Tutorial 3

Orthogonality

Learning Objectives

> In this tutorial you will explore orthogonality in depth.
>
> These problems relate to the following course learning objectives: *Work independently to understand concepts and procedures that have not been previously explained to you*, *translate between algebraic and geometric viewpoints to solve problems*, and *understand definitions that have been written by others*.

Definitions

Recall the vectors $\vec{a}$ and $\vec{b}$ are *orthogonal* if $\vec{a}\cdot\vec{b}=0$. We say the *sets A and B are orthogonal* if every vector in A is orthogonal to every vector in B.

Problems

Let $\vec{v}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$, $\vec{v}_3=\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$, $\vec{v}_4=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, $\vec{v}_5=\begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix}$, and $\vec{v}_6=\begin{bmatrix}1\\1\\2\\0\end{bmatrix}$.

1. (a) Identify all pairs of orthogonal vectors among $\vec{v}_1, \dots, \vec{v}_6$.
   (b) Let $A=\{\vec{v}_1,\vec{v}_2\}$ and $B=\{\vec{v}_3,\vec{v}_4\}$. Are A and B orthogonal sets? Why or why not?
   (c) Let $P=\{\vec{v}_1,\vec{v}_6\}$ and $Q=\{\vec{v}_3,\vec{v}_5\}$. Are P and Q orthogonal sets? Why or why not?
   (d) Can you split the vectors $\vec{v}_1, \dots, \vec{v}_6$ into two non-empty sets that are orthogonal to each other? Explain.

2. (a) Using guess-and-check, find two vectors that are orthogonal to both $\vec{v}_1$ and $\vec{v}_2$.
   (b) Set up and solve a system of equations to find all vectors orthogonal to $\vec{v}_1$ and $\vec{v}_2$.

3. The dot product is *commutative* and *distributive*. That is $\vec{v}\cdot(\alpha\vec{a}+\vec{b})=(\alpha\vec{a}+\vec{b})\cdot\vec{v}=\alpha(\vec{a}\cdot\vec{v})+\vec{b}\cdot\vec{v}$. Use this to show that if the set $X=\{\vec{x}\}$ is orthogonal to the set $Y=\{\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4\}$, then X is also orthogonal to $\operatorname{span} Y$.

4. We say that $\vec{a}$ and $\vec{b}$ are *close* if $\|\vec{a}-\vec{b}\|$ is small. We will see if we can extend this concept to lines.

   Let the lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 be given by the equations $y=x$, $y=1.001x$, $y=2000x$, $y=3000x$.

   (a) Out of $\ell_1, \dots, \ell_4$, which lines would you call "close"? Can you come up with a mathematical definition to justify your conclusion?
   (b) For $\ell_1, \dots, \ell_4$, find unit normal vectors $\vec{n}_1, \dots, \vec{n}_4$. For consistency, ensure each unit normal vector points towards the upper left (i.e., has negative first coordinate and positive second coordinate).
   (c) Compute the distances between $\vec{n}_1, \dots, \vec{n}_4$. Do these distances coincide with your intuition about closeness? Why might comparing normal vectors be preferable to comparing direction vectors?

1. (a) $\vec{v}_1 \perp \vec{v}_3$, $\vec{v}_1 \perp \vec{v}_5$, $\vec{v}_2 \perp \vec{v}_4$, $\vec{v}_3 \perp \vec{v}_4$, $\vec{v}_3 \perp \vec{v}_6$, $\vec{v}_5 \perp \vec{v}_6$.

   (b) No. $\vec{v}_2 \cdot \vec{v}_3 \neq 0$.

   (c) Yes. $\vec{v}_1 \cdot \vec{v}_3 = \vec{v}_1 \cdot \vec{v}_5 = 0$ and $\vec{v}_6 \cdot \vec{v}_3 = \vec{v}_6 \cdot \vec{v}_5 = 0$.

   (d) No. Whatever set contains $\vec{v}_1$ must also contain $\vec{v}_2$ and $\vec{v}_4$ and $\vec{v}_6$. However $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_6\} = \mathbb{R}^4$, so the only non-empty set orthogonal to $\{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_6\}$ is $\{\vec{0}\}$, which isn't a possibility in this case.

