Logic, Computability and Incompleteness

Completeness, Compactness and Löwenheim-Skolem

Validity in FOL

Validity: $\Phi_1, \ldots, \Phi_n \vDash \Psi$

iff for all (FOL) interpretations $\mathcal{I}$, if $\mathcal{I}$ satisfies each of $\Phi_1, \ldots, \Phi_n$ then $\mathcal{I}$ satisfies (or is a model of) Ψ,

iff $\vDash ((\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \Psi)$.

This is an essentially **semantical** notion, and can be established by (informal) **semantical** proof.

The above rendition of validity is equivalent to the statement that

it's impossible for all the premises to be

true and the conclusion **false**.

And this impossibility can be proved using a *reductio* strategy

Example

- For example, here's a semantic proof that the argument

  $\exists x \forall y Rxy$ *therefore* $\forall x \exists y Ryx$ is **valid**.
- Suppose the argument were <u>not</u> valid.

  Then there must exist an interpretation $\mathfrak{I}$ such that $\mathfrak{I}(\exists x \forall y Rxy)$ = **true**, and $\mathfrak{I}(\forall x \exists y Ryx)$ = **false**.

  According to the definition of truth for the quantifiers,

  $\mathfrak{I}(R)$ must then be such that, given the domain D of $\mathfrak{I}$,

  there is an element $e \in D$

  such that for all elements $e' \in D$, the pair $< e, e'> \in \mathfrak{I}(R)$.

  But then, for all elements e' of D, there is an element of D, namely this same e, such that $< e, e'> \in \mathfrak{I}(R)$.

Example

Thus according to the definition of truth for the quantifiers,

it must be the case that $\mathfrak{I}(\forall x \exists y Ryx) =$ **true**,

contrary to hypothesis.

Hence it is <u>impossible</u> for there to be such a counter-model $\mathfrak{I}$, and so the argument is **valid** ■

- This is a perfectly rigorous and legitimate proof,

  but this method becomes progressively more unwieldy as arguments become more complex.

- Hence the need for a **mechanical**, **syntactical** method of

  proof that captures the underlying semantical facts.

Formal, Syntactical Proof

- There are a number of alternative Formal Proof techniques, and all of them are co-extensive

  in terms of capturing **exactly the same** set of

  underlying semantical facts.
- But whatever particular proof method is chosen for FOL,

  we want it to have the following 2 essential characteristics…

Soundness and Completeness

Soundness and **Completeness** are two basic <u>metalogical</u> properties of logical systems, intimately relating the

semantical notion of **validity** and the

syntactical notion of **provability**.

Validity: $\Phi_1,\ldots,\Phi_n \vDash \Psi$ iff **for all** <u>interpretations</u> $\mathcal{J}$, if $\mathcal{J}$ satisfies each of $\Phi_1,\ldots,\Phi_n$ then $\mathcal{J}$ satisfies Ψ,

iff $\vDash ((\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \Psi)$

Provability: $\Phi_1,\ldots,\Phi_n \vdash \Psi$ iff **there is** a <u>proof</u> or formal, rule governed <u>derivation</u> of Ψ from sentences $\Phi_1,\ldots,\Phi_n$

iff $\vdash ((\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \Psi)$

Soundness and Completeness

Soundness is a correctness property of formal proof systems and establishes that **only** valid arguments (or sentences) are provable:

$$\text{if } \Phi_1,\ldots, \Phi_n \vdash \Psi \text{ then } \Phi_1,\ldots, \Phi_n \vDash \Psi$$

Completeness is an adequacy property of formal proof systems and establishes that **all** valid arguments (or sentences) are provable:

$$\text{if } \Phi_1,\ldots, \Phi_n \vDash \Psi \text{ then } \Phi_1,\ldots, \Phi_n \vdash \Psi$$

If a system of Logic is both sound and complete, then the model-theoretic and proof-theoretic notions coalesce, and

$$\Phi_1,\ldots, \Phi_n \vdash \Psi \textbf{ if and only if } \Phi_1,\ldots, \Phi_n \vDash \Psi$$

In which case we can treat them as more or less interchangeable.

