Learning Objectives

> Projections are used in mathematics, physics, computer science, and statistics. In this tutorial you will obtain a deeper understanding of projections.
>
> These problems relate to the following course learning objectives: *Translate between algebraic and geometric viewpoints to solve problems*, and *work independently to understand concepts and procedures that have not been previously explained to you*.

Problems

Let R be the square with side-length 1 and lower-left corner at $(0,0)$; let C be the corners of R; and, let S be the circle of radius $\sqrt{2}$ centered at $(0,0)$.

1. Write down a mathematically precise definition of the projection of the vector $\vec{v}$ onto the set X.

2. Let $\vec{v}_1 = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix} 1/3 \\ 2 \end{bmatrix}$.
   (a) Draw R, C, and S on separate grids.
   (b) Using your drawings, estimate the projections of $\vec{v}_1, \dots, \vec{v}_4$ onto R, C, and S.
   (c) Compute exactly the projections of $\vec{v}_1, \dots, \vec{v}_4$ onto R, C, and S.

3. Let ℓ be the line given in vector form by $\vec{x} = t\vec{d}$ where $\vec{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let $\vec{r} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.
   (a) Find a formula for the distance between $t\vec{d}$ and $\vec{r}$ in terms of t.
   (b) Compute $\mathrm{proj}_\ell \vec{r}$.
   (c) What is the angle between $\vec{d}$ and $\vec{r} - \mathrm{proj}_\ell \vec{r}$?
   (d) Explain the link between $\mathrm{proj}_\ell \vec{r}$ and $\mathrm{comp}_{\vec{d}} \vec{r}$.

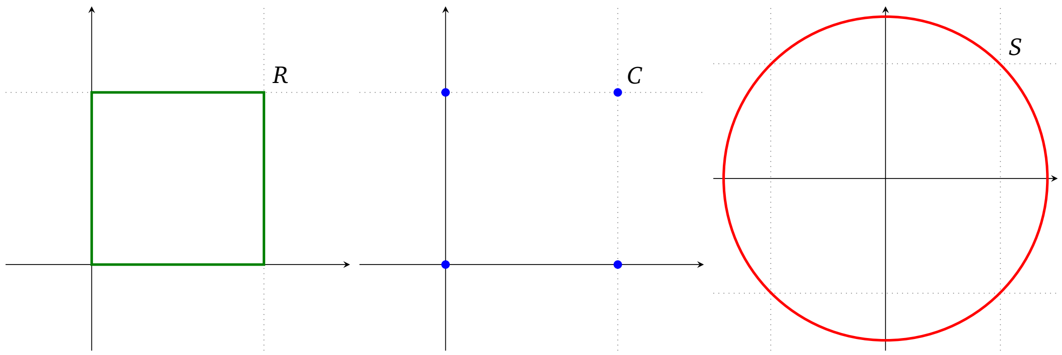
4. In $\mathbb{R}^3$, the projection of a vector onto the xy-plane can be thought of as the shadow of the vector at high noon. But where would this shadow be when it's not high noon?

   To simplify things, let's work in $\mathbb{R}^2$. Assume the x-axis is the "ground" and the y-axis is pointing straight up. If the sun is located far off in the distance in the direction of $\vec{e}_2$, the shadow of a vector will be its projection onto the x-axis.

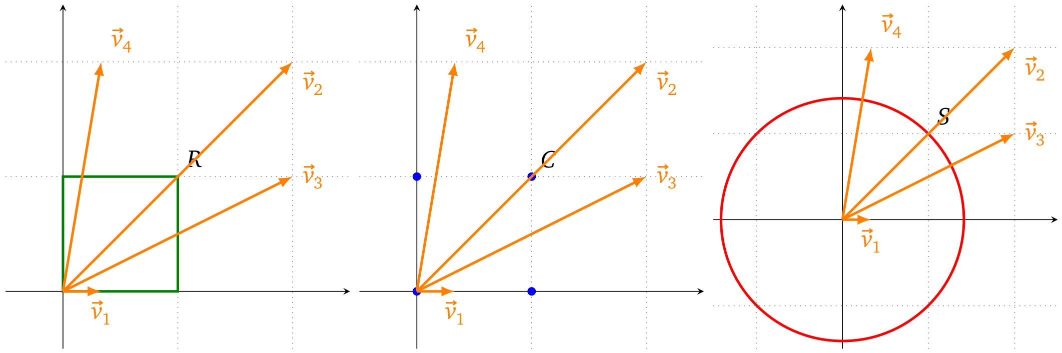
   (a) Find a formula for where the shadow of the vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ will be at high noon.
   (b) Suppose the ground is modeled with the equation $y = \frac{1}{2}x$ and that at 4:00pm, the sun shines perpendicularly to the ground. That is, the sun is far away in the direction $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find a formula for the shadow of the vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ on the slanted ground at 4:00pm.
   (c) The vector $\vec{y} = \begin{bmatrix} a \\ b \end{bmatrix}$ is above flat ground (i.e., the ground is modeled by the x-axis). Find a formula for the position of the shadow of $\vec{y}$ at 4:00pm.

