Logic, Computability and Incompleteness

Gödel’s Second Theorem, Löb’s Theorem and the Logic of Provability

Hilbert's Program (again)

- As we saw before, Hilbert's Formalist Program for the foundation of mathematics advocated an approach in which all of mathematics is deducible in an axiomatizable formal theory where the axioms themselves are provably consistent.
- In particular, Hilbert sought an 'internal' and finitary consistency proof for the axioms of elementary number theory.
- Gödel's First Theorem can be seen as refuting Hilbert's goal of reducing arithmetic to an 'inventory of provable formulas',

while Gödel's Second Theorem can be seen as undermining Hilbert's quest for an internal consistency proof for the axioms of arithmetic.

The Unprovability of Consistency

Gödel's First Incompleteness Theorem (roughly): if formal arithmetic is consistent, then neither $\boldsymbol{S}$ nor $\neg\, \boldsymbol{S}$ is provable,

where $\boldsymbol{S}$ is constructed such that $\vdash \boldsymbol{S} \leftrightarrow \neg\, \boldsymbol{Prov}(\ulcorner \boldsymbol{S} \urcorner)$.

So if arithmetic is consistent then $\boldsymbol{S}$ is unprovable, hence **true**

Is formal arithmetic consistent?

Gödel's Second Incompleteness Theorem (roughly):

if formal arithmetic <u>is</u> consistent,

then it <u>cannot</u> prove its own consistency.

A **basic fact** of Classical Logic: if formal arithmetic is a consistent theory, then at least one sentence is <u>unprovable</u>.

The Unprovability of Consistency

So let the consistency of arithmetic be expressed

in arithmetic (**!**) by the sentence:

$\neg\, \boldsymbol{Prov}\, (\ulcorner \mathbf{o} = \mathbf{o}' \urcorner)$ (**con**)

Gödel's Second Incompleteness Theorem:

if $\vdash$ **con** *then* arithmetic is inconsistent.

As above, Gödel's First Theorem states that

(#) *if* arithmetic is consistent, *then* it is *not* provable that ***S***

Since **con** is a sentence of the object language that expresses the consistency of arithmetic, then a formalization of the First Theorem, in terms of (#), would yield:

$\mathbf{con} \rightarrow \neg\, \boldsymbol{Prov}\, (\ulcorner \boldsymbol{S} \urcorner)$ (**!**)

The Unprovability of Consistency

or equivalently **con** $\rightarrow$ ***S***

And *if* this formalization of the First Theorem were provable in formal arithmetic, *then* it follows (on the assumption of consistency) that **not** $\vdash$ **con**.

Why? Because $\vdash$ **con** $\rightarrow$ ***S*** entails that

if $\vdash$ **con** *then* $\vdash$ ***S***

And if arithmetic is consistent, then (by 1st Theorem)

not $\vdash$ ***S***, and contraposition on the above yields **not** $\vdash$ **con**.

So *if* the conditional **con** $\rightarrow$ ***S*** is a theorem of arithmetic,

then if arithmetic is consistent,

then the consistency sentence **con** cannot be provable in arithmetic.

Gödel's Second Incompleteness Theorem

Thus to prove Gödel's Second Incompleteness Theorem need to show that $\vdash$ **con** $\to$ ***S***, and the underivability of the consequent ***S*** will yield the underivability of the antecedent **con**.

proof: will require the diagonal lemma and characteristics (i) – (iii) of a provability predicate.

As before, the diagonal lemma gives

$\vdash\ \boldsymbol{S} \leftrightarrow \neg\, \boldsymbol{Prov}(\ulcorner \boldsymbol{S} \urcorner)$, which yields $\vdash\ \boldsymbol{S} \to \neg\, \boldsymbol{Prov}(\ulcorner \boldsymbol{S} \urcorner)$

and then $\vdash\ \neg\neg\, \boldsymbol{Prov}(\ulcorner \boldsymbol{S} \urcorner) \to \neg\, \boldsymbol{S}$ and finally

