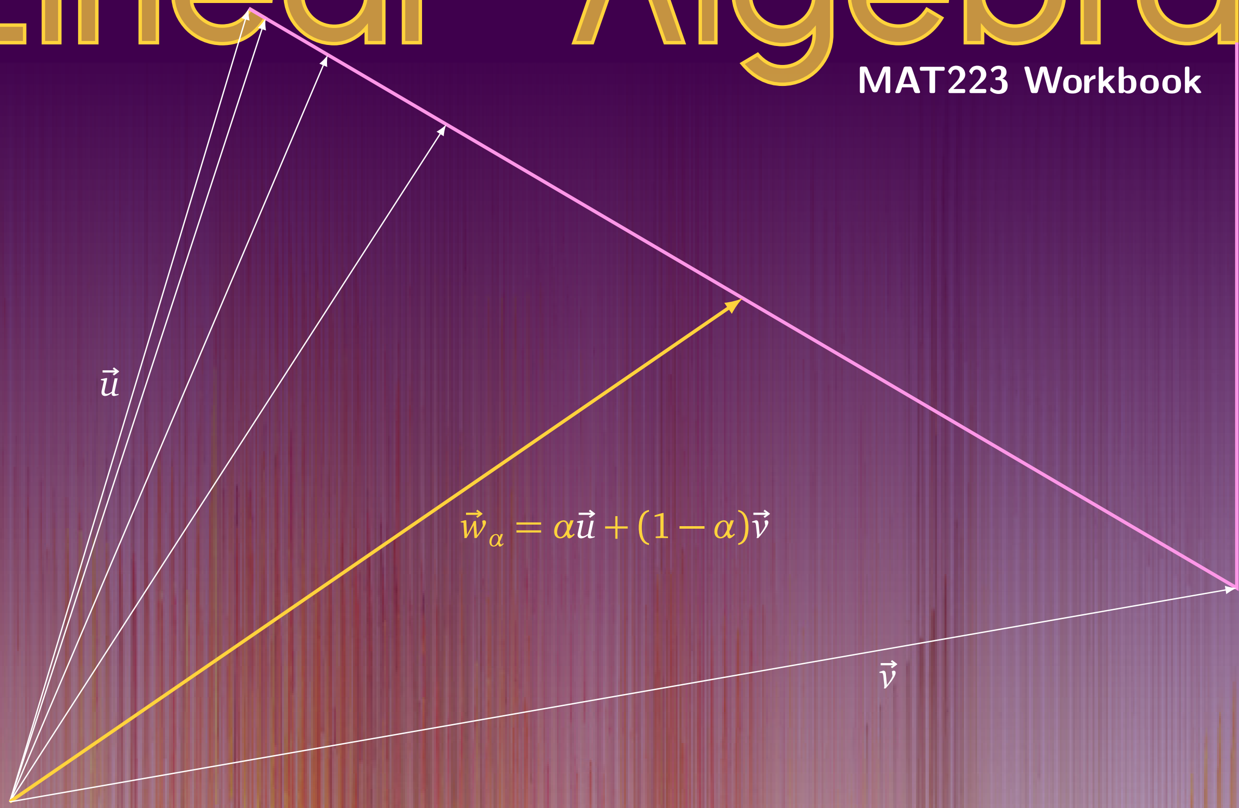
Linear Algebra

MAT223 Workbook



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Inquiry Based Linear Algebra

About the Document

This document is a hybrid of many linear algebra resources, including those of the IOLA (Inquiry Oriented Linear Algebra) project and Jason Siefken's IBL Linear Algebra project.

This document is a mix of short problems and more involved exploratory question. A typical class day looks like:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.

4. **Repeat step 2.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question especially primed to hear the insights of the instructor.

These problems are geared towards concepts instead of computation, though some problems focus on simple computation. The questions also have a geometric lean. Vectors are initially introduced with familiar coordinate notation, but eventually, coordinates are understood to be *representations* of vectors rather than "true" geometric vectors, and objects like the determinant are defined via oriented volumes rather than formulas involving matrix entries.

Module 1

Sets, Vectors & Notation

In this module you will learn

- The basics of sets and set-builder notation.
- The definition of vectors and how they relate to points.
- Column vector notation and to how represent vectors in drawings.
- How to compute linear combinations of vectors and use systems of linear equations to answer questions about linear combinations of vectors.

Sets

Modern mathematics makes heavy use of *sets*. A set is an unordered collection of distinct objects. We won't try and pin it down more than this—our intuition about collections of objects will suffice.[1] We write a set with curly-braces { and } and list the objects inside. For instance

$$\{1,2,3\}.$$

This would be read aloud as "the set containing the elements 1, 2, and 3." Things in a set are called *elements*, and the symbol $\in$ is used to specify that something is an element of a set. In contrast, $\notin$ is used to specify something is not an element of a set. For example,

$$3 \in \{1,2,3\} \qquad 4 \notin \{1,2,3\}.$$

Sets can contain mixtures of objects, including other sets. For example,

$$\{1,2,a,\{-70,\infty\},x\}$$

is a perfectly valid set.

It is tradition to use capital letters to name sets. So we might say $A = \{6,7,12\}$ or $X = \{7\}$. However there are some special sets which already have names/symbols associated with them. The *empty set* is the set containing no elements and is written $\emptyset$ or $\{\}$. Note that $\{\emptyset\}$ is *not* the empty set—it is the set containing the empty set! It is also traditional to call elements of a set *points* regardless of whether you consider them "point-like" objects.

Operations on Sets

If the set A contains all the elements that the set B does, we call B a *subset* of A. Conversely, we call A a *superset* of B.

> **Subset & Superset.** The set B is a ***subset*** of the set A, written $B \subseteq A$, if for all $b \in B$ we also have $b \in A$. In this case, A is called a ***superset*** of B.[a]
>
> [a] Some mathematicians use the symbol $\subset$ instead of $\subseteq$.

Some simple examples are $\{1,2,3\} \subseteq \{1,2,3,4\}$ and $\{1,2,3\} \subseteq \{1,2,3\}$. There's something funny about that last example, though. Those two sets are not only subsets/supersets of each other, they're *equal*. As surprising as it seems, we actually need to define what it means for two sets to be equal.

> **Set Equality.** The sets A and B are ***equal***, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Having a definition of equality to lean on will help us when we need to prove things about sets.

Example. Let A be the set of numbers that can be expressed as $2n$ for some whole number n, and let B be the set of numbers that can be expressed as $m+1$ where m is an odd whole number. We will show $A = B$.

First, let us show $A \subseteq B$. If $x \in A$ then $x = 2n$ for some whole number n. Therefore

$$x = 2n = 2(n-1)+1+1 = m+1$$

where $m = 2(n-1)+1$ is, by definition, an odd number. Thus $x \in B$, which proves $A \subseteq B$.

[1] When you pursue more rigorous math, you rely on definitions to get yourself out of philosophical jams. For instance, with our definition of set, consider "the set of all sets that don't contain themselves." Such a set cannot exist! This is called *Russel's Paradox*, and shows that if we start talking about sets of sets, we may need more than intuition.

Now we will show $B \subseteq A$. Let $x \in B$. By definition, $x = m + 1$ for some odd m and so by the definition of oddness, $m = 2k + 1$ for some whole number k. Thus

$$\begin{aligned} x = m + 1 &= (2k + 1) + 1 = 2k + 2 \\ &= 2(k + 1) = 2n, \end{aligned}$$

where $n = k + 1$, and so $x \in A$. Since $A \subseteq B$ and $B \subseteq A$, by definition $A = B$.

Set-builder Notation

Specifying sets by listing all their elements can be a hassle, and if there are an infinite number of elements, it's impossible! Fortunately, *set-builder notation* solves these problems. If X is a set, we can define a subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X *such that* some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$.[2] You may equivalently use "|" instead of ":", writing $Y = \{a \,|\, \text{some rule involving } a\}$.

There are also some common operations we can do with two sets.

Unions & Intersections. Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)} \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{-1, 0, 1, 2\}$, then $A \cap B = \{1, 2\}$ and $A \cup B = \{-1, 0, 1, 2, 3\}$. Set unions and intersections are *associative*, which means it doesn't matter how you apply parentheses to an expression involving just unions or just intersections. For example $(A \cup B) \cup C = A \cup (B \cup C)$, which means we can give an unambiguous meaning to an expression like $A \cup B \cup C$ (just put the parentheses wherever you like). But watch out, $(A \cup B) \cap C$ means something different than $A \cup (B \cap C)$!

Some common sets have special notation:

$$\begin{aligned} \emptyset &= \{\}, \text{ the empty set} \\ \mathbb{N} &= \{0, 1, 2, 3, \ldots\} = \{\text{natural numbers}\} \\ \mathbb{Z} &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{\text{integers}\} \\ \mathbb{Q} &= \{\text{rational numbers}\} \\ \mathbb{R} &= \{\text{real numbers}\} \\ \mathbb{R}^n &= \{\text{vectors in } n\text{-dimensional Euclidean space}\} \end{aligned}$$

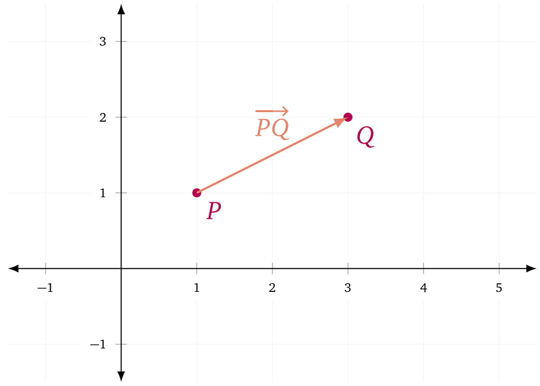
Vectors & Scalars

A *scalar* number (also referred to as a *scalar* or just an ordinary *number*) models a relationship between quantities. For example, a recipe might call for *six* times as much flour as sugar. In contrast, a *vector* models a relationship between points. For example, the store might be *2km East and 4km North* from my house. In this way, a vector may be thought of as a *displacement* with a *direction* and a *magnitude*.[3]

Given points $P = (1, 1)$ and $Q = (3, 2)$, we specify the *displacement* from P to Q as a vector $\overrightarrow{PQ}$ whose magnitude is $\sqrt{5}$ (as given by the Pythagorean theorem) and whose direction is specified by a directed line segment from P to Q.

[2] If you want to get technical, to make this notation unambiguous, you define a *universe of discourse*. That is, a set $\mathcal{U}$ containing every object you might want to talk about. Then $\{a : \text{some rule involving } a\}$ is short for $\{a \in \mathcal{U} : \text{some rule involving } a\}$

[3] Though in this book we will treat vectors as geometric objects relating to Euclidean space, they are much more general. For instance, someone's internet browsing habits could be described by a vector—the topics they find most interesting might be the "direction" and the amount of time they browse might be the "magnitude."

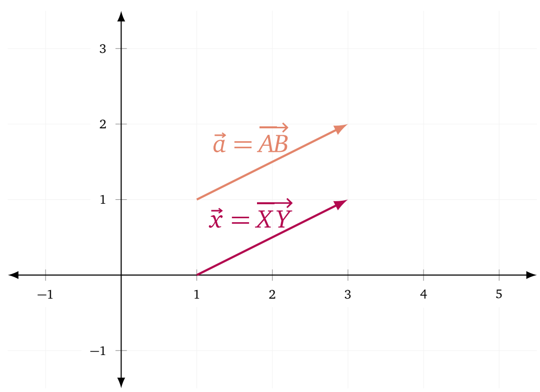


Vector Notation

There are many ways to represent vector quantities in writing. If we have two points, P and Q, we write $\overrightarrow{PQ}$ to represent the vector from P to Q. Absent points, a bold-faced letter (like **a**) or an arrow over a letter (like $\vec{a}$) are the most common vector notations. In this text, we will use $\vec{a}$ to represent a vector. The notation $\|\vec{a}\|$ represents the magnitude of the vector $\vec{a}$, which is sometimes called the *norm* or *length* of $\vec{a}$.

Graphically, we may represent vectors as directed line segments (a line segment with an arrow at one end), however we must take care to distinguish between the picture we draw and the "true" vector. For example, directed line segments always *start* somewhere, but a vector models a displacement and has no sense of "origin".

Consider the following: for the points $A=(1,1)$, $B=(3,2)$, $X=(1,0)$, and $Y=(3,1)$, define the vectors $\vec{a}=\overrightarrow{AB}$ and $\vec{x}=\overrightarrow{XY}$.



Are these the same or different vectors? As directed line segments, they are different because they are at different locations in space. However, both $\vec{a}$ and $\vec{x}$ have the same magnitude and direction. Thus, $\vec{a}=\vec{x}$ despite the fact that $A\neq X$.[4]

Takeaway. A vector is not the same as a line segment and a vector by itself has no "origin".

Vectors and Points

The distinction between vectors and points is sometimes nebulous because the two are so closely related. A *point* in Euclidean space specifies an absolute position whereas a vector specifies a displacement (i.e., a magnitude and direction). However, given a point P, one associates P with the vector $\vec{p}=\overrightarrow{OP}$, where O is the origin. Similarly, we associate the vector $\vec{v}$ with the point V so that $\overrightarrow{OV}=\vec{v}$. Thus, we have a way to unambiguously go back and forth between vectors and points.[5] As such, *we will treat vectors and points interchangeably*.

Takeaway. Vectors and points can and will be treated interchangeably.

[4] Some theories use *rooted vectors* instead of vectors as the fundamental object of study. A rooted vector represents a magnitude, direction, *and* a starting point. And, as rooted vectors, $\vec{a}\neq\vec{x}$ (from the example above). But for us, vectors will always be unrooted, even though our graphical representations of vectors might appear rooted.

[5] Mathematically, we say there is an *isomorphism* between vectors and points (once an origin is fixed, of course!).

