Inquiry Based Vector Calculus

About the Document

This document was originally designed in the spring of 2016 to guide students through an ten week Linear Algebra course (Math 281-3) at Northwestern University.

A typical class day using the problem-sets:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

Sets of Vectors

1 Write the following sets in set-builder notation

1.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

1.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Unions & Intersections

DEFINITION

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

2 Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4, 5\}$ and $Z = \{4, 5, 6\}$. Compute

2.1 $X \cup Y$

2.2 $X \cap Y$

2.3 $X \cup Y \cup Z$

2.4 $X \cap Y \cap Z$

3 Draw the following subsets of $\mathbb{R}^2$.

3.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.3 $J = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.4 $V \cup H$.

3.5 $V \cap H$.

3.6 Does $V \cup H = \mathbb{R}^2$?

Linear Combinations, Span, and Linear Independence

Linear Combination

DEFINITION

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars.

4 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

4.1 Write the coordinates of $\vec{w}$.

4.2 Draw a picture with $\vec{w}$, $\vec{v}_1$, and $\vec{v}_2$.

4.3 Is $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.4 Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.5 Is $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.7 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Span

DEF

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

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5 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

5.1 Draw span $\{\vec{v}_1\}$.

5.2 Draw span $\{\vec{v}_2\}$.

5.3 Describe span $\{\vec{v}_1, \vec{v}_2\}$.

5.4 Describe span $\{\vec{v}_1, \vec{v}_3\}$.

5.5 Describe span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

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6 Give an example of:

6.1 two vectors in $\mathbb{R}^3$ that span a plane;

6.2 two vectors in $\mathbb{R}^3$ that span a line;

6.3 four vectors in $\mathbb{R}^3$ that span a plane;

6.4 a set of 50 vectors in $\mathbb{R}^3$ whose span is the line through the origin and the point $\begin{bmatrix}1\\2\\-3\end{bmatrix}$.

In some sets, every vector is essential for computing a span. In others, there are “excess” vectors. This leads us to the concept of linear independence.

Linearly Dependent & Independent

DEFINITION

We say $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\},$$

and a set is ***linearly independent*** otherwise.

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7 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

7.1 Describe span $\{\vec{u}, \vec{v}, \vec{w}\}$.

7.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

7.3 Give a subset $Y \subseteq X$ so that span Y = span X and Y is linearly independent.

7.4 Give a subset $Z \subseteq X$ so that span Z = span X and Z is linearly independent and $Z \neq Y$.

Trivial Linear Combination

DEF

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

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8 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

8.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

8.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

Linearly Dependent & Independent

DEF

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector.

9

9.1 Explain how this new definition implies the old one.

9.2 Explain how the old definition implies this new one.

Since have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

10

Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

10.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

10.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

10.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

10.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Subspaces and Bases

Subspace

DEFINITION

A ***subspace*** $V \subseteq \mathbb{R}^n$ is a subset such that

(i) $\vec{u}, \vec{v} \in V$ implies $\vec{u} + \vec{v} \in V$.

(ii) $\vec{u} \in V$ implies $k\vec{u} \in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

11

For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

11.1 $A = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ for some $a \in \mathbb{Z}\}$.

11.2 $B = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$.

11.3 $C = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.4 $D = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.5 $E = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.6 $F = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.7 $G = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

11.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Basis (DEF)

A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

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12 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

12.1 Describe V.

12.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

12.3 Give a basis for V.

12.4 Give another basis for V.

12.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

Dimension (DEF)

The ***dimension*** of a subspace V is the number of elements in a basis for V.

12.6 What is the dimension of V?

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13 Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}7\\8\\8\end{bmatrix}$ and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

13.1 Give a basis for and the dimension of P.

13.2 Give a basis for and the dimension of Q.

13.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

13.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

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14 Consider the system

$$\begin{array}{rcrl} x & - & y & = 2 \\ 2x & + & y & = 1 \end{array} \tag{1}$$

14.1 Draw the lines in (1) on the same coordinate plane.

14.2 Algebraically solve the system (1). What does this solution represent on your graph?

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15 Let L be the line given by $x - y = 2$.

15.1 Write an equation of a line that doesn't intersect L.

15.2 Write an equation of a line that intersects L in

(a) one place.

(b) infinitely many places

(c) exactly two places

or explain why no such equation exists.

15.3 For each equation you came up with, solve the system algebraically. How can you tell algebraically how many solutions there are?