(0) $\vdash\ \boldsymbol{Prov}(\ulcorner \boldsymbol{S} \urcorner) \to \neg\, \boldsymbol{S}$

Recall (i) if $\vdash \boldsymbol{A}$, then $\vdash \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner)$, which applied to (0)

gives $\vdash \boldsymbol{Prov}(\ulcorner \underline{\boldsymbol{Prov}(\ulcorner \boldsymbol{S} \urcorner) \to \neg\, \boldsymbol{S}} \urcorner)$

Gödel's Second Incompleteness Theorem

Recall (ii) $\vdash \mathit{Prov}(\ulcorner A \rightarrow B \urcorner) \rightarrow (\mathit{Prov}(\ulcorner A \urcorner) \rightarrow \mathit{Prov}(\ulcorner B \urcorner))$

Which gives $\vdash \mathit{Prov}(\ulcorner \underline{\mathit{Prov}(\ulcorner S \urcorner) \rightarrow \neg S} \urcorner) \rightarrow$

$(\mathit{Prov}(\ulcorner \mathit{Prov}(\ulcorner S \urcorner) \urcorner) \rightarrow \mathit{Prov}(\ulcorner \neg S \urcorner))$

Now modus ponens gives

(1) $\vdash \mathit{Prov}(\ulcorner \mathit{Prov}(\ulcorner S \urcorner) \urcorner) \rightarrow \mathit{Prov}(\ulcorner \neg S \urcorner)$

Similarly, take $\vdash (S \wedge \neg S) \rightarrow \mathbf{o} = \mathbf{o}'$ and apply (i) to get

$\vdash \mathit{Prov}(\ulcorner \underline{(S \wedge \neg S) \rightarrow \mathbf{o} = \mathbf{o}'} \urcorner)$ then apply (ii) to get

$\vdash \mathit{Prov}(\ulcorner \underline{(S \wedge \neg S) \rightarrow \mathbf{o} = \mathbf{o}'} \urcorner) \rightarrow$

$(\mathit{Prov}(\ulcorner (S \wedge \neg S) \urcorner) \rightarrow \mathit{Prov}(\ulcorner \mathbf{o} = \mathbf{o}' \urcorner))$ and MP to get

($) $\vdash \mathit{Prov}(\ulcorner (S \wedge \neg S) \urcorner) \rightarrow \mathit{Prov}(\ulcorner \mathbf{o} = \mathbf{o}' \urcorner)$

Gödel's Second Incompleteness Theorem

Note that $\vdash \mathit{Prov}(\ulcorner (S \wedge \neg S) \urcorner) \leftrightarrow (\mathit{Prov}(\ulcorner S \urcorner) \wedge \mathit{Prov}(\ulcorner \neg S \urcorner))$

Substitution of provable equivalents in ($) yields

(2) $\vdash (\mathit{Prov}(\ulcorner S \urcorner) \wedge \mathit{Prov}(\ulcorner \neg S \urcorner)) \rightarrow \mathit{Prov}(\ulcorner \mathbf{o} = \mathbf{o}' \urcorner)$

Recall (iii) $\vdash \mathit{Prov}(\ulcorner A \urcorner) \rightarrow \mathit{Prov}(\ulcorner \mathit{Prov}(\ulcorner A \urcorner) \urcorner)$

which gives: (3) $\vdash \mathit{Prov}(\ulcorner S \urcorner) \rightarrow \underline{\mathit{Prov}(\ulcorner \mathit{Prov}(\ulcorner S \urcorner) \urcorner)}$

Recall (1) $\vdash \underline{\mathit{Prov}(\ulcorner \mathit{Prov}(\ulcorner S \urcorner) \urcorner)} \rightarrow \mathit{Prov}(\ulcorner \neg S \urcorner)$

which in combination with (3) yields

(4) $\vdash \mathit{Prov}(\ulcorner S \urcorner) \rightarrow (\mathit{Prov}(\ulcorner S \urcorner) \wedge \mathit{Prov}(\ulcorner \neg S \urcorner))$

With (4) and (2) we get $\vdash \mathit{Prov}(\ulcorner S \urcorner) \rightarrow \mathit{Prov}(\ulcorner \mathbf{o} = \mathbf{o}' \urcorner)$

And by contraposition

(5) $\vdash \neg \mathit{Prov}(\ulcorner \mathbf{o} = \mathbf{o}' \urcorner) \rightarrow \neg \mathit{Prov}(\ulcorner S \urcorner)$.