A Formal Proof System

- We will now examine a method of formal proof which will constitute a mechanical **positive** test for FOL validity, where this method is provably both sound and complete.
- The mechanical test for validity is designed as a positive test for unsatisfiability of sets of sentences Δ.
- The test will take the form of a systematic search for a refutation of Δ, such that ***if*** there is a refutation of Δ, then Δ is unstatisfiable.

  This will correspond to the soundness of the method.
- Conversely, ***if*** Δ is unstatisfiable then there is a refutation of Δ.

  This will correspond to the completeness of the method.

A Formal Proof System

- Let Δ be a set of sentences in prenex normal form, from which all vacuous quantifiers have been removed.
- A **refutation** of Δ is a derivation $\mathcal{D}$ from Δ in which

  some **finite set** of **quantifier-free sentences** in the derivation is **unsatisfiable**.
- In turn, a derivation $\mathcal{D}$ from Δ is a list of sentences (finite or denumerable), in which every entry is either a member of Δ

  or is obtainable from a previous entry in the list by one of the following two inference rules:

A Formal Proof System

UI

$\vdots$

(m) $\forall \mathbf{v} \Phi$

$\vdots$

(n) $\Phi \mathbf{v}/\boldsymbol{t}$ (m) annotation

where $\boldsymbol{t}$ may be any (closed) term

EI

$\vdots$

(m) $\exists \mathbf{v} \Phi$

$\vdots$

(n) $\Phi \mathbf{v}/\boldsymbol{t}$ (m) annotation

where $\boldsymbol{t}$ is a <u>name</u> which doesn't occur in Δ or in any other line earlier than n.

Example

- <u>Claim</u>: $\forall x\, L^2(x, f^1(x)) \vdash \forall x\, \exists y\, L^2(x,y)$

Counterexample set = $\{\ \forall x\, L^2(x, f^1(x))\ ,\ \neg\, \forall x \exists y\, L^2(x,y)\ \}$

$\Delta\ =\ \{\ \forall x\, L^2 x, f^1(x), \exists x \forall y\, \neg\, L^2(x,y)\ \}$

<u>derivation</u> $\mathcal{D}$ from Δ:

1. $\exists x \forall y\, \neg\, L^2 x,y$ Δ
2. $\forall y\, \neg\, L^2 a,y$ 1. EI a
3. $\forall x\, L^2(x, f^1(x))$ Δ
4. $L^2(a, f^1(a))$ 3. UI a
5. $\neg\, L^2(a, f^1(a))$ 2. UI $f^1(a)$

4. and 5. constitute a **finite set** of **quantifier-free sentences** that is **unsatisfiable.** Hence $\mathcal{D}$ is a **<u>refutation</u>** of Δ**.**

A Formal Proof System

The basic idea is that if $\mathcal{D}$ is a refutation of Δ,

then Δ has no model.

And if Δ is the counterexample set for some argument

$\Phi_1,\ldots, \Phi_n$ *therefore* Ψ,

then the argument is established as **valid**.

In which case $\mathcal{D}$ is a formal proof of validity, and hence

$\Phi_1,\ldots, \Phi_n \vdash \Psi$

- We will first establish the **correctness** of the formal method and then its **adequacy**….

Soundness

- **Soundness Theorem**: if there is a refutation of Δ, then Δ is unsatisfiable (where Δ is a set of sentences in prenex normal form, from which all vacuous quantifiers have been removed).
- Strong Soundness Theorem: if $\mathcal{I}$ is a model of Δ and $\mathcal{D}$ is a derivation from Δ, then the set of all sentences in $\mathcal{D}$ has a model $\mathcal{L}$, where $\mathcal{L}$ *differs* from $\mathcal{I}$ (at most)

  in what it assigns to names and function symbols which occur in sentences of $\mathcal{D}$ but not in Δ.
- Strong Soundness Theorem implies the (normal) Soundness Theorem, since if Δ were satisfiable it would have a model $\mathcal{I}$,

  and therefore so would all sentences in $\mathcal{D}$, in which case there could be no refutation.

