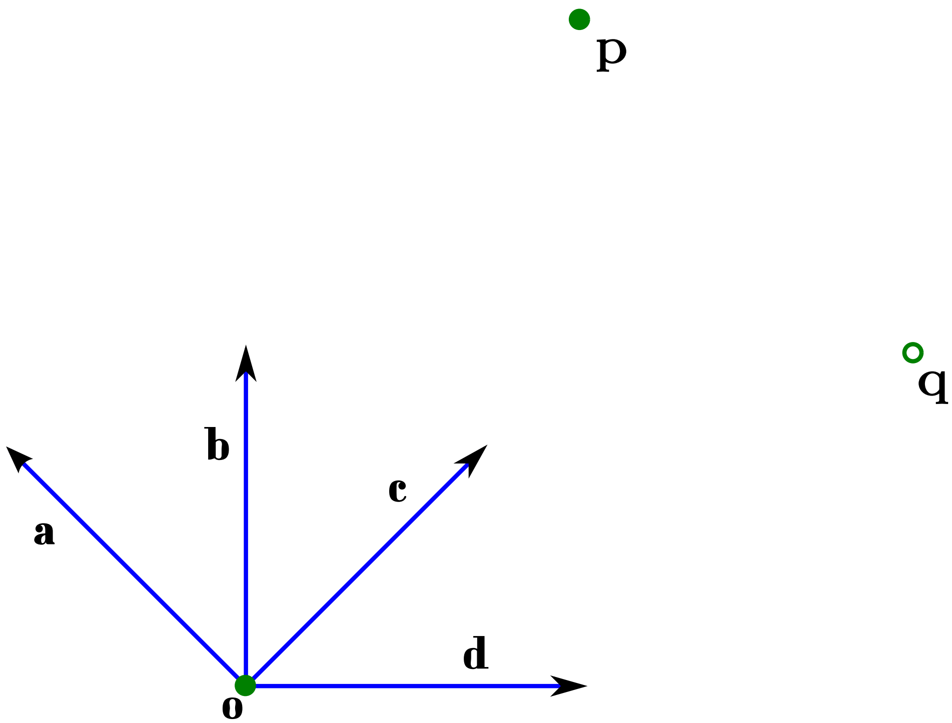
Vectors



Notice that all arrows in this diagram are the same length. We will call this length a *unit*.

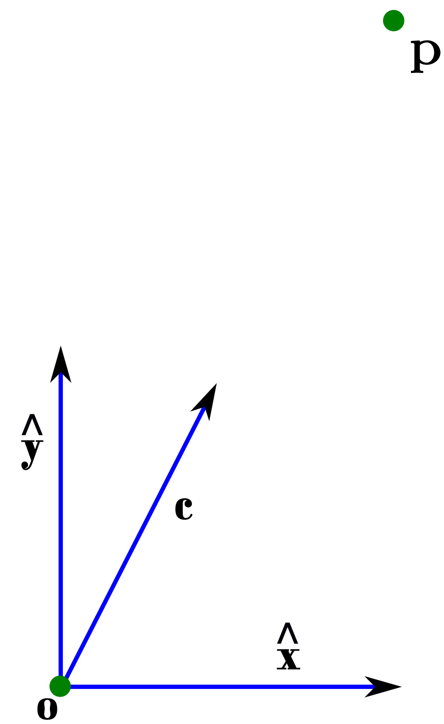
1.1 Give directions from **o** to **p** of the form "Walk ____units in the direction of arrow ____, then walk ____units in the direction of arrow ____."

1.2 Can you give directions with the two arrows you haven't used? Give such directions, or explain why it cannot be done.

1.3 Give directions from **o** to q.

1.4 Can you give directions from **o** to q using **c** and **a**? Give such directions, or explain why it cannot be done.

Unit Vectors



We are going to start using a more mathematical notation for giving directions. Our directions will now look like

$$p = \_\_\_\,\hat{\mathbf{x}} + \_\_\_\,\hat{\mathbf{y}}$$

which is read as "To get to p $(=)$ go ____units in the direction $\hat{\mathbf{x}}$ then $(+)$ go ____units in the direction $\hat{\mathbf{y}}$."

2.1 What is the difference between $p = \_\_\_\,\hat{\mathbf{x}} + \_\_\_\,\hat{\mathbf{y}}$ and $p = \_\_\_\,\hat{\mathbf{y}} + \_\_\_\,\hat{\mathbf{x}}$? Can they both give valid directions?

2.2 (a) Give directions to p using the new notation.
  (b) Give directions to p using $\mathbf{c}$.
  (c) What is the distance from $\mathbf{o}$ to p in units?

2.3 (a) $r = 1\mathbf{c}$. Give directions from $\mathbf{o}$ to r using $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.
  (b) What is the distance from $\mathbf{o}$ to r?

