Inquiry Based Linear Algebra

About the Document

This document was originally designed in the spring of 2016 to guide students through an ten week Linear Algebra course (Math 281-3) at Northwestern University.

A typical class day using the problem-sets:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.
2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.
4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

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Sets of Vectors

1 Write the following sets in set-builder notation

1.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

1.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Unions & Intersections (DEFINITION)

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)} \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

2 Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4, 5\}$ and $Z = \{4, 5, 6\}$. Compute

2.1 $X \cup Y$

2.2 $X \cap Y$

2.3 $X \cup Y \cup Z$

2.4 $X \cap Y \cap Z$

3 Draw the following subsets of $\mathbb{R}^2$.

3.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.3 $J = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.4 $V \cup H$.

3.5 $V \cap H$.

3.6 Does $V \cup H = \mathbb{R}^2$?

Linear Combinations, Span, and Linear Independence

Linear Combination (DEFINITION)

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars.

4 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

4.1 Write the coordinates of $\vec{w}$.

4.2 Draw a picture with $\vec{w}$, $\vec{v}_1$, and $\vec{v}_2$.

4.3 Is $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.4 Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.5 Is $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.7 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Span

DEF

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

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5 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

5.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

5.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

5.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

5.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

5.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

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6 Give an example of:

6.1 two vectors in $\mathbb{R}^3$ that span a plane;

6.2 two vectors in $\mathbb{R}^3$ that span a line;

6.3 four vectors in $\mathbb{R}^3$ that span a plane;

6.4 a set of 50 vectors in $\mathbb{R}^3$ whose span is the line through the origin and the point $\begin{bmatrix}1\\2\\-3\end{bmatrix}$.

In some sets, every vector is essential for computing a span. In others, there are "excess" vectors. This leads us to the concept of linear independence.

Linearly Dependent & Independent

DEFINITION

We say $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\},$$

and a set is ***linearly independent*** otherwise.

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7 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

7.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

7.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

7.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

7.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

Trivial Linear Combination

DEF

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

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8 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

8.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

8.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

Linearly Dependent & Independent

DEF

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector.

9

9.1 Explain how this new definition implies the old one.

9.2 Explain how the old definition implies this new one.

Since we have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

10

Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

10.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

10.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

10.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

10.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Subspaces and Bases

Subspace

DEFINITION

A ***subspace*** $V \subseteq \mathbb{R}^n$ is a subset such that

(i) $\vec{u}, \vec{v} \in V$ implies $\vec{u} + \vec{v} \in V$.

(ii) $\vec{u} \in V$ implies $k\vec{u} \in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

11

For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

11.1 $A = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ for some $a \in \mathbb{Z}\}$.

11.2 $B = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$.

11.3 $C = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.4 $D = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.5 $E = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.6 $F = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.7 $G = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

11.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Basis

DEF A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

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12 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

12.1 Describe V.

12.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

12.3 Give a basis for V.

12.4 Give another basis for V.

12.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

Dimension

DEF The ***dimension*** of a subspace V is the number of elements in a basis for V.

12.6 What is the dimension of V?

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13 Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}7\\8\\8\end{bmatrix}$ and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

13.1 Give a basis for and the dimension of P.

13.2 Give a basis for and the dimension of Q.

13.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

13.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

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14 Consider the system

$$\begin{aligned} x - y &= 2 \\ 2x + y &= 1 \end{aligned} \tag{1}$$

14.1 Draw the lines in (1) on the same coordinate plane.

14.2 Algebraically solve the system (1). What does this solution represent on your graph?

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15 Let L be the line given by $x - y = 2$.

15.1 Write an equation of a line that doesn't intersect L.

15.2 Write an equation of a line that intersects L in

(a) one place.

(b) infinitely many places

(c) exactly two places

or explain why no such equation exists.

15.3 For each equation you came up with, solve the system algebraically. How can you tell algebraically how many solutions there are?

The Row Reduction Algorithm

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16 16.1 Solve the system

$$\begin{aligned} x - y - 2z &= -5 \\ 2x + 3y + z &= 5 \\ 0x + 2y + 3z &= 8 \end{aligned} \tag{2}$$

any way you like.

16.2 Use an augmented matrix to solve the system (2).

The system (2) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (2) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

16.3 Rewrite (2) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

16.4 If (x, y, z) is a solution to (2), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

16.5 If (x, y, z) is a solution to (2), is $\vec{p} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

---

17 Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

17.1 Write the system of equations corresponding to A.

17.2 Solve the system of equations corresponding to A.

Infinite Solutions

18 Consider the system

$$\begin{aligned} x &+ 2y &= 3 \\ 2x &+ 4y &= 6 \end{aligned} \tag{3}$$

18.1 How many solutions does (3) have?

18.2 Write the solutions to (3) in vector form.

18.3 What happens when you use an augmented matrix to solve (3)?

Free Variables

19 Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make $2-1=1$ choices when writing a solution. Let's make the choice $y=t$.

