Inquiry Based Linear Algebra

About the Document

This document is a hybrid of many linear algebra resources, including those of the IOLA (Inquiry Oriented Linear Algebra) project, Jason Siefken's IBLLinearAlgebra project, and Asaki, Camfield, Moon, and Snipes' Radiograph and Tomography project.

This document is a mix of student projects, problem sets, and labs. A typical class day looks like:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

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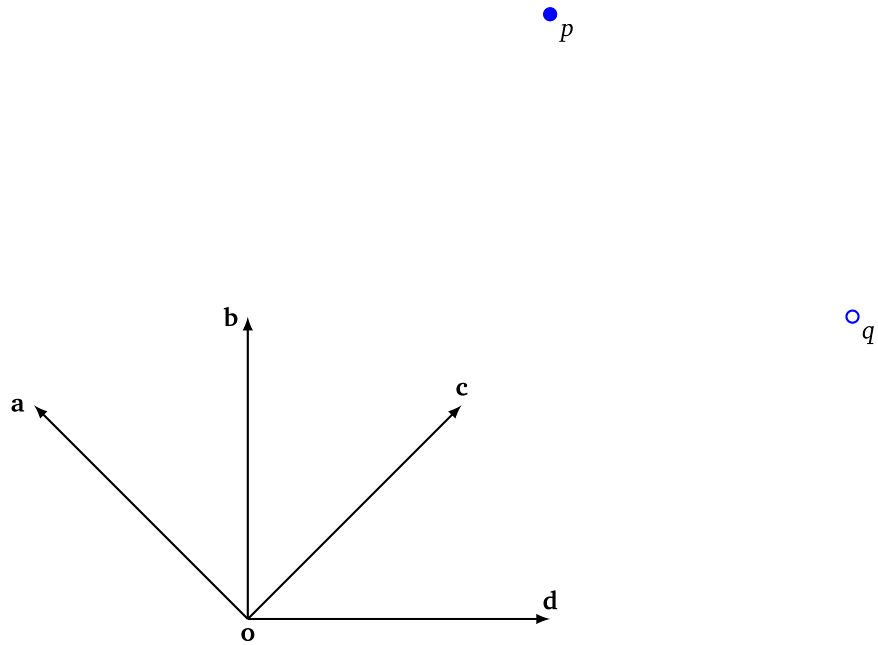
Vectors

1



Notice that all arrows in this diagram are the same length. We will call this length a *unit*.

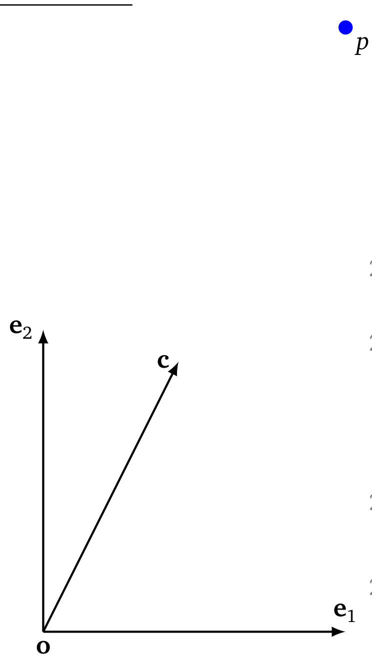
1.1 Give directions from $\mathbf{o}$ to p of the form "Walk ____units in the direction of arrow ____, then walk ____units in the direction of arrow ____."

1.2 Can you give directions with the two arrows you haven't used? Give such directions, or explain why it cannot be done.

1.3 Give directions from $\mathbf{o}$ to q.

1.4 Can you give directions from $\mathbf{o}$ to q using $\mathbf{c}$ and $\mathbf{a}$? Give such directions, or explain why it cannot be done.

2



We are going to start using a more mathematical notation for giving directions. Our directions will now look like

$$p = \underline{\qquad}\,\mathbf{e}_1 + \underline{\qquad}\,\mathbf{e}_2$$

which is read as "To get to p $(=)$ go ____units in the direction $\mathbf{e}_1$ then $(+)$ go ____units in the direction $\mathbf{e}_2$."

2.1 What is the difference between $p = \underline{\qquad}\,\mathbf{e}_1 + \underline{\qquad}\,\mathbf{e}_2$ and $p = \underline{\qquad}\,\mathbf{e}_2 + \underline{\qquad}\,\mathbf{e}_1$? Can they both give valid directions?

2.2 (a) Give directions to p using the new notation.

(b) Give directions to p using $\mathbf{c}$. (Notice that $\mathbf{c}$ is of unit length and points directly at p.)

(c) What is the distance from $\mathbf{o}$ to p in units?

2.3 (a) $r = 1\mathbf{c}$. Give directions from $\mathbf{o}$ to r using $\mathbf{e}_1$ and $\mathbf{e}_2$.

(b) What is the distance from $\mathbf{o}$ to r?

2.4 (a) $q = -2\mathbf{e}_1 + 3\mathbf{e}_2$; find the exact distance from $\mathbf{o}$ to q.

(b) $s = 2\mathbf{e}_1 + \mathbf{c}$; find the exact distance from $\mathbf{o}$ to s.

The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are called the *standard basis vectors* for $\mathbb{R}^2$ (the plane).

Column Vector Notation

We previously wrote $q = -2\mathbf{e}_1 + 3\mathbf{e}_2$. In column vector notation we write

$$q = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We may call q either a *vector* or a *point*. If we call q a vector, we are emphasizing that q gives direction of some sort. If we call q a point, we emphasize that q is some absolute location in space. (What's the philosophical difference between a location in space and directions from the origin to said location?)

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3 $r = 1\mathbf{c}$ and $s = 2\mathbf{e}_1 + \mathbf{c}$ where $\mathbf{c}$ is the vector from before.

3.1 Write r and s in column vector form.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1, 2, 3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an elemento of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X.

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}$.

$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}$.

$\mathbb{R} = \{\text{real numbers}\}$.

$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}$.

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4 4.1 Which of the following are true?

(a) $3 \in \{1, 2, 3\}$.
(b) $1.5 \in \{1, 2, 3\}$.
(c) $4 \in \{1, 2, 3\}$.
(d) "b"$\in \{x : x \text{ is an English letter}\}$.
(e) "ò"$\in \{x : x \text{ is an English letter}\}$.
(f) $\{1, 2\} \subseteq \{1, 2, 3\}$.
(g) For some $a \in \{1, 2, 3\}$, $a \geq 3$.
(h) For any $a \in \{1, 2, 3\}$, $a \geq 3$.
(i) $1 \subseteq \{1, 2, 3\}$.
(j) $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.
(k) $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$.

5 Write the following in set-builder notation

5.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

5.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Unions & Intersections

DEFINITION

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

6 Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

6.1 $X \cup Y$

6.2 $X \cap Y$

6.3 $X \cup Y \cup Z$

6.4 $X \cap Y \cap Z$

7 Draw the following subsets of $\mathbb{R}^2$.

7.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}.$

7.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}.$

7.3 $J = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}.$

7.4 $V \cup H$.

7.5 $V \cap H$.

7.6 Does $V \cup H = \mathbb{R}^2$?

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

8 Consider the system

$$\begin{aligned} x \; - \; y &= 2 \\ 2x \; + \; y &= 1 \end{aligned} \tag{1}$$

8.1 Draw the lines in (1) on the same coordinate plane.

8.2 Algebraically solve the system (1). What does this solution represent on your graph?

9 Let L be the line given by $x - y = 2$.

9.1 Write an equation of a line that doesn't intersect L.

9.2 Write an equation of a line that intersects L in

(a) one place.

(b) infinitely many places

(c) exactly two places

or explain why no such equation exists.

9.3 For each equation you came up with, solve the system algebraically. How can you tell algebraically how many solutions there are?

