Inquiry Based Linear Algebra

About the Document

This document is a hybrid of many linear algebra resources, including those of the IOLA (Inquiry Oriented Linear Algebra) project, Jason Siefken's IBLLinearAlgebra project, and Asaki, Camfield, Moon, and Snipes' Radiograph and Tomography project.

This document is a mix of student projects, problem sets, and labs. A typical class day looks like:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

Lesson 1: Linear Combinations

Textbook

Section 1.1

Objectives

- Internalize vectors as geometric objects representing displacements.
- Use column vector notation to write vectors.
- Relate points an vectors and be able to interpret a point as a vector and a vector as a point.
- Solve simple equations involving vectors.

Motivation

Students have differing levels of experience with vectors. We want to establish a common notation for vectors and use vector notation along with algebra to solve simple questions. E.g., "How can I get to location A given that I can only walk parallel to the lines $y = 4x$ and $y = -x$?"

We will use column vector notation and the idea of equating coordinates in order to solve problems.

Notes/Misconceptions

- We will use the language *component of $\vec{v}$ in the direction $\vec{u}$* in the future and it will be a *vector*. For this reason, try to refer to the entries of a column vector as *coordinates* or *entries* instead of components.
- Though we will almost exclusively use column vector notation in this course, students should be able to parse questions phrased in terms of row vectors.

Task 1.1: The Magic Carpet Ride

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Hands-on experience with vectors as displacements.

- Internalize vectors as geometric objects representing displacements.
- Use column vector notation to write vectors.
- Use pre-existing knowledge of algebra to answer vector questions.

Notes/Misconceptions

- There are many ways to solve this problem. Some students might start with equations. After they use their equations to solve the problem, make them draw a picture and come up with a graphical solution.
- When the students start coming up with vector equations, give them the vocabulary of *linear combinations* and *column vector notation*.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Lesson 2: Linear Combinations

Textbook

Section 1.2

Objectives

- Set up and solve vector equations $a\vec{v} + b\vec{u} = \vec{w}$. The solving method may be ad hoc.
- Use set notation and set operations/relations $\cup$, $\cap$, $\in$, $\subseteq$.
- Translate between set-builder notation and words in multiple ways.

Motivation

We revisit questions about linear combinations more formally and generate a need for algebra. The algebra we do to solve vector equations will become algorithmic when we learn row reduction, but at the moment, any method is fine.

As we talk about more complex objects, we need precise ways to talk about groups of vectors. I.e., we need sets and set-builder notation. This preview of set-builder notation will take some of difficulty away when we define span as a set of vectors.

In this course we will be using formal and precise language. Part of this lesson is that there are multiple correct ways (and multiple incorrect ways) to use formal language. Gone are the days of "there's only one right answer and it is 4"!

Notes/Misconceptions

You will have a mix of MAT135/136 and MAT137 students. The MAT137 students will be doing logic and sets in their class. The MAT135 students won't. Make sure not to leave them behind!

Task 1.2: The Magic Carpet Ride, Hide and Seek

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Address an existential question involving vectors: "Is it possible to find a linear combination that does. . . ?"

The goal of this problem is to

- Formalize geometric questions using the language of vectors.
- Find both geometric and algebraic arguments to support the same conclusion.
- Establish what a "negative multiple" of a vector should be.

Notes/Misconceptions

- Both *yes* and *no* are valid answers to this question depending on whether you are allowed to go backwards. Establish that "negative" multiples of a vector mean traveling backwards along that vector.
- This problem can be solved with algebra by finding a formula for the coefficients for an arbitrary position or with geometry, with arguments eventually hinging on the fact that non-parallel lines do not intersect.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X \;:\; \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \,|\, \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

1

1.1 Which of the following statements are true?

(a) $3 \in \{1,2,3\}$. True

(b) $1.5 \in \{1,2,3\}$. False

(c) $4 \in \{1,2,3\}$. False

(d) "b"$\in \{x \;:\; x \text{ is an English letter}\}$. True

(e) "ò"$\in \{x \;:\; x \text{ is an English letter}\}$. False

(f) $\{1,2\} \subseteq \{1,2,3\}$. True

(g) For some $a \in \{1,2,3\}$, $a \geq 3$. True

(h) For any $a \in \{1,2,3\}$, $a \geq 3$. False

(i) $1 \subseteq \{1,2,3\}$. False

(j) $\{1,2,3\} = \{x \in \mathbb{R} \;:\; 1 \leq x \leq 3\}$. False

(k) $\{1,2,3\} = \{x \in \mathbb{Z} \;:\; 1 \leq x \leq 3\}$. True

Practice reading sets and set-builder notation.

The goal of this problem is to

- Become familiar with $\in$, $\subseteq$, and $=$ in the context of sets.
- Distinguish between $\in$ and $\subseteq$.
- Use quantifiers with sets.

Notes/Misconceptions

- Most are easy up through (h).
- Make students "fix" (i) so it becomes true.
- (j) and (k) are an opportunity to use the definition of set equality. Students don't realize that $=$'s has a definition.

2

Write the following in set-builder notation

2.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

$$\{x \in \mathbb{R} \;:\; x > \sqrt{2}\}.$$

2.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

$$\left\{\vec{v} \in \mathbb{R}^2 \;:\; \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ with } a = 2b\right\} \text{ or } \left\{\vec{v} \in \mathbb{R}^2 \;:\; \vec{v} = \begin{bmatrix} 2t \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$$

$$\text{or } \left\{\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \;:\; a = 2b\right\}.$$

Practice writing sets using set-builder notation.

The goal of this problem is to

- Express English descriptions using math notation.
- Recognize there is more than one correct way to write formal math.
- Preview vector form of a line.

Notes/Misconceptions

- There are multiple correct ways to write each of these sets. It's a good opportunity to get man correct and incorrect sets up on the board for discussing.
- Don't worry about the geometry of B. That's coming in a later problem.

Unions & Intersections

DEFINITION

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$(\text{union})\;\; X \cup Y = \{a \;:\; a \in X \text{ or } a \in Y\}.$$

$$(\text{intersection})\;\; X \cap Y = \{a \;:\; a \in X \text{ and } a \in Y\}.$$

3 Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4, 5\}$ and $Z = \{4, 5, 6\}$. Compute

3.1 $X \cup Y$ $\{1, 2, 3, 4, 5\}$

3.2 $X \cap Y$ $\{2, 3\}$

3.3 $X \cup Y \cup Z$ $\{1, 2, 3, 4, 5, 6\}$

3.4 $X \cap Y \cap Z$ $\emptyset = \{\}$

Lesson 3: Visualizing Sets, Formal Language of Linear Combinations

Textbook

Section 1.2

Objectives

- Draw pictures of formally-described subsets of $\mathbb{R}^2$.
- Graphically represent $\cup$ and $\cap$ for subsets of $\mathbb{R}^2$.
- Graphically represent linear combinations and then come up with algebraic arguments to support graphical intuition.

Motivation

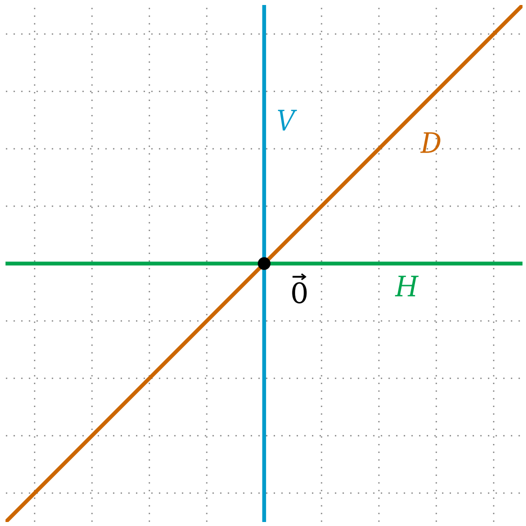
We want to build a bridge between the formal language of linear combinations and set-builder notation and geometric intuition. Where as last time the focus was on formal language, this time the focus is on linking geometry to formal descriptions.

4 Draw the following subsets of $\mathbb{R}^2$.

4.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}0\\t\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

4.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}t\\0\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

4.3 $D = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.



4.4 $N = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} \text{ for all } t \in \mathbb{R}\right\}$. $N = \{\}$.

4.5 $V \cup H$. $V \cup H$ looks like a "+" going through the origin.

4.6 $V \cap H$. $V \cap H = \{\vec{0}\}$ is just the origin.

4.7 Does $V \cup H = \mathbb{R}^2$?

No. $V \cup H$ does not contain $\begin{bmatrix}1\\1\end{bmatrix}$ while $\mathbb{R}^2$ does contain $\begin{bmatrix}1\\1\end{bmatrix}$.

Visualize sets of vectors.

The goal of this problem is to

- Apply set-builder notation in the context of vectors.
- Distinguish between "for all" and "for some" in set builder notation.
- Practice unions and intersections.
- Practice thinking about set equality.

Notes/Misconceptions

- 1–3 will be easy.
- Have a discussion about when you should draw vectors as arrows vs. as points.
- 4 gets at a subtle point that will come up again when we define span.
- Many will miss 7. Writing a proof for this is good practice.

Vector Combinations

Linear Combination (DEFINITION)

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called the ***coefficients*** of the linear combination.

5 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

5.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

$\vec{w} = \begin{bmatrix}3\\2\end{bmatrix}$; the coefficients are $(2, 1)$.

5.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix}3\\3\end{bmatrix} = 3\vec{v}_1 + 0\vec{v}_2$.

5.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$.

5.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix}4\\0\end{bmatrix} = 2\vec{v}_1 + 2\vec{v}_2$.

5.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

Practice linear combinations.

The goal of this problem is to

- Practice using the formal term *linear combination*.
- Foreshadow span.

Notes/Misconceptions

- In 2, the question should arise: "Is $3\vec{v}_1$ a linear combination of $\vec{v}_1$ *and* $\vec{v}_2$?" Address this.
- Refer to the magic carpet ride for 5. You don't need to do a full proof.

No. $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$ and $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$. Therefore

$$\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = a(\tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2) + b(\tfrac{1}{2}\vec{v}_1 - \tfrac{1}{2}\vec{v}_2) = (\tfrac{a+b}{2})\vec{v}_1 + (\tfrac{a-b}{2})\vec{v}_2.$$

Therefore any vector in $\mathbb{R}^2$ can be written as linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

5.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Yes. All linear combinations of $\vec{v}_1$ have equal x and y coordinates, therefore $\vec{w} = \begin{bmatrix}2\\1\end{bmatrix}$ is not a linear combination of $\vec{v}_1$.

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6 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix}3\\1\end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix}1\\2\end{bmatrix}$.

Practice formal writing.

Notes/Misconceptions

- Make everyone *write*. They will think they can do it, but they will find it hard if they try.

6.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

Gauss's location can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

6.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

Every vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

6.3 Rephrase the sentence "$\mathbb{R}^2$ *is the set of all linear combinations of* $\vec{h}$ *and* $\vec{m}$" using formal mathematical language.

$\mathbb{R}^2 = \{\vec{v} : \vec{v} = t\vec{m} + s\vec{h} \text{ for some } t, s \in \mathbb{R}\}$.

Lesson 4: Restricted Linear Combinations, Lines

Textbook

Section 1.2

Objectives

- Read and digest a new definition.
- Use pictures to explore a new concept.
- Convert from an equation-representation of a line to a set-representation.

Motivation

Part of doing math in the world is reading and understanding other people's definitions. Most students will not have heard of non-negative linear combinations or convex linear combinations. This is a chance for them to read and try to understand these formal definitions. They will need to draw pictures to get an intuition about what these concepts mean.

These concepts are useful in their own right, and in particular, convex linear combinations can be used to describe line segments. Adding these definitions to a student's toolbox serves the goal of *being able to describe the world with mathematics*.

To that end, we start working with lines. Lines are something students have used since grade school, but they worked with them in $y = mx + b$ form which is only applicable in $\mathbb{R}^2$. We want to convert this representation into vector form and set-based descriptions which apply to all dimensions.

Non-negative & Convex Linear Combinations

DEFINITION

The linear combination $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

7 Let

$$\vec{a} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \vec{b} = \begin{bmatrix}-1\\1\end{bmatrix} \qquad \vec{c} = \begin{bmatrix}0\\1\end{bmatrix} \qquad \vec{d} = \begin{bmatrix}0\\2\end{bmatrix} \qquad \vec{e} = \begin{bmatrix}-1\\-1\end{bmatrix}.$$

7.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$? All of them, since any vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$.

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$.

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$.

7.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that

(a) $\vec{a}$ and $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not.

Let $\vec{u} = \vec{a}$ and $\vec{v} = \vec{c}$.

(b) $\vec{a}$ and $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$.

Let $\vec{u} = \vec{a}$ and $\vec{v} = \vec{e}$.

(c) $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not.

Impossible. If $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$, then every non-negative linear combination of $\vec{a}$ and $\vec{b}$ is also a non-negative linear combination of $\vec{u}$ and $\vec{v}$. And, we already concluded that $\vec{d}$ is a non-negative linear combination of $\vec{a}$ and $\vec{b}$.

(d) $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$.

Impossible. Convex linear combinations all lie on the same line segment, but $\vec{a}$, $\vec{c}$, and $\vec{d}$ are not collinear.

Otherwise, explain why it's not possible.

Geometric meaning of *non-negative* and *convex* linear combinations.

The goal of this problem is to
- Read and apply the definition of non-negative and convex linear combinations.
- Gain geometric intuition for non-negative and convex linear combinations.
- Learn how to describe line segments using convex linear combinations.

Notes/Misconceptions

- This question is about reading and applying; emphasize that before they start.
- The geometry won't be obvious. Ask them to *draw* specific linear combinations (e.g., $(1/2, 1/2)$) to get an idea.
- They know $\vec{a}$ and $\vec{b}$ span all vectors from problem 5.
- In part 1, they will forget $\vec{a}$ and $\vec{b}$ are linear combinations of themselves.
- Part 2 (b) highlights a degeneracy that will come up again when discussing linear independependence and dependence. Explain how the picture for non-negative linear combinations almost always looks one way, but this case is an exception.

Lines and Planes

8 Let A be the set of points $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 1$.

8.1 Describe A using set-builder notation.

$$\left\{\vec{v} \in \mathbb{R}^2 \,:\, \vec{v} = \begin{bmatrix}t\\2t+1\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$$

$$\text{or } \left\{\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \,:\, y = 2x+1\right\} \text{ or } \left\{\begin{bmatrix}t\\2t+1\end{bmatrix} \in \mathbb{R}^2 \,:\, t \in \mathbb{R}\right\}$$

8.2 Draw A as a subset of $\mathbb{R}^2$.

XXXX

8.3 Add the vectors $\vec{a} = \begin{bmatrix}-1\\-1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\vec{d} = \vec{b} - \vec{a}$ to your drawing.

XXXX

8.4 For which $t \in \mathbb{R}$ is it true that $\vec{a} + t\vec{d} \in A$? Explain using your picture.

XXXX

Link prior knowledge to new notation/concepts.

The goal of this problem is to
- Convert between $y = mx + b$ form of a line and the set-builder definition of the same line.
- Think about lines in terms of vectors rather than equations.

Notes/Misconceptions

- This question is foreshadowing for vector form of a line.
- In part 3, some will draw $\vec{d}$ from the origin and some will draw it on the line. Both are fine, but make sure they understand that $\vec{d} \notin A$ by the end of part 4.

Lesson 5: Vector Form of Lines, Intersecting Lines

Textbook

Section 1.2

Objectives

- Fluency with vector form of a line in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Recognize that vector form of a line is not unique.
- Find the intersection of two lines in vector form.

Motivation

A single linear equation cannot describe a line in more than two dimensions. One way to describe a line that works in all dimensions is vector form, which is a shorthand for a particular set. Vector form has the upside that it makes it easy to produce points on a line, but it has the downside that it is not unique.

