Complex Numbers

A large part of mathematics is the process of abstraction. For instance, numbers are an abstraction of the concept of length. If we had two numbers a, b representing length, their product would be the area of the rectangle whose sides were a, b. Strictly speaking, with this interpretation, ab is an area and a, b are lengths and they cannot be compared (this is what the Greeks thought), but we have come to know numbers as a more abstract concept. After all, we can write both a, b and ab using decimals, so why shouldn't we compare them?

Taking this idea to solving polynomials gives us complex numbers. The solutions to $x^2 = a$ are $\pm\sqrt{a}$ formally, but if I said $a = -4$, we might claim that $\sqrt{-4}$ doesn't make any sense. However, with a simple definition, we can make a larger class of numbers where $\sqrt{-4}$ fits in perfectly.

Definition 1 *The imaginary unit i is defined as a solution to $x^2 = -1$. That is $i^2 = -1$.*

Definition 2 *A* complex number *is a number $a + bi$ where a is the* real part *and b is the* imaginary part.

You can think of complex numbers as vectors with two components, a real one and an imaginary one. You add complex numbers as you would vectors.

Compute the following:

1.1 $(1 + 2i)+(2 - 7i)$

1.2 $i - (6 + i)$

1.3 $24(3 + i) - \frac{1}{2}(2 + 2i)$

Unlike vectors however, complex numbers can be multiplied. Multiplication follows all the same rules as for real numbers, except $(i)(i) = -1$. So, for example, $i(2 - 3i) = 2i - 3i^2 = 3 + 2i$.

Compute the following products:

2.1 $i(7i)$

2.2 $(3 + i)(2 + 4i)$

2.3 $(7 + i)(7 - i)$

Definition 3 *The* conjugate *of a complex number $a + bi$ is written with a bar over top the number and is*

$$\overline{a + bi} = a - bi.$$

If c is a complex number then $c\bar{c}$ is always a real number. This means that while $1/c$ has complex numbers in the denominator of a fraction, $1/c = \bar{c}/(c\bar{c})$ does not. Now you know how to divide by a complex number!

Compute the following:

3.1 $\overline{2 - 7i}$

3.2 $\bar{i}$

3.3 $1/(2 - 7i)$

3.4 $(4 + 3i)/(-5 + i)$

Recall that for the square-root function $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Thus for example $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$. Use this and your knowledge of the quadratic formula to solve the following:

4.1 $x^2 + 3x + 9 = 0$

4.2 $-x^2 - 2 = 1$

4.3 $-x^2 + 5x - 12 = 7$

The amazing thing about complex numbers is that just by introducing the solution to the equation $x^2 = -1$ we can now write the solutions to any n-degree polynomial as complex numbers!

Theorem 1 *If* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *then we can factor* p *as*

$$p(x) = (x - c_1)(x - c_2)\cdots(x - c_n)$$

where $c_1, \ldots, c_n$ *are complex numbers.*

Theorem 1 will be important as we start solving equations involving determinants, but you can just think of Theorem 1 as telling us that we always have n solutions to an n-degree polynomial (though the solutions may not be all distinct).

Complex Numbers and Matrices

Using complex numbers as entries in a matrix allows us to represent things we couldn't before.

Compute the following:

5.1 $\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$

5.2 $\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

5.3 $\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

5.4 Compute $i \begin{bmatrix} i \\ 1 \end{bmatrix}$ and compare it with $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix}$. What do you notice? (How does this relate to eigen values and eigen vectors?)