2. (a) $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

   (b)

$$\begin{cases} x_1 + x_2 + x_3 + x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 &= 0 \end{cases}$$

   has complete solution

$$\vec{x} = t \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

3. By definition, if $\vec{u} \in \operatorname{span}(Y)$, then $\vec{u} = \alpha_1 \vec{y}_1 + \cdots + \alpha_4 \vec{y}_4$. By distributivity of the dot product, we have

$$\vec{x} \cdot \vec{u} = \vec{x} \cdot (\alpha_1 \vec{y}_1 + \cdots + \alpha_4 \vec{y}_4) = \alpha_1 (\vec{x} \cdot \vec{y}_1) + \cdots + \alpha_4 (\vec{x} \cdot \vec{y}_4) = 0,$$

   so $\vec{x}$ is orthogonal to every vector in $\operatorname{span}(Y)$.

4. (a) ℓ_1 and ℓ_2 are close and ℓ_3 and ℓ_4 are close.

   (b) $\vec{n}_1 \approx \begin{bmatrix} -0.70711 \\ 0.70711 \end{bmatrix}$, $\vec{n}_2 \approx \begin{bmatrix} -0.70675 \\ 0.70746 \end{bmatrix}$, $\vec{n}_3 \approx \begin{bmatrix} -0.0005 \\ 1 \end{bmatrix}$, and $\vec{n}_4 \approx \begin{bmatrix} -0.0003 \\ 1 \end{bmatrix}$.

   (c) $\|\vec{n}_1 - \vec{n}_2\| \approx 0.0005$, $\|\vec{n}_1 - \vec{n}_3\| \approx \|\vec{n}_1 - \vec{n}_4\| \approx \|\vec{n}_2 - \vec{n}_3\| \approx \|\vec{n}_2 - \vec{n}_4\| \approx 0.765$, and $\|\vec{n}_3 - \vec{n}_4\| \approx 0.0017$.

   These distances coincide with my intuitive idea of closeness. Using normal vectors might be preferable to using direction vectors, because it generalizes to planes. A plane has a unique normal direction but infinitely many direction vectors that might be hard to compare.

Learning Objectives

Students need to be able to...

- Apply the definition of orthogonality
- Interpret and apply a new definition, that of *orthogonal sets*
- Set up a system of equations to produce orthogonal vectors

Context

Students will have covered dot products, orthogonality, normal form of lines and planes, and projections in the previous week's lectures.[1]

What to Do

Introduce the learning objectives for the day's tutorial. Explain that we will be extending the notion of orthogonality they know from lecture to sets. Further, explain that one of the skills we're developing in this class is to be able to read, understand, and apply a new definition, and that's what the first tutorial question is all about.

After most groups have finished #1, go over it as a class. There is no need to write a formal proof for part (d), though everyone should be able to give a convincing argument, even if it's not a full "proof". Students might be puzzled on how to explain (b), resorting to "because they are...". Try to push them to refer to the definitions from which they can give a full argument (even if the argument seems silly).

Continue as usual, walking around the room and asking questions while letting students work on the next problem and gathering them together for discussion when most groups have finished.

7 minutes before class ends, pick a suitable problem to do as a wrap-up.

Notes

- Students will have a hard time explaining themselves for 1(d). Make sure you have some prompts in your back pocket.
- Students might get stuck on #2. Remind them to think about the definitions and to start by writing them down—it's amazing how impossible a problem seems before you write down the definition....
- For #3 we want an answer written as a proof. This will be hard for most students.
- #4 is there as a challenge question. You probably won't be talking about it.

[1] Note that if you use the word *projection*, it has a specific definition in this class: $\text{proj}_X \vec{a}$ is the closest point in X to $\vec{a}$. In particular, $\text{proj}_{\vec{b}} \vec{a}$ is not a defined notation, since $\vec{b}$ is not a set. Instead we use the notation $\text{comp}_{\vec{b}} \vec{a}$.