Vector form works because a line in any dimension can be defined by two points or, equivalently, a point and a direction. Though we don't yet have a systematic way to write solutions to a system of linear equations, if we have a system representing a line, all we need to do is guess two solutions to that system to find vector form of the line.

One thing vector form makes difficult is finding intersections, but intersections can be turned into just another algebra problem involving a system of equations.

Notes/Misconceptions

Giving a proper definition of vector form of a line is awkward and shouldn't be the focus. For vector form, that they "know it when they see it" and can "produce it" is good enough. (This is in contrast to other definitions which they must be able to correctly state).

Notes/Misconceptions

The biggest stumbling block for finding the intersection of two lines in vector form will be choosing different dummy variables before setting the lines equal.

DEFINITION

Vector Form of a Line

A line ℓ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d} + \vec{p}$$

for some vector $\vec{d}$ and point $\vec{p}$. That is, $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

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9 Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

9.1 Write ℓ in vector form. Is vector form of ℓ unique?

$$\vec{x} = t\begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}$$

The vector form is not unique, as any non-zero scalar multiple of $\begin{bmatrix}1\\-2\end{bmatrix}$ can serve as a direction vector. Additionally, any other point on the line can be used in place of $\begin{bmatrix}0\\3\end{bmatrix}$. For example, $\vec{x} = t\begin{bmatrix}-4\\8\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$ is another vector form of ℓ.

9.2 Write L in vector form. $\vec{x} = t\begin{bmatrix}1\\-2\\-2\end{bmatrix} + \begin{bmatrix}0\\3\\3\end{bmatrix}$. This is obtained by finding two points: one when $x = 0$ and one when $x = 1$ and subtracting them to find a direction vector for L.

9.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

$$\vec{x} = t\begin{bmatrix}-3\\6\\6\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Again, any non-zero scalar multiple of the direction vector will work for $\vec{d}$, as will any other point on the line work for $\vec{p}$.

Practice with vector form.

The goal of this problem is to
- Express lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ in vector form.
- Produce direction vectors by subtracting two points on a line.
- Recognize vector form is not unique.

Notes/Misconceptions
- If students get stuck on part 1, ask them to find a vector parallel to ℓ. If they're still stuck, ask them to find a vector connecting two points on ℓ.
- Many students will intuit part 1 but get stuck on part 2 because they can't draw it. Ask them to start by finding some points on L.

---

10 Let A, B, and C be given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B} \qquad \overbrace{\vec{x} = t\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

10.1 Do the lines A and B intersect? Justify your conclusion.

Yes. $(0)\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} = (-1)\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}$.

To find the intersection, if there is one, we must solve the vector equation:

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = s\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

One solution is when $t = 0$ and $s = -1$.

10.2 Do the lines A and C intersect? Justify your conclusion.

No. The vector equation

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = s\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Intersect lines in vector form.

The goal of this problem is to
- Practice computing the intersection between lines in vector form.
- Recognize "t" as a dummy variable as used in vector form and that, when comparing lines in vector form, "t" needs to be replaced with non-dummy variables.

Notes/Misconceptions
- The most common mistake is to set two lines in vector form equal and use "t" as the variable in each one. Make sure to have a discussion about this.

has no solutions. This is equivalent to saying that the following system of equations has no solutions:

$$\begin{aligned} t &= 2s+1 \\ 2t &= -s+1 \\ 3t+1 &= s+1 \end{aligned}$$

The third equation tells us that $s = 3t$, which when substituted into the first equation forces $t = -\frac{1}{5}$ and therefore $s = -\frac{3}{5}$. However, these two numbers don't satisfy the second equation.

10.3 Let $\vec{p} \neq \vec{q}$ and suppose X has vector form $\vec{x} = t\vec{d} + \vec{p}$ and Y has vector form $\vec{x} = t\vec{d} + \vec{q}$. Is it possible that X and Y intersect?

Yes. If $\vec{q} = \vec{p} + a\vec{d}$ for $a \neq 0$, then X and Y will actually be the same line, since in this case

$$\vec{x} = t\vec{d} + \vec{q} = t\vec{d} + (\vec{p} + a\vec{d}) = (t+a)\vec{d} + \vec{p}.$$

For example, the following two vector equations represent the same line.

$$\vec{x} = t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{and} \qquad \vec{x} = t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}7\\7\\7\end{bmatrix}.$$

Lesson 6: Planes, Span

Textbook

Section 1.2

Objectives

- Describe a plane in vector form.
- Visualize spans.
- Recognize the dimension of $\mathrm{span}(X)$ is not necessarily how many vectors are in X.
- Define *span*.

Motivation

Planes are just like lines but one dimension higher. Vector form of a plane is just like vector form of a plane with all the advantages and disadvantages. But, we now have *two* direction vectors.

Spans are similar to lines and planes; $\mathrm{span}\{\vec{a}, \vec{b}\}$ looks a lot like vector form of the plane $\vec{x} = t\vec{a} + s\vec{b}$. Except, $\mathrm{span}\{\vec{a}, \vec{b}\}$ may not always be a plane. We haven't defined linear independence and linear dependence yet, but we will continue to foreshadow it by seeing that the dimension of the span of a set is not always the size of that set.

Knowing definitions is an essential part of solving math problems. Span is the first definition that students will think they "know" but won't be able to write down.

Vector Form of a Plane

DEFINITION

A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

11 Recall the intersecting lines A and B given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

11.1 Find two direction vectors in $\mathcal{P}$.

Two possible answers are:

$$\vec{d}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \text{and} \qquad \vec{d}_2 = \begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

These are the two direction vectors we already know are in the plane—the ones from the two lines:

Note that neither of these is a multiple of the other, so they really are two unique direction vectors in $\mathcal{P}$.

11.2 Write $\mathcal{P}$ in vector form.

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + s\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

We already have two direction vectors, so we just needed a point on the plane. We used the point $\vec{p} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ that we already know is on line A.

11.3 Describe how vector form of a plane relates to linear combinations.

The vector form of a plane says that a vector $\vec{x}$ is on the plane exactly when it is equal to any linear combination of $\vec{d}_1$ and $\vec{d}_2$, plus $\vec{p}$.

Another way of saying the same thing is that the vector $\vec{x}$ is on the plane exactly when $\vec{x} - \vec{p}$ is equal to some linear combination of $\vec{d}_1$ and $\vec{d}_2$.

11.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

One possibile answer:

$$\vec{x} = t\begin{bmatrix}-1\\-2\\-3\end{bmatrix} + s\begin{bmatrix}-7\\7\\7\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

As with the equations of lines from before, we can use any non-zero scalar multiple of either direction vector and get the same plane. We also used the point $\vec{q} = \begin{bmatrix}-1\\1\\2\end{bmatrix}$ that we already knew is on line B.

12 Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be a plane with equation $x + y + z = 1$.

12.1 Find three points in $\mathcal{Q}$.

There are many choices here, of course. Three natural ones are:

$$\vec{p}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \vec{p}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{p}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

12.2 Find two direction vectors for $\mathcal{Q}$.

Now that we have three points on the plane, we can use the direction vectors joining any two pairs of them. For example:

$$\vec{d}_1 = \vec{p}_1 - \vec{p}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad \vec{d}_2 = \vec{p}_1 - \vec{p}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

12.3 Write $\mathcal{Q}$ in vector form.

Using the point $\vec{p}_1$ from above, one possible answer is:

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}_1 = t\begin{bmatrix}1\\-1\\0\end{bmatrix} + s\begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Span

Span

DEF The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

---

13 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

13.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

XXXX

13.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

XXXX

13.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

$\operatorname{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$

We can see this since for any $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$,

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{x}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right) + \frac{y}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{x+y}{2}\vec{v}_1 + \frac{x-y}{2}\vec{v}_2$$

13.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$. $\operatorname{span}\{\vec{v}_1, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1\}$, a line through the origin with direction vector $\vec{v}_1$.

13.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$

Lesson 7: Span, Translated Span

Textbook

Section 1.2

Objectives

- Explain why spans always go through the origin.
- Express lines or planes through the origin as spans.
- Express lines or planes not through the origin as translated spans.

Motivation

Translated spans link vector form of lines and planes with sets and spans. Soon we will have the vocabulary of linear independence and be able to talk about independent direction vectors of a plane, but right now just connecting the concepts and notation is enough.

14 Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

14.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

$\ell_1 = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, since $\begin{bmatrix}x\\y\end{bmatrix} \in \ell_1$ if and only if $x = y$, which in turn is true if and only if it is a scalar multiple of $\begin{bmatrix}1\\1\end{bmatrix}$.

14.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

This is not possible. $\begin{bmatrix}0\\0\end{bmatrix}$ is an element of the span of *any* set of vectors, since we can use all zeroes as the scalars in a linear combination, but $\begin{bmatrix}0\\0\end{bmatrix} \notin \ell_2$.

14.3 Does the expression $\operatorname{span}(\ell_1)$ make sense? If so, what is it? How about $\operatorname{span}(\ell_2)$?

Both of these expressions do make sense. One can compute the span of any set of vectors, and these lines are just special set of points in $\mathbb{R}^2$ which we are already used to thinking of as vectors.

$\operatorname{span}(\ell_1) = \ell_1$, since all of the vectors on the line ℓ_1 are already multiples of $\begin{bmatrix}1\\1\end{bmatrix}$, as we discovered earlier.

$\operatorname{span}(\ell_2)$ equals all of $\mathbb{R}^2$. It's easy to see that the vectors $v = \begin{bmatrix}4\\0\end{bmatrix}$ and $w = \begin{bmatrix}0\\-4\end{bmatrix}$ are both on ℓ_2, and the span of these two vectors alone is all of $\mathbb{R}^2$.

Set Addition

DEF

If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A+B$, is

$$A+B = \{\vec{x} \,:\, \vec{x} = \vec{a} + \vec{b} \text{ for some } \vec{a} \in A \text{ and } \vec{b} \in B\}.$$

15 Let $A = \left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell = \operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

15.1 Draw A, B, and $A+B$ in the same picture.

XXXX

15.2 Is $A+B$ the same as $B+A$?

Yes. Since A and B are such small sets we could just compute all the vectors in $A+B$ and $B+A$ and see that they're equal. However, we know that real numbers can be added up in any order, and the coordinates of an element of $A+B$ or $B+A$ are simply sums of the corresponding coordinates of elements of A and B.

15.3 Draw $\ell + A$.

XXXX

15.4 Consider the line ℓ_2 given in vector form by $\vec{x} = t\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

ℓ_2 cannot be described using only a span, for the same reason as the line ℓ_2 in Problem 14.2 couldn't be. We know that the origin $\begin{bmatrix}0\\0\end{bmatrix}$ must be an element of any span, but it is not a point on ℓ_2. xxx

ℓ_2 can be described as a span plus a set addition though. Specifically, it is $\ell + A$.

Lesson 8: Linear Independence & Dependence

Textbook

Section 1.2

Objectives

- Define linear independence/dependence using spans.
- Pick linearly independent subsets with the same span by inspection.
- Explain why having a "closed loop" or trivial linear combination means a set is linearly dependent.

Motivation

Linear independence/dependence is one of the biggest concepts in linear algebra. Linear independence/dependence tells us whether a set has redundant information in it with respect to spans. The idea of a having redundant information vs. not comes up all the time in the world (sometimes it's a plus, sometimes it's not).

Knowing a set is independent tells us what its span will look like (in terms of what dimension it will be). It is also an abstract concept that has both a "geometric" definition and an "algebraic" one. Geometrically, a set is linearly dependent if you can remove a vector without the span changing. Algebraically a set is linearly dependent if there is a non-trivial linear combination giving the zero vector. This lesson focuses on the geometric definition (with the algebraic definition coming next).

Though the algebraic definition is easier to work with in proofs, the geometric definition provides intuition about how to visualize linearly dependent sets.

Notes/Misconceptions

Don't define a linearly dependent *set*, define a linearly dependent *list*. Otherwise you cannot talk about $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ be linearly dependent since sets don't contain duplicates.

Task 1.3: The Magic Carpet, Getting Back Home

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$).

1. Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.
2. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?
3. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?
4. What is $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}$?

DEFINITION

Linearly Dependent & Independent

We say the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

16 Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

16.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

This is the set of all vectors in $\mathbb{R}^3$ with z-coordinate equal to zero.

16.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

No. $\vec{w} = \vec{u} + \vec{v}$, and so $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$.

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

16.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

$Y = \{\vec{u}, \vec{v}\}$ is one example that works.

16.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

$Z = \{\vec{u}, \vec{w}\}$ and $Z = \{\vec{v}, \vec{w}\}$ both have the same span as Y above.

DEF

Trivial Linear Combination

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

17 Recall $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

17.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set? $\vec{0} = \vec{u} + \vec{v} - \vec{w}$.

17.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

No. If we could do that, it would look like:

$$a_1\vec{u} + a_2\vec{v} = \vec{0}.$$

Since neither of $\vec{u}$ or $\vec{v}$ is the zero vector, this would mean neither of a_1 or a_2 is zero. But then we could rearrange the equation above to this:

$$\vec{u} = \frac{a_2}{a_1}\vec{v},$$

which is impossible since $\{\vec{u}, \vec{v}\}$ is linearly independent.

Lesson 9: Linear Independence & Dependence—Equivalent Definitions

Textbook

Section 1.2

Objectives

- Define linear independence/dependence in terms of trivial linear combinations.
- Explain how the geometric and algebraic definitions of linear independence/dependence relate.
- Explain the connection between a vector equation having multiple solutions and those vectors being linearly independent/dependent.
- Identify the largest linearly independent set that could exist in $\mathbb{R}^n$.

Motivation

We've done geometry, now let's do algebra. The geometric and algebraic definitions are equivalent, but they suggest different consequences. The geometric definition of linear independence tells us about the dimension of a span. The algebraic definition tells us about the number of solutions to a vector equation.

We now have an equivalent definition of linear dependence.

Linearly Dependent & Independent

DEF The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector.

18

18.1 Explain how this new definition implies the old one.

Suppose the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is linearly dependent in this new sense. That means there are scalars $a_1, a_2, \dots, a_n$, at least one of which is not zero, such that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0.$$

For the sake of example, let's suppose that a_1 is not zero. Then we can divide by it and rearrange this equation to this:

$$\vec{v}_1 = \frac{a_2}{a_1}\vec{v}_2 + cdots + \frac{a_n}{a_1}\vec{v}_n,$$

which means that v_1 is in $\operatorname{span}\{\vec{v}_2, \dots, \vec{v}_n\}$. This is precisely the old definition of linear dependence.

18.2 Explain how the old definition implies this new one.

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent according to the old definition. One way this can happen is if one of the $\vec{v}_i$ equals the zero vector, since $\vec{0}$ is in the span of any set of vectors. If for example $\vec{v}_1 = \vec{0}$, then the linear combination

$$7\vec{v}_1 + 0\vec{v}_2 + \cdots 0\vec{v}_n$$

is non-trivial and equals the zero vector, meaning the set of vectors is linearly dependent according to the new definition.

On the other hand, if none of the $\vec{v}_i$ equal the zero vector, then the old definition says that one of the vectors is in the span of the others. For the sake of example, suppose $\vec{v}_1$ is in the span of $\vec{v}_2, \dots \vec{v}_n$. That means there are scalars $a_2, \dots, a_n$ such that

$$\vec{v}_1 = a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n,$$

and since $\vec{v}_1 \neq \vec{0}$, at least one of the scalars a_i must not be zero. But then

$$\vec{v}_1 - a_2\vec{v}_2 - a_3\vec{v}_3 - \cdots - a_n\vec{v}_n = \vec{0},$$

and this linear combination is non-trivial.

