Logic, Computability and Incompleteness

Binary Relations

Binary Relations

2-place or **Binary Relations** are fundamental to human language, thought and reasoning.

(i) transitive verbs:

'*x* loves *y*', '*x* embraces *y*', '*x* is acquainted with *y*', etc.;

(ii) comparatives and adverbial comparisons:

'*x* is taller than *y*', '*x* is smarter than *y*', etc.;

(iii) family relationships:

'*x* is an uncle of *y*', *x* is a sister of *y*', etc.;

(iv) functional relationships (in mathematics):

'*x* is the square of *y*', '*x* is the tangent of the angle *y*',

'*x* is the (positive) square root of *y*', etc.

Properties of Binary Relations

We know **a priori** that:

Alice is not **taller than** herself;

If Alice is **taller than** Bill, then Bill is not **taller than** Alice;

If Alice is **taller than** Bill and Bill is **taller than** Carol, then Alice is **taller than** Carol.

The relation ***taller than*** is

- **irreflexive**, **asymmetric** and **transitive**.

Reflexivity

Definition A relation R is:

(i) **reflexive** iff $\forall x Rxx$,

(ii) **irreflexive** iff $\forall x \neg Rxx$

Examples of **reflexive** relations:

‘x is identical to y’, ‘x is the same age as y’.

irreflexive relations:

‘x is the sister of y’, ‘x is taller than y’,

‘the number x is smaller than the number y’.

Symmetry

Definition: A relation R is

(i) **symmetric** iff $\forall x \forall y(Rxy \rightarrow Ryx)$;

(ii) **asymmetric** iff $\forall x \forall y(Rxy \rightarrow \neg Ryx)$,

Examples of symmetry:

‘x is a spouse of y’, ‘x is a sibling of y’

‘x and y are 1 metre apart’.

Examples of asymmetry: ‘$x < y$’

Comparatives: ‘x is larger than y’, ‘x is taller than y’, etc.

Certain family relations are asymmetric: ‘x is the mother of y’.

Transitivity

Definition: A relation R is **transitive** iff

$$\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz).$$

Comparatives are **transitive**.

‘x is taller than y’, ‘x is older than y’,

‘the number x is less than the number y’ (i.e., ‘$x < y$’)

Example from the biological world:

‘x is an ancestor of y’.

Example from the physical world:

‘event x is earlier (in time) than event y’

Orderings

Ordering relations.

- **irreflexive**,
- **asymmetric**
- **transitive**.

Technical name: (strict) **partial orderings**.

The system **N** of natural numbers is **ordered**.

0 < 1 < 2 < 3 < ….

- the rationals **Q** (ratios of natural numbers)
- the real numbers **R** (rationals plus irrationals)

A (strict) **partial ordering** is any **transitive irreflexive relation** (asymmetry follows as a logical consequence).

Equivalence Relations

- **Definition**: A relation that is **reflexive**, **symmetric**, and **transitive** is said to be an **equivalence relation**.

Examples:

x **is the same height as** *y***,**

x **is the same age as** *y***,**

x **has the same surname as** *y***,**

x **is parallel to** *y***,**

Φ **is logically equivalent to** Ψ

Other Structural Properties

Connectedness: $\forall x \forall y (Rxy \lor Ryx)$

Density: $\forall x \forall y (Rxy \rightarrow \exists z (Rxz \land Rzy))$

Seriality: $\forall x \exists y Rxy$

Others can be expressed with the notion of **identity**:

Trichotomy: $\forall x \forall y (Rxy \lor x = y \lor Ryx)$

The Concept of Identity

Identity is a fundamental notion in human thought and reasoning. It's normally expressed using the 2-place predicate symbol '**=**' to mean *x* **is identical to** *y*

Identity is treated as a **binary relation**, but is **logically**/**semantically** privileged.

The defining properties of the identity relation are considered to be purely **logical**.

In this sense, the identity predicate **'***x* = *y***'** is the **only predicate** that itself belongs to logic.

All other predicates (like **'*x* is a brother of *y*'**) are treated as **non-logical**.

The Concept of Identity

- The concept of identity can be explained in terms of certain **a priori** logical principles.

(I) **The Principle of Self-Identity**:

Every object is identical to itself.

$$\forall x(x = x).$$

(II) **The Indiscernibility of Identicals**:

*If entities x and y are identical, and property **P** is true of x, then **P** is true of y.*

$$\forall x \forall y[(x = y \wedge (\boldsymbol{P}\, x)) \rightarrow \boldsymbol{P}(y)].$$

The Indiscernibility of Identicals was originally stated by the German mathematician and philosopher Gottfried Leibniz.