10 Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

10.1 Write the system of equations corresponding to A.

10.2 Solve the system of equations corresponding to A.

The Row Reduction Algorithm

11 11.1 Solve the system

$$\begin{array}{rcrcrl} x & - & y & - & 2z & = -5 \\ 2x & + & 3y & + & z & = 5 \\ 0x & + & 2y & + & 3z & = 8 \end{array} \tag{2}$$

any way you like.

11.2 Use an augmented matrix to solve the system (2).

The system (2) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (2) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

11.3 Find $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{p}$ so that you may rewrite (2) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

11.4 If (x, y, z) is a solution to (2), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

11.5 If (x, y, z) is a solution to (2), is $\vec{p}$ a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$?

DEFINITION

Vector Form of a Line

A line ℓ is written in ***vector form*** if it is expressed as

$$\vec{x} = \vec{d}t + \vec{p}$$

for some vector $\vec{x}$ and point $\vec{p}$. That is, $\ell = \{\vec{x} : \vec{x} = \vec{d}t + \vec{p} \text{ for some } t \in \mathbb{R}\}$.

Infinite Solutions

12 Consider the system

$$\begin{array}{rcrl} x & + & 2y & = 3 \\ 2x & + & 4y & = 6 \end{array} \tag{3}$$

12.1 How many solutions does (3) have?

12.2 Write the solutions to (3) in vector form.

12.3 What happens when you use an augmented matrix to solve (3)?

13 Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make $2-1=1$ choices when writing a solution. Let's make the choice $y=t$.

13.1 With the added equation $y=t$, solve the system represented by B.

14 Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

and call the variables in this system x_1, x_2, x_3, x_4, x_5.

14.1 Write the system of equations represented by C.

14.2 Identify how many choices you can make when writing down a solution corresponding to C.

14.3 Add one equation (of the form $x_i = t$ or $x_j = s$, etc.) for each choice you must make when solving the system.

14.4 Write in vector form all solutions to C.

15 15.1 An unknown system U is represented by an augmented matrix with 4 rows, 7 columns (one column is the augmented column). What is the minimum number of free variables U can have?

15.2 An unknown system V is represented by an augmented matrix with 6 rows, 5 columns (one column is the augmented column). What is the minimum number of free variables V can have?

16 **Homogeneous**

DEF A system is called ***homogeneous*** if all equations equal 0.

Let A be an unknown system of 3 equations and 3 variables and suppose $(x,y,z)=(1,2,1)$ and $(x,y,z)=(-1,1,1)$ are solutions to A.

16.1 Can you produce another solution to the system?

16.2 Can you produce a solution to the homogeneous version of A (the version of A where every equation equals 0)?

16.3 Suppose when you use an augmented matrix to solve the system A, you only have one free variable. Could A be homogeneous? Can you produce all solutions to the system A?

Dot Product

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

17

Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

17.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

17.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

17.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

17.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

Norm

DEF

The ***norm*** of a vector $\vec v \in \mathbb{R}^n$, denoted $\|\vec v\|$ is its length and is given by the formula

$$\|\vec v\| = \sqrt{\vec v \cdot \vec v}.$$

18

18.1 Let $\vec a = \begin{bmatrix}3\\2\end{bmatrix}$. Find $\|\vec a\|$ using the Pythagorean theorem and using the formula from the definition of the norm. How do these quantities relate?

18.2 Let $\vec b = \begin{bmatrix}1\\1\\-2\\2\end{bmatrix}$, and find $\|\vec b\|$. Did you know how to find 4-d lengths before?

18.3 Suppose $\vec u = \begin{bmatrix}x\\y\end{bmatrix}$ for some $x, y \in \mathbb{R}$. Could $\vec u \cdot \vec u$ be negative? Compute $\vec u \cdot \vec u$ algebraically and use this to prove your answer.

Distance

DEF

The ***distance*** between two vectors $\vec u$ and $\vec v$ is $\|\vec u - \vec v\|$.

Unit Vector

DEF

A vector $\vec v$ is called a ***unit vector*** if $\|\vec v\| = 1$.

19

Let $\vec u = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec v = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

19.1 Find the distance between $\vec u$ and $\vec v$.

19.2 Find a unit vector in the direction of $\vec u$.

19.3 Does there exists a *unit vector* $\vec x$ that is distance 1 from $\vec u$?

19.4 Suppose $\vec y$ is a unit vector and the distance between $\vec y$ and $\vec u$ is 2. What is the angle between $\vec y$ and $\vec u$?

Orthogonal

DEF

Two vectors $\vec u$ and $\vec v$ are ***orthogonal*** to each other if $\vec u \cdot \vec v = 0$. The word orthogonal is synonymous with the word perpendicular.

20

20.1 Find two vectors orthogonal to $\vec a = \begin{bmatrix}1\\-3\end{bmatrix}$. Can you find two such vectors that are not parallel?

20.2 Find two vectors orthogonal to $\vec b = \begin{bmatrix}1\\-3\\4\end{bmatrix}$. Can you find two such vectors that are not parallel?

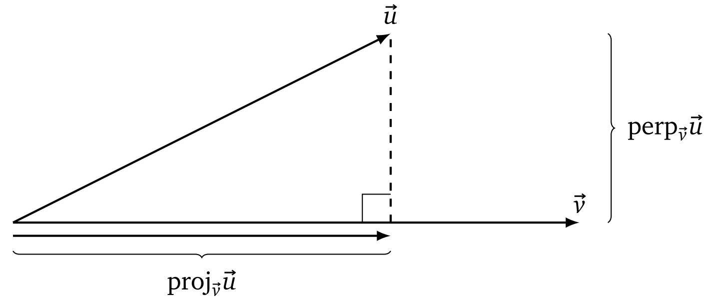
20.3 Suppose $\vec x$ and $\vec y$ are orthogonal to each other and $\|\vec x\| = 5$ and $\|\vec y\| = 3$. What is the distance between $\vec x$ and $\vec y$?

Projections

Projections (sometimes called orthogonal projections) are a way to measure how much one vector points in the direction of another.

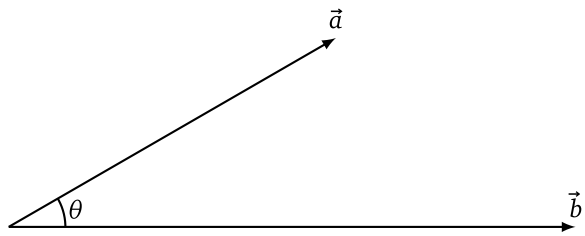
Projection

DEFINITION



The ***projection*** of $\vec{u}$ onto $\vec{v}$ is written $\mathrm{proj}_{\vec{v}}\vec{u}$ and is the vector in the direction of $\vec{v}$ such that $\vec{u}-\mathrm{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$. The vector $\vec{u}-\mathrm{proj}_{\vec{v}}$ is called the ***perpendicular component*** of $\vec{u}$ with respect to $\vec{v}$ and is notated as $\mathrm{perp}_{\vec{v}}\vec{u}$.

21



In this picture $\|\vec{a}\| = 4$, $\theta = \pi/6$, and $\vec{b} = \begin{bmatrix}6\\0\end{bmatrix}$.

21.1 Write $\vec{a}$ in column vector form.

21.2 Find $\|\mathrm{proj}_{\vec{b}}\vec{a}\|$ and $\|\mathrm{perp}_{\vec{b}}\vec{a}\|$.

21.3 Write down $\mathrm{proj}_{\vec{b}}\vec{a}$ and $\mathrm{perp}_{\vec{b}}\vec{a}$ in column vector form.

21.4 Let $\vec{c} = \begin{bmatrix}-4\\0\end{bmatrix}$. Write down $\mathrm{proj}_{\vec{c}}\vec{a}$ and $\mathrm{perp}_{\vec{c}}\vec{a}$ in column vector form.