Since we have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

19

Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

19.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

No. If both equations are true, they would combine to show

$$3\vec{u} + 2\vec{v} + \vec{w} = 2\vec{u} + \vec{v} - \vec{w}.$$

Collecting all the terms on the left side, we get:

$$\vec{u} + \vec{v} + 2\vec{w} = \vec{0},$$

which is a non-trivial linear combination of vectors in the given set equalling the zero vector.

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

19.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

Yes. If it were not, there would exist scalars a_1, a_2, a_3, not all of which are zero, such that:

$$a_1\vec{u} + a_2\vec{r} + a_3\vec{s} = \vec{0}.$$

But then

$$\vec{u} + 6\vec{r} - \vec{s} + (a_1\vec{u} + a_2\vec{r} + a_3\vec{s})$$

would be another way to write $\vec{a}$ using only the same three vectors.

19.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

Yes. If it were not, we would necessarily have $\vec{u} = a\vec{r}$ for some scalar a. But then

$$(a+6)\vec{r} - \vec{s}$$

would be another way to write $\vec{a}$ using only the same three vectors.

19.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

No. We know from earlier that $\vec{u} + \vec{v} + 2\vec{w} = \vec{0}$, and so $\vec{u} + \vec{v} + 2\vec{w} + 0\vec{r} = \vec{0}$ is a non-trivial linear combination of the vectors in this set that equals the zero vector.

Task 1.4: Linear Independence and Dependence, Creating Examples

1. Fill in the following chart keeping track of the strategies you used to generate examples.

	Linearly independent	Linearly dependent
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

2. Write at least two generalizations that can be made from these examples and the strategies you used to create them.

Lesson 10: Dot Product, Orthogonality

Textbook

Section 1.3

Objectives

- Compute the dot product of two vectors.
- Compute the length of a vector.
- Find the distance between two vectors.
- Define what it means for vectors to be orthogonal.
- Interpret the sign of the dot product geometrically.
- Create a unit vector in the direction of another.

Motivation

Studying $\mathbb{R}^n$ we're in a natural inner product space with lengths and angles. The dot product allows us to get at lengths and angles. It will also give an alternative way to compute matrix products (dot product with rows instead of linear combination of columns).

Most importantly, the dot product tells us how much two vectors point in the same direction as well as when they're orthogonal.

Dot Product

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

---

20 Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

20.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

XXXX

(b) Compute $\vec{a} \cdot \vec{b}$. $\vec{a} \cdot \vec{b} = (1)(3) + (1)(2) = 5.$

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

$\|\vec{a}\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$ and $\|\vec{b}\| = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$.

Using the two definitions of the dot product we have:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \|\vec{a}\|\|\vec{b}\| \cos\theta \\ \implies 5 &= (\sqrt{2})(\sqrt{13}) \cos\theta \\ \implies \theta &= \arccos\left(\frac{5}{\sqrt{26}}\right) \end{aligned}$$

20.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

INSERT GRAPH HERE

cos is positive for angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, as well as all shifts of this interval by a multiple of 2π in either direction.

cos is positive for angles in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, as well as all shifts of this interval by a multiple of 2π in either direction.

20.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

INSERT A PICTURE HERE

(d) $\vec{f}=\vec{0}$ is the only possibility. For any vector $\vec{f}=\begin{bmatrix}x\\y\end{bmatrix}$, we can compute:

$$\vec{f}\cdot\vec{a}=x+y \qquad \text{and} \qquad \vec{f}\cdot\vec{b}=3x+2y.$$

If these both equal zero, the first equation says that $y=-x$, and in turn the second one says $x=0$ (and so $y=0$ as well).

20.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

$\vec{v}=\begin{bmatrix}1\\1\\-3\end{bmatrix}$ is one such vector.

Since $\cos(\pi/2)=0$, from the second definition of the dot product above we know we are looking for a $\vec{v}$ such that $\vec{u}\cdot\vec{v}=0$. Using the first definition of the dot product, we can see that the $\vec{v}$ given above is one possibility.

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

$\vec{w}=\begin{bmatrix}-1\\1\\-1\end{bmatrix}$ is a possible answer.

$\vec{u}\cdot\vec{w}=0$, and $\vec{w}$ is clearly not parallel to $\vec{v}$ from above.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

Yes. $\begin{bmatrix}0\\2\\-4\end{bmatrix}$ is one possibility.

There are actually infinitely many such vectors; any linear combination of $\vec{w}$ and $\vec{v}$ will work.

To see this, note that any such vector $\vec{x}$ is of the form

$$\vec{x}=t\begin{bmatrix}1\\1\\-3\end{bmatrix}+s\begin{bmatrix}-1\\1\\-1\end{bmatrix}=\begin{bmatrix}t-s\\t+s\\-3t-s\end{bmatrix},$$

for scalars t and s. We can then compute

$$\vec{u}\cdot\vec{x}=(1)(t-s)+(2)(t+s)+(1)(-3t-s)=0,$$

and so any such vector $\vec{x}$ forms an angle of $\pi/2$ with $\vec{u}$.

THM

For a vector $\vec{v}\in\mathbb{R}^n$, the formula

$$\|\vec{v}\|=\sqrt{\vec{v}\cdot\vec{v}}$$

always holds.

DEF

Distance

The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u}-\vec{v}\|$.

DEF

Unit Vector

A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\|=1$.

21 Let $\vec{u}=\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}1\\1\\3\end{bmatrix}$.

21.1 Find the distance between $\vec{u}$ and $\vec{v}$.

$\vec{u}-\vec{v}=\begin{bmatrix}0\\1\\-2\end{bmatrix}$, and so $\|\vec{u}-\vec{v}\|=\sqrt{5}$.

21.2 Find a unit vector in the direction of $\vec{u}$.

$$\frac{1}{\sqrt{6}}\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

$\|\vec{u}\| = \sqrt{6}$, and so if we multiply $\vec{u}$ by $\frac{1}{\sqrt{6}}$, the length of the resulting vector will be 1.

21.3 Does there exists a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

No. $\|\vec{u}\| = \sqrt{6}$, and so the shortest length that a vector whose distance from $\vec{u}$ is 1 can have is $\sqrt{6}-1$, which is greater than 1.

21.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

The angle between $\vec{u}$ and $\vec{y}$ is $\arccos\left(-\frac{3}{2\sqrt{6}}\right)$.

By assumption, $2 = \|\vec{u}-\vec{y}\|$, and so

$$\begin{aligned} 4 &= \|\vec{u}-\vec{y}\|^2 \\ &= (\vec{u}-\vec{y})\cdot(\vec{u}-\vec{y}) \\ &= \vec{u}\cdot\vec{u} - 2(\vec{u}\cdot\vec{y}) + \vec{y}\cdot\vec{y} \\ &= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{y} + \|\vec{y}\|^2 \\ &= 6 - 2\vec{u}\cdot\vec{y} + 1. \end{aligned}$$

Then we rearrange to find that $\vec{u}\cdot\vec{y} = -\frac{3}{2}$.

Using this in the second definition of the dot product, we see:

$$-\frac{3}{2} = \left(\sqrt{6}\right)(1)\cos\theta,$$

where θ is the angle between $\vec{u}$ and $\vec{y}$.

Orthogonal

DEF Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

22

22.1 Find two vectors orthogonal to $\vec{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Can you find two such vectors that are not parallel?

Two such vectors are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ -2 \end{bmatrix}$.

It is impossible for two non-parallel vectors to both be orthogonal to $\vec{a}$. If $\vec{b} = \begin{bmatrix} x \\ y \end{bmatrix}$ is orthogonal to $\vec{a}$, then we must have that $x - 3y = 0$, or in other words that $x = 3y$. Any $\vec{b}$ satisfying this is a multiple of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

22.2 Find two vectors orthogonal to $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Can you find two such vectors that are not parallel?

Two such vectors are $\begin{bmatrix} 7 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

These two vectors are not parallel.

22.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\| = 5$ and $\|\vec{y}\| = 3$. What is the distance between $\vec{x}$ and $\vec{y}$?

The distance between them must be $\sqrt{34}$.

One way to see this is with Pythagoras' theorem. Two perpendicular line segments of lengths 3 and 5 form the two shorter sides of a right angle triangle, and so the length of the third side is $\sqrt{5^2+3^2} = \sqrt{34}$.

An equivalent way to see this is to use what we know about dot products to calculate $\|\vec{x}-\vec{y}\|$ as follows:

$$\|\vec{x}-\vec{y}\| = \sqrt{(\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})} = \sqrt{\|\vec{x}\|^2 - 2(\vec{x}\cdot\vec{y}) + \|\vec{y}\|^2} = \sqrt{5^2+2(0)+3^2},$$

where in the last step we've used the fact that $\vec{x}$ and $\vec{y}$ are orthogonal, so $\vec{x}\cdot\vec{y}=0$.

Lesson 11: Normal Form of Lines and Planes

Textbook

Section 1.2

Objectives

- Describe lines and planes in normal form.

Motivation

Physics often describes surfaces in terms of normal and tangential components. Normal form of lines and planes is one way to get at this decomposition. Further, thinking about lines and planes in terms of right angles will help when visualizing orthogonal projections.

23

23.1 Draw $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.

XXXX

23.2 If $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x}\cdot\vec{u}$? $\vec{x}\cdot\vec{u} = 0$, by the definition of orthogonality.

23.3 Expand the dot product $\vec{u}\cdot\vec{x}$ to get an equation for A.

A is the line with vector equation $\vec{x} = t\begin{bmatrix}3\\-2\end{bmatrix}$.

If $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix} \in A$, then $\vec{x}\cdot\vec{u} = 2x + 3y = 0$.

23.4 If possible, express A as a span. $A = \operatorname{span}\left\{\begin{bmatrix}3\\-2\end{bmatrix}\right\}$.

DEF **Normal Vector**

A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to it.

24

Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}1\\-1\end{bmatrix}$, and define the lines

$$\ell_1 = \operatorname{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \operatorname{span}\{\vec{p}\} + \operatorname{span}\{\vec{d}\}.$$

24.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

$\vec{n} = \begin{bmatrix}2\\-1\end{bmatrix}$ is one possibility.

This vector is orthogonal to $\vec{d}$, which is a direction vector for both lines.

24.2 Let $\vec{v} \in \ell_1$ and $\vec{u} \in \ell_2$. What is $\vec{n}\cdot\vec{v}$? What about $\vec{n}\cdot\vec{u}$?

$\vec{n}\cdot\vec{v} = 0$, since any $\vec{v} \in \ell_1$ is a multiple of $\vec{d}$.

$\vec{n}\cdot\vec{u} = 3$, since any such $\vec{u}$ is of the form $\vec{u} = \vec{p} + t\vec{d}$ for some scalar t, and so

$$\vec{n}\cdot\vec{u} = \vec{n}\cdot(\vec{p} + t\vec{d}) = \vec{n}\cdot\vec{p} + t(\vec{n}\cdot\vec{d}) = 3 + t(0) = 3.$$

24.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n}\cdot(\vec{x} - \vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

The normal form of ℓ_1 is $\begin{bmatrix}2\\-1\end{bmatrix}\cdot\vec{x} = 0$. test

The normal form of ℓ_2 is $\begin{bmatrix}2\\-1\end{bmatrix}\cdot\left(\vec{x} - \begin{bmatrix}1\\-1\end{bmatrix}\right) = 0$. In the previous part we saw that $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p}$ for all $\vec{x} \in \ell_2$, or in other words $\vec{n}\cdot(\vec{x} - \vec{p}) = 0$.

25

Let $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

25.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

$X = \left\{\vec{x} \in \mathbb{R}^3 \,:\, \vec{x}\cdot\vec{n} = 0\right\}$.

Geometrically, this is a plane through the origin and perpendicular to $\vec{n}$.

25.2 Describe X using an equation. $x + y + z = 0$.

25.3 Describe X as a span. $X = \operatorname{span}\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}$ is one way to do this.

Lesson 12: Projections

Textbook

Section 1.4

Objectives

- Project a vector onto lines and finite sets.
- Find the components of one vector in terms of another.

Motivation

Projection of a vector onto a set, defined as the closet point in the set to the vector, is a general operation used outside of linear algebra. However, in the land of linear algebra, we have exact formulas for the projection. Projections are a chance to explore a seemingly simple definition and see it relate to sets, lines, normal form, and vector form.

$\text{comp}_{\vec v}\,\vec u$ is the component of a vector in the direction of another, which is sometimes called the projection of $\vec u$ onto $\vec v$. It relates to how much one vector points in the direction of another and provides a decomposition of vectors in terms of orthogonal components.

Notes/Misconceptions

In this class, we don't write $\text{proj}_{\vec v}\,\vec u$, i.e., the projection of one vector onto another. We instead call this $\text{comp}_{\vec v}\,\vec u$. We do this so as not to confuse $\text{proj}_{\{\vec v\}}\,\vec u$ and $\text{comp}_{\vec v}\,\vec u$. One is projection onto a singleton. The other is $\frac{\vec u\cdot\vec v}{\vec v\cdot\vec v}\vec v$.

Projections

Projection

DEF Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\text{proj}_X \vec{v}$, is the closest point to $\vec{v}$ in X.

---

26 Let $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}2\\2\end{bmatrix}$ and $\ell = \text{span}\{\vec{a}\}$.

26.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.

XXXX

26.2 Find $\text{proj}_{\{\vec{b}\}} \vec{v}$, $\text{proj}_{\{\vec{a},\vec{b}\}} \vec{v}$.

$\text{proj}_{\{\vec{b}\}} \vec{v} = \vec{b}$. Since there is only one point in $\{\vec{b}\}$, it must be the closest point to $\vec{v}$.

$\text{proj}_{\{\vec{a},\vec{b}\}} \vec{v} = \vec{a}$. We can simply compute $\|\vec{v} - \vec{a}\| = \sqrt{5}$ and $\|\vec{v} - \vec{b}\| = \sqrt{13}$, so $\vec{a}$ is closer to $\vec{v}$.

26.3 Find $\text{proj}_\ell \vec{v}$. (Recall that a quadratic $at^2 + bt + c$ has a minimum at $t = -\frac{b}{2a}$).

$\text{proj}_\ell \vec{v} = 2\vec{a}$.

Any point in ℓ is of the form $t\vec{a}$ for some scalar t. The distance between such a point and $\vec{v}$ is

$$\|\vec{v} - t\vec{a}\| = \sqrt{\|v\|^2 - 2t(\vec{v}\cdot\vec{a}) + t^2\|a\|^2} = \sqrt{8 - 4t + t^2}$$

The quadratic inside the square root has a minimum at $t = 2$, so $2\vec{a}$ is the closest point in the line to $\vec{v}$.

26.4 Is $\vec{v} - \text{proj}_\ell \vec{v}$ a normal vector for ℓ? Why or why not?

Yes.

By the previous part, $\vec{v} - \text{proj}_\ell \vec{v} = \vec{v} - 2\vec{a} = \begin{bmatrix}0\\2\end{bmatrix}$. This vector is orthogonal to $\vec{a}$, and therefore to ℓ.

---

27 Let K be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}$ and let $\vec{c} = \begin{bmatrix}1\\3\end{bmatrix}$.

27.1 Make a sketch with $\vec{c}$, K, and $\text{proj}_K \vec{c}$ (you don't need to compute $\text{proj}_K \vec{c}$ exactly).

XXXX

27.2 What should $(\vec{c} - \text{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix}$ be? Explain.

$(\vec{c} - \text{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix} = 0.$

From our picture we can see that $c - \text{proj}_K \vec{c}$ is perpendicular to the line K, and so the dot product of this vector with any direction vector for K should be zero.