Proof of Strong Soundness Theorem

- **Proof** of Strong Soundness Theorem: by inductive construction of a model $\mathscr{L}$ of $\mathscr{D}$ on the basis of $\mathscr{I}$.

  Let $\Delta_0 = \Delta$ $\quad\quad$ $\mathscr{I}_0 = \mathscr{I}$

  $\Delta_n = \Delta \cup \{S_1,\ldots,S_n\}$, where $S_1,\ldots,S_n$ are the first n sentences in $\mathscr{D}$.

  Define a model $\mathscr{I}_{n+1}$ of Δ_{n+1}, where induction step is based on the annotation A_{n+1} used in the derivation to get S_{n+1}.

  Four cases:

  i) A_{n+1} is 'Δ', in which case $\mathscr{I}_{n+1} = \mathscr{I}_n$.

  ii) A_{n+1} is 'UI' and the instantial term $\boldsymbol{t}_{n+1}$ contains only names and function terms already occurring in Δ_n. Then $\mathscr{I}_{n+1} = \mathscr{I}_n$.

Proof of Strong Soundness Theorem

iii) A_{n+1} is 'UI' and the instantial term $\boldsymbol{t}_{n+1}$ contains names or function terms *not* occurring in Δ_n.

Take some element d in D (the domain of $\mathcal{I}$), and in every case let $\mathcal{I}_{n+1}$ assign d as the denotation of new names, and all new functions are interpreted as constant functions with d as value.

$\mathcal{I}_{n+1}$ is a model of Δ_n by continuity, and a model of Δ_{n+1} since UI is truth preserving.

iv) A_{n+1} is 'EI', in which case the instantial term $\boldsymbol{t}_{n+1}$ is new. Since the premise of this rule is in Δ_n, the premise must be true in $\mathcal{I}_n$. So there must be at least one element $e \in D$

such that $\mathcal{I}_e^{\boldsymbol{t}n+1}$ is a model of S_{n+1} and also of Δ_n.

So let $\mathcal{I}_{n+1} = \mathcal{I}_e^{\boldsymbol{t}n+1}$ for some such e.

Proof of Strong Soundness Theorem

- Now define the model $\mathcal{L}$ to be just like $\mathcal{I}$, except that for each function symbol or name appearing in $\mathcal{D}$ but not in Δ,

  $\mathcal{L}$ assigns whatever $\mathcal{I}_n$ assigns it,

  where S_n is the first entry in $\mathcal{D}$ in which the new term occurs.
- Hence if Δ had a model $\mathcal{I}$, then all the sentences in $\mathcal{D}$ would have a model $\mathcal{L}$, in which case $\mathcal{D}$ could not contain a refutation (Strong Soundness Theorem) ■
- Therefore if some derivation $\mathcal{D}$ from Δ **is** a refutation, then Δ has no model $\mathcal{I}$ and is unsatisfiable (**Soundness Theorem**) ■

Completeness Theorem

- **Completeness** of FOL will be the culmination of our **positive** metatheoretical results. First proved by Kurt Gödel in 1930.
- **Completeness Theorem**: if a set of sentences Δ is unsatisfiable, then it has a refutation.