19.1 With the added equation $y=t$, solve the system represented by B.

20 Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

and call the variables in this system x_1, x_2, x_3, x_4, x_5.

20.1 Write the system of equations represented by C.

20.2 Identify how many choices you can make when writing down a solution corresponding to C.

20.3 Add one equation (of the form $x_i = t$ or $x_j = s$, etc.) for each choice you must make when solving the system.

20.4 Write in vector form all solutions to C.

21 21.1 An unknown system U is represented by an augmented matrix with 4 rows and 6 columns. What is the minimum number of free variables solutions to U will have?

21.2 An unknown system V is represented by an augmented matrix with 6 rows and 4 columns. What is the minimum number of free variables solutions to V will have?

22 **Homogeneous**

DEF A system is called ***homogeneous*** if all equations equal 0.

Let A be an unknown system of 3 equations and 3 variables and suppose $(x,y,z)=(1,2,1)$ and $(x,y,z)=(-1,1,1)$ are solutions to A.

22.1 Can you produce another solution to the system?

22.2 Can you produce a solution to the homogeneous version of A (the version of A where every equation equals 0)?

22.3 Suppose when you use an augmented matrix to solve the system A, you only have one free variable. Could A be homogeneous? Can you produce all solutions to the system A?

Rank

Rank

DEF The ***rank*** of the matrix A is the number of leading ones in the reduced row echelon form of A.

23

23.1 Determine the rank of (a) $\begin{bmatrix}1&1\\2&2\end{bmatrix}$ (b) $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ (c) $\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$ (d) $\begin{bmatrix}3\\3\\2\end{bmatrix}$ (e) $\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

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24 Consider the homogeneous system

$$\begin{array}{cccc} x & +2y & +z & =0\\ x & +2y & +3z & =0\\ -x & -2y & +z & =0 \end{array} \tag{4}$$

and the non-augmented matrix of coefficients $A=\begin{bmatrix}1&2&1\\1&2&3\\-1&-2&1\end{bmatrix}$.

24.1 What is rank(A)?

24.2 Give the general solution to (4).

24.3 Are the column vectors of A linearly independent?

24.4 Give a non-homogeneous system with the same coefficients as (4) that has

(a) infinitely many solutions

(b) no solutions.

25

25.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

25.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Span Again

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26 Consider the system

$$\begin{array}{cccc} x & -y & -z & =0\\ 0x & +1y & +2z & =0\\ 3x & -3y & +3z & =0 \end{array} \tag{5}$$

which has the unique solution $(x,y,z)=(0,0,0)$.

26.1 Give vectors $\vec{u},\vec{v},\vec{w}$ so that the system (5) corresponds to the vector equation $x\vec{u}+y\vec{v}+z\vec{w}=\vec{0}$.

26.2 Is $\vec{w}\in\operatorname{span}\{\vec{u},\vec{v}\}$? If so, write it as a linear combination of $\vec{u}$ and $\vec{v}$.

The matrix M is the non-augmented matrix corresponding to a homogeneous system of linear equations. M also corresponds to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$. Further, we know

$$\operatorname{rref}(M)=\begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\end{bmatrix}.$$

26.3 Give a solution to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$.

26.4 Is $\vec{c}\in\operatorname{span}\{\vec{a},\vec{b}\}$? If so, write it as a linear combination of $\vec{a}$ and $\vec{b}$.

26.5 Do you have enough information to tell if $\{\vec{a},\vec{b}\}$ is linearly independent? Why or why not?

27 Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$ you have no free variables.

27.1 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$, the second column corresponds to a free variable.

27.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

27.3 Is $\{\vec{u},\vec{w}\}$ linearly independent?

27.4 Is $\{\vec{u},\vec{v}\}$ linearly independent?

Maximal Linearly Independent Subset

DEF Given a set of vectors X, a ***maximal linearly independent subset*** of X is a linearly independent subset $V\subseteq X$ with the most possible vectors in it (i.e., if you took any subset of X with more vectors, it would be linearly dependent).

28

28.1 Give a maximal linearly independent subset, T, of $\left\{\begin{bmatrix}a\\b\\c\end{bmatrix} : a,b,c\in\mathbb{R}\right\}$.

