Inquiry Based Linear Algebra

About the Document

This document is a hybrid of many linear algebra resources, including those of the IOLA (Inquiry Oriented Linear Algebra) project, Jason Siefken's IBLLinearAlgebra project, and Asaki, Camfield, Moon, and Snipes' Radiograph and Tomography project.

This document is a mix of student projects, problem sets, and labs. A typical class day looks like:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

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Task 1.1: The Magic Carpet Ride

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Task 1.2: The Magic Carpet Ride, Hide and Seek

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1, 2, 3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

1

1.1 Which of the following statements are true?

(a) $3 \in \{1, 2, 3\}$.

(b) $1.5 \in \{1, 2, 3\}$.

(c) $4 \in \{1, 2, 3\}$.

(d) "b"$\in \{x : x \text{ is an English letter}\}$.

(e) "ò"$\in \{x : x \text{ is an English letter}\}$.

(f) $\{1, 2\} \subseteq \{1, 2, 3\}$.

(g) For some $a \in \{1, 2, 3\}$, $a \geq 3$.

(h) For any $a \in \{1, 2, 3\}$, $a \geq 3$.

(i) $1 \subseteq \{1, 2, 3\}$.

(j) $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.

(k) $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$.

2

Write the following in set-builder notation

2.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

2.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

DEFINITION

Unions & Intersections

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)}\quad X\cup Y=\{a : a\in X \text{ or } a\in Y\}.$$

$$\text{(intersection)}\quad X\cap Y=\{a : a\in X \text{ and } a\in Y\}.$$

3 Let $X=\{1,2,3\}$ and $Y=\{2,3,4,5\}$ and $Z=\{4,5,6\}$. Compute

3.1 $X\cup Y$

3.2 $X\cap Y$

3.3 $X\cup Y\cup Z$

3.4 $X\cap Y\cap Z$

4 Draw the following subsets of $\mathbb{R}^2$.

4.1 $V=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}=\begin{bmatrix}0\\t\end{bmatrix} \text{ for some } t\in\mathbb{R}\right\}.$

4.2 $H=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}=\begin{bmatrix}t\\0\end{bmatrix} \text{ for some } t\in\mathbb{R}\right\}.$

4.3 $J=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}=t\begin{bmatrix}1\\1\end{bmatrix} \text{ for some } t\in\mathbb{R}\right\}.$

4.4 $J=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}=t\begin{bmatrix}1\\1\end{bmatrix} \text{ for all } t\in\mathbb{R}\right\}.$

4.5 $V\cup H$.

4.6 $V\cap H$.

4.7 Does $V\cup H=\mathbb{R}^2$?

Vector Combinations

DEFINITION

Linear Combination

A ***linear combination*** of the vectors $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n$ is a vector

$$\vec{w}=\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\cdots+\alpha_n\vec{v}_n.$$

The scalars $\alpha_1,\alpha_2,\dots,\alpha_n$ are called the ***coefficients*** of the linear combination.

5 Let $\vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w}=2\vec{v}_1+\vec{v}_2$.

5.1 Write the coordinates of $\vec{w}$. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

5.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

6 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

6.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

6.3 Rephrase the sentence *"$\mathbb{R}^2$ is the set of all linear combinations of $\vec{h}$ and $\vec{m}$"* using formal mathematical language.

Non-negative & Convex Linear Combinations

DEFINITION

The linear combination $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

7 Let

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \vec{e} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

7.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$?

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$?

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$?

7.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that $\vec{a}$, $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not. Otherwise, explain why it's not possible.

7.3 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that $\vec{a}$, $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$. Otherwise, explain why it's not possible.

7.4 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that $\vec{a}$, $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not. Otherwise, explain why it's not possible.

7.5 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$. Otherwise, explain why it's not possible.

Lines and Planes

8 Let A be the set of points $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 1$.

8.1 Describe A using set-builder notation.

8.2 Draw A as a subset of $\mathbb{R}^2$.

8.3 Add the vectors $\vec{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{d} = \vec{b} - \vec{a}$ to your drawing.

8.4 For which $t \in \mathbb{R}$ is it true that $\vec{a} + t\vec{d} \in A$? Explain using your picture.

Vector Form of a Line

DEFINITION

A line ℓ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d} + \vec{p}$$

for some vector $\vec{d}$ and point $\vec{p}$. That is, $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

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9 Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

9.1 Write ℓ in vector form. Is vector form of ℓ unique?

9.2 Write L in vector form.

9.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

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10 Let A, B, and C be given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B} \qquad \overbrace{\vec{x} = t\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

10.1 Do the lines A and B intersect? Justify your conclusion.

10.2 Do the lines A and C intersect? Justify your conclusion.

10.3 Let $\vec{p} \neq \vec{q}$ and suppose X has vector form $\vec{x} = t\vec{d} + \vec{p}$ and Y has vector form $\vec{x} = t\vec{d} + \vec{q}$. Is it possible that X and Y intersect?

Vector Form of a Plane

DEFINITION

A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

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11 Recall the lines A and B given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

11.1 Find two direction vectors in $\mathcal{P}$.

11.2 Write $\mathcal{P}$ in vector form.

11.3 Describe how vector form of a plane relates to linear combinations.

11.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

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12 Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be a plane with equation $x + y + z = 1$.

12.1 Find three points in $\mathcal{Q}$.

12.2 Find two direction vectors for $\mathcal{Q}$.

12.3 Write $\mathcal{Q}$ in vector form.

Span

Span

DEF

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

13 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

13.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

13.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

13.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

13.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

13.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

14 Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

14.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

14.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

14.3 Does the expression $\operatorname{span}(\ell_1)$ make sense? If so, what is it? How about $\operatorname{span}(\ell_2)$?

Set Addition

DEF

If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A + B$, is

$$A + B = \{\vec{x} : \vec{x} = \vec{a} + \vec{b} \text{ for some } \vec{a} \in A \text{ and } \vec{b} \in B\}.$$

15 Let $A = \left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell = \operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

15.1 Draw A, B, and $A + B$ in the same picture.

15.2 Is $A + B$ the same as $B + A$?

15.3 Draw $\ell + A$.

15.4 Consider the line ℓ_2 given in vector form by $\vec{x} = t\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

Task 1.3: The Magic Carpet, Getting Back Home

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$).

1. Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.
2. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?
3. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?
4. What is span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}$?

DEFINITION

Linearly Dependent & Independent

We say the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

16

Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

16.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

16.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

16.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

16.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

DEF

Trivial Linear Combination

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

17

Recall $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

17.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

17.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

DEF

Linearly Dependent & Independent

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector.

18

18.1 Explain how this new definition implies the old one.

18.2 Explain how the old definition implies this new one.

Since we have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

19

Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

19.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

19.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

19.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

19.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Task 1.4: Linear Independence and Dependence: Creating Examples

1. Fill in the following chart keeping track of the strategies you used to generate examples.

	Linearly independent	Linearly dependent
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

2. Write at least two generalizations that can be made from these examples and the strategies you used to create them.

Dot Product

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

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20 Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

20.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

20.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

20.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

20.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

THM

For a vector $\vec{v}\in\mathbb{R}^n$, the formula
$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$
always holds.

Distance

DEF

The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u}-\vec{v}\|$.

Unit Vector

DEF

A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

21

Let $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

21.1 Find the distance between $\vec{u}$ and $\vec{v}$.

21.2 Find a unit vector in the direction of $\vec{u}$.

21.3 Does there exists a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

21.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Orthogonal

DEF

Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

22

22.1 Find two vectors orthogonal to $\vec{a} = \begin{bmatrix}1\\-3\end{bmatrix}$. Can you find two such vectors that are not parallel?

22.2 Find two vectors orthogonal to $\vec{b} = \begin{bmatrix}1\\-3\\4\end{bmatrix}$. Can you find two such vectors that are not parallel?

22.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\| = 5$ and $\|\vec{y}\| = 3$. What is the distance between $\vec{x}$ and $\vec{y}$?