Completeness Proof: Canonical Derivation

- **Proof** : first need to define a canonical derivation from Δ, such that, if Δ is unsatisfiable, then any canonical derivation from Δ will be a refutation.
- Definition: $\mathcal{D}$ is a canonical derivation from Δ iff it satisfies the following 5 conditions:

  i) every sentence $\Phi \in \Delta$ occurs in $\mathcal{D}$.

  ii) if $\exists \mathbf{v}\Phi \in \mathcal{D}$, then for some term t, $\Phi\mathbf{v}/t \in \mathcal{D}$.

  iii) if $\forall \mathbf{v}\Phi \in \mathcal{D}$, then for some term t, $\Phi\mathbf{v}/t \in \mathcal{D}$.

  iv) if $\forall \mathbf{v}\Phi \in \mathcal{D}$, then for every term t that can be constructed from names and function symbols occurring in $\mathcal{D}$, $\Phi\mathbf{v}/t \in \mathcal{D}$.

  v) all function symbols occurring in $\mathcal{D}$ appear in Δ.

Completeness Proof: Canonical Derivation

Program for constructing a canonical derivation from Δ:

Let S_1, S_2, S_3, … be an enumeration of the sentences in Δ.

Stage 1(a): enter S_1 as the first line in $\mathcal{D}$.

1(b): add as many entries to $\mathcal{D}$ as possible using EI with restrictions (to be stated momentarily)

1(c): add as many entries to $\mathcal{D}$ as possible using UI

- restrictions: no sentence is the premise of more than one application of EI, no sentence occurs twice, and each instantial term has fewer than N (= current stage number) occurrences of function symbols, and is formed from names and function symbols already in $\mathcal{D}$.

Stage 2(a): enter S_2 as the n^{th} line in $\mathcal{D}$ and repeat…

Example

$\Delta = \{S_1, S_2\} = \{\forall x\, L^2 x, f^1(x), \exists x \forall y \neg L^2 x,y\}$

1.	$\forall x\, L^2 x, f^1(x)$	Δ	(1a)	
2.	$L^2 a, f^1(a)$	1. UI	(1c)	a
3.	$\exists x \forall y \neg L^2 x,y$	Δ	(2a)	
4.	$\forall y \neg L^2 b,y$	3. EI	(2b)	b
5.	$L^2 b, f^1(b)$	1. UI	(2c)	b
6.	$L^2 f^1(a), f^1(f^1(a))$	1. UI	(2c)	$f^1(a)$
7.	$L^2 f^1(b), f^1(f^1(b))$	1. UI	(2c)	$f^1(b)$
8.	$\neg L^2 b,a$	4. UI	(2c)	a
9.	$\neg L^2 b,b$	4. UI	(2c)	b
10.	$\neg L^2 b, f^1(a)$	4. UI	(2c)	$f^1(a)$
11.	$\neg L^2 b, f^1(b)$	4. UI	(2c)	$f^1(b)$

11. and 5. **unsatisfiable.** Hence $\mathcal{D}$ is a **refutation** of Δ

Completeness Proof: Lemma II and Matching

- Definition of Matching: suppose $\mathbf{\Gamma}$ is a set of quantifier-free sentences.

  An interpretation $\mathcal{I}$ matches $\mathbf{\Gamma}$ iff $\mathcal{I}$ is a model of $\mathbf{\Gamma}$ (written $\mathcal{I} \vDash \mathbf{\Gamma}$), and there are no elements in the domain of $\mathcal{I}$ not named by terms in $\mathbf{\Gamma}$.
- **Lemma II**. Suppose $\mathcal{D}$ is a canonical derivation from Δ, $\mathbf{\Gamma}$ is the set of all quantifier-free sentences in $\mathcal{D}$, and $\mathcal{I}$ matches $\mathbf{\Gamma}$.

  Then $\mathcal{I}$ is a model of $\mathcal{D}$ and hence of Δ.
- **Proof** by reductio (see B&J p. 134-5).

Completeness Proof: an OK set of sentences

Now all that remains to be proved is that if every finite subset of $\mathbf{\Gamma}$ is satisfiable, then some interpretation $\mathcal{I}$ matches $\mathbf{\Gamma}$.

