Logic, Computability and Incompleteness

Completeness, Compactness and Löwenheim-Skolem

Soundness and Completeness

Soundness and **Completeness** are two basic <u>metalogical</u> properties of logical systems, intimately relating the semantical notion of **validity** and the syntactical notion of **provability**.

Validity: $\Phi_1,\ldots,\Phi_n \vDash \Psi$ iff for all interpretations $\mathcal{I}$, if $\mathcal{I}$ satisfies each of $\Phi_1,\ldots,\Phi_n$ then $\mathcal{I}$ satisfies Ψ,

iff $\vDash ((\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \Psi)$

Provability: $\Phi_1,\ldots,\Phi_n \vdash \Psi$ iff there is a proof or formal, rule governed derivation of Ψ from sentences $\Phi_1,\ldots,\Phi_n$

iff $\vdash ((\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \Psi)$

Soundness and Completeness

Soundness is a <u>correctness</u> property of formal proof systems and establishes that **only** valid arguments (or sentences) are provable: if $\Phi_1,\ldots,\Phi_n \vdash \Psi$ then $\Phi_1,\ldots,\Phi_n \vDash \Psi$

Completeness is an <u>adequacy</u> property of formal proof systems and establishes that **all** valid arguments (or sentences) are provable: if $\Phi_1,\ldots,\Phi_n \vDash \Psi$ then $\Phi_1,\ldots,\Phi_n \vdash \Psi$

If a system of Logic is both <u>sound</u> and <u>complete</u>, then the semantical and proof-theoretic concepts coincide exactly and

$$\Phi_1,\ldots,\Phi_n \vdash \Psi \text{ if and only if } \Phi_1,\ldots,\Phi_n \vDash \Psi$$

In which case we can treat them as more or less interchangeable.

A Formal Proof System

- We will now examine a method of <u>formal</u> <u>proof</u> which will constitute a mechanical **<u>positive</u>** test for FOL validity, where this method is provably both sound and complete.

- The mechanical test for validity is designed as a positive test for unsatisfiability of sets of sentences Δ.

- The test will take the form of a systematic search for a refutation of Δ, such that if there is a refutation of Δ, then Δ is unstatisfiable.

  This will correspond to the <u>soundness</u> of the method.

- Conversely, if Δ is unstatisfiable then there is a refutation of Δ.

  This will correspond to the <u>completeness</u> of the method.

A Formal Proof System

- Let Δ be a set of sentences in prenex normal form, from which all vacuous quantifiers have been removed.

- A **refutation** of Δ is a <u>derivation</u> $\mathcal{D}$ from Δ in which

  some **finite set** of **quantifier-free sentences** in the derivation is **unsatisfiable**.

- In turn, a <u>derivation</u> $\mathcal{D}$ from Δ is a list of sentences (finite or denumerable), in which every entry is either a <u>member</u> of Δ or is obtainable from a previous entry in the list by one of the following two <u>inference</u> <u>rules</u>:

A Formal Proof System

UI

$\vdots$

(m) $\forall \mathbf{v} \Phi$

$\vdots$

(n) $\Phi \mathbf{v}/\boldsymbol{t}$ (m) annotation

where $\boldsymbol{t}$ may be any (closed) term

EI

$\vdots$

(m) $\exists \mathbf{v} \Phi$

$\vdots$

(n) $\Phi \mathbf{v}/\boldsymbol{t}$ (m) annotation

where $\boldsymbol{t}$ is a <u>name</u> which doesn't occur in Δ

or in any other line earlier than n.

A Formal Proof System

The basic idea is that if $\mathcal{D}$ is a derivation from Δ in which a finite set of quantifier-free sentences in $\mathcal{D}$ has no model

(i.e. Δ is a refutation), then Δ has no model.

- In other words…

Soundness

- **Soundness Theorem**: if there is a refutation of Δ, then Δ is unsatisfiable (where Δ is a set of sentences in prenex normal form, from which all vacuous quantifiers have been removed).

