Inquiry Based Linear Algebra

About the Document

This document is a hybrid of many linear algebra resources, including those of the IOLA (Inquiry Oriented Linear Algebra) project, Jason Siefken's IBLLinearAlgebra project, and Asaki, Camfield, Moon, and Snipes' Radiograph and Tomography project.

This document is a mix of student projects, problem sets, and labs. A typical class day looks like:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

Task 1.1: The Magic Carpet Ride

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Hands-on experience with vectors as displacements.

- Internalize vectors as geometric objects representing displacements.
- Use column vector notation to write vectors.
- Use pre-existing knowledge of algebra to answer vector questions.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Task 1.2: The Magic Carpet Ride, Hide and Seek

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Address an existential question involving vectors: "Is it possible to find a linear combination that does...?"

The goal of this problem is to

- Formalize geometric questions using the language of vectors.
- Find both geometric and algebraic arguments to support the same conclusion.
- Establish what a "negative multiple" of a vector should be.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1, 2, 3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

1

1.1 Which of the following statements are true?

(a) $3 \in \{1, 2, 3\}$. True

(b) $1.5 \in \{1, 2, 3\}$. False

(c) $4 \in \{1, 2, 3\}$. False

(d) "b"$\in \{x : x \text{ is an English letter}\}$. True

(e) "ò"$\in \{x : x \text{ is an English letter}\}$. False

(f) $\{1, 2\} \subseteq \{1, 2, 3\}$. True

(g) For some $a \in \{1, 2, 3\}$, $a \geq 3$. True

(h) For any $a \in \{1, 2, 3\}$, $a \geq 3$. False

(i) $1 \subseteq \{1, 2, 3\}$. False

(j) $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. False

(k) $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$. True

Practice reading sets and set-builder notation.

The goal of this problem is to
- Become familiar with $\in$, $\subseteq$, and $=$ in the context of sets.
- Distinguish between $\in$ and $\subseteq$.
- Use quantifiers with sets.

2

Write the following in set-builder notation

2.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

$$\{x \in \mathbb{R} : x > \sqrt{2}\}.$$

2.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ with } a = 2b\right\} \text{ or } \left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} 2t \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$$

$$\text{or } \left\{\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 : a = 2b\right\}.$$

Practice writing sets using set-builder notation.

The goal of this problem is to
- Express English descriptions using math notation.
- Recognize there is more than one correct way to write formal math.
- Preview vector form of a line.

Unions & Intersections

DEFINITION

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$(\text{union}) \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$(\text{intersection}) \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

3 Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

3.1 $X \cup Y$ $\{1,2,3,4,5\}$

3.2 $X \cap Y$ $\{2,3\}$

3.3 $X \cup Y \cup Z$ $\{1,2,3,4,5,6\}$

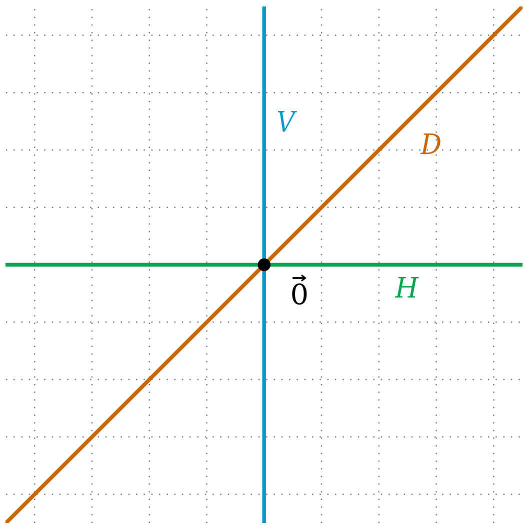
3.4 $X \cap Y \cap Z$ $\emptyset = \{\}$

4 Draw the following subsets of $\mathbb{R}^2$.

4.1 $V = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

4.2 $H = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

4.3 $D = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.



4.4 $N = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R} \right\}$. $N = \{\}$.

4.5 $V \cup H$. $V \cup H$ looks like a "+" going through the origin.

4.6 $V \cap H$. $V \cap H = \{\vec{0}\}$ is just the origin.

4.7 Does $V \cup H = \mathbb{R}^2$?

No. $V \cup H$ does not contain $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ while $\mathbb{R}^2$ does contain $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Visualize sets of vectors.

The goal of this problem is to
- Apply set-builder notation in the context of vectors.
- Distinguish between "for all" and "for some" in set builder notation.
- Practice unions and intersections.
- Practice thinking about set equality.

Vector Combinations

Linear Combination (DEFINITION)

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the ***coefficients*** of the linear combination.

5 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

5.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

$\vec{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$; the coefficients are $(2,1)$.

Practice linear combinations.

The goal of this problem is to
- Practice using the formal term *linear combination*.
- Foreshadow span.

5.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix}3\\3\end{bmatrix} = 3\vec{v}_1 + 0\vec{v}_2$.

5.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$.

5.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix}4\\0\end{bmatrix} = 2\vec{v}_1 + 2\vec{v}_2$.

5.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

No. $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$ and $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$. Therefore

$$\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = a(\tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2) + b(\tfrac{1}{2}\vec{v}_1 - \tfrac{1}{2}\vec{v}_2) = (\tfrac{a+b}{2})\vec{v}_1 + (\tfrac{a-b}{2})\vec{v}_2.$$

Therefore any vector in $\mathbb{R}^2$ can be written as linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

5.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Yes. All linear combinations of $\vec{v}_1$ have equal x and y coordinates, therefore $\vec{w} = \begin{bmatrix}2\\1\end{bmatrix}$ is not a linear combination of $\vec{v}_1$.

Practice formal writing.

6 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix}3\\1\end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix}1\\2\end{bmatrix}$.

6.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

Gauss's location can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

6.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

Every vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

6.3 Rephrase the sentence "*$\mathbb{R}^2$ is the set of all linear combinations of $\vec{h}$ and $\vec{m}$*" using formal mathematical language.

$\mathbb{R}^2 = \{\vec{v} \,:\, \vec{v} = t\vec{m} + s\vec{h} \text{ for some } t, s \in \mathbb{R}\}$.

Non-negative & Convex Linear Combinations

DEFINITION

The linear combination $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Geometric meaning of *non-negative* and *convex* linear combinations.

The goal of this problem is to

- Read and apply the definition of non-negative and convex linear combinations.
- Gain geometric intuition for non-negative and convex linear combinations.
- Learn how to describe line segments using convex linear combinations.

7 Let

$$\vec{a} = \begin{bmatrix}1\\1\end{bmatrix} \quad \vec{b} = \begin{bmatrix}-1\\1\end{bmatrix} \quad \vec{c} = \begin{bmatrix}0\\1\end{bmatrix} \quad \vec{d} = \begin{bmatrix}0\\2\end{bmatrix} \quad \vec{e} = \begin{bmatrix}-1\\-1\end{bmatrix}.$$

7.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$? All of them, since any vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$.

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$.

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$.

7.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that

(a) $\vec{a}$ and $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not.

Let $\vec{u} = \vec{a}$ and $\vec{v} = \vec{c}$.

(b) $\vec{a}$ and $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$.

Let $\vec{u} = \vec{a}$ and $\vec{v} = \vec{e}$.

(c) $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not.

Impossible. If $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$, then every non-negative linear combination of $\vec{a}$ and $\vec{b}$ is also a non-negative linear combination of $\vec{u}$ and $\vec{v}$. And, we already concluded that $\vec{d}$ is a non-negative linear combination of $\vec{a}$ and $\vec{b}$.

(d) $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$.

Impossible. Convex linear combinations all lie on the same line segment, but $\vec{a}$, $\vec{c}$, and $\vec{d}$ are not collinear.

Otherwise, explain why it's not possible.

Lines and Planes

8 Let A be the set of points $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 1$.

Link prior knowledge to new notation/concepts.

The goal of this problem is to
- Convert between $y = mx + b$ form of a line and the set-builder definition of the same line.
- Think about lines in terms of vectors rather than equations.

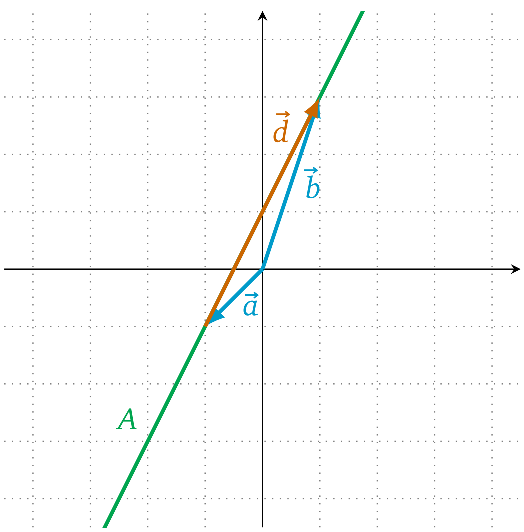
8.1 Describe A using set-builder notation.

$$\left\{\vec{v} \in \mathbb{R}^2 \;:\; \vec{v} = \begin{bmatrix} t \\ 2t+1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$$

$$\text{or } \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;:\; y = 2x+1\right\} \text{ or } \left\{\begin{bmatrix} t \\ 2t+1 \end{bmatrix} \in \mathbb{R}^2 \;:\; t \in \mathbb{R}\right\}$$

8.2 Draw A as a subset of $\mathbb{R}^2$.

8.3 Add the vectors $\vec{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{d} = \vec{b} - \vec{a}$ to your drawing.



8.4 For which $t \in \mathbb{R}$ is it true that $\vec{a} + t\vec{d} \in A$? Explain using your picture.

$\vec{a} + t\vec{d} \in A$ for all $t \in \mathbb{R}$. We can see this because if we start at the vector $\vec{a}$ and the displace by $t\vec{d}$, we will always be on the line A.

Vector Form of a Line

DEFINITION

A line ℓ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d} + \vec{p}$$

for some vector $\vec{d}$ and point $\vec{p}$. That is, $\ell = \{\vec{x} \;:\; \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

9 Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

Practice with vector form.

The goal of this problem is to
- Express lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ in vector form.
- Produce direction vectors by subtracting two points on a line.
- Recognize vector form is not unique.

9.1 Write ℓ in vector form. Is vector form of ℓ unique?

$$\vec{x} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

The vector form is not unique, as any non-zero scalar multiple of $\begin{bmatrix}1\\-2\end{bmatrix}$ can serve as a direction vector. Additionally, any other point on the line can be used in place of $\begin{bmatrix}0\\3\end{bmatrix}$. For example, $\vec{x} = t\begin{bmatrix}-4\\8\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$ is another vector form of ℓ.

9.2 Write L in vector form. $\vec{x} = t\begin{bmatrix}1\\-2\\-2\end{bmatrix} + \begin{bmatrix}0\\3\\3\end{bmatrix}$. This is obtained by finding two points: one when $x = 0$ and one when $x = 1$ and subtracting them to find a direction vector for L.

9.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

$\vec{x} = t\begin{bmatrix}-3\\6\\6\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Again, any non-zero scalar multiple of the direction vector will work for $\vec{d}$, as will any other point on the line work for $\vec{p}$.

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10 Let A, B, and C be given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B} \qquad \overbrace{\vec{x} = t\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

Intersect lines in vector form.

The goal of this problem is to

- Practice computing the intersection between lines in vector form.
- Recognize "t" as a dummy variable as used in vector form and that, when comparing lines in vector form, "t" needs to be replaced with non-dummy variables.

10.1 Do the lines A and B intersect? Justify your conclusion.

Yes. $(0)\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} = (-1)\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}$.

To find the intersection, if there is one, we must solve the vector equation:

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = s\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

One solution is when $t = 0$ and $s = -1$.

10.2 Do the lines A and C intersect? Justify your conclusion.

No. The vector equation

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = s\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}$$

has no solutions. This is equivalent to saying that the following system of equations has no solutions:

$$\begin{aligned} t &= 2s + 1\\ 2t &= -s + 1\\ 3t + 1 &= s + 1 \end{aligned}$$

The third equation tells us that $s = 3t$, which when substituted into the first equation forces $t = -\frac{1}{5}$ and therefore $s = -\frac{3}{5}$. However, these two numbers don't satisfy the second equation.

10.3 Let $\vec{p} \neq \vec{q}$ and suppose X has vector form $\vec{x} = t\vec{d} + \vec{p}$ and Y has vector form $\vec{x} = t\vec{d} + \vec{q}$. Is it possible that X and Y intersect?

Yes. If $\vec{q} = \vec{p} + a\vec{d}$ for $a \neq 0$, then X and Y will actually be the same line, since in this case

$$\vec{x} = t\vec{d} + \vec{q} = t\vec{d} + (\vec{p} + a\vec{d}) = (t + a)\vec{d} + \vec{p}.$$

For example, the following two vector equations represent the same line.

$$\vec{x} = t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{and} \qquad \vec{x} = t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}7\\7\\7\end{bmatrix}.$$

Vector Form of a Plane

DEFINITION

A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

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11 Recall the intersecting lines A and B given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

Apply vector form of a plane.

The goal of this problem is to

- Use direction vectors for lines given in vector form.
- Think about planes in terms of vectors rather than equations.
- Combine direction vectors in a plane to produce new direction vectors.

11.1 Find two direction vectors in $\mathcal{P}$.

Two possible answers are:

$$\vec{d}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \text{and} \qquad \vec{d}_2 = \begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

These are the two direction vectors we already know are in the plane—the ones from the two lines:

Note that neither of these is a multiple of the other, so they really are two unique direction vectors in $\mathcal{P}$.

11.2 Write $\mathcal{P}$ in vector form.

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + s\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

We already have two direction vectors, so we just needed a point on the plane. We used the point $\vec{p} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ that we already know is on line A.

11.3 Describe how vector form of a plane relates to linear combinations.

The vector form of a plane says that a vector $\vec{x}$ is on the plane exactly when it is equal to any linear combination of $\vec{d}_1$ and $\vec{d}_2$, plus $\vec{p}$.

Another way of saying the same thing is that the vector $\vec{x}$ is on the plane exactly when $\vec{x} - \vec{p}$ is equal to some linear combination of $\vec{d}_1$ and $\vec{d}_2$.

11.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

One possible answer:

$$\vec{x} = t\begin{bmatrix}-1\\-2\\-3\end{bmatrix} + s\begin{bmatrix}-7\\7\\7\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

As with the equations of lines from before, we can use any non-zero scalar multiple of either direction vector and get the same plane. We also used the point $\vec{q} = \begin{bmatrix}-1\\1\\2\end{bmatrix}$ that we already knew is on line B.

12 Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be a plane with equation $x+y+z=1$.

12.1 Find three points in $\mathcal{Q}$.

There are many choices here, of course. Three natural ones are:

$$\vec{p}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \vec{p}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{p}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

12.2 Find two direction vectors for $\mathcal{Q}$.

Now that we have three points on the plane, we can use the direction vectors joining any two pairs of them. For example:

$$\vec{d}_1 = \vec{p}_1 - \vec{p}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad \vec{d}_2 = \vec{p}_1 - \vec{p}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

12.3 Write $\mathcal{Q}$ in vector form.

Using the point $\vec{p}_1$ from above, one possible answer is:

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}_1 = t\begin{bmatrix}1\\-1\\0\end{bmatrix} + s\begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Span

Span

DEF

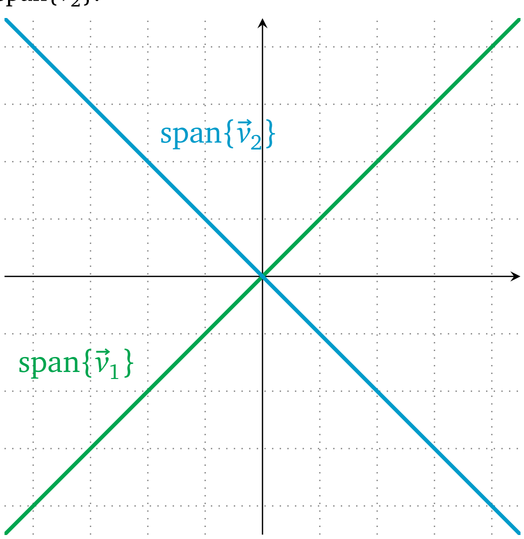
The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

13 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

13.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

13.2 Draw $\operatorname{span}\{\vec{v}_2\}$.



Apply the definition of *span*.