23

23.1 Draw $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.

23.2 If $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x}\cdot\vec{u}$?

23.3 Expand the dot product $\vec{u}\cdot\vec{x}$ to get an equation for A.

23.4 If possible, express A as a span.

Normal Vector

DEF

A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to it.

24

Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}1\\-1\end{bmatrix}$, and define the lines
$$\ell_1 = \operatorname{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \{\vec{p}\} + \operatorname{span}\{\vec{d}\}.$$

24.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

24.2 Let $\vec{v}\in\ell_1$ and $\vec{u}\in\ell_2$. What is $\vec{n}\cdot\vec{v}$? What about $\vec{n}\cdot\vec{u}$?

24.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n}\cdot(\vec{x}-\vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

25 Let $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

25.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

25.2 Describe X using an equation.

25.3 Describe X as a span.

Projections

Projection

DEF

Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\text{proj}_X \vec{v}$, is the closest point to $\vec{v}$ in X.

26 Let $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\ell = \text{span}\{\vec{a}\}$.

26.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.

26.2 Find $\text{proj}_{\{\vec{b}\}} \vec{v}$, $\text{proj}_{\{\vec{a},\vec{b}\}} \vec{v}$.

26.3 Find $\text{proj}_\ell \vec{v}$. (Recall that a quadratic $at^2 + bt + c$ has a minimum at $t = -\frac{b}{2a}$).

26.4 Is $\vec{v} - \text{proj}_\ell \vec{v}$ a normal vector for ℓ? Why or why not?

27 Let K be the line given in vector form by $\vec{x} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and let $\vec{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

27.1 Make a sketch with $\vec{c}$, K, and $\text{proj}_K \vec{c}$ (you don't need to compute $\text{proj}_K \vec{c}$ exactly).

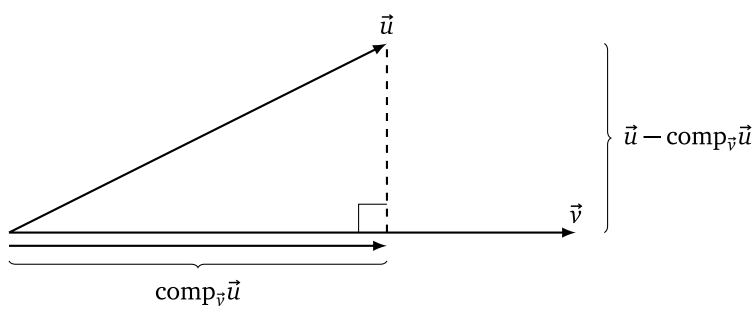
27.2 What should $(\vec{c} - \text{proj}_K \vec{c}) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be? Explain.

27.3 Use your formula from the previous part to find $\text{proj}_K \vec{c}$ *without* computing any distances.

Component

DEFINITION

Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***component of $\vec{u}$ in the $\vec{v}$ direction***, written $\text{comp}_{\vec{v}} \vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \text{comp}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.



28 Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

28.1 List two conditions that $\text{comp}_{\vec{b}} \vec{a}$ must satisfy.

28.2 Find a formula for $\text{comp}_{\vec{b}} \vec{a}$.

29 Let $\vec{d}=\begin{bmatrix}3\\3\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}1\\2\end{bmatrix}$.

29.1 Draw $\vec{d}$, $\vec{u}$, $\operatorname{span}\{\vec{d}\}$, and $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}}\vec{u}$ in the same picture.

29.2 How do $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}}\vec{u}$ and $\operatorname{comp}_{\vec{d}}\vec{u}$ relate?

29.3 Compute $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}}\vec{u}$ and $\operatorname{comp}_{\vec{d}}\vec{u}$.

29.4 Compute $\operatorname{comp}_{-\vec{d}}\vec{u}$. Is this the same as or different from $\operatorname{comp}_{\vec{d}}\vec{u}$? Explain.

Subspaces and Bases

Subspace

DEFINITION

A ***subspace*** $V\subseteq\mathbb{R}^n$ is a non-empty subset such that

(i) $\vec{u},\vec{v}\in V$ implies $\vec{u}+\vec{v}\in V$.

(ii) $\vec{u}\in V$ implies $k\vec{u}\in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

30 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

30.1 $A=\{\vec{x}\in\mathbb{R}^2:\vec{x}=\begin{bmatrix}a\\0\end{bmatrix}$ for some $a\in\mathbb{Z}\}$.

30.2 $B=\{\vec{x}\in\mathbb{R}^2:\vec{x}\neq\begin{bmatrix}0\\0\end{bmatrix}\}$.

30.3 $C=\{\vec{x}\in\mathbb{R}^2:\vec{x}=\begin{bmatrix}0\\t\end{bmatrix}$ for some $t\in\mathbb{R}\}$.

30.4 $D=\{\vec{x}\in\mathbb{R}^2:\vec{x}=\begin{bmatrix}0\\t\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}$ for some $t\in\mathbb{R}\}$.

30.5 $E=\{\vec{x}\in\mathbb{R}^2:\vec{x}=\begin{bmatrix}0\\t\end{bmatrix}$ or $\vec{x}=\begin{bmatrix}t\\0\end{bmatrix}$ for some $t\in\mathbb{R}\}$.

30.6 $F=\{\vec{x}\in\mathbb{R}^2:\vec{x}=t\begin{bmatrix}3\\1\end{bmatrix}$ for some $t\in\mathbb{R}\}$.

30.7 $G=\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

30.8 $H=\operatorname{span}\{\vec{u},\vec{v}\}$ for some unknown vectors $\vec{u},\vec{v}\in\mathbb{R}^2$.

Basis

DEF

A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B}=V$.

Dimension

DEF

The ***dimension*** of a subspace V is the number of elements in a basis for V.

31 Let $\vec{u}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V=\operatorname{span}\{\vec{u},\vec{v},\vec{w}\}$.

31.1 Describe V.

31.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ a basis for V? Why or why not?

31.3 Give a basis for V.

31.4 Give another basis for V.

31.5 Is $\operatorname{span}\{\vec{u},\vec{v}\}$ a basis for V? Why or why not?

31.6 What is the dimension of V?

32 Let $\vec a = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec b = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec c = \begin{bmatrix}7\\8\\8\end{bmatrix}$ (notice these vectors are linearly independent) and let $P = \operatorname{span}\{\vec a, \vec b\}$ and $Q = \operatorname{span}\{\vec b, \vec c\}$.

32.1 Give a basis for and the dimension of P.

32.2 Give a basis for and the dimension of Q.

32.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

32.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Matrices

33 Let $A = \begin{bmatrix}1 & 2\\3 & 3\end{bmatrix}$, $\vec x = \begin{bmatrix}x\\y\end{bmatrix}$, and $\vec b = \begin{bmatrix}-2\\-1\end{bmatrix}$.

33.1 Compute the product $A\vec x$.

33.2 Write down a system of equations that corresponds to the matrix equation $A\vec x = \vec b$.

33.3 Let $\begin{bmatrix}x_0\\y_0\end{bmatrix}$ be a solution to $A\vec x = \vec b$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

33.4 Let $\begin{bmatrix}x_0\\y_0\end{bmatrix}$ be a solution to $A\vec x = \vec b$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

34 Let $\vec u = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec v = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec w = \begin{bmatrix}7\\8\\9\end{bmatrix}$.

34.1 How could you determine if $\{\vec u, \vec v, \vec w\}$ was a linearly independent set?

34.2 Can your method be rephrased in terms of a matrix equation? Explain.

35 Consider the system represented by

$$\begin{bmatrix}1 & -3 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec b.$$

35.1 If $\vec b = \begin{bmatrix}1\\2\\3\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

35.2 If $\vec b = \begin{bmatrix}1\\1\\0\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

36 Let $\vec d_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\vec d_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$. Let $\mathcal P$ be the plane given in vector form by $\vec x = t\vec d_1 + s\vec d_2$. Further, suppose M is a matrix so that $M\vec r \in \mathcal P$ for any $\vec r$.

