Logic, Computability and Incompleteness.
Exercise Set 2: Due Thursday 5 April. Turn in during class, or to TO, ground floor DSB.

PART I

1) Consider the Turing machine quadruple $q_2 S_1 R q_3$. Adopting the conventions used in the Lecture Slides (and B&J ch 10), including the restriction that the Turing machine in question can read/write only the symbols S_0 and S_1, construct a sentence in first-order logic that formalizes this quadruple.

2) Let the configuration of a Turing machine computation at time *s* be the following:
...00$\underline{1}$011100...
2
where there are only blank squares remaining on the left and right of the tape, and where the number of the currently scanned square is *p*.

(i) Provide the formal description of time *s* in FOL.
(ii) Provide the formal description of time *s* +1, on the basis of the quadruple instruction in question 1).

3) Given the enumeration of Turing machines developed in the Lecture Slides (and B&J ch 5), determine the 3rd, 4th and 5th Turing machines in the list, give the number of the corresponding 'word', and compute the values of $\boldsymbol{u}(3)$, $\boldsymbol{u}(4)$ and $\boldsymbol{u}(5)$, where $\boldsymbol{u}$ is the antidiagonal function based on the list.

4) Induction on the structural complexity of a formula is a form of proof by mathematical induction, wherein the basis step for the minimal case is to prove that the statement or property holds for the set of atomic formulas, and the induction step is to prove that, *if* the statement or property holds for all formulas constructed from *n* or fewer applications of the formation rules for compound formulas, then it holds for $n + 1$ applications. In other words, the induction step requires proving that the result of applying a formation rule will preserve the property, if the inputs to the rule have the property.
Consider the propositional fragment $\mathbf{L}_P$ of FOL defined with sentence letters P_1, P_2,... and the formations rules: all sentence letters of $\mathbf{L}_P$ are atomic formulas of $\mathbf{L}_P$ and

(i) if $\mathbf{A}$ is a formula of $\mathbf{L}_P$, then $\neg\mathbf{A}$ is a formula of $\mathbf{L}_P$
(ii) if $\mathbf{A}$, $\mathbf{B}$ are formulas of $\mathbf{L}_P$, then $(\mathbf{A} \wedge \mathbf{B})$, $(\mathbf{A} \vee \mathbf{B})$, $(\mathbf{A} \rightarrow \mathbf{B})$, and $(\mathbf{A} \leftrightarrow \mathbf{B})$ are formulas of $\mathbf{L}_P$.

Prove by induction on complexity that if two truth value assignments $\mathcal{I}_1$ and $\mathcal{I}_2$ assign the same truth values to the sentence letters in a formula $\boldsymbol{S}$, then $\boldsymbol{S}$ has the same truth value in $\mathcal{I}_1$ and $\mathcal{I}_2$.

5) Using the derivation system developed in the lecture slides (and B&J chapter 9), give proofs of the following two claims. Be sure to specify the relevant set Δ, include annotations, and establish in each case that your derivation is a refutation.

i) $\vdash \neg\exists x Px \rightarrow \forall x(Px \rightarrow Sx)$
ii) $\exists x\forall y(x = y), P\mathrm{a} \vdash \forall x(x = \mathrm{a})$

6) Show that if a set of sentences Δ has models with arbitrarily large finite domains, then Δ has a model with an infinite domain.

PART II

7) To each of the subsets of the set of natural numbers there corresponds the distinct truth that 0 is or is not a member of that subset. Establish that there must then exist truths of the full theory of the natural numbers that are not theorems of formal arithmetic.

8) What formula in the language ***L*** of Robinson Arithmetic represents the recursive function **sum**(x,y)? Establish that this formula has the required properties. Express this formula in terms of the base level syntax of the object language (i.e. using the primitive vocabulary items displayed on B&J p. 171 and the official formation rules for FOL).

9) Using the Gödel numbering scheme from the lecture slides and B&J, determine gn[$\forall x \exists y A^2_1(x , f^2_2(x, y))$].

10) Let ***F*** be the formula $\forall y A^2_1(x , y)$. What is the diagonalization of ***F*** ? Use the derivation system from question 5) to establish that the diagonalization of ***F*** is logically equivalent to $\forall y A^2_1(\ulcorner \boldsymbol{F} \urcorner , y)$.

11) Following the proof of the diagonal lemma, construct the sentence ***G*** such that

$$\vdash_Q \boldsymbol{G} \leftrightarrow \forall y A^2_1(\ulcorner \boldsymbol{G} \urcorner , y)$$

12) Suppose that Robinson Arithmetic is augmented with a 1-place predicate symbol ***K*** intended to formalize *knowledge* of arithmetic, perhaps for the purpose of Knowledge Representation in some AI system. Suppose further that this predicate is part of the Gödel numbering scheme, and that the logic of the knowledge predicate includes the principles:
(i) $\vdash \boldsymbol{K}(\ulcorner \Phi \urcorner) \rightarrow \Phi$, since knowledge is considered to be factive, and
(ii) if $\vdash \Phi$, then $\vdash \boldsymbol{K}(\ulcorner \Phi \urcorner)$, which is intended to capture the idea that if a statement is proven then it is known. Show that the resulting formal theory of Knowledge Representation is inconsistent.