The goal of this problem is to

- Practice applying a new definition in a familiar context ($\mathbb{R}^2$).
- Recognize spans as lines and planes through the origin.

13.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

$\operatorname{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$

We can see this since for any $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$,

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{x}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right) + \frac{y}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{x+y}{2}\vec{v}_1 + \frac{x-y}{2}\vec{v}_2$$

13.4 Describe $\mathrm{span}\{\vec v_1, \vec v_3\}$. $\mathrm{span}\{\vec v_1, \vec v_3\} = \mathrm{span}\{\vec v_1\}$, a line through the origin with direction vector $\vec v_1$.

13.5 Describe $\mathrm{span}\{\vec v_1, \vec v_2, \vec v_3\}$. $\mathrm{span}\{\vec v_1, \vec v_2, \vec v_3\} = \mathrm{span}\{\vec v_1, \vec v_2\} = \mathbb{R}^2$

Connect geometric figures to spans.

The goal of this problem is to
- Identify a relationship between lines and spans.
- Describe a line through the origin as a span.
- Identify when a line cannot be described as a span.
- Apply the definition of $\mathrm{span}\,X$ even when X is infinite.

14 Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

14.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

$\ell_1 = \mathrm{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, since $\begin{bmatrix}x\\y\end{bmatrix} \in \ell_1$ if and only if $x = y$, which in turn is true if and only if it is a scalar multiple of $\begin{bmatrix}1\\1\end{bmatrix}$.

14.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

This is not possible. $\begin{bmatrix}0\\0\end{bmatrix}$ is an element of the span of *any* set of vectors, since we can use all zeroes as the scalars in a linear combination, but $\begin{bmatrix}0\\0\end{bmatrix} \notin \ell_2$.

14.3 Does the expression $\mathrm{span}(\ell_1)$ make sense? If so, what is it? How about $\mathrm{span}(\ell_2)$?

Both of these expressions do make sense. One can compute the span of any set of vectors, and these lines are just special set of points in $\mathbb{R}^2$ which we are already used to thinking of as vectors.

$\mathrm{span}(\ell_1) = \ell_1$, since all of the vectors on the line ℓ_1 are already multiples of $\begin{bmatrix}1\\1\end{bmatrix}$, as we discovered earlier.

$\mathrm{span}(\ell_2)$ equals all of $\mathbb{R}^2$. It's easy to see that the vectors $v = \begin{bmatrix}4\\0\end{bmatrix}$ and $w = \begin{bmatrix}0\\-4\end{bmatrix}$ are both on ℓ_2, and the span of these two vectors alone is all of $\mathbb{R}^2$.

DEF **Set Addition**

If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A + B$, is

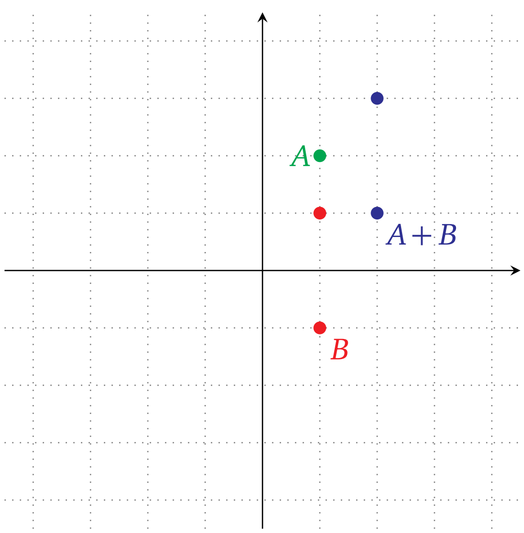
$$A + B = \{\vec x \,:\, \vec x = \vec a + \vec b \text{ for some } \vec a \in A \text{ and } \vec b \in B\}.$$

Describing geometry using sets.

The goal of this problem is to
- Practice applying a new definition in a familiar context ($\mathbb{R}^2$).
- Gain an intuitive understanding of set addition.
- Describe lines that don't pass through $\vec 0$ using a combination of set addition and spans.

15 Let $A = \left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell = \mathrm{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

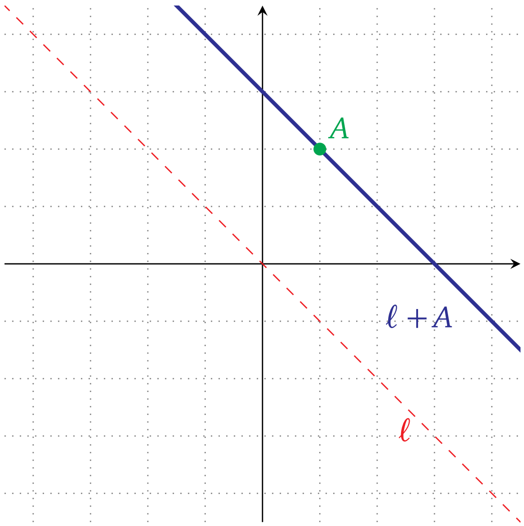
15.1 Draw A, B, and $A + B$ in the same picture.



15.2 Is $A + B$ the same as $B + A$?

Yes. Since A and B are such small sets we could just compute all the vectors in $A + B$ and $B + A$ and see that they're equal. However, we know that real numbers can be added up in any order, and the coordinates of an element of $A + B$ or $B + A$ are simply sums of the corresponding coordinates of elements of A and B.

15.3 Draw $\ell + A$.



15.4 Consider the line ℓ_2 given in vector form by $\vec{x} = t\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

ℓ_2 cannot be described using only a span, for the same reason as the line ℓ_2 in Problem 14.2 couldn't be. We know that the origin $\begin{bmatrix}0\\0\end{bmatrix}$ must be an element of any span, but it is not a point on ℓ_2.

ℓ_2 can be described as a span plus a set addition though. Specifically, $\ell_2 = \ell + A$.

Task 1.3: The Magic Carpet, Getting Back Home

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$).

Span in higher dimensions.

The goal of this problem is to

- Examine subtleties that exist in three dimensions that are missing in two dimensions.
- Apply linear algebra tools to answer open-ended questions.

1. Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.
2. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?
3. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?
4. What is $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}$?

Linearly Dependent & Independent (Geometric)

DEFINITION

We say the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

Geometric definition of linear independence/dependence.

16

Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

16.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

The xy-plane in $\mathbb{R}^3$. That is, the set of all vectors in $\mathbb{R}^3$ with z-coordinate equal to zero.

16.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

No. $\vec{w} = \vec{u} + \vec{v}$, and so $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$.

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

16.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

$Y = \{\vec{u}, \vec{v}\}$ is one example that works.

16.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

$Z = \{\vec{u}, \vec{w}\}$ and $Z = \{\vec{v}, \vec{w}\}$ both have the same span as Y above.

Apply the (geometric) definition of linear independence/dependence.

The goal of this problem is to

- Develop a mental picture linking linear dependence and "redundant" vectors.
- Practice applying a new definition.
- Find multiple linearly independent subsets of a linearly dependent set.

Trivial Linear Combination

DEF

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

17

Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

17.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set? $\vec{0} = \vec{u} + \vec{v} - \vec{w}$.

17.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

No. Suppose

$$a_1\vec{u} + a_2\vec{v} = \vec{0}$$

was a non-trivial linear combination. Then at least one of a_1 or a_2 is non-zero. If a_1 is non-zero, then

$$\vec{u} = -\frac{a_2}{a_1}\vec{v}$$

and so $\vec{u} \in \operatorname{span}\{\vec{v}\}$. If a_2 is non-zero, then

$$\vec{v} = -\frac{a_1}{a_2}\vec{u}.$$

and so $\vec{v} \in \operatorname{span}\{\vec{u}\}$. In either case, $\{\vec{u}, \vec{v}\}$ would be linearly dependent.

We now have an equivalent definition of linear dependence.

Link trivial/non-trivial linear combinations to linear independence/dependence.

Linearly Dependent & Independent (Algebraic)

DEF

The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector.

18

18.1 Explain how this new definition implies the old one.

Suppose the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is linearly dependent in this new sense. That means there are scalars $a_1, a_2, \dots, a_n$, at least one of which is non-zero, such that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0.$$

Link algebraic and geometric definitions of linear independence/dependence.

The goal of this problem is to

- Understand how the algebraic and geometric definitions of linear independence/dependence relate.
- Practice writing mathematical arguments.

Suppose $a_i \neq 0$. Then

$$\vec{v}_i = \tfrac{a_1}{a_i}\vec{v}_1 + \cdots + \tfrac{a_{i-1}}{a_i}\vec{v}_{i-1} + \tfrac{a_{i+1}}{a_i}\vec{v}_{i+1} + \cdots + \tfrac{a_n}{a_i}\vec{v}_n.$$

This means $\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}$, which is precisely the old definition of linear dependence.

18.2 Explain how the old definition implies this new one.

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent according to the old definition. Fix i so that $\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}$.

By the definition of span, we know that

$$\vec{v}_i = \beta_1\vec{v}_1 + \cdots + \beta_{i-1}\vec{v}_{i-1} + \beta_{i+1}\vec{v}_{i+1} + \cdots + \beta_n\vec{v}_n.$$

Thus

$$\vec{0} = -\vec{v}_i + \beta_1\vec{v}_1 + \cdots + \beta_{i-1}\vec{v}_{i-1} + \beta_{i+1}\vec{v}_{i+1} + \cdots + \beta_n\vec{v}_n,$$

and this is a non-trivial linear combination since the coefficient of $\vec{v}_i$ is $-1 \neq 0$.

Since we have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

---

19 Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

Linear dependence and infinite solutions.

The goal of this problem is to

- Connect linear dependence with infinite solutions.
- Connect linear independence with unique solutions.

19.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

No. If both equations are true, they would combine to show

$$3\vec{u} + 2\vec{v} + \vec{w} = 2\vec{u} + \vec{v} - \vec{w}.$$

Collecting all the terms on the left side, we get:

$$\vec{u} + \vec{v} + 2\vec{w} = \vec{0},$$

which is a non-trivial linear combination of vectors in the given set equalling the zero vector.

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

19.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

Yes. If it were not, there would exist scalars a_1, a_2, a_3, not all of which are zero, such that:

$$a_1\vec{u} + a_2\vec{r} + a_3\vec{s} = \vec{0}.$$

But then

$$\vec{u} + 6\vec{r} - \vec{s} + (a_1\vec{u} + a_2\vec{r} + a_3\vec{s})$$

would be another way to write $\vec{a}$ using only the same three vectors.

19.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

Yes. If it were not, we would necessarily have $\vec{u} = \beta\vec{r}$ for some scalar β. But then

$$(\beta + 6)\vec{r} - \vec{s}$$

would be another way to write $\vec{a}$ using only the same three vectors.

19.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

No. We know from earlier that $\vec{u} + \vec{v} + 2\vec{w} = \vec{0}$, and so $\vec{u} + \vec{v} + 2\vec{w} + 0\vec{r} = \vec{0}$ is a non-trivial linear combination of the vectors in this set that equals the zero vector.

Task 1.4: Linear Independence and Dependence, Creating Examples

1. Fill in the following chart keeping track of the strategies you used to generate examples.

	Linearly independent	Linearly dependent
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

2. Write at least two generalizations that can be made from these examples and the strategies you used to create them.

Dot Product

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

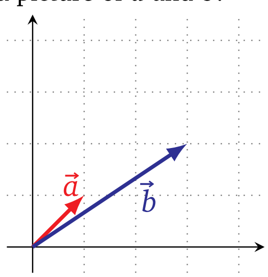
20 Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Practicing dot products.

The goal of this problem is to

- Use both the algebraic and geometric definitions of the dot product as appropriate to compute dot products.
- Gain an intuition that positive dot product means "pointing in similar directions", negative dot product means "pointing in opposite directions", and zero dot product means "pointing in orthogonal directions".

20.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.



(b) Compute $\vec{a} \cdot \vec{b}$. $\vec{a} \cdot \vec{b} = (1)(3) + (1)(1) = 4.$

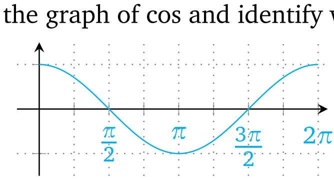
(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

$\|\vec{a}\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$ and $\|\vec{b}\| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}$.

Using the two definitions of the dot product we have:

$$\begin{aligned} &\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta \\ \implies &4 = (\sqrt{2})(\sqrt{10}) \cos\theta \\ \implies &\theta = \arccos\left(\frac{2}{\sqrt{5}}\right) \end{aligned}$$

20.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.



Cosine is positive for angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, as well as all shifts of this interval by a multiple of 2π in either direction.

cos is positive for angles in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, as well as all shifts of this interval by a multiple of 2π in either direction.

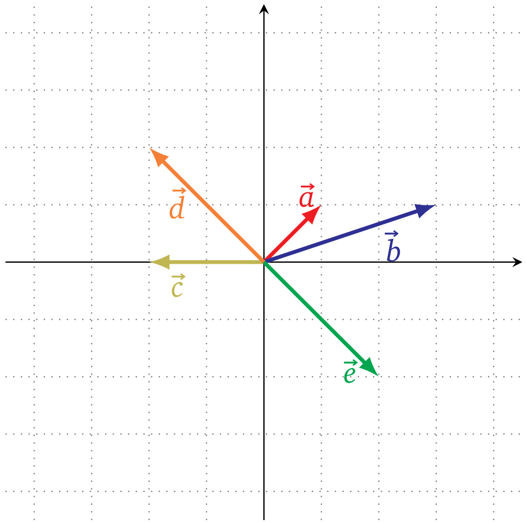
20.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.



(d) $\vec{f} = \vec{0}$ is the only possibility. For any vector $\vec{f} = \begin{bmatrix} x \\ y \end{bmatrix}$, we can compute:

$$\vec{f} \cdot \vec{a} = x + y \qquad \text{and} \qquad \vec{f} \cdot \vec{b} = 3x + 2y.$$

If these both equal zero, the first equation says that $y = -x$, and in turn the second one says $x = 0$ (and so $y = 0$ as well).

20.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ is one such vector.

Since $\cos(\pi/2) = 0$, from the second definition of the dot product above we know we are looking for a $\vec{v}$ such that $\vec{u} \cdot \vec{v} = 0$. Using the first definition of the dot product, we can see that the $\vec{v}$ given above is one possibility.

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

$\vec{w} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ is a possible answer.

$\vec{u} \cdot \vec{w} = 0$, and $\vec{w}$ is clearly not parallel to $\vec{v}$ from above.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

Yes. $\begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$ is one possibility.

There are actually infinitely many such vectors; any linear combination of $\vec{w}$ and $\vec{v}$ will work.