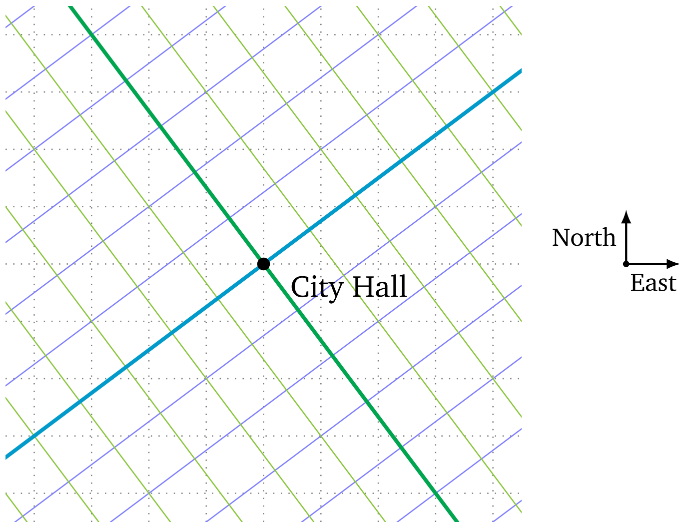
36.1 How many rows does M have?

36.2 Find such an M.

36.3 Find necessary and sufficient conditions (phrased as equations) for $\vec n$ to be a normal vector for $\vec p$.

36.4 Find a matrix K so that solutions to $K\vec x = \vec 0$ are normal vectors for $\mathcal P$. How do K and M relate?

37 The fictional town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \frac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

37.1 The points $A = (2, 1)$ and $B = (3, -1)$ are given in street coordinates. Find their east-north coordinates.

37.2 The points $X = (4, 3)$ and $Y = (1, 7)$ are given in east-north coordinates. Find their street coordinates.

37.3 Define $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$. Does $\operatorname{span}\{\vec{e}_1, \vec{e}_2\} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$?

37.4 Notice that $Y = 5\vec{e}_1 + 5\vec{e}_2 = \vec{d}_1 + 7\vec{d}_2$. Is the point Y better represented by the pair $(5, 5)$ or by the pair $(1, 7)$? Explain.

Representation in a Basis

DEFINITION

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notate $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

such that $\vec{v} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$.

Similarly,

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{B}} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \ldots, \vec{b}_n$ with coefficients $\alpha_1, \ldots, \alpha_n$.

38 Let $\mathcal{E}=\{\vec{e}_1,\vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C}=\{\vec{c}_1,\vec{c}_2\}$ where $\vec{c}_1=\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2=\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

38.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$.

38.2 Express $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$.

38.3 Let $\vec{v}=2\vec{e}_1+2\vec{e}_2$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

38.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}}=[\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

38.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}}=[\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

38.6 What is YX?

Orientation of a Basis

DEF

The ordered basis $\mathcal{B}=\{\vec{b}_1,\ldots,\vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i\mapsto\vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

39 Let $\{\vec{e}_1,\vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\vec{e}_1$ measured counter clockwise.

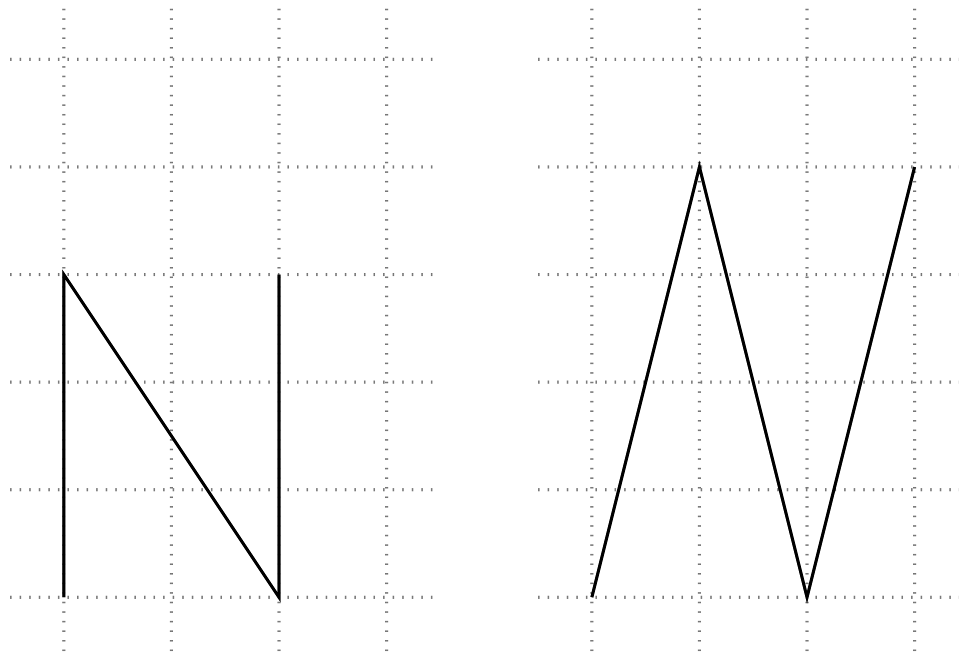
39.1 For which θ is $\{\vec{e}_1,\vec{u}_\theta\}$ a linearly independent set?

39.2 For which θ can $\{\vec{e}_1,\vec{u}_\theta\}$ be continuously transformed into $\{\vec{e}_1,\vec{e}_2\}$ and remain linearly independent the whole time?