22 Let $\|\vec{a}\| = 4$, $\vec{d} = \begin{bmatrix}3\\1\end{bmatrix}$ and let $\theta = \pi/6$ be the angle between $\vec{a}$ and $\vec{d}$.

22.1 Write down $\mathrm{proj}_{\vec{d}}\vec{a}$ and $\mathrm{perp}_{\vec{d}}\vec{a}$ in column vector form.

22.2 Compute $\mathrm{proj}_{\mathbf{e}_1}\vec{d}$ and $\mathrm{proj}_{\mathbf{e}_2}\vec{d}$. How do these projections relate to the coordinates of $\vec{d}$? What can you say in general about projections onto $\mathbf{e}_1$ and $\mathbf{e}_2$?

Matrix Equations

Let M be an $n \times m$ matrix with rows $\vec{r}_1, \ldots, \vec{r}_n$ and let $\vec{v}$ be a $1 \times m$ vector. Then

$$M\vec{v} = \begin{bmatrix}\vec{r}_1 \cdot \vec{v}\\ \vdots \\ \vec{r}_n \cdot \vec{v}\end{bmatrix}.$$

23

23.1 Let $M = \begin{bmatrix}1 & 5 & 2\\2 & 1 & 1\\0 & 1 & 0\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}8\\1\\1\end{bmatrix}$. Compute $M\vec{v}$.

24 Consider the system

$$\begin{array}{rrrl} x & +2y & +z & =1 \\ x & +2y & +3z & =2 \\ -x & -2y & +z & =3 \end{array} \qquad (4)$$

24.1 Write (4) as a vector equation.

24.2 Write (4) as a matrix equation (i.e., one of the form $M\vec{x}=\vec{b}$).

25 Let M be an $n\times m$ matrix with columns $\vec{c}_1,\dots\vec{c}_m$ and rows $\vec{r}_1,\dots,\vec{r}_n$. Let $\vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix}$.

25.1 Express $M\vec{x}$ in terms of $\vec{r}_1,\dots,\vec{r}_n$ and $\vec{x}$.

25.2 Express $M\vec{x}$ in terms of $\vec{c}_1,\dots,\vec{c}_m$ and $\vec{x}$.

Linear Combinations, Span, and Linear Independence

DEFINITION

Linear Combination

A ***linear combination*** of the vectors $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$ is a vector

$$\vec{w}=\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\cdots+\alpha_n\vec{v}_n$$

where $\alpha_1,\alpha_2,\dots,\alpha_n$ are scalars.

26 Let $\vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w}=2\vec{v}_1+\vec{v}_2$.

26.1 Write the coordinates of $\vec{w}$.

26.2 Draw a picture with $\vec{w}$, $\vec{v}_1$, and $\vec{v}_2$.

26.3 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

26.4 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

26.5 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

26.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

26.7 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

DEF

Span

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V=\{\vec{v}:\vec{v}=\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\cdots+\alpha_n\vec{v}_n \text{ for some } \vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\in V \text{ and scalars } \alpha_1,\alpha_2,\dots,\alpha_n\}.$$

27 Let $\vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3=\begin{bmatrix}2\\2\end{bmatrix}$.

27.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

27.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

27.3 Describe $\operatorname{span}\{\vec{v}_1,\vec{v}_2\}$.

27.4 Describe $\operatorname{span}\{\vec{v}_1,\vec{v}_3\}$.

27.5 Describe $\operatorname{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$.

In some sets, every vector is essential for computing a span. In others, there are "excess" vectors. This leads us to the concept of linear independence.

DEFINITION

Linearly Dependent & Independent

We say $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\},$$

and a set is ***linearly independent*** otherwise.

28 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

28.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

28.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

28.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

28.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

Task 1.1: The Magic Carpet Ride

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Task 1.2: The Magic Carpet Ride, Hide and Seek

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Task 1.3: The Magic Carpet, Getting Back Home

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.

1. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?

2. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?

3. What is span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}$?

Trivial Linear Combination

DEF We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

---

29 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

29.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

29.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

Linearly Dependent & Independent

DEF $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector.

---

30 30.1 Explain how this new definition implies the old one.

30.2 Explain how the old definition implies this new one.

Since we have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

---

31 Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

31.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

31.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

31.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

31.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

---

32 Consider the system

$$\begin{array}{rrrl} x & -y & -z & = 0 \\ 0x & +1y & +2z & = 0 \\ 3x & -3y & +3z & = 0 \end{array} \tag{5}$$

which has the unique solution $(x, y, z) = (0, 0, 0)$.

32.1 Give vectors $\vec{u}, \vec{v}, \vec{w}$ so that the system (5) corresponds to the vector equation $x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}$.

32.2 Is $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$? If so, write it as a linear combination of $\vec{u}$ and $\vec{v}$.

33 The matrix M is the non-augmented matrix corresponding to a homogeneous system of linear equations. M also corresponds to the vector equation $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$. Further, we know

$$\mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

33.1 Give a solution to the vector equation $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$.

33.2 Is $\vec{c} \in \mathrm{span}\{\vec{a}, \vec{b}\}$? If so, write it as a linear combination of $\vec{a}$ and $\vec{b}$.

33.3 Do you have enough information to tell if $\{\vec{a}, \vec{b}\}$ is linearly independent? Why or why not?

Finding Linearly Independent Subsets

34 Suppose when you use an augmented matrix to solve $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ you have no free variables.

34.1 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

Suppose when you use an augmented matrix to solve $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$, the second column (and only the second column) corresponds to a free variable.

34.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

34.3 Is $\{\vec{u}, \vec{w}\}$ linearly independent?

34.4 Is $\{\vec{u}, \vec{v}\}$ linearly independent?

Maximal Linearly Independent Subset

DEF

Given a set of vectors X, a ***maximal linearly independent subset*** of X is a linearly independent subset $V \subseteq X$ with the most possible vectors in it (i.e., if you took any subset of X with more vectors, it would be linearly dependent).

35

35.1 Give a maximal linearly independent subset, T, of $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

35.2 What is the size of T?

36 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \vec{v}_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and the matrices

$$A = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 2 & -1 & 1 & 2 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

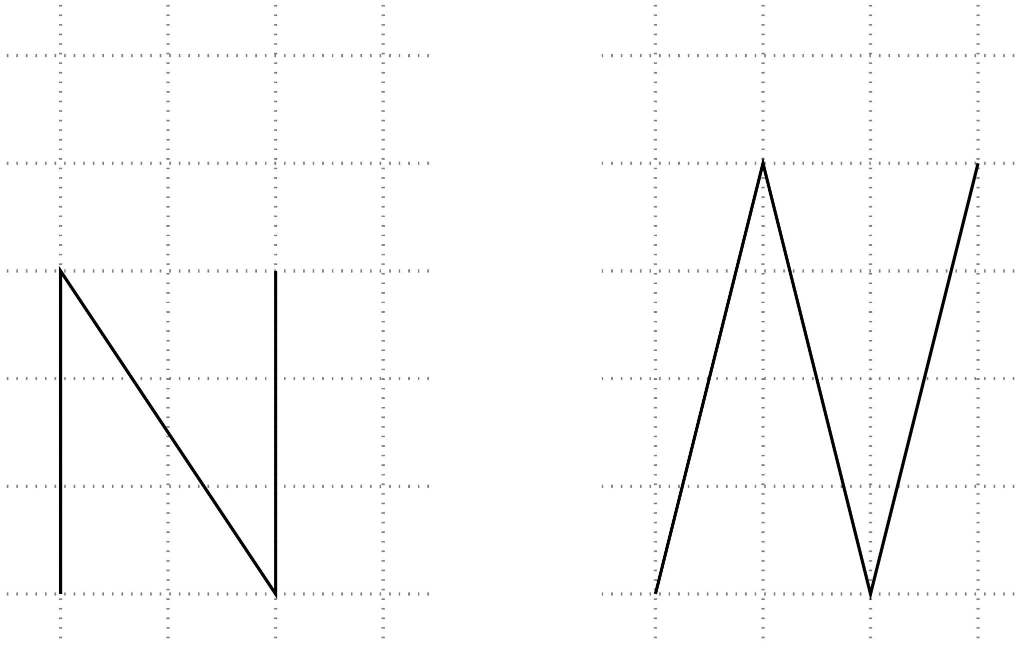
(Notice that the columns of A are the vectors $\vec{v}_1, \ldots, \vec{v}_5$)