To see this, note that any such vector $\vec{x}$ is of the form

$$\vec{x} = t \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t - s \\ t + s \\ -3t - s \end{bmatrix},$$

for scalars t and s. We can then compute

$$\vec{u} \cdot \vec{x} = (1)(t - s) + (2)(t + s) + (1)(-3t - s) = 0,$$

and so any such vector $\vec{x}$ forms an angle of $\pi/2$ with $\vec{u}$.

THM

For a vector $\vec{v} \in \mathbb{R}^n$, the formula
$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$
always holds.

DEF

Distance

The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

DEF

Unit Vector

A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

21

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

21.1 Find the distance between $\vec{u}$ and $\vec{v}$.

$\vec{u} - \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and so $\|\vec{u} - \vec{v}\| = \sqrt{5}$.

21.2 Find a unit vector in the direction of $\vec{u}$.

$\frac{1}{\sqrt{6}}\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$.

$\|\vec{u}\| = \sqrt{6}$, and so if we multiply $\vec{u}$ by $\frac{1}{\sqrt{6}}$, the length of the resulting vector will be 1.

21.3 Does there exists a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

No. $\|\vec{u}\| = \sqrt{6}$, and so the shortest length that a vector whose distance from $\vec{u}$ is 1 can have is $\sqrt{6} - 1$, which is greater than 1.

21.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

The angle between $\vec{u}$ and $\vec{y}$ is $\arccos\left(-\frac{3}{2\sqrt{6}}\right)$.

By assumption, $2 = \|\vec{u} - \vec{y}\|$, and so

$$\begin{aligned} 4 &= \|\vec{u} - \vec{y}\|^2 \\ &= (\vec{u} - \vec{y}) \cdot (\vec{u} - \vec{y}) \\ &= \vec{u} \cdot \vec{u} - 2(\vec{u} \cdot \vec{y}) + \vec{y} \cdot \vec{y} \\ &= \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{y} + \|\vec{y}\|^2 \\ &= 6 - 2\vec{u} \cdot \vec{y} + 1. \end{aligned}$$

Then we rearrange to find that $\vec{u} \cdot \vec{y} = -\frac{3}{2}$.

Using this in the second definition of the dot product, we see:

$$-\frac{3}{2} = \left(\sqrt{6}\right)(1)\cos\theta,$$

where θ is the angle between $\vec{u}$ and $\vec{y}$.

Practice using norms.

The goal of this problem is to
- Practice finding the distance between two vectors.
- Produce a unit vector pointing in the same direction as another vector.
- Intuitively apply the triangle inequality: $\|\vec{a} + \vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$.

DEF

Orthogonal

Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u} \cdot \vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

22

22.1 Find two vectors orthogonal to $\vec{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Can you find two such vectors that are not parallel?

Two such vectors are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ -2 \end{bmatrix}$.

Apply the definition of *orthogonal*.

The goal of this problem is to
- Gain an intuitive understanding of *orthogonal vectors*.
- Produce orthogonal vectors via guess-and-check.
- Apply the Pythagorean theorem to orthogonal vectors to find lengths.

It is impossible for two non-parallel vectors to both be orthogonal to $\vec{a}$. If $\vec{b} = \begin{bmatrix} x \\ y \end{bmatrix}$ is orthogonal to $\vec{a}$, then we must have that $x - 3y = 0$, or in other words that $x = 3y$. Any $\vec{b}$ satisfying this is a multiple of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

22.2 Find two vectors orthogonal to $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Can you find two such vectors that are not parallel?

Two such vectors are $\begin{bmatrix} 7 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

These two vectors are not parallel.

22.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\| = 5$ and $\|\vec{y}\| = 3$. What is the distance between $\vec{x}$ and $\vec{y}$?

The distance between them must be $\sqrt{34}$.

One way to see this is with Pythagoras' theorem. Two perpendicular line segments of lengths 3 and 5 form the two shorter sides of a right angle triangle, and so the length of the third side is $\sqrt{5^2 + 3^2} = \sqrt{34}$.

An equivalent way to see this is to use what we know about dot products to calculate $\|\vec{x} - \vec{y}\|$ as follows:

$$\|\vec{x} - \vec{y}\| = \sqrt{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})} = \sqrt{\|\vec{x}\|^2 - 2(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2} = \sqrt{5^2 + 2(0) + 3^2},$$

where in the last step we've used the fact that $\vec{x}$ and $\vec{y}$ are orthogonal, so $\vec{x} \cdot \vec{y} = 0$.

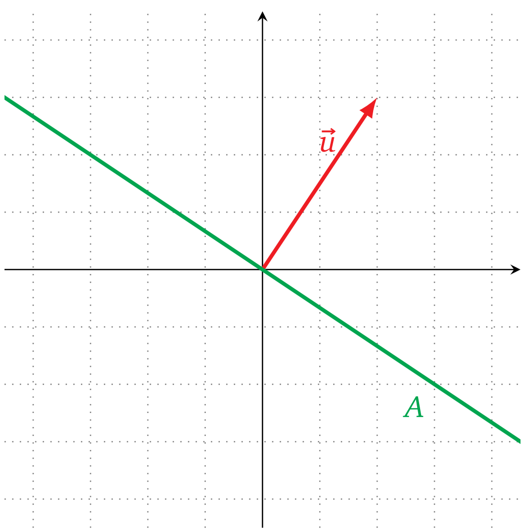
23

Generate lines using orthogonality.

The goal of this problem is to

- Visually see how the set of all vectors orthogonal to a given vector forms a line.
- Given a line defined as the set of all vectors orthogonal to a given vector, express the line using an equation or span.

23.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.



23.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$? $\vec{x} \cdot \vec{u} = 0$, by the definition of orthogonality.

23.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for A.

A is the line with vector equation $\vec{x} = t\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in A$, then $\vec{x} \cdot \vec{u} = 2x + 3y = 0$.

23.4 If possible, express A as a span. $A = \operatorname{span}\left\{\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right\}$.

Normal Vector

DEF

A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to it.

24 Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}1\\-1\end{bmatrix}$, and define the lines

$$\ell_1 = \operatorname{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \operatorname{span}\{\vec{p}\} + \{\vec{d}\}.$$

24.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

$\vec{n} = \begin{bmatrix}2\\-1\end{bmatrix}$ is one possibility.

This vector is orthogonal to $\vec{d}$, which is a direction vector for both lines.

24.2 Let $\vec{v} \in \ell_1$ and $\vec{u} \in \ell_2$. What is $\vec{n} \cdot \vec{v}$? What about $\vec{n} \cdot \vec{u}$?

$\vec{n} \cdot \vec{v} = 0$, since any $\vec{v} \in \ell_1$ is a multiple of $\vec{d}$.

$\vec{n} \cdot \vec{u} = 3$, since any such $\vec{u}$ is of the form $\vec{u} = \vec{p} + t\vec{d}$ for some scalar t, and so

$$\vec{n} \cdot \vec{u} = \vec{n} \cdot (\vec{p} + t\vec{d}) = \vec{n} \cdot \vec{p} + t(\vec{n} \cdot \vec{d}) = 3 + t(0) = 3.$$

24.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n} \cdot (\vec{x} - \vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

The normal form of ℓ_1 is $\begin{bmatrix}2\\-1\end{bmatrix} \cdot \vec{x} = 0$. test

The normal form of ℓ_2 is $\begin{bmatrix}2\\-1\end{bmatrix} \cdot \left(\vec{x} - \begin{bmatrix}1\\-1\end{bmatrix}\right) = 0$. In the previous part we saw that $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ for all $\vec{x} \in \ell_2$, or in other words $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$.

25 Let $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Planes in normal form.

The goal of this problem is to
- Observe that the set of all vectors orthogonal to another in $\mathbb{R}^3$ is a plane.
- Translate descriptions of sets into precise mathematical statements using set-builder notation.
- Express a plane in multiple ways.

25.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

$X = \left\{\vec{x} \in \mathbb{R}^3 : \vec{x} \cdot \vec{n} = 0\right\}$.

Geometrically, this is a plane through the origin and perpendicular to $\vec{n}$.

25.2 Describe X using an equation. $x + y + z = 0$.

25.3 Describe X as a span. $X = \operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ is one way to do this.

Projections

Projection

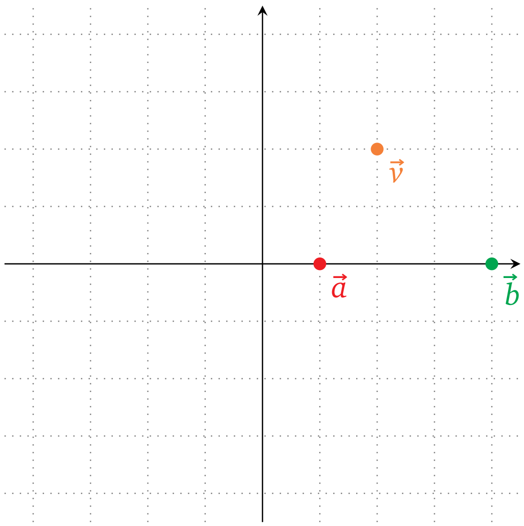
DEF Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\operatorname{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

26 Let $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}2\\2\end{bmatrix}$ and $\ell = \operatorname{span}\{\vec{a}\}$.

Apply the definition of projection.

The goal of this problem is to
- Use the definition of projection to compute projections onto finite sets and lines.
- Pick an appropriate representation of a line to solve a projection problem.

26.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.



26.2 Find $\text{proj}_{\{\vec b\}}\vec v$, $\text{proj}_{\{\vec a,\vec b\}}\vec v$.

$\text{proj}_{\{\vec b\}}\vec v = \vec b$. Since there is only one point in $\{\vec b\}$, it must be the closest point to $\vec v$.

$\text{proj}_{\{\vec a,\vec b\}}\vec v = \vec a$. We can simply compute $\|\vec v - \vec a\| = \sqrt{5}$ and $\|\vec v - \vec b\| = \sqrt{13}$, so $\vec a$ is closer to $\vec v$.

26.3 Find $\text{proj}_\ell \vec v$. (Recall that a quadratic $at^2 + bt + c$ has a minimum at $t = -\frac{b}{2a}$).

$\text{proj}_\ell \vec v = 2\vec a = \begin{bmatrix}2\\0\end{bmatrix}$.

Any point in ℓ is of the form $t\vec a$ for some scalar t. The distance between such a point and $\vec v$ is

$$\|\vec v - t\vec a\| = \sqrt{\|v\|^2 - 2t(\vec v\cdot\vec a) + t^2\|a\|^2} = \sqrt{8 - 4t + t^2}$$

The quadratic inside the square root has a minimum at $t = 2$, so $2\vec a$ is the closest point in the line to $\vec v$.

26.4 Is $\vec v - \text{proj}_\ell \vec v$ a normal vector for ℓ? Why or why not?

Yes.

By the previous part, $\vec v - \text{proj}_\ell \vec v = \vec v - 2\vec a = \begin{bmatrix}0\\2\end{bmatrix}$. This vector is orthogonal to $\vec a$, and therefore to ℓ.

---

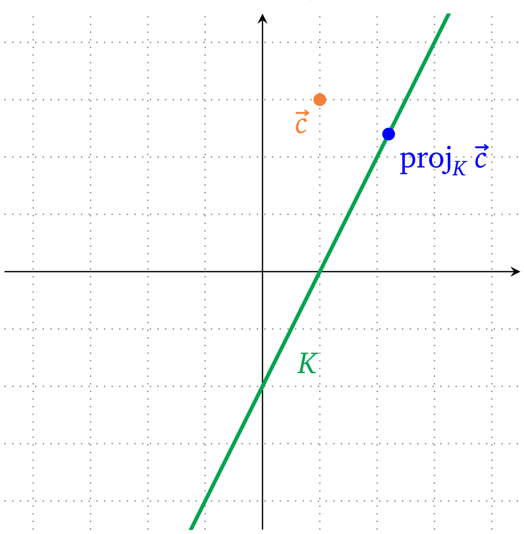
27 Let K be the line given in vector form by $\vec x = t\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}$ and let $\vec c = \begin{bmatrix}1\\3\end{bmatrix}$.

Project onto lines.

The goal of this problem is to

- Use orthogonality to compute the projection onto a line.
- Project onto lines that don't pass through $\vec 0$.

27.1 Make a sketch with $\vec c$, K, and $\text{proj}_K \vec c$ (you don't need to compute $\text{proj}_K \vec c$ exactly).



27.2 What should $(\vec c - \text{proj}_K \vec c)\cdot\begin{bmatrix}1\\2\end{bmatrix}$ be? Explain.

$(\vec c - \text{proj}_K \vec c)\cdot\begin{bmatrix}1\\2\end{bmatrix} = 0$.

From our picture we can see that $c - \text{proj}_K \vec{c}$ is perpendicular to the line K, and so the dot product of this vector with any direction vector for K should be zero.

27.3 Use your formula from the previous part to find $\text{proj}_K \vec{c}$ *without* computing any distances.

$\text{proj}_K \vec{c} = \frac{1}{5}\begin{bmatrix}11\\12\end{bmatrix}$

If $\text{proj}_K \vec{c} = \begin{bmatrix}x\\y\end{bmatrix}$, the formula from the previous part tells us

$$\left(\begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}x\\y\end{bmatrix}\right)\cdot\begin{bmatrix}1\\2\end{bmatrix} = 1 - x + 6 - 2y = 0 \qquad \iff \qquad x + 2y = 7$$

So we need a point on K that satisfies this equation. In other words, we need

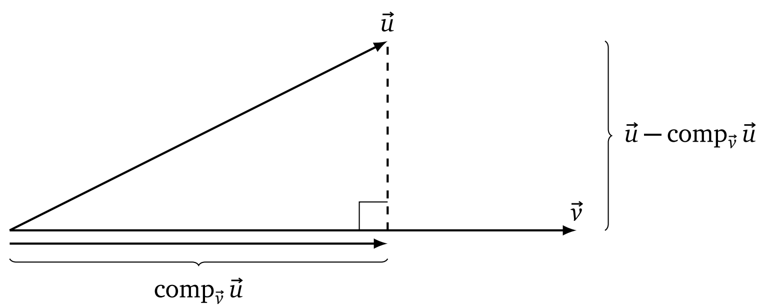
$$(t+1) + 2(2t) = 7 \qquad \implies \qquad t = \frac{6}{5}.$$

The point on K for this value of t is $\frac{1}{5}\begin{bmatrix}11\\12\end{bmatrix}$.

Component

DEFINITION

Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***component of $\vec{u}$ in the $\vec{v}$ direction***, written $\text{comp}_{\vec{v}}\,\vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \text{comp}_{\vec{v}}\,\vec{u}$ is orthogonal to $\vec{v}$.



28 Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

Component of a vector in the direction of another.

The goal of this problem is to

- Read and apply a new definition.
- Use orthogonality to obtain a formula for components in terms of dot products.

28.1 List two conditions that $\text{comp}_{\vec{b}}\,\vec{a}$ must satisfy.

$\text{comp}_{\vec{b}}\,\vec{a}$ must be a scalar multiple of $\vec{b}$.

$\vec{a} - \text{comp}_{\vec{b}}\,\vec{a}$ must be orthogonal to $\vec{b}$, or in other words $(\vec{a} - \text{comp}_{\vec{b}}\,\vec{a})\cdot\vec{b} = 0$.

28.2 Find a formula for $\text{comp}_{\vec{b}}\,\vec{a}$.

$\text{comp}_{\vec{b}}\,\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}$.

From the previous part, we should have $\text{comp}_{\vec{b}}\,\vec{a} = t\vec{b}$ for some scalar t, and $(\vec{a} - \text{comp}_{\vec{b}}\,\vec{a})\cdot\vec{b} = 0$.