28.2 What is the size of T?

29 Consider the vectors

$$\vec{v}_1=\begin{bmatrix}1\\2\\1\end{bmatrix}\qquad \vec{v}_2=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\qquad \vec{v}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}\qquad \vec{v}_4=\begin{bmatrix}-1\\2\\0\end{bmatrix}\qquad \vec{v}_5=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

and the matrices

$$A=\begin{bmatrix}1&-1&0&-1&1\\2&-1&1&2&-1\\1&-1&0&0&1\end{bmatrix}\qquad \mathrm{rref}(A)=\begin{bmatrix}1&0&1&0&-2\\0&1&1&0&-3\\0&0&0&1&0\end{bmatrix}.$$

(Notice that the columns of A are the vectors $\vec{v}_1,\dots,\vec{v}_5$)

29.1 Is $V=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4,\vec{v}_5\}$ linearly independent?

29.2 Pick a maximal linearly independent subset of V.

29.3 Pick another (different) maximal linearly independent subset of V.

29.4 Give a basis for $\mathrm{span}(V)$.

29.5 What is the dimension of $\mathrm{span}(V)$?

30

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

30.1 Write the shape of the matrices A, B, C (i.e., for each one, write the dimensions in $m \times n$ form).

30.2 List *all* products between the matrices A, B, C that are defined. (Your list will be some subset of AB, AC, BA, CA, BC, CB.)

30.3 Compute AC and CA.

31 31.1 If the matrices X and Y are both square $n \times n$ matrices, does $XY = YX$? Explain.

31.2 If the matrices X and Y are both square $n \times n$ matrices, does $X + Y = Y + X$? Explain.

32 Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

32.1 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

32.2 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

33 The entries of a matrix are specified by (row,column) pairs of integers. If a_{ij} is the (i, j) entry of a matrix A, we may write $A = [a_{ij}]$.

33.1 Write the 2×2 matrix A with entries $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 7$ and $a_{22} = 9$.

33.2 Let $B = [b_{ij}]$ be the 3×3 matrix where $b_{ij} = i + j$. Write B.

33.3 Let $C = [c_{ij}]$ be the 3×4 matrix where $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ if $i \neq j$.

34 DEF The ***transpose*** of a matrix $A = [a_{ij}]$ is the matrix $A^T = [a_{ji}]$.

Visually, the transpose of a matrix swaps rows and columns.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

34.1 What is the shape of A and A^T?

34.2 Write down A^T.

B and D are 4×6 matrices and C is a 6×4 matrix.

34.3 Does $(BC)^T = B^T C^T$? Explain.

34.4 Does $(B + D)^T = B^T + D^T$? Explain.

34.5 Compute AA^T and $A^T A$ (where A is the matrix defined earlier). What do you notice?

35

DEF A matrix X is called ***symmetric*** if $X = X^T$.

Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

35.1 Prove that if W is a square matrix, then $V = W^T W + W + W^T$ is a symmetric matrix.

36

DEF A ***zero matrix*** is a matrix whose entries are all zeros. An ***identity matrix*** is a square matrix whose diagonal entries are 1 and non-diagonal entries are 0.

We write the $m \times n$ zero matrix as $0_{m\times n}$ or just 0 if the shape is determined by context. The $n \times n$ identity matrix is notated $I_{n\times n}$ or just I if the shape is determined by context.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

36.1 Write down the 3×3 identity matrix and the 3×3 zero matrix.

36.2 Compute $I_{3\times3}A$, $AI_{3\times3}$, $0_{3\times3}A$, and $A0_{3\times3}$.

36.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

37

37.1 Solve the matrix equation

$$I_{4\times4}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

38 $\mathcal{R}:\mathbb{R}^2\to\mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

38.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

38.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

38.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

38.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter clockwise by 90°.

39 $\mathcal{S}:\mathbb{R}^3\to\mathbb{R}^3$ stretches in the $\hat{\mathbf{z}}$ direction by a factor of 2 and contracts in the $\hat{\mathbf{y}}$ direction by a factor of 3.

39.1 Write a matrix representation of $\mathcal{S}$.

Linear Transformation

DEFINITION

If V and W are vector spaces, a function $T:V\to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v})=T\vec{u}+T\vec{v}\qquad\text{and}\qquad T(\alpha\vec{v})=\alpha T\vec{v}$$

for all vectors $\vec{u},\vec{v}\in V$ and all scalars α.

40 40.1 Classify the following as linear transformation or not

(a) $\mathcal{R}$ from above.

(b) $\mathcal{S}$ from above.

(c) $W:\mathbb{R}^2\to\mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x^2\\y\end{bmatrix}$.

(d) $T:\mathbb{R}^2\to\mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2\\y\end{bmatrix}$.

(e) $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix}=\mathrm{proj}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

It turns out every linear transformation can be written as a matrix (in fact this is why matrix multiplication was invented).

41 Define $\mathcal{P}$ to be projection onto $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

41.1 Write down a matrix for $\mathcal{P}$.

41.2 What is the rank of the matrix corresponding to $\mathcal{P}$?

Matrix multiplication was designed to exactly model composition of linear transformations.