39.3 For which θ is $\{\vec{e}_1,\vec{u}_\theta\}$ right-handed? Left-handed?

39.4 For which θ is $\{\vec{u}_\theta,\vec{e}_1\}$ (in that order) right-handed? Left-handed?

Task 2.1: Italicizing N

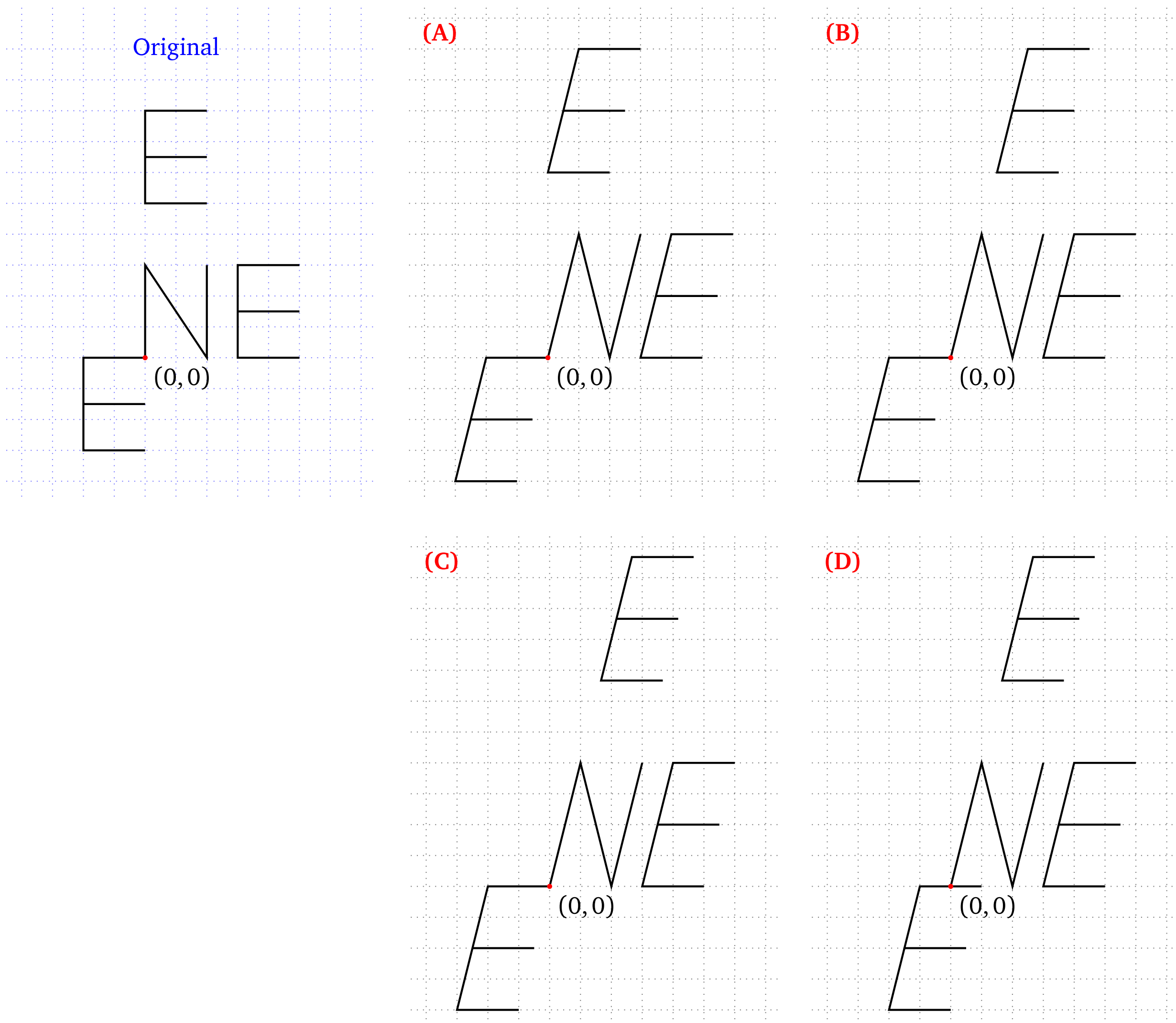


Suppose that the “N” on the left is written in regular 12-point font. Find a matrix A that will transform the “N” into the letter on the right which is written in an *italic* 16-point font.

Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

Task 2.2: Beyond the N

A few students were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If an "E" is placed around the "N," the students argued over four different possible results for the transformed E's. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?



Linear Transformations

40 $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

40.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

40.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

40.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

40.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter clockwise by 90°.

Linear Transformation

DEFINITION

Let V and W be subspaces. A function $T : V \to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v}) = T\vec{u} + T\vec{v} \qquad \text{and} \qquad T(\alpha\vec{v}) = \alpha T\vec{v}$$

for all vectors $\vec{u}, \vec{v} \in V$ and all scalars α.

41 41.1 Classify the following as linear transformation or not

(a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

(b) $W : \mathbb{R}^2 \to \mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x^2\\y\end{bmatrix}$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+2\\y\end{bmatrix}$.

(d) $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix} = \operatorname{comp}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$.

Image of a Set

DEFINITION

Let $L : V \to W$ be a transformation and let $X \subset V$ be a set. The ***image of the set*** V ***under*** L, denoted $L(V)$, is the set

$$L(V) = \{\vec{x} : \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in V\}.$$

42 Let $S = \{\begin{bmatrix}x\\y\end{bmatrix} : 0 \le x, y \le 1\} \subseteq \mathbb{R}^2$ be the filled-in unit square and let $C = \{\vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\} \subseteq \mathbb{R}^2$ be the corners of the unit square.

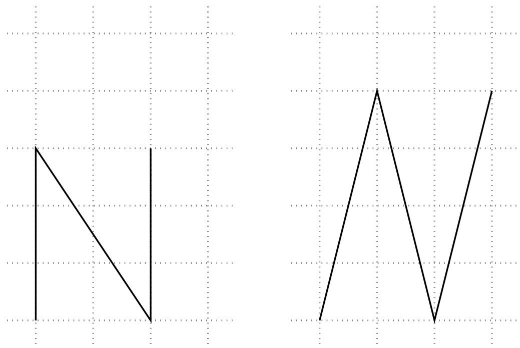
42.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

42.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).

42.3 Suppose that $\ell = \{$ all convex combinations of $\vec{a}$ and $\vec{b}\}$ is a line segment with endpoints $\vec{a}$ and $\vec{b}$ and A is a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

42.4 Explain how images of sets relate to the *Italicising N* task.

Task 2.3: Pat and Jamie



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

1. Do you think Pat and Jamie's approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?
2. Try Pat and Jamie's approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

43 Define $\mathcal{P}$ to be projection onto span $\{\vec{u}\}$ where $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

43.1 Find a matrix P so that $P\vec{x}=\mathcal{P}(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$.

43.2 Find a matrix R so that $R\vec{x}=\mathcal{R}(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$.

43.3 Write down matrices A and B for $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{P}$.

43.4 How do the matrices A and B relate to the matrices P and R?

Range (DEF)

The ***range*** (or ***image***) of a linear transformation $T:V\to W$ is the set of vectors that T can output. That is,

$$\mathrm{range}(T)=\{\vec{y}\in W:\vec{y}=T\vec{x}\text{ for some }\vec{x}\in V\}.$$

Null Space (DEFINITION)

The ***null space*** (or ***kernel***) of a linear transformation $T:V\to W$ is the set of vectors that get mapped to zero under T. That is,

$$\mathrm{null}(T)=\{\vec{x}\in V:T\vec{x}=\vec{0}\}.$$

44 Let $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ be projection onto span $\{\vec{u}\}$ where $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$ (like before).

44.1 What is the range of $\mathcal{P}$?

44.2 What is the null space of $\mathcal{P}$?

45 Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be an arbitrary linear transformation.

45.1 Show that the null space of T is a subspace.

45.2 Show that the range of T is a subspace.

Induced Transformation (DEFINITION)

Let M be an $n\times m$ matrix. We say M ***induces*** a linear transformation $T_M:\mathbb{R}^m\to\mathbb{R}^n$ defined by

$$[T_M\vec{v}]_{\mathcal{E}'}=M[\vec{v}]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

46 Let M be a 2×2 matrix and let $\vec{v}\in\mathbb{R}^2$. Further, let T_M be the transformation induced by M.

46.1 What is the difference between "$M\vec{v}$" and $M[\vec{v}]_{\mathcal{E}}$"?

46.2 What is $[T_M\vec{e}_1]_{\mathcal{E}}$?

46.3 Can you relate the columns of M to the range of T_M?

Fundamental Subspaces (DEF)

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M is the span of the rows of M; the ***column space*** of M is the span of the columns of M; and the ***null space*** of M is the set of solutions to $M\vec{x}=\vec{0}$.

47 Consider $A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$.

47.1 Describe the row space of A.

47.2 Describe the column space of A.

47.3 Is the row space of A the same as the column space of A?

47.4 Describe the set of all vectors perpendicular to the rows of A.

47.5 Describe the null space of A.

47.6 Describe the range and null space of T_A, the transformation induced by A.

48

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

48.1 How does the row space of B relate to the row space of C?

48.2 How does the null space of B relate to the null space of C?

48.3 Compute the null space of B.

49

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \mathrm{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

49.1 How does the column space of P relate to the column space of Q?

49.2 Describe the column space of P and the column space of Q.

Rank

DEF

For a linear transformation $T: V \to W$, the ***rank*** of T, denoted $\operatorname{rank}(T)$, is the dimension of the range of T.

For an $n \times m$ matrix M, the ***rank*** of M, denoted $\operatorname{rank}(M)$, is the number of pivots in $\operatorname{rref}(M)$.

---

50 Let $\mathcal{P}$ be projection onto span $\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by $90°$.

50.1 Describe range $(\mathcal{P})$ and range $(\mathcal{R})$.

50.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

50.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

50.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

51

51.1 Determine the rank of (a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

---

52 Consider the homogeneous system

$$\begin{aligned} x &\ +2y &\ +z &\ =0 \\ x &\ +2y &\ +3z &\ =0 \\ -x &\ -2y &\ +z &\ =0 \end{aligned} \tag{1}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$.

52.1 What is $\operatorname{rank}(A)$?

52.2 Give the general solution to (1).

52.3 Are the column vectors of A linearly independent?

52.4 Give a non-homogeneous system with the same coefficients as (1) that has

(a) infinitely many solutions

(b) no solutions.