- Strong Soundness Theorem: if $\mathcal{I}$ is a model of Δ and $\mathcal{D}$ is a derivation from Δ, then the set of all sentences in $\mathcal{D}$ has a model $\mathcal{L}$, in which $\mathcal{L}$ differs from $\mathcal{I}$ (at most) in what it assigns to names and function symbols which occur in sentences of $\mathcal{D}$ but not in Δ.

- Strong Soundness Theorem implies the (normal) Soundness Theorem, since if Δ were satisfiable it would have a model $\mathcal{I}$, and therefore so would all sentences in $\mathcal{D}$, in which case there could be no refutation.

Proof of Strong Soundness Theorem

- **Proof** of Strong Soundness Theorem: by recursive construction of a model $\mathcal{L}$ of $\mathcal{D}$ on the basis of $\mathcal{I}$.

  Let $\Delta_0 = \Delta$ $\mathcal{I}_0 = \mathcal{I}$

  $\Delta_n = \Delta \cup \{S_1,\ldots,S_n\}$, where $S_1,\ldots,S_n$ are the first n sentences in $\mathcal{D}$.

  Define a model $\mathcal{I}_{n+1}$ of Δ_{n+1}, where induction step is based on the annotation A_{n+1} used in the derivation to get S_{n+1}.

  Four cases:

  i) A_{n+1} is 'Δ', in which case $\mathcal{I}_{n+1} = \mathcal{I}_n$.

  ii) A_{n+1} is 'UI' and the instantial term t_{n+1} contains only names and function terms already occurring in Δ_n. Then $\mathcal{I}_{n+1} = \mathcal{I}_n$.

Proof of Strong Soundness Theorem

iii) A_{n+1} is 'UI' and the instantial term t_{n+1} contains names or function terms *not* occurring in Δ_n. Take some $d \in D$ of $\mathcal{I}$, and in every case let $\mathcal{I}_{n+1}$ assign d as the denotation of new names, and all new functions are constant functions with d as value. $\mathcal{I}_{n+1}$ is a model of Δ_n by continuity, and a model of Δ_{n+1} since UI is truth preserving.

iv) A_{n+1} is 'EI', in which case the instantial term t_{n+1} is new. Since the premise of this rule is in Δ_n, the premise must be true in $\mathcal{I}_n$. So there must be at least one element $e \in D$ such that $\mathcal{I}_e^{tn+1}$ is a model of S_{n+1} and also of Δ_n. So let $\mathcal{I}_{n+1} = \mathcal{I}_e^{tn+1}$ for some such e.

Proof of Strong Soundness Theorem

- Now define the model $\mathcal{L}$ to be just like $\mathcal{I}$, except that for each function symbol or name appearing in $\mathcal{D}$ but not in Δ, $\mathcal{L}$ assigns whatever $\mathcal{I}_n$ assigns it, where S_n is the first entry in $\mathcal{D}$ in which the new term occurs.

- Hence if Δ had a model $\mathcal{I}$, then all the sentences in $\mathcal{D}$ would have a model $\mathcal{L}$, in which case $\mathcal{D}$ could not contain a refutation (**Strong Soundness Theorem**) □

- Therefore if some derivation $\mathcal{D}$ from Δ **is** a refutation, then Δ has no model $\mathcal{I}$ and is unsatisfiable (**Soundness Theorem**) □

Completeness Theorem

- **Completeness** of FOL will be the culmination of our **positive** metatheoretical results. First proved by Kurt Gödel in 1930.

- **Completeness Theorem**: if a set of sentences Δ is unsatisfiable, then it has a refutation.

Completeness Proof: Canonical Derivation

- **<u>Proof</u>** : first need to define a <u>canonical derivation</u> from Δ, such that, if Δ is unsatisfiable, then any <u>canonical derivation</u> from Δ will be a refutation.

