Learning Objectives

> In this tutorial you will be constructing matrices and linear transformations that satisfy given conditions, or explaining why they don't exist.
>
> These problems relate to the following course learning objectives: *Use matrices to solve problems*, *translate between algebraic and geometric viewpoints to solve problems*, and *clearly and correctly express the mathematical ideas of linear algebra to others*.

1. Write mathematically precise definitions of the rank of a matrix A and the rank of a linear transformation $\mathcal{T}$.

2. Give an example of a 2×3 matrix A with the specified rank, or explain why it cannot exist.

   (a) $\text{rank}(A) = 1$

   (b) $\text{rank}(A) = 2$

   (c) $\text{rank}(A) = 3$

   (d) $\text{rank}(A) = 0$

3. For a linear transformation $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^m$, explain how $\text{rank}(\mathcal{L})$ relates to m or n under the following conditions:

   (a) $\mathcal{L}$ is one-to-one.

   (b) $\mathcal{L}$ is **not** one-to-one.

   (c) $\mathcal{L}$ is onto.

   (d) $\mathcal{L}$ is **not** onto.

4. Give examples of linear transformations $\mathcal{T}, \mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfy the following, or explain why they cannot exist.

   (a) $\text{rank}(\mathcal{T}) = \text{rank}(\mathcal{S}) = \text{rank}(\mathcal{S}\circ\mathcal{T}) = 2$

   (b) $\text{rank}(\mathcal{T}) = \text{rank}(\mathcal{S}) = 2$, and $\text{rank}(\mathcal{S}\circ\mathcal{T}) = 1$

   (c) $\text{rank}(\mathcal{T}) = \text{rank}(\mathcal{S}) = 2$, and $\text{rank}(\mathcal{S}\circ\mathcal{T}) = 3$

   (d) $\text{rank}(\mathcal{T}) = 2$, $\text{rank}(\mathcal{S}) = 1$ and $\text{rank}(\mathcal{S}\circ\mathcal{T}) = 0$

5. Tommy has returned once again and is working on similar matrices.

   (a) Write down a mathematically precise definition for two matrices to be similar.

   (b) Tommy began with a 3×3 matrix A and multiplied by a change of basis matrix X to find $B = XAX^{-1}$. His matrix computations gave

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Unfortunately, Tommy lost his paper containing X. Can you help him by finding a change of basis matrix X that gives this solution or explaining to Tommy why no such matrix exists?

1. The rank of a matrix A is the number of pivots in rref(A); the rank of a linear transformation $\mathcal{T}$ is the dimension of the range of $\mathcal{T}$.

2. (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is one example. There are many others with linearly dependent rows, not both zero.

   (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is one example. There are many others with linearly independent rows.

   (c) Such a matrix does not exist. A only has two rows, so rref(A) can have at most two pivot positions. Equivalently, the columnspace of A is a subspace of $\mathbb{R}^2$, so it can't be 3 dimensional.

   (d) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the only example. There are no others.

3. A linear transformation $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if its nullity is 0, and it is onto if its range is all of $\mathbb{R}^m$.

   (a) Using the rank-nullity theorem, we have $\operatorname{rank}(\mathcal{L}) = n$.

   (b) Similarly, if $\mathcal{L}$ is not one-to-one, we have $\operatorname{rank}(\mathcal{L}) < n$. Note that we can never have $\operatorname{rank}(\mathcal{L}) > n$ (why?).

   (c) $\mathcal{L}$ is onto if $\operatorname{rank}(\mathcal{L}) = m$.

   (d) $\mathcal{L}$ is not onto if $\operatorname{rank}(\mathcal{L}) < m$.

4. (a) Since the ranks of $\mathcal{T}$ and $\mathcal{S}$ are both 2, their ranges are both planes in $\mathbb{R}^3$. Let $\mathcal{T} = \mathcal{S}$ be projection onto the xy-plane. Then, $\mathcal{S} = \mathcal{S} \circ \mathcal{T}$ is rank 2.

   (b) Again, the ranges of $\mathcal{T}$ and $\mathcal{S}$ must be planes, but this time we want the range of $\mathcal{S} \circ \mathcal{T}$ to be a line. Let $\mathcal{T}$ be projection onto the xy-plane and let $\mathcal{S}$ be projection onto the xz-plane. Then $\mathcal{S} \circ \mathcal{T}$ is projection onto the x-axis, which has rank 1.

   (c) This is impossible. Since $\operatorname{range}(\mathcal{S} \circ \mathcal{T}) \subseteq \operatorname{range}(\mathcal{S})$, we cannot have

   $$3 = \operatorname{rank}(\mathcal{S} \circ \mathcal{T}) \ > \ \operatorname{rank}(\mathcal{S}) = 2.$$

   (d) The range of $\mathcal{T}$ is a plane, while the range of $\mathcal{S}$ is a line, and we would like $\mathcal{S} \circ \mathcal{T}$ to send every vector to 0. Let $\mathcal{S}$ be projection onto the xy-plane and let $\mathcal{T}$ be projection onto the z-axis. Then $\operatorname{null}(\mathcal{S}) = \operatorname{range}(\mathcal{T})$ and so $\mathcal{S} \circ \mathcal{T} = \mathbf{0}$, the transformation which sends every vector to zero.

5. Sorry Tommy, there must be a mistake somewhere in your lost notes. Comparing ranks, $\operatorname{rank}(A) = 3$ and $\operatorname{rank}(B) = 2$. Since X is invertible, it does not change the dimension of the image of any subspace, so for similar matrices $A \sim B$, we must have $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Learning Objectives

Students need to be able to...

- Define rank (algebraically for matrices and geometrically for linear transformations)
- Produce examples of linear transformations
- Describe connection between matrix multiplication and composition of functions.
- Compute the image of a set of vectors under a transformation.

Context

What to Do

- Give the students a couple of minutes to think about problem 1. Then invite four of them up to the board. Have two define "rank" of a matrix and two define "rank" of a linear transformation. Discuss any issues that their definitions may have. Make sure to end this portion with correct, clear definitions on the board. And make sure that the students understand them.
- Next, arrange the students in groups and give them 10-12 minutes to work on problem 2 and 4. Be sure to walk around the room and try to get the students engaged and working. Once time is up, spend a few minutes at the board addressing any glaring issues you might've noticed while you were circling the room.
- Repeat the same process with either problem 3 or 5. Ask the students to see if they prefer one problem over the other. Wrap-up the tutorial by discussing any issues you noticed while circling the room.

Notes

- Be mindful of late-comers. During group work, try to approach them and make sure they are aware of what is going on, and that they are engaged and working in a group.
- Try to keep an organized board. A disorganized and chaotic board is just as annoying to the students as a jumbled mess on a test paper that you're trying to grade is to you.