36.1 Is $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ linearly independent?

36.2 Pick a maximal linearly independent subset of V.

36.3 Pick another (different) maximal linearly independent subset of V.

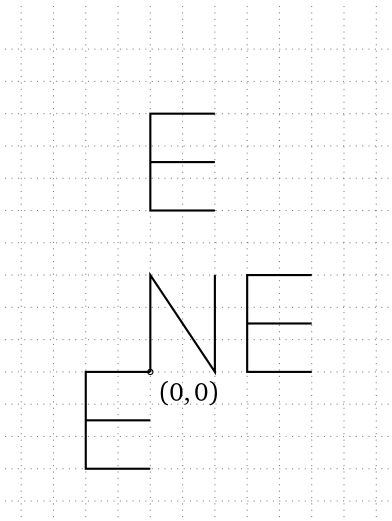
Task 2.1: Italicizing N



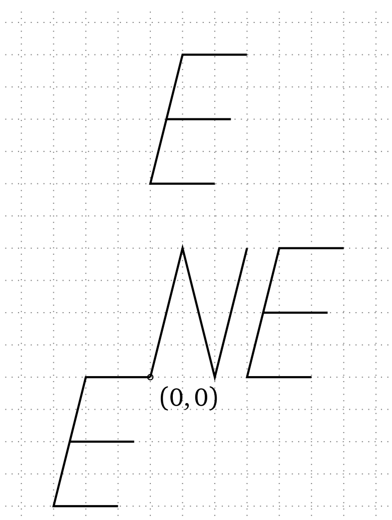
Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

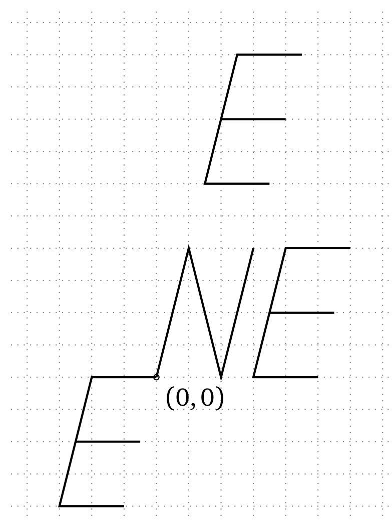
Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

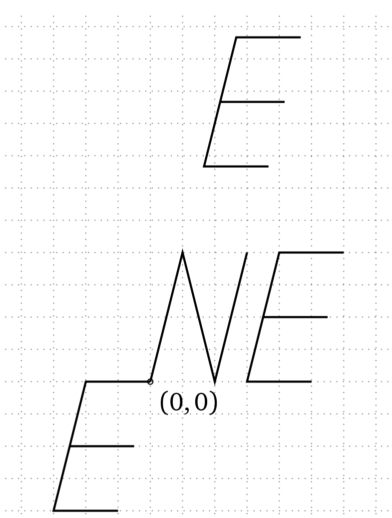
Task 2.2: Beyond the N

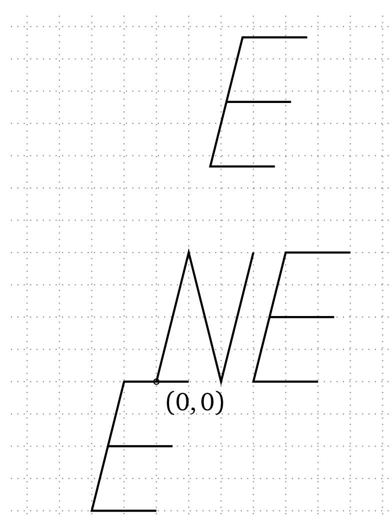


A few students were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If an "E" is placed around the "N," the students argued over four different possible results for the transformed E's. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?

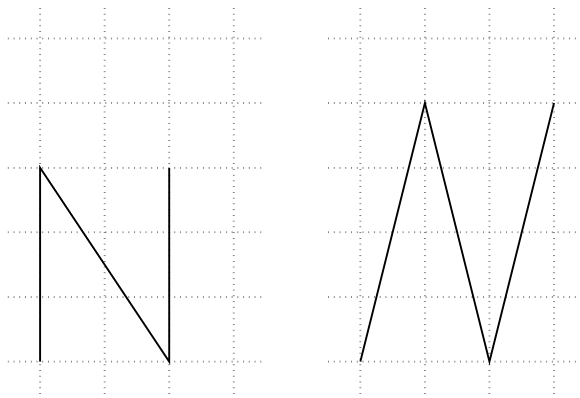








Task 2.3: Pat and Jamie



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

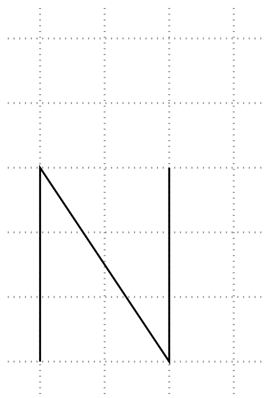
Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

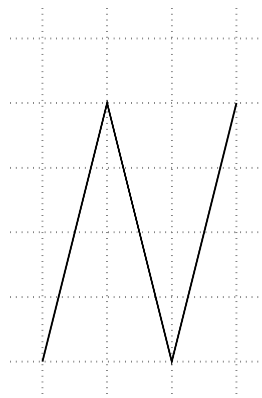
> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

1. Do you think Pat and Jamie's approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?

2. Try Pat and Jamie's approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

Task 2.4: Getting back N





Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

Consider the new task: find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.

2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

37 $\mathcal{R}:\mathbb{R}^2\to\mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

37.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

37.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

37.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

37.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter clockwise by 90°.

38 $\mathcal{S}:\mathbb{R}^3\to\mathbb{R}^3$ stretches in the $\mathbf{e}_3$ direction by a factor of 2 and contracts in the $\mathbf{e}_2$ direction by a factor of 3.

38.1 Write a matrix representation of $\mathcal{S}$.

Linear Transformation

DEFINITION

If V and W are vector spaces, a function $T:V\to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v})=T\vec{u}+T\vec{v} \qquad \text{and} \qquad T(\alpha\vec{v})=\alpha T\vec{v}$$

for all vectors $\vec{u},\vec{v}\in V$ and all scalars α.

39 39.1 Classify the following as linear transformation or not

(a) $\mathcal{R}$ from before.

(b) $\mathcal{S}$ from before.

(c) $W:\mathbb{R}^2\to\mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x^2\\y\end{bmatrix}$.

(d) $T:\mathbb{R}^2\to\mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2\\y\end{bmatrix}$.

(e) $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix}=\mathrm{proj}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

It turns out every (finite-dimensional) linear transformation can be written as a matrix (in fact this is why matrix multiplication was invented).

40 Define $\mathcal{P}$ to be projection onto $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

40.1 Write down a matrix for $\mathcal{P}$.

Matrix multiplication was designed to exactly model composition of linear transformations.

