

# Logic, Computability and Incompleteness

First-order Logic Revisited

# What is Logic?

A standard characterization:

Logic is the ‘science’ of valid arguments.

Modern ‘symbolic’ or formal logic is the *mathematical theory* of valid arguments

What is an argument?

Intuitively, an argument can be thought of as an inference, or piece of reasoning,

where certain statements are meant to support or establish a conclusion.

# What is an Argument?

More precisely,

an argument is a finite set of (declarative) sentences,  
where one sentence is singled out as the **conclusion**  
and the other sentences are the *premises*.

**Standard Argument Form:**

	<b>Premise 1</b>
	<b>Premise 2</b>
	...
	<b>Premise <math>n</math></b>
<i>Therefore,</i>	<hr/>
	<b>Conclusion</b>

# What is Validity?

An argument is valid iff it is not possible for *all* the premises to be **true** and the conclusion **false**.

Alternatively, *if* all the premises *were* **true**,  
*then* the conclusion *would have to be* **true** as well.

Some arguments are valid and some are not:

If it is snowing, then it is cold outside.

It is snowing.

**Therefore**, it is cold outside.

If the earth is round then the sky is blue.

The sky is blue.

**Therefore** the earth is round.

# Some Examples

All politicians are human.

Some humans are wise.

**Therefore**, some politicians are wise.

If the sky is blue then the earth is flat.

The earth is not flat.

**Therefore**, the sky is not blue.

All politicians are corrupt.

Arnold is corrupt.

**Therefore**, Arnold is a politician.

Either today is Wednesday or pigs can fly.

Pigs can't fly.

**Therefore**, today is Wednesday.

# Formal Logic

As rational beings, we have *intuitions* about which arguments are valid and which are not.

Modern logic provides a **mathematical theory** of validity whereby it is possible to **prove** that an argument is valid.

Although logic began as a branch of Philosophy and has been studied since ancient times, it underwent dramatic mathematical development in the 19<sup>th</sup> and 20<sup>th</sup> centuries.

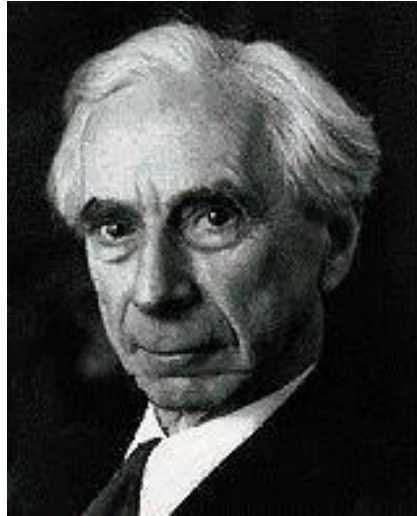
The biggest advance in logic for 2000 years was due to Gottlob Frege, who developed a system of quantifiers and variables to capture the logic of **generality** first explored by Aristotle.

Frege's system is now known as **First-order Logic**.

## Some Famous Logicians



Gottlob  
Frege



Bertrand  
Russell



Kurt  
Gödel



Alfred  
Tarski

# Natural vs Artificial Languages

The *medium* of logic is language.

When studying the grammar of a natural language such as English, we try to devise rules that accurately characterize a *pre-existing* phenomenon.

In contrast, artificial (or formal) languages, of the kind used in logic (and computer science), are **defined** by the grammatical rules we give.

Logic requires a precisely defined artificial **object language** in which formal arguments can be expressed and analyzed.

This step of ‘idealization’ is necessary to obtain mathematically rigorous results.