27.3 Use your formula from the previous part to find $\text{proj}_K \vec{c}$ *without* computing any distances.

$\text{proj}_K \vec{c} = \frac{1}{5}\begin{bmatrix}11\\12\end{bmatrix}$

If $\text{proj}_K \vec{c} = \begin{bmatrix}x\\y\end{bmatrix}$, the formula from the previous part tells us

$$\left(\begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}x\\y\end{bmatrix}\right)\cdot\begin{bmatrix}1\\2\end{bmatrix} = 1 - x + 6 - 2y = 0 \quad\iff\quad x + 2y = 7$$

So we need a point on K that satisfies this equation. In other words, we need

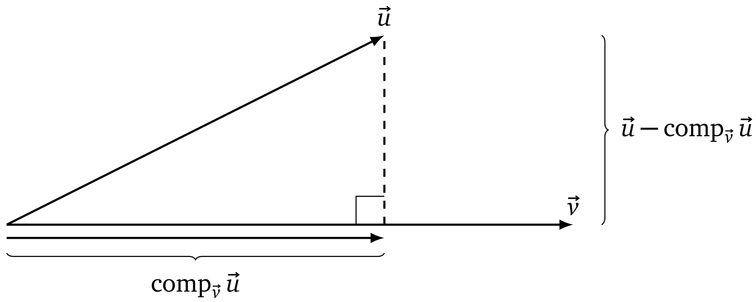
$$(t+1) + 2(2t) = 7 \quad\implies\quad t = \frac{6}{5}.$$

The point on K for this value of t is $\frac{1}{5}\begin{bmatrix}11\\12\end{bmatrix}$.

DEFINITION

Component

Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***component of $\vec{u}$ in the $\vec{v}$ direction***, written $\text{comp}_{\vec{v}}\,\vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \text{comp}_{\vec{v}}\,\vec{u}$ is orthogonal to $\vec{v}$.



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28 Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

28.1 List two conditions that $\text{comp}_{\vec{b}}\,\vec{a}$ must satisfy.

$\text{comp}_{\vec{b}}\,\vec{a}$ must be a scalar multiple of $\vec{b}$.

$\vec{a} - \text{comp}_{\vec{b}}\,\vec{a}$ must be orthogonal to $\vec{b}$, or in other words $(\vec{a} - \text{comp}_{\vec{b}}\,\vec{a}) \cdot \vec{b} = 0$.

28.2 Find a formula for $\text{comp}_{\vec{b}}\,\vec{a}$.

$\text{comp}_{\vec{b}}\,\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\vec{b}$.

From the previous part, we should have $\text{comp}_{\vec{b}}\,\vec{a} = t\vec{b}$ for some scalar t, and $(\vec{a} - \text{comp}_{\vec{b}}\,\vec{a}) \cdot \vec{b} = 0$.

Combining these, we get:

$$0 = (\vec{a} - t\vec{b}) \cdot \vec{b} = \vec{a} \cdot \vec{b} - t\vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{b} - t(\vec{b} \cdot \vec{b}).$$

Solving for t, we get $t = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}$.

Lesson 13: Projections, Subspaces

Textbook

Sections 1.2, 1.4

Objectives

- Identify $\mathrm{proj}_{\mathrm{span}\{\vec v\}} \vec u$ with $\mathrm{comp}_{\vec v} \vec u$.
- Identify $\mathrm{comp}_{\vec v} \vec u$ and $\mathrm{comp}_{\alpha \vec v} \vec u$ for all $\alpha \neq 0$, including negative α.
- Define subspace.
- Distinguish subspaces and non-subspaces of $\mathbb{R}^2$.

Motivation

Spans are a constructive way to describe lines, planes, and other flat objects. Subspaces are a categorical way of defining flat objects. Instead of explaining how to find the vectors in a set, we list their properties. This is a really powerful idea that facilitates abstraction.

Since we do not do abstract vector spaces in this course, subspaces are the first place (unless you count projections) students will encounter a set defined by its properties. Subspaces are suitable for a first-encounter because 1) the properties are simple and familiar and 2) subspaces of $\mathbb{R}^n$ have a concrete geometric interpretation.

Notes/Misconceptions

- Philosophically, a subspace should be defined as a non-empty set closed under linear combinations. However, defining it as closed under addition and scalar multiplication gives students new to proofs something explicit to hang on to when attempting a proof.
- Some people define a subspace as a set containing $\vec 0$ and satisfying closure. We define a subspace as a non-empty set satisfying closure. We won't be trying to trick students by asking if an empty set is a subspace, so don't belabor the point.

29 Let $\vec{d}=\begin{bmatrix}3\\3\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}1\\2\end{bmatrix}$.

29.1 Draw $\vec{d}$, $\vec{u}$, $\text{span}\{\vec{d}\}$, and $\text{proj}_{\text{span}\{\vec{d}\}}\vec{u}$ in the same picture.

XXXX

29.2 How do $\text{proj}_{\text{span}\{\vec{d}\}}\vec{u}$ and $\text{comp}_{\vec{d}}\vec{u}$ relate? They are equal.

29.3 Compute $\text{proj}_{\text{span}\{\vec{d}\}}\vec{u}$ and $\text{comp}_{\vec{d}}\vec{u}$.

Using our formula from the previous problem

$$\text{proj}_{\text{span}\{\vec{d}\}}\vec{u}=\text{comp}_{\vec{d}}\vec{u}=\frac{\vec{u}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d}=\frac{9}{18}\vec{d}=\frac{1}{2}\begin{bmatrix}3\\3\end{bmatrix}.$$

29.4 Compute $\text{comp}_{-\vec{d}}\vec{u}$. Is this the same as or different from $\text{comp}_{\vec{d}}\vec{u}$? Explain.

$$\text{comp}_{-\vec{d}}\vec{u}=\frac{\vec{u}\cdot(-\vec{d})}{\|-\vec{d}\|^2}(-\vec{d})=\frac{-9}{18}(-\vec{d})=\frac{1}{2}\begin{bmatrix}3\\3\end{bmatrix}=\text{comp}_{\vec{d}}\vec{u}.$$

We expect them to be equal since $\vec{d}$ and $-\vec{d}$ are in the same direction as one another.

Subspaces and Bases

Subspace

DEFINITION

A ***subspace*** $V\subseteq\mathbb{R}^n$ is a non-empty subset such that

(i) $\vec{u},\vec{v}\in V$ implies $\vec{u}+\vec{v}\in V$.

(ii) $\vec{u}\in V$ implies $k\vec{u}\in V$ for all scalars k.

Subspaces give a mathematically precise definition of a “flat space through the origin.”

30 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

30.1 $A=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}=\begin{bmatrix}a\\0\end{bmatrix}\text{ for some } a\in\mathbb{Z}\right\}$.

XXXX

A is not a subspace, since for example $\begin{bmatrix}1\\0\end{bmatrix}\in A$ but $\frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}\notin A$.

30.2 $B=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}\neq\begin{bmatrix}0\\0\end{bmatrix}\right\}$.

XXXX

B is not a subspace, since for example $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\end{bmatrix}$ are both in B, but their sum is $\begin{bmatrix}0\\0\end{bmatrix}$ which is not in B.

30.3 $C=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}=\begin{bmatrix}0\\t\end{bmatrix}\text{ for some } t\in\mathbb{R}\right\}$.

XXXX

C is a subspace.

(i) Let $\vec{u},\vec{v}\in C$. Then $\vec{u}=\begin{bmatrix}0\\t\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}0\\s\end{bmatrix}$ for some $s,t\in\mathbb{R}$.

But then $\vec{u}+\vec{v}=\begin{bmatrix}0\\s+t\end{bmatrix}\in C$.

(ii) Let $\vec{u}=\begin{bmatrix}0\\t\end{bmatrix}\in C$. For any scalar α we have $\alpha\vec{u}=\begin{bmatrix}0\\\alpha t\end{bmatrix}\in C$.

30.4 $D=\left\{\vec{x}\in\mathbb{R}^2 : \vec{x}=\begin{bmatrix}0\\t\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\text{ for some } t\in\mathbb{R}\right\}$.

XXXX

D is not a subspace, since for example $\begin{bmatrix}1\\1\end{bmatrix}\in D$, but $0\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\notin D$.

30.5 $E = \left\{\vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = \begin{bmatrix}0\\t\end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix}t\\0\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

XXXX

E is not a subspace, since for example $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are both in E, but their sum is $\begin{bmatrix}1\\1\end{bmatrix}$ which is not in E.

30.6 $F = \left\{\vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = t\begin{bmatrix}3\\1\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

XXXX

F is a subspace.

(i) Let $\vec{u}, \vec{v} \in F$. Then $\vec{u} = t\begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{v} = s\begin{bmatrix}3\\1\end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = (s+t)\begin{bmatrix}3\\1\end{bmatrix} \in F$.

(ii) Let $\vec{u} = t\begin{bmatrix}3\\1\end{bmatrix} \in F$. For any scalar α we have $\alpha\vec{u} = (\alpha t)\begin{bmatrix}3\\1\end{bmatrix} \in F$.

30.7 $G = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

XXXX

G is a subspace.

By definition of a span, $G = \left\{\vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

The proof that G is a subspace now proceeds similarly to the proof for F above.

(i) Let $\vec{u}, \vec{v} \in G$. Then $\vec{u} = t\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v} = s\begin{bmatrix}1\\1\end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = (s+t)\begin{bmatrix}1\\1\end{bmatrix} \in G$.

(ii) Let $\vec{u} = t\begin{bmatrix}1\\1\end{bmatrix} \in G$. For any scalar α we have $\alpha\vec{u} = (\alpha t)\begin{bmatrix}1\\1\end{bmatrix} \in G$.

30.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

XXXX

H is a subspace.

(i) Let $\vec{x}, \vec{y} \in H$. Then $\vec{x} = \alpha_1\vec{u} + \alpha_2\vec{v}$ and $\vec{y} = \beta_1\vec{u} + \beta_2\vec{v}$ for some scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$. But then

$$\vec{x} + \vec{y} = \alpha_1\vec{u} + \alpha_2\vec{v} + \beta_1\vec{u} + \beta_2\vec{v} = (\alpha_1 + \beta_1)\vec{u} + (\alpha_2 + \beta_2)\vec{v} \in H.$$

(ii) Let $\vec{x} = \alpha_1\vec{u} + \alpha_2\vec{v} \in H$. For any scalar β we have $\beta\vec{u} = (\beta\alpha_1)\vec{u} + (\beta\alpha_2)\vec{v} \in H$.

Lesson 14: Basis, Dimension

Textbook

Sections 1.2, 4.3

Objectives

- Define Basis.
- Define Dimension.
- Find a basis for a subspace.
- Find the dimension of a subspace.
- Explain why every vector has a unique representation as a linear combination of basis vectors.

Motivation

Bases are sets of just enough vectors to describe every vector in a subspace. An additional consequence of a basis is that every vector can be *uniquely* represented as a linear combination of basis vectors. Using this fact we will be able to consider objects in multiple different coordinate systems. However, now is the time to get familiar with what a basis is and how to find one.

Dimension ties the abstract notion of subspace to our intuition about Euclidean space. We already know a plane in $\mathbb{R}^3$ is two dimensional, but now we know where that number *two* comes from.

Basis (DEF)

A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

Dimension (DEF)

The ***dimension*** of a subspace V is the number of elements in a basis for V.

---

31 Let $\vec u = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec v = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec w = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec u, \vec v, \vec w\}$.

31.1 Describe V. V is the xy-plane in $\mathbb{R}^3$.

31.2 Is $\{\vec u, \vec v, \vec w\}$ a basis for V? Why or why not?

No. The set $\{\vec u, \vec v, \vec w\}$ is linearly dependent since $\vec w = \vec u + \vec v$.

31.3 Give a basis for V. $\{\vec u, \vec v\}$.

31.4 Give another basis for V. $\{\vec u, \vec w\}$ or $\{\vec v, \vec w\}$.

31.5 Is $\operatorname{span}\{\vec u, \vec v\}$ a basis for V? Why or why not?

No. $\operatorname{span}\{\vec u, \vec v\}$ is an infinite set of vectors which includes $\vec 0$, so it cannot be linearly independent and therefore isn't a basis.

31.6 What is the dimension of V?

A basis for V has two vectors so it is two-dimensional. We also know this because V is the xy-plane in $\mathbb{R}^3$ and all planes are two-dimensional.

Apply the definitions of basis and dimension to an easy example.

The goal of this problem is to learn

- To apply the definition of basis and dimension.
- Intuition that a plane is two dimensional.
- A basis is not unique, but always has the same size (this is not proved).
- Spans are never bases—you must not confuse a subspace with its basis!

Notes/Misconceptions

- Students will claim V is $\mathbb{R}^2$ and fail to distinguish $\mathbb{R}^2$ and the xy-plane in $\mathbb{R}^3$.
- Parts 2, 3, 4, 6 will be easy; don't belabor them.
- Students will fail to distinguish $\operatorname{span}\{\vec u, \vec v\}$ from $\{\vec u, \vec v\}$. Make sure this distinction comes out.

---

32 Let $\vec a = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec b = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec c = \begin{bmatrix}7\\8\\8\end{bmatrix}$ (notice these vectors are linearly independent) and let $P = \operatorname{span}\{\vec a, \vec b\}$ and $Q = \operatorname{span}\{\vec b, \vec c\}$.

32.1 Give a basis for and the dimension of P.

$\{\vec a, \vec b\}$ is a basis for P, and so its dimension is 2.

32.2 Give a basis for and the dimension of Q.

$\{\vec b, \vec c\}$ is a basis for Q, and so its dimension is 2.

32.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

Yes. $\{\vec b\}$ is a basis for $P \cap Q$, and so its dimension is 1.

P and Q are both planes and are not parallel (since $\vec a, \vec b, \vec c$ are linearly independent). The intersection of any two non-parallel planes is a line. We know that $\vec 0$ and $\vec c$ are on this line, and therefore the line is $\operatorname{span}\{\vec c\}$

32.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

No. For example $\vec a$ and $\vec c$ are both in $P \cup Q$, but $\vec a + \vec c \notin P \cup Q$.

A vector is in $P \cup Q$ if it is in P or Q, so we must show that $\vec a + \vec c \notin P$ and $\vec a + \vec c \notin Q$

$\vec a + \vec c \notin P$ since if it were, we would also have $(\vec a + \vec c) - \vec a = \vec c \in P$. We know this is impossible since the vectors $\vec a, \vec b, \vec c$ are linearly independent, and so $\vec c$ does not equal a linear combination of $\vec a$ and $\vec b$.

An analogous argument shows that $\vec a + \vec c \notin Q$.

Lesson 15: Matrices

Textbook

Section 3.1

Objectives

- Write a system of linear equations as a matrix equation.
- Write a matrix equation as a system of linear equations.
- Pose familiar problems (e.g., "find a normal vector", or "do these planes intersect?") as matrix-equation questions.

Motivation

Matrices will soon become a powerful tool to study linear transformations. However, we will start out viewing them as a notation to represent systems of linear equations. The fact that matrix-vector multiplication has two interpretations, as a linear combination of columns or as a dot product with rows, already connects geometry and angles with questions about linear combinations.

33 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

33.1 Compute the product $A\vec{x}$.

$$A\vec{x} = \begin{bmatrix} x + 2y \\ 3x + 3y \end{bmatrix}.$$

33.2 Write down a system of equations that corresponds to the matrix equation $A\vec{x} = \vec{b}$.

$$\begin{aligned} x + 2y &= -2 \\ 3x + 3y &= -1 \end{aligned}$$

33.3 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

x_0 and y_0, when used as scalars in a linear combination of the columns of A, make the vector $\vec{b}$. In other words:

$$x_0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y_0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

33.4 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

The lines represented by the equations $x + 2y = -2$ and $3x + 3y = -1$ from the system of equations above intersect at the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

34 Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

34.1 How could you determine if $\{\vec{u}, \vec{v}, \vec{w}\}$ was a linearly independent set?

The set is linearly independent if and only if no non-trivial linear combination of the vectors $\vec{u}, \vec{v}, \vec{w}$ equals $\vec{0}$. That is, if x, y, z are scalars such that $x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}$, then $x = y = z = 0$.