---

53 53.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

53.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of columns in } A.$$

---

54 54.1 Is here a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

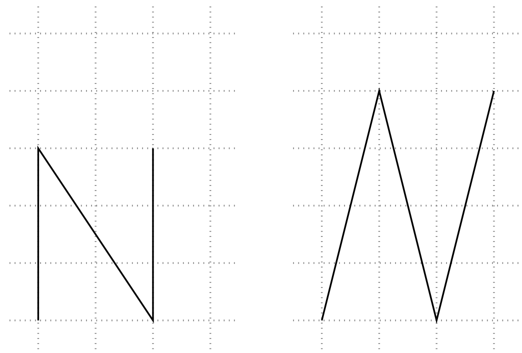
---

55 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

55.1 What is the rank and nullity of A^T?

55.2 What is the rank and nullity of A?

Task 2.4: Getting back N



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

Consider the new task: find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.
2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

Inverses

56

56.1 Apply the row operation $R_3 \to R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

56.2 Apply the row operation $R_3 \to R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

DEF An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

56.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

56.4 Without computing, what should the result of applying the row operation $R_3 \to R_3 - 2R_1$ to E_1 be? Compute and verify.

56.5 Without computing, what should E_1E_2 be? What about E_2E_1? Now compute and verify.

DEF The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated by A^{-1}.

57

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

57.1 Which pairs of matrices above are inverses of each other?

58

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

58.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

58.2 What is E_2E_1B?

58.3 Find B^{-1}.

58.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

59

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

59.1 What is $A^{-1}A$?

59.2 What is $\mathrm{rref}(A)$?

59.3 What is $\mathrm{rref}(C)$? (Hint, there is no need to actually do row reduction!)

59.4 Solve the system $A\vec{x} = \vec{b}$.

60 60.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

60.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

More Change of Basis

61 Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ and let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

61.1 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

61.2 Compute $X[\vec{e}_1]_{\mathcal{B}}$ and $X[\vec{e}_2]_{\mathcal{B}}$. What do you notice?

61.3 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

62 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}.$$

62.1 Should X^{-1} exist? Explain.

62.2 Consider the equation

$$X^{-1}[\vec{v}]_? = [\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

62.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

63 Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

63.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

63.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}5\\3\end{bmatrix}$.

63.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

63.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

63.5 Find a matrix for T in the $\mathcal{C}$ basis.

63.6 Find a matrix for T in the standard basis.

DEFINITION

Similar Matrices

A matrices A and B are called ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

$$A = XBX^{-1}.$$

Unit n-cube

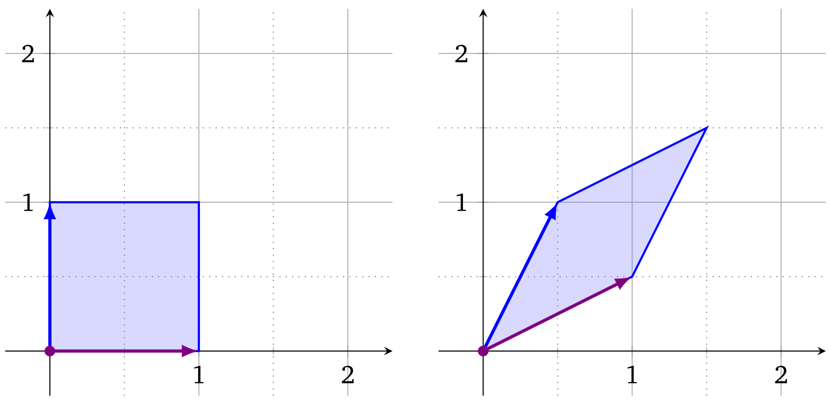
DEFINITION

The unit n-cube is the n-dimensional cube with side length 1 and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The volume of the unit n-cube is always 1.

64 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



64.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

64.2 Write down a matrix for T.

64.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may need to use trigonometry.

Determinant

DEFINITION

The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors (or the row vectors).

65 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

65.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

65.2 Compute the area of $A(C_2)$.

65.3 Compute $\det(A)$.

66 Suppose R is a rotation counterclockwise by 30°.

66.1 Draw C_2 and $R(C_2)$.

66.2 Compute the area of $R(C_2)$.

66.3 Compute $\det(R)$.

67 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

67.1 What is $\det(F)$?

68 Let $D=\{\vec{x} : \|\vec{x}\|\leq 1\}$ be the unit disk. You know the following about the linear transformations M, T, and S: M is defined by $\vec{x}\mapsto 2\vec{x}$; T has determinant 2; and S has determinant 3.

68.1 Find the oriented volumes of $M(C_2)$, $T(C_2)$, and $S(C_2)$.

68.2 How does the volume of $T(C_2+\{\vec{e}_1\})$ compare to the volume of $T(C_2)$?

68.3 What is the oriented volume of $T\circ M(C_2)$? What is $\det(T\circ M)$?

68.4 What is the oriented volume of $S(D)$?

69

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1\to R_1+2R_2$ applied.

69.1 What is $\det(E_f)$?

69.2 What is $\det(E_m)$?

69.3 What is $\det(E_a)$?

69.4 What is $\det(E_f E_m)$?

69.5 What is $\det(4I_{3\times 3})$?

69.6 What is $\det(W)$ where $W=E_f E_a E_f E_m E_m$?

70

$$U=\begin{bmatrix}1&2&1&2\\0&3&-2&4\\0&0&-1&0\\0&0&0&4\end{bmatrix}$$

70.1 What is $\det(U)$?

70.2 V is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

70.3 P is projection onto the vector $\begin{bmatrix}-1\\-1\end{bmatrix}$. What is $\det(P)$?

71 Suppose you know $\det(X)=4$.

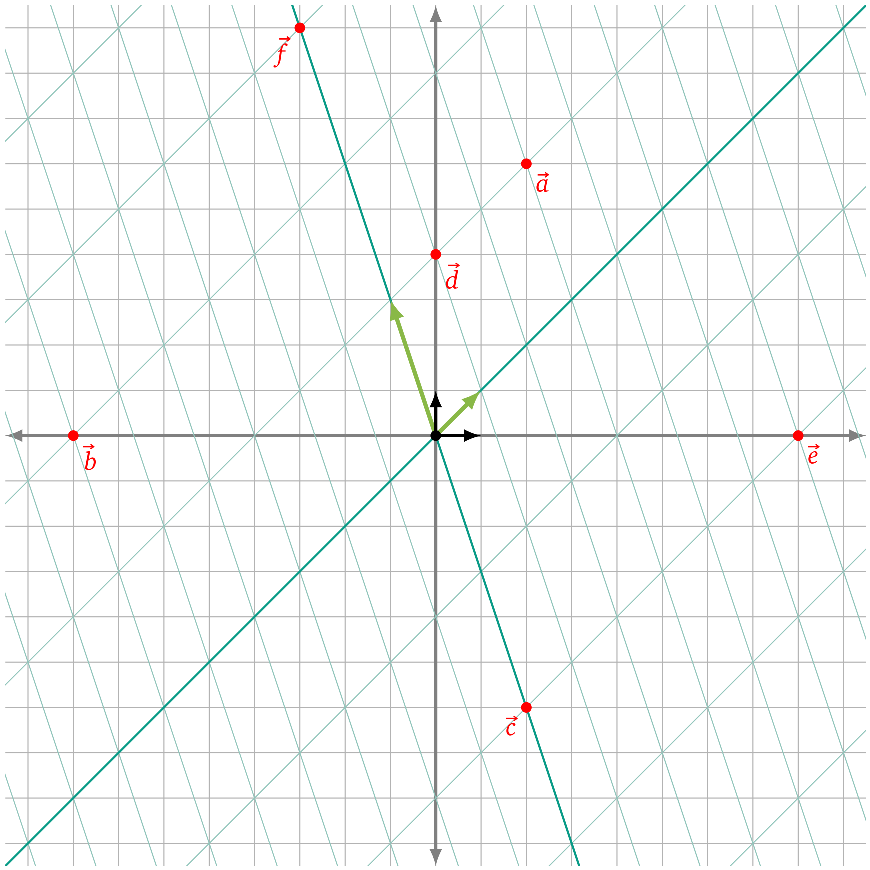
71.1 What is $\det(X^{-1})$?