2.4 (a) $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$; find the exact distance from $\mathbf{o}$ to q.
  (b) $s = 2\hat{\mathbf{x}} + \mathbf{c}$; find the exact distance from $\mathbf{o}$ to s.

We've been learning vector addition. $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are called the *standard basis vectors* for $\mathbb{R}^2$ (the plane). Everyone has agreed that if we give directions from the origin to some point and we don't specify otherwise, we will give directions in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

Column Vector Notation

We previously wrote $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$. In column vector notation we write

$$q = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We may call q either a *vector* or a *point*. If we call q a vector, we are emphasizing that q gives direction of some sort. If we call q a point, we emphasize that q is some absolute location in space. (What's the philosophical difference between a location in space and directions from the origin to said location?)
$r = 1\mathbf{c}$; $s = 2\hat{\mathbf{x}} + \mathbf{c}$.

3.1 Write r and s in column vector form.

Vector Length

The *length* or *norm* of a vector $\vec{w}$ is denoted $\|\vec{w}\|$ and is the distance from $\mathbf{o}$ to the point you end up at if you follow $\vec{w}$'s instructions.

4.1 Find $\|\vec{a}\|$, $\|\vec{b}\|$, $\|\vec{c}\|$ where

- (a) $\vec{a} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$
- (b) $\vec{b} = 2\vec{a}$
- (c) $\vec{c} = -\vec{a}/2$

4.2 $\hat{\mathbf{z}}$ points perpendicular to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ into the 3rd dimension.
Let $\vec{v} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$ and $\vec{w} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}}$.

- (a) Write $\vec{v}$ in terms of $\vec{w}$ and $\hat{\mathbf{z}}$ and draw a picture showing the relationship between the three vectors (3-d pictures are a hard but essential skill in this course).
- (b) Find $\|\vec{w}\|$ and $\|\vec{v}\|$. (Hint, look at your picture and see if there are any right triangles to exploit).

4.3 Let $\vec{u} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$.

- (a) Find $\|\vec{u}\|$.
- (b) Find $\|k\vec{u}\|$ where k is some unknown constant.
- (c) What value(s) of k makes $\|k\vec{u}\| = 1$?
- (d) Write down a vector in column form that points in the same direciton as $\vec{u}$ and has length 1.

Unit Vectors

Vectors that have length 1 are called *unit vectors*.

5.1 $\vec{a} = -\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$. Find a unit vector in the direction of $\vec{a}$, and call this vector $\vec{u}$ (u for $u$nit, get it?).

5.2 Write $\vec{a}$ in terms of $\vec{u}$. Does $\|\vec{a}\|$ show up in your formula at all?

5.3 Write $3\vec{u}$ in column vector form and find its length.

5.4 Write $7.5\vec{u}$ in column vector form and find its length.

5.5 $\vec{v}$ is a different unit vector (I won't tell you its exact form). Find $\|9\vec{v}\|$ Why do we like unit vectors so much?

Dot Product

The *dot product* is incredible because it is easy to compute and has a useful geometric meaning.

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the dot product of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

We also have a geometry-related formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

6.1 Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

6.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

6.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

6.4 $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

We've explored how dot products relate to angles, but how do they relate to lengths?

7.1 Let $\vec{a} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$

(a) Find $\|\vec{a}\|$ and $\vec{a} \cdot \vec{a}$. How do the two quantities relate?

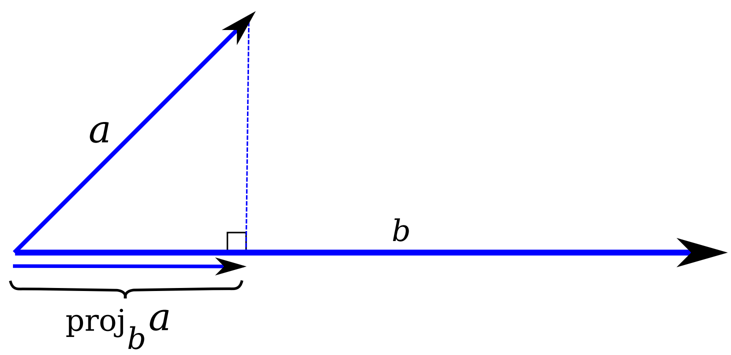
(b) Write down an equation for the length of a vector $\vec{v}$ in terms of dot products.

7.2 Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}$, and find $\|\vec{b}\|$. Did you know how to find 4-d lengths before?

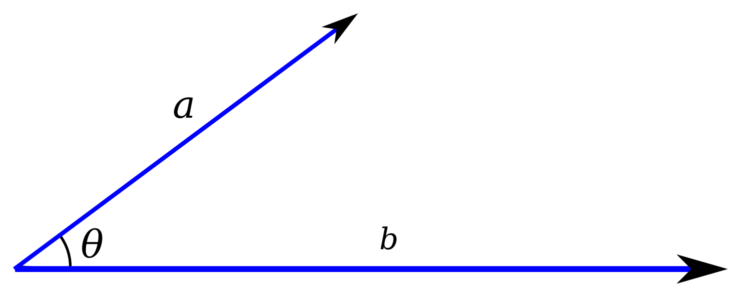
7.3 Suppose $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ for $x, y \in \mathbb{R}$. Could $\vec{u} \cdot \vec{u}$ be negative? Compute $\vec{u} \cdot \vec{u}$ algebraically and use this to justify your answer.