41.3 Write down a matrix for $\mathcal{P}$ and for $\mathcal{R}$, the counter-clockwise rotation by 90°.

41.4 Write down matrices for $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{P}$.

Range

DEF

The ***range*** (or ***image***) of a linear transformation $T:V\to W$ is the set of vectors that T can output. That is,

$$\mathrm{range}(T)=\{\vec{y}\in W:\vec{y}=T\vec{x}\text{ for some }\vec{x}\in V\}.$$

Null Space

DEF

The ***null space*** (or ***kernel***) of a linear transformation $T:V\to W$ is the set of vectors that get mapped to zero under T. That is,

$$\mathrm{null}(T)=\{\vec{x}\in V:T\vec{x}=\vec{0}\}.$$

42 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the vector $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

42.1 What is the range of $\mathcal{P}$?

42.2 What is the null space of $\mathcal{P}$?

43 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

43.1 Show that the null space of T is a subspace.

43.2 Show that the range of T is a subspace.

Fundamental Subspaces

DEF

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M is the span of the rows of M; the ***column space*** of M is the span of the columns of M; and the ***null space*** of M is the set of solutions to $M\vec{x} = \vec{0}$.

44 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

44.1 Describe the row space of A.

44.2 Describe the column space of A.

44.3 Is the row space of A the same as the column space of A?

44.4 Describe the set of all vectors perpendicular to the rows of A.

44.5 Describe the null space of A.

45

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

45.1 How does the row space of B relate to the row space of C?

45.2 How does the null space of B relate to the null space of C?

45.3 Compute the null space of B.

46

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \operatorname{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

46.1 How does the column space of P relate to the column space of Q?

46.2 Describe the column space of P and the column space of Q.

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of columns in } A.$$

47 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

47.1 What is the rank and nullity of A^T?

47.2 What is the rank and nullity of A?

Matrix Inverses

48 48.1 Apply the row operation $R_3 \to R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

48.2 Apply the row operation $R_3 \to R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

DEF An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

48.3 Compute $E_1 A$ and $E_2 A$. How do the resulting matrices relate to row operations?

48.4 Without computing, what should the result of applying the row operation $R_3 \to R_3 - 2R_1$ to E_1 be? Compute and verify.

48.5 Without computing, what should $E_1 E_2$ be? What about $E_2 E_1$? Now compute and verify.

DEF The ***inverse*** of an $n \times n$ matrix A is an $n \times n$ matrix B such that $AB = I_{n\times n} = BA$. In this case, B is called the inverse of A and is notated as A^{-1}.

49 Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

49.1 Which pairs of matrices above are inverses of each other?

50

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

50.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

50.2 What is $E_2 E_1 B$?

50.3 Find B^{-1}.

50.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

51

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

51.1 What is $A^{-1}A$?

51.2 What is rref(A)?

51.3 What is rref(C)? (Hint, there is no need to actually do row reduction!)

51.4 Solve the system $A\vec{x} = \vec{b}$.

52 52.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

52.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

Change of Basis

53 Let $\vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, $\vec{c} = \begin{bmatrix}4\\0\end{bmatrix}$, and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$.

53.1 Is $\mathcal{B}$ a basis for $\mathbb{R}^2$?

53.2 Find coefficients α_1 and α_2 so that $\vec{c} = \alpha_1\vec{b}_1 + \alpha_2\vec{b}_2$.

We call the vector $\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$ the representation of $\vec{c}$ in the $\mathcal{B}$ basis and notate this by $[\vec{c}]_{\mathcal{B}} = \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$.

53.3 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

Let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

53.4 Compute $X[\vec{c}]_{\mathcal{B}}$. What do you notice?

54 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}.$$

54.1 Should X^{-1} exist? Explain.

54.2 Consider the equation

$$X^{-1}[\vec{v}]_? = [\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

54.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

55 Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

55.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

55.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}5\\3\end{bmatrix}$.

55.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

55.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

55.5 Find a matrix for T in the $\mathcal{C}$ basis.

55.6 Find a matrix for T in the standard basis.

DEFINITION — Similar Matrices

A matrix A and a matrix B are ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

$$A = XBX^{-1}.$$

Unit n-cube

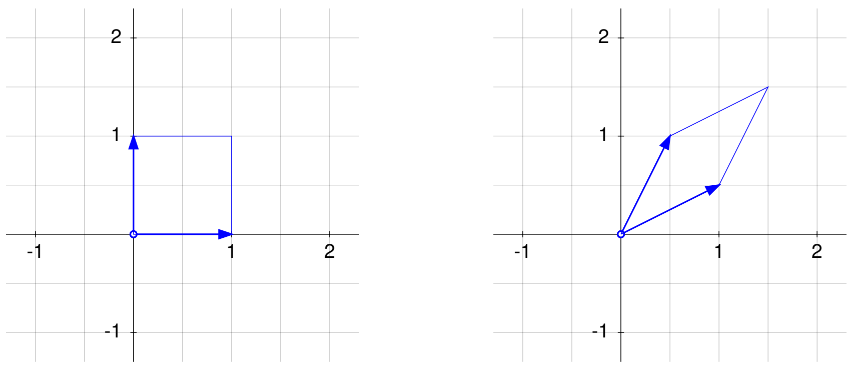
DEFINITION

The unit n-cube is the n-dimensional cube with side length 1 and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The volume of the unit n-cube is always 1.