# First-order Language

- **Generic First-order Syntax:**
- **Vocabulary:**
- Logical symbols: constants  $\neg \vee \wedge \rightarrow \leftrightarrow \exists \forall =$   
variables  $x, y, z, \dots$  [and  $), ($  for punctuation]
- Non-logical symbols:  
individual constants, function symbols  
sentence letters, predicate letters
- A **language**  $L$  possesses a denumerable supply of non-logical symbols (and where each of the 4 categories above possess denumerably many elements).
- These **combine** with the logical symbols, according to the **Formation Rules**, to yield **terms** and **formulas** of the language:

# Generic First-order Syntax

- **Formation Rules:**
- Definition of Terms and Formulas

Terms of  $L$

(1) **Atomic terms**

(i) all individual constants of  $L$  are terms of  $L$

(ii) all variables of  $L$  are terms of  $L$

(2) **Compound terms**

if  $f^n$  is an  $n$ -place function symbol and  $t_1, \dots, t_n$  are terms, then  $f^n(t_1, \dots, t_n)$  is a term of  $L$

- **Nothing else** is a term of  $L$ !

# Generic First-order Syntax

## Formulas of $L$

### (1) Atomic formulas:

- (i) all sentence letters of  $L$  are formulas of  $L$
- (ii) if  $P^n$  is an  $n$ -place predicate letter and  $t_1, \dots, t_n$  are terms, then  $P^n t_1, \dots, t_n$  is a formula of  $L$ .

### (2) Compound formulas

- (i) if  $\Phi$  is a formula of  $L$ , then  $\neg \Phi$  is a formula of  $L$
- (ii) if  $\Phi, \Psi$  are formulas of  $L$ , then  $(\Phi \wedge \Psi)$ ,  $(\Phi \vee \Psi)$ ,  $(\Phi \rightarrow \Psi)$ , and  $(\Phi \leftrightarrow \Psi)$  are formulas of  $L$ .
- (iii) if  $\Phi$  is a formula of  $L$  and  $\mathbf{v}$  is a variable, then  $\forall \mathbf{v} \Phi$  and  $\exists \mathbf{v} \Phi$  are formulas of  $L$ .

# Generic First-order Syntax

- **Nothing else** is a formula of  $L$ !
- Note that there are an infinite number of formulas, but **no** formula is infinitely long.
- Since formulas are **f**inite sequences of symbols from a denumerable set,  
a *language*  $L$  can have only denumerably many formulas.

# Generic First-order Semantics

- **Generic First-order Semantics**
- An **interpretation**  $\mathcal{I}$  (or structure or model) of the language  $L$  is a way of “giving meaning” to the symbols of  $L$ .
- In particular, an interpretation  $\mathcal{I}$  of  $L$  specifies the following:
  - (i) a **non-empty set**  $D$  (the domain or universe of discourse).
  - (ii) for each individual constant  $c$ ,  
 $\mathcal{I}(c)$  is an **object**  $e \in D$ .
  - (iii) for each  $n$ -ary function symbol  $f^n$ ,  
 $\mathcal{I}(f^n)$  is an  **$n$ -ary function**  $F^n: D^n \rightarrow D$   
(where  $D^n$  is the  $n^{\text{th}}$  Cartesian product of  $D$ )

# Generic First-order Semantics

(iv) for each sentence letter  $S$

$\mathcal{I}(S)$  is a truth value, either **0** or **1**

(v) we treat '=' as a privileged 2-place predicate symbol,

where  $\mathcal{I}(=)$  is the set of all pairs  $\langle e, e \rangle$

such that  $e \in D$ .

(vi) for each  $n$ -ary predicate letter  $P^n$  [other than '=']

$\mathcal{I}(P^n)$  is a set of ordered  $n$ -tuples

such that  $\mathcal{I}(P^n) \subseteq D^n$ .

# Generic First-order Semantics

The notation used in clause (iii), where  $\mathcal{I}(f^n)$  is  $F^n: D^n \bullet \rightarrow D$ , connotes the fact that the interpretation of a function symbol  $f^n$  is an  $n$ -ary **function**

mapping  $n$ -tuples of elements of  $D$  to elements of  $D$

while (vi) indicates that  $\mathcal{I}(P^n)$  is an  $n$ -ary **relation**.