Gödel's Second Incompleteness Theorem

Since **con** has been defined as the sentence $\neg\, \mathit{Prov}\,(\ulcorner \mathbf{o} = \mathbf{o}' \urcorner)$

and $\vdash\ \boldsymbol{S} \leftrightarrow \neg\, \mathit{Prov}\,(\ulcorner \boldsymbol{S} \urcorner)$,

Rewriting (5) $\vdash\ \neg\, \mathit{Prov}\,(\ulcorner \mathbf{o} = \mathbf{o}' \urcorner) \rightarrow \neg\, \mathit{Prov}\,(\ulcorner \boldsymbol{S} \urcorner)$ as

$\vdash$ **con** $\rightarrow \boldsymbol{S}$ yields the desired result

and if arithmetic is consistent then not $\vdash$ **con** ■

This is another incompleteness result, because if arithmetic is consistent then **con** is ***true***,

so if arithmetic is consistent then **con** is yet another **unprovable truth**, and the wedge between

truth and provability that started with the First Theorem is driven even deeper.

Löb's Theorem

We've just seen a 'direct' proof of Gödel's Second Incompleteness Theorem.

However, it is also possible to prove this theorem as a corollary of the closely related but more general Löb's Theorem, which is motivated as follows.

Another way to think of provable consistency is in terms of the characteristic

for all sentences $\boldsymbol{A}$,

(v) $\vdash \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A}$

Which 'asserts that' if a sentence is provable, then it is true,

so that it's **provable** in the formal theory that only truths are provable.

Löb's Theorem

Löb's Theorem: if $\boldsymbol{B}(y)$ is a provability predicate for some theory **T** that extends $\boldsymbol{Q}$, then for any sentence $\boldsymbol{A}$ in the language of **T**

$$\textit{if} \quad \vdash_{\mathbf{T}} \boldsymbol{B}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A} \quad \textit{then} \vdash_{\mathbf{T}} \boldsymbol{A}$$

proof: assume (1) $\vdash_{\mathbf{T}} \boldsymbol{B}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A}$.

Let $\boldsymbol{D}(y)$ be the formula $\boldsymbol{B}(y) \rightarrow \boldsymbol{A}$.

The diagonal lemma guarantees a sentence $\boldsymbol{C}$ such that

$$\vdash_{\mathbf{T}} \boldsymbol{C} \leftrightarrow \boldsymbol{D}(\ulcorner \boldsymbol{C} \urcorner), \quad \text{i.e.}$$

(2) $\vdash_{\mathbf{T}} \boldsymbol{C} \leftrightarrow (\boldsymbol{B}(\ulcorner \boldsymbol{C} \urcorner) \rightarrow \boldsymbol{A})$

(1) and (2) in combination with (i) – (iii) yield $\vdash_{\mathbf{T}} \boldsymbol{A}$ ■

(see B&J p. 187 for the detailed steps).

Gödel’s Second Incompleteness Theorem Again

Reformulation of Gödel’s Second Incompleteness Theorem:

if $\boldsymbol{B}(y)$ is a provability predicate for some consistent theory **T** that extends $\boldsymbol{Q}$, then **T** cannot prove its own consistency

i.e. **not** $\vdash_{\mathbf{T}} \neg \boldsymbol{B}(\ulcorner \mathbf{0} = \mathbf{0}' \urcorner)$

proof: suppose $\vdash_{\mathbf{T}} \neg \boldsymbol{B}(\ulcorner \mathbf{0} = \mathbf{0}' \urcorner)$.