Combining these, we get:

$$0 = (\vec{a} - t\vec{b})\cdot\vec{b} = \vec{a}\cdot\vec{b} - t\vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{b} - t(\vec{b}\cdot\vec{b}).$$

Solving for t, we get $t = \dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}$.

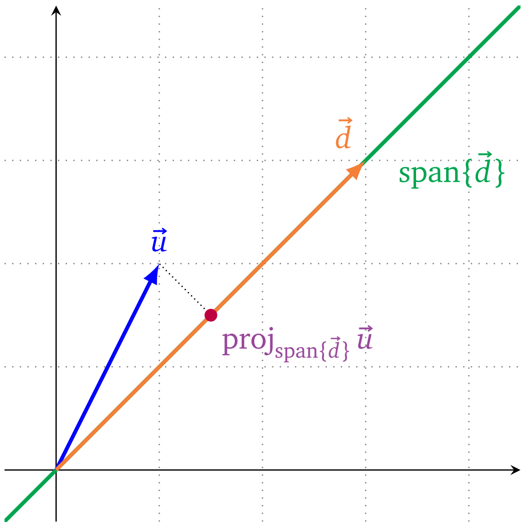
29 Let $\vec{d} = \begin{bmatrix}3\\3\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$.

Relate components and projections.

The goal of this problem is to

- Find a connection between components and projections onto spans.
- Recognize that $\text{comp}_{\vec{u}}\,\vec{v} = \text{comp}_{-\vec{u}}\,\vec{v}$.

29.1 Draw $\vec{d}$, $\vec{u}$, $\text{span}\{\vec{d}\}$, and $\text{proj}_{\text{span}\{\vec{d}\}}\,\vec{u}$ in the same picture.



29.2 How do $\text{proj}_{\text{span}\{\vec d\}}\,\vec u$ and $\text{comp}_{\vec d}\,\vec u$ relate? They are equal.

29.3 Compute $\text{proj}_{\text{span}\{\vec d\}}\,\vec u$ and $\text{comp}_{\vec d}\,\vec u$.

Using our formula from the previous problem

$$\text{proj}_{\text{span}\{\vec d\}}\,\vec u = \text{comp}_{\vec d}\,\vec u = \frac{\vec u\cdot\vec d}{\|\vec d\|^2}\vec d = \tfrac{9}{18}\vec d = \tfrac{1}{2}\begin{bmatrix}3\\3\end{bmatrix}.$$

29.4 Compute $\text{comp}_{-\vec d}\,\vec u$. Is this the same as or different from $\text{comp}_{\vec d}\,\vec u$? Explain.

$$\text{comp}_{-\vec d}\,\vec u = \frac{\vec u\cdot(-\vec d)}{\|-\vec d\|^2}(-\vec d) = \tfrac{-9}{18}(-\vec d) = \tfrac{1}{2}\begin{bmatrix}3\\3\end{bmatrix} = \text{comp}_{\vec d}\,\vec u.$$

We expect them to be equal since $\vec d$ and $-\vec d$ are in the same direction as one another.

Subspaces and Bases

DEFINITION

Subspace

A ***subspace*** $V \subseteq \mathbb{R}^n$ is a non-empty subset such that

(i) $\vec u, \vec v \in V$ implies $\vec u + \vec v \in V$.

(ii) $\vec u \in V$ implies $k\vec u \in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

30 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

30.1 $A = \left\{\vec x \in \mathbb{R}^2 : \vec x = \begin{bmatrix}a\\0\end{bmatrix} \text{ for some } a \in \mathbb{Z}\right\}$.

XXXX

A is not a subspace, since for example $\begin{bmatrix}1\\0\end{bmatrix} \in A$ but $\frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} \notin A$.

30.2 $B = \left\{\vec x \in \mathbb{R}^2 : \vec x \neq \begin{bmatrix}0\\0\end{bmatrix}\right\}$.

XXXX

B is not a subspace, since for example $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\end{bmatrix}$ are both in B, but their sum is $\begin{bmatrix}0\\0\end{bmatrix}$ which is not in B.

30.3 $C = \left\{\vec x \in \mathbb{R}^2 : \vec x = \begin{bmatrix}0\\t\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

XXXX

C is a subspace.

Visualizing subspaces.

The goal of this problem is to

- Read and apply the definition of subspace.
- Identify from a picture whether or not a set is a subspace.
- Write formal arguments showing whether or not certain sets are subspaces.

(i) Let $\vec{u}, \vec{v} \in C$. Then $\vec{u} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ s \end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = \begin{bmatrix} 0 \\ s+t \end{bmatrix} \in C$.

(ii) Let $\vec{u} = \begin{bmatrix} 0 \\ t \end{bmatrix} \in C$. For any scalar α we have $\alpha\vec{u} = \begin{bmatrix} 0 \\ \alpha t \end{bmatrix} \in C$.

30.4 $D = \left\{\vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

XXXX

D is not a subspace, since for example $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in D$, but $0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin D$.

30.5 $E = \left\{\vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

XXXX

E is not a subspace, since for example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both in E, but their sum is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not in E.

30.6 $F = \left\{\vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

XXXX

F is a subspace.

(i) Let $\vec{u}, \vec{v} \in F$. Then $\vec{u} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = (s+t)\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in F$.

(ii) Let $\vec{u} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in F$. For any scalar α we have $\alpha\vec{u} = (\alpha t)\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in F$.

30.7 $G = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

XXXX

G is a subspace.

By definition of a span, $G = \left\{\vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

The proof that G is a subspace now proceeds similarly to the proof for F above.

(i) Let $\vec{u}, \vec{v} \in G$. Then $\vec{u} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = (s+t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in G$.

(ii) Let $\vec{u} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in G$. For any scalar α we have $\alpha\vec{u} = (\alpha t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in G$.

30.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

XXXX

H is a subspace.

(i) Let $\vec{x}, \vec{y} \in H$. Then $\vec{x} = \alpha_1\vec{u} + \alpha_2\vec{v}$ and $\vec{y} = \beta_1\vec{u} + \beta_2\vec{v}$ for some scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$. But then

$$\vec{x} + \vec{y} = \alpha_1\vec{u} + \alpha_2\vec{v} + \beta_1\vec{u} + \beta_2\vec{v} = (\alpha_1 + \beta_1)\vec{u} + (\alpha_2 + \beta_2)\vec{v} \in H.$$

(ii) Let $\vec{x} = \alpha_1\vec{u} + \alpha_2\vec{v} \in H$. For any scalar β we have $\beta\vec{u} = (\beta\alpha_1)\vec{u} + (\beta\alpha_2)\vec{v} \in H$.

Basis (DEF)

A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

Dimension (DEF)

The ***dimension*** of a subspace V is the number of elements in a basis for V.

31 Let $\vec{u}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V=\operatorname{span}\{\vec{u},\vec{v},\vec{w}\}$.

31.1 Describe V. V is the xy-plane in $\mathbb{R}^3$.

31.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ a basis for V? Why or why not?

No. The set $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent since $\vec{w}=\vec{u}+\vec{v}$.

31.3 Give a basis for V. $\{\vec{u},\vec{v}\}$.

31.4 Give another basis for V. $\{\vec{u},\vec{w}\}$ or $\{\vec{v},\vec{w}\}$.

31.5 Is $\operatorname{span}\{\vec{u},\vec{v}\}$ a basis for V? Why or why not?

No. $\operatorname{span}\{\vec{u},\vec{v}\}$ is an infinite set of vectors which includes $\vec{0}$, so it cannot be linearly independent and therefore isn't a basis.

31.6 What is the dimension of V?

A basis for V has two vectors so it is two-dimensional. We also know this because V is the xy-plane in $\mathbb{R}^3$ and all planes are two-dimensional.

32 Let $\vec{a}=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b}=\begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c}=\begin{bmatrix}7\\8\\8\end{bmatrix}$ (notice these vectors are linearly independent) and let $P=\operatorname{span}\{\vec{a},\vec{b}\}$ and $Q=\operatorname{span}\{\vec{b},\vec{c}\}$.

The relationship between subspaces, bases, unions, and intersections.

The goal of this problem is to learn

- Recognize intersections of subspaces as subspaces.
- Recognize the union of subspaces need not be a subspace.
- Visualize planes in $\mathbb{R}^3$ to solve problems without computations.

32.1 Give a basis for and the dimension of P.

$\{\vec{a},\vec{b}\}$ is a basis for P, and so its dimension is 2.

32.2 Give a basis for and the dimension of Q.

$\{\vec{b},\vec{c}\}$ is a basis for Q, and so its dimension is 2.

32.3 Is $P\cap Q$ a subspace? If so, give a basis for it and its dimension.

Yes. $\{\vec{b}\}$ is a basis for $P\cap Q$, and so its dimension is 1.

P and Q are both planes and are not parallel (since $\vec{a},\vec{b},\vec{c}$ are linearly independent). The intersection of any two non-parallel planes is a line. We know that $\vec{0}$ and $\vec{c}$ are on this line, and therefore the line is $\operatorname{span}\{\vec{c}\}$

32.4 Is $P\cup Q$ a subspace? If so, give a basis for it and its dimension.

No. For example $\vec{a}$ and $\vec{c}$ are both in $P\cup Q$, but $\vec{a}+\vec{c}\notin P\cup Q$.

A vector is in $P\cup Q$ if it is in P or Q, so we must show that $\vec{a}+\vec{c}\notin P$ and $\vec{a}+\vec{c}\notin Q$

$\vec{a}+\vec{c}\notin P$ since if it were, we would also have $(\vec{a}+\vec{c})-\vec{a}=\vec{c}\in P$. We know this is impossible since the vectors $\vec{a},\vec{b},\vec{c}$ are linearly independent, and so $\vec{c}$ does not equal a linear combination of $\vec{a}$ and $\vec{b}$.

An analogous argument shows that $\vec{a}+\vec{c}\notin Q$.

Matrices

33 Let $A=\begin{bmatrix}1&2\\3&3\end{bmatrix}$, $\vec{x}=\begin{bmatrix}x\\y\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}-2\\-1\end{bmatrix}$.

33.1 Compute the product $A\vec{x}$.

$A\vec{x}=\begin{bmatrix}x+2y\\3x+3y\end{bmatrix}$.

33.2 Write down a system of equations that corresponds to the matrix equation $A\vec{x}=\vec{b}$.

$$\begin{aligned}x+2y&=-2\\3x+3y&=-1\end{aligned}$$

33.3 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

x_0 and y_0, when used as scalars in a linear combination of the columns of A, make the vector $\vec{b}$. In other words:

$$x_0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y_0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

33.4 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

The lines represented by the equations $x + 2y = -2$ and $3x + 3y = -1$ from the system of equations above intersect at the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

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34 Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

34.1 How could you determine if $\{\vec{u}, \vec{v}, \vec{w}\}$ was a linearly independent set?

The set is linearly independent if and only if no non-trivial linear combination of the vectors $\vec{u}, \vec{v}, \vec{w}$ equals $\vec{0}$. That is, if x, y, z are scalars such that $x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}$, then $x = y = z = 0$.

In other words, the only solution of the following system of equations is $x = y = z = 0$.

$$\begin{aligned} x + 4y + 7z &= 0 \\ 2x + 5y + 8z &= 0 \\ 3x + 6y + 9z &= 0 \end{aligned}$$

34.2 Can your method be rephrased in terms of a matrix equation? Explain.

The system of linear equations above can be represented by the matrix equation $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

So another way to say the above is that the set is linearly independent if and only if the only solution to the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

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35 Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

35.1 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

This system has no solutions, since the third equation would be $0 = 3$, which is impossible.

35.2 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

A line. The system would be

$$\begin{aligned} x - 3y &= 1 \\ z &= 1 \\ 0 &= 0 \end{aligned}$$

A vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that satisfies this system must have $z = 1$, and by the first equation in the system any value of x determines the value of y, and vice versa. In other words the system has one free variable, and so its set of solutions is a line.

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36 Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Let $\mathcal{P}$ be the plane given in vector form by $\vec{x} = t\vec{d}_1 + s\vec{d}_2$. Further, suppose M is a matrix so that $M\vec{r} \in \mathcal{P}$ for any $\vec{r} \in \mathbb{R}^2$.

36.1 How many rows does M have?

Three. It must have three rows in order for $M\vec{r}$ to be an element of $\mathbb{R}^3$.

36.2 Find such an M.

$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$ is one possible answer, since if $\vec{r} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $M\vec{r} = a\vec{d}_1 + b\vec{d}_2$.

Another less interesting answer is the 3×2 zero matrix.

36.3 Find necessary and sufficient conditions (phrased as equations) for $\vec{n}$ to be a normal vector for $\mathcal{P}$.

$\vec{n}$ is normal to $\mathcal{P}$ if and only if $\vec{n} \cdot \vec{d}_1 = 0$ *and* $\vec{n} \cdot \vec{d}_2 = 0$

36.4 Find a matrix K so that solutions to $K\vec{x} = \vec{0}$ are normal vectors for $\mathcal{P}$. How do K and M relate?

$K = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$. K and M are transposes of one another.

The conditions $\vec{n} \cdot \vec{d}_1 = 0$ and $\vec{n} \cdot \vec{d}_2 = 0$ from the previous part translate to the following system of equations:

$$\begin{aligned} x + y + 2z &= 0 \\ -x + y &= 0. \end{aligned}$$

This system of equations can be represented by the matrix equation

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Change of Basis & Coordinates

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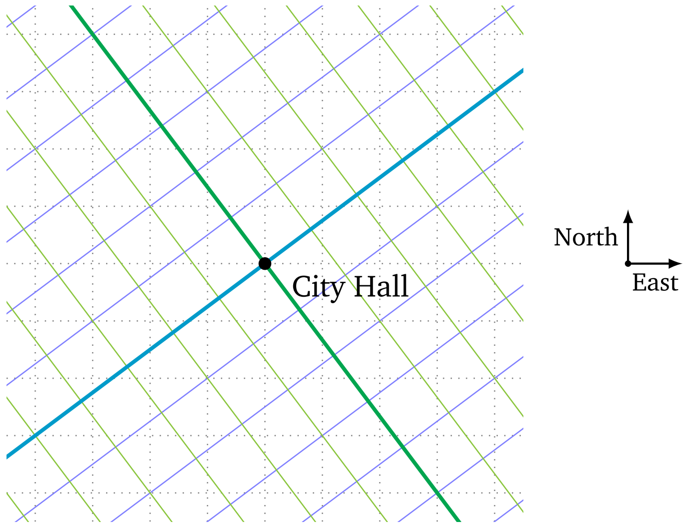
37 The fictional town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \frac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

37.1 The points $A = (2, 1)$ and $B = (3, -1)$ are given in street coordinates. Find their east-north coordinates.

$A = \frac{1}{5}(5, -4)$ and $B = (3, 1)$ in east-north coordinates.

We obtain A for example by finding the vector $2\vec{d}_1 + \vec{d}_2$.

37.2 The points $X = (4, 3)$ and $Y = (1, 7)$ are given in east-north coordinates. Find their street coordinates. $X = (5, 0)$ and $Y = (5, 5)$ in street coordinates.

37.3 Define $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$. Does $\operatorname{span}\{\vec{e}_1, \vec{e}_2\} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$?

Yes. Both of these sets spans all of $\mathbb{R}^2$.

37.4 Notice that $Y = 5\vec{e}_1 + 5\vec{e}_2 = \vec{d}_1 + 7\vec{d}_2$. Is the point Y better represented by the pair $(5, 5)$ or by the pair $(1, 7)$? Explain.

It is equally well represented by either pair. For example, the street coordinates might be more useful for a resident of Oronto, while the east-north coordinates might be more useful for someone looking at Oronto on a world map.