The Row Reduction Algorithm

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16 16.1 Solve the system

$$\begin{array}{rcrcrl} x & - & y & - & 2z & = -5 \\ 2x & + & 3y & + & z & = 5 \\ 0x & + & 2y & + & 3z & = 8 \end{array} \tag{2}$$

any way you like.

16.2 Use an augmented matrix to solve the system (2).

The system (2) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (2) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

16.3 Rewrite (2) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

16.4 If (x, y, z) is a solution to (2), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

16.5 If (x, y, z) is a solution to (2), is $\vec{p} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

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17 Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

17.1 Write the system of equations corresponding to A.

17.2 Solve the system of equations corresponding to A.

Infinite Solutions

18 Consider the system

$$\begin{aligned} x &+ 2y &= 3 \\ 2x &+ 4y &= 6 \end{aligned} \tag{3}$$

18.1 How many solutions does (3) have?

18.2 Write the solutions to (3) in vector form.

18.3 What happens when you use an augmented matrix to solve (3)?

Free Variables

19 Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make $2-1=1$ choices when writing a solution. Let's make the choice $y=t$.

19.1 With the added equation $y=t$, solve the system represented by B.

20 Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

and call the variables in this system x_1, x_2, x_3, x_4, x_5.

20.1 Write the system of equations represented by C.

20.2 Identify how many choices you can make when writing down a solution corresponding to C.

20.3 Add one equation (of the form $x_i = t$ or $x_j = s$, etc.) for each choice you must make when solving the system.

20.4 Write in vector form all solutions to C.

21

21.1 An unknown system U is represented by an augmented matrix with 4 rows and 6 columns. What is the minimum number of free variables solutions to U will have?

21.2 An unknown system V is represented by an augmented matrix with 6 rows and 4 columns. What is the minimum number of free variables solutions to V will have?

22

Homogeneous

DEF A system is called ***homogeneous*** if all equations equal 0.

Let A be an unknown system of 3 equations and 3 variables and suppose $(x,y,z)=(1,2,1)$ and $(x,y,z)=(-1,1,1)$ are solutions to A.

22.1 Can you produce another solution to the system?

22.2 Can you produce a solution to the homogeneous version of A (the version of A where every equation equals 0)?

22.3 Suppose when you use an augmented matrix to solve the system A, you only have one free variable. Could A be homogeneous? Can you produce all solutions to the system A?

Rank

Rank

DEF The ***rank*** of the matrix A is the number of leading ones in the reduced row echelon form of A.

23

23.1 Determine the rank of (a) $\begin{bmatrix}1&1\\2&2\end{bmatrix}$ (b) $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ (c) $\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$ (d) $\begin{bmatrix}3\\3\\2\end{bmatrix}$ (e) $\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

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24 Consider the homogeneous system

$$\begin{array}{rrrl} x & +2y & +z & =0\\ x & +2y & +3z & =0\\ -x & -2y & +z & =0\end{array} \tag{4}$$

and the non-augmented matrix of coefficients $A=\begin{bmatrix}1&2&1\\1&2&3\\-1&-2&1\end{bmatrix}$.

24.1 What is rank(A)?

24.2 Give the general solution to (4).

24.3 Are the column vectors of A linearly independent?

24.4 Give a non-homogeneous system with the same coefficients as (4) that has

(a) infinitely many solutions

(b) no solutions.

25

25.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

25.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Span Again

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26 Consider the system

$$\begin{array}{rrrl} x & -y & -z & =0\\ 0x & +1y & +2z & =0\\ 3x & -3y & +3z & =0\end{array} \tag{5}$$

which has the unique solution $(x,y,z)=(0,0,0)$.

26.1 Give vectors $\vec u,\vec v,\vec w$ so that the system (5) corresponds to the vector equation $x\vec u+y\vec v+z\vec w=\vec 0$.

26.2 Is $\vec w\in\operatorname{span}\{\vec u,\vec v\}$? If so, write it as a linear combination of $\vec u$ and $\vec v$.

The matrix M is the non-augmented matrix corresponding to a homogeneous system of linear equations. M also corresponds to the vector equation $x\vec a+y\vec b+z\vec c=\vec 0$. Further, we know

$$\operatorname{rref}(M)=\begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\end{bmatrix}.$$

26.3 Give a solution to the vector equation $x\vec a+y\vec b+z\vec c=\vec 0$.

26.4 Is $\vec c\in\operatorname{span}\{\vec a,\vec b\}$? If so, write it as a linear combination of $\vec a$ and $\vec b$.