56 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



56.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

56.2 Write down a matrix for T.

56.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may need to use trigonometry.

Determinant

DEF

The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors or the row vectors.

57 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

57.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

57.2 Compute the area of $A(C_2)$.

57.3 Compute $\det(A)$.

58 Suppose R is a rotation counterclockwise by 30°.

58.1 Draw C_2 and $R(C_2)$.

58.2 Compute the area of $R(C_2)$.

58.3 Compute $\det(R)$.

59 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\qquad\text{and}\qquad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

59.1 What is $\det(F)$?

60

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \to R_1 + 2R_2$ applied.

60.1 What is $\det(E_f)$?

60.2 What is $\det(E_m)$?

60.3 What is $\det(E_a)$?

60.4 What is $\det(E_f E_m)$?

60.5 What is $\det(4I_{3\times 3})$?

60.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

61
$$U=\begin{bmatrix}1&2&1&2\\0&3&-2&4\\0&0&-1&0\\0&0&0&4\end{bmatrix}$$

61.1 What is $\det(U)$?

61.2 V is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

61.3 P is projection onto the vector $\begin{bmatrix}-1\\-1\end{bmatrix}$. What is $\det(P)$?

62 Suppose you know $\det(X) = 4$.

62.1 What is $\det(X^{-1})$?

62.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

62.3 Suppose Y is not invertible. What is $\det(Y)$?

After all this work with determinants, we see that (like dot products) there is a geometric and an algebraic way of thinking about them, and they *determine* if a matrix is invertible.

63

63.1 The linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a change of coordinates and $\det(L) = -4$. What is the volume form for this change of coordinates?

63.2 Suppose $P : \mathbb{R}^2 \to \mathbb{R}^2$ is the parameterization defined by $P\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1&2\\3&9\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}$. Find the volume form for P.

63.3 Suppose $p : \mathbb{R}^2 \to \mathbb{R}^2$ is the parameterization defined by $p(r,\theta) = (r\cos\theta, r\sin\theta)$. Find a linear approximation to p at the point (r_0,θ_0). Use determinants to compute the volume form for p at (r_0,θ_0).

DEFINITION

Jacobian

Let $p : \mathbb{R}^n \to \mathbb{R}^n$ be a parameterization. Let $L_{\vec{x}_0}(\vec{x}) = J_{\vec{x}_0}\vec{x} + \vec{q}_{\vec{x}_0}$ be the linear approximation to p at the point $\vec{x}_0$. The ***Jacobian*** of p at the point $\vec{x}_0$ is defined to be

$$\mathrm{Jacob}_{\vec{x}_0}(p) = \det(J_{\vec{x}_0}).$$

Eigenvectors

Eigenvector

DEFINITION

For a linear transformation X, an ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

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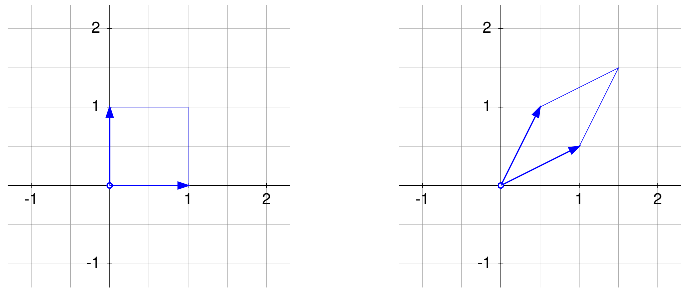
64 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



64.1 Give an eigenvector for T. What is the eigenvalue?

64.2 Can you give another?

---

65 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

65.1 Give an eigenvector and a corresponding eigenvalue for A.

65.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

65.3 What is the dimension of $\mathrm{null}(B)$?

65.4 What is $\det(B)$?

---

66 Let $C = \begin{bmatrix}-1 & 2\\ 1 & 0\end{bmatrix}$ and $E_\lambda = C - \lambda I$.

66.1 For what values of λ does E_λ have a non-trivial null space?

66.2 What are the eigenvalues of C?

66.3 Find the eigenvectors of C.

Characteristic Polynomial

DEF

For a matrix A, the ***characteristic polynomial*** of A is

$$\mathrm{char}(A) = \det(A - \lambda I).$$

---

67 Let $D = \begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$.

67.1 Compute $\mathrm{char}(D)$.

67.2 Find the eigenvalues of D.

---

68 Suppose $\mathrm{char}(E) = \lambda(\lambda-2)(\lambda+3)$ for some unknown 3×3 matrix E.

68.1 What are the eigenvalues of E?

68.2 Is E invertible?

68.3 What is $\mathrm{nullity}(E)$, $\mathrm{nullity}(E-3I)$, $\mathrm{nullity}(E+3I)$?

69 Consider

$$A=\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}\qquad \vec v_1=\begin{bmatrix}1\\1\\1\end{bmatrix}\qquad \vec v_2=\begin{bmatrix}1\\1\\-2\end{bmatrix}\qquad \vec v_3=\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

and notice that $\vec v_1,\vec v_2,\vec v_3$ are eigenvectors for A.

