Answer all questions from part A, exactly two questions from part B, and exactly two questions from part C.

Part A is worth 40%, and all questions from parts B and C are equally weighted (i.e. 15% for each question selected).

Part A:

1. Consider the restricted propositional language $\boldsymbol{L}$ with the single sentence letter $\boldsymbol{P}$ and the single truth-functional connective $\wedge$. Prove by induction on structural complexity that for all formulas ϕ of $\boldsymbol{L}$, $\phi \equiv \boldsymbol{P}$.

2. Define a *word* as any finite string from some enumerably infinite alphabet $a_1, a_2, a_3, \ldots$. Prove that the set of two-letter words is enumerable.

3. Consider the Turing machine quadruple $q_1 S_0 L q_3$. Adopting the conventions used in the lecture slides (and Boolos and Jeffery), including the restriction that the Turing machine in question can read/write only the symbols S_0 and S_1, construct a sentence in first-order logic that formalizes this quadruple.

4. Suppose the two-place function on natural numbers $\boldsymbol{f}$ is defined such that $\boldsymbol{f}(x, y) = x^y$. Give the two equations that informally capture $\boldsymbol{f}$ as a primitive recursive function, and then, assuming that **prod**(x,y) is the primitive recursive function yielding the product of x and y, provide the formal specification of $\boldsymbol{f}$ in full primitive recursive format.

5. Using the derivation system developed in the lecture slides (and Boolos and Jeffery), give a proof of the following. Be sure to specify the relevant set Δ, include annotations, and establish that your derivation is a refutation.

$$\exists x \forall y (Py \leftrightarrow y = x) \vdash \exists x Px$$

Part B:

1. Given that the set of all Turing machines is enumerable, define a 1-place (total) function on positive integers, $u(x)$, that cannot be computed by any Turing machine, and show why this is the case. Explain the connection between $u(x)$ and the halting problem. Is there an effective positive test for halting?

2. Show how to construct a first-order sentence that is true only in interpretations with exactly n-element domains, where n is any positive integer. Is it possible to construct a first-order sentence true in only those interpretations with finite domains? Explain in detail why or why not. Is it possible to construct a first-order sentence true in exactly those interpretations with infinite domains?

3. Consider the propositional language $\boldsymbol{L}$ with denumerably many sentence letters $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \ldots$ and the two connectives $\neg$, $\rightarrow$. If Δ is a set of sentences of $\boldsymbol{L}$, define what is meant by the statements "Δ is a theory in $\boldsymbol{L}$" and "Δ is complete". What is the smallest theory Δ in $\boldsymbol{L}$? What is the largest? Suppose that Δ is a theory in $\boldsymbol{L}$ and that for each sentence letter $\boldsymbol{P}_i$, either $\boldsymbol{P}_i \in \Delta$ or $\neg\boldsymbol{P}_i \in \Delta$. Prove that Δ is complete.

4. Suppose Γ is a set of sentences of first-order logic, ϕ is a single sentence, and $\Gamma \vDash \phi$. Show in detail how the completeness proof for first-order logic yields the result that there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vDash \phi$.

Part C:

1. What does it mean for a property $\mathscr{P}$ to be decidable? Explain what decidability means in the context of first-order logic, and how the halting problem for Turing machines leads to the result that first-order logic is undecidable. Explain what completeness means in the context of first-order logic. How do the completeness and undecidability of first-order logic relate?

2. Show that there is an effective procedure for deciding the validity of prenex sentences in which no function symbols occur, and in which no existential quantifier is to the left of any universal quantifier.

3. Let $\boldsymbol{F}$ be the formula $\forall y R(x, y)$. What is the diagonalization of $\boldsymbol{F}$? Suppose the Gödel number of $\boldsymbol{F}$ is k and **diag**(x) represents the diagonal function in Q. What is **diag**(k)? State the diagonal lemma and give an outline of how the lemma is used to obtain Tarski's result that the set of Gödel numbers of the sentences true in Q is not definable in Q.

4. Let the relation **proof** be specified such that:

proof = $\{< j, k > : j$ is the Gödel number of a proof of the sentence with Gödel number $k\}$.

proof is a recursive relation and hence is definable in Q. Let the (complex) formula $\boldsymbol{Pr}$ (x,y) define the relation **proof** in $\boldsymbol{Q}$.

(i) Using $\boldsymbol{Pr}$ (x,y), specify the 1-place 'proof predicate' $\boldsymbol{Prov}(y)$.

(ii) Explain why it follows that for all sentences $\boldsymbol{A}$ in the language of Q

if $\vdash_Q \boldsymbol{A}$, then $\vdash_Q \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner)$, and if $\vdash_Q \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner)$, then $\vdash_Q \boldsymbol{A}$.

(iii) Assuming the diagonal lemma, specify the Gödel Sentence $\boldsymbol{S}$

(iv) Using $\boldsymbol{S}$ and the information in (ii) above, show that if Q is consistent then it is incomplete.