1. The *projection* of $\vec{v}$ onto X is the closest point in X to $\vec{v}$.

2. (a)



(b)



(c)

$$\text{proj}_R\vec{v}_1 = \vec{v}_1 \qquad \text{proj}_R\vec{v}_2 = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{proj}_R\vec{v}_3 = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{proj}_R\vec{v}_4 = \begin{bmatrix}1/3\\1\end{bmatrix}$$

$$\text{proj}_C\vec{v}_1 = \begin{bmatrix}0\\0\end{bmatrix} \qquad \text{proj}_C\vec{v}_2 = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{proj}_C\vec{v}_3 = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{proj}_C\vec{v}_4 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\text{proj}_S\vec{v}_1 = \begin{bmatrix}\sqrt{2}\\0\end{bmatrix} \qquad \text{proj}_S\vec{v}_2 = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{proj}_S\vec{v}_3 = \frac{\sqrt{2}\vec{v}_3}{\|\vec{v}_3\|} \qquad \text{proj}_S\vec{v}_4 = \frac{\sqrt{2}\vec{v}_4}{\|\vec{v}_4\|}$$

3. (a) $f(t) = \sqrt{(t-2)^2 + (2t-2)^2} = \sqrt{5t^2 - 12t + 8}$.

   (b) $\text{proj}_\ell\vec{r}$ must be the multiple of $\vec{d}$ that is closest to $\vec{r}$. Notice that $f(t)$ is minimized when $5t^2 - 12t + 8$ is minimized which happens when $t = 6/5$. Therefore

$$\text{proj}_\ell\vec{r} = \tfrac{6}{5}\begin{bmatrix}1\\2\end{bmatrix}.$$

   (c) 90°.

   (d) They are the same thing! To find the closest point on a line to a point, you draw a perpendicular to the line passing through the point. The intersection with the line is the closest point and is therefore the projection. However, the line in this case is given by all multiples of $\vec{d}$, so geometrically, $\text{comp}_{\vec{d}}\vec{r}$ is defined in the exact same way as $\text{proj}_\ell\vec{r}$ (in this case).

4. (a) $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$.

(b) Let $\vec{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $\vec{x} \mapsto \text{comp}_{\vec{d}}\vec{x} = \frac{2x+y}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) Let $\vec{s} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ be the direction of the sunlight and let $\vec{x}$ be as before. We are looking for when $\vec{x} + t\vec{s}$ has a zero y-coordinate, which happens when $t = -y/2$. Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} + \frac{-y}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x + y/2 \\ 0 \end{bmatrix}.$$

Learning Objectives

Students need to be able to...

- Compute projections from the definition without memorizing a formula
- Exploit right angles to compute projections that would otherwise be difficult

Context

Students have covered projections and components in class two weeks ago. Last week they started working with matrices/matrix multiplication.

In this class, $\operatorname{proj}_X \vec{v}$ is defined as the closest vector in X to $\vec{v}$. It is *not* defined in terms of a formula. Also, to avoid confusion, we *don't write* $\operatorname{proj}_{\vec{u}} \vec{v}$, since this is easily confused with projection onto a singleton. Instead, the "component of $\vec{v}$ in the direction $\vec{u}$", written $\operatorname{comp}_{\vec{u}} \vec{v}$ is defined to be the vector in the direction of $\vec{u}$ so that $\vec{v} - \operatorname{comp}_{\vec{u}} \vec{v}$ is orthogonal to $\vec{u}$. There is a formula for this operation (unlike projections), and they should all know it since they've used it on their homework.

What to Do

Introduce the learning objectives for the day's tutorial. Explain that, like most things in life, there is no algorithm to compute projections—the only way to do it is to understand the idea. Since we're practicing learning and understanding non-algorithmic tasks in this class, projections provide the perfect place to practice.

Have students pair up and write the definition of projection. Again, make them write it. These definitions will show up on the midterms and many will write them wrong. Again, you might have some groups come up to the board and write their definitions and then have a short class discussion on whether they are right. This definition will be easier for them since there aren't any quantifiers.

After everyone is on the same page with the definition, have them continue on #2. The point of #2 is to have them cement in their minds the link between projections and orthogonality. Don't give this point away too readily–it's best if they discover it themselves, so they *own* it.

After most of finished #2, have a class discussion and repeat with #3. Remember, your goal is not to get through problems. This tutorial has *way* more problems than the students will be able to get through.

7 minutes before the end of class, pick a problem that most students have started working on to do as a wrap-up.

Notes

- The definition of $\operatorname{proj}_X \vec{v}$ is written mostly with words instead of with set notation. This might throw some students off—they won't be able to tell that it's precise.
- Projections relate to orthogonality, but not always. #2 is designed to tease this out—that for non-smooth shapes projections might not relate to orthogonality.

- With our definition, $\text{proj}_X \vec{v}$ might not be unique and so might not be well defined. We will never try to trick a student by giving them a non-unique projection. If a student asks about this, you can tell them that on tests the projection will always be unique.
- For #3, student *should* know how to minimize a parabola, but many won't. However, almost all have taken calculus in high school, so feel free to use basic calculus to answer that question.
- You won't get to #4, but if students are working on it, make sure they draw a picture. It's very hard to do in one's head.