Representation in a Basis

DEFINITION

Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notate $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

such that $\vec{v} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$.

Similarly,

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{B}} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \dots, \vec{b}_n$ with coefficients $\alpha_1, \dots, \alpha_n$.

38 Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

38.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$. $\vec{c}_1 = 2\vec{e}_1 + \vec{e}_2$ and $\vec{c}_2 = 5\vec{e}_1 + 3\vec{e}_2$.

38.2 Express $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$. $\vec{e}_1 = 3\vec{c}_1 - \vec{c}_2$ and $\vec{e}_2 = -5\vec{c}_1 + 2\vec{c}_2$.

38.3 Let $\vec{v} = 2\vec{e}_1 + 2\vec{e}_2$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

$[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

The second one is since

$$\vec{v} = 2\vec{e}_1 + 2\vec{e}_2 = 2(3\vec{c}_1 - \vec{c}_2) + 2(-5\vec{c}_1 + 2\vec{c}_2) = -4\vec{c}_1 + 2\vec{c}_2.$$

38.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}} = [\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

$X = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is such a matrix.

We know X must be a 2×2 matrix, so suppose $X = \begin{bmatrix} a & b \\ c & b \end{bmatrix}$ for some $a, b, c, d \in \mathbb{R}$.

From the first part above, we know

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}},$$

and so we need X to satisfy

$$X\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad X\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

But $X\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $X\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$, so we can now immediately solve for a, b, c, d to find that X must be the matrix $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

38.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}} = [\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

$Y = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is such a matrix.

Using similar reasoning to the previous part, we know Y must be a 2×2 matrix, so suppose $Y = \begin{bmatrix} a & b \\ c & b \end{bmatrix}$ for some $a, b, c, d \in \mathbb{R}$.

From the second part above, we know

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_{\mathcal{C}} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}_{\mathcal{C}},$$

and so we need Y to satisfy

$$Y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad Y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix},$$

But $Y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $Y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$, so we can now immediately solve for a, b, c, d to find that Y must be the matrix $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

38.6 What is YX?

$YX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Orientation of a Basis

DEFINITION

The ordered basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i \mapsto \vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

39 Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\vec{e}_1$ measured counter-clockwise starting at $\vec{e}_1$.

39.1 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ a linearly independent set? Every θ that is not a multiple of π.

39.2 For which θ can $\{\vec{e}_1, \vec{u}_\theta\}$ be continuously transformed into $\{\vec{e}_1, \vec{e}_2\}$ and remain linearly independent the whole time?

Every $\theta \in (0, \pi)$.

For $\theta \in (\pi, 2\pi)$, a continuous transformation of $\vec{u}_\theta$ to $\vec{e}_2$ would have to cross the x-axis, at which point $\{\vec{e}_1, \vec{u}_\theta\}$ would cease to be linearly independent.

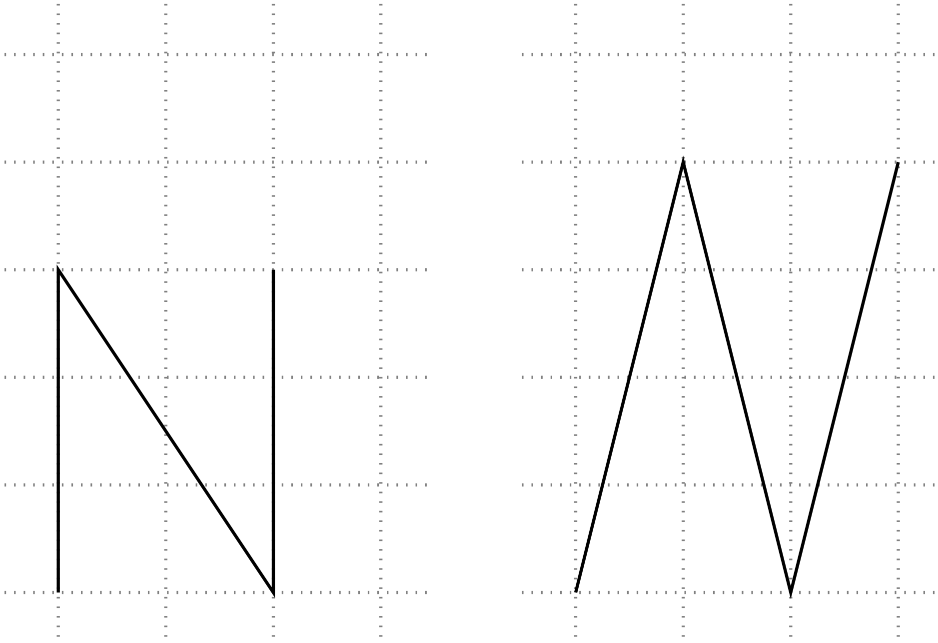
39.3 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ right-handed? Left-handed?

It is right-handed for $\theta \in (0, \pi)$, and left handed for $\theta \in (\pi, 2\pi)$.

39.4 For which θ is $\{\vec{u}_\theta, \vec{e}_1\}$ (in that order) right-handed? Left-handed?

It is right-handed for $\theta \in (\pi, 2\pi)$, and left handed for $\theta \in (0, \pi)$.

Task 2.1: Italicizing N

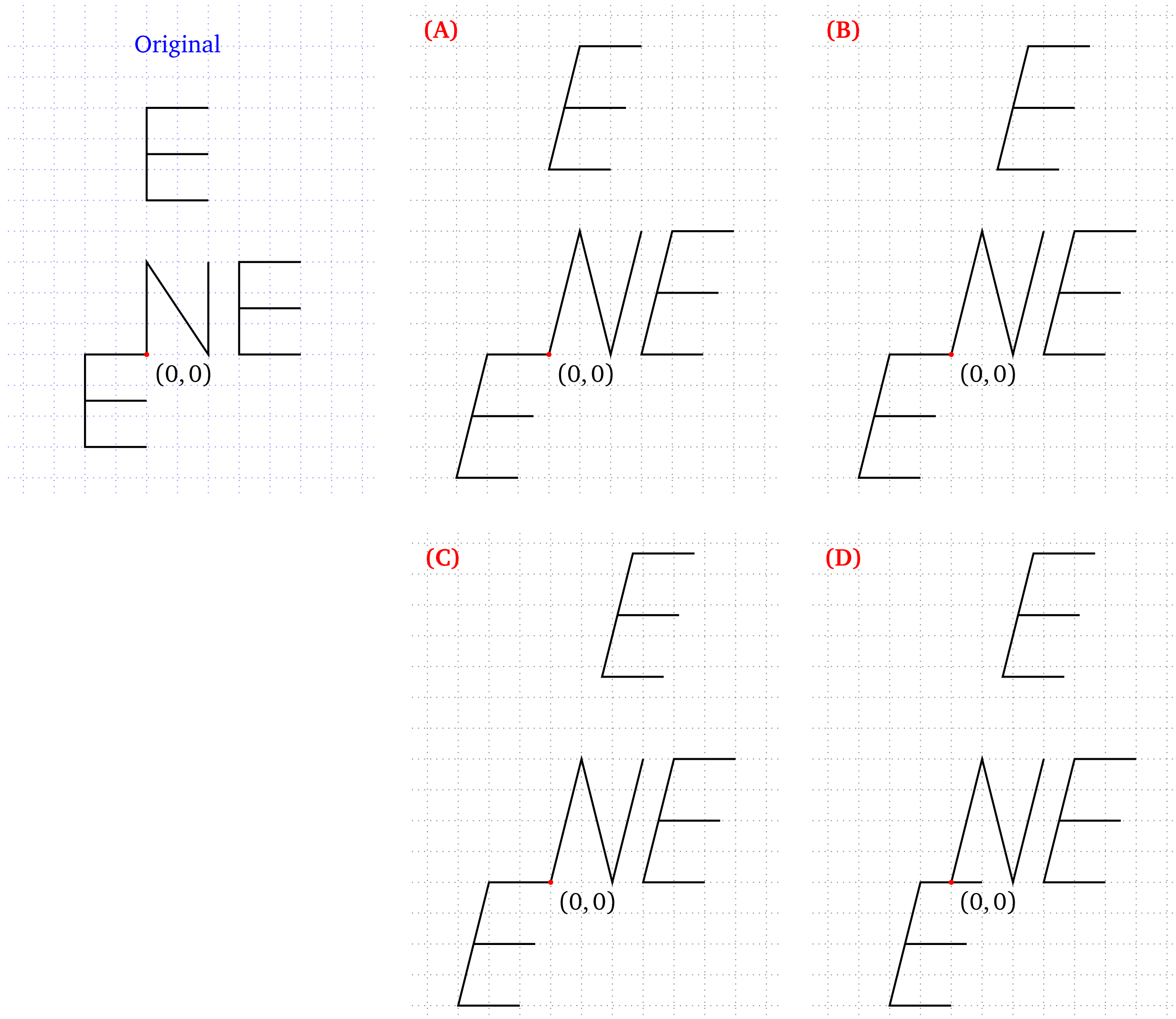


Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

Task 2.2: Beyond the N

A few students were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If an "E" is placed around the "N," the students argued over four different possible results for the transformed E's. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?



40 $\mathcal{R}:\mathbb{R}^2\to\mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

40.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$. $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$.

40.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

$$\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}=\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}+\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}.$$

40.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

$$\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)=a\begin{bmatrix}0\\1\end{bmatrix}+b\begin{bmatrix}-1\\0\end{bmatrix}.$$

Rotating a vector and then multiplying by a scalar gives the same result as multiplying first then rotating. Similarly, adding two vectors and then rotating their sum gives the same result as rotating them and then adding.

40.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter-clockwise by 90°.

$R=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is such a matrix.

Linear Transformation

DEFINITION

Let V and W be subspaces. A function $T:V\to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v})=T\vec{u}+T\vec{v}\qquad\text{and}\qquad T(\alpha\vec{v})=\alpha T\vec{v}$$

for all vectors $\vec{u},\vec{v}\in V$ and all scalars α.

41 41.1 Classify the following as linear transformations or not.

(a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

A linear transformation. We proved this in the previous problem.

(b) $W:\mathbb{R}^2\to\mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x^2\\y\end{bmatrix}$.

Not a linear transformation, since for example $W\left(2\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\end{bmatrix}\neq\begin{bmatrix}2\\0\end{bmatrix}=2W\begin{bmatrix}1\\0\end{bmatrix}$.

(c) $T:\mathbb{R}^2\to\mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2\\y\end{bmatrix}$.

Not a linear transformation, since for example $T\left(2\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\end{bmatrix}\neq 2T\begin{bmatrix}1\\0\end{bmatrix}$.

(d) $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix}=\operatorname{comp}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

A linear transformation.

We found a general formula for $\operatorname{comp}_{\vec{u}}$ in a previous exercise:

$$\operatorname{comp}_{\vec{u}}\vec{x}=\frac{\vec{u}\cdot\vec{x}}{\vec{u}\cdot\vec{u}}\vec{u}=\frac{\vec{u}\cdot\vec{x}}{13}\vec{u}.$$

For any two vectors $\vec{x}$ and $\vec{y}$, we have

$$\begin{aligned}\operatorname{comp}_{\vec{u}}(\vec{x}+\vec{y})&=\frac{\vec{u}\cdot(\vec{x}+\vec{y})}{13}\vec{u}\\&=\frac{\vec{u}\cdot\vec{x}}{13}\vec{u}+\frac{\vec{u}\cdot\vec{y}}{13}\vec{u}\\&=\operatorname{comp}_{\vec{u}}\vec{x}+\operatorname{comp}_{\vec{u}}\vec{y}.\end{aligned}$$

For any $\vec{x}$ and scalar α, we have

$$\operatorname{comp}_{\vec{u}}(\alpha\vec{x})=\frac{\vec{u}\cdot(\alpha\vec{x})}{13}\vec{u}=\frac{\alpha(\vec{u}\cdot\vec{x})}{13}\vec{u}=\alpha\operatorname{comp}_{\vec{u}}\vec{x}.$$

DEFINITION

Image of a Set

Let $L: V \to W$ be a transformation and let $X \subset V$ be a set. The ***image of the set*** V ***under*** L, denoted $L(V)$, is the set

$$L(V) = \{\vec{x} \in W : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in V\}.$$

42 Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \le x, y \le 1 \right\} \subseteq \mathbb{R}^2$ be the filled-in unit square and let $C = \{\vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\} \subseteq \mathbb{R}^2$ be the corners of the unit square.

42.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

$\mathcal{R}(C) = \{\vec{0}, \vec{e}_2, -\vec{e}_1, -\vec{e}_1 + \vec{e}_2\}$.

$W(C) = C$.

$T(C) = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$.

42.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).

XXXX

42.3 Let $\ell = \{\text{all convex combinations of } \vec{a} \text{ and } \vec{b}\}$ be a line segment with endpoints $\vec{a}$ and $\vec{b}$ and let A be a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

$A(\ell)$ must be a line segment, with endpoints $A(\vec{a})$ and $A(\vec{b})$.

For any scalars α_1 and α_2, by the linearity of A we have: $A(\alpha_1\vec{a}+\alpha_2\vec{b}) = \alpha_1 A(\vec{a})+\alpha_2 A(\vec{b})$.

If $\alpha_1 + \alpha_2 = 1$, then the linear combination on the right is also convex, and so $A(\ell)$ is the set of convex combinations of $A(\vec{a})$ and $A(\vec{b})$. This is precisely the straight line segment joining $A(\vec{a})$ and $A(\vec{b})$.

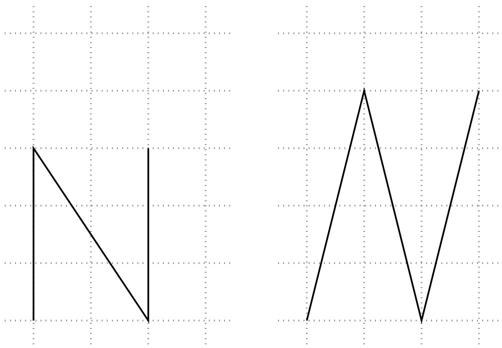
Note that if $A(\vec{a}) = A(\vec{b})$ (for example, if A is the zero transformation), then $A(\ell)$ will consist of the single point, which we think of as a "degenerate" line segment in this situation.

42.4 Explain how images of sets relate to the *Italicizing N* task.

The task asked us to find a linear transformation such that the image of the regular "N" is the italicized "N".

By the previous exercise, we now know it suffices to find a linear transformation that sends the four endpoints of line segments on the regular "N" to the corresponding four endpoints on the italicized "N".

Task 2.3: Pat and Jamie



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

1. Do you think Pat and Jamie's approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?
2. Try Pat and Jamie's approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

43 Define $\mathcal{P}$ to be projection onto span$\{\vec{u}\}$ where $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

43.1 Find a matrix P so that $P\vec{x}=\mathcal{P}(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$.

$P=\frac{1}{13}\begin{bmatrix}4&6\\6&9\end{bmatrix}$ is such a matrix.

The matrix P corresponding to $\mathcal{P}$ is a 2×2 matrix, so suppose $P=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ for some $a,b,c,d\in\mathbb{R}$. Then we know that if $\{\vec{e}_1,\vec{e}_2\}$ is the standard basis for $\mathbb{R}^2$,

$$P(\vec{e}_1)=\begin{bmatrix}a\\c\end{bmatrix}\quad\text{and}\quad P(\vec{e}_2)=\begin{bmatrix}b\\d\end{bmatrix}.$$

We know from an earlier exercise that $\mathcal{P}(\vec{x})=\dfrac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$. Therefore, the first column of P is

$$\begin{bmatrix}a\\c\end{bmatrix}=\mathcal{P}(\vec{e}_1)=\frac{2}{13}\vec{u}=\frac{1}{13}\begin{bmatrix}4\\6\end{bmatrix}$$

and the second column of P is

$$\begin{bmatrix}b\\d\end{bmatrix}=\mathcal{P}(\vec{e}_2)=\frac{3}{13}\vec{u}=\frac{1}{13}\begin{bmatrix}6\\9\end{bmatrix}.$$

43.2 Find a matrix R so that $R\vec{x}=\mathcal{R}(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$.

$R=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is such a matrix.