Vector Arithmetic

Vectors provide a natural way to give directions. For example, suppose $\vec{e}_1$ points one kilometer eastwards and $\vec{e}_2$ points one kilometer northwards. Now, if you were standing at the origin and wanted to move to a location 3 kilometers east and 2 kilometers north, you might say: "Walk 3 times the length of $\vec{e}_1$ in the $\vec{e}_1$ direction and 2 times the length of $\vec{e}_2$ in the $\vec{e}_2$ direction." Mathematically, we express this as

$$3\vec{e}_1 + 2\vec{e}_2.$$

Of course, we've incidentally described a new vector. Namely, let P be the point at 3-east and 2-north. Then

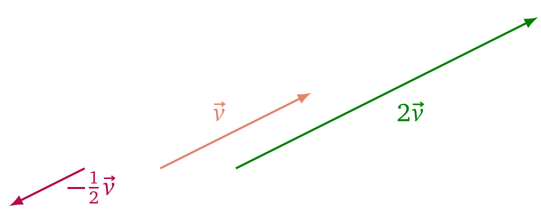
$$\overrightarrow{OP} = 3\vec{e}_1 + 2\vec{e}_2.$$

If the vector $\vec{r}$ points north but has a length of 10 kilometers, we have a similar formula:

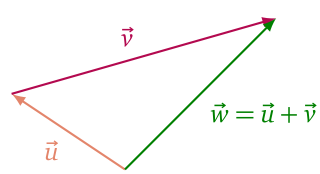
$$\overrightarrow{OP} = 3\vec{e}_1 + \tfrac{1}{5}\vec{r},$$

and we have the relationship $\vec{r} = 10\vec{e}_2$. Our notation here is very suggestive. Indeed, if we could make sense of "$\alpha\vec{v}$" (scalar multiplication) and "$\vec{v}+\vec{w}$" (vector addition) for any scalar α and any vectors $\vec{v}$ and $\vec{w}$, we could do algebra with vectors.

We will define scalar multiplication and vector addition intuitively: For a vector $\vec{v}$ and a scalar $\alpha > 0$, the vector $\vec{w} = \alpha\vec{v}$ is the vector pointing in the same direction as $\vec{v}$ but with length scaled up by α. That is, $\|\vec{w}\| = \alpha\|\vec{v}\|$. Similarly, $-\vec{v}$ is the vector of the same length as $\vec{v}$ but pointing in the exact opposite direction.



For two vectors $\vec{u}$ and $\vec{v}$, the sum $\vec{w} = \vec{u} + \vec{v}$ represents the displacement vector created by first displacing along $\vec{u}$ and then displacing along $\vec{v}$.



Takeaway. You add vectors tip to tail and you scale vectors by changing their length.

Now, there is one snag. What should $\vec{v} + (-\vec{v})$ be? Well, first we displace along $\vec{v}$ and then we displace in the exact opposite direction by the same amount. So, we have gone nowhere. This corresponds to a displacement with zero magnitude. But, what direction did we displace? Here we make a philosophical stand.

Zero Vector. The ***zero vector***, notated as $\vec{0}$, is the vector with no magnitude.

We will be pragmatic about the direction of the zero vector and say, *the zero vector does not have a well-defined direction.*[6] That means sometimes we consider the zero vector to point in every direction and sometimes we consider it to point in no directions. It depends on our mood—but we must never talk about *the* direction of the zero vector, since it's not defined.

Formalizing, for vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and scalars α and β are scalars, the following are laws are always satisfied:

$$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w}) \qquad \text{(Associativity)}$$
$$\vec{u}+\vec{v} = \vec{v}+\vec{u} \qquad \text{(Commutativity)}$$
$$\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v} \qquad \text{(Distributivity)}$$

[6] In the mathematically precise definition of vector, the idea of "magnitude" and "direction" are dropped. Instead, a set of vectors is defined to be a set over which you can reasonably define addition and scalar multiplication.

and

$$(\alpha\beta)\vec v = \alpha(\beta\vec v) \qquad \text{(Associativity II)}$$
$$(\alpha+\beta)\vec v = \alpha\vec v + \beta\vec v \qquad \text{(Distributivity II)}$$

Indeed, if we intuitively think about vectors in flat (Euclidean) space, all of these properties are satisfied.[7] From now on, these properties of vector operations will be considered the *laws (or axioms) of vector arithmetic*.

We group scalar multiplication and vector addition under one name: *linear combinations*.

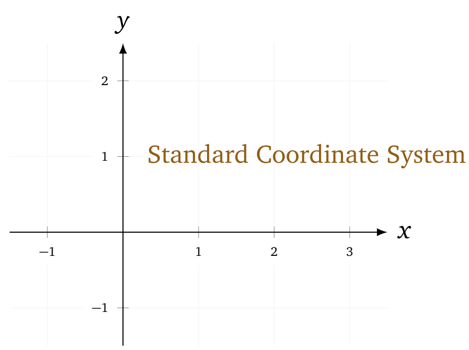
Linear Combination. A ***linear combination*** of the vectors $\vec v_1, \vec v_2, \ldots, \vec v_n$ is a vector

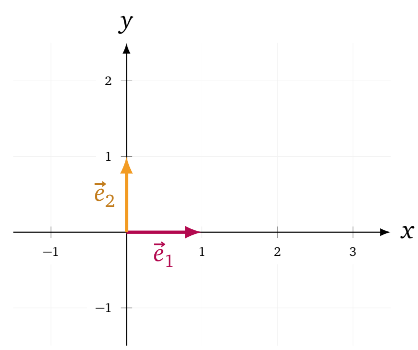
$$\vec w = \alpha_1\vec v_1 + \alpha_2\vec v_2 + \cdots + \alpha_n\vec v_n.$$

The scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the ***coefficients*** of the linear combination.

Coordinates and the Standard Basis

Consider the standard, flat, Euclidean plane (which is notated by $\mathbb{R}^2$). A coordinate system for $\mathbb{R}$ is a way to assign a unique pair of numbers to every point in $\mathbb{R}^2$. Though there are infinitely many coordinate systems we could choose for the plane, there is one standard one: the xy coordinate system depicted below, which you're already familiar with.





In conjunction with the standard coordinate system, there are also *standard basis vectors*. The vector $\vec e_1$ always points one unit in the direction of the positive x-axis and $\vec e_2$ aways points one unit in the direction of the positive y-axis.

Using the standard basis, we can represent every point (or vector) in the plane as a linear combination. If the point P has xy-coordinates (α, β), then $\overrightarrow{OP} = \alpha\vec e_1 + \beta\vec e_2$. Not only that, but this is the *only* way to represent the vector $\overrightarrow{OP}$ as a linear combination of α and β.

Takeaway. Every vector in $\mathbb{R}^2$ can be written uniquely as a linear combination of the standard basis vectors.

For a vector $\vec w = \alpha\vec e_1 + \beta\vec e_2$, we call the pair (α, β) the *standard coordinates* of the vector $\vec w$. There are many equivalent notations used to represent a vector in coordinates.

(α, β)	parenthesis
$\langle\alpha, \beta\rangle$	angle brackets
$\begin{bmatrix}\alpha & \beta\end{bmatrix}$	square brackets in a row (a row matrix)
$\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$	square brackets in a column (a column matrix)

Coordinates and vectors go hand in hand, and we will often write

$$\vec v = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

[7] If we deviate from flat space, some of these rules are no longer respected. Consider moving 100 kilometers north then 100 kilometers east on a sphere. Is this the same as moving 100 kilometers east and then 100 kilometers north?

as a shorthand for "$\vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2$".

Solving Problems with Coordinates

Coordinates allow for vector arithmetic to be carried out in a mechanical way. Suppose $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then,

$$\vec{u} = \vec{v} \qquad \iff \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \iff \qquad a = x \text{ and } b = y.$$

Further,

$$\vec{u} + \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+x \\ b+y \end{bmatrix} \qquad \text{and} \qquad t\vec{v} = t\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ta \\ tb \end{bmatrix}$$

for any scalar t.

Using these rules, otherwise complicated questions about vectors can be reduced to simple algebra questions.[8]

Example. Let $\vec{x} = \vec{e}_1 - \vec{e}_2$, $\vec{y} = 3\vec{e}_1 - \vec{e}_2$, and $\vec{r} = 2\vec{e}_1 + 2\vec{e}_2$. Is $\vec{r}$ a linear combination of $\vec{x}$ and $\vec{y}$?

XXX Finish

Higher Dimensions

We coordinatize three dimensional space (notated by $\mathbb{R}^3$) by constructing x, y, and z axes. Again, $\mathbb{R}^3$ has standard basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ which each point one unit along the x, y, and z axes, respectively.

Since we live in three dimensional space, its study has a long history and many notations for the standard basis are in use. This text will use $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, but other common notations include:

$$\begin{array}{ccc} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \hat{\imath} & \hat{\jmath} & \hat{k} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}$$

Beyond three dimensions, drawing pictures becomes hard, but we can still use vectors. We use $\mathbb{R}^n$ to notate n-dimensional Euclidean space. The standard basis for $\mathbb{R}^n$ is $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. Again, every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of the standard basis, and a coordinate representation of a vector in $\mathbb{R}^n$ is the list of n scalars.

Example. Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be given by $\vec{x} = 2\vec{e}_1 - \vec{e}_3$ and $\vec{y} = 6\vec{e}_2 + 3\vec{e}_3$. Compute $\vec{z} = \vec{x} + 2\vec{y}$.

$$\vec{z} = \vec{x} + 2\vec{y} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 5 \end{bmatrix} = 2\vec{e}_1 + 12\vec{e}_2 + 6\vec{e}_3$$

[8] So simple, that computers are able to answer billions of such question a second as you play your favorite video game!

Task 1.1: The Magic Carpet Ride

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Task 1.2: The Magic Carpet Ride, Hide and Seek

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X \ : \ \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

1

1.1 Which of the following statements are true?

(a) $3 \in \{1,2,3\}$.
(b) $1.5 \in \{1,2,3\}$.
(c) $4 \in \{1,2,3\}$.
(d) "b"$\in \{x \ : \ x \text{ is an English letter}\}$.
(e) "ò"$\in \{x \ : \ x \text{ is an English letter}\}$.
(f) $\{1,2\} \subseteq \{1,2,3\}$.
(g) For some $a \in \{1,2,3\}$, $a \geq 3$.
(h) For any $a \in \{1,2,3\}$, $a \geq 3$.
(i) $1 \subseteq \{1,2,3\}$.
(j) $\{1,2,3\} = \{x \in \mathbb{R} \ : \ 1 \leq x \leq 3\}$.
(k) $\{1,2,3\} = \{x \in \mathbb{Z} \ : \ 1 \leq x \leq 3\}$.

2

Write the following in set-builder notation

2.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

2.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Unions & Intersections

DEFINITION

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)} \quad X \cup Y = \{a \ : \ a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a \ : \ a \in X \text{ and } a \in Y\}.$$

3

Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

3.1 $X \cup Y$

3.2 $X \cap Y$

3.3 $X \cup Y \cup Z$

3.4 $X \cap Y \cap Z$

4 Draw the following subsets of $\mathbb{R}^2$.

4.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}0\\t\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

4.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}t\\0\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

4.3 $D = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

4.4 $N = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} \text{ for all } t \in \mathbb{R}\right\}$.

4.5 $V \cup H$.

4.6 $V \cap H$.

4.7 Does $V \cup H = \mathbb{R}^2$?

Vector Combinations

Linear Combination

DEFINITION

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called the ***coefficients*** of the linear combination.

5 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

5.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

5.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

6 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix}3\\1\end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix}1\\2\end{bmatrix}$.

6.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

6.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

6.3 Rephrase the sentence "$\mathbb{R}^2$ *is the set of all linear combinations of* $\vec{h}$ *and* $\vec{m}$" using formal mathematical language.

Sets of Vectors, Lines & Planes

In this module you will learn

- How to draw a set of vectors making an appropriate choice of when to use line segments and when to use dots to represent vectors.
- The *vector form* of lines and planes, including how to determine the intersection of lines and planes in vector form.
- Restricted linear combinations and how to use them to represent common geometric objects (like line segments or polygons).

With a handle on vectors, we can now use them to describe some common geometric objects: lines and planes.

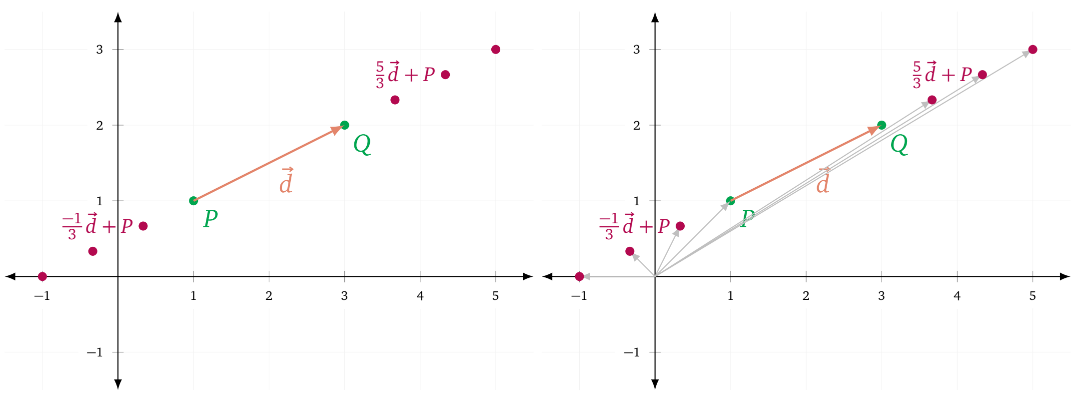
Lines

Consider for a moment the line ℓ through the points P and Q. When $P, Q \in \mathbb{R}^2$, we can describe ℓ with an equation of the form $y = mx + b$ (provided it isn't a vertical line), but if $P, Q \in \mathbb{R}^3$, it's much harder to describe ℓ with an equation. Using vectors provides an easier way.

