Learning Objectives

In this tutorial you work with *span*, using its definition to justify your claims.

These problems relate to the following Course Learning Objectives: *Clearly and correctly express the mathematical ideas of linear algebra to others, and understand and apply logical arguments and definitions that have been written by others*.

Problems

1. Write down a mathematically-precise definition of "span $\{\vec{u}, \vec{v}, \vec{w}\}$".

2. Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Let $S = \operatorname{span}\{\vec{a}, \vec{b}\}$.

   For each of the following, use the definition of span to justify whether they are in S.

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3. Let $\mathcal{P} \subseteq \mathbb{R}^3$ be the plane with equation $x + y + z = 0$.

   (a) Find a linearly independent set that spans $\mathcal{P}$.

   (b) Find a linearly dependent set that spans $\mathcal{P}$.

4. Correct the statement: *The span of two vectors $\vec{p}, \vec{q} \in \mathbb{R}^3$ is a plane through $\vec{0}$ containing both $\vec{p}$ and $\vec{q}$.*

5. In economics vectors are used to describe consumer preferences. Often times, it is assumed that these preference vectors cannot be multiplied by negative scalars. Thus, for an economist, "the set of all linear combinations" might look different than the span.

   Define the *positive span* of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ as the set of all linear combinations of $\vec{u}$, $\vec{v}$, and $\vec{w}$ with non-negative coefficients.

   (a) Write down the positive span of $\{\vec{u}, \vec{v}, \vec{w}\}$ using set-builder notation.

   (b) Describe the positive span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. How is it different than the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

   (c) Could the positive span of a linearly independent set ever be all of $\mathbb{R}^2$? What about the positive span of a linearly dependent set?

1. $\operatorname{span}\{\vec{u},\vec{v},\vec{w}\}=\{\vec{x}:\vec{x}=\alpha\vec{u}+\beta\vec{v}+\gamma\vec{w} \text{ for some } \alpha,\beta,\gamma\in\mathbb{R}\}$.

2. $\vec{x}\in S$ because $\vec{x}=\vec{a}+\vec{b}$.

   $\vec{y}\notin S$. The last coordinate of $\vec{y}$ is 0, so if $\vec{y}$ were a linear combination of $\vec{a}$ and $\vec{b}$, it would have to take the form $t\vec{a}+0\vec{b}=\begin{bmatrix}t\\t\\0\end{bmatrix}$. But the first and second coordinates of $\vec{y}$ differ, so this is impossible.

   $\vec{z}\notin S$ because $\vec{z}$ has two coordinates and every vector in S has three coordinates.

   $\vec{0}\in S$ because $\vec{0}=0\vec{a}+0\vec{b}$.

3. (a) $\mathcal{P}=\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$.

   (b) $\mathcal{P}=\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\-1\end{bmatrix}\right\}$.

4. "The span of two *linearly independent* vector $\vec{p},\vec{q}\in\mathbb{R}^3$ is *the* plane through the origin containing them."

5. (a) Positive span $\{\vec{u},\vec{v},\vec{w}\}=\{\vec{x}:\vec{x}=\alpha\vec{u}+\beta\vec{v}+\gamma\vec{w} \text{ for some } \alpha,\beta,\gamma\geq 0\}$.

   (b) The *positive span* of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ is the first quadrant of the xy-plane, whereas the *span* is the entire xy-plane.

   (c) The positive span of linearly independent vectors could never be all of $\mathbb{R}^2$. Let $\vec{u},\vec{v}\in\mathbb{R}^2$ be linearly independent. In this case, $\vec{u}\neq 0$ and $\vec{u}\neq t\vec{v}$ for any t. Therefore, the positive span of $\vec{u}$ and $\vec{v}$ cannot contain the vector $-\vec{u}\in\mathbb{R}^2$, and so the positive span of $\vec{u}$ and $\vec{v}$ is not all of $\mathbb{R}^2$.

   However, the positive span of a linearly dependent set could be all of $\mathbb{R}^2$. For example, $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix}\right\}$ has positive span equal to $\mathbb{R}^2$.

Learning Objectives

Students need to be able to...

- write precise mathematical definitions of concepts used in class
- distinguish between a concept being “familiar” and actually knowing it
- determine whether vectors are in a span or not
- understand that you can take the span of linearly dependent sets (which might be a point of confusion when bases come around).

Context

Students have spent several days on span and have gone over the definition of linearly independent and dependent. They have just started dot products and orthogonality.

What to Do

Introduce the learning objectives for the day’s tutorial. Explain that linear algebra relies heavily on knowing precise definitions and even though you might “feel” like you know a concept, you don’t really know it if you cannot write it down.

Have students pair up and ask them to actually write down their definition for problem 1. Many will be tempted to “do it in their head”. Don’t let them!

When many groups have something written, partition the board and invite several groups to write their definition on the board. Make sure to include a wrong definition (which will be most of them). Have the class read each definition and take a moment to vote “totally correct”, “somewhat correct”, or ‘totally incorrect” for each definition. Then, discuss what’s wrong with each definition and how it might be fixed (if possible). *Make it clear that on the midterms only “totally correct” definitions will get points; even a definition with 99% of the correct words might get a zero if it is not totally correct.*

After this discussion, have students move on to number 2. When most students have finished, have a quick discussion of number 2, rinse, and repeat.

7 minutes before the end of tutorial, pick a problem most groups have started working on and do it as a wrap-up. Remember, the goal of tutorial is not to get through all the problems, and don’t go over a problem the groups have not started yet—that won’t be helpful to them.

Notes

1. Most groups will have something wrong with their definition of span. Don’t let them off the hook—make sure they understand that for definitions, there is no “close”.
2. Students wont know the difference between

$$\{\vec{x} : \vec{x} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} \text{ for some } \alpha, \beta, \gamma \in \mathbb{R}\}$$

and

$$\{\vec{x} : \vec{x} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} \text{ for all } \alpha, \beta, \gamma \in \mathbb{R}\}.$$

If this point comes up, emphasize it.

3. If they're stuck on problem 2, remind them they can always use systems of linear equations to answer this question. They know how to solve *any* system by now.

4. If they're stuck on problem 3, ask them to write the plane down in vector form and then ask how the direction vectors might relate to the question.