Learning Objectives

In this tutorial you will be working with eigenvectors and eigenvalues.

These problems relate to the following course learning objectives: *Clearly and correctly express the mathematical ideas of linear algebra to others* and *translate between algebraic and geometric viewpoints to solve problems*.

1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

    (a) Write a mathematically precise definition of what it means for $\vec{v}$ to be an eigenvector for T.

    (b) Describe in plain English what it means for $\vec{v}$ to be an eigenvector for T.

2. Consider

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \qquad C = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{v}_6 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   For each $\vec{v}_i$, determine whether $\vec{v}_i$ is an eigenvector for A, B, or C. If so, identify the corresponding eigenvalue.

3. Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the x-axis and $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter-clockwise by 90°.

   Find all eigenvectors and eigenvalues for $\mathcal{P}$ and $\mathcal{R}$.

4. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and that $\vec{v}$ is an eigenvector for T with eigenvalue 2. Is it possible that $7\vec{v}$ is an eigenvector for T with eigenvalue 14? Prove your answer.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. T has an eigenvector $\vec{v}_1$ with eigenvalue 1 and an eigenvector $\vec{v}_2$ with eigenvalue 1/2. Further, $\|\vec{v}_1\|, \|\vec{v}_2\| \leq 100$.

    (a) Let $\vec{w} = \vec{v}_1 + \vec{v}_2$. Approximate $T^{100}\vec{w}$.

    (b) Is $\{\vec{v}_1, \vec{v}_2\}$ a basis for $\mathbb{R}^2$? Prove your answer.

    (c) Suppose $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. With this information, can you approximate $T^{100}\begin{bmatrix} a \\ b \end{bmatrix}$?

    (d) Can you generalize your procedure from (c) to any linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ that has two positive, distinct eigenvalues?

1. (a) $\vec{v}$ is an eigenvector for T if $\vec{v} \neq \vec{0}$ and $T\vec{v} = \lambda\vec{v}$ for some scalar λ.
   (b) $\vec{v}$ is an eigenvector for T if it doesn't change directions when T is applied.

2. For A: $\vec{v}_1$ is an eigenvector with eigenvalue -1; $\vec{v}_2$ is an eigenvector with eigenvalue 2.

   For B: $\vec{v}_2$ is an eigenvector with eigenvalue 2; $\vec{v}_4$ is an eigenvector with eigenvalue -1.

   For C: $\vec{v}_4$ is an eigenvector with eigenvalue 0.

3. $\mathcal{P}$ has eigenvalues of 0 and 1. All non-zero multiples of $\begin{bmatrix}0\\1\end{bmatrix}$ have eigenvalue 0 and non-zero multiples of $\begin{bmatrix}1\\0\end{bmatrix}$ have eigenvalue 1.

   $\mathcal{R}$ has no real eigenvalues and no real eigenvectors. Every non-zero vector changes direction when $\mathcal{R}$ is applied.

4. It is not possible.

   Proof: Since $\vec{v}$ is an eigenvector for T with eigenvalue 2, we know $T\vec{v} = 2\vec{v}$. Since T is linear, we know
   $$T(7\vec{v}) = 7T\vec{v} = 14\vec{v},$$
   and so $7\vec{v}$ is an eigenvector for T with eigenvalue 2. There is no other possibility.