In general,  $n$ -place functions are equivalent to a (proper) subset of the set of  $(n+1)$ -place relations:

- So  $\mathcal{I}(f^n) \subset D^{n+1}$ , with the constraint that  
for every  $n$ -tuple  $\langle d_1, \dots, d_n \rangle \in D^n$ ,  
there exists **exactly one**  $d_{n+1} \in D$  such that  
 $\langle d_1, \dots, d_n, d_{n+1} \rangle \in \mathcal{I}(f^n)$ .

# Example

- Consider the following very simple (fragment of an) interpretation  $\mathcal{I}$  as an illustration:
- Let the domain  $D$  of  $\mathcal{I}$  be the 2-member set  $\{\text{Jack}, \text{Jill}\}$ .

So  $\mathcal{I}(=)$  is the set  $\{ \langle \text{Jack}, \text{Jack} \rangle, \langle \text{Jill}, \text{Jill} \rangle \}$

- Consider the first two individual constants  $c_1$  and  $c_2$

Let  $\mathcal{I}(c_1) = \text{Jack}$  and  $\mathcal{I}(c_2) = \text{Jill}$

- Consider the 1-place predicate symbol  $P^1$

The definition requires that  $\mathcal{I}(P^1) \subseteq D^1$

$D^1 = \{\text{Jack}, \text{Jill}\}$  (which is equivalent to  $\{ \langle \text{Jack} \rangle, \langle \text{Jill} \rangle \}$ )

So let  $\mathcal{I}(P^1) = \{\text{Jack}\}$



# Example

- Consider the 2-place predicate symbol  $L^2$

The definition requires that  $\mathcal{I}(L^2) \subseteq D^2$

$$D^2 = \{ \langle \text{Jack}, \text{Jack} \rangle, \langle \text{Jack}, \text{Jill} \rangle, \langle \text{Jill}, \text{Jack} \rangle, \langle \text{Jill}, \text{Jill} \rangle \}$$

$$\text{So let } \mathcal{I}(L^2) = \{ \langle \text{Jack}, \text{Jack} \rangle, \langle \text{Jack}, \text{Jill} \rangle \}$$

- Consider the 1-place function symbol  $f^1$

The definition requires that  $\mathcal{I}(f^1)$  is a 1-place function

$$F^1: D^1 \rightarrow D.$$

$$\text{As before, } D^1 = \{ \text{Jack}, \text{Jill} \}$$

$$\text{So let } \mathcal{I}(f^1) = \{ \langle \text{Jack}, \text{Jack} \rangle, \langle \text{Jill}, \text{Jack} \rangle \}$$

- Note that the interpreted 2-place predicate  $L^2$  cannot be used to define a corresponding 1-place function (why?).

# Generic First-order Semantics

- In accord with clauses (ii)-(vi) above, the non-logical symbols of  $L$  (plus '=' ) are given some fixed interpretation by  $\mathcal{I}$ .
- This then fixes the **truth value** of every **sentence** (i.e. **closed** formula – contains no *free variables*)  $\Phi$  of  $L$ , **relative** to the interpretation  $\mathcal{I}$ .
- The formation rules entail that all sentences of  $L$  are either **atomic** or have one of the following **7 forms** (as determined by the main 'connective'):

$\neg \Phi$ ,  $(\Phi \wedge \Psi)$ ,  $(\Phi \vee \Psi)$ ,  $(\Phi \rightarrow \Psi)$ ,  $(\Phi \leftrightarrow \Psi)$   $\forall \mathbf{v}\Phi$ ,  $\exists \mathbf{v}\Phi$

So the following **Rules of Truth** give the exhaustive procedures for computing the **truth value** (either **1** for **True** or **0** for **False**) for every sentence of  $L$  **relative** to a given interpretation  $\mathcal{I}$ :

# Truth in an Interpretation

(I) If  $\Phi$  is **atomic**, then it's either a sentence letter  $S$  or has the form  $P^n t_1, \dots, t_n$

(i) If  $\Phi$  is a sentence letter  $S$ , then the truth value of  $\Phi$  relative to  $\mathcal{J}$ , written  $\mathcal{J}(\Phi)$  is simply  $\mathcal{J}(S)$ .