69.1 Find the eigenvalues of A.

69.2 Find the characteristic polynomial of A.

69.3 Compute $A\vec w$ where $w=2\vec v_1-\vec v_2$.

69.4 Compute $A\vec u$ where $\vec u=a\vec v_1+b\vec v_2+c\vec v_3$ for unknown scalar coefficients a,b,c.

Notice that $\mathcal{V}=\{\vec v_1,\vec v_2,\vec v_3\}$ is a basis for $\mathbb{R}^3$.

69.5 If $[\vec x]_{\mathcal{V}}=\begin{bmatrix}1\\3\\4\end{bmatrix}$ is $\vec x$ written in the $\mathcal{V}$ basis, compute $A\vec x$ in the $\mathcal{V}$ basis.

70 The transformation P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation (where $\mathcal{V}$ is from above).

70.1 Describe in words what P does.

70.2 Describe how you can use P and P^{-1} to easily compute $A\vec y$ for any $\vec y\in\mathbb{R}^3$.

70.3 Can you find a matrix D so that

$$PDP^{-1}=A?$$

70.4 $[\vec x]_{\mathcal{V}}=\begin{bmatrix}1\\3\\4\end{bmatrix}$. Compute $A^{100}\vec x$.

71 For an $n\times n$ matrix T, suppose its eigenvectors $\{\vec v_1,\dots\vec v_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1,\dots,\lambda_n$ be the corresponding eigenvalues.

71.1 Is T diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

71.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

71.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Eigenspace

DEFINITION

Let A be a matrix with eigenvalues $\{\lambda_1,\dots,\lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A-\lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x-\lambda_i$ occurs as a factor).

72 Define $F=\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

72.1 Is F diagonalizable? Why or why not?

72.2 What is the geometric and algebraic multiplicity of each eigenvalue of F?

72.3 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?

Orthogonality

Orthogonal & Orthonormal

DEF A set of vectors is ***orthogonal*** if every pair of vectors in the set is orthogonal. A set of vectors is ***orthonormal*** if it is both an orthogonal set and every vector is a unit vector.

73

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} \qquad \vec{b}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

The matrix $A = [\vec{b}_1 | \vec{b}_2]$ takes vectors in the $\mathcal{B}$ basis and rewrites them in the standard basis.

73.1 What does A^{-1} do?

73.2 Find a matrix B that takes vectors in the standard basis and rewrites them in the $\mathcal{B}$ basis.

73.3 Write $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the $\mathcal{B}$ basis.

73.4 What is the relationship between A and B?

Orthogonal Matrix

DEF An ***orthogonal matrix*** is a square matrix whose columns are orthonormal (Yes, a better name would be orthonormal matrix, but that is not the term the rest of the world uses).

74 Suppose $X = [\vec{x}_1 | \vec{x}_2 | \vec{x}_3 | \vec{x}_4]$ is an orthogonal matrix.

74.1 What is the shape of X (i.e., it is a what×what matrix)?

74.2 Compute $X^T X$.

74.3 What is X^{-1}?

75

$$Y = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

75.1 Is Y an orthogonal matrix?

75.2 Fix Y so it is an orthogonal matrix. Call the new matrix X.

75.3 Compute X^{-1}.

75.4 Compute Y^{-1}.

75.5 Compute $|\det(X)|$ and $|\det(Y)|$ (the absolute value of the determinant of X and Y).

Matrix equations involving orthogonal matrices are easy to solve because the inverse of an orthogonal matrix is so easy to compute!

76 Let $A = [\vec{a}_1 | \vec{a}_2 | \vec{a}_3 | \vec{a}_4]$ be an orthogonal matrix.

76.1 Explain why $\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b} \\ \vec{a}_2 \cdot \vec{b} \\ \vec{a}_3 \cdot \vec{b} \\ \vec{a}_4 \cdot \vec{b} \end{bmatrix}$ is a solution to $A\vec{x} = \vec{b}$.

76.2 Find scalars a, b, c, d so $\vec{b} = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3 + d\vec{a}_4$ (your answers will have variables in them).

Orthogonal matrices also allow us to compute projections quite easily.

Orthogonal Projection

DEF If V is a subspace of $\mathbb{R}^n$, the ***projection*** (sometimes called the orthogonal projection) of $\vec{x}$ onto V is the closest point in V to $\vec{x}$. We notate the projection of $\vec{x}$ onto V as $\text{proj}_V\vec{x}$.

