THE UNIVERSITY of EDINBURGH

PHILOSOPHY, PSYCHOLOGY AND LANGUAGE SCIENCES

LOGIC, COMPUTABILITY AND INCOMPLETENESS

PHIL10133

Exam Date:	**Friday, 24th May 2019**	From and To:	**09:30-11:30**	Exam Diet: **May**

Please read full instructions before commencing writing

Exam paper information

- Answer **ALL** questions from **part A**, exactly **TWO** questions from **part B**, and exactly **TWO** questions from **part C**; NINE questions in total.
- Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each one selected).

Special instructions

- Answer questions in the exam booklet.
- Make sure you write the numbers of the questions answered on the **FRONT** of the booklet.

Special items

- None

Convenor of the Board of Examiners: Dr David Levy

Course Organiser: Dr Paul Schweizer

External: Dr Davide Rizza, University of East Anglia

This examination will be marked anonymously

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Part A

1. Answer all parts of (i), (ii) and (iii) below.

(i) Consider a 1-place predicate P^1. Construct a sentence in first-order logic with identity which asserts that the predicate is true of *at most two* objects in the domain.

(ii) Consider an interpretation $\Im$ with the 3-element domain $D = \{5,3,9\}$ and the interpreted 2-place relation symbol R^2 such that
$\Im(R^2) = \{<5,3>, <3,9>, <9,9>, <3,5>, <5,9>, <9,5>\}$.
Please state whether or not R^2 is symmetric, irreflexive and transitive, and in each case explain why or why not.

(iii) Could R^2 be used as the basis to define a corresponding 1-place function? Explain why or why not.

2. Consider the propositional language ***L*** with denumerably many sentence letters P_1, P_2, ... and the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$. Prove by induction on structural complexity that each formula of ***L*** must have exactly the same number of right parentheses as left parentheses.

3. Show that the set of compound formulas of the language ***L*** as described above is enumerable.

4. The predecessor of a natural number *x*, written **pred**(*x*), is the number immediately preceding it, except in the case of 0, where we let **pred**(0) = 0. Give the two equations that informally define **pred** as a primitive recursive function, and then provide its specification in official primitive recursive format.

5. Using the derivation system developed in the text and lecture slides, give a proof of the following. Be sure to specify the relevant set Δ, include annotations, and establish that your derivation is a refutation.

$$\exists x \forall y Rxy \;\vdash\; \forall x \exists y Ryx$$

[Paper continues on next page]

Part B:

1. Answer all parts of (i), (ii) and (iii) below.

   (i) The statement $\forall x \exists y Lxy \vDash \exists y \forall x Lxy$ is false. Specify a simple interpretation $\mathfrak{I}$ which establishes this fact.

   (ii) The formula $\neg\exists x Px \wedge \forall y(Py \rightarrow Qy)$ is satisfiable. Specify a simple interpretation $\mathfrak{I}$ which establishes this fact.

   (iii) Construct a formula Θ in prenex form such that $\Theta \equiv \exists x Px \rightarrow \forall y \neg Qy$

2. Define a 1-place function of positive integers, $u(x)$, that cannot be computed by any Turing machine, and establish that it has this property. Explain the connection between $u(x)$ and the halting problem. Is there an effective positive test for halting?

3. Church's theorem on the undecidability of first-order logic can be proved relative to the undecidability of the theory *Q*. Give a detailed exposition of the proof. Does the conclusion rely upon the Church-Turing thesis? Explain why or why not.

4. Give a proof of Cantor's theorem that for any set **Γ**, the power set **P(Γ)** must have greater cardinality than **Γ** itself.

Part C:

1. Answer all of parts (i) and (ii) below.

   (i) Consider the Turing machine quadruple $q_1 S_0 S_1 q_3$. Adopting the conventions used in the text and lecture slides, including the restriction that the Turing machine in question can read/write only the symbols S_0 and S_1, construct a sentence in first-order logic that formalizes this quadruple.

   (ii) Assuming the same conventions as above, give the quadruple instructions for a Turing machine that computes the two-place function $f(x,y) = (x + y) + 1$ on the positive integers. Exhibit the sequence of configurations for computing the value $f(2,3)$.

2. Describe and explain an effective procedure for deciding the validity of first-order sentences in prenex form, in which no function symbols occur, and in which no existential quantifier is to the left of any universal quantifier.

[Paper continues on next page]

3. Answer all of parts (i), (ii), (iii) and (iv) below.

(i) As in the lecture slides, let the (complex) formula ***Pr*** (x,y) define the relation **proof** in ***Q***. What is the intuitive interpretation or meaning of ***Pr*** (x,y)?

(ii) Using ***Pr*** (x,y), specify the 1-place 'proof predicate' ***Prov*** (y).

(iii) Assuming the diagonal lemma, specify the Gödel Sentence ***S***.

(iv) Since ***Prov*** (y) is a proof predicate, it follows that for all sentences ***A*** in the language of Q

if $\vdash_Q$ ***A***, then $\vdash_Q$ ***Prov*** ($\ulcorner$***A***$\urcorner$), and if $\vdash_Q$ ***Prov*** ($\ulcorner$***A***$\urcorner$), then $\vdash_Q$ ***A***.

Using the information above, show that if Q is consistent then it is incomplete.

4. State Gödel's second incompleteness theorem as proved in the lecture slides, and give an exposition of the basic conceptual strategy of the proof. What is the relationship between Gödel's first theorem and the second theorem? In what sense is the second theorem an 'incompleteness' result?

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