Using the same reasoning as the previous part, we can compute

$$\mathcal{R}(\vec{e}_1)=\vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}\quad\text{and}\quad\mathcal{R}(\vec{e}_2)=-\vec{e}_1=\begin{bmatrix}-1\\0\end{bmatrix}.$$

Therefore, the matrix R for $\mathcal{R}$ is the matrix with the two vectors above as its respective columns.

43.3 Write down matrices A and B for $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{P}$.

$A=\frac{1}{13}\begin{bmatrix}6&-4\\9&-6\end{bmatrix}$ and $B=\frac{1}{13}\begin{bmatrix}-6&-9\\4&6\end{bmatrix}$ are two such matrices.

Using the same reasoning as above, we can compute

$$(\mathcal{P}\circ\mathcal{R})(\vec{e}_1)=\mathcal{P}(\mathcal{R}(\vec{e}_1))=\mathcal{P}(\vec{e}_2)=\frac{1}{13}\begin{bmatrix}6\\9\end{bmatrix}\qquad\text{and}\qquad(\mathcal{P}\circ\mathcal{R})(\vec{e}_2)=\mathcal{P}(\mathcal{R}(\vec{e}_2))=\mathcal{P}(-\vec{e}_1)=\frac{1}{13}\begin{bmatrix}-4\\-6\end{bmatrix}.$$

Therefore, the matrix A for $\mathcal{P}\circ\mathcal{R}$ is the matrix with the two vectors above as its respective columns.

Similarly, for $\mathcal{R}\circ\mathcal{P}$, we can compute:

$$(\mathcal{R}\circ\mathcal{P})(\vec{e}_1)=\mathcal{R}(\mathcal{P}(\vec{e}_1))=\mathcal{R}\left(\frac{1}{13}\begin{bmatrix}4\\6\end{bmatrix}\right)=\frac{1}{13}\begin{bmatrix}-6\\4\end{bmatrix}$$
$$(\mathcal{R}\circ\mathcal{P})(\vec{e}_2)=\mathcal{R}(\mathcal{P}(\vec{e}_2))=\mathcal{R}\left(\frac{1}{13}\begin{bmatrix}6\\9\end{bmatrix}\right)=\frac{1}{13}\begin{bmatrix}-9\\6\end{bmatrix}.$$

Therefore, the matrix B for $\mathcal{R}\circ\mathcal{P}$ is the matrix with these two vectors as its respective columns.

43.4 How do the matrices A and B relate to the matrices P and R?

$A=PR$ and $B=RP$.

We can simply compute these matrix products to see this, but from the previous parts we that for any vector $\vec{x}$

$$A\vec{x}=(\mathcal{P}\circ\mathcal{R})(\vec{x})=\mathcal{P}(\mathcal{R}(\vec{x}))=\mathcal{P}(R\vec{x})=PR\vec{x}.\quad\text{and}\quad B\vec{x}=(\mathcal{R}\circ\mathcal{P})(\vec{x})=\mathcal{R}(\mathcal{P}(\vec{x})=\mathcal{R}(P\vec{x})=RP\vec{x}.$$

Using $\vec{x}=\vec{e}_1$ shows that first column of A must equal the first column of PR, and using $\vec{x}=\vec{e}_2$ shows that the second column fo A must equal the second column of PR, and therefore $A=PR$. For the same reason, we must also have $B=RP$.

Range

DEF

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\operatorname{range}(T) = \{\vec y \in W \;:\; \vec y = T\vec x \text{ for some } \vec x \in V\}.$$

Null Space

DEFINITION

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$\operatorname{null}(T) = \{\vec x \in V \;:\; T\vec x = \vec 0\}.$$

---

44 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto $\operatorname{span}\{\vec u\}$ where $\vec u = \begin{bmatrix}2\\3\end{bmatrix}$ (like before).

44.1 What is the range of $\mathcal{P}$?

$\operatorname{range}(\mathcal{P}) = \operatorname{span}\{\vec u\}$.

$\mathcal{P}(\vec x)$ is by definition the vector in $\operatorname{span}\{\vec u\}$ that is closest to $\vec x$, so in particular $\mathcal{P}(\vec x) \in \operatorname{span}\{\vec u\}$ for all $\vec x \in \mathbb{R}^2$. Therefore $\operatorname{range}(\mathcal{P}) \subseteq \operatorname{span}\{\vec u\}$.

On the other hand, $\mathcal{P}(\alpha\vec u) = \alpha\mathcal{P}(\vec u) = \alpha\vec u$ for any scalar α, and so $\operatorname{range}(\mathcal{P}) = \operatorname{span}\{\vec u\}$.

44.2 What is the null space of $\mathcal{P}$?

$$\operatorname{null}(\mathcal{P}) = \operatorname{span}\left\{\begin{bmatrix}3\\-2\end{bmatrix}\right\}.$$

A vector $\vec x$ projects to $\vec 0$ if and only if $\vec x$ is on the line perpendicular to $\operatorname{span}\{\vec u\}$ passing through the origin.

---

45 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

45.1 Show that the null space of T is a subspace.

(i) Let $\vec u, \vec v \in \operatorname{null}(T)$. Applying the linearity of T we see $T(\vec u + \vec v) = T(\vec u) + T(\vec v) = \vec 0 + \vec 0 = \vec 0$, and so $\vec u + \vec v \in \operatorname{null}(T)$.

(ii) Let $\vec u \in \operatorname{null}(T)$ and let α be any scalar. Again using the linearity of T we see $T(\alpha\vec u) = \alpha T(\vec u) = \alpha\vec 0 = \vec 0$, and so $\alpha\vec u \in \operatorname{null}(T)$.

45.2 Show that the range of T is a subspace.

(i) Let $\vec y, \vec z \in \operatorname{range}(T)$. Then there exist $\vec u, \vec v \in \mathbb{R}^n$ such that $T(\vec u) = \vec y$ and $T(\vec v) = \vec z$. Then $\vec y + \vec z = T(\vec u) + T(\vec v) = T(\vec u + \vec v)$, since T is linear, and so $\vec y + \vec z \in \operatorname{range}(T)$.

(ii) Let $\vec y \in \operatorname{range}(T)$ and let α be any scalar. Then there exists $\vec u \in \mathbb{R}^n$ such that $T(\vec u) = \vec y$, and $\alpha\vec y = \alpha T(\vec u) = T(\alpha\vec u)$, since T is linear, and so $\alpha\vec y \in \operatorname{range}(T)$.

Induced Transformation

DEFINITION

Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $T_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[T_M\vec v]_{\mathcal{E}'} = M[\vec v]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

---

46 Let M be a 2×2 matrix and let $\vec v \in \mathbb{R}^2$. Further, let T_M be the transformation induced by M.

46.1 What is the difference between "$M\vec v$" and "$M[\vec v]_{\mathcal{E}}$"?

"$M\vec v$" is ambiguous notation, as it is only defined if $\vec v$ is a specific column vector. There are infinitely many different bases of $\mathbb{R}^2$, and so a given vector $\vec v$ has infinitely many different representations as a column vector, each in a different basis.

"$M[\vec v]_{\mathcal{E}}$" is unambiguous, as $[\vec v]_{\mathcal{E}}$ is an explicit representation of $\vec v$ in a particular basis.

46.2 What is $[T_M\vec e_1]_{\mathcal{E}}$?

It is the first column of M.

By definition, $[T_M\vec e_1]_{\mathcal{E}} = M[\vec e_1]_{\mathcal{E}} = M\begin{bmatrix}1\\0\end{bmatrix}$, which equals the first column of M.

46.3 Can you relate the columns of M to the range of T_M?

The range of T_M equals the span of the columns of M.

By the previous part, the first column of M is in the range of T_M. By a similar argument, the second column of M is also in the range of T_M, since it equals $[T_M\vec{e}_2]_{\mathcal{E}}$. Therefore the span of the columns of M is a subset of the range of T_M.

On the other hand, if $\vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ are the columns of M and $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$ is an element of $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$, then

$$[\vec{x}]_{\mathcal{E}} = \alpha_1 \begin{bmatrix} a \\ c \end{bmatrix} + \alpha_2 \begin{bmatrix} b \\ d \end{bmatrix} = \alpha_1[T_M\vec{e}_1]_{\mathcal{E}} + \alpha_2[T_M\vec{e}_2]_{\mathcal{E}} = [T_M(\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2)]_{\mathcal{E}}.$$

Therefore $\vec{x}$ is in the range of T_M.

Fundamental Subspaces

DEFINITION

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\operatorname{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\operatorname{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\operatorname{null}(M)$, is the set of solutions to $M\vec{x} = \vec{0}$.

---

47 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

47.1 Describe the row space of A.

$\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, which is the xy-plane in $\mathbb{R}^3$.

47.2 Describe the column space of A.

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2.$$

47.3 Is the row space of A the same as the column space of A?

No.

Although they are both two dimensional spaces, $\operatorname{row}(A)$ is a subspace of $\mathbb{R}^3$ and all vectors in it have three coordinates (with the third always being zero), while $\operatorname{col}(A)$ is a subspace of $\mathbb{R}^2$ and all vectors in it have two coordinates. Therefore, these two spaces are different.

47.4 Describe the set of all vectors perpendicular to the rows of A.

The z-axis in $\mathbb{R}^3$.

A vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is perpendicular to the rows of A if and only if its dot product with both rows is zero. That is

$$\vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = y = 0.$$

$\vec{x}$ satisfies these equations if and only if $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$ for some real number t, or in other words if $\vec{x}$ is on the z-axis.

47.5 Describe the null space of A.

The z-axis in $\mathbb{R}^3$.

A vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in $\operatorname{null}(A)$ if and only if

$$A\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}.$$

These are the same conditions as in the previous part, so the set of vectors satisfying this is the z-axis.

47.6 Describe the range and null space of T_A, the transformation induced by A.

range$(T_A)=\mathrm{col}(A)=\mathbb{R}^2$ and null$(T_A)=\mathrm{null}(A)$, which is the z-axis in $\mathbb{R}^3$.

By Problem 46.3, the range of an induced transformation equals the span of the columns of the matrix. In other words, range$(T_A)=\mathrm{col}(A)$.

Next, by definition $\vec{v}\in\mathrm{null}(T_A)$ when $[T_A\vec{v}]_{\mathcal{E}}=A[\vec{v}]_{\mathcal{E}}=\vec{0}$. In other words, $\vec{v}\in\mathrm{null}(T_A)$ if and only if $[\vec{v}]_{\mathcal{E}}\in\mathrm{null}(A)$. We know from the previous part that null(A) is the z-axis in $\mathbb{R}^3$.

48

$$B=\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\qquad C=\mathrm{rref}(B)=\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$$

48.1 How does the row space of B relate to the row space of C?

They are equal.

Row operations replace rows with linear combinations of rows. Therefore, since C is the matrix B after the application of some row operations, row$(C)\subseteq$ row(B).

Since row operations are all reversible, we also know that B can be obtained from C by applying row operations, so row$(B)\subseteq$ row(C).

Therefore, row$(B)=$ row(C).

48.2 How does the null space of B relate to the null space of C?

They are equal.

A vector is in null(B) or null(C) if and only if it is orthogonal to all vectors in row(B) or all vectors in row(C), respectively. But row$(B)=$ row(C) by the previous part, so their null spaces must also be equal.

48.3 Compute the null space of B.

$$\mathrm{null}(C)=\mathrm{span}\left\{\begin{bmatrix}2\\-1\\2\end{bmatrix}\right\}.$$

We compute null(C), since it equals null(B) by the previous part.

$\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$ is in null(C) if and only if $C\vec{x}=\begin{bmatrix}x-z\\y+2z\end{bmatrix}=\vec{0}$. The complete solution to this matrix equation is

$$\mathrm{null}(C)=\left\{\begin{bmatrix}t\\-\frac{t}{2}\\t\end{bmatrix}\in\mathbb{R}^3\ :\ t\in\mathbb{R}\right\}=\mathrm{span}\left\{\begin{bmatrix}2\\-1\\2\end{bmatrix}\right\}.$$

49

$$P=\begin{bmatrix}0&0\\1&2\end{bmatrix}\qquad Q=\mathrm{rref}(P)=\begin{bmatrix}1&2\\0&0\end{bmatrix}$$

49.1 How does the column space of P relate to the column space of Q?

They are not equal, but have the same dimension.

49.2 Describe the column space of P and the column space of Q.

$\mathrm{col}(P)=\mathrm{span}\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\right\}=\mathrm{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$, which is the y-axis in $\mathbb{R}^2$.

$\mathrm{col}(Q)=\mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix}\right\}=\mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$, which is the x-axis in $\mathbb{R}^2$.

Rank

DEF

For a linear transformation $T:V\to W$, the ***rank*** of T, denoted rank(T), is the dimension of the range of T.

For an $n\times m$ matrix M, the ***rank*** of M, denoted rank(M), is the number of pivots in rref(M).

50 Let $\mathcal{P}$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by $90°$.

50.1 Describe range($\mathcal{P}$) and range($\mathcal{R}$).

range($\mathcal{P}$) = span$\{\vec{u}\}$, and range($\mathcal{R}$) = $\mathbb{R}^2$.

For $\mathcal{P}$, by the definition of projection $\mathcal{P}(\vec{x})$ is the vector in span$\{\vec{u}\}$ that is closest to $\vec{x}$, so in particular $\mathcal{P}(\vec{x}) \in$ span$\{\vec{u}\}$ for all $\vec{x} \in \mathbb{R}^2$. Therefore range($\mathcal{P}$) $\subseteq$ span$\{\vec{u}\}$.

On the other hand, $\mathcal{P}(\alpha\vec{u}) = \alpha\mathcal{P}(\vec{u}) = \alpha\vec{u}$ for any scalar α, and so range($\mathcal{P}$) = span$\{\vec{u}\}$.

For $\mathcal{Q}$, we have that any vector $\vec{x} \in \mathbb{R}^2$, $\vec{x} = \mathcal{Q}(\vec{y})$, where $\vec{y}$ is the rotation of $\vec{x}$ clockwise by $90°$. Therefore range($\mathcal{Q}$) = $\mathbb{R}^2$.

50.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

rank($\mathcal{P}$) = 1 and rank($\mathcal{R}$) = 2.

By the previous part, we know range($\mathcal{P}$) is 1-dimensional and range($\mathcal{Q}$) is 2-dimensional.

50.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

rank(P) = 1 and rank(R) = 2.

By Problem 43, $P = \frac{1}{13}\begin{bmatrix}4 & 6\\6 & 9\end{bmatrix}$ and $R = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ are the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. Then we compute:

$$\text{rref}(P) = \begin{bmatrix}1 & \frac{3}{2}\\0 & 0\end{bmatrix} \quad \text{and} \quad \text{rref}(R) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

These matrices have 1 and 2 pivots, respectively.

50.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

They are equal.

By Problem 46.3, the range of a transformation is equal to the column space of its corresponding matrix, and therefore the dimensions of these two spaces are equal. In other words, the rank of a transformation is equal to the dimension of the the column space of its corresponding matrix.

