Multiple Linear Regression: Global tests and Multiple Testing

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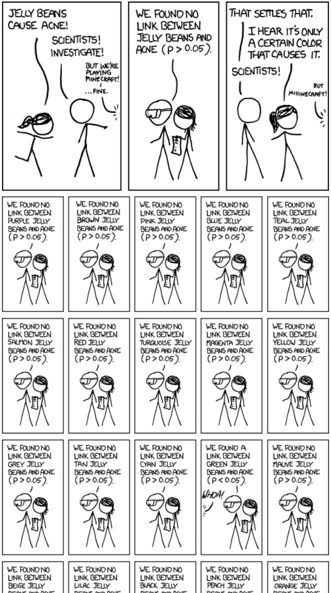
Today's Lecture

- Multiple testing - preserving your Type I error rate.



Inference about multiple coefficients

Our model contains multiple parameters; often we want to perform multiple tests:

$$
\begin{aligned}
H_{01} : \beta_1 &= 0 \\
H_{02} : \beta_2 &= 0 \\
\vdots &= \vdots \\
H_{0k} : \beta_k &= 0
\end{aligned}
$$

where each test has a size of α

- For any individual test, $P(\text{reject } H_{0i} | H_{0i}) = \alpha$

Inference about multiple coefficients

For any individual test, $P(\text{reject } H_{0i} | H_{0i}) = \alpha$.
But this DOES NOT MEAN that

$$P(\text{reject at least one } H_{0i} | \text{all } H_{0i} \text{are true}) = \alpha$$

. This is called the Family-wise error rate (FWER). Ignore it at your own peril!

Family-wise error rate

To calculate the FWER

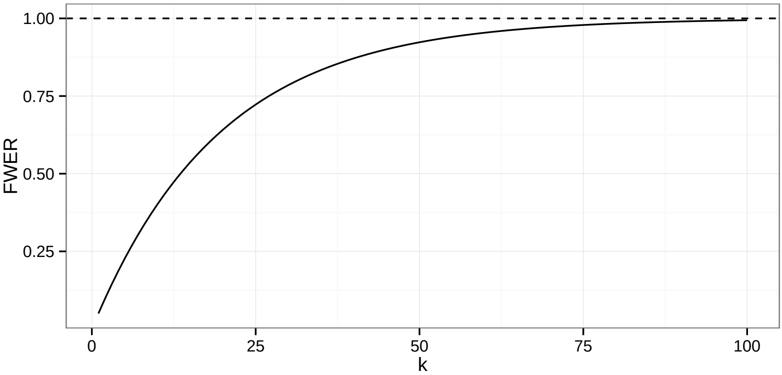
- First note $P(\text{no rejections}|\text{all } H_{0i}\text{are true}) = (1-\alpha)^k$
- It follows that

$$\begin{aligned} \text{FWER} &= P(\text{at least one rejection}|\text{all } H_{0i}\text{are true}) \\ &= 1-(1-\alpha)^k \end{aligned}$$

Family-wise error rate

$$\mathrm{FWER} = 1 - (1 - \alpha)^k$$

```
alpha <- .05
k <- 1:100
FWER <- 1-(1-alpha)^k
qplot(k, FWER, geom="line") + geom_hline(yintercept = 1, lty=2)
```



Addressing multiple comparisons

Three general approaches

- Do nothing in a reasonable way
  - Don't trust scientifically implausible results
  - Don't over-emphasize isolated findings
- Correct for multiple comparisons
  - Often, use the Bonferroni correction and use $\alpha_i = \alpha/k$ for each test
  - Thanks to the Bonferroni inequality, this gives an overall $FWER \leq \alpha$
- Use a global test

Global tests

Compare a smaller "null" model to a larger "alternative" model

- Smaller model must be nested in the larger model
- That is, the smaller model must be a special case of the larger model
- For both models, the RSS gives a general idea about how well the model is fitting
- In particular, something like

$$\frac{RSS_S - RSS_L}{RSS_L}$$

  compares the relative RSS of the models

Nested models

- These models are nested:

$$\begin{aligned} \text{Smaller} &= \text{Regression of } Y \text{ on } X_1 \\ \text{Larger} &= \text{Regression of } Y \text{ on } X_1, X_2, X_3, X_4 \end{aligned}$$

- These models are not:

$$\begin{aligned} \text{Smaller} &= \text{Regression of } Y \text{ on } X_2 \\ \text{Larger} &= \text{Regression of } Y \text{ on } X_1, X_3 \end{aligned}$$

Global F tests

- Compute the test statistic

$$F_{obs} = \frac{(RSS_S - RSS_L)/(df_S - df_L)}{RSS_L/df_L}$$

- If H_0 (the null model) is true, then $F_{obs} \sim F_{df_S - df_L, df_L}$
- Note $df_s = n - p_S - 1$ and $df_L = n - p_L - 1$
- We reject the null hypothesis if the p-value is above α, where

$$\text{p-value} = P(F_{df_S - df_L, df_L} > F_{obs})$$

Global F tests

There are a couple of important special cases for the F test

- The null model contains the intercept only
  - When people say ANOVA, this is often what they mean (although all F tests are based on an analysis of variance)
- The null model and the alternative model differ only by one term
  - Gives a way of testing for a single coefficient
  - Turns out to be equivalent to a two-sided t-test: $t^2_{df_L} \sim F_{1,df_L}$

Lung data: multiple coefficients simultaneously

You can test multiple coefficients simultaneously using the F test

```
mlr_null <- lm(disease ~ nutrition, data=dat)
mlr1 <- lm(disease ~ nutrition+ airqual + crowding + smoking, data=dat)
anova(mlr_null, mlr1)

## Analysis of Variance Table
##
## Model 1: disease ~ nutrition
## Model 2: disease ~ nutrition + airqual + crowding + smoking
##   Res.Df    RSS Df Sum of Sq      F    Pr(>F)
## 1     97 9192.7
## 2     94 1248.0  3    7944.7 199.47 < 2.2e-16 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Lung data: single coefficient test

The F test is equivalent to the t test when there's only one parameter of interest

```
mlr_null <- lm(disease ~ nutrition, data=dat)
mlr1 <- lm(disease ~ nutrition + airqual, data=dat)
anova(mlr_null, mlr1)

## Analysis of Variance Table
##
## Model 1: disease ~ nutrition
## Model 2: disease ~ nutrition + airqual
##   Res.Df    RSS Df Sum of Sq      F    Pr(>F)
## 1     97 9192.7
## 2     96 5969.5  1    3223.1 51.833 1.347e-10 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

summary(mlr1)$coef

##                Estimate Std. Error   t value     Pr(>|t|)
## (Intercept) 37.62538251 2.43946243 15.423637 9.946294e-28
## nutrition   -0.03469855 0.01692446 -2.050202 4.307101e-02
## airqual      0.36114435 0.05016218  7.199535 1.346935e-10
```

Today's Big Ideas

F tests can control for multiple comparisons!

- hands-on example