(ii) If  $\Phi$  has the form  $P^n t_1, \dots, t_n$ , then, because  $\Phi$  is **closed**, the terms  $t_1, \dots, t_n$  must also be **closed**, and

- $\mathcal{J}(\Phi) = \mathbf{1}$  iff  $\langle \mathcal{J}(t_1), \dots, \mathcal{J}(t_n) \rangle \in \mathcal{J}(P^n)$  (and  $\mathbf{0}$  otherwise), where, for each term  $t_i$  in the series  $t_1, \dots, t_n$ , if  $t_i$  is a constant  $\mathbf{c}$  then  $\mathcal{J}(t_i) = \mathcal{J}(\mathbf{c})$ ;

otherwise  $t_i$  is a  $k$ -place function term  $f^k(t_1, \dots, t_k)$  applied to closed terms  $t_1, \dots, t_k$  and  $\mathcal{J}(t_i) = \mathcal{J}(f^k)(\mathcal{J}(t_1), \dots, \mathcal{J}(t_k))$

# Truth in an Interpretation

(II) For **compound** formulas:

(1)  $\mathcal{J}(\neg \Phi) = 1$  iff  $\mathcal{J}(\Phi) = 0$  (and 0 otherwise).

(2)  $\mathcal{J}(\Phi \wedge \Psi) = 1$  iff  $\mathcal{J}(\Phi) = 1$  and  $\mathcal{J}(\Psi) = 1$  (and 0 otherwise).

(3)  $\mathcal{J}(\Phi \vee \Psi) = 1$  iff  $\mathcal{J}(\Phi) = 1$  or  $\mathcal{J}(\Psi) = 1$  (and 0 otherwise).

(4)  $\mathcal{J}(\Phi \rightarrow \Psi) = 1$  iff  $\mathcal{J}(\Phi) = 0$  or  $\mathcal{J}(\Psi) = 1$  (and 0 otherwise).

(5)  $\mathcal{J}(\Phi \leftrightarrow \Psi) = 1$  iff  $\mathcal{J}(\Phi) = \mathcal{J}(\Psi)$  (and 0 otherwise).

# Truth Table Format

$\Phi$	$\neg \Phi$
--------	-------------

1	0
---	---

0	1
---	---

$\Phi$	$\Psi$	$(\Phi \wedge \Psi)$
--------	--------	----------------------

1	1	1
---	---	---

1	0	0
---	---	---

0	1	0
---	---	---

0	0	0
---	---	---

$\Phi$	$\Psi$	$(\Phi \vee \Psi)$
--------	--------	--------------------

1	1	1
---	---	---

1	0	1
---	---	---

0	1	1
---	---	---

0	0	0
---	---	---

$\Phi$	$\Psi$	$(\Phi \rightarrow \Psi)$
--------	--------	---------------------------

1	1	1
---	---	---

1	0	0
---	---	---

0	1	1
---	---	---

0	0	1
---	---	---

$\Phi$	$\Psi$	$(\Phi \leftrightarrow \Psi)$
--------	--------	-------------------------------

1	1	1
---	---	---

1	0	0
---	---	---

0	1	0
---	---	---

0	0	1
---	---	---

# Truth in an Interpretation

(6)  $\mathcal{J}(\forall \mathbf{v}\Phi) = 1$  iff  $\mathcal{J}_e^a(\Phi\mathbf{v}/a) = 1$  for every  $e \in D$ ,  
where  $a$  is a new individual constant,  $\mathcal{J}_e^a$  is the  
interpretation exactly like  $\mathcal{J}$  except that  $\mathcal{J}_e^a(a) = e$ ,  
and  $\Phi\mathbf{v}/a$  is the result of substituting  $a$  for every free  
occurrence of  $\mathbf{v}$  in  $\Phi$  (and 0 otherwise).