Projections

Projections (sometimes called orthogonal projections) are a way to measure how much one vector points in the direction of another.



The projection of $\vec{a}$ onto $\vec{b}$ is written $\text{proj}_{\vec{b}}\vec{a}$ and is a vector in the direction of $\vec{b}$.



8.1 In this picture $\|\vec{a}\| = 4$ and $\theta = \pi/6$. Find $\|\text{proj}_{\vec{b}}\vec{a}\|$.

8.2 If $\vec{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, write down $\text{proj}_{\vec{b}}\vec{a}$ in column vector form. How do the coordinates relate to $\|\text{proj}_{\vec{b}}\vec{a}\|$?

8.3 Consider $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Compute $\text{proj}_{\hat{\mathbf{x}}}\vec{u}$ and $\text{proj}_{\hat{\mathbf{y}}}\vec{u}$. How do these projections relate to the coordinates of $\vec{u}$? What can you say in general about projections onto $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$?

$$\vec{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

9.1 Find θ, the angle between $\vec{w}$ and $\vec{v}$.

9.2 Use θ to compute $\text{proj}_{\vec{v}}\vec{w}$ and $\text{proj}_{\vec{w}}\vec{v}$.

9.3 Write down a formula for $\text{proj}_{\vec{b}}\vec{a}$ where $\vec{a}$ and $\vec{b}$ are arbitrary vectors.

10.1 For the arbitrary vector $\vec{a}$, what is $\text{proj}_{3\vec{a}}\vec{a}$?

10.2 If $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) vectors, what is $\text{proj}_{\vec{b}}\vec{a}$ $\text{proj}_{\vec{a}}\vec{b}$?

Lines, Planes, Normals, Equations

11.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors perpendicular to it.

11.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is perpendicular to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

11.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for a line. This is called *normal form*

A *normal vector* to a line is one that is orthogonal to it.

11.4 Rewrite the line $\vec{u} \cdot \vec{x} = 0$ in $y = mx + b$ form and verify it matches the line you drew above.

We can also write a line in *parametric form* by introducing a parameter that traces out the line as the parameter runs over all real numbers.

12.1 Draw the line L with x, y coordinates given by

$$\begin{aligned} x &= t \\ y &= 2t \end{aligned}$$

as t ranges over $\mathbb{R}$.

12.2 Write the line $\vec{u} \cdot \vec{x} = 0$ (where $\vec{u}$ is the same as before) in parametric form.

Vector form is the same as parametric form but written in vector notation. For example, the line L from earlier could be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

13.1 Write $\vec{u} \cdot \vec{x} = 0$ in vector form. That is, find a vector $\vec{v}$ so the line $\vec{u} \cdot \vec{x} = 0$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\vec{v}$$

as t ranges over $\mathbb{R}$.

13.2 What is $\vec{v} \cdot \vec{u}$? Why? Will this always happen?

Moving to Planes

When solving equations, sometimes we get to make choices. For example, if $x + 2y = 0$, we can find solutions by fixing either x or y and solving for the other. e.g., if $x = 2$, then $y = -1$ nad if $y = 3$ then $x = -6$.

14.1 Write down three solutions $\vec{a}$, $\vec{b}$, $\vec{c}$ to

$$2x + y - z = 0. \tag{1}$$

14.2 Is $2\vec{a} - \vec{b}$ a solution? Is any linear combination of solutions a solution? Justify why or why not.

14.3 Rewrite equation (1) in normal form $\vec{n} \cdot \vec{x} = 0$ where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

14.4 What do you notice about the angle between solutions to equation (1) and $\vec{n}$?

14.5 You've already seen that scalars come out of dot products (e.g., $\vec{a} \cdot (3\vec{b}) = 3(\vec{a} \cdot \vec{b})$. Use this combined with normal form to prove a linear combination of solutions is still a solution.

When writing down solutions to equation (1), you got to choose two coordinates before the remaining coordinate became determined. This means the solutions have two parameters (and consequently form a two dimensional space).

14.6 Write down parametric form of a line of solutions to equation (1).

14.7 Write down parametric form of a different line of solutions to equation (1).

14.8 Write down all solutions to equation (1) in parametric form. That is, find $a_x, a_y, a_z, b_x, b_y, b_z$ so that

$$\begin{aligned} x &= a_x t + b_x s \\ y &= a_y t + b_y s \\ z &= a_z t + b_z s \end{aligned}$$

gives all solutions as t, s vary over all of $\mathbb{R}$.

14.9 Write all solutions to equation (1) in vector form.