40.2 Write down a matrix for $\mathcal{P}$ and for $\mathcal{R}$, the counter-clockwise rotation by 90°.

40.3 Write down matrices for $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{P}$.

Range

DEF

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\mathrm{range}(T) = \{\vec{y} \in W : \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

Null Space

DEF

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$\mathrm{null}(T) = \{\vec{x} \in V : T\vec{x} = \vec{0}\}.$$

---

41 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the vector $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$ (like before).

41.1 What is the range of $\mathcal{P}$?

41.2 What is the null space of $\mathcal{P}$?

Fundamental Subspaces

DEF

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M is the span of the rows of M; the ***column space*** of M is the span of the columns of M; and the ***null space*** of M is the set of solutions to $M\vec{x} = \vec{0}$.

---

42 Consider $A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$.

42.1 Describe the row space of A.

42.2 Describe the column space of A.

42.3 Is the row space of A the same as the column space of A?

42.4 Describe the set of all vectors perpendicular to the rows of A.

42.5 Describe the null space of A.

43

$$B = \begin{bmatrix}1 & 2 & 3\\1 & 1 & 1\end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\end{bmatrix}$$

43.1 How does the row space of B relate to the row space of C?

43.2 How does the null space of B relate to the null space of C?

43.3 Compute the null space of B.

44

$$P = \begin{bmatrix}0 & 0\\1 & 2\end{bmatrix} \qquad Q = \mathrm{rref}(P) = \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

44.1 How does the column space of P relate to the column space of Q?

44.2 Describe the column space of P and the column space of Q.

One-to-One & Onto

DEF A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called ***one-to-one*** if $T(\vec{a}) = T(\vec{b})$ only when $\vec{a} = \vec{b}$. The transformation T is called ***onto*** if its range is all of $\mathbb{R}^m$.

---

45 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be matrix transformations and suppose

$$\text{null}(T) = \text{span}\{\mathbf{e}_1\} \qquad \text{and} \qquad \text{null}(S) = \{\vec{0}\}.$$

45.1 Could T be one-to-one? Explain.

45.2 Could S be one-to-one? Explain.

45.3 Could S be onto? Explain.

45.4 Could T be onto? Explain.

Matrix Inverses

46

46.1 Apply the row operation $R_3 \to R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

46.2 Apply the row operation $R_3 \to R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

DEF An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

46.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

46.4 Without computing, what should the result of applying the row operation $R_3 \to R_3 - 2R_1$ to E_1 be? Compute and verify.

46.5 Without computing, what should E_1E_2 be? What about E_2E_1? Now compute and verify.

DEF The ***inverse*** of an $n \times n$ matrix A is an $n \times n$ matrix B such that $AB = I_{n\times n} = BA$. In this case, B is called the inverse of A and is notated as A^{-1}.

---

47 Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

47.1 Which pairs of matrices above are inverses of each other?

48

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

48.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

48.2 What is $E_2 E_1 B$?

48.3 Find B^{-1}.

48.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

49

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

49.1 What is $A^{-1}A$?

49.2 What is $\mathrm{rref}(A)$?

49.3 What is $\mathrm{rref}(C)$? (Hint, there is no need to actually do row reduction!)

49.4 Solve the system $A\vec{x} = \vec{b}$.

50

50.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

50.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

Subspaces and Bases

Subspace

DEFINITION

A ***subspace*** $V \subseteq \mathbb{R}^n$ is a non-empty subset such that

(i) $\vec{u}, \vec{v} \in V$ implies $\vec{u} + \vec{v} \in V$.

(ii) $\vec{u} \in V$ implies $k\vec{u} \in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

51

For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

51.1 $A = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ for some $a \in \mathbb{Z}\}$.

51.2 $B = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$.

51.3 $C = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

51.4 $D = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

51.5 $E = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

51.6 $F = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

51.7 $G = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

51.8 $H = \mathrm{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

52 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

52.1 Show that the null space of T is a subspace.

52.2 Show that the range of T is a subspace.

Basis

DEF A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

53 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

53.1 Describe V.

53.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

53.3 Give a basis for V.

53.4 Give another basis for V.

53.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

Dimension

DEF The ***dimension*** of a subspace V is the number of elements in a basis for V.

53.6 What is the dimension of V?

54 Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}7\\8\\8\end{bmatrix}$ and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

54.1 Give a basis for and the dimension of P.

54.2 Give a basis for and the dimension of Q.

54.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

54.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Rank

Rank

DEF The ***rank*** of the matrix A is the number of leading ones in the reduced row echelon form of A.

55

55.1 Determine the rank of (a) $\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ (c) $\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}3\\3\\2\end{bmatrix}$ (e) $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

56 Consider the homogeneous system

$$\begin{array}{rrrl} x & +2y & +z & =0 \\ x & +2y & +3z & =0 \\ -x & -2y & +z & =0 \end{array} \tag{6}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$.

56.1 What is rank(A)?

56.2 Give the general solution to (6).

56.3 Are the column vectors of A linearly independent?

56.4 Give a non-homogeneous system with the same coefficients as (6) that has

(a) infinitely many solutions

(b) no solutions.

57 57.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

57.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of columns in } A.$$

58 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

58.1 What is the rank and nullity of A^T?

58.2 What is the rank and nullity of A?

Orthogonality

Orthogonal Projection

DEF

If V is a subspace of $\mathbb{R}^n$, the ***projection*** (sometimes called the orthogonal projection) of $\vec{x}$ onto V is the closest point in V to $\vec{x}$. We notate the projection of $\vec{x}$ onto V as $\operatorname{proj}_V \vec{x}$.

Projections are normally hard to compute and a priori might require some sort of calculus-style optimization to find. However, from geometry we know that if we travel from $\operatorname{proj}_V \vec{x}$ to $\vec{x}$, we should always trace out a path perpendicular to V. Otherwise, we could find a point in V that was slightly closer to $\vec{x}$, violating the definition of $\operatorname{proj}_V \vec{x}$. Thus, orthogonality will be our savior.

59 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis.

59.1 If $\vec{x} = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$, find the projection of $\vec{x}$ onto the xy-plane.

Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

59.2 If $\vec{y} = 3\vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$, find the projection of $\vec{y}$ onto span$\{\vec{b}_1, \vec{b}_3\}$.

Suppose $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is a basis for $\mathbb{R}^3$ with

$$\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1 \qquad \vec{c}_1 \cdot \vec{c}_2 = 0 \qquad \vec{c}_1 \cdot \vec{c}_3 = 0 \qquad \vec{c}_2 \cdot \vec{c}_3 = \sqrt{2}/2.$$

59.3 If $\vec{z} = 5\vec{c}_1 + 2\vec{c}_2 - \vec{c}_3$, find the projection of $\vec{z}$ onto span $\{\vec{c}_1, \vec{c}_2\}$.

60 Let's put this all together. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Let $\mathcal{P}$ be the plane defined by

$$0x + y - z = 0.$$

60.1 Write $\mathcal{P}$ in vector form (Hint: think about the vectors listed in the $\mathcal{B}$ basis).

60.2 Find an orthonormal basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ for $\mathbb{R}^3$ so $\mathcal{P} = \operatorname{span}\{\vec{c}_1, \vec{c}_2\}$.

60.3 Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\operatorname{proj}_{\mathcal{P}} \vec{x}$.

Gram-Schmidt Orthogonalization

We've seen how useful orthonormal bases are. The incredible thing is that we can turn any basis into an orthonormal basis through a process called Gram-Schmidt orthogonalization.

61 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

61.1 Draw $\vec{a}$ and $\vec{b}$ and find $\vec{w} = \operatorname{proj}_{\vec{b}} \vec{a}$.

61.2 Add $\vec{c} = \vec{a} - \vec{w}$ to your drawing. What is the angle between $\vec{c}$ and $\vec{b}$.

61.3 Can you write $\vec{a}$ as the sum of two vectors, one in the direction of $\vec{b}$ and one orthogonal to $\vec{b}$? If so, do it.