(7)  $\mathcal{J}(\exists \mathbf{v}\Phi) = 1$  iff  $\mathcal{J}_e^a(\Phi\mathbf{v}/a) = 1$  for some  $e \in D$ ,  
again where  $a$  is a new individual constant,  $\mathcal{J}_e^a$  is the  
interpretation exactly like  $\mathcal{J}$  except that  $\mathcal{J}_e^a(a) = e$ ,  
and  $\Phi\mathbf{v}/a$  is the result of substituting  $a$  for every free  
occurrence of  $\mathbf{v}$  in  $\Phi$  (and 0 otherwise).

# Example

- Consider the previous interpretation  $\mathcal{I}$ , where the domain  $D$  is the 2-member set  $\{\text{Jack}, \text{Jill}\}$ ,

$$\mathcal{I}(c_1) = \text{Jack} \text{ and } \mathcal{I}(c_2) = \text{Jill}, \quad \mathcal{I}(P^1) = \{\text{Jack}\}$$

$$\mathcal{I}(L^2) = \{\langle \text{Jack}, \text{Jack} \rangle, \langle \text{Jack}, \text{Jill} \rangle\}$$

$$\mathcal{I}(f^1) = \{\langle \text{Jack}, \text{Jack} \rangle, \langle \text{Jill}, \text{Jack} \rangle\}$$

- Now consider the **sentence**  $P^1c_1$ .

$$\mathcal{I}(c_1) = \text{Jack}, \quad \text{and} \quad \text{Jack} \in \{\text{Jack}\}$$

$$\text{Hence } \mathcal{I}(c_1) \in \mathcal{I}(P^1), \quad \text{so } \mathcal{I}(P^1c_1) = \mathbf{1}$$

- Consider the **sentence**  $P^1c_2$ .

$$\mathcal{I}(c_2) = \text{Jill}, \quad \text{and } \text{Jill} \notin \{\text{Jack}\}$$

$$\text{Hence } \mathcal{I}(c_2) \notin \mathcal{I}(P^1), \quad \text{so } \mathcal{I}(P^1c_2) = \mathbf{0}$$

# Example

- Consider the **sentence**  $P^1 f^1 (c_2)$

$$\mathcal{J}(f^1 (c_2)) = \mathcal{J}(f^1) (\mathcal{J}(c_2)),$$

where  $(\mathcal{J}(c_2)) = \text{Jill}$  and  $\mathcal{J}(f^1) (\text{Jill}) = \text{Jack}$

As before,  $\text{Jack} \in \{\text{Jack}\}$ , hence  $\mathcal{J}(f^1 (c_2)) \in \mathcal{J}(P^1)$ ,

so  $\mathcal{J}(P^1 f^1 (c_2)) = \mathbf{1}$

- Consider the **sentence**  $L^2 c_2 c_1$

$$\mathcal{J}(c_2) = \text{Jill}, \mathcal{J}(c_1) = \text{Jack}$$

and  $\langle \text{Jill}, \text{Jack} \rangle \notin \{\langle \text{Jack}, \text{Jack} \rangle, \langle \text{Jack}, \text{Jill} \rangle\}$

Hence  $\langle \mathcal{J}(c_2), \mathcal{J}(c_1) \rangle \notin \mathcal{J}(L^2)$

so  $\mathcal{J}(L^2 c_2 c_1) = \mathbf{0}$



# Example

- Consider the sentence  $\forall x P^1x$

$$\mathcal{I}(\forall x P^1x) = ?$$

$$\text{i) } \mathcal{I}_{Jack}^a(P^1a) = ?$$

$$\mathcal{I}_{Jack}^a(a) = Jack$$

and  $Jack \in \{Jack\}$ , so  $\mathcal{I}_{Jack}^a(a) \in \mathcal{I}(P^1)$ ,

$$\text{so } \mathcal{I}(P^1a) = 1$$

$$\text{ii) } \mathcal{I}_{Jill}^a(P^1a) = ?$$

$$\mathcal{I}_{Jill}^a(a) = Jill$$

and  $Jill \notin \{Jack\}$ , so  $\mathcal{I}_{Jill}^a(a) \notin \mathcal{I}(P^1)$ ,

