is a single sentence, and $\Gamma \vDash \phi$. Show in some detail how the completeness proof for first-order logic yields the result that there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vDash \phi$.

Part C

1. Church originally proved the undecidability of first-order logic relative to the undecidability of the theory *Q*. Given an exposition of the proof. Does the conclusion rely upon the Church-Turing Thesis? Explain why or why not. Is there an effective positive test for validity in first-order logic?

2. State the diagonal lemma, and explain in some detail how it is used to construct the Gödel Sentence *S*, and why this sentence leads to the conclusion that *Q* is incomplete. Let *Q** be the theory obtained by adding *S* as an axiom to supplement *Q*. Why does *Q** not thereby escape Gödel's incompleteness result?

3. First-order logic is provably complete, and the formal theory of arithmetic Q is closed under first-order logical consequence. Yet Gödel's theorems show that Q is incomplete. Explain why these two theoretical facts are not incompatible. Why do considerations of brute cardinality reveal that there must be truths of the full theory of natural numbers that are not theorems of formal arithmetic? How does this type of 'incompleteness' compare with Gödel's incompleteness results?

4. Gödel's second incompleteness theorem is sometimes described as a formalization of his first theorem. Explain why this is the case. State Löb's Theorem, and explain why it is relevant to an 'internal' consistency proof for formal arithmetic. How does Gödel's second incompleteness theorem follow as a direct corollary of Löb's Theorem?