In other words, the only solution of the following system of equations is $x = y = z = 0$.

$$\begin{aligned} x + 4y + 7z &= 0 \\ 2x + 5y + 8z &= 0 \\ 3x + 6y + 9z &= 0 \end{aligned}$$

34.2 Can your method be rephrased in terms of a matrix equation? Explain.

The system of linear equations above can be represented by the matrix equation $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

So another way to say the above is that the set is linearly independent if and only if the only solution to the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

35 Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

35.1 If $\vec{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

This system has no solutions, since the third equation would be $0 = 3$, which is impossible.

35.2 If $\vec{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

A line. The system would be

$$\begin{aligned} x - 3y &= 1 \\ z &= 1 \\ 0 &= 0 \end{aligned}$$

A vector $\vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ that satisfies this system must have $z = 1$, and by the first equation in the system any value of x determines the value of y, and vice versa. In other words the system has one free variable, and so its set of solutions is a line.

---

36 Let $\vec{d}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$. Let $\mathcal{P}$ be the plane given in vector form by $\vec{x} = t\vec{d}_1 + s\vec{d}_2$. Further, suppose M is a matrix so that $M\vec{r} \in \mathcal{P}$ for any $\vec{r} \in \mathbb{R}^2$.

36.1 How many rows does M have?

Three. It must have three rows in order for $M\vec{r}$ to be an element of $\mathbb{R}^3$.

36.2 Find such an M.

$M = \begin{bmatrix}1 & -1\\1 & 1\\2 & 0\end{bmatrix}$ is one possible answer, since if $\vec{r} = \begin{bmatrix}a\\b\end{bmatrix}$, then $M\vec{r} = a\vec{d}_1 + b\vec{d}_2$.

Another less interesting answer is the 3×2 zero matrix.

36.3 Find necessary and sufficient conditions (phrased as equations) for $\vec{n}$ to be a normal vector for $\mathcal{P}$.

$\vec{n}$ is normal to $\mathcal{P}$ if and only if $\vec{n} \cdot \vec{d}_1 = 0$ *and* $\vec{n} \cdot \vec{d}_2 = 0$

36.4 Find a matrix K so that solutions to $K\vec{x} = \vec{0}$ are normal vectors for $\mathcal{P}$. How do K and M relate?

$K = \begin{bmatrix}1 & 1 & 2\\-1 & 1 & 0\end{bmatrix}$. K and M are transposes of one another.

The conditions $\vec{n} \cdot \vec{d}_1 = 0$ and $\vec{n} \cdot \vec{d}_2 = 0$ from the previous part translate to the following system of equations:

$$\begin{aligned} x + y + 2z &= 0 \\ -x + y &= 0. \end{aligned}$$

This system of equations can be represented by the matrix equation

$$\begin{bmatrix}1 & 1 & 2\\-1 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Lesson 16: Change of Basis I

Textbook

Section 4.4

Objectives

- Write a vector in multiple bases.
- Explain what the notation $[\vec{v}]_{\mathcal{B}}$ means.
- Explain what the notation $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{\mathcal{B}}$ means.

Motivation

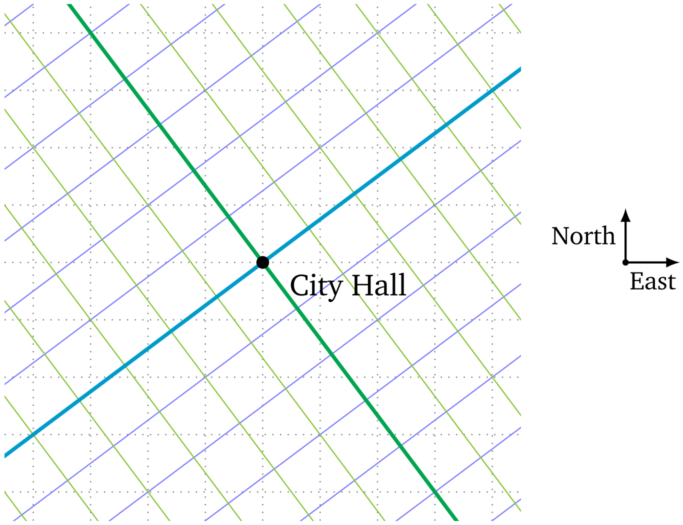
One of the most useful ideas in linear algebra is that you can represent a vector, a geometric object, with a list of numbers. This is done by picking a basis. So far we've implicitly used the standard basis, but now we're going to use other bases.

Now lists of numbers can mean many different things and can be identified with vectors in many ways, so we need some notation to keep things straight. It's important now to distinguish when something is a list of numbers (a matrix) and when it is a vector. This distinction will arise again when we talk about linear transformations and their matrix representations.

Notes/Misconceptions

So far, we have written $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to mean $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}}$ where $\mathcal{E}$ is the standard basis. We will continue to do this as convenient, but if multiple bases are ever involved, we will be careful to specify the basis.

37 The fictional town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \frac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

37.1 The points $A = (2, 1)$ and $B = (3, -1)$ are given in street coordinates. Find their east-north coordinates.

$A = \frac{1}{5}(5, -4)$ and $B = (3, 1)$ in east-north coordinates.

We obtain A for example by finding the vector $2\vec{d}_1 + \vec{d}_2$.

37.2 The points $X = (4, 3)$ and $Y = (1, 7)$ are given in east-north coordinates. Find their street coordinates. $X = (5, 0)$ and $Y = (5, 5)$ in street coordinates.

37.3 Define $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$. Does $\operatorname{span}\{\vec{e}_1, \vec{e}_2\} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$?

Yes. Both of these sets spans all of $\mathbb{R}^2$.

37.4 Notice that $Y = 5\vec{e}_1 + 5\vec{e}_2 = \vec{d}_1 + 7\vec{d}_2$. Is the point Y better represented by the pair $(5, 5)$ or by the pair $(1, 7)$? Explain.

It is equally well represented by either pair. For example, the street coordinates might be more useful for a resident of Oronto, while the east-north coordinates might be more useful for someone looking at Oronto on a world map.

Representation in a Basis

DEFINITION

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notate $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

such that $\vec{v} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$.

Similarly,

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{B}} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \ldots, \vec{b}_n$ with coefficients $\alpha_1, \ldots, \alpha_n$.

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38 Let $\mathcal{E} = \{\hat{x}, \hat{y}\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

38.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\hat{x}$ and $\hat{y}$. $\vec{c}_1 = 2\hat{x} + \hat{y}$ and $\vec{c}_2 = 5\hat{x} + 3\hat{y}$.

38.2 Express $\hat{x}$ and $\hat{y}$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$. $\hat{x} = 3\vec{c}_1 - \vec{c}_2$ and $\hat{y} = -5\vec{c}_1 + 2\vec{c}_2$.

38.3 Let $\vec{v} = 2\hat{x} + 2\hat{y}$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

$[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

The second one is since

$$\vec{v} = 2\hat{x} + 2\hat{y} = 2(3\vec{c}_1 - \vec{c}_2) + 2(-5\vec{c}_1 + 2\vec{c}_2) = -4\vec{c}_1 + 2\vec{c}_2.$$

38.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}} = [\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

$X = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is such a matrix.

We know X must be a 2×2 matrix, so suppose $X = \begin{bmatrix} a & b \\ c & b \end{bmatrix}$ for some $a, b, c, d \in \mathbb{R}$.

From the first part above, we know

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}},$$

and so we need X to satisfy

$$X\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad X\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

But $X\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $X\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$, so we can now immediately solve for a, b, c, d to find that X must be the matrix $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

38.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}} = [\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

$Y = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is such a matrix.

Using similar reasoning to the previous part, we know Y must be a 2×2 matrix, so suppose $Y = \begin{bmatrix} a & b \\ c & b \end{bmatrix}$ for some $a, b, c, d \in \mathbb{R}$.

From the second part above, we know

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_{\mathcal{C}} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}_{\mathcal{C}},$$

and so we need Y to satisfy

$$Y\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}\quad\text{and}\quad Y\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-5\\2\end{bmatrix},$$

But $Y\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a\\c\end{bmatrix}$ and $Y\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}b\\d\end{bmatrix}$, so we can now immediately solve for a,b,c,d to find that Y must be the matrix $\begin{bmatrix}3&-5\\-1&2\end{bmatrix}$.

38.6 What is YX?

$YX=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

Lesson 17: Orientation, Matrix Transformations

Textbook

Section 3.2

Objectives

- Identify the orientation of ordered bases in $\mathbb{R}^2$.
- Given a set of input and output vectors for a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$, find a matrix for the transformation.
- Given a picture X and its image under a linear transformation, find a matrix for the transformation.

Motivation

Orientation is a topic that comes up in physics and explains the sign of the determinant. We define orientation with an existential statement about whether or not certain homeomorphisms exist. The goal is not to prove anything rigorously about orientation, but to get students to make pictures for themselves of vectors moving.

The Idea is simple: $n-1$ vectors span a space that partitions $\mathbb{R}^n$ in two. Add a vector in the top partition (appropriately ordered) and you get a positive orientation; add to the bottom and you get a negative orientation. There's no way to get from one to the other without passing through the hyperplane. We focus on $\mathbb{R}^2$ so that the pictures are easy to draw. Eventually we will compute orientation from the determinant, but it's nice to have a grounding in where it comes from.

While we're thinking dynamically, we can start thinking about transformation. We already know how to multiply a matrix and a vector and interpret it in two different ways. Now we will add a third: multiplication by a given matrix is a transformation from vectors to vectors.

Most of our study of matrix transformations will be of transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, even though non-square matrices can describe other transformations. For now we stick with pictures of $\mathbb{R}^2$ since they are easy to draw. Then we will generalize to linear transformations.

DEFINITION

Orientation of a Basis

The ordered basis $\mathcal{B}=\{\vec{b}_1,\ldots,\vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i\mapsto\vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

---

39 Let $\{\hat{x},\hat{y}\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\hat{x}$ measured counter-clockwise starting at $\hat{x}$.

39.1 For which θ is $\{\hat{x},\vec{u}_\theta\}$ a linearly independent set? Every θ that is not a multiple of π.

39.2 For which θ can $\{\hat{x},\vec{u}_\theta\}$ be continuously transformed into $\{\hat{x},\hat{y}\}$ and remain linearly independent the whole time?

Every $\theta\in(0,\pi)$.

For $\theta\in(\pi,2\pi)$, a continuous transformation of $\vec{u}_\theta$ to $\hat{y}$ would have to cross the x-axis, at which point $\{\hat{x},\vec{u}_\theta\}$ would cease to be linearly independent.

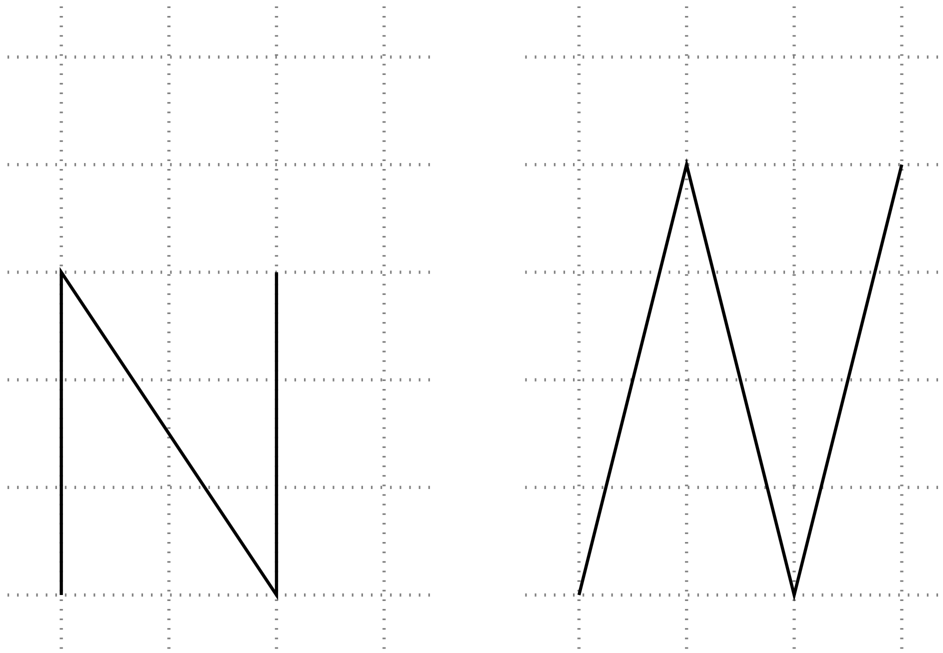
39.3 For which θ is $\{\hat{x},\vec{u}_\theta\}$ right-handed? Left-handed?

It is right-handed for $\theta\in(0,\pi)$, and left handed for $\theta\in(\pi,2\pi)$.

39.4 For which θ is $\{\vec{u}_\theta,\hat{x}\}$ (in that order) right-handed? Left-handed?

It is right-handed for $\theta\in(\pi,2\pi)$, and left handed for $\theta\in(0,\pi)$.

Task 2.1: Italicizing N



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

Lesson 18: Linear Transformations I

Textbook

Section 3.2

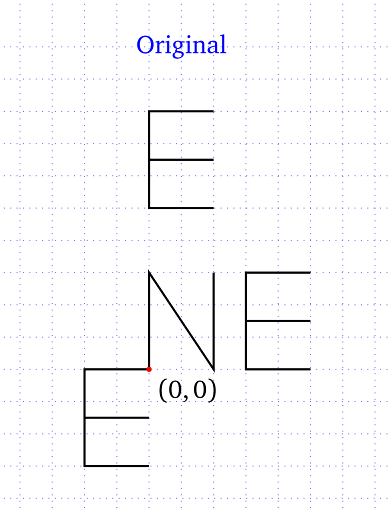
Objectives

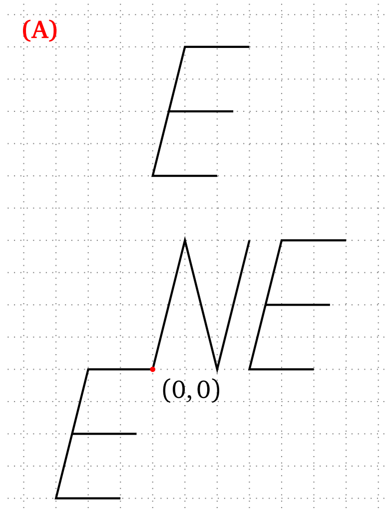
-

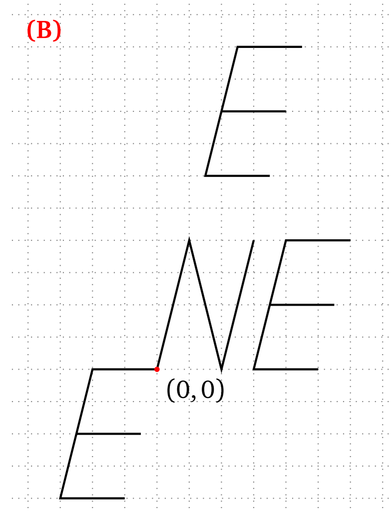
Motivation

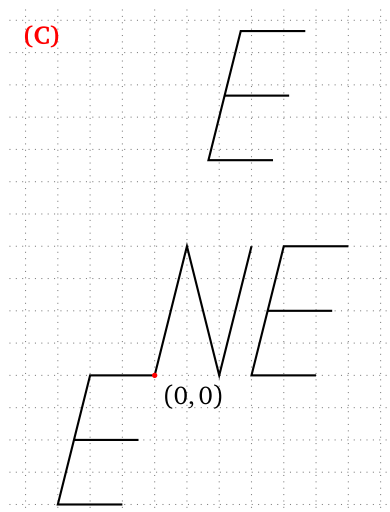
Task 2.2: Beyond the N

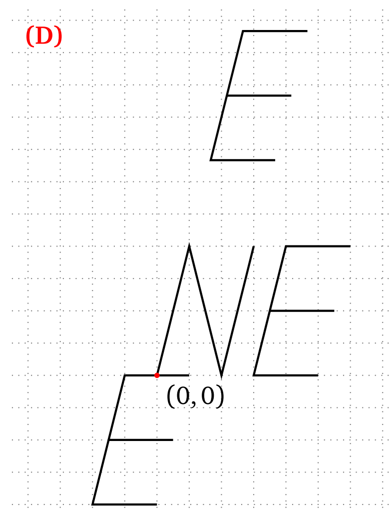
A few students were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If an "E" is placed around the "N," the students argued over four different possible results for the transformed E's. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?