71.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

71.3 Suppose Y is not invertible. What is $\det(Y)$?

Task 3.1: The Green and the Black

Consider the following two bases for $\mathbb{R}^2$: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$ and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.



1. Write each point above in both the green and the black bases.
2. Find a change-of-basis matrix X so that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{u}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

4. When working with the transformation T, which basis do you prefer vectors be represented in?

Eigenvectors

Eigenvector

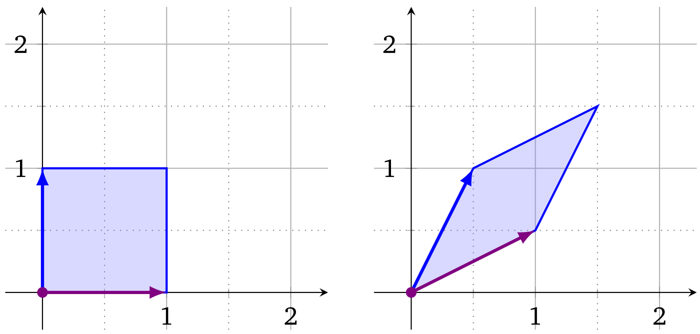
DEFINITION

Let X be a linear transformation. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

72 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



72.1 Give an eigenvector for T. What is the eigenvalue?

72.2 Can you give another?

73 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

73.1 Give an eigenvector and a corresponding eigenvalue for A.

73.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

73.3 What is the dimension of null(B)?

73.4 What is det(B)?

74 Let $C = \begin{bmatrix}-1 & 2\\1 & 0\end{bmatrix}$ and $E_\lambda = C - \lambda I$.

74.1 For what values of λ does E_λ have a non-trivial null space?

74.2 What are the eigenvalues of C?

74.3 Find the eigenvectors of C.

Characteristic Polynomial

DEF

For a matrix A, the ***characteristic polynomial*** of A is

$$\text{char}(A) = \det(A - \lambda I).$$

75 Let $D = \begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}$.

75.1 Compute char(D).

75.2 Find the eigenvalues of D.

76 Suppose char$(E) = \lambda(\lambda - 2)(\lambda + 3)$ for some unknown 3×3 matrix E.

76.1 What are the eigenvalues of E?

76.2 Is E invertible?

76.3 What is nullity(E), nullity($E-3I$), nullity($E+3I$)?

---

77 Consider

$$A=\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}\qquad \vec v_1=\begin{bmatrix}1\\1\\1\end{bmatrix}\qquad \vec v_2=\begin{bmatrix}1\\1\\-2\end{bmatrix}\qquad \vec v_3=\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

and notice that $\vec v_1,\vec v_2,\vec v_3$ are eigenvectors for A.

77.1 Find the eigenvalues of A.

77.2 Find the characteristic polynomial of A.

77.3 Compute $A\vec w$ where $w=2\vec v_1-\vec v_2$.

77.4 Compute $A\vec u$ where $\vec u=a\vec v_1+b\vec v_2+c\vec v_3$ for unknown scalar coefficients a,b,c.

Notice that $\mathcal{V}=\{\vec v_1,\vec v_2,\vec v_3\}$ is a basis for $\mathbb{R}^3$.

77.5 If $[\vec x]_{\mathcal{V}}=\begin{bmatrix}1\\3\\4\end{bmatrix}$ is $\vec x$ written in the $\mathcal{V}$ basis, compute $A\vec x$ in the $\mathcal{V}$ basis.

---

78 The transformation P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation (where $\mathcal{V}$ is from above).

78.1 Describe in words what P does.

78.2 Describe how you can use P and P^{-1} to easily compute $A\vec y$ for any $\vec y\in\mathbb{R}^3$.

78.3 Can you find a matrix D so that

$$PDP^{-1}=A?$$

78.4 $[\vec x]_{\mathcal{V}}=\begin{bmatrix}1\\3\\4\end{bmatrix}$. Compute $A^{100}\vec x$.

---

79 For an $n\times n$ matrix T, suppose its eigenvectors $\{\vec v_1,\dots\vec v_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1,\dots,\lambda_n$ be the corresponding eigenvalues.

79.1 Is T diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

79.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

79.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Eigenspace

DEFINITION

Let A be a matrix with eigenvalues $\{\lambda_1,\dots,\lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A-\lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x-\lambda_i$ occurs as a factor).

---

80 Define $F=\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

80.1 Is F diagonalizable? Why or why not?

80.2 What is the geometric and algebraic multiplicity of each eigenvalue of F?

80.3 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?

Span Again

81 Consider the system

$$\begin{array}{rrrl} x & -y & -z & =0 \\ 0x & +1y & +2z & =0 \\ 3x & -3y & +3z & =0 \end{array} \tag{2}$$

which has the unique solution $(x,y,z)=(0,0,0)$.

81.1 Give vectors $\vec{u},\vec{v},\vec{w}$ so that the system (2) corresponds to the vector equation $x\vec{u}+y\vec{v}+z\vec{w}=\vec{0}$.

81.2 Is $\vec{w}\in\operatorname{span}\{\vec{u},\vec{v}\}$? If so, write it as a linear combination of $\vec{u}$ and $\vec{v}$.

The matrix M is the non-augmented matrix corresponding to a homogeneous system of linear equations. M also corresponds to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$. Further, we know

$$\operatorname{rref}(M)=\begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\end{bmatrix}.$$

81.3 Give a solution to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$.

81.4 Is $\vec{c}\in\operatorname{span}\{\vec{a},\vec{b}\}$? If so, write it as a linear combination of $\vec{a}$ and $\vec{b}$.

81.5 Do you have enough information to tell if $\{\vec{a},\vec{b}\}$ is linearly independent? Why or why not?

Finding Linearly Independent Subsets

82 Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$ you have no free variables.

82.1 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$, the second column corresponds to a free variable.

82.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

82.3 Is $\{\vec{u},\vec{w}\}$ linearly independent?

82.4 Is $\{\vec{u},\vec{v}\}$ linearly independent?

Maximal Linearly Independent Subset

DEF

Given a set of vectors X, a ***maximal linearly independent subset*** of X is a linearly independent subset $V\subseteq X$ with the most possible vectors in it (i.e., if you took any subset of X with more vectors, it would be linearly dependent).

83

83.1 Give a maximal linearly independent subset, T, of $\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}: a,b,c\in\mathbb{R}\right\}$.

83.2 What is the size of T?

84 Consider the vectors

$$\vec{v}_1=\begin{bmatrix}1\\2\\1\end{bmatrix}\qquad \vec{v}_2=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\qquad \vec{v}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}\qquad \vec{v}_4=\begin{bmatrix}-1\\2\\0\end{bmatrix}\qquad \vec{v}_5=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

and the matrices

$$A=\begin{bmatrix}1&-1&0&-1&1\\2&-1&1&2&-1\\1&-1&0&0&1\end{bmatrix}\qquad \operatorname{rref}(A)=\begin{bmatrix}1&0&1&0&-2\\0&1&1&0&-3\\0&0&0&1&0\end{bmatrix}.$$

(Notice that the columns of A are the vectors $\vec{v}_1, \dots, \vec{v}_5$)

84.1 Is $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ linearly independent?

84.2 Pick a maximal linearly independent subset of V.

84.3 Pick another (different) maximal linearly independent subset of V.

84.4 Give a basis for $\operatorname{span}(V)$.

84.5 What is the dimension of $\operatorname{span}(V)$?

85

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

85.1 Write the shape of the matrices A, B, C (i.e., for each one, write the dimensions in $m \times n$ form).

85.2 List *all* products between the matrices A, B, C that are defined. (Your list will be some subset of AB, AC, BA, CA, BC, CB.)

85.3 Compute AC and CA.

86

86.1 If the matrices X and Y are both square $n \times n$ matrices, does $XY = YX$? Explain.

86.2 If the matrices X and Y are both square $n \times n$ matrices, does $X + Y = Y + X$? Explain.

87

The entries of a matrix are specified by (row,column) pairs of integers. If a_{ij} is the (i, j) entry of a matrix A, we may write $A = [a_{ij}]$.