5. (a) $T^{100}\vec{w} = \vec{v}_1 + \frac{1}{2^{100}}\vec{v}_2 \approx \vec{v}_1$.
   (b) Yes. Since $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors, neither is zero. Suppose $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent. Then, $\vec{v}_1 = t\vec{v}_2$ for some t (because $\vec{v}_2 \neq \vec{0}$). Computing,
   $$T\vec{v}_1 = T(t\vec{v}_2) = tT\vec{v}_2 = \tfrac{t}{2}\vec{v}_2 = \tfrac{1}{2}\vec{v}_1,$$
   and so $\vec{v}_1$ would have an eigenvalue of $\frac{1}{2}$. But, this is a contradiction since the eigenvalue of $\vec{v}_1$ is 1. Thus $\{\vec{v}_1, \vec{v}_2\}$ must be linearly independent.
   Finally, a linearly independent set of two vectors in $\mathbb{R}^2$ must span all of $\mathbb{R}^2$, and so $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$.
   (c) Yes. First write $\begin{bmatrix}a\\b\end{bmatrix}$ in the $\{\vec{v}_1, \vec{v}_2\}$ basis. Then throw away the $\vec{v}_2$ component.
   The matrix that converts from the standard basis to the $\{\vec{v}_1, \vec{v}_2\}$ basis is
   $$C = \begin{bmatrix}-5 & 3\\ 2 & -1\end{bmatrix}.$$
   The matrix that drops second coordinate of a vector is
   $$D = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$
   Thus
   $$T^{100}\begin{bmatrix}a\\b\end{bmatrix} \approx C^{-1}DC\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}-5a+3b\\ -10a+6b\end{bmatrix}.$$
   (d) Yes. Suppose S has a unique, largest, positive eigenvalue λ. Then $S' = \frac{1}{\lambda}S$ behaves similarly to T. Approximate the same way, then multiply by λ.

Learning Objectives

Students need to be able to...

- Define eigenvector and eigenvalue
- Have an intuitive geometric understanding of eigenvectors
- Use the definition of eigenvector to classify vectors as eigenvectors or not
- Apply eigenvectors to solve problems

Context

Students have been introduced to determinants, characteristic polynomials, and the definition of eigenvectors/values in lecture. Some students may have started diagonalization in lecture.

In this course we will only pursue real eigenvalues/vectors. Some students may know complex numbers from high school, but most will not.

What to Do

Start the tutorial by stating the day's learning objectives. Emphasize that eigenvectors/values are the capstone of this course and they take a while to get used to. Today we are getting used to them.

Have students start in groups on #1. They should be used to the style of part (a) by now, but they will probably struggle with part (b). Try to get them to produce something better than reading the mathematical definition aloud.

#2 is a computational question that follows straight from the definition. If you like, divide the class into three teams. One team will check for A, another for B, and the last for C. Write up (or have each team write up) a summary of their results. Then have everyone verify each team's result.

Before finishing #2, make sure to have a discussion about $\vec{v}_6$. Is it an eigenvector? Does it go to a multiple of itself?

It is likely that #1 and #2 will take most of class time. If you do have extra time, continue as usual, letting students work in small groups on #3 and then having a mini-discussion when half the groups have figured it out.

6 minutes before the end of class, pick a suitable problem to do as a wrap-up.

Notes

- For #1(b), many students will write "$\vec{v}$ is an eigenvector for T if $\vec{v}$ is non-zero and T of $\vec{v}$ is λ times $\vec{v}$ for some scalar λ." This is reading the mathematical definition aloud and is not what we want. We want students to rephrase $T\vec{v} = \lambda\vec{v}$ in terms of a vector not changing direction.
- For #2, some students will get confused between $\vec{v} \neq \vec{0}$ and $\lambda \neq 0$. $C, \vec{v}_4$ has eigenvalue 0 precisely so this confusion will come up. Make sure to have a conversation about this whether or not it comes up naturally.

- Encourage students to draw pictures for #3. They will naturally try to start by writing down a matrix for $\mathcal{P}$ and $\mathcal{R}$, but that short-circuits thinking.
- When talking about eigenvalues of a transformation T not existing, always make sure to say "T has no real eigenvalues". Students who know about complex numbers can have a deeper conversation with you about them after class.
- In #4, some might find it so "obvious" that they don't know how to prove it. That's largely the point. They should be able to prove obvious things!
- #5 culminates in diagonalization, but it asks students to make a judgement about whether a vector is "approximated". The fact that $\|\vec{v}_1\|, \|\vec{v}_2\| \leq 100$ is essential for this, but isn't the point of the problem. The point is that there is a unique, largest, positive eigenvalue.