40 $\mathcal{R}: \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

40.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$. $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$.

40.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

$\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}=\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}+\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

40.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

$\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)=a\begin{bmatrix}0\\1\end{bmatrix}+b\begin{bmatrix}-1\\0\end{bmatrix}$.

Rotating a vector and then multiplying by a scalar gives the same result as multiplying first then rotating. Similarly, adding two vectors and then rotating their sum gives the same result as rotating them and then adding.

40.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter-clockwise by 90°.

$R=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is such a matrix.

DEFINITION

Linear Transformation

Let V and W be subspaces. A function $T: V \to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v})=T\vec{u}+T\vec{v} \qquad \text{and} \qquad T(\alpha\vec{v})=\alpha T\vec{v}$$

for all vectors $\vec{u},\vec{v}\in V$ and all scalars α.

41 41.1 Classify the following as linear transformations or not.

(a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

A linear transformation. We proved this in the previous problem.

(b) $W: \mathbb{R}^2 \to \mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x^2\\y\end{bmatrix}$.

Not a linear transformation, since for example $W\left(2\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\end{bmatrix}\neq\begin{bmatrix}2\\0\end{bmatrix}=2W\begin{bmatrix}1\\0\end{bmatrix}$.

(c) $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2\\y\end{bmatrix}$.

Not a linear transformation, since for example $T\left(2\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\end{bmatrix}\neq 2T\begin{bmatrix}1\\0\end{bmatrix}$.

(d) $\mathcal{P}: \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix}=\operatorname{comp}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

A linear transformation.
We found a general formula for $\operatorname{comp}_{\vec{u}}$ in a previous exercise:

$$\operatorname{comp}_{\vec{u}}\vec{x}=\frac{\vec{u}\cdot\vec{x}}{\vec{u}\cdot\vec{u}}\vec{u}=\frac{\vec{u}\cdot\vec{x}}{13}\vec{u}.$$

For any two vectors $\vec{x}$ and $\vec{y}$, we have

$$\begin{aligned}\operatorname{comp}_{\vec{u}}(\vec{x}+\vec{y})&=\frac{\vec{u}\cdot(\vec{x}+\vec{y})}{13}\vec{u}\\&=\frac{\vec{u}\cdot\vec{x}}{13}\vec{u}+\frac{\vec{u}\cdot\vec{y}}{13}\vec{u}\\&=\operatorname{comp}_{\vec{u}}\vec{x}+\operatorname{comp}_{\vec{u}}\vec{y}.\end{aligned}$$

For any $\vec{x}$ and scalar α, we have

$$\operatorname{comp}_{\vec{u}}(\alpha\vec{x})=\frac{\vec{u}\cdot(\alpha\vec{x})}{13}\vec{u}=\frac{\alpha(\vec{u}\cdot\vec{x})}{13}\vec{u}=\alpha\operatorname{comp}_{\vec{u}}\vec{x}.$$

Lesson 19: Linear Transformations II, Composition of Linear Transformations

Textbook

Section 1.2

Objectives

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Motivation

DEFINITION

Image of a Set

Let $L : V \to W$ be a transformation and let $X \subset V$ be a set. The ***image of the set** V **under** L*, denoted $L(V)$, is the set

$$L(V) = \{\vec{x} \in W \ : \ \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in V\}.$$

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42 Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\} \subseteq \mathbb{R}^2$ be the filled-in unit square and let $C = \{\vec{0}, \hat{x}, \hat{y}, \hat{x} + \hat{y}\} \subseteq \mathbb{R}^2$ be the corners of the unit square.

42.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

$\mathcal{R}(C) = \{\vec{0}, \hat{y}, -\hat{x}, -\hat{x} + \hat{y}\}$.

$W(C) = C$.

$T(C) = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$.

42.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).

XXXX

42.3 Let $\ell = \{\text{all convex combinations of } \vec{a} \text{ and } \vec{b}\}$ be a line segment with endpoints $\vec{a}$ and $\vec{b}$ and let A be a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

$A(\ell)$ must be a line segment, with endpoints $A(\vec{a})$ and $A(\vec{b})$.

For any scalars α_1 and α_2, by the linearity of A we have: $A(\alpha_1\vec{a} + \alpha_2\vec{b}) = \alpha_1 A(\vec{a}) + \alpha_2 A(\vec{b})$.

If $\alpha_1 + \alpha_2 = 1$, then the linear combination on the right is also convex, and so $A(\ell)$ is the set of convex combinations of $A(\vec{a})$ and $A(\vec{b})$. This is precisely the straight line segment joining $A(\vec{a})$ and $A(\vec{b})$.

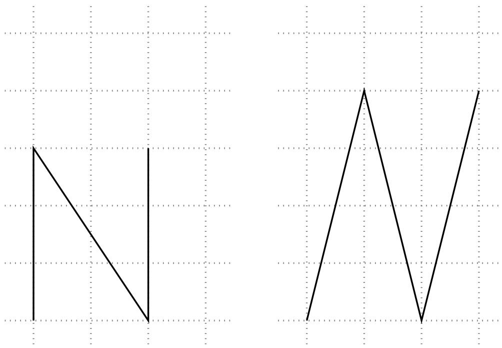
Note that if $A(\vec{a}) = A(\vec{b})$ (for example, if A is the zero transformation), then $A(\ell)$ will consist of the single point, which we think of as a "degenerate" line segment in this situation.

42.4 Explain how images of sets relate to the *Italicizing N* task.

The task asked us to find a linear transformation such that the image of the regular "N" is the italicized "N".

By the previous exercise, we now know it suffices to find a linear transformation that sends the four endpoints of line segments on the regular "N" to the corresponding four endpoints on the italicized "N".

Task 2.3: Pat and Jamie



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

1. Do you think Pat and Jamie's approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?

2. Try Pat and Jamie's approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

Lesson 20: Range, Nullspace

Textbook

Section 3.4

Objectives

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Motivation

43 Define $\mathcal{P}$ to be projection onto $\operatorname{span}\{\vec{u}\}$ where $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

43.1 Find a matrix P so that $P\vec{x}=\mathcal{P}(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$.

$P=\frac{1}{13}\begin{bmatrix}4&6\\6&9\end{bmatrix}$ is such a matrix.

The matrix P corresponding to $\mathcal{P}$ is a 2×2 matrix, so suppose $P=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ for some $a,b,c,d\in\mathbb{R}$. Then we know that if $\{\hat{x},\hat{y}\}$ is the standard basis for $\mathbb{R}^2$,

$$P(\hat{x})=\begin{bmatrix}a\\c\end{bmatrix}\quad\text{and}\quad P(\hat{y})=\begin{bmatrix}b\\d\end{bmatrix}.$$

We know from an earlier exercise that $\mathcal{P}(\vec{x})=\dfrac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$. Therefore, the first column of P is

$$\begin{bmatrix}a\\c\end{bmatrix}=\mathcal{P}(\hat{x})=\frac{2}{13}\vec{u}=\frac{1}{13}\begin{bmatrix}4\\6\end{bmatrix}$$

and the second column of P is

$$\begin{bmatrix}b\\d\end{bmatrix}=\mathcal{P}(\hat{y})=\frac{3}{13}\vec{u}=\frac{1}{13}\begin{bmatrix}6\\9\end{bmatrix}.$$

43.2 Find a matrix R so that $R\vec{x}=\mathcal{R}(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$.

$R=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is such a matrix.

Using the same reasoning as the previous part, we can compute

$$\mathcal{R}(\hat{x})=\hat{y}=\begin{bmatrix}0\\1\end{bmatrix}\quad\text{and}\quad\mathcal{R}(\hat{y})=-\hat{x}=\begin{bmatrix}-1\\0\end{bmatrix}.$$

Therefore, the matrix R for $\mathcal{R}$ is the matrix with the two vectors above as its respective columns.

43.3 Write down matrices A and B for $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{P}$.

$A=\frac{1}{13}\begin{bmatrix}6&-4\\9&-6\end{bmatrix}$ and $B=\frac{1}{13}\begin{bmatrix}-6&-9\\4&6\end{bmatrix}$ are two such matrices.

Using the same reasoning as above, we can compute

$$(\mathcal{P}\circ\mathcal{R})(\hat{x})=\mathcal{P}(\mathcal{R}(\hat{x}))=\mathcal{P}(\hat{y})=\frac{1}{13}\begin{bmatrix}6\\9\end{bmatrix}\qquad\text{and}\qquad(\mathcal{P}\circ\mathcal{R})(\hat{y})=\mathcal{P}(\mathcal{R}(\hat{y}))=\mathcal{P}(-\hat{x})=\frac{1}{13}\begin{bmatrix}-4\\-6\end{bmatrix}.$$

Therefore, the matrix A for $\mathcal{P}\circ\mathcal{R}$ is the matrix with the two vectors above as its respective columns.

Similarly, for $\mathcal{R}\circ\mathcal{P}$, we can compute:

$$(\mathcal{R}\circ\mathcal{P})(\hat{x})=\mathcal{R}(\mathcal{P}(\hat{x}))=\mathcal{R}\left(\frac{1}{13}\begin{bmatrix}4\\6\end{bmatrix}\right)=\frac{1}{13}\begin{bmatrix}-6\\4\end{bmatrix}$$
$$(\mathcal{R}\circ\mathcal{P})(\hat{y})=\mathcal{R}(\mathcal{P}(\hat{y}))=\mathcal{R}\left(\frac{1}{13}\begin{bmatrix}6\\9\end{bmatrix}\right)=\frac{1}{13}\begin{bmatrix}-9\\6\end{bmatrix}.$$

Therefore, the matrix B for $\mathcal{R}\circ\mathcal{P}$ is the matrix with these two vectors as its respective columns.

43.4 How do the matrices A and B relate to the matrices P and R?

$A=PR$ and $B=RP$.

We can simply compute these matrix products to see this, but from the previous parts we that for any vector $\vec{x}$

$$A\vec{x}=(\mathcal{P}\circ\mathcal{R})(\vec{x})=\mathcal{P}(\mathcal{R}(\vec{x}))=\mathcal{P}(R\vec{x})=PR\vec{x}.\quad\text{and}\quad B\vec{x}=(\mathcal{R}\circ\mathcal{P})(\vec{x})=\mathcal{R}(\mathcal{P}(\vec{x})=\mathcal{R}(P\vec{x})=RP\vec{x}.$$

Using $\vec{x}=\hat{x}$ shows that first column of A must equal the first column of PR, and using $\vec{x}=\hat{y}$ shows that the second column fo A must equal the second column of PR, and therefore $A=PR$. For the same reason, we must also have $B=RP$.

DEF

Range

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\operatorname{range}(T) = \{\vec y \in W \,:\, \vec y = T\vec x \text{ for some } \vec x \in V\}.$$

DEFINITION

Null Space

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$\operatorname{null}(T) = \{\vec x \in V \,:\, T\vec x = \vec 0\}.$$

---

44 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto $\operatorname{span}\{\vec u\}$ where $\vec u = \begin{bmatrix}2\\3\end{bmatrix}$ (like before).

44.1 What is the range of $\mathcal{P}$?

$\operatorname{range}(\mathcal{P}) = \operatorname{span}\{\vec u\}$.

$\mathcal{P}(\vec x)$ is by definition the vector in $\operatorname{span}\{\vec u\}$ that is closest to $\vec x$, so in particular $\mathcal{P}(\vec x) \in \operatorname{span}\{\vec u\}$ for all $\vec x \in \mathbb{R}^2$. Therefore $\operatorname{range}(\mathcal{P}) \subseteq \operatorname{span}\{\vec u\}$.

On the other hand, $\mathcal{P}(\alpha\vec u) = \alpha\mathcal{P}(\vec u) = \alpha\vec u$ for any scalar α, and so $\operatorname{range}(\mathcal{P}) = \operatorname{span}\{\vec u\}$.

44.2 What is the null space of $\mathcal{P}$?

$$\operatorname{null}(\mathcal{P}) = \operatorname{span}\left\{\begin{bmatrix}3\\-2\end{bmatrix}\right\}.$$

A vector $\vec x$ projects to $\vec 0$ if and only if $\vec x$ is on the line perpendicular to $\operatorname{span}\{\vec u\}$ passing through the origin.

---

45 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

45.1 Show that the null space of T is a subspace.

(i) Let $\vec u, \vec v \in \operatorname{null}(T)$. Applying the linearity of T we see $T(\vec u + \vec v) = T(\vec u) + T(\vec v) = \vec 0 + \vec 0 = \vec 0$, and so $\vec u + \vec v \in \operatorname{null}(T)$.

(ii) Let $\vec u \in \operatorname{null}(T)$ and let α be any scalar. Again using the linearity of T we see $T(\alpha\vec u) = \alpha T(\vec u) = \alpha\vec 0 = \vec 0$, and so $\alpha\vec u \in \operatorname{null}(T)$.

45.2 Show that the range of T is a subspace.

(i) Let $\vec y, \vec z \in \operatorname{range}(T)$. Then there exist $\vec u, \vec v \in \mathbb{R}^n$ such that $T(\vec u) = \vec y$ and $T(\vec v) = \vec z$. Then $\vec y + \vec z = T(\vec u) + T(\vec v) = T(\vec u + \vec v)$, since T is linear, and so $\vec y + \vec z \in \operatorname{range}(T)$.

(ii) Let $\vec y \in \operatorname{range}(T)$ and let α be any scalar. Then there exists $\vec u \in \mathbb{R}^n$ such that $T(\vec u) = \vec y$, and $\alpha\vec y = \alpha T(\vec u) = T(\alpha\vec u)$, since T is linear, and so $\alpha\vec y \in \operatorname{range}(T)$.

DEFINITION

Induced Transformation

Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $T_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[T_M\vec v]_{\mathcal{E}'} = M[\vec v]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

Lesson 21: Fundamental Subspaces

Textbook

Section 3.4

Objectives

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Motivation

46 Let M be a 2×2 matrix and let $\vec{v}\in\mathbb{R}^2$. Further, let T_M be the transformation induced by M.

46.1 What is the difference between "$M\vec{v}$" and "$M[\vec{v}]_{\mathcal{E}}$"?

"$M\vec{v}$" is ambiguous notation, as it is only defined if $\vec{v}$ is a specific column vector. There are infinitely many different bases of $\mathbb{R}^2$, and so a given vector $\vec{v}$ has infinitely many different representations as a column vector, each in a different basis.

"$M[\vec{v}]_{\mathcal{E}}$" is unambiguous, as $[\vec{v}]_{\mathcal{E}}$ is an explicit representation of $\vec{v}$ in a particular basis.

46.2 What is $[T_M\vec{\hat{x}}]_{\mathcal{E}}$?

It is the first column of M.

By definition, $[T_M\hat{x}]_{\mathcal{E}} = M[\hat{x}]_{\mathcal{E}} = M\begin{bmatrix}1\\0\end{bmatrix}$, which equals the first column of M.

46.3 Can you relate the columns of M to the range of T_M?

The range of T_M equals the span of the columns of M.

By the previous part, the first column of M is in the range of T_M. By a similar argument, the second column of M is also in the range of T_M, since it equals $[T_M\hat{y}]_{\mathcal{E}}$. Therefore the span of the columns of M is a subset of the range of T_M.