87.1 Write the 2×2 matrix A with entries $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 7$ and $a_{22} = 9$.

87.2 Let $B = [b_{ij}]$ be the 3×3 matrix where $b_{ij} = i + j$. Write B.

87.3 Let $C = [c_{ij}]$ be the 3×4 matrix where $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ if $i \neq j$.

88

DEF The ***transpose*** of a matrix $A = [a_{ij}]$ is the matrix $A^T = [a_{ji}]$.

Visually, the transpose of a matrix swaps rows and columns.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

88.1 What is the shape of A and A^T?

88.2 Write down A^T.

B and D are 4×6 matrices and C is a 6×4 matrix.

88.3 Does $(BC)^T = B^T C^T$? Explain.

88.4 Does $(B + D)^T = B^T + D^T$? Explain.

88.5 Compute AA^T and $A^T A$ (where A is the matrix defined earlier). What do you notice?

89

DEF A matrix X is called ***symmetric*** if $X = X^T$.

Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

89.1 Prove that if W is a square matrix, then $V = W^T W + W + W^T$ is a symmetric matrix.

90

DEF A ***zero matrix*** is a matrix whose entries are all zeros. An ***identity matrix*** is a square matrix whose diagonal entries are 1 and non-diagonal entries are 0.

We write the $m \times n$ zero matrix as $0_{m\times n}$ or just 0 if the shape is determined by context. The $n \times n$ identity matrix is notated $I_{n\times n}$ or just I if the shape is determined by context.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

90.1 Write down the 3×3 identity matrix and the 3×3 zero matrix.

90.2 Compute $I_{3\times3}A$, $AI_{3\times3}$, $0_{3\times3}A$, and $A0_{3\times3}$.

90.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

91

91.1 Solve the matrix equation

$$I_{4\times4}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

Orthogonality

Orthogonal & Orthonormal

DEF A set of vectors is ***orthogonal*** if every pair of vectors in the set is orthogonal. A set of vectors is ***orthonormal*** if it is both an orthogonal set and every vector is a unit vector.

92

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} \qquad \vec{b}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

The matrix $A = [\vec{b}_1|\vec{b}_2]$ takes vectors in the $\mathcal{B}$ basis and rewrites them in the standard basis.

92.1 What does A^{-1} do?

92.2 Find a matrix B that takes vectors in the standard basis and rewrites them in the $\mathcal{B}$ basis.

92.3 Write $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the $\mathcal{B}$ basis.

92.4 What is the relationship between A and B?

Orthogonal Matrix

DEF An ***orthogonal matrix*** is a square matrix whose columns are orthonormal (Yes, a better name would be orthonormal matrix, but that is not the term the rest of the world uses).

---

93 Suppose $X = [\vec{x}_1|\vec{x}_2|\vec{x}_3|\vec{x}_4]$ is an orthogonal matrix.

93.1 What is the shape of X (i.e., it is a what×what matrix)?

93.2 Compute $X^T X$.

93.3 What is X^{-1}?

94

$$Y = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

94.1 Is Y an orthogonal matrix?

94.2 Fix Y so it is an orthogonal matrix. Call the new matrix X.

94.3 Compute X^{-1}.

94.4 Compute Y^{-1}.

94.5 Compute $|\det(X)|$ and $|\det(Y)|$ (the absolute value of the determinant of X and Y).

Matrix equations involving orthogonal matrices are easy to solve because the inverse of an orthogonal matrix is so easy to compute!

---

95 Let $A = [\vec{a}_1|\vec{a}_2|\vec{a}_3|\vec{a}_4]$ be an orthogonal matrix.

95.1 Explain why $\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b} \\ \vec{a}_2 \cdot \vec{b} \\ \vec{a}_3 \cdot \vec{b} \\ \vec{a}_4 \cdot \vec{b} \end{bmatrix}$ is a solution to $A\vec{x} = \vec{b}$.

95.2 Find scalars a, b, c, d so $\vec{b} = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3 + d\vec{a}_4$ (your answers will have variables in them).

Orthogonal matrices also allow us to compute projections quite easily.

Orthogonal Projection

DEF If V is a subspace of $\mathbb{R}^n$, the ***projection*** (sometimes called the orthogonal projection) of $\vec{x}$ onto V is the closest point in V to $\vec{x}$. We notate the projection of $\vec{x}$ onto V as $\text{proj}_V \vec{x}$.

Projections are normally hard to compute and a priori might require some sort of calculus-style optimization to find. However, from geometry we know that if we travel from $\mathrm{proj}_V \vec{x}$ to $\vec{x}$, we should always trace out a path perpendicular to V. Otherwise, we could find a point in V that was slightly closer to $\vec{x}$, violating the definition of $\mathrm{proj}_V \vec{x}$. Thus, orthogonality will be our savior.

---

96 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis.

96.1 If $\vec{x} = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$, find the projection of $\vec{x}$ onto the xy-plane.

Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

96.2 If $\vec{y} = 3\vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$, find the projection of $\vec{y}$ onto $\mathrm{span}\{\vec{b}_1, \vec{b}_3\}$.

Suppose $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is a basis for $\mathbb{R}^3$ with

$$\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1 \qquad \vec{c}_1 \cdot \vec{c}_2 = 0 \qquad \vec{c}_1 \cdot \vec{c}_3 = 0 \qquad \vec{c}_2 \cdot \vec{c}_3 = \sqrt{2}/2.$$

96.3 If $\vec{z} = 5\vec{c}_1 + 2\vec{c}_2 - \vec{c}_3$, find the projection of $\vec{z}$ onto $\mathrm{span}\{\vec{c}_1, \vec{c}_2\}$.

---

97 Let's put this all together. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Let $\mathcal{P}$ be the plane defined by

$$0x + y - z = 0.$$

97.1 Write $\mathcal{P}$ in vector form (Hint: think about the vectors listed in the $\mathcal{B}$ basis).

97.2 Find an orthonormal basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ for $\mathbb{R}^3$ so $\mathcal{P} = \mathrm{span}\{\vec{c}_1, \vec{c}_2\}$.

97.3 Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\mathrm{proj}_{\mathcal{P}} \vec{x}$.

Gram-Schmidt Orthogonalization

We've seen how useful orthonormal bases are. The incredible thing is that we can turn any basis into an orthonormal basis through a process called Gram-Schmidt orthogonalization.

---

98 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

98.1 Draw $\vec{a}$ and $\vec{b}$ and find $\vec{w} = \mathrm{proj}_{\vec{b}} \vec{a}$.

98.2 Add $\vec{c} = \vec{a} - \vec{w}$ to your drawing. What is the angle between $\vec{c}$ and $\vec{b}$.

98.3 Can you write $\vec{a}$ as the sum of two vectors, one in the direction of $\vec{b}$ and one orthogonal to $\vec{b}$? If so, do it.

---

99 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

99.1 Write $\vec{a} = \vec{u} + \vec{v}$ where $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is orthogonal to $\vec{b}$.

99.2 Find an orthonormal basis for $\mathrm{span}\{\vec{a}, \vec{b}\}$.

With two vectors, making an orthonormal set without changing the span is quite easy. With more vectors, it is only slightly harder.

Gram-Schmidt Process

DEFINITION

The ***Gram-Schmidt*** orthogonalization procedure takes in a set of vectors and outputs a set of orthonormal vectors with the same span. The idea is to iteratively produce a set of vectors where each new vector you produce is orthogonal to the previous vectors.

The algorithm is as follows: Let $\{v_1, \ldots, v_n\}$ be a set of vectors. Produce a set $\{v_2', \ldots, v_n'\}$ that is orthogonal to v_1 by subtracting off the respective projections of $v_2, \ldots, v_n$ onto v_1. Next, produce a set $\{v_3'', \ldots, v_n''\}$ orthogonal to both v_1 and v_2' by subtracting off the respective projections onto v_2'. Continue this process until you have a set $V = \{v_1, v_2', v_3'', v_4''', \ldots\}$ that is orthogonal. Finally, normalize V so all vectors have unit length.