- <u>Definition</u>: $\mathcal{D}$ is a <u>canonical derivation</u> from Δ iff it satisfies the following 5 conditions:

  i) every sentence $\Phi \in \Delta$ occurs in $\mathcal{D}$.

  ii) if $\exists\mathbf{v}\Phi \in \mathcal{D}$, then for some term t, $\Phi\mathbf{v}/t \in \mathcal{D}$.

  iii) if $\forall\mathbf{v}\Phi \in \mathcal{D}$, then for some term t, $\Phi\mathbf{v}/t \in \mathcal{D}$.

  iv) if $\forall\mathbf{v}\Phi \in \mathcal{D}$, then for every term t that can be constructed from names and function symbols occurring in $\mathcal{D}$, $\Phi\mathbf{v}/t \in \mathcal{D}$.

  v) all function symbols occurring in $\mathcal{D}$ appear in Δ.

Completeness Proof: Canonical Derivation

Program for constructing a canonical derivation from Δ:

Let S_1, S_2, S_3, … be an enumeration of the sentences in Δ.

Stage 1(a): enter S_1 as the first line in $\mathcal{D}$.

1(b): add as many entries to $\mathcal{D}$ as possible using EI with restrictions (to be stated momentarily)

1(c): add as many entries to $\mathcal{D}$ as possible using UI

- restrictions: no sentence is the premise of more than one application of EI, no sentence occurs twice, and each instantial term has fewer than N (= current stage number) occurrences of function symbols, and is formed from names and function symbols already in $\mathcal{D}$.

Stage 2(a): enter S_2 as the nth line in $\mathcal{D}$ and repeat…

Completeness Proof: Lemma II and Matching

- Definition of Matching: suppose Γ is a set of quantifier-free sentences.

  An interpretation $\mathcal{I}$ matches Γ iff $\mathcal{I} \vDash \Gamma$ and there are no elements in the domain of $\mathcal{I}$ not named by terms in Γ.

- **Lemma II**. Suppose $\mathcal{D}$ is a canonical derivation from Δ, Γ is the set of all quantifier-free sentences in $\mathcal{D}$, and $\mathcal{I}$ matches Γ.

  Then $\mathcal{I}$ is a model of $\mathcal{D}$ and hence of Δ.

- **Proof** by reductio (see B&J p. 134-5)

Completeness Proof: an OK set of sentences

Now all that remains to be proved is that if every finite subset of Γ is satisfiable, then some interpretation $\mathcal{I}$ matches Γ.

- This will show that if Δ is unsatisfiable, then a canonical derivation $\mathcal{D}$ is a refutation, since it must possess a finite subset of quantifier-free sentences which is unsatisfiable.

- So this is really the <u>contraposition</u> of **completeness** −

  if $\mathcal{D}$ is <u>not</u> a refutation then Δ <u>is</u> satisfiable.

- Will prove this by constructing an $\mathcal{I}$ which must be a model of Δ if every finite subset of Γ is satisfiable.

- To do this, first need to introduce the concept of an OK set of sentences: a set of sentences Σ is OK iff

  every finite subset of Σ is satisfiable.

Completeness Proof: Lemma III

- So to prove **completeness**, only need to prove
- **Lemma III**: if Γ is an enumerable, OK set of quantifier-free sentences, then there is an $\mathcal{I}$ which matches Γ.
- **Proof:** must define such an $\mathcal{I}$ using the given information:

General procedure:

First enumerate all atomic sentences A_1, A_2,… which are either

    (i) sentence letters (propositional) occurring in Γ, or

    (ii) first-order sentences formed by filling in the argument places of the '=' sign, and predicate letters occurring in Γ, using terms appearing in Γ.

Completeness Proof: Construct an $\mathcal{I}$ which matches $\mathbf{\Gamma}$

Now define the sequence $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, … and verify that all members in the sequence are OK:

Let $\mathbf{\Gamma}_1 = \mathbf{\Gamma}$ OK by hypothesis.

Now suppose $\mathbf{\Gamma}_n$ has been defined and is OK.

Then at least one of the sets $\mathbf{\Gamma}_n \cup \{\mathbf{A}_n\}$ or $\mathbf{\Gamma}_n \cup \{\neg\mathbf{A}_n\}$ is OK.