Let $\vec{d} = \overrightarrow{PQ}$ and consider the set of points (or vectors) $\vec{x}$ that can be expressed as

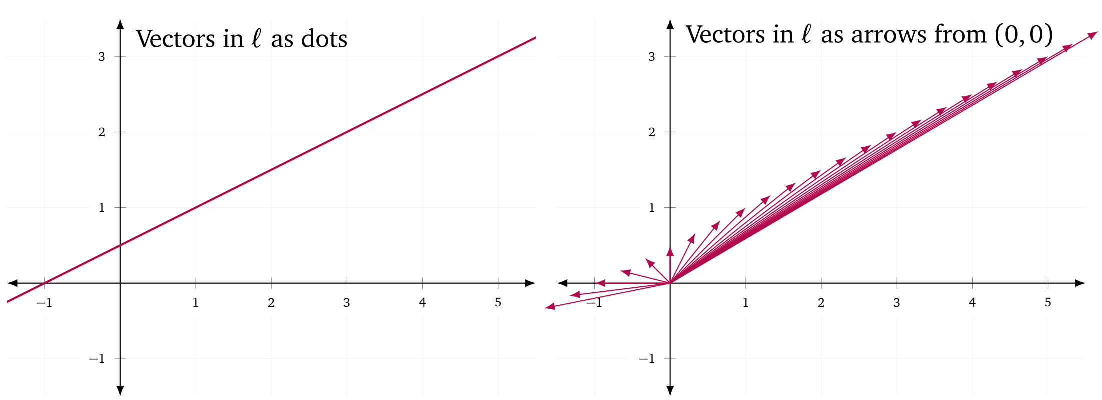
$$\vec{x} = t\vec{d} + P$$

for $t \in \mathbb{R}$. Geometrically, this is the set of all points we get by starting at P and displacing by some multiple of $\vec{d}$. This is a line!



We simultaneously interpret this line as a set of points (the points that make up the line) and as a set of vectors rooted at the origin (the vectors pointing from the origin to the line). Note that sometimes we draw vectors as directed line segments. Other times, drawing drawing vectors as line segments makes it hard to see what is going on, and so it is better to draw each vector by marking only its ending point.

Which picture below do you think best represents ℓ?



Takeaway. When drawing a picture depicting several vectors, make an appropriate choice (arrows, dots, or a mix) so that the picture is clear.

The line ℓ described above can be written in set-builder notation as:

$$\ell = \{\vec{x} : \vec{x} = t\vec{d} + P \text{ for some } t \in \mathbb{R}\}.$$

Notice that in set-builder notation, we write "for some $t \in \mathbb{R}$." Make sure you understand why replacing "for some $t \in \mathbb{R}$" with "for all $t \in \mathbb{R}$" would be incorrect.

Writing lines with set-builder notation all the time can be overkill, so we will allow ourselves to describe lines in a shorthand called *vector form*.[9]

Vector Form of a Line. Let ℓ be a line and let $\vec{d}$ and $\vec{p}$ be vectors. If $\ell = \{\vec{x} : \vec{x} = t\vec{d}+\vec{p} \text{ for some } t \in \mathbb{R}\}$ we say the vector equation

$$\vec{x} = t\vec{d} + \vec{p}$$

is ℓ expressed in ***vector form***. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

We can also use coordinates when writing a line in vector form. For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

corresponds to the line passing through $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ with $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ as a direction vector.

The "t" that appears in a vector form is called the *parameter variable*, and for this reason, some textbooks use the term *parametric form* in place of "vector form".

Writing a line in vector form only requires a point on the line and a direction for the line. Thus, converting to vector form from another form is straightforward.

Example. Find vector form of the line $\ell \subseteq \mathbb{R}^2$ with equation $y = 2x + 3$.

First, we find two points on the line. By guess-and-check we see $P = (0,3)$ and $Q = (1,5)$ are on ℓ. Thus, a direction vector for ℓ is given by

$$\vec{d} = (1,5) - (0,3) = (1,2).$$

We may now express ℓ in vector form as

$$\vec{x} = t\vec{d} + P$$

or, in components,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

It's important to note that when we write a line in vector form, it is a *specific shorthand* notation. If we augment the notation, we no longer have written a line in "vector form".

Example. Let ℓ and let $\vec{d}$ be a direction vector for ℓ and let $\vec{p} \in \ell$ be a point on ℓ. Writing

$$\vec{x} = t\vec{d} + \vec{p}$$

or

$$\vec{x} = t\vec{d} + \vec{p} \quad \text{where} \quad t \in \mathbb{R}$$

specifies ℓ in vector form and are both shorthands for $\{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$. But,

$$\vec{x} = t\vec{d} + \vec{p} \quad \text{for some} \quad t \in \mathbb{R}$$

and

$$\vec{x} = t\vec{d} + \vec{p} \quad \text{for all} \quad t \in \mathbb{R}$$

are logical statements about the vectors $\vec{x}$, $\vec{d}$, and $\vec{p}$. These statements are either true or false; they do *not* specify ℓ in vector form.

[9] $y = mx + b$ form of a line is also shorthand. The line ℓ described by the equation $y = mx + b$ is actually the set $\{(x, y) \in \mathbb{R}^2 : y = mx + b\}$.

Similarly, the statement

$$\ell = t\vec{d} + \vec{p}$$

is mathematically nonsensical and does not specify ℓ in vector form. (On the left is a *set* and on the right is a *vector*!)

But, why is vector form useful? For starters, every line can be expressed in vector form (you cannot write a vertical line in $y = mx + b$ form, and in $\mathbb{R}^3$, you would need two linear equations to represent a line). But, the most useful thing about expressing a line in vector form is that you can easily generate points on that line.

Suppose ℓ can be reprsented in vector form as $\vec{x} = t\vec{d} + \vec{p}$. Then, for every $t \in \mathbb{R}$, the vector $t\vec{d} + \vec{p} \in \ell$. Not only that, but as t ranges over $\mathbb{R}$, all points on ℓ are "traced out". Thus, we can find points on ℓ without having to "solve" any equations.

The downside to using vector form is that it is not unique. There are multiple direction vectors and multiple points for every line. Thus, merely by looking at the vector equation for two lines, it can be hard to tell if they're equal.

For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

all represent the same line. In the second equation, the direction vector is parallel but scaled, and in the third equation, a different point on the line was chosen.

Recall that in vector form, the variable t is called the *parameter variable*. It is an instance of a *dummy variable*. In other words, t is a placeholder—just because "t" appears in two different vector forms, doesn't mean it's the same quantity.

To drive this point home, we need to remember that vector form is shorthand for a set described in set-builder notation.

Let $\vec{d}_1, \vec{d}_2 \neq \vec{0}$ and $\vec{p}_1, \vec{p}_2$ be vectors and define the lines

$$\ell_1 = \{\vec{x} \,:\, \vec{x} = t\vec{d}_1 + \vec{p}_1 \text{ for some } t \in \mathbb{R}\}$$

$$\ell_2 = \{\vec{x} \,:\, \vec{x} = t\vec{d}_2 + \vec{p}_2 \text{ for some } t \in \mathbb{R}\}.$$

These lines have vector forms $\vec{x} = t\vec{d}_1 + \vec{p}_1$ and $\vec{x} = t\vec{d}_2 + \vec{p}_2$. However, declaring that $\ell_1 = \ell_2$ if and only if $t\vec{d}_1 + \vec{p}_1 = t\vec{d}_2 + \vec{p}_2$ does *not* make sense. Instead, as per the definition, $\ell_1 = \ell_2$ if $\ell_1 \subseteq \ell_2$ and $\ell_2 \subseteq \ell_1$. If $\vec{x} \in \ell_1$ then $\vec{x} = t\vec{d}_1 + \vec{p}_1$ for some $t \in \mathbb{R}$. If $\vec{x} \in \ell_2$ then $\vec{x} = t\vec{d}_2 + \vec{p}_2$ for some *possibly different* $t \in \mathbb{R}$. This can get confusing really quickly. The easiest way to avoid confusion is to use different parameter variables when comparing different vector forms.

Example. Determine if the lines ℓ_1 and ℓ_2, given in vector form as

$$\vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \vec{x} = t\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

are the same line.

To determine this, we need to figure out if $\vec{x} \in \ell_1$ implies $\vec{x} \in \ell_2$ and if $\vec{x} \in \ell_2$ implies $\vec{x} \in \ell_1$.

If $\vec{x} \in \ell_1$, then $\vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}$. If $\vec{x} \in \ell_2$, then $\vec{x} = s\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ for some $s \in \mathbb{R}$. Thus if

$$t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{x} = s\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

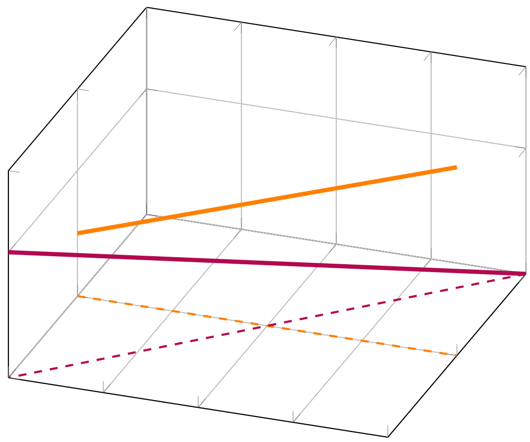
always has a solution, $\ell_1 = \ell_2$. Moving everything to one side we see

$$\begin{aligned} \vec{0} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 2 \end{bmatrix} - t\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} + s\begin{bmatrix} 2 \\ 2 \end{bmatrix} - t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (s+1)\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \tfrac{t}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= (s+1-\tfrac{t}{2})\begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned}$$

This has a solution whenever $0 = s + 1 - t/2$. Since for every $t \in \mathbb{R}$ we can find an $s \in \mathbb{R}$ satisfying this equation and for every $s \in \mathbb{R}$ we can find a $t \in \mathbb{R}$ satisfying this equation, we know $\ell_1 = \ell_2$.

Vector Form in Higher Dimensions

The geometry of lines in space ($\mathbb{R}^3$ and above) is a bit more complicated than that of lines in the plane. Lines in the plane either intersect or are parallel. In space, we have to be a more careful about what we mean by "parallel lines," since lines with entirely different directions can still fail to intersect.[10]



Example. Consider the lines described by

$$\vec{x} = t(1,3,-2) + (1,2,1)$$
$$\vec{x} = t(-2,-6,4) + (3,1,0).$$

They have parallel directions since $(-2,-6,4) = -2(1,3,-2)$. Hence, in this case, we say the lines are *parallel*. (How can we be sure the lines are not the same?)

Example. Consider the lines described by

$$\vec{x} = t(1,3,-2) + (1,2,1)$$
$$\vec{x} = t(0,2,3) + (0,3,9).$$

They are not parallel because neither of the direction vectors is a multiple of the other. They may or may not intersect. (If they don't, we say the lines are *skew*.) How can we find out? Mirroring our earlier approach, we can set their equations equal and see if we can solve for the point of intersection *after ensuring we give their parametric variables different names*. We'll keep one parametric variable named t and name the other one s. Thus, we want

$$\vec{x} = t(1,3,-2) + (1,2,1) = s(0,2,3) + (0,3,9),$$

which after collecting terms yields

$$(t+1, 3t+2, -2t+1) = (0, 2s+3, 3s+9).$$

Picking out the components yields three equations

$$t + 1 = 0$$
$$3t + 2 = 2s + 3$$
$$-2t + 1 = 3s + 9$$

in 2 unknowns s and t. This is an *overdetermined* system, and it may or may not have a consistent solution. The first two equations yield $t = -1$ and $s = -2$. Putting these values in the last equation yields $(-2)(-1) + 1 = 3(-2) + 9$, which is indeed true. Hence, the equations are consistent, and the lines intersect. To find the point of intersection, put $t = -1$ in the equation for the first line (or $s = -2$ in that for the second) to obtain $(0, -1, 3)$.

[10] Recall that in Euclidean geometry two lines are defined to be parallel if they coincide or never intersect.

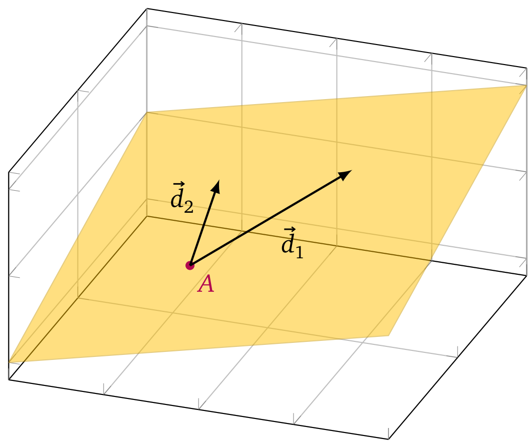
Planes

Any two distinct points define a line. To define a plane, we need three points. But there's a caveat: the three points cannot be on the same line, otherwise they'd define a line and not a plane. Let $A, B, C \in \mathbb{R}^3$ be three points that are not collinear and let $\mathcal{P}$ be the plane that passes through A, B, and C.

Just like lines, planes have direction vectors. For $\mathcal{P}$, both $\vec{d}_1 = \overrightarrow{AB}$ and $\vec{d}_2 = \overrightarrow{AC}$ are direction vectors for $\mathcal{P}$. Of course, $\vec{d}_1$, $\vec{d}_2$ and their multiples are not the only direction vectors for $\mathcal{P}$. There are infinitely many more, including $\vec{d}_1 + \vec{d}_2$, and $\vec{d}_1 - 7\vec{d}_2$, and so on. However, since a plane is a *two*-dimensional object, we only need two different direction vectors to describe it.

Like lines, planes have a vector form. Using the direction vectors $\vec{d}_1 = \overrightarrow{AB}$ and $\vec{d}_2 = \overrightarrow{AC}$, the plane $\mathcal{P}$ can be written in vector form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\vec{d}_1 + s\vec{d}_2 + A.$$



Vector Form of a Plane. A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

Example. Describe the plane $\mathcal{P} \subseteq \mathbb{R}^3$ with equation $z = 2x + y + 3$ in vector form.

