MATH 110, Fall 2013
Tutorial #11
November 27, 2013

Today's main problems

$\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ where

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

1. Apply the Gram-Schmidt process to $\mathcal{V}$ to find an orthonormal basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ for $\mathbb{R}^4$. Note that some of the vectors in V are already orthogonal to eachother. You may reorder the vectors before you apply Gram-Schmidt to minimize the amount of work you need to do.

Further Questions

Let $B = [\vec{b}_1|\vec{b}_2|\vec{b}_3]$ where $\vec{b}_i \in \mathcal{B}$ from problem 1. Define

$$C = \operatorname{span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \qquad \vec{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \vec{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. (a) Is the system $B\vec{x} = \vec{c}_1$ consistent?
   (b) Is the system $B\vec{x} = \vec{c}_2$ consistent?
   (c) Is the system $B\vec{x} = \text{proj}_C\vec{y}$ consistent for any choice of $\vec{y}$?

3. Even if a system $A\vec{x} = \vec{b}$ is inconsistent, we can attempt to find a best solution. That is, we can attempt to find a vector $\vec{x}$ so that $A\vec{x}$ is as close to $\vec{b}$ as it can be while still being in the column space. Find the best solution to the system $B\vec{x} = \vec{c}_1$ (also called the *least squares* solution) by projecting $\vec{c}_1$ onto the column space of B and then solving $B\vec{x} = \text{proj}_C\vec{c}_1$.

MATH 110, Fall 2013
Tutorial #11
November 27, 2013

Challenge questions

4. Let A and B be orthogonal matrices.

   (a) Prove $A(A^T + B^T)B = A + B$.

   (b) Use part (a) to prove that if $\det A + \det B = 0$ then $A + B$ is not invertible.

5. We can define a dot product on polynomials. If $p(x)$ and $q(x)$ are polynomials, then

$$p \cdot q = \int_{-1}^{1} p(x)q(x)\,\mathrm{d}x.$$

   Use the Gram-Schmidt process on the basis $\mathcal{P} = \{1, x, x^2, x^3, x^4\}$ to come up with an orthonormal basis for polynomials of degree at most 4. If you graph the polynomials that make up your basis, what do you notice? (These polynomials are called the Legendre polynomials.)

MATH 110, Fall 2013
Tutorial #11. Instructions for TAs

Objectives

Students need practice doing Gram-Schmidt and especially doing careful and methodical arithmetic.

Hidden objectives

Suggestions

Though they have seen Gram-Schmidt in class, most of them won't remember the procedure. It is worth reminding them of the idea behind Gram-Schmidt at the beginning: "take a set of vectors that is already orthogonal to one another (this could be a single vector) and add a new vector by subtracting off its projection onto each vector already in your set." If you can draw a 3-d picture to illustrate the idea, more power to you, but don't get too carried away. They seen this in class and your job isn't to give a lecture on the topic.

If students are having computational trouble with Gram-Schmidt, make them be methodical. Make them get out a clean sheet of paper and write all the intermediate calculations down. I know fractions are super hard, but sometimes they're just there. Also, have them resist the urge to normalize until they've orthogonalized everything.

Wrapup

The objective of this tutorial is question 1. If people got question 1, but they struggled with the arithmetic and bookkeeping, it is worth you doing this methodically as a wrapup. A mathematician's skill of being systematic, organized, and renaming things where appropriate isn't learned over night. If everyone had no organizational trouble with the computations, you can consider doing a different problem as a wrapup.

Solutions

1.