---

100 Let $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$.

100.1 Use the Gram-Schmidt procedure to find an orthonormal basis for $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

100.2 Find an orthonormal basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ for $\mathbb{R}^4$ so that $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Let $R = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$.

100.3 Find an orthonormal basis for the row space of R.

100.4 Find the null space of R (Hint, you've already done the work, so there is no need to row reduce).

---

101 Let

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{y}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}.$$

101.1 Find an orthonormal basis $\mathcal{W}$ so that $\operatorname{span}\mathcal{W} = \operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$.

Orthogonal Complement

DEF

The ***orthogonal complement*** of a subspace V is written $V^\perp$ and defined as

$$V^\perp = \{\vec{x} : \vec{x} \text{ is orthogonal to } V\}.$$

101.2 Find the orthogonal complement of $\operatorname{span}\mathcal{W}$.

101.3 Write $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ in the form $\vec{v} = \vec{r} + \vec{n}$ where $\vec{r} \in \operatorname{span}\mathcal{W}$ and $\vec{n} \in (\operatorname{span}\mathcal{W})^\perp$.

QR Decomposition

QR Decomposition

DEF

For a matrix A, we can rewrite $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Writing A as QR is called the ***QR decomposition*** of A.

---

102 Suppose A, B, C are square matrices and $C = AB$.

102.1 How do the column spaces of A and C relate?

102.2 How do the column spaces of B and C relate?

103 $\mathcal{V}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ forms a basis for $\mathbb{R}^3$. When we apply the Gram-Schmidt process to $\mathcal{V}$, we get

$$\begin{aligned} q_1' &= \vec{v} \\ q_2' &= \vec{v}_2 - \tfrac{1}{2}\vec{v}_2 \\ q_3' &= \vec{v}_3 - \vec{v}_1 + 2\vec{v}_2 \end{aligned}$$

form an orthogonal set. Normalizing we get

$$\begin{aligned} \vec{q}_1 &= 2q_1' \\ \vec{q}_2 &= 3q_2' \\ \vec{q}_3 &= \tfrac{1}{2}q_3' \end{aligned}$$

form an orthonormal set.

103.1 Write $\vec{v}_1$ as a linear combination of $\vec{q}_1,\vec{q}_2,\vec{q}_3$.

103.2 Write $\vec{v}_2$ as a linear combination of $\vec{q}_1,\vec{q}_2,\vec{q}_3$.

103.3 Write $\vec{v}_3$ as a linear combination of $\vec{q}_1,\vec{q}_2,\vec{q}_3$.

Define $A=[\vec{v}_1|\vec{v}_2|\vec{v}_2]$ and $Q=[\vec{q}_1|\vec{q}_2|\vec{q}_3]$.

103.4 Find a matrix R so that $A=QR$.

We've just discovered one process to find the QR decomposition of a matrix. It's really as simple as doing Gram-Schmidt and keeping track of your coefficients. Now, we have another way to the matrix equation $A\vec{x}=\vec{b}$. If we do a QR decomposition and exploit the fact that $Q^{-1}=Q^T$, we have

$$A\vec{x}=QR\vec{x}=\vec{b} \qquad \Longrightarrow \qquad R\vec{x}=Q^T\vec{b}$$

and R is a triangular matrix, so we can just do back substitution! (It turns out that if you solve systems this way, there is less rounding error than if you use row reduction.)

Symmetric Matrices

When you're new to Linear Algebra, learning lots of new concepts and algorithms, it's sometimes hard to grasp the significance of certain properties of a matrix.

Symmetric matrices are easy to forget at first, but they have many profound properties (not to mention they are one of the key concepts of Quantum Mechanics).

104 Let A be a symmetric matrix and let $\vec{v}$ be an eigenvector with eigenvalue 3 and $\vec{w}$ be an eigenvector with eigenvalue 4. Note, for this problem, we are thinking of $\vec{v}$ and $\vec{w}$ as column vectors.

104.1 Write $A\vec{v}$, $\vec{v}^TA^T$, $\vec{v}^TA$, $A\vec{w}$, $\vec{w}^TA^T$, and $\vec{w}^TA$ in terms of $\vec{v}$, $\vec{w}$ and scalars.

104.2 How do $\vec{v}^T\vec{w}$ and $\vec{w}^T\vec{v}$ relate?

104.3 What should $\vec{v}^TA\vec{w}$ be in terms of $\vec{v}^T$ and $\vec{w}$? (Note, you could compute $(\vec{v}^TA)\vec{w}$ or $\vec{v}^T(A\vec{w})$. Better do both to be safe).

104.4 What can you conclude about $\vec{v}^T\vec{w}$? How about $\vec{v}\cdot\vec{w}$?

We've just deduced that all eigenspaces of a symmetric matrix are orthogonal! On top of that, symmetric matrices always have a basis of eigenvectors. That means that not only can you always diagonalize a symmetric matrix, but you can *orthogonally* diagonalize a symmetric matrix. (i.e. if A is symmetric, then $A=QDQ^T$ where Q is orthogonal and D is diagonal). This is like the best of all worlds in one!

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

105 Consider the system

$$\begin{array}{rcrl} x & - & y & = 2 \\ 2x & + & y & = 1 \end{array} \tag{3}$$

105.1 Draw the lines in (3) on the same coordinate plane.

105.2 Algebraically solve the system (3). What does this solution represent on your graph?

106 Let L be the line given by $x - y = 2$.

106.1 Write an equation of a line that doesn't intersect L.

106.2 Write an equation of a line that intersects L in

(a) one place.
(b) infinitely many places
(c) exactly two places

or explain why no such equation exists.

106.3 For each equation you came up with, solve the system algebraically. How can you tell algebraically how many solutions there are?

The Row Reduction Algorithm

107 107.1 Solve the system

$$\begin{array}{rcrcrl} x & - & y & - & 2z & = -5 \\ 2x & + & 3y & + & z & = 5 \\ 0x & + & 2y & + & 3z & = 8 \end{array} \tag{4}$$

any way you like.

107.2 Use an augmented matrix to solve the system (4).

The system (4) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (4) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

107.3 Rewrite (4) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

107.4 If (x, y, z) is a solution to (4), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

107.5 If (x, y, z) is a solution to (4), is $\vec{p} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

108 Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

108.1 Write the system of equations corresponding to A.

108.2 Solve the system of equations corresponding to A.

Infinite Solutions

109 Consider the system

$$\begin{array}{rcrl} x & + & 2y & = 3 \\ 2x & + & 4y & = 6 \end{array} \tag{5}$$

109.1 How many solutions does (5) have?

109.2 Write the solutions to (5) in vector form.

109.3 What happens when you use an augmented matrix to solve (5)?

Free Variables

110 Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make $2-1=1$ choices when writing a solution. Let's make the choice $y=t$.

110.1 With the added equation $y=t$, solve the system represented by B.

111 Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

and call the variables in this system x_1, x_2, x_3, x_4, x_5.

111.1 Write the system of equations represented by C.

111.2 Identify how many choices you can make when writing down a solution corresponding to C.

111.3 Add one equation (of the form $x_i = t$ or $x_j = s$, etc.) for each choice you must make when solving the system.

111.4 Write in vector form all solutions to C.

112 112.1 An unknown system U is represented by an augmented matrix with 4 rows and 6 columns. What is the minimum number of free variables solutions to U will have?

112.2 An unknown system V is represented by an augmented matrix with 6 rows and 4 columns. What is the minimum number of free variables solutions to V will have?

113 **Homogeneous**

DEF A system is called ***homogeneous*** if all equations equal 0.

Let A be an unknown system of 3 equations and 3 variables and suppose $(x,y,z)=(1,2,1)$ and $(x,y,z)=(-1,1,1)$ are solutions to A.

113.1 Can you produce another solution to the system?

113.2 Can you produce a solution to the homogeneous version of A (the version of A where every equation equals 0)?

113.3 Suppose when you use an augmented matrix to solve the system A, you only have one free variable. Could A be homogeneous? Can you produce all solutions to the system A?