- This will show that if Δ is unsatisfiable, then a canonical derivation $\mathcal{D}$ is a refutation, since it must possess a finite subset of quantifier-free sentences which is unsatisfiable.
- So this is really the contraposition of **completeness** –

  if $\mathcal{D}$ is not a refutation then Δ is satisfiable.
- Will prove this by constructing an $\mathcal{I}$ which must be a model of Δ if every finite subset of $\mathbf{\Gamma}$ is satisfiable.
- To do this, first need to introduce the concept of an **OK** set of sentences: a set of sentences $\mathbf{\Sigma}$ is **OK** iff

  every finite subset of $\mathbf{\Sigma}$ is satisfiable.

Completeness Proof: Lemma III

- So to prove **completeness**, only need to prove
- **Lemma III**: if $\mathbf{\Gamma}$ is an enumerable, **OK** set of quantifier-free sentences, then there is an $\mathcal{I}$ which matches $\mathbf{\Gamma}$.
- **Proof**: must define such an $\mathcal{I}$ using the given information:

  General procedure:

  First enumerate all atomic sentences $\mathbf{A}_1$, $\mathbf{A}_2$,… which are

  (i) sentence letters (propositional) occurring in $\mathbf{\Gamma}$, ***or***

  (ii) formed by filling in the argument places of the '**=**' sign, using terms appearing in $\mathbf{\Gamma}$, ***or***

  (iii) formed by filling in the argument places of predicate letters occurring in $\mathbf{\Gamma}$, using terms appearing in $\mathbf{\Gamma}$.

Completeness Proof: Construct an $\mathscr{I}$ which matches $\mathbf{\Gamma}$

Now define the sequence $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, … and verify that all members in the sequence are **OK**:

Let $\mathbf{\Gamma}_1 = \mathbf{\Gamma}$ **OK** by hypothesis.

Now suppose $\mathbf{\Gamma}_n$ has been defined and is **OK**.

Then at least one of the sets $\mathbf{\Gamma}_n \cup \{\mathbf{A}_n\}$ or $\mathbf{\Gamma}_n \cup \{\neg\mathbf{A}_n\}$ is **OK**.

Define $\mathbf{\Gamma}_{n+1}$ as the **OK** one if just one is,

and $\mathbf{\Gamma}_n \cup \{\mathbf{A}_n\}$ if both are **OK**.

Let $\mathbf{B}_i$ be whichever of $\mathbf{A}_i$, $\neg\mathbf{A}_i$ is in the expansion $\mathbf{\Gamma}_{i+1}$.

Completeness Proof: Construct an $\mathscr{I}$ which matches $\mathbf{\Gamma}$

Now, if **r**, **s** are terms in $\mathbf{\Gamma}$, exactly one of **r** = **s**, ¬**(r** = **s)** is in the sequence of **B**'s.

Definition: **r ~ s** iff **r** = **s** is one of the **B**'s.

~ is an equivalence relation on the set of terms in $\mathbf{\Gamma}$.

Now to define $\mathscr{I}$ which matches $\mathbf{\Gamma}$: want $\mathscr{I}$ to assign each term ***t*** its own equivalence class [***t***] as denotation (!)

and want $\mathscr{I}(\mathbf{B}_i) = 1$ for each i. So…

A) let the domain D of $\mathscr{I}$ be the set of all equivalence classes of terms in $\mathbf{\Gamma}$.

B) let $\mathscr{I}(\boldsymbol{t}) = [\boldsymbol{t}]$ for each individual constant ***t***

Completeness Proof: Construct an $\mathcal{J}$ which matches $\boldsymbol{\Gamma}$

C) for each n-place function symbol f^n, let $\mathcal{J}(f^n)$ be the function g^n such that for all $[\boldsymbol{t}_1], \ldots, [\boldsymbol{t}_n]$, in D,

$g^n([\boldsymbol{t}_1], \ldots, [\boldsymbol{t}_n]) = [f^n(\mathbf{s}_1,\ldots,\mathbf{s}_n)]$ if there are terms $\mathbf{s}_1,\ldots,\mathbf{s}_n$ in $[\boldsymbol{t}_1], \ldots, [\boldsymbol{t}_n]$ such that $f^n(\mathbf{s}_1,\ldots,\mathbf{s}_n)$ is a term appearing in $\boldsymbol{\Gamma}$.