$$\text{so } \mathcal{I}_{Jill}^a(P^1a) = 0$$

$$\text{Hence } \mathcal{I}(\forall x P^1x) = 0$$

# Example

- Consider the sentence  $\exists x P^1x$

$$\mathcal{I}(\exists x P^1x) = ?$$

$$\text{i) } \mathcal{I}^a_{\text{Jack}}(P^1a) = ?$$

$$\mathcal{I}^a_{\text{Jack}}(a) = \text{Jack}$$

$$\text{and } \text{Jack} \in \{\text{Jack}\}, \text{ so } \mathcal{I}^a_{\text{Jack}}(a) \in \mathcal{I}(P^1),$$

$$\text{so } \mathcal{I}(P^1a) = 1$$

$$\text{Hence } \mathcal{I}(\exists x P^1x) = 1$$

- Consider the sentence  $\forall x \exists y L^2xy$
- Consider the sentence  $\exists x \forall y L^2xy$

# Truth in an Interpretation

- The ‘Mates Quantification’ scheme in clauses (6) and (7) uses substitution to attain the same semantical results as *variable interpretation sequences*.

On the Mates approach we don’t need to assign values to variables and we only ever need to consider the truth values of **closed formulas**.

- For the interpretation of  $n$ -ary predicate letters  $P^n$ , B&J explicitly assign a **characteristic function**, say  $\mathcal{C}^n$ , of  $n$ -tuples of elements of the domain.

Thus  $\mathcal{J}(P^n) = \mathcal{C}^n$  and  $\mathcal{J}(P^n t_1, \dots, t_n) = \mathcal{C}^n(\mathcal{J}(t_1), \dots, \mathcal{J}(t_n))$ .

This is equivalent to assigning a set of  $n$ -tuples, as above.

# Some Standard Model-Theoretic Notions

- Some standard **model-theoretic notions**:

(i)  $\mathcal{I}$  **satisfies** (or is a model of)  $\Phi$  **iff**  $\mathcal{I}(\Phi) = 1$

(ii)  $\Phi$  is **satisfiable** (or consistent) **iff**  $\mathcal{I}(\Phi) = 1$  **for some** interpretation  $\mathcal{I}$

(iii)  $\Phi$  is **valid** (or a logical truth) **iff**  $\mathcal{I}(\Phi) = 1$  **for every** interpretation  $\mathcal{I}$ . In this case we write  $\models \Phi$

(iv) **logical implication**:  $\Phi \models \Psi$  **iff** **for every** interpretation  $\mathcal{I}$  such that  $\mathcal{I}(\Phi) = 1$ , it's the case that  $\mathcal{I}(\Psi) = 1$

in other words, **for all** interpretations  $\mathcal{I}$ ,  $\mathcal{I}(\Phi) \leq \mathcal{I}(\Psi)$

Clearly  $\Phi \models \Psi$  **iff**  $\models (\Phi \rightarrow \Psi)$ , so logical implication can be expressed in terms of the validity of the material conditional.

# Some Standard Model-Theoretic Notions

(v) **logical equivalence**:  $\Phi \equiv \Psi$  iff for all interpretations  $\mathcal{J}$ ,  $\mathcal{J}(\Phi) = \mathcal{J}(\Psi)$ .

- Clearly  $\Phi \equiv \Psi$  iff  $\Phi \models \Psi$  and  $\Psi \models \Phi$ , iff  $\models (\Phi \leftrightarrow \Psi)$
- Familiar generalization of logical implication to multiple premises:  $\Phi_1, \dots, \Phi_n \models \Psi$  iff for all interpretations  $\mathcal{J}$ , if  $\mathcal{J}$  satisfies each of  $\Phi_1, \dots, \Phi_n$  then  $\mathcal{J}$  satisfies  $\Psi$ ,

$$\text{iff } \models ((\Phi_1 \wedge \dots \wedge \Phi_n) \rightarrow \Psi)$$

- For a **set of sentences**  $\Gamma$ ,  $\mathcal{J}$  satisfies  $\Gamma$  iff  $\mathcal{J}(\Theta) = 1$  for every  $\Theta \in \Gamma$ .  $\mathcal{J}$  is then a **model** of  $\Gamma$ .