62 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

62.1 Write $\vec{a} = \vec{u} + \vec{v}$ where $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is orthogonal to $\vec{b}$.

62.2 Find an orthonormal basis for $\operatorname{span}\{\vec{a}, \vec{b}\}$.

With two vectors, making an orthonormal set without changing the span is quite easy. With more vectors, it is only slightly harder.

DEFINITION

Gram-Schmidt Process

The ***Gram-Schmidt*** orthogonalization procedure takes in a set of vectors and outputs a set of orthonormal vectors with the same span. The idea is to iteratively produce a set of vectors where each new vector you produce is orthogonal to the previous vectors.

The algorithm is as follows: Let $\{v_1, \ldots, v_n\}$ be a set of vectors. Produce a set $\{v_2', \ldots, v_n'\}$ that is orthogonal to v_1 by subtracting off the respective projections of $v_2, \ldots, v_n$ onto v_1. Next, produce a set $\{v_3'', \ldots, v_n''\}$ orthogonal to both v_1 and v_2' by subtracting off the respective projections onto v_2'. Continue this process until you have a set $V = \{v_1, v_2', v_3'', v_4''', \ldots\}$ that is orthogonal. Finally, normalize V so all vectors have unit length.

---

63 Let $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$.

63.1 Use the Gram-Schmidt procedure to find an orthonormal basis for $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

63.2 Find an orthonormal basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ for $\mathbb{R}^4$ so that $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Let $R = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$.

63.3 Find an orthonormal basis for the row space of R.

63.4 Find the null space of R (Hint, you've already done the work, so there is no need to row reduce).

---

64 Let

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{y}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \qquad \text{and} \qquad \mathcal{W} = \operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}.$$

64.1 Find an orthonormal basis $\mathcal{B}$ for $\mathcal{W}$.

DEF

Orthogonal Complement

The ***orthogonal complement*** of a subspace V is written $V^\perp$ and defined as

$$V^\perp = \{\vec{x} : \vec{x} \text{ is orthogonal to all vectors in } V\}.$$

64.2 Find the orthogonal complement of $\mathcal{W}$.

64.3 Write $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in the form $\vec{v} = \vec{r} + \vec{n}$ where $\vec{r} \in \mathcal{W}$ and $\vec{n} \in \mathcal{W}^\perp$.

THM

Orthogonal Decomposition

If $\mathcal{V} \subseteq \mathbb{R}^n$ is a subspace than any vector $\vec{x} \in \mathbb{R}^n$ can be written uniquely as $\vec{x} = \vec{v} + \vec{w}$ where $\vec{v} \in \mathcal{V}$ and $\vec{w} \in \mathcal{V}^\perp$.

---

65 Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. For each of the following $\mathcal{V}$, decompose $\vec{x}$ into the sum of a vector in $\mathcal{V}$ and one in $\mathcal{V}^\perp$.

65.1 $\mathcal{V}$ is the x-axis.

65.2 $\mathcal{V} = \{\vec{0}\}$.

65.3 $\mathcal{V}$ is the plane $x + y + z = 0$.

65.4 $\mathcal{V} = \mathbb{R}^3$.

DEF

Direct Sum

Suppose $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ are two subspaces and every vector $\vec{x} \in \mathbb{R}^n$ can be uniquely expressed as $\vec{x} = \vec{v} + \vec{w}$ for $\vec{v} \in \mathcal{V}$ and $\vec{w} \in \mathcal{W}$. Then we say $\mathbb{R}^n$ is the ***direct sum*** of $\mathcal{V}$ and $\mathcal{W}$ and write

$$\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W}.$$

THEOREM

Fundamental Theorem of Linear Algebra

Let A be an $n \times m$ matrix. Then

$$\mathrm{row}(A) = \mathrm{null}(A)^\perp \qquad \text{and} \qquad \mathbb{R}^m = \mathrm{row}(A) \oplus \mathrm{null}(A)$$

and

$$\mathrm{col}(A) = \mathrm{null}(A^T)^\perp \qquad \text{and} \qquad \mathbb{R}^n = \mathrm{col}(A) \oplus \mathrm{null}(A^T).$$

66 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and let $\vec{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

66.1 Verify that $\vec{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -6 \\ 10 \end{bmatrix}$ are all solutions to $A\vec{x} = \vec{b}$.

66.2 Decompose $\vec{u}$, $\vec{v}$, and $\vec{w}$ into the sum of vectors in $\mathrm{row}(A)$ and $\mathrm{null}(A)$. What do you notice?

66.3 Suppose $\vec{x} \in \mathrm{row}(A)$ and $A\vec{x} = \vec{b}$. What is $\vec{x}$? Is it unique?

66.4 Consider the transformation $T : \mathrm{row}(A) \to \mathbb{R}^2$ where $T(\vec{y}) = A\vec{y}$. Is T invertible? Is A invertible? Explain.

Determinants

DEFINITION

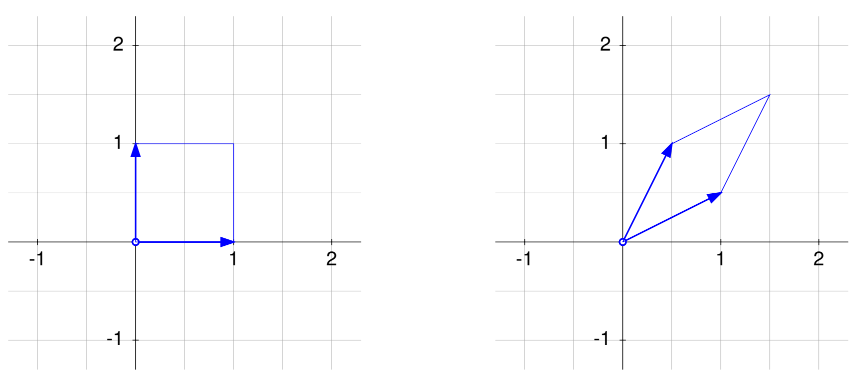
Unit n-cube

The unit n-cube is the n-dimensional cube with side length 1 and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The volume of the unit n-cube is always 1.

67 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



67.1 What is $T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

67.2 Write down a matrix for T.

67.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may need to use trigonometry.

Determinant

DEF The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors or the row vectors.

---

68 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

68.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

68.2 Compute the area of $A(C_2)$.

68.3 Compute $\det(A)$.

---

69 Suppose R is a rotation counterclockwise by $30°$.

69.1 Draw C_2 and $R(C_2)$.

69.2 Compute the area of $R(C_2)$.

69.3 Compute $\det(R)$.

---

70 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

70.1 What is $\det(F)$?

71

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \to R_1 + 2R_2$ applied.