Define $\mathbf{\Gamma}_{n+1}$ as the OK one if just one is,

and $\mathbf{\Gamma}_n \cup \{\mathbf{A}_n\}$ if both are OK.

Let $\mathbf{B}_i$ be whichever of $\mathbf{A}_i$, $\neg\mathbf{A}_i$ is in the expansion $\mathbf{\Gamma}_{i+1}$.

Completeness Proof: Construct an $\mathcal{I}$ which matches $\boldsymbol{\Gamma}$

Now, if $\mathbf{r}$, $\mathbf{s}$ are terms in $\boldsymbol{\Gamma}$, exactly one of $\mathbf{r} = \mathbf{s}$, $\neg(\mathbf{r} = \mathbf{s})$ is in the sequence of $\mathbf{B}$'s.

Definition: $\mathbf{r} \sim \mathbf{s}$ iff $\mathbf{r} = \mathbf{s}$ is one of the $\mathbf{B}$'s.

$\sim$ is an equivalence relation on the set of terms in $\boldsymbol{\Gamma}$.

Now to define $\mathcal{I}$ which matches $\boldsymbol{\Gamma}$: want $\mathcal{I}$ to assign each term t its own equivalence class $[t]$ as denotation (!)

and want $\mathcal{I}(\mathbf{B}_i) = 1$ for each i. So…

A) let the domain D of $\mathcal{I}$ be the set of all equivalence classes of terms in $\boldsymbol{\Gamma}$.

B) let $\mathcal{I}(t) = [t]$ for each individual constant t.

Completeness Proof: Construct an $\mathcal{I}$ which matches Γ

C) for each n-place function symbol f^n, let $\mathcal{I}(f^n)$ be the function g^n such that for all $[t_1], \ldots, [t_n]$, in D,

$g^n([t_1], \ldots, [t_n]) = [f^n(s_1, \ldots, s_n)]$ if there are terms $s_1, \ldots, s_n$ in $[t_1], \ldots, [t_n]$ such that $f^n(s_1, \ldots, s_n)$ is a term appearing in Γ.

Otherwise $g^n([t_1], \ldots, [t_n]) = [t]$, for any term t in Γ.

D) a sentence letter is true in $\mathcal{I}$ iff it is one of the **B**'s.

E) for each n-place predicate letter P^n occurring in Γ

$< [t_1], \ldots, [t_n] > \in \mathcal{I}(P^n)$ iff $P^n(t_1, \ldots, t_n)$ is one of the **B**'s.

It follows by induction (using B, C) that each complex term t occurring in Γ denotes its own equivalence class $[t]$.

Completeness Proof: Construct an $\mathcal{I}$ which matches $\mathbf{\Gamma}$

Also, it follows that each of the **B**'s is true in $\mathcal{I}$, since by cases (three), $\mathbf{A}_i$ is true in $\mathcal{I}$ iff $\mathbf{A}_i = \mathbf{B}_i$.

To see that $\mathcal{I}$ matches $\mathbf{\Gamma}$, first, it's clear that every object in D is named by a term of $\mathbf{\Gamma}$.

So just need to show that $\mathcal{I}$ is a model of $\mathbf{\Gamma}$:

Suppose sentence $\mathbf{S} \in \mathbf{\Gamma}$.

$\mathbf{S}$ is a truth-functional combination of some finite set $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$ of the **A**'s.

In any interpretation $\mathcal{J}$ in which all of $\mathbf{B}_1, \ldots, \mathbf{B}_k$ are true, each $\mathbf{A}_1, \ldots, \mathbf{A}_k$ has the same truth value as in $\mathcal{I}$, and hence $\mathbf{S}$ has the same value as in $\mathcal{I}$.

Completeness Proof: Construct an $\mathcal{I}$ which matches Γ

All of $\mathbf{B}_1,\ldots,\mathbf{B}_k$ are in Γ_{k+1}, as is $\mathbf{S}$, which is in Γ_1.

Thus $\{\mathbf{B}_1,\ldots,\mathbf{B}_k, \mathbf{S}\} \subseteq \Gamma_{k+1}$.