Projections are normally hard to compute and a priori might require some sort of calculus-style optimization to find. However, from geometry we know that if we travel from $\text{proj}_V\vec{x}$ to $\vec{x}$, we should always trace out a path perpendicular to V. Otherwise, we could find a point in V that was slightly closer to $\vec{x}$, violating the definition of $\text{proj}_V\vec{x}$. Thus, orthogonality will be our savior.

---

77 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis.

77.1 If $\vec{x} = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$, find the projection of $\vec{x}$ onto the xy-plane.

Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

77.2 If $\vec{y} = 3\vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$, find the projection of $\vec{y}$ onto $\text{span}\{\vec{b}_1, \vec{b}_3\}$.

Suppose $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is a basis for $\mathbb{R}^3$ with

$$\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1 \qquad \vec{c}_1\cdot\vec{c}_2 = 0 \qquad \vec{c}_1\cdot\vec{c}_3 = 0 \qquad \vec{c}_2\cdot\vec{c}_3 = \sqrt{2}/2.$$

77.3 If $\vec{z} = 5\vec{c}_1 + 2\vec{c}_2 - \vec{c}_3$, find the projection of $\vec{z}$ onto $\text{span}\,\{\vec{c}_1, \vec{c}_2\}$.

---

78 Let's put this all together. $\mathcal{B} = \left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^3$. Let $\mathcal{P}$ be the plane defined by

$$0x + y - z = 0.$$

78.1 Write $\mathcal{P}$ in vector form (Hint: think about the vectors listed in the $\mathcal{B}$ basis).

78.2 Find an orthonormal basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ for $\mathbb{R}^3$ so $\mathcal{P} = \text{span}\,\{\vec{c}_1, \vec{c}_2\}$.

78.3 Let $\vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Find $\text{proj}_{\mathcal{P}}\vec{x}$.

Gram-Schmidt Orthogonalization

We've seen how useful orthonormal bases are. The incredible thing is that we can turn any basis into an orthonormal basis through a process called Gram-Schmidt orthogonalization.

---

79 Let $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}3\\1\end{bmatrix}$.

79.1 Draw $\vec{a}$ and $\vec{b}$ and find $\vec{w} = \text{proj}_{\vec{b}}\vec{a}$.

79.2 Add $\vec{c} = \vec{a} - \vec{w}$ to your drawing. What is the angle between $\vec{c}$ and $\vec{b}$.

79.3 Can you write $\vec{a}$ as the sum of two vectors, one in the direction of $\vec{b}$ and one orthogonal to $\vec{b}$? If so, do it.

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80 Let $\vec{a} = \begin{bmatrix}1\\2\\6\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$.

80.1 Write $\vec{a} = \vec{u} + \vec{v}$ where $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is orthogonal to $\vec{b}$.

80.2 Find an orthonormal basis for $\text{span}\{\vec{a}, \vec{b}\}$.

With two vectors, making an orthonormal set without changing the span is quite easy. With more vectors, it is only slightly harder.

Gram-Schmidt Process

DEFINITION

The ***Gram-Schmidt*** orthogonalization procedure takes in a set of vectors and outputs a set of orthonormal vectors with the same span. The idea is to iteratively produce a set of vectors where each new vector you produce is orthogonal to the previous vectors.

The algorithm is as follows: Let $\{v_1,\dots,v_n\}$ be a set of vectors. Produce a set $\{v_2',\dots,v_n'\}$ that is orthogonal to v_1 by subtracting off the respective projections of $v_2,\dots,v_n$ onto v_1. Next, produce a set $\{v_3'',\dots,v_n''\}$ orthogonal to both v_1 and v_2' by subtracting off the respective projections onto v_2'. Continue this process until you have a set $V=\{v_1,v_2',v_3'',v_4''',\dots\}$ that is orthogonal. Finally, normalize V so all vectors have unit length.

81 Let $\vec{x}_1=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$, $\vec{x}_2=\begin{bmatrix}2\\1\\0\\1\end{bmatrix}$, and $\vec{x}_3=\begin{bmatrix}2\\2\\1\\2\end{bmatrix}$.

81.1 Use the Gram-Schmidt procedure to find an orthonormal basis for $\operatorname{span}\{\vec{x}_1,\vec{x}_2,\vec{x}_3\}$.

81.2 Find an orthonormal basis $\mathcal{V}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\}$ for $\mathbb{R}^4$ so that $\operatorname{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}=\operatorname{span}\{\vec{x}_1,\vec{x}_2,\vec{x}_3\}$.

Let $R=\begin{bmatrix}1&-1&-1&1\\2&1&0&1\\2&2&1&2\end{bmatrix}$.