To describe $\mathcal{P}$ in vector form, we need a point on $\mathcal{P}$ and two direction vectors for $\mathcal{P}$. By guess-and-check, we see the points

$$A = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

are all on $\mathcal{P}$. Thus

$$\vec{d}_1 = B - A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \vec{d}_2 = C - A = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

are both direction vectors for $\mathcal{P}$. Therefore, we can express $\mathcal{P}$ in vector form as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + A = t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

Example. Find the line of intersection between $\mathcal{P}_1$ and $\mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are expressed in vector form as ...

XXX Finish

Restricted Linear Combinations

Using vectors, we can describe more than just lines and planes—we can describe all sorts of geometric objects.

Recall that when we write $\vec{x} = t\vec{d} + \vec{p}$ to describe the line ℓ, what we mean is

$$\ell = \{\vec{x} \,:\, \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}.$$

The line ℓ stretches off infinitely in both directions. But, what if we wanted to describe just a part of ℓ? We can do this by placing additional restrictions on t. For example, consider the ray R and the line segment S:

$$R = \{\vec{x} \,:\, \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \geq 0\}$$
$$S = \{\vec{x} \,:\, \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in [0,2]\}$$

XXX Figure

We can also make polygons by adding restrictions to the vector form of a plane. Let $\vec{a} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}-1\\1\end{bmatrix}$ and consider the unit square U and the parallelogram P defined by

$$U = \{\vec{x} \,:\, \vec{x} = t\vec{e}_1 + s\vec{e}_2 \text{ for some } t, s \in [0,1]\}$$
$$P = \{\vec{x} \,:\, \vec{x} = t\vec{a} + s\vec{b} \text{ for some } t \in [0,1] \text{ and } s \in [-1,1]\}$$

XXX Figure

Each set so far is a set of linear combinations, and we have made different shapes by restricting the coefficients of those linear combinations. There are two ways of restricting linear combinations that arise up often enough to get their own names.

Non-negative & Convex Linear Combinations.
Let $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$. The vector $\vec{w}$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0.$$

The vector $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1.$$

You can think of a non-negative linear combinations as vector you can arrive at by only displacing "forward" along your vectors. Convex linear combinations can be thought of as weighted averages of vectors (the average of $\vec{v}_1, \ldots, \vec{v}_n$ would be the convex linear combination with coefficients $\alpha_i = \frac{1}{n}$). A convex linear combination of two vectors gives a point on the line segment connecting them.

Example. Let $\vec{a} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}-1\\1\end{bmatrix}$ and define

$$\begin{aligned} A &= \{\vec{x} \,:\, \vec{x} \text{ is a convex linear combination of } \vec{a} \text{ and } \vec{b}\} \\ &= \{\vec{x} \,:\, \vec{x} = \alpha\vec{a} + (1-\alpha)\vec{b} \text{ for some } \alpha \in [0,1]\}. \end{aligned}$$

Draw A.

XXX Finish

XXX Figure

Non-negative & Convex Linear Combinations

DEFINITION

Let $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$. The vector $\vec{w}$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \dots, \alpha_n \geq 0.$$

The vector $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \dots, \alpha_n \geq 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1.$$

---

7 Let

$$\vec{a} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \vec{b} = \begin{bmatrix}-1\\1\end{bmatrix} \qquad \vec{c} = \begin{bmatrix}0\\1\end{bmatrix} \qquad \vec{d} = \begin{bmatrix}0\\2\end{bmatrix} \qquad \vec{e} = \begin{bmatrix}-1\\-1\end{bmatrix}.$$

7.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$?

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$?

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$?

7.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that

(a) $\vec{a}$ and $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not.

(b) $\vec{a}$ and $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$.

(c) $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not.

(d) $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$.

Otherwise, explain why it's not possible.

Lines and Planes

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8 Let L be the set of points $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 1$.

8.1 Describe L using set-builder notation.

8.2 Draw L as a subset of $\mathbb{R}^2$.

8.3 Add the vectors $\vec{a} = \begin{bmatrix}-1\\-1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\vec{d} = \vec{b} - \vec{a}$ to your drawing.

8.4 Is $\vec{d} \in L$? Explain.

8.5 For which $t \in \mathbb{R}$ is it true that $\vec{a} + t\vec{d} \in L$? Explain using your picture.

Vector Form of a Line

DEFINITION

Let ℓ be a line and let $\vec{d}$ and $\vec{p}$ be vectors. If $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$ we say the vector equation

$$\vec{x} = t\vec{d} + \vec{p}$$

is ℓ expressed in ***vector form***. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

---

9 Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

9.1 Write ℓ in vector form. Is vector form of ℓ unique?

9.2 Write L in vector form.

9.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

10 Let A, B, and C be given in vector form by

$$\overbrace{\vec{x}=t\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x}=t\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B} \qquad \overbrace{\vec{x}=t\begin{bmatrix}2\\-1\\1\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

10.1 Do the lines A and B intersect? Justify your conclusion.

10.2 Do the lines A and C intersect? Justify your conclusion.

10.3 Let $\vec{p}\neq\vec{q}$ and suppose X has vector form $\vec{x}=t\vec{d}+\vec{p}$ and Y has vector form $\vec{x}=t\vec{d}+\vec{q}$. Is it possible that X and Y intersect?

Vector Form of a Plane

DEFINITION

A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x}=t\vec{d}_1+s\vec{d}_2+\vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P}=\{\vec{x}:\vec{x}=t\vec{d}_1+s\vec{d}_2+\vec{p}$ for some $t,s\in\mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

11 Recall the intersecting lines A and B given in vector form by

$$\overbrace{\vec{x}=t\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x}=t\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

11.1 Find two direction vectors for $\mathcal{P}$.

11.2 Write $\mathcal{P}$ in vector form.

11.3 Describe how vector form of a plane relates to linear combinations.

11.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

12 Let $\mathcal{Q}\subseteq\mathbb{R}^3$ be a plane with equation $x+y+z=1$.

12.1 Find three points in $\mathcal{Q}$.

12.2 Find two direction vectors for $\mathcal{Q}$.

12.3 Write $\mathcal{Q}$ in vector form.

Spans, Translated Spans, and Linear Independence/Dependence

In this module you will learn

- The definition of span and how to visualize spans.
- How to express lines/planes/volumes through the origin as spans.
- How to express lines/planes/volumes *not* through the origin as *translated* spans using set addition.
- Geometric and algebraic definitions of linear independence and linear dependence.
- How to find linearly independent subsets.

Let $\vec u = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec v = \begin{bmatrix}1\\-2\end{bmatrix}$. Can the vectors $\vec w = \begin{bmatrix}2\\5\end{bmatrix}$ be obtained as a linear combination of $\vec u$ and $\vec v$?

By drawing a picture, the answer appears to be *yes*.

XXX Figure

Algebraically, we can use the definition of a *linear combination* to set up a system of equations. We know $\vec w$ can be expressed as a linear combination of $\vec u$ and $\vec v$ if and only if the vector equation

$$\vec w = \begin{bmatrix}2\\5\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-2\end{bmatrix} = \alpha\vec u + \beta\vec v$$

has a solution. By inspection, we see $\alpha = 3$ and $\beta = 1$ solve this equation.

After initial success, we might ask the following: *what are all the locations in $\mathbb{R}^2$ that can be obtained as a linear combination of $\vec u$ and $\vec v$?* Geometrically, it appears any location can be reached. To verify this algebraically, consider the vector equation

$$\vec x = \begin{bmatrix}x\\y\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-2\end{bmatrix} = \alpha\vec u + \beta\vec v. \tag{1}$$

Here $\vec x$ represents an arbitrary point in $\mathbb{R}^2$. Thus, if equation (1) always has a solution,[11] any vector in $\mathbb{R}^2$ can be obtained as a linear combination of α and β.

We can solve this equation for α and β by considering the equations arising from the first and second coordinates. Namely,

$$\begin{aligned} x &= \alpha + \beta \\ y &= \alpha - 2\beta. \end{aligned}$$

Subtracting the second equation from the first, we get $x - y = 3\beta$ and so $\beta = (x-y)/3$. Plugging β into the first equation and solving, we get $\alpha = (2x+y)/3$. Thus, equation (1) *always* has a solution. Namely,

$$\begin{aligned} \alpha &= \tfrac{1}{3}(2x+y) \\ \beta &= \tfrac{1}{3}(x-y). \end{aligned}$$

There is a formal term for the set of vectors that can be obtained as linear combinations of others: *span*.

Span. The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec v : \vec v = \alpha_1\vec v_1 + \alpha_2\vec v_2 + \cdots + \alpha_n\vec v_n \text{ for some } \vec v_1, \vec v_2, \ldots, \vec v_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Additionally, we define $\operatorname{span}\{\} = \{\vec 0\}$.

We just showed above that $\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\} = \mathbb{R}^2$.

Example. Let $\vec u = \begin{bmatrix}-1\\2\end{bmatrix}$ and $\vec v = \begin{bmatrix}1\\-2\end{bmatrix}$. Find $\operatorname{span}\{\vec u, \vec v\}$.

XXX Finish

[11] The official terminology would be to say that the equations is always *consistent*.

Example. Let $\vec{a}=\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\vec{b}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{c}=\begin{bmatrix}1\\1\\1\end{bmatrix}$. Show that $\mathbb{R}^3=\operatorname{span}\{\vec{a},\vec{b},\vec{c}\}$.

XXX Finish

Representing Lines & Planes as Spans

If spans remind you of vector forms of lines and planes, your intuition is keen. Consider the line ℓ given in vector form by

$$\vec{x}=t\vec{d}+\vec{0}.$$

The line ℓ passes through the origin, and if we unravel its definition, we see

$$\ell=\{\vec{x} : \vec{x}=t\vec{d}+\vec{0}\text{ for some }t\in\mathbb{R}\}=\{\vec{x} : \vec{x}=t\vec{d}\text{ for some }t\in\mathbb{R}\}=\operatorname{span}\{\vec{d}\}.$$

Similarly, if $\mathcal{P}$ is a plane given in vector form by

$$\vec{x}=t\vec{d}_1+s\vec{d}_2+\vec{0},$$

then

$$\mathcal{P}=\{\vec{x} : \vec{x}=t\vec{d}_1+s\vec{d}_2\text{ for some }t,s\in\mathbb{R}\}=\operatorname{span}\{\vec{d}_1,\vec{d}_2\}.$$

If the "$\vec{p}$" in our vector form is $\vec{0}$, then that vector form actually defines a span. This means, if you accept that every line/plane through the origin has a vector form, then every line/plane through the origin can be written as a span. Conversely, if $X=\operatorname{span}\{\vec{v}_1,\ldots,\vec{v}_n\}$ is a span, we know $\vec{0}=0\vec{v}_1+\cdots+0\vec{v}_n\in X$, and so every span contains the origin.

As it turns out, spans exactly describe points, lines, planes, and volumes[12] through the origin.

Example. The line $\ell_1\subseteq\mathbb{R}^2$ is described by the equation $x+2y=0$ and the line $\ell_2\subseteq\mathbb{R}^2$ is described by the equation $4x-2y=6$. If possible, describe ℓ_1 and ℓ_2 using spans.

XXX Finish

However, not all points, lines, planes, and volumes pass through the origin and so we can't describe every such object directly as a span.

Example. Show the line $\ell_2\subseteq\mathbb{R}^2$ described by the equation $4x-2y=6$ cannot be written as a span.

XXX Finish

Takeaway. Lines and planes through the origin, and only lines and planes through the origin, can be expressed as spans.

Set Addition

We're going to work around the fact that only objects which pass through the origin can be written as spans, but first let's take a detour and learn about *set addition*.

Set Addition. If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A+B$, is

$$A+B=\{\vec{x} : \vec{x}=\vec{a}+\vec{b}\text{ for some }\vec{a}\in A\text{ and }\vec{b}\in B\}.$$

Set sums are very different than regular sums despite using the same symbol, "+".[13] However, they are very useful. Let $C=\{\vec{x}\in\mathbb{R}^2 : \|\vec{x}\|=1\}$ be the unit circle centered at the origin, and consider the sets

$$X=C+\{\vec{e}_2\}\qquad Y=C+\{3\vec{e}_1,\vec{e}_2\}\qquad Z=C+\{\vec{0},\vec{e}_1,\vec{e}_2\}.$$

Rewriting, we see $X=\{\vec{x}+\vec{e}_2 : \|\vec{x}\|=1\}$ is just C translated by $\vec{e}_2$. Similarly, $Y=\{\vec{x}+\vec{v} : \|\vec{x}\|=1\text{ and }\vec{v}=3\vec{e}_1\text{ or }\vec{v}=\vec{e}_2\}=(C+\{3\vec{e}_1\})\cup(C+\{\vec{e}_1\})$, and so Y is the union of two translated copies of C.[14]

XXX Figure

Translated Spans

[12] We use the word *volume* to indicate the higher-dimensional analogue of a plane.

[13] For example, $A+\{\}=\{\}$, which might seem counterintuitive for an "addition" operation.

[14] If you want to stretch your mind, consider what $C+C$ is as a set.

Set addition allows us to easily create parallel lines and planes by translation. For example, consider the lines ℓ_1 and ℓ_2 given in vector form as $\vec{x} = t\vec{d}$ and $\vec{x} = t\vec{d} + \vec{p}$, respectively, where $\vec{d} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}-1\\1\end{bmatrix}$. These lines differ from each other by a translation, and using the idea of set addition we can write

$$\ell_2 = \ell_1 + \{\vec{p}\}.$$

XXX Figure (with lots of copies of $\vec{p}$ translating ℓ_1).

Note that it would be incorrect to write "$\ell_2 = \ell_1 + \vec{p}$". Because ℓ_1 is a set and $\vec{p}$ is not a set, "$\ell_1 + \vec{p}$" does not make mathematical sense.