71.1 What is $\det(E_f)$?

71.2 What is $\det(E_m)$?

71.3 What is $\det(E_a)$?

71.4 What is $\det(E_f E_m)$?

71.5 What is $\det(4I_{3\times 3})$?

71.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

72

$$U=\begin{bmatrix}1&2&1&2\\0&3&-2&4\\0&0&-1&0\\0&0&0&4\end{bmatrix}$$

72.1 What is $\det(U)$?

72.2 V is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

72.3 P is projection onto the vector $\begin{bmatrix}-1\\-1\end{bmatrix}$. What is $\det(P)$?

73 Suppose you know $\det(X) = 4$.

73.1 What is $\det(X^{-1})$?

73.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

73.3 Suppose Y is not invertible. What is $\det(Y)$?

Eigenvectors

Eigenvector

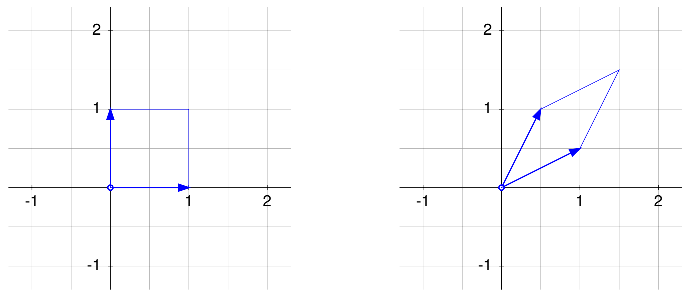
DEFINITION

For a linear transformation X, an ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

74 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



74.1 Give an eigenvector for T. What is the eigenvalue?

74.2 Can you give another?

75 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

75.1 Give an eigenvector and a corresponding eigenvalue for A.

75.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

75.3 What is the dimension of $\text{null}(B)$?

75.4 What is $\det(B)$?

76 Let $C = \begin{bmatrix}-1 & 2\\ 1 & 0\end{bmatrix}$ and $E_\lambda = C - \lambda I$.

76.1 For what values of λ does E_λ have a non-trivial null space?

76.2 What are the eigenvalues of C?

76.3 Find the eigenvectors of C.

Characteristic Polynomial

DEF

For a matrix A, the ***characteristic polynomial*** of A is

$$\mathrm{char}(A) = \det(A - \lambda I).$$

77 Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

77.1 Compute $\mathrm{char}(D)$.

77.2 Find the eigenvalues of D.

78 Suppose $\mathrm{char}(E) = \lambda(\lambda - 2)(\lambda + 3)$ for some unknown 3×3 matrix E.

78.1 What are the eigenvalues of E?

78.2 Is E invertible?

78.3 What is $\mathrm{nullity}(E)$, $\mathrm{nullity}(E - 3I)$, $\mathrm{nullity}(E + 3I)$?

79 Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for A.

79.1 Find the eigenvalues of A.

79.2 Find the characteristic polynomial of A.

79.3 Compute $A\vec{w}$ where $w = 2\vec{v}_1 - \vec{v}_2$.

79.4 Compute $A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

79.5 If $\vec{x} = \vec{v}_1 + 3\vec{v}_2 + 4\vec{v}_4$ is $\vec{x}$ written as a linear combination of vectors in $\mathcal{V}$, compute $A\vec{x}$ as a linear combination of vectors in $\mathcal{V}$.

80 Let A, $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be as in the previous problem, let $P = [\vec{v}_1|\vec{v}_2|\vec{v}_3]$ be the matrix with columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, and let $\vec{b} \in \mathbb{R}^3$ be a fixed vector.

80.1 Describe what a solution to $P\vec{x} = \vec{b}$ means in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

80.2 Describe how to interpret the output of the linear transformation P^{-1} in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

80.3 Describe how you can use P and P^{-1} to easily compute $A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

80.4 Can you find a matrix D so that

$$PDP^{-1} = A?$$

80.5 Suppose $P^{-1}\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $A^{100}\vec{b}$.

81 For an $n \times n$ matrix T, suppose its eigenvectors $\{\vec{v}_1, \dots \vec{v}_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues.

81.1 Is T diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

81.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

81.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Eigenspace

DEFINITION

Let A be a matrix with eigenvalues $\{\lambda_1, \dots, \lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

---

82 Define $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

82.1 Is F diagonalizable? Why or why not?

82.2 What is the geometric and algebraic multiplicity of each eigenvalue of F?

82.3 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?

Span Again

83

$$A=\begin{bmatrix}1 & 2\\ 3 & 1\\ 0 & -1\end{bmatrix} \qquad B=\begin{bmatrix}-1 & -1\\ 0 & 1\\ 1 & -2\end{bmatrix} \qquad C=\begin{bmatrix}1 & 2 & 0\\ -1 & -1 & -1\end{bmatrix}$$

83.1 Write the shape of the matrices A, B, C (i.e., for each one, write the dimensions in $m \times n$ form).

83.2 List *all* products between the matrices A, B, C that are defined. (Your list will be some subset of AB, AC, BA, CA, BC, CB.)

83.3 Compute AC and CA.

84

84.1 If the matrices X and Y are both square $n \times n$ matrices, does $XY = YX$? Explain.

84.2 If the matrices X and Y are both square $n \times n$ matrices, does $X + Y = Y + X$? Explain.

85

Consider the system represented by

$$\begin{bmatrix}1 & -3 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \vec{b}.$$

85.1 If $\vec{b} = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

85.2 If $\vec{b} = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

86

The entries of a matrix are specified by (row,column) pairs of integers. If a_{ij} is the (i, j) entry of a matrix A, we may write $A = [a_{ij}]$.

86.1 Write the 2×2 matrix A with entries $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 7$ and $a_{22} = 9$.

86.2 Let $B = [b_{ij}]$ be the 3×3 matrix where $b_{ij} = i + j$. Write B.

86.3 Let $C = [c_{ij}]$ be the 3×4 matrix where $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ if $i \neq j$.

87

DEF The ***transpose*** of a matrix $A = [a_{ij}]$ is the matrix $A^T = [a_{ji}]$.

Visually, the transpose of a matrix swaps rows and columns.

$$A=\begin{bmatrix}1 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$$

87.1 What is the shape of A and A^T?

87.2 Write down A^T.

B and D are 4×6 matrices and C is a 6×4 matrix.

87.3 Does $(BC)^T = B^T C^T$? Explain.

87.4 Does $(B + D)^T = B^T + D^T$? Explain.

87.5 Compute AA^T and $A^T A$ (where A is the matrix defined earlier). What do you notice?

88

DEF A matrix X is called ***symmetric*** if $X = X^T$.

Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

88.1 Prove that if W is a square matrix, then $V = W^TW + W + W^T$ is a symmetric matrix.

---

89 DEF A ***zero matrix*** is a matrix whose entries are all zeros. An ***identity matrix*** is a square matrix whose diagonal entries are 1 and non-diagonal entries are 0.

We write the $m \times n$ zero matrix as $0_{m\times n}$ or just 0 if the shape is determined by context. The $n \times n$ identity matrix is notated $I_{n\times n}$ or just I if the shape is determined by context.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

89.1 Write down the 3×3 identity matrix and the 3×3 zero matrix.

89.2 Compute $I_{3\times 3}A$, $AI_{3\times 3}$, $0_{3\times 3}A$, and $A0_{3\times 3}$.

89.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

---

90 90.1 Solve the matrix equation

$$I_{4\times 4}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

Change of Basis

---

91 Let $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$.

91.1 Is $\mathcal{B}$ a basis for $\mathbb{R}^2$?

91.2 Find coefficients α_1 and α_2 so that $\vec{c} = \alpha_1\vec{b}_1 + \alpha_2\vec{b}_2$.

We call the vector $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ the representation of $\vec{c}$ in the $\mathcal{B}$ basis and notate this by $[\vec{c}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$.

91.3 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

Let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

91.4 Compute $X[\vec{c}]_{\mathcal{B}}$. What do you notice?

---

92 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}.$$

92.1 Should X^{-1} exist? Explain.

92.2 Consider the equation

$$X^{-1}[\vec{v}]_? = [\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

92.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

93 Let $\vec{c}_1=\begin{bmatrix}2\\1\end{bmatrix}$, $\vec{c}_2=\begin{bmatrix}5\\3\end{bmatrix}$, $\mathcal{C}=\{\vec{c}_1,\vec{c}_2\}$, and $A=\begin{bmatrix}2&5\\1&3\end{bmatrix}$. Note that $A^{-1}=\begin{bmatrix}3&-5\\-1&2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

93.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

93.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}5\\3\end{bmatrix}$.