26.5 Do you have enough information to tell if $\{\vec a,\vec b\}$ is linearly independent? Why or why not?

27 Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$ you have no free variables.

27.1 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$, the second column corresponds to a free variable.

27.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

27.3 Is $\{\vec{u},\vec{w}\}$ linearly independent?

27.4 Is $\{\vec{u},\vec{v}\}$ linearly independent?

Maximal Linearly Independent Subset

DEF

Given a set of vectors X, a ***maximal linearly independent subset*** of X is a linearly independent subset $V\subseteq X$ with the most possible vectors in it (i.e., if you took any subset of X with more vectors, it would be linearly dependent).

28

28.1 Give a maximal linearly independent subset, T, of $\left\{\begin{bmatrix}a\\b\\c\end{bmatrix} : a,b,c\in\mathbb{R}\right\}$.

28.2 What is the size of T?

29 Consider the vectors

$$\vec{v}_1=\begin{bmatrix}1\\2\\1\end{bmatrix}\qquad \vec{v}_2=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\qquad \vec{v}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}\qquad \vec{v}_4=\begin{bmatrix}-1\\2\\0\end{bmatrix}\qquad \vec{v}_5=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

and the matrices

$$A=\begin{bmatrix}1&-1&0&-1&1\\2&-1&1&2&-1\\1&-1&0&0&1\end{bmatrix}\qquad \operatorname{rref}(A)=\begin{bmatrix}1&0&1&0&-2\\0&1&1&0&-3\\0&0&0&1&0\end{bmatrix}.$$

(Notice that the columns of A are the vectors $\vec{v}_1,\dots,\vec{v}_5$)

29.1 Is $V=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4,\vec{v}_5\}$ linearly independent?

29.2 Pick a maximal linearly independent subset of V.

29.3 Pick another (different) maximal linearly independent subset of V.

29.4 Give a basis for span(V).

29.5 What is the dimension of span(V)?

30

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

30.1 Write the shape of the matrices A, B, C (i.e., for each one, write the dimensions in $m \times n$ form).

30.2 List *all* products between the matrices A, B, C that are defined. (Your list will be some subset of AB, AC, BA, CA, BC, CB.)

30.3 Compute AC and CA.

31

31.1 If the matrices X and Y are both square $n \times n$ matrices, does $XY = YX$? Explain.

31.2 If the matrices X and Y are both square $n \times n$ matrices, does $X + Y = Y + X$? Explain.

32

Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

32.1 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

32.2 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

33

The entries of a matrix are specified by (row,column) pairs of integers. If a_{ij} is the (i, j) entry of a matrix A, we may write $A = [a_{ij}]$.

33.1 Write the 2×2 matrix A with entries $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 7$ and $a_{22} = 9$.

33.2 Let $B = [b_{ij}]$ be the 3×3 matrix where $b_{ij} = i + j$. Write B.

33.3 Let $C = [c_{ij}]$ be the 3×4 matrix where $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ if $i \neq j$.

34

DEF

The ***transpose*** of a matrix $A = [a_{ij}]$ is the matrix $A^T = [a_{ji}]$.

Visually, the transpose of a matrix swaps rows and columns.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

34.1 What is the shape of A and A^T?

34.2 Write down A^T.

B and D are 4×6 matrices and C is a 6×4 matrix.

34.3 Does $(BC)^T = B^T C^T$? Explain.

34.4 Does $(B + D)^T = B^T + D^T$? Explain.

34.5 Compute AA^T and $A^T A$ (where A is the matrix defined earlier). What do you notice?

35

DEF A matrix X is called ***symmetric*** if $X = X^T$.

Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

35.1 Prove that if W is a square matrix, then $V = W^T W + W + W^T$ is a symmetric matrix.

36

DEF A ***zero matrix*** is a matrix whose entries are all zeros. An ***identity matrix*** is a square matrix whose diagonal entries are 1 and non-diagonal entries are 0.

We write the $m \times n$ zero matrix as $0_{m\times n}$ or just 0 if the shape is determined by context. The $n \times n$ identity matrix is notated $I_{n\times n}$ or just I if the shape is determined by context.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

36.1 Write down the 3×3 identity matrix and the 3×3 zero matrix.

36.2 Compute $I_{3\times3}A$, $AI_{3\times3}$, $0_{3\times3}A$, and $A0_{3\times3}$.

36.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

37

37.1 Solve the matrix equation

$$I_{4\times4} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$