Example. Recall $\ell_2 \subseteq \mathbb{R}^2$ is the line described by the equation $4x - 2y = 6$. Describe ℓ_2 as a translated span.

XXX Finish

We can now see translated spans provide an alternative notation to vector form for specifying lines and planes. If Q is described in vector form by

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p},$$

then

$$Q = \operatorname{span}\{\vec{d}_1, \vec{d}_2\} + \{\vec{p}\}.$$

Takeaway. All lines and planes, whether through the origin or not, can be expressed as translated spans.

Linear Independence & Linear Dependence

Let

$$\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

Since $\vec{w} = \vec{u} + \vec{v}$, we know that $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$. Geometrically, this is also clear because $\operatorname{span}\{\vec{u}, \vec{v}\}$ is the xy-plane in $\mathbb{R}^3$ and $\vec{w}$ lies on that plane.

What about $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$? Intuitively, since $\vec{w}$ is already a linear combination of $\vec{u}$ and $\vec{v}$, we can't get anywhere *new* by taking linear combinations of $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ compared to linear combinations of just $\vec{u}$ and $\vec{v}$. So $\operatorname{span}\{\vec{u}, \vec{v}\} = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

Can we prove this from the definitions? Yes! Suppose $\vec{r} \in \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$. By definition,

$$\vec{r} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. Since $\vec{w} = \vec{u} + \vec{v}$, we see

$$\vec{r} = \alpha\vec{u} + \beta\vec{v} + \gamma(\vec{u} + \vec{v}) = (\alpha + \gamma)\vec{u} + (\beta + \gamma)\vec{v} \in \operatorname{span}\{\vec{u}, \vec{v}\}.$$

Thus, $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\} \subseteq \operatorname{span}\{\vec{u}, \vec{v}\}$. Conversely, if $\vec{s} \in \operatorname{span}\{\vec{u}, \vec{v}\}$, by definition,

$$\vec{s} = a\vec{u} + b\vec{v} = a\vec{u} + b\vec{v} + 0\vec{w}$$

for some $a, b \in \mathbb{R}$, and so $\vec{s} \in \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$. Thus $\operatorname{span}\{\vec{u}, \vec{v}\} \subseteq \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$. We conclude $\operatorname{span}\{\vec{u}, \vec{v}\} = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

In this case, $\vec{w}$ was a redundant vector—it wasn't needed for the span.

Linearly Dependent & Independent (Geometric).
We say the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

We will also refer to sets of vectors (for example $\{\vec{v}_1,\dots,\vec{v}_n\}$) as being linearly independent or linearly dependent. For technical reasons, we didn't state the definition in terms of sets.[15]

The geometric definition of linear dependence says that the vectors $\vec{v}_1,\dots,\vec{v}_n$ are linearly dependent if you can remove at least one vector without changing the span. In other words, $\vec{v}_1,\dots,\vec{v}_n$ are linearly dependent *if there is a redundant vector*.

Example. Let $\vec{u}=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}=\begin{bmatrix}2\\3\end{bmatrix}$, and $\vec{w}=\begin{bmatrix}4\\5\end{bmatrix}$. Determine whether $\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent or linearly dependent.

XXX Finish

Example. Determine whether the planes ... (given in vector form) are the same.

XXX Finish

We can also think of linear independence/dependence from an algebraic perspective. Suppose the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ satisfy

$$\vec{w}=\vec{u}+\vec{v}. \tag{2}$$

The set $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent since $\vec{w}\in\operatorname{span}\{\vec{u},\vec{v}\}$, but equation (2) can be rearranged to get

$$\vec{0}=\vec{u}+\vec{v}-\vec{w}. \tag{3}$$

Here we have expressed $\vec{0}$ as a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$. By itself, this is nothing special. After all, without any thinking we know $\vec{0}=0\vec{u}+0\vec{v}+0\vec{w}$ is a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$. However, the right side of equation (3) has non-zero coefficients, which makes the linear combination *non-trivial*.

Trivial Linear Combination.
The linear combination $\alpha_1\vec{v}_1+\cdots+\alpha_n\vec{v}_n$ is called ***trivial*** if $\alpha_1=\cdots=\alpha_n=0$. If at least one $\alpha_i\neq 0$, the linear combination is called ***non-trivial***.

We can always write $\vec{0}$ as a linear combination of vectors if we let all the coefficients be zero, but it turns out we can only write $\vec{0}$ as a *non-trivial* linear combination of vectors if those vectors are linearly dependent. This is the inspiration for another definition of linear independence/dependence.

Linear Independence & Linear Dependence (Algebraic). The vectors $\vec{v}_1,\dots,\vec{v}_n$ are called ***linearly independent*** if for the only linear combination satisfying

$$\vec{0}=\alpha_1\vec{v}_1+\cdots+\alpha_n\vec{v}_n$$

is the trivial linear combination (where $\alpha_1=\cdots=\alpha_n=0$).

The idea of a "redundant vector" coming from the geometric definition of linear dependence is easy to visualize, but it can be hard to prove things with—checking for linear independence with the geometric definition involves verifying for every vector that it is not in the span of the others. The algebraic definition on the other hand is less obvious, but checking for linear independence or dependence of a set involves reasoning about solutions to just one equation.

Example. Let $\vec{u}=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v}=\begin{bmatrix}2\\3\end{bmatrix}$, and $\vec{w}=\begin{bmatrix}4\\5\end{bmatrix}$. Use the algebraic definition of linear independence to determine whether $\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent or dependent.

XXX Finish

Theorem. The geometric and algebraic definitions of linear independence are equivalent.

Proof. To show the two definitions are equivalent, we need to show that geometric $\implies$ algebraic and algebraic $\implies$ geometric.

(geometric $\implies$ algebraic) Suppose $\vec{v}_1,\dots,\vec{v}_n$ are linearly dependent by the geometric definition. That means that for some i, we have

$$\vec{v}_i\in\operatorname{span}\{\vec{v}_1,\dots,\vec{v}_{i-1},\vec{v}_{i+1},\dots,\vec{v}_n\}.$$

[15] The issue is, every element of a set is unique. Clearly, the vectors $\vec{v}$ and $\vec{v}$ are linearly dependent, but $\{\vec{v},\vec{v}\}=\{\vec{v}\}$, and so $\{\vec{v},\vec{v}\}$ is technically a linearly independent set. This issue would be resolved by talking about *multisets* instead of sets, but it isn't worth the hassle.

Fix such an i. Then, by the definition of span we know

$$\vec{v}_i = \alpha_1\vec{v}_1 + \cdots \alpha_{i-1}\vec{v}_{i-1} + \alpha_{i+1}\vec{v}_{i+1} + \cdots + \alpha_n\vec{v}_n,$$

and so

$$\vec{0} = \alpha_1\vec{v}_1 + \cdots \alpha_{i-1}\vec{v}_{i-1} - \vec{v}_i + \alpha_{i+1}\vec{v}_{i+1} + \cdots + \alpha_n\vec{v}_n.$$

This must be a non-trivial linear combination because the coefficient of $\vec{v}_i$ is $-1 \neq 0$. Therefore, $\vec{v}_1, \ldots, \vec{v}_n$ is linearly dependent by the algebraic definition.

(geometric $\implies$ algebraic) Suppose $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent by the algebraic definition. That means there exist $\alpha_1, \ldots, \alpha_n$, not all zero, so that

$$\vec{0} = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n.$$

Fix i so that $\alpha_i \neq 0$ (why do we know there is such an i?). Rearranging we get

$$-\alpha_i\vec{v}_i = \alpha_1\vec{v}_1 + \cdots \alpha_{i-1}\vec{v}_{i-1} + \alpha_{i+1}\vec{v}_{i+1} + \cdots + \alpha_n\vec{v}_n,$$

and since $\alpha_i \neq 0$, we can multiply both sides by $\frac{-1}{\alpha_i}$ to get

$$\vec{v}_i = \tfrac{-\alpha_1}{\alpha_i}\vec{v}_1 + \cdots + \tfrac{-\alpha_{i-1}}{\alpha_i}\vec{v}_{i-1} + \tfrac{-\alpha_{i+1}}{\alpha_i}\vec{v}_{i+1} + \cdots + \tfrac{-\alpha_n}{\alpha_i}\vec{v}_n.$$

This shows that

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\},$$

and so $\vec{v}_1, \ldots, \vec{v}_n$ is linearly dependent by the geometric definition. ■

Linear Independence and Unique Solutions

The algebraic definition of linear independence can teach us something about solutions to systems of equations.

Recall the linearly dependent vectors

$$\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

which satisfy the non-trivial relationship $\vec{u} + \vec{v} - \vec{w} = \vec{0}$. Since $\vec{u} + \vec{v} - \vec{w} = \vec{0}$ is a non-trivial relationship giving $\vec{0}$, we can use it to generate others. For example,

$$\begin{aligned} 17(\vec{u} + \vec{v} - \vec{w}) &= 17\vec{u} + 17\vec{v} - 17\vec{w} &= 17\vec{0} &= \vec{0} \\ -3(\vec{u} + \vec{v} - \vec{w}) &= -3\vec{u} - 3\vec{v} + 3\vec{w} &= -3\vec{0} &= \vec{0} \\ &\vdots \end{aligned}$$

are all different non-trivial linear combinations that give $\vec{0}$. In other words, if the equation $\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0}$ has a non-trivial solution, it has *infinitely many* non-trivial solutions. Conversely, if the equation $\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0}$ has infinitely many solutions, one of them has to be non-trivial!

Homogeneous System.
A system of linear equations or a vector equation in the variables $\alpha_1, \ldots, \alpha_n$ is called ***homogeneous*** if it takes the form

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0},$$

where the right side of the equation is $\vec{0}$.

This insight links linear independence and homogeneous systems together, and is encapsulated in the following theorem.

Theorem. The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if and only if the homogeneous equation

$$\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = \vec{0}$$

has a unique solution.

This theorem has a practical application: suppose you wanted to decide if the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ were linearly independent. You could (i) find a non-trivial solution to $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$, or (ii) merely show that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$ has more than one solution. Sometimes one is easier than the other.

Linear Independence and Vector Form

The equation

$$\vec{x}=t_1\vec{d}_1+t_2\vec{d}_2$$

represents a plane in vector form of a plane whenever $\vec{d}_1$ and $\vec{d}_2$ are non-zero, non-parallel vectors. In other words, $\vec{x}=t_1\vec{d}_1+t_2\vec{d}_2$ represents a plane whenever $\{\vec{d}_1,\vec{d}_2\}$ is linearly independent.

Does this reasoning work for lines too? The equation

$$\vec{x}=t\vec{d}$$

represents a line in vector form precisely when $\vec{d}\neq\vec{0}$. And $\{\vec{d}\}$ is linearly independent exactly when $\vec{d}\neq 0$.

This reasoning generalizes to volumes. The equation

$$\vec{x}=t_1\vec{d}_1+t_2\vec{d}_2+t_3\vec{d}_3$$

represents a *volume* in vector form exactly when $\{\vec{d}_1,\vec{d}_2,\vec{d}_3\}$ is linearly independent. To see this, suppose $\{\vec{d}_1,\vec{d}_2,\vec{d}_3\}$ were linearly dependent. That means one or more vectors could be removed from $\{\vec{d}_1,\vec{d}_2,\vec{d}_3\}$ without changing its span. Therefore, if $\{\vec{d}_1,\vec{d}_2,\vec{d}_3\}$ is linearly independent $\vec{x}=t_1\vec{d}_1+t_2\vec{d}_2+t_3\vec{d}_3$ at best represents a plane (though it could be a line or a point).

Takeaway. When writing an object in vector form, the direction vectors must always be linearly independent.

Span

Span

DEFINITION

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Additionally, we define $\operatorname{span}\{\} = \{\vec{0}\}$.

---

13 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

13.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

13.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

13.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

13.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

13.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

---

14 Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

14.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

14.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

14.3 Does the expression $\operatorname{span}(\ell_1)$ make sense? If so, what is it? How about $\operatorname{span}(\ell_2)$?

Set Addition

DEF

If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A + B$, is

$$A + B = \{\vec{x} : \vec{x} = \vec{a} + \vec{b} \text{ for some } \vec{a} \in A \text{ and } \vec{b} \in B\}.$$

---

15 Let $A = \left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell = \operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

15.1 Draw A, B, and $A + B$ in the same picture.

15.2 Is $A + B$ the same as $B + A$?

15.3 Draw $\ell + A$.

15.4 Consider the line ℓ_2 given in vector form by $\vec{x} = t\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

Task 1.3: The Magic Carpet, Getting Back Home

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$).

1. Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.
2. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?
3. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?
4. What is span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}$?

DEFINITION

Linearly Dependent & Independent (Geometric)

We say the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

---

16 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

16.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

16.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

16.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

16.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

DEF

Trivial Linear Combination

The linear combination $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ is called ***trivial*** if $\alpha_1 = \cdots = \alpha_n = 0$. If at least one $\alpha_i \neq 0$, the linear combination is called ***non-trivial***.

---

17 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

17.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

17.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

DEF

Linearly Dependent & Independent (Algebraic)

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector.

---

18 18.1 Explain how this algebraic definition (new) implies the geometric one (original).

18.2 Explain how the geometric definition (original) implies this algebraic one (new).

Since we have geometric def $\implies$ algebraic def, and algebraic def $\implies$ geometric def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as algebraic def $\iff$ geometric def).

---

19 Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

19.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

19.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

19.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

19.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Task 1.4: Linear Independence and Dependence, Creating Examples

1. Fill in the following chart keeping track of the strategies you used to generate examples.

	Linearly independent	Linearly dependent
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

2. Write at least two generalizations that can be made from these examples and the strategies you used to create them.

Dot Products & Normal Forms

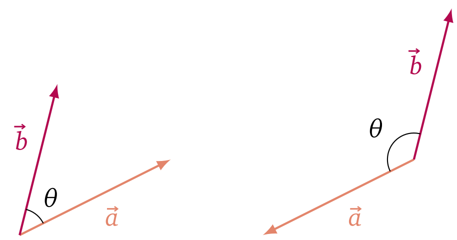
In this module you will learn

- Geometric an algebraic definitions of the dot product.
- How dot products relate to the length of a vector and the angle between two vectors.
- The *normal form* of lines, planes, and hyperplanes.