On the other hand, if $\vec{v}_1=\begin{bmatrix}a\\c\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}b\\d\end{bmatrix}$ are the columns of M and $\vec{x}=\alpha_1\vec{v}_1+\alpha_2\vec{v}_2$ is an element of $\operatorname{span}\{\vec{v}_1,\vec{v}_2\}$, then

$$[\vec{x}]_{\mathcal{E}} = \alpha_1\begin{bmatrix}a\\c\end{bmatrix}+\alpha_2\begin{bmatrix}b\\d\end{bmatrix} = \alpha_1[T_M\hat{x}]_{\mathcal{E}}+\alpha_2[T_M\hat{y}]_{\mathcal{E}} = [T_M(\alpha_1\hat{x}+\alpha_2\hat{y})]_{\mathcal{E}}.$$

Therefore $\vec{x}$ is in the range of T_M.

Fundamental Subspaces

DEFINITION

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\operatorname{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\operatorname{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\operatorname{null}(M)$, is the set of solutions to $M\vec{x}=\vec{0}$.

47 Consider $A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$.

47.1 Describe the row space of A.

$\operatorname{row}(A)=\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$, which is the xy-plane in $\mathbb{R}^3$.

47.2 Describe the column space of A.

$\operatorname{col}(A)=\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}=\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}=\mathbb{R}^2$.

47.3 Is the row space of A the same as the column space of A?

No.

Although they are both two dimensional spaces, $\operatorname{row}(A)$ is a subspace of $\mathbb{R}^3$ and all vectors in it have three coordinates (with the third always being zero), while $\operatorname{col}(A)$ is a subspace of $\mathbb{R}^2$ and all vectors in it have two coordinates. Therefore, these two spaces are different.

47.4 Describe the set of all vectors perpendicular to the rows of A.

The z-axis in $\mathbb{R}^3$.

A vector $\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$ is perpendicular to the rows of A if and only if its dot product with both rows is zero. That is

$$\vec{x}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}=x=0 \quad\text{and}\quad \vec{x}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}=y=0.$$

$\vec{x}$ satisfies these equations if and only if $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$ for some real number t, or in other words if $\vec{x}$ is on the z-axis.

47.5 Describe the null space of A.

The z-axis in $\mathbb{R}^3$.

A vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in null(A) if and only if

$$A\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}.$$

These are the same conditions as in the previous part, so the set of vectors satisfying this is the z-axis.

47.6 Describe the range and null space of T_A, the transformation induced by A.

range$(T_A) = $ col$(A) = \mathbb{R}^2$ and null$(T_A) = $ null(A), which is the z-axis in $\mathbb{R}^3$.

By Problem 46.3, the range of an induced transformation equals the span of the columns of the matrix. In other words, range$(T_A) = $ col(A).

Next, by definition $\vec{v} \in$ null(T_A) when $[T_A\vec{v}]_{\mathcal{E}} = A[\vec{v}]_{\mathcal{E}} = \vec{0}$. In other words, $\vec{v} \in$ null(T_A) if and only if $[\vec{v}]_{\mathcal{E}} \in$ null(A). We know from the previous part that null(A) is the z-axis in $\mathbb{R}^3$.

48

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

48.1 How does the row space of B relate to the row space of C?

They are equal.

Row operations replace rows with linear combinations of rows. Therefore, since C is the matrix B after the application of some row operations, row$(C) \subseteq$ row(B).

Since row operations are all reversible, we also know that B can be obtained from C by applying row operations, so row$(B) \subseteq$ row(C).

Therefore, row$(B) = $ row(C).

48.2 How does the null space of B relate to the null space of C?

They are equal.

A vector is in null(B) or null(C) if and only if it is orthogonal to all vectors in row(B) or all vectors in row(C), respectively. But row$(B) = $ row(C) by the previous part, so their null spaces must also be equal.

48.3 Compute the null space of B.

null$(C) = \mathrm{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right\}$.

We compute null(C), since it equals null(B) by the previous part.

$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in null(C) if and only if $C\vec{x} = \begin{bmatrix} x - z \\ y + 2z \end{bmatrix} = \vec{0}$. The complete solution to this matrix equation is

$$\mathrm{null}(C) = \left\{\begin{bmatrix} t \\ -\frac{t}{2} \\ t \end{bmatrix} \in \mathbb{R}^3 \;:\; t \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right\}.$$

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \operatorname{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

49.1 How does the column space of P relate to the column space of Q?

They are not equal, but have the same dimension.

49.2 Describe the column space of P and the column space of Q.

$\operatorname{col}(P) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, which is the y-axis in $\mathbb{R}^2$.

$\operatorname{col}(Q) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, which is the x-axis in $\mathbb{R}^2$.

Lesson 22: Rank

Textbook

Section 3.4

Objectives

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Motivation

Rank

DEF

For a linear transformation $T : V \to W$, the ***rank*** of T, denoted rank(T), is the dimension of the range of T.

For an $n \times m$ matrix M, the ***rank*** of M, denoted rank(M), is the number of pivots in rref(M).

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50 Let $\mathcal{P}$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

50.1 Describe range($\mathcal{P}$) and range($\mathcal{R}$).

range($\mathcal{P}$) = span$\{\vec{u}\}$, and range($\mathcal{R}$) = $\mathbb{R}^2$.

For $\mathcal{P}$, by the definition of projection $\mathcal{P}(\vec{x})$ is the vector in span$\{\vec{u}\}$ that is closest to $\vec{x}$, so in particular $\mathcal{P}(\vec{x}) \in$ span$\{\vec{u}\}$ for all $\vec{x} \in \mathbb{R}^2$. Therefore range($\mathcal{P}$) $\subseteq$ span$\{\vec{u}\}$.

On the other hand, $\mathcal{P}(\alpha\vec{u}) = \alpha\mathcal{P}(\vec{u}) = \alpha\vec{u}$ for any scalar α, and so range($\mathcal{P}$) = span$\{\vec{u}\}$.

For $\mathcal{Q}$, we have that any vector $\vec{x} \in \mathbb{R}^2$, $\vec{x} = \mathcal{Q}(\vec{y})$, where $\vec{y}$ is the rotation of $\vec{x}$ clockwise by 90°. Therefore range($\mathcal{Q}$) = $\mathbb{R}^2$.

50.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

rank($\mathcal{P}$) = 1 and rank($\mathcal{R}$) = 2.

By the previous part, we know range($\mathcal{P}$) is 1-dimensional and range($\mathcal{Q}$) is 2-dimensional.

50.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

rank(P) = 1 and rank(R) = 2.

By Problem 43, $P = \frac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. Then we compute:

$$\text{rref}(P) = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{rref}(R) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These matrices have 1 and 2 pivots, respectively.

50.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

They are equal.

By Problem 46.3, the range of a transformation is equal to the column space of its corresponding matrix, and therefore the dimensions of these two spaces are equal. In other words, the rank of a transformation is equal to the dimension of the the column space of its corresponding matrix.

We already know that the dimension of the column space of a matrix is equal to the number of pivots in its reduced row echelon form, and that is by definition the rank of the matrix.

51

51.1 Determine the rank of (a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

For each part, we compute the reduced row echelon form of the matrix and count the number of pivots.

(a) rank$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\right) = 1$, since rref$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has one pivot.

(b) rank$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 2$, since rref$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has two pivots

(c) rank$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 2$. This matrix is already in reduced row echelon form, and has two pivots.

(d) rank$\left(\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}\right) = 1$, since rref$\left(\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has one pivot.

(e) rank$\left(\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\right)=3$, since rref$\left(\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ has three pivots.

---

52 Consider the homogeneous system

$$\begin{array}{rrrl} x & +2y & +z & =0\\ x & +2y & +3z & =0\\ -x & -2y & +z & =0 \end{array} \tag{1}$$

and the non-augmented matrix of coefficients $A=\begin{bmatrix}1&2&1\\1&2&3\\-1&-2&1\end{bmatrix}$.

52.1 What is rank(A)?

rank$(A)=2$, since rref$(A)=\begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$ as two pivots.

52.2 Give the general solution to (2b).

$\vec{x}$ is a solution to the system if $\vec{x}=t\begin{bmatrix}-2\\1\\0\end{bmatrix}$ for some real number t.

If $\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$ is a solution to the system, then we must have $\begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x+2y\\z\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, from which it follows that $z=0$ and $x=-2y$. In other words, any scalar multiple of $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ is a solution.

52.3 Are the column vectors of A linearly independent?

No. The second column is two times the first column.

52.4 Give a non-homogeneous system with the same coefficients as (2b) that has

(a) infinitely many solutions

(b) no solutions.

(a)

$$\begin{array}{rrrl} x & +2y & +z & =1\\ x & +2y & +3z & =1\\ -x & -2y & +z & =-1 \end{array}$$

(b)

$$\begin{array}{rrrl} x & +2y & +z & =0\\ x & +2y & +3z & =0\\ -x & -2y & +z & =1 \end{array}$$

---

53 53.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

No. A 3×4 matrix has four columns, each of which are vectors in $\mathbb{R}^3$. It is not possible for four different vectors in $\mathbb{R}^3$ to be linearly indepedent.

53.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Yes. Since rank$(B)=3$, there are three pivots in rref(B). Pivot positions in rref(B) indicate a maximal linearly independent subset of the columns of B. Since there are three columns in B and three pivots, the three columns of B must be linearly independent.

Lesson 23: Rank-nullity Theorem, Inverses I

Textbook

Section 1.2

Objectives

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Motivation

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \# \text{ of columns in } A.$$

---

54 54.1 Is here a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

Yes. If $T : V \to W$ is a linear transformation, then $\operatorname{rank}(T) + \dim(\operatorname{null}(T)) = \dim(V)$.

If A is the matrix corresponding to T, then $\operatorname{rank}(T) = \operatorname{rank}(A)$ by Problem 50.4.

$\operatorname{null}(T) = \operatorname{null}(A)$ by Problem 47.6, since $T = T_A$, and so $\dim(\operatorname{null}(T)) = \operatorname{nullity}(A)$.

Finally, the number of columns of A is equal to the dimension of the domain of T.

---

55 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

55.1 What is the rank and nullity of A^T?

$\operatorname{rank}(A^T) = 2$ and $\operatorname{nullity}(A^T) = 1$.

A^T is the matrix with rows $\vec{u}, \vec{v}$, and $\vec{w}$. Since $\vec{w} = 2\vec{u} - \vec{v}$, the third row of A^T can be reduced to a row of zeros by the row operation $R_3 \mapsto R_3 - 2R_1 + R_2$. Neither of the first two rows can be reduced to rows of zeros since they are linearly independent. Therefore $\operatorname{rref}(A^T)$ has two pivots, meaning $\operatorname{rank}(A^T) = 2$.

The rank-nullity theorem then says that $2 + \operatorname{nullity}(A^T) = 3$, and so $\operatorname{nullity}(A^T) = 1$

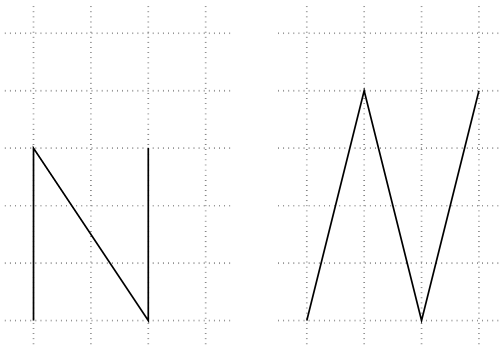
55.2 What is the rank and nullity of A?

$\operatorname{rank}(A) = 2$ and $\operatorname{nullity}(A) = 1$.

We know that $\operatorname{rank}(A)$ equals the number of pivots in $\operatorname{rref}(A)$, which in turn equals the dimension of $\operatorname{col}(A)$. Since A has two linearly independent columns, $\dim(\operatorname{col}(A)) = 2$.

Again, the rank-nullity theorem then says that $2 + \operatorname{nullity}(A) = 3$, and so $\operatorname{nullity}(A) = 1$

Task 2.4: Getting back N



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

Consider the new task: find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.
2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

Lesson 24: Inverses II, Elementary Matrices

Textbook

Section 3.5

Objectives

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Motivation

56 56.1 Apply the row operation $R_3 \mapsto R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3 \mapsto R_3+2R_1} \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_1.$$

56.2 Apply the row operation $R_3 \mapsto R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3 \mapsto R_3-2R_1} \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = E_2.$$

DEF An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

56.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

$$E_1A = \begin{bmatrix}1&2&3\\4&5&6\\9&12&15\end{bmatrix} \text{ and } E_2A = \begin{bmatrix}1&2&3\\4&5&6\\5&4&3\end{bmatrix}.$$

E_1A is the result applying the row operation $R_3 \mapsto R_3 + 2R_1$ to A, and similarly E_2A is the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to A.

56.4 Without computing, what should the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to E_1 be? Compute and verify.

It should be the identity matrix, since the row operation $R_3 \mapsto R_3 - 2R_1$ should undo the operation $R_3 \mapsto R_3 + 2R_1$.

56.5 Without computing, what should E_1E_2 be? What about E_2E_1? Now compute and verify.

They should both be the identity matrix.

The solution to part 3 above lead us to believe that applying E_1 to a matrix has the effect of applying the row operation $R_3 \mapsto R_3 + 2R_1$ to it. Applying that row operation to E_2 would produce the identity matrix, so we expect that E_1E_2 should equal the identity matrix.

Similar reasoning leads us to believe that E_2E_1 should also equal the identity matrix.

Indeed, we can compute

$$E_1E_2 = \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_2E_1.$$

DEF The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated by A^{-1}.

57 Consider the matrices

$$A = \begin{bmatrix}1&2&0\\0&1&0\\-3&-6&1\end{bmatrix} \quad B = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix} \quad C = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

$$D = \begin{bmatrix}1&-2&0\\0&1&0\\3&0&1\end{bmatrix} \quad E = \begin{bmatrix}1&0&0\\0&2&0\\0&1&1\end{bmatrix} \quad F = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

57.1 Which pairs of matrices above are inverses of each other?

A and D are inverses of each other, and F is its own inverse.

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

58.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_2 \mapsto \frac{1}{2}R_2} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - 4R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The two elementary matrices are $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$.

58.2 What is E_2E_1B?

$$E_2E_1B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

58.3 Find B^{-1}.

$$B^{-1} = E_2E_1 = \begin{bmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}.$$

By the previous part we already know that $(E_2E_1)B = I$. We can also check that $B(E_2E_1) = I$, meaning E_2E_1 is the inverse of B.

58.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

Suppose A is a matrix that can be row reduced to the identity. Let $E_1, E_2, \ldots, E_n$ be the elementary matrices corresponding to the sequence of row operations that reduces A to I. Then as we have seen, we have $E_nE_{n-1}\cdots E_2E_1A = I$.

Thus $E_nE_{n-1}\cdots E_2E_1$ is the inverse of A.

Lesson 25: Applications of Inverses I

Textbook

Section 3.5

Objectives

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Motivation

59

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

59.1 What is $A^{-1}A$?

$A^{-1}A = I$. This is true by the definition of an inverse, but we can also verify it by hand.

59.2 What is rref(A)?

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

59.3 What is rref(C)? (Hint, there is no need to actually do row reduction!)

$$\text{rref}(C) = \left[I \middle| A^{-1}\vec{b}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

We know that the reduced row echelon form of C must be of the form $[I|\vec{c}]$ for some $\vec{c}$, and we know that multiplying on the left by A^{-1} is equivalent to applying the sequence of row operations that reduces A to rref(A) $= I$. So the same sequence of row operations applied to $\vec{b}$, the last column of C, will produce the vector $\vec{c} = A^{-1}\vec{b} = \begin{bmatrix} -9 \\ 6 \\ 2 \end{bmatrix}$.