We already know that the dimension of the column space of a matrix is equal to the number of pivots in its reduced row echelon form, and that is by definition the rank of the matrix.

51

51.1 Determine the rank of (a) $\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ (c) $\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}3\\3\\2\end{bmatrix}$ (e) $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

For each part, we compute the reduced row echelon form of the matrix and count the number of pivots.

(a) $\text{rank}\left(\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}\right) = 1$, since $\text{rref}\left(\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ has one pivot.

(b) $\text{rank}\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right) = 2$, since $\text{rref}\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ has two pivots

(c) $\text{rank}\left(\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = 2$. This matrix is already in reduced row echelon form, and has two pivots.

(d) $\text{rank}\left(\begin{bmatrix}3\\3\\2\end{bmatrix}\right) = 1$, since $\text{rref}\left(\begin{bmatrix}3\\3\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$ has one pivot.

(e) $\text{rank}\left(\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = 3$, since $\text{rref}\left(\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ has three pivots.

52 Consider the homogeneous system

$$\begin{array}{rrrl} x & +2y & +z & =0 \\ x & +2y & +3z & =0 \\ -x & -2y & +z & =0 \end{array} \tag{1}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$.

52.1 What is rank(A)?

rank(A) = 2, since rref(A) = $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ as two pivots.

52.2 Give the general solution to system (1).

$\vec{x}$ is a solution to the system if $\vec{x} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ for some real number t.

If $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution to the system, then we must have $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, from which it follows that $z = 0$ and $x = -2y$. In other words, any scalar multiple of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is a solution.

52.3 Are the column vectors of A linearly independent?

No. The second column is two times the first column.

52.4 Give a non-homogeneous system with the same coefficients as (1) that has

(a) infinitely many solutions

(b) no solutions.

(a)

$$\begin{array}{rrrl} x & +2y & +z & =1 \\ x & +2y & +3z & =1 \\ -x & -2y & +z & =-1 \end{array}$$

(b)

$$\begin{array}{rrrl} x & +2y & +z & =0 \\ x & +2y & +3z & =0 \\ -x & -2y & +z & =1 \end{array}$$

53 53.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

No. A 3×4 matrix has four columns, each of which are vectors in $\mathbb{R}^3$. It is not possible for four different vectors in $\mathbb{R}^3$ to be linearly indepedent.

53.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Yes. Since rank(B) = 3, there are three pivots in rref(B). Pivot positions in rref(B) indicate a maximal linearly independent subset of the columns of B. Since there are three columns in B and three pivots, the three columns of B must be linearly independent.

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\text{rank}(A) + \text{nullity}(A) = \#\text{ of columns in } A.$$

54 54.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

Yes. If $T : V \to W$ is a linear transformation, then $\text{rank}(T) + \dim(\text{null}(T)) = \dim(V)$.

If A is the matrix corresponding to T, then $\text{rank}(T) = \text{rank}(A)$ by Problem 50.4.

$\text{null}(T) = \text{null}(A)$ by Problem 47.6, since $T = T_A$, and so $\dim(\text{null}(T)) = \text{nullity}(A)$.

Finally, the number of columns of A is equal to the dimension of the domain of T.

55 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

55.1 What is the rank and nullity of A^T?

$\text{rank}(A^T) = 2$ and $\text{nullity}(A^T) = 1$.

A^T is the matrix with rows $\vec{u}, \vec{v}$, and $\vec{w}$. Since $\vec{w} = 2\vec{u} - \vec{v}$, the third row of A^T can be reduced to a row of zeros by the row operation $R_3 \mapsto R_3 - 2R_1 + R_2$. Neither of the first two rows can be reduced to rows of zeros since they are linearly independent. Therefore $\text{rref}(A^T)$ has two pivots, meaning $\text{rank}(A^T) = 2$.

The rank-nullity theorem then says that $2 + \text{nullity}(A^T) = 3$, and so $\text{nullity}(A^T) = 1$

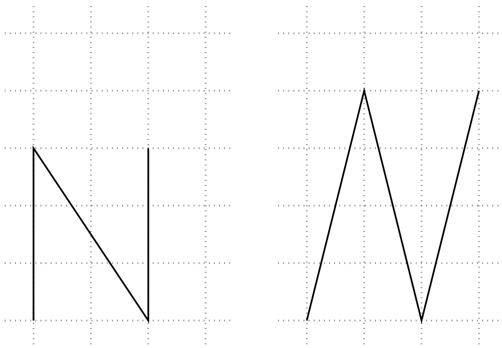
55.2 What is the rank and nullity of A?

$\text{rank}(A) = 2$ and $\text{nullity}(A) = 1$.

We know that $\text{rank}(A)$ equals the number of pivots in $\text{rref}(A)$, which in turn equals the dimension of $\text{col}(A)$. Since A has two linearly independent columns, $\dim(\text{col}(A)) = 2$.

Again, the rank-nullity theorem then says that $2 + \text{nullity}(A) = 3$, and so $\text{nullity}(A) = 1$

Task 2.4: Getting back N



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

Consider the new task: find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.
2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

56 56.1 Apply the row operation $R_3 \mapsto R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3 \mapsto R_3+2R_1} \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_1.$$

56.2 Apply the row operation $R_3 \mapsto R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3 \mapsto R_3-2R_1} \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = E_2.$$

DEF An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

56.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

$$E_1A = \begin{bmatrix}1&2&3\\4&5&6\\9&12&15\end{bmatrix} \text{ and } E_2A = \begin{bmatrix}1&2&3\\4&5&6\\5&4&3\end{bmatrix}.$$

E_1A is the result applying the row operation $R_3 \mapsto R_3 + 2R_1$ to A, and similarly E_2A is the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to A.

56.4 Without computing, what should the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to E_1 be? Compute and verify.

It should be the identity matrix, since the row operation $R_3 \mapsto R_3 - 2R_1$ should undo the operation $R_3 \mapsto R_3 + 2R_1$.

56.5 Without computing, what should E_1E_2 be? What about E_2E_1? Now compute and verify.

They should both be the identity matrix.

The solution to part 3 above lead us to believe that applying E_1 to a matrix has the effect of applying the row operation $R_3 \mapsto R_3 + 2R_1$ to it. Applying that row operation to E_2 would produce the identity matrix, so we expect that E_1E_2 should equal the identity matrix.

Similar reasoning leads us to believe that E_2E_1 should also equal the identity matrix.

Indeed, we can compute

$$E_1E_2 = \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_2E_1.$$

DEF The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated by A^{-1}.

57 Consider the matrices

$$A = \begin{bmatrix}1&2&0\\0&1&0\\-3&-6&1\end{bmatrix} \qquad B = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix} \qquad C = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

$$D = \begin{bmatrix}1&-2&0\\0&1&0\\3&0&1\end{bmatrix} \qquad E = \begin{bmatrix}1&0&0\\0&2&0\\0&1&1\end{bmatrix} \qquad F = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

57.1 Which pairs of matrices above are inverses of each other?

A and D are inverses of each other, and F is its own inverse.

58

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

58.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_2 \mapsto \frac{1}{2}R_2} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - 4R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The two elementary matrices are $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$.

58.2 What is E_2E_1B?

$$E_2E_1B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

58.3 Find B^{-1}.

$$B^{-1} = E_2E_1 = \begin{bmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}.$$

By the previous part we already know that $(E_2E_1)B = I$. We can also check that $B(E_2E_1) = I$, meaning E_2E_1 is the inverse of B.

58.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

Suppose A is a matrix that can be row reduced to the identity. Let $E_1, E_2, \ldots, E_n$ be the elementary matrices corresponding to the sequence of row operations that reduces A to I. Then as we have seen, we have $E_nE_{n-1}\cdots E_2E_1A = I$.

Thus $E_nE_{n-1}\cdots E_2E_1$ is the inverse of A.

59

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

59.1 What is $A^{-1}A$?

$A^{-1}A = I$. This is true by the definition of an inverse, but we can also verify it by hand.

59.2 What is $\mathrm{rref}(A)$?

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

59.3 What is $\mathrm{rref}(C)$? (Hint, there is no need to actually do row reduction!)

$$\mathrm{rref}(C) = \left[I \middle| A^{-1}\vec{b}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

We know that the reduced row echelon form of C must be of the form $[I|\vec{c}]$ for some $\vec{c}$, and we know that multiplying on the left by A^{-1} is equivalent to applying the sequence of row operations that reduces A to $\mathrm{rref}(A) = I$. So the same sequence of row operations applied to $\vec{b}$, the last column of C, will produce the vector $\vec{c} = A^{-1}\vec{b} = \begin{bmatrix} -9 \\ 6 \\ 2 \end{bmatrix}$.

59.4 Solve the system $A\vec{x} = \vec{b}$.

The system has one solution: $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -9 \\ 6 \\ 2 \end{bmatrix}$.

We can read this solution from the reduced row echelon form of the augmented matrix C representing this system. We can also multiply both sides of the equation on the left by A^{-1}:

$$A\vec{x} = \vec{x} \implies A^{-1}A\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}.$$

60 60.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

No.

By the definition of an inverse we need $(XY)^{-1}(XY) = I$, so that multiplying by $(XY)^{-1}$ undoes multiplication by XY. To do this, we must first undo multiplication by X, then undo multiplication by Y. That is, we must first multiply by X^{-1} then multiply by Y^{-1}.

In other words, we expect that $(XY)^{-1} = Y^{-1}X^{-1}$. We can then verify this by computing

$$(XY)(Y^{-1}X^{-1}) = XYY^{-1}X^{-1} = XIX^{-1} = XX^{-1} = I$$

and

$$(Y^{-1}X^{-1})(XY) = Y^{-1}X^{-1}XY = Y^{-1}IY = Y^{-1}Y = I.$$

60.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

No.

Suppose M^{-1} exists. Then $M^{-1}M = MM^{-1} = I$. Let S be the linear transformation induced by M^{-1}. Since M is the matrix for T we must have $ST = TS = \mathrm{id}$. But then S would be the inverse of T, which is impossible.

More Change of Basis

61 Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ and let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

61.1 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

$[\vec{e}_1]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$ and $[\vec{e}_2]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$.

This is because $\vec{e}_1 = \frac{1}{2}(\vec{b}_1 + \vec{b}_2)$ and $\vec{e}_2 = \frac{1}{2}(\vec{b}_1 - \vec{b}_2)$

61.2 Compute $X[\vec{e}_1]_{\mathcal{B}}$ and $X[\vec{e}_2]_{\mathcal{B}}$. What do you notice?

$X[\vec{e}_1]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\vec{b}_1 + \frac{1}{2}\vec{b}_2 = \begin{bmatrix}1\\0\end{bmatrix}$ and

$X[\vec{e}_2]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\vec{b}_1 - \frac{1}{2}\vec{b}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

We notice that multiplying by X turns the representations of these two vectors in the basis $\mathcal{B}$ into representations in the standard basis.

61.3 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

$X^{-1} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$.

X^{-1} should undo what X does. In the previous part we saw that X takes vectors represented in $\mathcal{B}$ and represents them in the standard basis. So X^{-1} should do the reverse, and take vectors represented in the standard basis and represent them in the basis $\mathcal{B}$.

62 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}.$$

62.1 Should X^{-1} exist? Explain.

Yes. X converts vectors from the standard basis into the $\mathcal{B}$ basis, and this process can be undone. X^{-1} is the matrix that does this.

62.2 Consider the equation

$$X^{-1}[\vec{v}]_? = [\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

$X^{-1}[\vec{v}]_{\mathcal{S}} = [\vec{v}]_{\mathcal{B}}$.

As we said in the previous part X^{-1} should undo what X does, meaning it should convert vectors from the standard basis into the $\mathcal{B}$ basis.

62.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

$[\vec{b}_1]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$, and $[\vec{b}_2]_{\mathcal{B}} = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, where each of these vectors have n coordinates.

In general, $[\vec{b}_i]_{\mathcal{B}}$ should be the column vector with zeroes in all coordinates except for a 1 in the i^{th} coordinate.

---

63 Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

63.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$. $[\vec{c}_1]_{\mathcal{C}} = \begin{bmatrix}1\\0\end{bmatrix}$ and $[\vec{c}_2]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

63.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}5\\3\end{bmatrix}$. $T\begin{bmatrix}2\\1\end{bmatrix} = T\vec{c}_1 = 2\vec{c}_1 = \begin{bmatrix}4\\2\end{bmatrix}$ and $T\begin{bmatrix}5\\3\end{bmatrix} = T\vec{c}_2 = \vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}$.

63.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$. $[T\vec{c}_1]_{\mathcal{C}} = [2\vec{c}_1]_{\mathcal{C}} = \begin{bmatrix}2\\0\end{bmatrix}$ and $[T\vec{c}_2]_{\mathcal{C}} = [\vec{c}_2]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}$.

63.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

$T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}2\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$.

If $\vec{v}$ is a vector such that $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix}\alpha\\\beta\end{bmatrix}$, then $\vec{v} = \alpha\vec{c}_1 + \beta\vec{c}_2$. Since T is linear, we can then compute

$$T\vec{v} = T(\alpha\vec{c}_1 + \beta\vec{c}_2) = \alpha T(\vec{c}_1) + \beta T(\vec{c}_2) = 2\alpha\vec{c}_1 + \beta\vec{c}_2 = \begin{bmatrix}2\alpha\\\beta\end{bmatrix}_{\mathcal{C}}.$$

63.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

$[T]_{\mathcal{C}} = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$.

From the results of the previous parts, we know that we must have $[T]_{\mathcal{C}}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ and $[T]_{\mathcal{C}}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$, so these must be the first and second columns of $[T]_{\mathcal{C}}$, respectively.

63.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

$[T]_{\mathcal{E}} = \begin{bmatrix}7 & -10\\3 & -4\end{bmatrix}$

There are two methods to determine this.

Method 1: Since $\vec{e}_1 = 3\vec{c}_1 - \vec{c}_2$ and $\vec{e}_2 = -5\vec{c}_1 + 2\vec{c}_2$, we compute

$$[T\vec{e}_1]_{\mathcal{E}} = [T(3\vec{c}_1 - \vec{c}_2)]_{\mathcal{E}} = 3[T(\vec{c}_1)]_{\mathcal{E}} - [T(\vec{c}_2)]_{\mathcal{E}} = 3\begin{bmatrix}4\\2\end{bmatrix} - \begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}7\\3\end{bmatrix}$$

and

$$[T\vec{e}_2]_{\mathcal{E}} = [T(-5\vec{c}_1 + 2\vec{c}_2)]_{\mathcal{E}} = -5[T(\vec{c}_1)]_{\mathcal{E}} + 2[T(\vec{c}_2)]_{\mathcal{E}} = -5\begin{bmatrix}4\\2\end{bmatrix} + 2\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}-10\\-4\end{bmatrix}.$$

These two vectors are the respective columns of $[T]_{\mathcal{E}}$, as usual.