Then $\vdash_{\mathbf{T}} \boldsymbol{B}(\ulcorner \mathbf{0} = \mathbf{0}' \urcorner) \rightarrow \mathbf{0} = \mathbf{0}'$ (by prop. logic)

and by Löb’s Theorem $\vdash_{\mathbf{T}} \mathbf{0} = \mathbf{0}'$

But $\vdash_{\boldsymbol{Q}} \neg\, \mathbf{0} = \mathbf{0}'$, and thus **T** is inconsistent

So *if* **T** *is* consistent, *then* it can’t prove its own consistency ■

The Henkin Sentence

Historically, Löb's Theorem was used to answer a question posed by Henkin with regard to the the 'Henkin Sentence' ***H***.

Unlike the Gödel sentence ***S***, ***H*** 'asserts its own **provability**'.

Given the proof predicate ***Prov*** (y)

The diagonal lemma guarantees an ***H*** such that

$$\vdash \ \boldsymbol{H} \leftrightarrow \boldsymbol{Prov}\,(\ulcorner \boldsymbol{H} \urcorner)$$

Is ***H*** <u>provable</u> (and hence **true**)?

It follows directly from Löb's Theorem that $\vdash \ \boldsymbol{H}$ ■

Modal Logic of Provability

The defining characteristics of a provability predicate have very clear analogues in modal logic:

(i) *if* $\vdash \boldsymbol{A}$, *then* $\vdash \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner)$

corresponds to the modal inference rule of *Necessitation*

if $\vdash \boldsymbol{A}$, *then* $\vdash \Box \boldsymbol{A}$

(ii) $\vdash \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \rightarrow \boldsymbol{B} \urcorner) \rightarrow (\boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{Prov}(\ulcorner \boldsymbol{B} \urcorner))$

corresponds to the **K** axiom schema

$\Box (\mathbf{A} \rightarrow \mathrm{C}) \rightarrow (\Box \mathbf{A} \rightarrow \Box \mathbf{C})$

Modal Logic of Provability

(iii) $\vdash \textbf{\textit{Prov}}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \textbf{\textit{Prov}}(\ulcorner \textbf{\textit{Prov}}(\ulcorner \boldsymbol{A} \urcorner) \urcorner)$

corresponds to the **S4** axiom schema

$$\Box \boldsymbol{A} \rightarrow \Box \Box \boldsymbol{A}$$

(v) $\vdash \textbf{\textit{Prov}}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A}$

corresponds to the **T** axiom schema

$$\Box \boldsymbol{A} \rightarrow \boldsymbol{A}$$

The modal theory **S4** is the closure of all the **K**, **S4** and **T** axioms under logical consequence and the rule of *Necessitation*.

Modal Logic of Provability

Hence **S4** is too strong to represent the logic of arithmetical proof, as shown by the Gödel-Löb results.

Instead, need to replace **T** axiom schema with the <u>formalized</u> and then <u>modalized</u> version of Löb's Theorem.

Recall Löb's Theorem:

$$\textit{if} \quad \vdash \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A} \quad \textit{then} \vdash \boldsymbol{A}$$

Löb's Theorem **formalized** <u>in</u> <u>arithmetic</u>:

$$\vdash \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A} \quad \textit{only if} \quad \vdash \boldsymbol{A}$$

$$\vdash \boldsymbol{Prov}(\ulcorner \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{Prov}(\ulcorner \boldsymbol{A} \urcorner)$$

Modal Logic of Provability

The **modal** version of the <u>formalization</u>

$$\vdash \quad \boldsymbol{Prov}\,(\ulcorner \boldsymbol{Prov}\,(\ulcorner \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{A} \urcorner) \rightarrow \boldsymbol{Prov}\,(\ulcorner \boldsymbol{A} \urcorner)$$

yields the **G** axiom schema:

$$\Box\,(\Box\,\boldsymbol{A} \rightarrow \boldsymbol{A}) \rightarrow \Box\,\boldsymbol{A}$$

- The **modal theory** **G** is the closure of all the **K**, **S4** and **G** axioms under logical consequence and the rule of *Necessitation*.
- Hence the modal theory **G** represents the logic of provability in formal arithmetic.