Second-order Logic

- Note that in the statement (II) above we utilize the meta-linguistic variable '***P***', which is not part of our formal object language.
- Implicitly, we are making a universal quantification over all properties of individuals. This transcends the boundaries of first-order logic, in which the quantifiers range over individuals in the domain of discourse.
- However, the Indiscernibility of Identicals can be formalized in second-order logic.
- Let ***P*** be a second-order variable ranging over properties of individuals. Then (II) is rendered as the closed second-order formula: $\forall \boldsymbol{P} \forall x \forall y((x = y \wedge \boldsymbol{P}(x)) \rightarrow \boldsymbol{P}(y))$.

The Concept of Identity

- Given the Indiscernibility of Identicals, if there is a property ***P*** had by some object ***a*** that is not had by an object ***b***, then it follows by *deductive logic* that ***a*** ≠ ***b***.

In this case ***P*** is a discerning property for **a** and **b**.

For example, suppose you want to *prove* that

Donald Trump *is* **not identical to** Boris Johnson.

The following is a logically valid argument:

Boris Johnson attended Eton College.
Donald Trump did *not* attend Eton College.
Therefore Donald Trump ≠ Boris Johnson.

The Concept of Identity

As mentioned above, a relation R is an **equivalence relation** just in case it is

- reflexive $\forall x Rxx$.
- symmetric $\forall x \forall y (Rxy \rightarrow Ryx)$.
- transitive $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$.

So identity is an equivalence relation (and can be defined as the "smallest" equivalence relation.)

Expressing Cardinality in FOL

- With the use of **=** we can construct sentences which are true only in models of some specific finite cardinality.

  For example, $\exists x \forall y\ (y = x)$ is an 'axiom' true only in interpretations with 1-element domains

  $\exists x \exists y (x \neq y)$ is an axiom true only in interpretations with with at least 2 element domains.

  $\exists x \exists y \forall z (z = x \vee z = y)$ is true only in interpretations with with at most 2 element domains.

  while $\exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$ is true only in interpretations with with exactly 2 element domains.

  Obviously these patterns generalize to any positive integer n.

Expressing Cardinality in FOL

- Russell showed that we can construct a sentence that is true for no finite n, and hence forces the domain to be **infinite.**

  $\forall x \forall y \forall z((Rxy \land Ryz) \to Rxz)$ expresses the property of transitivity,

  $\forall x \forall y(Rxy \to \neg Ryx)$ expresses the property of being asymmetric, and

  $\forall x \exists y\, Rxy$ expresses the property of being serial.

  The set Δ containing these 3 sentences as elements is satisfiable,

  but Δ is modelled by no interpretation with a finite domain.

Expressing Cardinality in FOL

- Equivalently, we can conjoin them into a single sentence to get Russell's **Axiom of Infinity**:

  $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz) \wedge \forall x \forall y(Rxy \rightarrow \neg Ryx) \wedge$
  $\forall x \exists y\, Rxy$

  If $\mathcal{J}$ is a model of $\boldsymbol{\Delta}$, then the extension $\mathcal{J}(R)$ of the binary predicate symbol R must be a transitive, asymmetric and serial relation.

  And if $\mathcal{J}$ is a model of Δ, then its domain D must be **infinite**.

  Proof: Let $\boldsymbol{a}_1$ be one of the objects in D.

  By seriality, there is an object, $\boldsymbol{a}_2$ say, in the domain such that $R\boldsymbol{a}_1\boldsymbol{a}_2$.

Expressing Cardinality in FOL

Any asymmetric relation is irreflexive (will be able to prove this in our formal system):

so $\boldsymbol{a}_1 \neq \boldsymbol{a}_2$.

By seriality again, there's an object $\boldsymbol{a}_3$ in the domain such that $R\boldsymbol{a}_2\boldsymbol{a}_3$.

By transitivity, $R\boldsymbol{a}_1\boldsymbol{a}_3$.

By irreflexivity, we have it that $\boldsymbol{a}_1 \neq \boldsymbol{a}_3$ and $\boldsymbol{a}_2 \neq \boldsymbol{a}_3$.

[**Summary so far**: the domain must contain at least **3** distinct objects: $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$.]

By seriality again, there's an object a_4 in the domain such that $R\boldsymbol{a}_3\boldsymbol{a}_4$.

Expressing Cardinality in FOL

By transitivity, $R\boldsymbol{a}_1\boldsymbol{a}_4$ and $R\boldsymbol{a}_2\boldsymbol{a}_4$.

By irreflexivity again, $\boldsymbol{a}_1 \neq \boldsymbol{a}_4$, $\boldsymbol{a}_2 \neq \boldsymbol{a}_4$, and $\boldsymbol{a}_3 \neq \boldsymbol{a}_4$.

[**Summary so far**: the domain must contain at least **4** distinct objects: $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$.]

And so on….

By repeating this reasoning, there must be **infinitely many distinct objects**:

$$\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \ldots, \boldsymbol{a}_n, \ldots \blacksquare$$