81.3 Find an orthonormal basis for the row space of R.

81.4 Find the null space of R (Hint, you've already done the work, so there is no need to row reduce).

82 Let

$$\vec{y}_1=\begin{bmatrix}1\\1\\2\end{bmatrix}\qquad \vec{y}_2=\begin{bmatrix}-1\\-1\\2\end{bmatrix}\qquad \vec{y}_3=\begin{bmatrix}1\\1\\6\end{bmatrix}.$$

82.1 Find an orthonormal basis $\mathcal{W}$ so that $\operatorname{span}\mathcal{W}=\operatorname{span}\{\vec{y}_1,\vec{y}_2,\vec{y}_3\}$.

Orthogonal Complement

DEF

The ***orthogonal complement*** of a subspace V is written $V^\perp$ and defined as

$$V^\perp=\{\vec{x}:\vec{x}\text{ is orthogonal to }V\}.$$

82.2 Find the orthogonal complement of $\operatorname{span}\mathcal{W}$.

82.3 Write $\vec{v}=\begin{bmatrix}1\\0\\1\end{bmatrix}$ in the form $\vec{v}=\vec{r}+\vec{n}$ where $\vec{r}\in\operatorname{span}\mathcal{W}$ and $\vec{n}\in(\operatorname{span}\mathcal{W})^\perp$.

QR Decomposition

QR Decomposition

DEF

For a matrix A, we can rewrite $A=QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Writing A as QR is called the ***QR decomposition*** of A.

83 Suppose A,B,C are square matrices and $C=AB$.

83.1 How do the column spaces of A and C relate?

83.2 How do the column spaces of B and C relate?

84 $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms a basis for $\mathbb{R}^3$. When we apply the Gram-Schmidt process to $\mathcal{V}$, we get

$$\begin{aligned} q_1' &= \vec{v} \\ q_2' &= \vec{v}_2 - \tfrac{1}{2}\vec{v}_2 \\ q_3' &= \vec{v}_3 - \vec{v}_1 + 2\vec{v}_2 \end{aligned}$$

form an orthogonal set. Normalizing we get

$$\begin{aligned} \vec{q}_1 &= 2q_1' \\ \vec{q}_2 &= 3q_2' \\ \vec{q}_3 &= \tfrac{1}{2}q_3' \end{aligned}$$

form an orthonormal set.

84.1 Write $\vec{v}_1$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

84.2 Write $\vec{v}_2$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

84.3 Write $\vec{v}_3$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

Define $A = [\vec{v}_1|\vec{v}_2|\vec{v}_2]$ and $Q = [\vec{q}_1|\vec{q}_2|\vec{q}_3]$.

84.4 Find a matrix R so that $A = QR$.

We've just discovered one process to find the QR decomposition of a matrix. It's really as simple as doing Gram-Schmidt and keeping track of your coefficients. Now, we have another way to the matrix equation $A\vec{x} = \vec{b}$. If we do a QR decomposition and exploit the fact that $Q^{-1} = Q^T$, we have

$$A\vec{x} = QR\vec{x} = \vec{b} \qquad \Longrightarrow \qquad R\vec{x} = Q^T\vec{b}$$

and R is a triangular matrix, so we can just do back substitution! (It turns out that if you solve systems this way, there is less rounding error than if you use row reduction.)

Symmetric Matrices

When you're new to Linear Algebra, learning lots of new concepts and algorithms, it's sometimes hard to grasp the significance of certain properties of a matrix.

Symmetric matrices are easy to forget at first, but they have many profound properties (not to mention they are one of the key concepts of Quantum Mechanics).

85 Let A be a symmetric matrix and let $\vec{v}$ be an eigenvector with eigenvalue 3 and $\vec{w}$ be an eigenvector with eigenvalue 4. Note, for this problem, we are thinking of $\vec{v}$ and $\vec{w}$ as column vectors.

85.1 Write $A\vec{v}$, $\vec{v}^TA^T$, $\vec{v}^TA$, $A\vec{w}$, $\vec{w}^TA^T$, and $\vec{w}^TA$ in terms of $\vec{v}$, $\vec{w}$ and scalars.

85.2 How do $\vec{v}^T\vec{w}$ and $\vec{w}^T\vec{v}$ relate?

85.3 What should $\vec{v}^TA\vec{w}$ be in terms of $\vec{v}^T$ and $\vec{w}$? (Note, you could compute $(\vec{v}^TA)\vec{w}$ or $\vec{v}^T(A\vec{w})$. Better do both to be safe).

85.4 What can you conclude about $\vec{v}^T\vec{w}$? How about $\vec{v}\cdot\vec{w}$?

We've just deduced that all eigenspaces of a symmetric matrix are orthogonal! On top of that, symmetric matrices always have a basis of eigenvectors. That means that not only can you always diagonalize a symmetric matrix, but you can *orthogonally* diagonalize a symmetric matrix. (i.e. if A is symmetric, then $A = QDQ^T$ where Q is orthogonal and D is diagonal). This is like the best of all worlds in one!