Let $\vec{a}$ and $\vec{b}$ be vectors rooted at the same point and let θ denote the *smaller* of the two angles between them, so $0 \leq \theta \leq \pi$. The *dot product* of $\vec{a}$ and $\vec{b}$ is defined to be

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta.$$

We will call this the *geometric definition of the dot product*. The dot product is also sometimes called the *scalar product* because the result is a scalar. Note that $\vec{a} \cdot \vec{b} = 0$ when either $\vec{a}$ or $\vec{b}$ is zero or, more interestingly, if their directions are perpendicular.



Algebraically, we can define the dot product in terms of coordinates:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

We will call this the *algebraic definition of the dot product*.

By switching between algebraic and geometric definitions, we can use the dot product to find quantities that are otherwise difficult to find.

Example. Find the angle between the vectors $\vec{v} = (1, 2, 3)$ and $\vec{w} = (1, 1, -2)$.

From the algebraic definition of the dot product, we know

$$\vec{v} \cdot \vec{w} = 1(1) + 2(1) + 3(-2) = -3.$$

From the geometric definition, we know

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = \sqrt{14}\sqrt{6} \cos\theta = 2\sqrt{21} \cos\theta.$$

Equating the two definitions of $\vec{v} \cdot \vec{w}$, we see

$$\cos\theta = \frac{-3}{2\sqrt{21}}$$

and so $\theta = \arccos\left(\frac{-3}{2\sqrt{21}}\right)$.

The dot product has several interesting properties. Since the angle between $\vec{a}$ and itself is 0, the geometric definition of the dot product tells us

$$\vec{a} \cdot \vec{a} = \|\vec{a}\|\|\vec{a}\| \cos 0 = \|\vec{a}\|^2.$$

In other words,

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}},$$

and so dot products can be used to compute the length of vectors.[16]

From the algebraic definition of the dot product, we can deduce several distributive laws. Namely, for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ and any scalar k we have

$$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} \qquad \vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

$$(k\vec{a})\cdot\vec{b} = k(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(k\vec{b})$$

and

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$$

Orthogonality

Recall that for vectors $\vec{a}$ and $\vec{b}$, the relationship $\vec{a}\cdot\vec{b} = 0$ can hold for two reasons: (i) either $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$, or both or (ii) $\vec{a}$ and $\vec{b}$ meet at 90°. Thus, the dot product can be used to tell if two vectors are perpendicular. There is some strangeness with the zero vector here, but it turns out this strangeness simplifies our lives mathematically.

Orthogonal. Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

The definition of orthogonal encapsulates both the idea of two vectors forming a right angle and the idea of one of them being $\vec{0}$.

Before we continue, let's pin down the idea of one vector pointing in the *direction* of another. There are many ways we could define this idea, but we'll go with this one.

Direction. The vector $\vec{u}$ points in the ***direction*** of the vector $\vec{v}$ if $k\vec{u} = \vec{v}$ for some scalar k. The vector $\vec{u}$ points in the ***positive direction*** of $\vec{v}$ if $k\vec{u} = \vec{v}$ for some positive scalar k.

The vector $2\vec{e}_1$ points in the direction of $\vec{e}_1$ since $\frac{1}{2}(2\vec{e}_1) = \vec{e}_1$. Since $\frac{1}{2} > 0$, $2\vec{e}_1$ also points in the positive direction of $\vec{e}_1$. In contrast, $-\vec{e}_1$ points in the direction $\vec{e}_1$ but not the positive direction of $\vec{e}_1$.

When it comes to the relationship between two vectors, there are two extremes: they point in the same direction, or they are orthogonal. The dot product can be used to tell you which of these cases you're in, and more than that, it can tell you to what extent one vector points in the direction of another (even if they don't point in the same direction).

Example. Let $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{b} = \begin{bmatrix}3\\3\end{bmatrix}$, $\vec{c} = \begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{v} = \begin{bmatrix}3\\4\end{bmatrix}$. Which vector out of $\vec{a}$, $\vec{b}$, and $\vec{c}$ has a direction closest to the direction of $\vec{v}$?

XXX Finish

Normal Form of Lines and Planes

Let $\vec{n} = \begin{bmatrix}1\\2\end{bmatrix}$. If a vector $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ is orthogonal to $\vec{n}$, then

$$\vec{n}\cdot\vec{v} = v_1 + 2v_2 = 0,$$

and so $v_1 = -2v_2$. In other words, $\vec{v}$ is orthogonal to $\vec{n}$ exactly when $\vec{v} \in \operatorname{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$. What have we learned? The set of all vectors orthogonal to $\vec{n}$ forms a line $\ell = \operatorname{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$. In this case, we call $\vec{n}$ a *normal vector* for ℓ.

XXX Figure

Normal Vector. A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

In $\mathbb{R}^2$, normal vectors provide yet another way to describe lines, including lines which don't pass through the origin.

[16] Oftentimes in non-geometric settings, the dot product between two vectors is defined first and then the length of $\vec{a}$ is actually *defined* to be $\sqrt{\vec{a}\cdot\vec{a}}$.

Let $n = \begin{bmatrix}1\\2\end{bmatrix}$ as before, and fix $\vec{p} = \begin{bmatrix}1\\1\end{bmatrix}$. If we draw the set of all vectors orthogonal to $\vec{n}$ but *root all the vectors at* $\vec{p}$, again we get a line, but this time the line passes through $\vec{p}$.

XXX Figure

In fact, the line we get is $\ell_2 = \operatorname{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\} + \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} = \ell + \{\vec{p}\}$, which is just ℓ (the parallel line through the origin) translated by $\vec{p}$.

Let's relate this to dot products and normal vectors. By definition, for every $\vec{v} \in \ell$, we have $\vec{n} \cdot \vec{v} = 0$. Since ℓ_2 is a translate of ℓ by $\vec{p}$, we deduce the relationship that for every $\vec{v} \in \ell_2$,

$$\vec{n} \cdot (\vec{v} - \vec{p}) = 0,$$

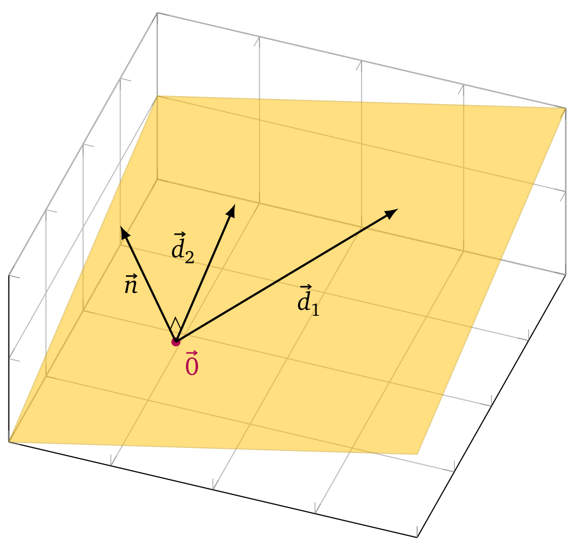
and this is the *normal form* of a line in $\mathbb{R}^2$.

Normal Form of a Line. A line $\ell \subseteq \mathbb{R}^2$ is expressed in ***normal form*** if ℓ is the solution set to the equation

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

where $\vec{n}$ and $\vec{p}$ are fixed vectors.

What about in $\mathbb{R}^3$? Fix a non-zero vector $\vec{n} \in \mathbb{R}^3$ and let $\mathcal{Q} \subseteq \mathbb{R}^3$ be the set of vectors orthogonal to $\vec{n}$. $\mathcal{Q}$ is a plane through the origin, and again, we call $\vec{n}$ a *normal vector* of the plane $\mathcal{Q}$.



In a similar way to the case of a line, $\mathcal{Q}$ is the set of solutions to $\vec{n} \cdot \vec{x} = 0$. And, for any $\vec{p} \in \mathbb{R}^3$, the translated plane $\mathcal{Q} + \{\vec{p}\}$ is the solution set to

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0.$$

Thus, we see planes in $\mathbb{R}^3$ have a normal form just like lines in $\mathbb{R}^2$ do.

Example. Find vector form and normal form of the plane $\mathcal{P}$ passing through the point $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$.

To find vector form of $\mathcal{P}$, we need a point on the plane and two direction vectors. We have three points on the plane, so we can obtain two direction vectors by subtracting these points in different ways. Let

$$\vec{d}_1 = \overrightarrow{AB} = \begin{bmatrix}-1\\1\\0\end{bmatrix} \qquad \vec{d}_2 = \overrightarrow{AC} = \begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

Using the point A, we may now write vector form of $\mathcal{P}$ as

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}-1\\1\\0\end{bmatrix} + s\begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

To write normal form we need to find a normal vector to $\mathcal{P}$. By inspection, we can see that $\vec{n} = (1,1,1)$ is a normal vector to $\mathcal{P}$. If we weren't so insightful, we could also solve the system $\vec{n}\cdot\vec{d}_1 = 0$ and $\vec{n}\cdot\vec{d}_2 = 0$ to find a normal vector. Now, we may express $\mathcal{P}$ in normal form as

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\left(\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=0.$$

In $\mathbb{R}^2$, only lines have a normal form, and in $\mathbb{R}^3$ only planes have a normal form. In general, we call objects in $\mathbb{R}^n$ which have a normal form *hyperplanes*.

Hyperplane. The set $X \subseteq \mathbb{R}^n$ is called a ***hyperplane*** if there exists $\vec{n} \neq \vec{0}$ and $\vec{p}$ so that X is the set of solutions to the equation

$$\vec{n}\cdot(\vec{x}-\vec{p}) = 0.$$

Hyperplanes always have dimension one less than the space they're contained in. So, hyperplanes in $\mathbb{R}^2$ are (one-dimensional) lines, hyperplanes in $\mathbb{R}^3$ are regular (two-dimensional) planes, and hyperplanes in $\mathbb{R}^4$ are (three-dimensional) volumes.

Hyperplanes and Linear Equations

Suppose $\vec{n},\vec{p} \in \mathbb{R}^3$ and $\vec{n} \neq \vec{0}$. Then, solutions to

$$\vec{n}\cdot(\vec{x}-\vec{p}) = 0$$

define a plane $\mathcal{P}$. But, $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$ if and only if

$$\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p} = \alpha.$$

Since $\vec{n}$ and $\vec{p}$ are fixed, α is a constant. Expanding using coordinates, we see

$$\vec{n}\cdot(\vec{x}-\vec{p}) = \vec{n}\cdot\vec{x} - \alpha = n_x x + n_y y + n_z z - \alpha = 0$$

and so $\mathcal{P}$ is the set of solutions to

$$n_x x + n_y y + n_z z = \alpha. \tag{4}$$

Equation (4) is sometimes called *scalar form* of a plane. For us, it will not be important to distinguish between scalar and normal form, but what is important is that we can use the row reduction algorithm to write the complete solution to (4), and this complete solution will necessarily be written in vector form.

Example. Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be the plane passing through $\vec{p}$ and with normal vector $\vec{n}$ where

$$\vec{p} = \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \text{and} \qquad \vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Write $\mathcal{Q}$ in vector form.

We know $\mathcal{Q}$ is the set of solutions to $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$. In scalar form, this equation becomes

$$\vec{n}\cdot(\vec{x}-\vec{p}) = \vec{n}\cdot\vec{x} - \vec{n}\cdot\vec{p} = x + y + z - 2 = 0.$$

Rearranging, we see $\mathcal{Q}$ is the set of all solutions to

$$x + y + z = 2.$$

Using the row reduction algorithm to write the complete solution,[17] we get

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}-1\\1\\0\end{bmatrix} + s\begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}2\\0\\0\end{bmatrix}.$$

[17] In some sense, this is overkill because the equation corresponds to the augmented matrix $\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\end{array}\right]$, which is already row reduced.

Dot Product

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

20

Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

20.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

20.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

20.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

20.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

THM

For a vector $\vec{v} \in \mathbb{R}^n$, the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

always holds.

Distance

DEF

The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

Unit Vector

DEF

A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

21

Let $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

21.1 Find the distance between $\vec{u}$ and $\vec{v}$.

21.2 Find a unit vector in the direction of $\vec{u}$.

21.3 Does there exist a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

21.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Orthogonal

DEF Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

22

22.1 Find two vectors orthogonal to $\vec{a} = \begin{bmatrix}1\\-3\end{bmatrix}$. Can you find two such vectors that are not parallel?

22.2 Find two vectors orthogonal to $\vec{b} = \begin{bmatrix}1\\-3\\4\end{bmatrix}$. Can you find two such vectors that are not parallel?

22.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\| = 5$ and $\|\vec{y}\| = 3$. What is the distance between $\vec{x}$ and $\vec{y}$?

23

23.1 Draw $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.

23.2 If $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x}\cdot\vec{u}$?

23.3 Expand the dot product $\vec{u}\cdot\vec{x}$ to get an equation for A.

23.4 If possible, express A as a span.

Normal Vector

DEF A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

24

Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}1\\-1\end{bmatrix}$, and define the lines

$$\ell_1 = \operatorname{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \operatorname{span}\{\vec{d}\} + \{\vec{p}\}.$$

24.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

24.2 Let $\vec{v} \in \ell_1$ and $\vec{u} \in \ell_2$. What is $\vec{n}\cdot\vec{v}$? What about $\vec{n}\cdot(\vec{u}-\vec{p})$? Explain using a picture.

24.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n}\cdot(\vec{x}-\vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

24.4 Some textbooks would claim that ℓ_2 could be expressed in normal form as $\begin{bmatrix}2\\-1\end{bmatrix}\cdot\vec{x} = 3$. How does this relate to the $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$ normal form? Where does the 3 come from?