Since Γ_{k+1} is OK, this finite subset must be satisfiable,

and hence all its members true in some interpretation $\mathcal{J}$.

And since all of $\mathbf{B}_1,\ldots,\mathbf{B}_k$ and $\mathbf{S}$ are true in $\mathcal{J}$, $\mathbf{S}$ <u>is true in $\mathcal{I}$</u>.

- So, if Γ is the set of quantifier-free sentences in a canonical derivation $\mathcal{D}$ from Δ, then if $\mathcal{D}$ is <u>not</u> a refutation (i.e. Γ is OK) then $\mathcal{I}$ is a model of Δ (**Completeness Theorem**) $\square$

- Immediate consequences of Soundness and Completeness:

Compactness Theorem

- **Compactness Theorem**: A set of sentences Σ is unsatisfiable iff some <u>finite</u> subset $\Sigma_0 \subseteq \Sigma$ is unsatisfiable.

- **Proof**: if Σ is unsatisfiable, then by Completeness a canonical derivation $\mathcal{D}$ from Σ is a refutation. Let $\{A_1,\ldots, A_m\}$ be the finite set of quantifier-free sentences in $\mathcal{D}$ that is unsatisfiable. Truncate $\mathcal{D}$ at A_m to obtain $\mathcal{D}_0$, which is finite.

  Let $\{S_1,\ldots, S_n\}$ be the members of Σ occurring in $\mathcal{D}_0$, and let $\Sigma_0 = \{S_1,\ldots, S_n\}$. Σ_0 is unsatisfiable (by Soundness) $\square$

- Direct result: finite entailment:

  if $\Delta \vDash \Psi$, then a finite subset $\Delta_0 \subseteq \Delta$ is such that

  $\Delta_0 \vDash \Psi$

Löwenheim-Skolem Theorem

- **Löwenheim-Skolem Theorem**: If a set of sentences Δ has a model, then it has a model with an enumerable domain.

- **Proof**: if Δ is satisfiable, then the canonical derivation $\mathcal{D}$ from Δ is <u>not</u> a refutation (by Soundness). Hence <u>every</u> finite subset of Γ (the set of quantifier-free sentences in $\mathcal{D}$) is satisfiable (by Compactness). By Lemma III there is a model $\mathcal{I}$ that matches Γ, and by Lemma II $\mathcal{I}$ is a model of Δ.

  Since $\mathcal{I}$ matches Γ, every object in the domain of $\mathcal{I}$ is named by some term in Γ. Since Γ is enumerable, there are only enumerably many such terms.

  Therefore $\mathcal{I}$ has an enumerable domain □

Cardinality

- The Löwenheim-Skolem Theorem reveals a fundamental fact about the expressive power of sentences in FOL with respect to the cardinality of their models.

- This version is the 'downward' Löwenheim-Skolem Theorem and shows that you can't force there to be only models with a domain of cardinality greater than $\aleph_0$.

- Hence any (consistent) set of sentences (e.g. a formal theory of numbers) will be satisfied by an interpretation with a countable domain.

Cardinality

- We've seen the 'downward' Löwenheim-Skolem Theorem, showing that you can't force there to be only models with a domain of cardinality greater than $\aleph_0$.

- There is also an 'upward' version of the theorem:

  if a set Δ has an infinite model, then it has a model with uncountably many elements.

- This shows we also can't force only less than uncountable models.

- In other words, FOL cannot distinguish between different levels of infinity.

What is First-Order Logic?

- It turns out that compactness plus upward and downward Löwenheim-Skolem (L-S) metalogically capture FOL…

- **Lindströms Theorem** (1969):

  Let $\mathcal{L}$ be any 'extension' of FOL with the two properties:

  i) downward L-S

  and

  ii) either upward L-S or compactness.

  Then $\mathcal{L}$ is no 'stronger' than FOL, in the sense that <u>every</u> sentence of $\mathcal{L}$ has exactly the same models as some sentence of FOL.