Method 2: Since A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis, we know $[T]_{\mathcal{E}} = A[T]_{\mathcal{C}}A^{-1}$. Using $[T]_{\mathcal{C}}$ from the previous part, we compute

$$[T]_{\mathcal{E}} = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix} = \begin{bmatrix}7 & -10\\3 & -4\end{bmatrix}.$$

DEFINITION

Similar Matrices

A matrices A and B are called ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

$$A = XBX^{-1}.$$

Determinants

DEFINITION

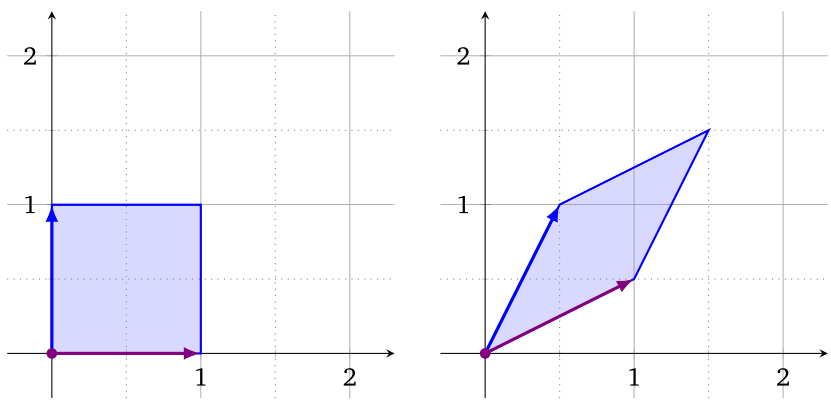
Unit n-cube

The unit n-cube is the n-dimensional cube with side length 1 and lower-left corner located at the origin. That is

$$C_n = \left\{\vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1]\right\} = [0,1]^n.$$

The volume of the unit n-cube is always 1.

64 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



64.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}$, and $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}$?

We can see first two directly in the picture.

Using the linearity of T, we can compute

$$T\begin{bmatrix}1\\1\end{bmatrix} = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = T\begin{bmatrix}1\\0\end{bmatrix} + T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\\frac{1}{2}\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}.$$

64.2 Write down a matrix for T.

The matrix for T in the standard basis is $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$.

64.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may need to use trigonometry.

The volume is $\frac{3}{4}$.

Determinant

DEFINITION

The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors (or the row vectors).

---

65 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

65.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

XXXX

65.2 Compute the area of $A(C_2)$. The area of this parallelogram is 2.

65.3 Compute $\det(A)$.

$\det(A) = 2$.

The paralellogram with sides $\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ is positively oriented, so $\det(A) = +2$.

---

66 Suppose R is a rotation counter-clockwise by 30°.

66.1 Draw C_2 and $R(C_2)$.

XXXX

66.2 Compute the area of $R(C_2)$.

The area is 1.

R rotates the entire unit square, which does not change its area.

66.3 Compute $\det(R)$.

Since R preserves orientation, $\det(R)$ must be positive. Since R does not change the area of the unit square, $\det(R) = +1$.

---

67 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

67.1 What is $\det(F)$?

$\det(F) = -1$.

F does not change the area of the unit square, but reverses its orientation, so $\det(F) = -1$.

---

68 Let $D = \{\vec{x} : \|\vec{x}\| \leq 1\}$ be the unit disk. You know the following about the linear transformations M, T, and S: M is defined by $\vec{x} \mapsto 2\vec{x}$; T has determinant 2; and S has determinant 3.

68.1 Find the oriented volumes of $M(C_2)$, $T(C_2)$, and $S(C_2)$.

The volumes are 4, 2, and 3, respectively.

We can compute the the volume of $M(C_2)$ directly, since M transforms C_2 into a square with side length 2. For the other two, we are given their determinants and we know that the volume of the image of the unit square equals the determinant of each transformation.

68.2 How does the volume of $T(C_2 + \{\vec{e}_1\})$ compare to the volume of $T(C_2)$?

They are equal. The transformation T scales the volumes of all regions by 2.

68.3 What is the oriented volume of $T \circ M(C_2)$? What is $\det(T \circ M)$?

They are both equal to 8.

$T \circ M(C_2) = T(M(C_2))$. We already know $M(C_2)$ has a volume of 4, and so $T(M(C_2))$ has a volume of 8, since T scales the volumes of all regions by 2. The volume of $T \circ M(C_2)$ is the determinant of $T \circ M$ by definition.

68.4 What is the oriented volume of $S(D)$?

The volume is 3π.

69

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

69.1 What is $\det(E_f)$?

$\det(E_f) = -1$.

$\det(I_{3\times 3}) = 1$, and swapping one pair of rows of a matrix changes the sign of its determinant.

69.2 What is $\det(E_m)$?

$\det(E_m) = 6$.

Multiplying one row of a matrix by a constant multiplies its determinant by the same constant.

69.3 What is $\det(E_a)$?

$\det(E_a) = 1$.

Adding a multiple of one row of a matrix to another row has no effect on its determinant.

69.4 What is $\det(E_f E_m)$? $\det(E_f E_m) = \det(E_f)\det(E_m) = (-1)(6) = -6$.

69.5 What is $\det(4I_{3\times 3})$? $\det(4I_{3\times 3}) = 4^3 = 64$.

69.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

$\det(W) = \det(E_f)\det(E_a)\det(E_f)\det(E_m)\det(E_m) = (-1)(1)(-1)(6)(6) = 36$.

70

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

70.1 What is $\det(U)$?

$\det(U) = -12$.

70.2 V is a square matrix and $\operatorname{rref}(V)$ has a row of zeros. What is $\det(V)$?

$\det(V) = 0$.

70.3 P is projection onto the vector $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. What is $\det(P)$?

$\det(P) = 0$.

The image of the unit square under P is a line segment, which has zero volume.

---

71 Suppose you know $\det(X) = 4$.

71.1 What is $\det(X^{-1})$?

$\det(X^{-1}) = \frac{1}{4}$.

We know that $XX^{-1} = I$. Therefore we must have that $\det(XX^{-1}) = \det(X)\det(X^{-1}) = \det(I) = 1$, and so $\det(X^{-1}) = \frac{1}{4}$.

71.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

$\det(Y^{-1}) = \frac{1}{\det(Y)}$.

Using the same reasoning as the previous part, we know that $YY^{-1} = I$. Therefore we must have $\det(Y)\det(Y^{-1}) = \det(YY^{-1}) = \det(I) = 1$, and so $\det(Y^{-1}) = \frac{1}{\det(Y)}$.

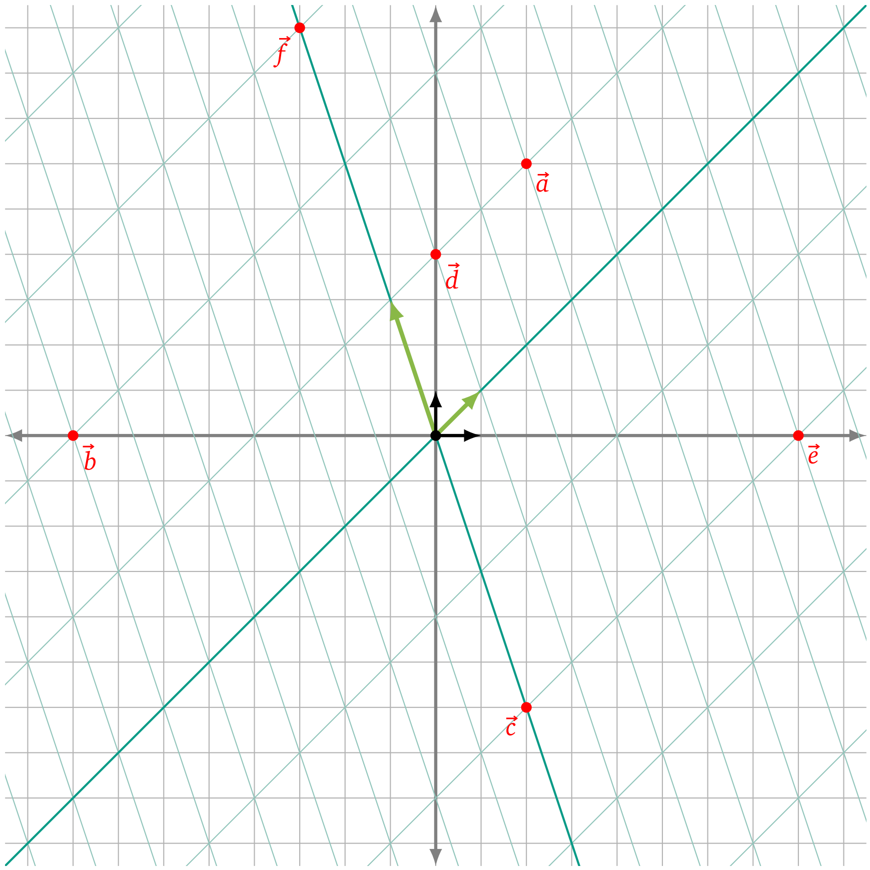
71.3 Suppose Y is not invertible. What is $\det(Y)$?

$\det(Y) = 0$.

This makes sense because for a square matrix Y, $\det(Y^{-1}) = \frac{1}{\det(Y)}$ by the previous part. This formula always works, except when $\det(Y) = 0$.

Task 3.1: The Green and the Black

Consider the following two bases for $\mathbb{R}^2$: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$ and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.



1. Write each point above in both the green and the black bases.
2. Find a change-of-basis matrix X that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{u}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

4. When working with the transformation T, which basis do you prefer vectors be represented in?

Eigenvectors

Eigenvector

DEFINITION

Let X be a linear transformation. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

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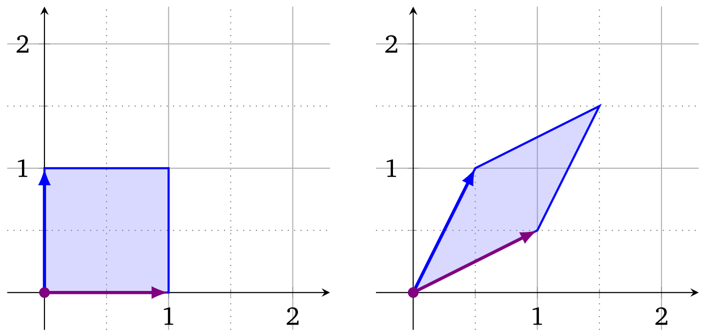
72 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



72.1 Give an eigenvector for T. What is the eigenvalue?

$\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for T, with corresponding eigenvalue $\frac{3}{2}$.

We can see from the image that $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}$.

72.2 Can you give another?

Any scalar multiple of $\begin{bmatrix}1\\1\end{bmatrix}$ is also an eigenvector for T.

For any scalar α, we have $T\left(\alpha\begin{bmatrix}1\\1\end{bmatrix}\right) = \alpha T\begin{bmatrix}1\\1\end{bmatrix} = \alpha\frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}$, meaning $\alpha\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for T with eigenvalue $\frac{3}{2}\alpha$.

More interestingly, since $T\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\end{bmatrix}$ is an eigenvector with corresponding eigenvalue $\frac{1}{2}$.

---

73 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

73.1 Give an eigenvector and a corresponding eigenvalue for A.

$\begin{bmatrix}3\\3\\1\end{bmatrix}$ is an eigenvector for A, with corresdponding eigenvalue $\frac{2}{3}$.

73.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

$B\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

We compute

$$B\begin{bmatrix}3\\3\\1\end{bmatrix} = (A - \frac{2}{3}I)\begin{bmatrix}3\\3\\1\end{bmatrix} = A\begin{bmatrix}3\\3\\1\end{bmatrix} - \frac{2}{3}I\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} - \begin{bmatrix}2\\2\\2/3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

73.3 What is the dimension of null(B)?

The most we can say is that nullity$(B) \geq 1$.

We know $\begin{bmatrix}3\\3\\1\end{bmatrix} \in \text{null}(B)$ by the previous part, and so the dimension of null(B) is at least 1. It could be larger, but we do not have enough information to say for sure.

73.4 What is det(B)? det$(B) = 0$.

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74 Let $C = \begin{bmatrix}-1 & 2\\ 1 & 0\end{bmatrix}$ and $E_\lambda = C - \lambda I$.

74.1 For what values of λ does E_λ have a non-trivial null space?

74.2 What are the eigenvalues of C?

74.3 Find the eigenvectors of C.

Characteristic Polynomial

DEF

For a matrix A, the ***characteristic polynomial*** of A is

$$\text{char}(A) = \det(A - \lambda I).$$

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75 Let $D = \begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$.

75.1 Compute char(D).

75.2 Find the eigenvalues of D.

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76 Suppose char$(E) = \lambda(\lambda-2)(\lambda+3)$ for some unknown 3×3 matrix E.

76.1 What are the eigenvalues of E?

76.2 Is E invertible?

76.3 What is null(E), null$(E-3I)$, null$(E+3I)$?

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77 Consider

$$A = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{bmatrix} \qquad \vec v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \vec v_2 = \begin{bmatrix}1\\1\\-2\end{bmatrix} \qquad \vec v_3 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

and notice that $\vec v_1, \vec v_2, \vec v_3$ are eigenvectors for A.

77.1 Find the eigenvalues of A.

The eigenvalues of A are 2, -1, and 1.

We compute that $A\vec v_1 = 2\vec v_1$, $A\vec v_2 = -\vec v_2$, and $A\vec v_3 = \vec v_3$, so 2, -1, and 1 are eigenvalues of A. By the last part of the previous problem, there are no other eigenvalues.

77.2 Find the characteristic polynomial of A.

char$(A) = -(\lambda-2)(\lambda+1)(\lambda-1)$.

We compute

$$\text{char}(A) = \det(A-\lambda I) = \det\left(\begin{bmatrix}1-\lambda & 0 & 1\\ 0 & 1-\lambda & 1\\ 1 & 1 & -\lambda\end{bmatrix}\right) = -\lambda^3+2\lambda^2+\lambda-2 = -(\lambda-2)(\lambda+1)(\lambda-1).$$

77.3 Compute $A\vec w$ where $w = 2\vec v_1 - \vec v_2$.

$A\vec w = 4\vec v_1 + \vec v_2$.

Using the computations we did in the first part above, we find

$$A\vec w = A(2\vec v_1 - \vec v_2) = 2A\vec v_1 - A\vec v_2 = 2(2\vec v_1) - (-\vec v_2) = 4\vec v_1 + \vec v_2.$$

77.4 Compute $A\vec u$ where $\vec u = a\vec v_1 + b\vec v_2 + c\vec v_3$ for unknown scalar coefficients a, b, c.

$A\vec u = 2a\vec v_1 - b\vec v_2 + c\vec v_3$.

Using the same reasoning as the previous part, we compute

$$A\vec u = aA\vec v_1 + bA\vec v_2 + cA\vec v_3 = 2a\vec v_1 - b\vec v_2 + c\vec v_3.$$

Notice that $\mathcal{V} = \{\vec v_1, \vec v_2, \vec v_3\}$ is a basis for $\mathbb{R}^3$.

77.5 If $[\vec x]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is $\vec x$ written in the $\mathcal{V}$ basis, compute $A\vec x$ in the $\mathcal{V}$ basis.

$[A\vec x]_{\mathcal{V}} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

If $[\vec x]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, then $\vec x = \vec v_1 + 3\vec v_2 + 4\vec v_3$. Using the previous part, we then have that $A\vec x = 2\vec v_1 - 3\vec v_2 + 4\vec v_3$, so $[A\vec x]_{\mathcal{V}} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

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78 The transformation P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation (where $\mathcal{V}$ is from above).

78.1 Describe in words what P does.

78.2 Describe how you can use P and P^{-1} to easily compute $A\vec y$ for any $\vec y \in \mathbb{R}^3$.

78.3 Can you find a matrix D so that

$$PDP^{-1} = A?$$

78.4 $[\vec x]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $A^{100}\vec x$.

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79 For an $n \times n$ matrix T, suppose its eigenvectors $\{\vec v_1, \dots \vec v_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues.

79.1 Is T diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

79.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

79.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Eigenspace

DEFINITION

Let A be a matrix with eigenvalues $\{\lambda_1, \dots, \lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

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80 Define $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

80.1 Is F diagonalizable? Why or why not?

80.2 What is the geometric and algebraic multiplicity of each eigenvalue of F?

80.3 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?