25

Let $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

25.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

25.2 Describe X using an equation.

25.3 Describe X as a span.

Projections & Vector Components

In this module you will learn

- The definition of the projection of a vector and the vector component of one vector in the direction of another.
- The relationship between projection, orthogonality, and vector components.
- How to project a vector onto a line.

Consider the following situation: you're designing a 3d video game, but your uses only have 2d screens. Or, you have a 900 dimensional dataset, but you want to visualize it on a continuum (i.e., as a line). Each of these is an example of finding the best approximation to a set of points given some restrictions. In general, this operation is called a *projection* and in the world of linear algebra, it has a very particular meaning.

Projection. Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\operatorname{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

Let $\mathcal{P}_{xy} \subseteq \mathbb{R}^3$ be the xy-plane in $\mathbb{R}^3$ and let $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Intuitively, $\operatorname{proj}_{\mathcal{P}_{xy}} \vec{v}$ is the "shadow" that $\vec{v}$ would casts on $\mathcal{P}_{xy}$ if the sun were directly overhead. In other words,

$$\operatorname{proj}_{\mathcal{P}_{xy}} \vec{v} = \begin{bmatrix}1\\2\\0\end{bmatrix}.$$

XXX Figure

Let $\ell_y \subseteq \mathbb{R}^3$ be the x-axis in $\mathbb{R}^3$. It's a little bit harder to visualize what $\operatorname{proj}_{\ell_y} \vec{v}$ is, so let's appeal to some definitions.

By definition, every vector in ℓ_y takes the form $\vec{u}_t = \begin{bmatrix}0\\t\\0\end{bmatrix}$ for some $t \in \mathbb{R}$. The distance between $\vec{u}_t$ and $\vec{v}$ is

$$\|\vec{u}_t - \vec{v}\| = \left\| \begin{bmatrix}0\\t\\0\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} \right\| = \sqrt{1^2 + (t-2)^2 + 3^2}.$$

Since $(t-2)^2$ is always positive, the quantity $\sqrt{1^2 + (t-2)^2 + 3^2}$ is minimized when $(t-2)^2 = 0$. That is, when $t = 2$. Thus, we see $\vec{u}_2$ is the closest vector in ℓ_y to $\vec{v}$ and so,

$$\operatorname{proj}_{\ell_y} \vec{v} = \vec{u}_2 = \begin{bmatrix}0\\2\\0\end{bmatrix}.$$

Here, we appealed directly to the definition of projection to find the answer.

Example. Let $\ell \subseteq \mathbb{R}^2$ be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix}$, and let $\vec{v} = \begin{bmatrix}-1\\-1\end{bmatrix}$. Use the definition of projection to find $\operatorname{proj}_\ell \vec{v}$.

XXX Finish

Every example of a projection so far shares a geometric property. In the case of lines and planes, the vector from the projection to the original point is a normal vector for the line or plane.

XXX Figure

Stated precisely, if X is a line or plane and $\vec{v} \notin X$ is a vector, then $\vec{v} - \operatorname{proj}_X \vec{v}$ is a normal vector for X. Using this fact, we can find projections onto lines and planes without solving non-linear equations.

Example. Let $\ell \subseteq \mathbb{R}^2$ be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix}$, and let $\vec{v} = \begin{bmatrix}-1\\-1\end{bmatrix}$. Use the fact that $\vec{v} - \operatorname{proj}_\ell \vec{v}$ is a normal vector to ℓ to find Find $\operatorname{proj}_\ell \vec{v}$.

XXX Finish

Projections Onto Other Sets

For projections onto lines and planes, we can use what we know about normal vectors to simplify our life. The same is true when projecting onto other sets, but we must always keep the definition in mind.

Example. Let $\mathcal{T} \subseteq \mathbb{R}^2$ be the filled in triangle with vertices $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\2\end{bmatrix}$, and let

$$\vec{a} = \begin{bmatrix}1/4\\1/4\end{bmatrix} \qquad \vec{b} = \begin{bmatrix}3\\1/2\end{bmatrix} \qquad \vec{c} = \begin{bmatrix}1\\1\end{bmatrix}.$$

Find $\operatorname{proj}_\mathcal{T} \vec{a}$, $\operatorname{proj}_\mathcal{T} \vec{b}$, and $\operatorname{proj}_\mathcal{T} \vec{c}$.

XXX Finish XXX Figure

Subtleties of Projections

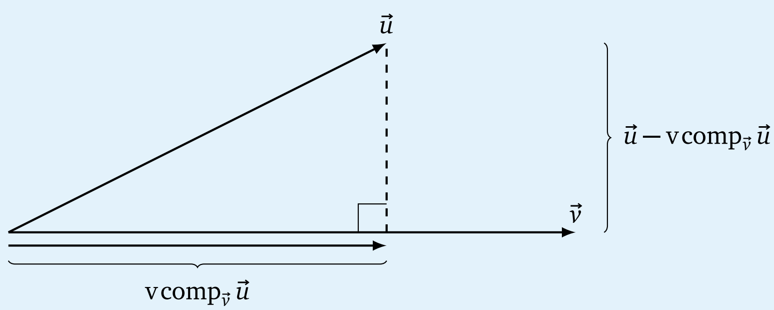
You might be wondering, what is $\operatorname{proj}_X \vec{v}$ if $\vec{v}$ is equidistant from *two* closest points in X. Or, what if X is an *open* set (for example, an open interval in $\mathbb{R}^1$). Then there might not be a closest point in X to $\vec{v}$. In both these cases, we say $\operatorname{proj}_X \vec{v}$ is *undefined*.

Formally, for a fixed set X, we consider $P(\vec{v}) = \operatorname{proj}_X \vec{v}$ as a function that inputs and outputs vectors. And, as a function, P has a domain consisting of exactly the vectors $\vec{v}$ for which $P(\vec{v})$ is defined. As it happens, if X is a line or a plane, the domain of P is all of $\mathbb{R}^n$, and in this text, we will be sensible and only ask about projections restricted to an appropriate domain.

Vector Components

We've seen before that dot products can be used to measure how much one vector points in the direction of another. But, we can go further. Suppose $\vec{v} \neq \vec{0}$ and $\vec{u}$ are vectors. We might want to *decompose* $\vec{u}$ into the sum of two vectors, one which is in the direction of $\vec{v}$ and the other which is orthogonal to $\vec{v}$. The tool to that does this is the vector component.

Vector Components. Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***vector component of $\vec{u}$ in the $\vec{v}$ direction***, written $\operatorname{vcomp}_{\vec{v}} \vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.



From the definition, it's obvious that

$$\vec{u} = \operatorname{vcomp}_{\vec{v}} \vec{u} + (\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u})$$

is a decomposition of $\vec{u}$ into the sum of two vectors, one, $\operatorname{vcomp}_{\vec{v}} \vec{u}$, is in the direction of $\vec{v}$, and the other, $\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$, is orthogonal to $\vec{v}$.

Example. Find the component of $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$ in the direction of $\vec{b} = \begin{bmatrix}1\\1\end{bmatrix}$.

XXX Finish

Since we'll be computing vector components often, let's try to find a formula for $\operatorname{vcomp}_{\vec{v}} \vec{u}$.

By definition $\operatorname{vcomp}_{\vec{v}} \vec{u}$ is a vector in the direction of $\vec{v}$, so

$$\operatorname{vcomp}_{\vec{v}} \vec{u} = k\vec{v}.$$

Further, from the definition $\vec{u} - \mathrm{vcomp}_{\vec{v}}\, \vec{u}$ is orthogonal to $\vec{v}$, and so

$$\vec{v} \cdot (\vec{u} - \mathrm{vcomp}_{\vec{v}}\, \vec{u}) = \vec{v} \cdot (\vec{u} - k\vec{v}) = \vec{v} \cdot \vec{u} - k\vec{v} \cdot \vec{v} = 0,$$

Because $\vec{v} \neq \vec{0}$, we know $\vec{v} \cdot \vec{v} \neq 0$. Therefore, we may rearrange and solve for k to find

$$k = \frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}},$$

which means

$$\mathrm{vcomp}_{\vec{v}}\, \vec{u} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}}\right)\vec{v}.$$

The Relationship Between Vector Components and Projections

Vector components and projections onto lines are closely related. So closely related that many textbooks use the single word *projection* to talk about both vector components and projections. Let's take a moment to explore this relationship.

Let $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and let $\ell = \mathrm{span}\{\vec{v}\}$. Drawing a picture of ℓ, $\vec{u}$, and $\mathrm{proj}_\ell\, \vec{u}$, we see that $\mathrm{proj}_\ell\, \vec{u}$ satisfies all the properties of $\mathrm{vcomp}_{\vec{v}}\, \vec{u}$.

XXX Figure

Since $\ell = \mathrm{span}\{\vec{v}\}$ and $\mathrm{proj}_\ell\, \vec{u} \in \ell$, we know that $\mathrm{proj}_\ell\, \vec{u}$ is in the direction of $\vec{v}$. Further, from geometric arguments, we know $\vec{u} - \mathrm{proj}_\ell\, \vec{u}$ is a normal vector for ℓ and is therefore orthogonal to its direction vector, $\vec{v}$! What's more, we didn't use anything in particular about $\vec{u}$ and $\vec{v}$ when making this argument (other than $\vec{v} \neq \vec{0}$). This means, we may establish a general formula.

Theorem. For vectors $\vec{u}$ and $\vec{v} \neq 0$, we have

$$\mathrm{proj}_{\mathrm{span}\{\vec{v}\}}\, \vec{u} = \mathrm{vcomp}_{\vec{v}}\, \vec{u}.$$

This is great news because vector components are easy to compute using dot products while projections are usually hard to compute.

Example. Compute the projection of $\vec{a} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ onto $\ell = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\}$.

XXX Finish

It's worth noting, however, that vector components are equal to projections *only in the case when you're projecting onto a span*. In general, projections and vector components are unrelated.

Example. Let $\vec{a} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, and let ℓ be the line given in vector form by $\vec{x} = t\vec{b} + \vec{a}$. Show that $\mathrm{proj}_\ell\, \vec{a} \neq \mathrm{vcomp}_{\vec{v}}\, \vec{a}$.

XXX Finish

Projections

DEF

Projection

Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\operatorname{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

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26 Let $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}2\\2\end{bmatrix}$ and $\ell = \operatorname{span}\{\vec{a}\}$.

26.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.

26.2 Find $\operatorname{proj}_{\{\vec{b}\}} \vec{v}$, $\operatorname{proj}_{\{\vec{a},\vec{b}\}} \vec{v}$.

26.3 Find $\operatorname{proj}_\ell \vec{v}$. (Recall that a quadratic $at^2 + bt + c$ has a minimum at $t = -\frac{b}{2a}$).

26.4 Is $\vec{v} - \operatorname{proj}_\ell \vec{v}$ a normal vector for ℓ? Why or why not?

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27 Let K be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}$ and let $\vec{c} = \begin{bmatrix}1\\3\end{bmatrix}$.

27.1 Make a sketch with $\vec{c}$, K, and $\operatorname{proj}_K \vec{c}$ (you don't need to compute $\operatorname{proj}_K \vec{c}$ exactly).

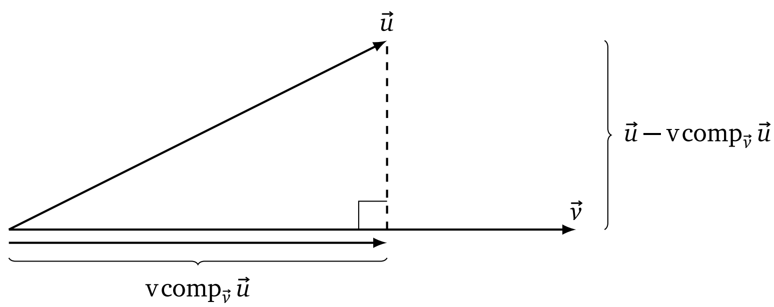
27.2 What should $(\vec{c} - \operatorname{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix}$ be? Explain.

27.3 Use your formula from the previous part to find $\operatorname{proj}_K \vec{c}$ *without* computing any distances.

DEFINITION

Vector Components

Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***vector component of $\vec{u}$ in the $\vec{v}$ direction***, written $\operatorname{vcomp}_{\vec{v}} \vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.



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28 Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

28.1 List two conditions that $\operatorname{vcomp}_{\vec{b}} \vec{a}$ must satisfy.

28.2 Find a formula for $\operatorname{vcomp}_{\vec{b}} \vec{a}$.

---

29 Let $\vec{d} = \begin{bmatrix}3\\3\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$.

29.1 Draw $\vec{d}$, $\vec{u}$, $\operatorname{span}\{\vec{d}\}$, and $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}} \vec{u}$ in the same picture.

29.2 How do $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}} \vec{u}$ and $\operatorname{vcomp}_{\vec{d}} \vec{u}$ relate?

29.3 Compute $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}} \vec{u}$ and $\operatorname{vcomp}_{\vec{d}} \vec{u}$.

29.4 Compute $\operatorname{vcomp}_{-\vec{d}} \vec{u}$. Is this the same as or different from $\operatorname{vcomp}_{\vec{d}} \vec{u}$? Explain.

Subspaces & Bases

In this module you will learn

- Formal and intuitive definitions of subspaces.
- The relationship between subspaces and spans.
- How to prove whether or not a set is a subspace.
- How to find a basis for and the dimension of a subspace.

Lines or planes through the origin can be written as spans of their direction vectors. However, a line or plane that doesn't pass through the origin cannot be written as a span—it must be expressed as a *translated* span.

XXX Figure

There's something special about sets that can be expressed as (untranslated) spans. In particular, since a linear combination of linear combinations is still a linear combination, a span is *closed* with respect to linear combinations. That is, by taking linear combinations of vectors in a span, you cannot escape the span. In general, sets that have this property are called *subspaces*.