59.4 Solve the system $A\vec{x} = \vec{b}$.

The system has one solution: $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -9 \\ 6 \\ 2 \end{bmatrix}$.

We can read this solution from the reduced row echelon form of the augmented matrix C representing this system. We can also multiply both sides of the equation on the left by A^{-1}:

$$A\vec{x} = \vec{x} \implies A^{-1}A\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}.$$

60

60.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

No.

By the definition of an inverse we need $(XY)^{-1}(XY) = I$, so that multiplying by $(XY)^{-1}$ undoes multiplication by XY. To do this, we must first undo multiplication by X, then undo multiplication by Y. That is, we must first multiply by X^{-1} then multiply by Y^{-1}.

In other words, we expect that $(XY)^{-1} = Y^{-1}X^{-1}$. We can then verify this by computing

$$(XY)(Y^{-1}X^{-1}) = XYY^{-1}X^{-1} = XIX^{-1} = XX^{-1} = I$$

and

$$(Y^{-1}X^{-1})(XY) = Y^{-1}X^{-1}XY = Y^{-1}IY = Y^{-1}Y = I.$$

60.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

No.

Suppose M^{-1} exists. Then $M^{-1}M = MM^{-1} = I$. Let S be the linear transformation induced by M^{-1}. Since M is the matrix for T we must have $ST = TS = \text{id}$. But then S would be the inverse of T, which is impossible.

More Change of Basis

61 Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ and let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

61.1 Compute $[\hat{x}]_{\mathcal{B}}$ and $[\hat{y}]_{\mathcal{B}}$.

$[\hat{x}]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$ and $[\hat{y}]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$.

This is because $\hat{x} = \frac{1}{2}(\vec{b}_1 + \vec{b}_2)$ and $\hat{y} = \frac{1}{2}(\vec{b}_1 - \vec{b}_2)$

61.2 Compute $X[\hat{x}]_{\mathcal{B}}$ and $X[\hat{y}]_{\mathcal{B}}$. What do you notice?

$X[\hat{x}]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\vec{b}_1 + \frac{1}{2}\vec{b}_2 = \begin{bmatrix}1\\0\end{bmatrix}$ and

$X[\hat{y}]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\vec{b}_1 - \frac{1}{2}\vec{b}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

We notice that multiplying by X turns the representations of these two vectors in the basis $\mathcal{B}$ into representations in the standard basis.

61.3 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

$X^{-1} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$.

X^{-1} should undo what X does. In the previous part we saw that X takes vectors represented in $\mathcal{B}$ and represents them in the standard basis. So X^{-1} should do the reverse, and take vectors represented in the standard basis and represent them in the basis $\mathcal{B}$.

Lesson 26: Applications of Inverses II, Change of Basis

Textbook

Section 4.4

Objectives

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Motivation

62 Let $\mathcal{S}=\{\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B}=\{\vec{b}_1,\vec{b}_2,\ldots,\vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X=[\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}}=[\vec{v}]_{\mathcal{S}}.$$

62.1 Should X^{-1} exist? Explain.

Yes. X converts vectors from the standard basis into the $\mathcal{B}$ basis, and this process can be undone. X^{-1} is the matrix that does this.

62.2 Consider the equation

$$X^{-1}[\vec{v}]_?=[\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

$X^{-1}[\vec{v}]_{\mathcal{S}}=[\vec{v}]_{\mathcal{B}}$.

As we said in the previous part X^{-1} should undo what X does, meaning it should convert vectors from the standard basis into the $\mathcal{B}$ basis.

62.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

$[\vec{b}_1]_{\mathcal{B}}=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$, and $[\vec{b}_2]_{\mathcal{B}}=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, where each of these vectors have n coordinates.

In general, $[\vec{b}_i]_{\mathcal{B}}$ should be the column vector with zeroes in all coordinates except for a 1 in the i^{th} coordinate.

63 Let $\vec{c}_1=\begin{bmatrix}2\\1\end{bmatrix}$, $\vec{c}_2=\begin{bmatrix}5\\3\end{bmatrix}$, $\mathcal{C}=\{\vec{c}_1,\vec{c}_2\}$, and $A=\begin{bmatrix}2&5\\1&3\end{bmatrix}$. Note that $A^{-1}=\begin{bmatrix}3&-5\\-1&2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

63.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

63.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}5\\3\end{bmatrix}$.

63.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

63.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

63.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

63.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

Similar Matrices

DEFINITION

A matrices A and B are called ***similar matrices***, denoted $A\sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A\sim B$ if there is an invertible matrix X so that

$$A=XBX^{-1}.$$

Lesson 27: Determinants

Textbook

Section 5.4

Objectives

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Motivation

DEFINITION

Unit n-cube

The unit n-cube is the n-dimensional cube with side length 1 and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The volume of the unit n-cube is always 1.

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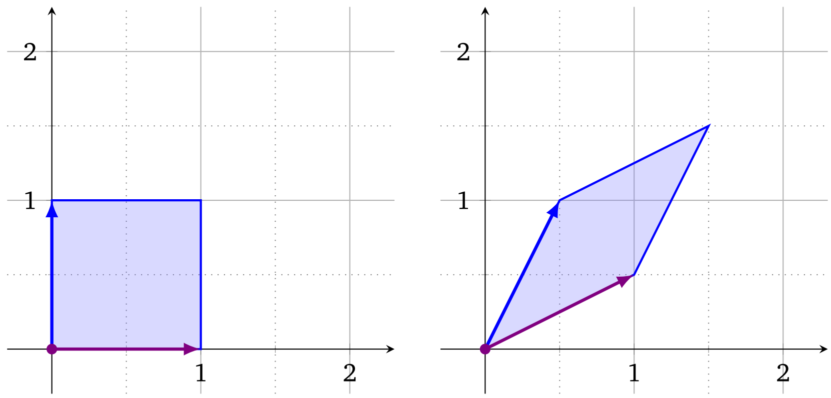
64 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



64.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}, T\begin{bmatrix}0\\1\end{bmatrix}, T\begin{bmatrix}1\\1\end{bmatrix}$?

$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}$, and $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}$?

We can see first two directly in the picture.

Using the linearity of T, we can compute

$$T\begin{bmatrix}1\\1\end{bmatrix} = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = T\begin{bmatrix}1\\0\end{bmatrix} + T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\\frac{1}{2}\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}.$$

64.2 Write down a matrix for T.

The matrix for T in the standard basis is $\begin{bmatrix}1 & \frac{1}{2}\\\frac{1}{2} & 1\end{bmatrix}$.

64.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may need to use trigonometry.

The volume is $\frac{3}{4}$.

DEFINITION

Determinant

The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors (or the row vectors).

---

65 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

65.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

XXXX

65.2 Compute the area of $A(C_2)$. The area of this parallelogram is 2.

65.3 Compute $\det(A)$.

$\det(A) = 2$.

The paralellogram with sides $\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ is positively oriented, so $\det(A) = +2$.

---

66 Suppose R is a rotation counter-clockwise by 30°.

66.1 Draw C_2 and $R(C_2)$.

XXXX

66.2 Compute the area of $R(C_2)$.

The area is 1.

R rotates the entire unit square, which does not change its area.

66.3 Compute $\det(R)$.

Since R preserves orientation, $\det(R)$ must be positive. Since R does not change the area of the unit square, $\det(R) = +1$.

---

67 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}.$$

67.1 What is $\det(F)$?

$\det(F) = -1$.

F does not change the area of the unit square, but reverses its orientation, so $\det(F) = -1$.

Lesson 28: Determinants and Compositions

Textbook

Section 5.2

Objectives

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Motivation

68 Let $D = \{\vec{x} : \|\vec{x}\| \leq 1\}$ be the unit disk. You know the following about the linear transformations M, T, and S: M is defined by $\vec{x} \mapsto 2\vec{x}$; T has determinant 2; and S has determinant 3.

68.1 Find the oriented volumes of $M(C_2)$, $T(C_2)$, and $S(C_2)$.

The volumes are 4, 2, and 3, respectively.

We can compute the the volume of $M(C_2)$ directly, since M transforms C_2 into a square with side length 2. For the other two, we are given their determinants and we know that the volume of the image of the unit square equals the determinant of each transformation.

68.2 How does the volume of $T(C_2 + \{\vec{e}_1\})$ compare to the volume of $T(C_2)$?

They are equal. The transformation T scales the volumes of all regions by 2.

68.3 What is the oriented volume of $T \circ M(C_2)$? What is $\det(T \circ M)$?

They are both equal to 8.

$T \circ M(C_2) = T(M(C_2))$. We already know $M(C_2)$ has a volume of 4, and so $T(M(C_2))$ has a volume of 8, since T scales the volumes of all regions by 2. The volume of $T \circ M(C_2)$ is the determinant of $T \circ M$ by definition.

68.4 What is the oriented volume of $S(D)$?

The volume is 3π.

69

- E_f is $I_{3\times3}$ with the first two rows swapped.
- E_m is $I_{3\times3}$ with the third row multiplied by 6.
- E_a is $I_{3\times3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

69.1 What is $\det(E_f)$?

69.2 What is $\det(E_m)$?

69.3 What is $\det(E_a)$?

69.4 What is $\det(E_f E_m)$?

69.5 What is $\det(4I_{3\times3})$?

69.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

70

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

70.1 What is $\det(U)$?

70.2 V is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

70.3 P is projection onto the vector $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. What is $\det(P)$?

71 Suppose you know $\det(X) = 4$.

71.1 What is $\det(X^{-1})$?

71.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

71.3 Suppose Y is not invertible. What is $\det(Y)$?

Lesson 29: Eigenstuff I

Textbook

Section 6.1

Objectives

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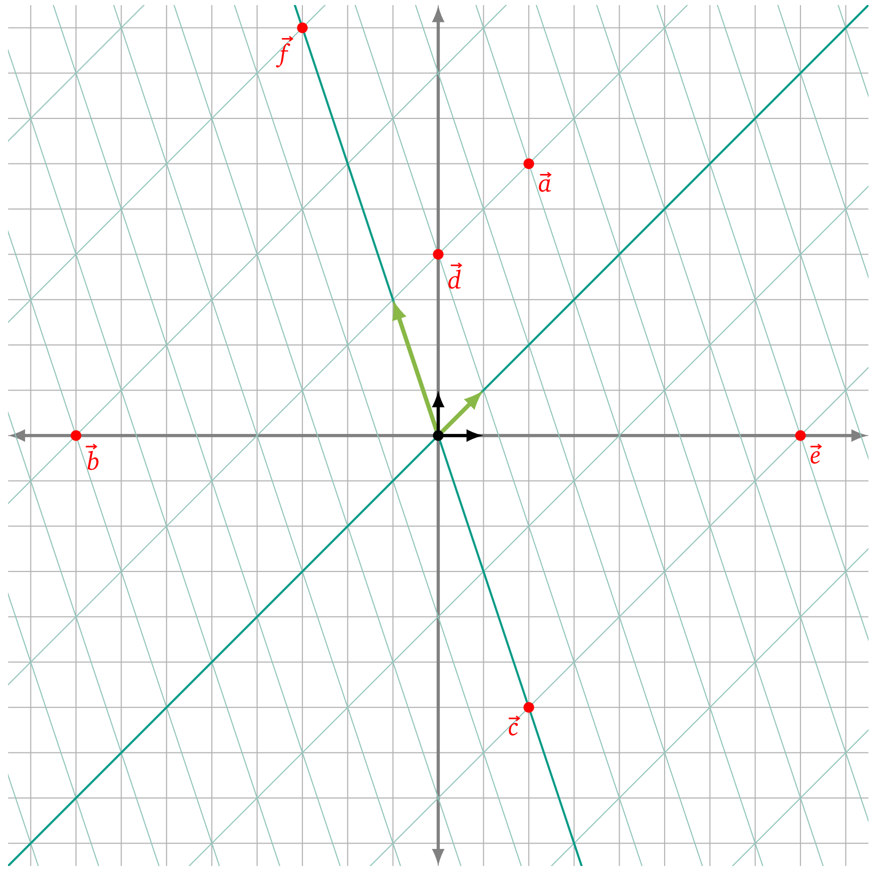
Motivation

Task 3.1: The Green and the Black

Consider the following two bases for $\mathbb{R}^2$: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$ and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.



1. Write each point above in both the green and the black bases.

2. Find a change-of-basis matrix X that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{u}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

4. When working with the transformation T, which basis do you prefer vectors be represented in?

Lesson 30: Eigenstuff II

Textbook

Section 6.1

Objectives

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Motivation

Eigenvectors

DEFINITION

Eigenvector

Let X be a linear transformation. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

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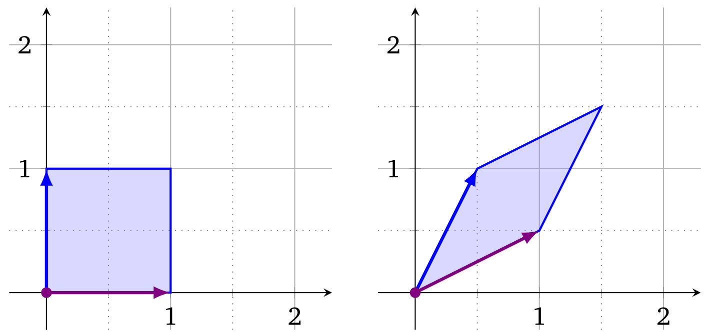
72 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



72.1 Give an eigenvector for T. What is the eigenvalue?

72.2 Can you give another?

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73 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

73.1 Give an eigenvector and a corresponding eigenvalue for A.

73.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

73.3 What is the dimension of null(B)?

73.4 What is det(B)?

---

74 Let $C = \begin{bmatrix}-1 & 2\\1 & 0\end{bmatrix}$ and $E_\lambda = C - \lambda I$.

74.1 For what values of λ does E_λ have a non-trivial null space?

74.2 What are the eigenvalues of C?

74.3 Find the eigenvectors of C.

Lesson 31: Characteristic Polynomial, Diagonalization I

Textbook

Section 6.2

Objectives

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Motivation

Characteristic Polynomial

DEF

For a matrix A, the ***characteristic polynomial*** of A is

$$\operatorname{char}(A) = \det(A - \lambda I).$$

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75 Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

75.1 Compute $\operatorname{char}(D)$.

75.2 Find the eigenvalues of D.

---

76 Suppose $\operatorname{char}(E) = \lambda(\lambda - 2)(\lambda + 3)$ for some unknown 3×3 matrix E.

76.1 What are the eigenvalues of E?

76.2 Is E invertible?

76.3 What is $\operatorname{null}(E)$, $\operatorname{null}(E - 3I)$, $\operatorname{null}(E + 3I)$?

---

77 Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for A.

77.1 Find the eigenvalues of A.

77.2 Find the characteristic polynomial of A.

77.3 Compute $A\vec{w}$ where $w = 2\vec{v}_1 - \vec{v}_2$.

77.4 Compute $A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

77.5 If $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is $\vec{x}$ written in the $\mathcal{V}$ basis, compute $A\vec{x}$ in the $\mathcal{V}$ basis.

Lesson 32: Diagonalization II

Textbook

Section 6.2

Objectives

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Motivation

78 The transformation P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation (where $\mathcal{V}$ is from above).

78.1 Describe in words what P does.

78.2 Describe how you can use P and P^{-1} to easily compute $A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

78.3 Can you find a matrix D so that

$$PDP^{-1} = A?$$

78.4 $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $A^{100}\vec{x}$.

79 For an $n \times n$ matrix T, suppose its eigenvectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues.

79.1 Is T diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

79.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

79.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Eigenspace

DEFINITION

Let A be a matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

Lesson 33: Diagonalization III

Textbook

Section 6.2

Objectives

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Motivation

80 Define $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

80.1 Is F diagonalizable? Why or why not?

80.2 What is the geometric and algebraic multiplicity of each eigenvalue of F?

80.3 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?