Otherwise $g^n([\boldsymbol{t}_1], \ldots, [\boldsymbol{t}_n]) = [\boldsymbol{t}]$, for any term $\boldsymbol{t}$ in $\boldsymbol{\Gamma}$.

D) a sentence letter is true in $\mathcal{J}$ iff it is one of the **B**'s.

E) for each n-place predicate letter P^n occurring in $\boldsymbol{\Gamma}$

$< [\boldsymbol{t}_1], \ldots, [\boldsymbol{t}_n] > \in \mathcal{J}(P^n)$ iff $P^n(\boldsymbol{t}_1,\ldots, \boldsymbol{t}_n)$ is one of the **B**'s.

The definition of $\mathcal{J}$ is now finished. It follows by induction (using B, C) that each complex term $\boldsymbol{t}$ occurring in $\boldsymbol{\Gamma}$ denotes its own equivalence class $[\boldsymbol{t}]$. (B&J p.139).

Completeness Proof: Construct an $\mathscr{I}$ which matches $\mathbf{\Gamma}$

Also, it follows that each of the **B**'s is true in $\mathscr{I}$, since by cases ((i)-(iii) above), $\mathbf{A}_i$ is true in $\mathscr{I}$ iff $\mathbf{A}_i = \mathbf{B}_i$ (B&J p.140)

To see that $\mathscr{I}$ matches $\mathbf{\Gamma}$, first, it's clear that every object in *D* is named by a term of $\mathbf{\Gamma}$.

So just need to show that $\mathscr{I}$ **is a model** of $\mathbf{\Gamma}$:

Suppose sentence $\mathbf{S} \in \mathbf{\Gamma}$.

S is a truth-functional combination of some finite set $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$ of the **A**'s.

In any interpretation $\mathscr{J}$ in which all of $\mathbf{B}_1, \ldots, \mathbf{B}_k$ are **true**,

each $\mathbf{A}_1, \ldots, \mathbf{A}_k$ has the same truth value as in $\mathscr{I}$,

and hence **S** has the same value as in $\mathscr{I}$.

Finally, need to show that this value = **1**

Completeness Proof: Construct an $\mathcal{J}$ which matches $\boldsymbol{\Gamma}$

All of $\mathbf{B}_1,\ldots,\mathbf{B}_k$ are in $\boldsymbol{\Gamma}_{k+1}$, as is $\mathbf{S}$, which is in $\boldsymbol{\Gamma}_1$.

Thus $\{\mathbf{B}_1,\ldots,\mathbf{B}_k, \mathbf{S}\} \subseteq \boldsymbol{\Gamma}_{k+1}$.

Since $\boldsymbol{\Gamma}_{k+1}$ is OK, this finite subset must be satisfiable, and hence all its members true in some interpretation $\mathcal{J}$.

And since all of $\mathbf{B}_1,\ldots,\mathbf{B}_k$ and $\mathbf{S}$ are true in $\mathcal{J}$, $\mathbf{S}$ **is true in $\mathcal{J}$**.

- Thus $\mathcal{J}$ matches $\boldsymbol{\Gamma}$, and by **Lemma II** $\mathcal{J} \vDash \Delta$. So,
- **Completeness Theorem:** if $\boldsymbol{\Gamma}$ is the set of quantifier-free sentences in a canonical derivation $\mathcal{D}$ from Δ, then if $\mathcal{D}$ is **not** a refutation (i.e. $\boldsymbol{\Gamma}$ is **OK**), then $\mathcal{J}$ is a model of Δ ■