93.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

93.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

93.5 Find a matrix for T in the $\mathcal{C}$ basis.

93.6 Find a matrix for T in the standard basis.

DEFINITION

Similar Matrices

A matrix A and a matrix B are ***similar matrices***, denoted $A\sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A\sim B$ if there is an invertible matrix X so that

$$A=XBX^{-1}.$$

After all this work with determinants, we see that (like dot products) there is a geometric and an algebraic way of thinking about them, and they *determine* if a matrix is invertible.

94 94.1 The linear transformation $L:\mathbb{R}^3\to\mathbb{R}^3$ is a change of coordinates and $\det(L)=-4$. What is the volume form for this change of coordinates?

94.2 Suppose $P:\mathbb{R}^2\to\mathbb{R}^2$ is the parameterization defined by $P\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&9\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}$. Find the volume form for P.

94.3 Suppose $p:\mathbb{R}^2\to\mathbb{R}^2$ is the parameterization defined by $p(r,\theta)=(r\cos\theta,r\sin\theta)$. Find a linear approximation to p at the point (r_0,θ_0). Use determinants to compute the volume form for p at (r_0,θ_0).

DEFINITION

Jacobian

Let $p:\mathbb{R}^n\to\mathbb{R}^n$ be a parameterization. Let $L_{\vec{x}_0}(\vec{x})=J_{\vec{x}_0}\vec{x}+\vec{q}_{\vec{x}_0}$ be the linear approximation to p at the point $\vec{x}_0$. The ***Jacobian*** of p at the point $\vec{x}_0$ is defined to be

$$\mathrm{Jacob}_{\vec{x}_0}(p)=\det(J_{\vec{x}_0}).$$

Orthogonality

Orthogonal & Orthonormal

DEF A set of vectors is ***orthogonal*** if every pair of vectors in the set is orthogonal. A set of vectors is ***orthonormal*** if it is both an orthogonal set and every vector is a unit vector.

95

$$\mathcal{B}=\{\vec b_1,\vec b_2\}\qquad \vec b_1=\begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix}\qquad \vec b_2=\begin{bmatrix}-\sqrt{3}/2\\ 1/2\end{bmatrix}$$

The matrix $A=[\vec b_1|\vec b_2]$ takes vectors in the $\mathcal{B}$ basis and rewrites them in the standard basis.

95.1 What does A^{-1} do?

95.2 Find a matrix B that takes vectors in the standard basis and rewrites them in the $\mathcal{B}$ basis.

95.3 Write $\vec x=\begin{bmatrix}1\\2\end{bmatrix}$ in the $\mathcal{B}$ basis.

95.4 What is the relationship between A and B?

Orthogonal Matrix

DEF An ***orthogonal matrix*** is a square matrix whose columns are orthonormal (Yes, a better name would be orthonormal matrix, but that is not the term the rest of the world uses).

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96 Suppose $X=[\vec x_1|\vec x_2|\vec x_3|\vec x_4]$ is an orthogonal matrix.

96.1 What is the shape of X (i.e., it is a what×what matrix)?

96.2 Compute X^TX.

96.3 What is X^{-1}?

97

$$Y=\begin{bmatrix}1&1&1&-1\\1&-1&-1&-1\\1&1&-1&1\\1&-1&1&1\end{bmatrix}$$

97.1 Is Y an orthogonal matrix?

97.2 Fix Y so it is an orthogonal matrix. Call the new matrix X.

97.3 Compute X^{-1}.

97.4 Compute Y^{-1}.

97.5 Compute $|\det(X)|$ and $|\det(Y)|$ (the absolute value of the determinant of X and Y).

Matrix equations involving orthogonal matrices are easy to solve because the inverse of an orthogonal matrix is so easy to compute!

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98 Let $A=[\vec a_1|\vec a_2|\vec a_3|\vec a_4]$ be an orthogonal matrix.

98.1 Explain why $\vec x=\begin{bmatrix}\vec a_1\cdot\vec b\\ \vec a_2\cdot\vec b\\ \vec a_3\cdot\vec b\\ \vec a_4\cdot\vec b\end{bmatrix}$ is a solution to $A\vec x=\vec b$.

98.2 Find scalars a,b,c,d so $\vec b=a\vec a_1+b\vec a_2+c\vec a_3+d\vec a_4$ (your answers will have variables in them).

Orthogonal matrices also allow us to compute projections quite easily.

QR Decomposition

DEF — **QR Decomposition**

For a matrix A, we can rewrite $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Writing A as QR is called the ***QR decomposition*** of A.

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99 Suppose A, B, C are square matrices and $C = AB$.

99.1 How do the column spaces of A and C relate?

99.2 How do the column spaces of B and C relate?

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100 $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms a basis for $\mathbb{R}^3$. When we apply the Gram-Schmidt process to $\mathcal{V}$, we get

$$\begin{aligned} q_1' &= \vec{v} \\ q_2' &= \vec{v}_2 - \tfrac{1}{2}\vec{v}_2 \\ q_3' &= \vec{v}_3 - \vec{v}_1 + 2\vec{v}_2 \end{aligned}$$

form an orthogonal set. Normalizing we get

$$\begin{aligned} \vec{q}_1 &= 2q_1' \\ \vec{q}_2 &= 3q_2' \\ \vec{q}_3 &= \tfrac{1}{2}q_3' \end{aligned}$$

form an orthonormal set.

100.1 Write $\vec{v}_1$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

100.2 Write $\vec{v}_2$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

100.3 Write $\vec{v}_3$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

Define $A = [\vec{v}_1|\vec{v}_2|\vec{v}_2]$ and $Q = [\vec{q}_1|\vec{q}_2|\vec{q}_3]$.

100.4 Find a matrix R so that $A = QR$.

We've just discovered one process to find the QR decomposition of a matrix. It's really as simple as doing Gram-Schmidt and keeping track of your coefficients. Now, we have another way to the matrix equation $A\vec{x} = \vec{b}$. If we do a QR decomposition and exploit the fact that $Q^{-1} = Q^T$, we have

$$A\vec{x} = QR\vec{x} = \vec{b} \qquad \Longrightarrow \qquad R\vec{x} = Q^T\vec{b}$$

and R is a triangular matrix, so we can just do back substitution! (It turns out that if you solve systems this way, there is less rounding error than if you use row reduction.)

Symmetric Matrices

When you're new to Linear Algebra, learning lots of new concepts and algorithms, it's sometimes hard to grasp the significance of certain properties of a matrix.

Symmetric matrices are easy to forget at first, but they have many profound properties (not to mention they are one of the key concepts of Quantum Mechanics).

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101 Let A be a symmetric matrix and let $\vec{v}$ be an eigenvector with eigenvalue 3 and $\vec{w}$ be an eigenvector with eigenvalue 4. Note, for this problem, we are thinking of $\vec{v}$ and $\vec{w}$ as column vectors.

101.1 Write $A\vec{v}$, $\vec{v}^TA^T$, $\vec{v}^TA$, $A\vec{w}$, $\vec{w}^TA^T$, and $\vec{w}^TA$ in terms of $\vec{v}$, $\vec{w}$ and scalars.

101.2 How do $\vec{v}^T\vec{w}$ and $\vec{w}^T\vec{v}$ relate?

101.3 What should $\vec{v}^TA\vec{w}$ be in terms of $\vec{v}^T$ and $\vec{w}$? (Note, you could compute $(\vec{v}^TA)\vec{w}$ or $\vec{v}^T(A\vec{w})$. Better do both to be safe).

101.4 What can you conclude about $\vec{v}^T\vec{w}$? How about $\vec{v}\cdot\vec{w}$?

We've just deduced that all eigenspaces of a symmetric matrix are orthogonal! On top of that, symmetric matrices always have a basis of eigenvectors. That means that not only can you always diagonalize a symmetric matrix, but you can *orthogonally* diagonalize a symmetric matrix. (i.e. if A is symmetric, then $A = QDQ^T$ where Q is orthogonal and D is diagonal). This is like the best of all worlds in one!