The logical consequence relation extended to **sets of sentences**:

# Some Standard Model-Theoretic Notions

- $\Gamma \models \Psi$  iff for every interpretation  $\mathcal{I}$ , if  $\mathcal{I}$  satisfies  $\Gamma$  then  $\mathcal{I}$  satisfies  $\Psi$
- It is in the above format that we will think of valid arguments, recasting the ‘standard argument form’

**Premise 1**

**Premise 2**

...

**Premise  $n$**

*Therefore,*

**Conclusion**

so that  $\Gamma$  is the set of premises and  $\Psi$  is the conclusion.

A **sentence**  $\Phi$  is valid iff  $\emptyset \models \Phi$

# Some Standard Model-Theoretic Notions

- Essential connection between **implication** and **satisfiability**:  
 $\Gamma \models \Psi$  iff the set  $\Gamma \cup \{\neg \Psi\}$  is **unsatisfiable**.

If the set  $\Gamma \cup \{\neg \Psi\}$  did have a model then it would be a counterexample to the claim  $\Gamma \models \Psi$ .

- **Formal Theories:**

A Formal Theory  $T$  is a set of sentences (in some formal language  $L$ ) which is closed under the relation of logical consequence.

So for all sentences  $\Phi$  of  $L$ , if  $T \models \Phi$  then  $\Phi \in T$

# Prenex Normal Form

- **Prenex Form:**

$Q_1 \mathbf{v}_1, \dots, Q_n \mathbf{v}_n \Phi$  where  $Q_i = \forall$  or  $\exists$  and  $\Phi$  is **quantifier free**

Theorem: for every formula  $\Psi$  there is some formula  $\Theta$  such that  $\Theta$  is in prenex form and  $\Psi \equiv \Theta$ .

Proof: give rules for successively moving quantifiers to the left which preserve logical equivalence:

(i) quantifier duality:  $\neg Q \mathbf{v} \Phi \equiv Q' \mathbf{v} \neg \Phi$

where  $Q = \forall$  or  $\exists$ , and  $\forall' = \exists$ ,  $\exists' = \forall$

(ii) can directly pull quantifiers out from conjunctions, disjunctions, and the consequents of conditionals (provided  $\mathbf{v}$  does not occur free in  $\Psi$ ):



# Prenex Normal Form

(1)  $(Q_{\mathbf{v}} \Phi \wedge \Psi) \equiv Q_{\mathbf{v}} (\Phi \wedge \Psi)$

$$(\Psi \wedge Q_{\mathbf{v}} \Phi) \equiv Q_{\mathbf{v}} (\Psi \wedge \Phi)$$

(2)  $(Q_{\mathbf{v}} \Phi \vee \Psi) \equiv Q_{\mathbf{v}} (\Phi \vee \Psi)$

$$(\Psi \vee Q_{\mathbf{v}} \Phi) \equiv Q_{\mathbf{v}} (\Psi \vee \Phi)$$

(3)  $(\Psi \rightarrow Q_{\mathbf{v}} \Phi) \equiv Q_{\mathbf{v}} (\Psi \rightarrow \Phi)$

(4) But must **reverse** quantifier in the antecedent

$$(Q_{\mathbf{v}} \Phi \rightarrow \Psi) \equiv Q'_{\mathbf{v}} (\Phi \rightarrow \Psi)$$

Because  $(Q_{\mathbf{v}} \Phi \rightarrow \Psi) \equiv (\neg Q_{\mathbf{v}} \Phi \vee \Psi) \equiv (Q'_{\mathbf{v}} \neg \Phi \vee \Psi) \equiv$

$$Q'_{\mathbf{v}} (\neg \Phi \vee \Psi) \equiv Q'_{\mathbf{v}} (\Phi \rightarrow \Psi)$$