Completeness of the Formalism

- We have now demonstrated that our method of formal, syntactic proof is <u>complete</u>.
- This shows that if some formula $\mathbf{\Psi}$ follows as a **logical consequence** of a set of formulas $\mathbf{\Gamma}$, then our proof method is strong enough to yield a **formal demonstration** of this fact.
- In particular, if $\mathbf{\Gamma}$ is a set of **axioms** for some <u>formal</u> <u>theory</u> **T**, then our deductive apparatus is strong enough to yield a **proof** of every sentence in the language which follows as a **logical consequence** of these **axioms**.
- Some immediate consequences of the Soundness and Completeness proofs:

Compactness Theorem

- **Compactness Theorem**: A set of sentences $\boldsymbol{\Sigma}$ is unsatisfiable iff some finite subset $\boldsymbol{\Sigma}_0 \subseteq \boldsymbol{\Sigma}$ is unsatisfiable.
- **Proof**: if $\boldsymbol{\Sigma}$ is unsatisfiable, then by Completeness a canonical derivation $\mathcal{D}$ from $\boldsymbol{\Sigma}$ is a refutation.
- Let $\{A_1, \ldots, A_m\}$ be the finite set of quantifier-free sentences in $\mathcal{D}$ that is unsatisfiable.
- Let j be the number of the line in $\mathcal{D}$ at which A_m occurs, and truncate $\mathcal{D}$ at line j to obtain $\mathcal{D}_0$, which is finite.

  Let $\{S_1, \ldots, S_n\}$ be the members of $\boldsymbol{\Sigma}$ occurring in $\mathcal{D}_0$, and let $\boldsymbol{\Sigma}_0 = \{S_1, \ldots, S_n\}$.

  $\boldsymbol{\Sigma}_0$ is unsatisfiable (by Soundness) ■

Compactness Theorem

- Direct result: finite entailment:

  if $\Delta \models \Psi$, then a finite subset $\Delta_0 \subseteq \Delta$ is such that

  $$\Delta_0 \models \Psi$$

Löwenheim-Skolem Theorem

- **Löwenheim-Skolem Theorem**: If a set of sentences Δ has a model, then it has a model with an enumerable domain.
- **Proof**: if Δ is satisfiable, then the canonical derivation $\mathscr{D}$ from Δ is not a refutation (by Soundness). Hence every finite subset of **Γ** (the set of quantifier-free sentences in $\mathscr{D}$) is satisfiable (by Compactness).

By Lemma III there is a model $\mathscr{I}$ that matches **Γ**, and by Lemma II $\mathscr{I}$ is a model of Δ.

Since $\mathscr{I}$ matches **Γ**, every object in the domain of $\mathscr{I}$ is named by some term in **Γ**. Since **Γ** is enumerable, there are only enumerably many such terms.

Therefore $\mathscr{I}$ has an enumerable domain ■

Cardinality

- The Löwenheim-Skolem Theorem reveals a fundamental fact about the expressive power of sentences in FOL with respect to the cardinality of their models.
- This version is the 'downward' Löwenheim-Skolem Theorem and shows that you can't force there to be only models with a domain of cardinality greater than $\aleph_0$.
- Hence any (consistent) set of sentences (e.g. a formal theory of numbers) will be satisfied by an interpretation with a countable domain.

Cardinality

- We’ve seen the ‘downward’ Löwenheim-Skolem Theorem, showing that you can’t force there to be only models with a domain of cardinality greater than $\aleph_0$.
- There is also an ‘upward’ version of the theorem:

  if a set Δ has an infinite model, then it has a model with uncountably many elements.
- This shows we also can’t force only less than uncountable models.
- In other words, FOL cannot distinguish between different levels of infinity.

What is First-Order Logic?

- It turns out that compactness plus upward and downward Löwenheim-Skolem (L-S) metalogically capture FOL…
- **Lindströms Theorem** (1969):

Let $\mathcal{L}$ be any 'extension' of FOL with the two properties:

i) downward L-S

and

ii) either upward L-S or compactness.

Then $\mathcal{L}$ is no 'stronger' than FOL, in the sense that every sentence of $\mathcal{L}$ has exactly the same models as some sentence of FOL.