Subspace. A non-empty subset $V \subseteq \mathbb{R}^n$ is called a ***subspace*** if for all $\vec{u}, \vec{v} \in V$ and all scalars k we have

(i) $\vec{u} + \vec{v} \in V$; and

(ii) $k\vec{u} \in V$.

In the definition of subspace, property (i) is called begin *closed with respect to vector addition* and property (ii) is called being *closed with respect to scalar multiplication*.

Subspaces generalize the idea of *flat spaces through the origin*. They encompass lines, planes, volumes and more.

Example. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be the complete solution to $x + 2y = 0$. Show that $\mathcal{V}$ is a subspace.

XXX Finish

Example. Let $\mathcal{W} \subseteq \mathbb{R}^2$ be the line expressed in vector form as

$$\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}.$$

Determine whether $\mathcal{W}$ is a subspace.

XXX Finish

As mentioned earlier, subspaces and spans are deeply connected by the following theorem.

Theorem. Every subspace is a span and every span is a subspace. More precisely, $\mathcal{V} \subseteq \mathbb{R}^n$ is a subspace if and only if $\mathcal{V} = \operatorname{span} \mathcal{X}$ for some set $\mathcal{X}$.

Proof. We will start by showing every span is a subspace. Fix $\mathcal{X} \subseteq \mathbb{R}^2$ and let $\mathcal{V} = \operatorname{span} \mathcal{X}$. First note that if $\mathcal{X} \neq \{\}$, then $\mathcal{V}$ is non-empty because $\mathcal{X} \subseteq \mathcal{V}$ and if $\mathcal{X} = \{\}$, then $\mathcal{V} = \{\vec{0}\}$, and so is still non-empty.

Fix $\vec{v}, \vec{u} \in \mathcal{V}$. By definition there are $\vec{x}_1, \vec{x}_2, \ldots, \vec{y}_1, \vec{y}_2, \ldots \in \mathcal{X}$ and scalars $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ so that

$$\vec{v} = \sum \alpha_i \vec{x}_i \qquad \vec{u} = \sum \beta_i \vec{y}_i.$$

To verify property (i), observe that

$$\vec{u} + \vec{v} = \sum \alpha_i \vec{x}_i + \sum \beta_i \vec{y}_i$$

is also a linear combination of vectors in $\mathcal{X}$ (because all $\vec{x}_i$ and $\vec{y}_i$ are in $\mathcal{X}$), and so $\vec{u} + \vec{v} \in \operatorname{span} \mathcal{X} = \mathcal{V}$.

To verify property (ii), observe that for any scalar α,

$$\alpha\vec{v} = \alpha \sum \alpha_i \vec{x}_i = \sum (\alpha\alpha_i)\vec{x}_i \in \operatorname{span} \mathcal{X} = \mathcal{V}.$$

Since $\mathcal{V}$ is non-empty and satisfies both properties (i) and (ii), it is a subspace.

Now we will prove that every subspace is a span. Let $\mathcal{V}$ be a subspace and consider $\mathcal{V}' = \operatorname{span}\mathcal{V}$. Since taking a span may only enlarge a set, we know $\mathcal{V} \subseteq \mathcal{V}'$. If we establish that $\mathcal{V}' \subseteq \mathcal{V}$, then $\mathcal{V} = \mathcal{V}' = \operatorname{span}\mathcal{V}$, which would complete the proof.

Fix $\vec{x} \in \mathcal{V}'$. By definition, there are $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathcal{V}$ and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ so that

$$\vec{x} = \sum \alpha_i \vec{v}_i.$$

Observe that $\alpha_i \vec{v}_i \in \mathcal{V}$ for all i, since $\mathcal{V}$ is closed under scalar multiplication. It follows that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 \in \mathcal{V}$ because $\mathcal{V}$ is closed under sums. Continuing, $(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) + \alpha_3\vec{v}_3 \in \mathcal{V}$ because $\mathcal{V}$ is closed under sums. By finite induction we see

$$\vec{x} = \sum \alpha_i \vec{v}_i = \Big(\big((\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) + \alpha_3\vec{v}_3\big) + \cdots + \alpha_{n-1}\vec{v}_{n-1}\Big) + \alpha_n\vec{v}_n \in \mathcal{V}.$$

Thus $\mathcal{V}' \subseteq \mathcal{V}$, which completes the proof. ∎

The previous theorem is saying that spans and subspaces are two ways of talking about the same thing. Spans provide a *constructive* definition of lines/planes/volumes/etc. through the origin. That is, when you describe a line/plane/etc. through the origin as a span, you're saying "this is a line/plane/etc. through the origin because every point in it is a linear combination of *these specific vectors*". In contrast, subspaces provide a *categorical* definition of lines/planes/etc. through the origin. When you describe a line/plane/etc. through the origin as a subspace, you're saying "this is a line/plane/etc. through the origin because these *properties* are satisfied" even though you might not be able to specify any of the vectors inside.

Takeaway. Spans and subspaces are two different ways of talking about the same objects: points/lines/planes/etc. through the origin.

Special Subspaces

When thinking about $\mathbb{R}^n$ there are two special subspaces that are always available. The first is $\mathbb{R}^n$ itself. $\mathbb{R}^n$ is obviously non-empty, and linear combinations of vectors in $\mathbb{R}^n$ remain in $\mathbb{R}^n$. The second is the *trivial subspace*, $\{\vec{0}\}$.

Trivial Subspace. The subset $\{\vec{0}\} \subseteq \mathbb{R}^n$ is called the ***trivial subspace***.

Theorem. The trivial subspace is a subspace.

Proof. First note that $\{\vec{0}\}$ is non-empty since $\vec{0} \in \{\vec{0}\}$. Now, since $\vec{0}$ is the only vector in $\{\vec{0}\}$, properties (i) and (ii) follow quickly:

$$\vec{0} + \vec{0} = \vec{0} \in \{\vec{0}\}$$

and

$$\alpha\vec{0} = \vec{0} \in \{\vec{0}\}.$$

∎

Bases

Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and consider $\ell = \operatorname{span}\{\vec{d}\}$.

XXX Figure

We know that ℓ is a subspace and, for instance, $\ell = \operatorname{span}\{\vec{d}, -2\vec{d}, \frac{1}{2}\vec{d}\}$. However, the simplest descriptions of ℓ involve the span of only one vector.

Analogously, let $\mathcal{P} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$ be the plane through the origin with direction vectors $\vec{d}_1$ and $\vec{d}_2$. There are many ways to write $\mathcal{P}$ as a span, but the simplest ones involve exactly two vectors. The idea of a *basis* comes from trying to find the simplest description of a subspace.

Basis. A ***basis*** for a subspace $\mathcal{V}$ is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = \mathcal{V}$.

In short, a basis for a subspace is a linearly independent set that spans that subspace.

Example. Let $\ell = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\-4\end{bmatrix}, \begin{bmatrix}1/2\\1\end{bmatrix}\right\}$. Find two different bases for ℓ.

XXX Finish

Unpacking the definition of basis a bit more, we can see that for a subspace $\mathcal{V}$, a basis for $\mathcal{V}$ is a set of vectors that is *just the right size* to describe everything in $\mathcal{V}$. It's not too big—because it is linearly independent, there are no redundancies. It's not too small—because we require it to span $\mathcal{V}$.[18]

There are several facts everyone should know about bases:

- Bases are not unique. Every subspace (except the trivial subspace) has multiple bases.
- Given a basis for a subspace, every vector in the subspace can be written as a *unique* linear combination of vectors in that subspace.
- Any two bases for the same subspace have the same number of elements.

You can prove the first fact by observing that if $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots\}$ is a basis with at least one element,[19] then $\{2\vec{b}_1, 2\vec{b}_2, \ldots\}$ is a different basis. The second fact is a consequence of all bases being linearly independent. The third fact is less obvious and takes some legwork to prove, so we will accept it as is.

Dimension

Let $\mathcal{V}$ be a subspace. Though there are many bases for $\mathcal{V}$, they all have the same number of vectors in them. And, this number says something fundamental about $\mathcal{V}$: it tells us the maximum number of linearly independent vectors that can simultaneously exist in $\mathcal{V}$. We call this number the *dimension* of $\mathcal{V}$.

Dimension. The ***dimension*** of a subspace V is the number of elements in a basis for V.

This definition agrees with our intuition about lines and planes: the dimension of a line through $\vec{0}$ is 1, and the dimension of a plane through $\vec{0}$ is 2. It even tells us the dimension of the single point $\{\vec{0}\}$ is 0.[20]

Example. Find the dimension of $\mathbb{R}^2$.

Since $\{\vec{e}_1, \vec{e}_2\}$ is a basis for $\mathbb{R}^2$, we know $\mathbb{R}^2$ is two dimensional.

Example. Let $\ell = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\-4\end{bmatrix}, \begin{bmatrix}1/2\\1\end{bmatrix}\right\}$.

XXX Finish

Example. Let $A = \{(x_1, x_2, x_3, x_4) \;:\; x_1 + 2x_2 - x_3 = 0 \text{ and } x_1 + 6x_4 = 0\}$. Find a basis for and the dimension of A.

XXX Finish

Like $\mathbb{R}^2$ and $\mathbb{R}^3$, whenever we discuss $\mathbb{R}^n$, we always have a standard basis that comes along for the ride.

Standard Basis. The ***standard basis*** for $\mathbb{R}^n$ is the set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ where

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\\\vdots\end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\end{bmatrix} \qquad \vec{e}_3 = \begin{bmatrix}0\\0\\1\\\vdots\end{bmatrix} \qquad \cdots.$$

That is $\vec{e}_i$ is the vector with a 1 in its ith coordinate and zeros elsewhere.

The notation $\vec{e}_i$ is context specific. If we say $\vec{e}_i \in \mathbb{R}^2$, then $\vec{e}_i$ must have exactly two components. If we say $\vec{e}_i \in \mathbb{R}^{45}$, then $\vec{e}_i$ must have 45 components.

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[18] If you're into British fairy tales, you might call a basis a *Goldilocks set*.

[19] The empty set is a basis for the trivial subspace.

[20] The dimension of a line, plane, or point not through the origin is defined to be the dimension of the subspace obtained when translating it to the origin.

Subspaces and Bases

Subspace

DEFINITION

A non-empty subset $V \subseteq \mathbb{R}^n$ is called a ***subspace*** if for all $\vec{u}, \vec{v} \in V$ and all scalars k we have

(i) $\vec{u} + \vec{v} \in V$; and

(ii) $k\vec{u} \in V$.

Subspaces give a mathematically precise definition of a "flat space through the origin."

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30 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

30.1 $A = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for some } a \in \mathbb{Z}\right\}$.

30.2 $B = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$.

30.3 $C = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

30.4 $D = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

30.5 $E = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

30.6 $F = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

30.7 $G = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

30.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Basis

DEF

A ***basis*** for a subspace $\mathcal{V}$ is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = \mathcal{V}$.

Dimension

DEF

The ***dimension*** of a subspace V is the number of elements in a basis for V.

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31 Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

31.1 Describe V.

31.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

31.3 Give a basis for V.

31.4 Give another basis for V.

31.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

31.6 What is the dimension of V?

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32 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 7 \\ 8 \\ 8 \end{bmatrix}$ (notice these vectors are linearly independent) and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

32.1 Give a basis for and the dimension of P.

32.2 Give a basis for and the dimension of Q.

32.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

32.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Matrix Representations of Systems of Linear Equations

In this module you will learn

- How to represent a system of linear equations as a matrix equation.
- Multiple ways to interpret solutions of systems of linear equations.
- How linear independence/dependence relates to solutions to matrix equations.
- How to use matrix equations to find normal vectors to lines or planes.

Matrix-vector multiplication gives a compact way to represent systems of linear equations.

Consider the system

$$\begin{cases} x+2y-2z=-15 \\ 2x+\ \ y-5z=-21 \\ x-4y+\ \ z=18 \end{cases}, \tag{5}$$

which is equivalent to the vector equation

$$\begin{bmatrix} x+2y-2z \\ 2x+y-5z \\ x-4y+z \end{bmatrix} = \begin{bmatrix} -15 \\ -21 \\ 18 \end{bmatrix}.$$

But, we can also rewrite (5) using matrix-vector multiplication:

$$\underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -5 \\ 1 & -4 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -15 \\ -21 \\ 18 \end{bmatrix}.$$

The matrix A on the left is called the *coefficient matrix* because it is made up of the coefficients from equation (5).

By using coefficient matrices, every system of linear equations can be rewritten as a single matrix equation of the form

$$A\vec{x}=\vec{b}$$

where A is a coefficient matrix, $\vec{x}$ is a column vector of variables, and $\vec{b}$ is a column vector of constants.

Example. Consider the one equation system

$$\{x-4y+z=5 \tag{6}$$

and the two-equation system

$$\begin{cases} x-4y+z=5 \\ \qquad y-z=9 \end{cases}. \tag{7}$$

Rewrite each system as a single matrix equation.

XXX Finish

Interpretations of Matrix Equations

The solution set to a system of linear equations, like

$$\begin{cases} x+2y-2z=-15 \\ 2x+\ \ y-5z=-21 \\ x-4y+\ \ z=18 \end{cases} \tag{8}$$

can be interpreted as the intersection of three planes (or hyperplanes if there were more variables)—each individual equation specifies one plane and so solutions to the system are in the intersection of the planes specified by each equation.

However, when this system is rewritten in matrix form, the two ways to interpret matrix-vector multiplication give rise to two additional ways to interpret the solution set.

The *Column* Picture

Using the column interpretation of matrix-vector multiplication, we see that system (8) is equivalent to

$$\begin{bmatrix}1&2&-2\\2&1&-5\\1&-4&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=x\begin{bmatrix}1\\2\\1\end{bmatrix}+y\begin{bmatrix}2\\1\\-4\end{bmatrix}+z\begin{bmatrix}-2\\-5\\1\end{bmatrix}=\begin{bmatrix}-15\\-21\\18\end{bmatrix}.$$

This means the question, "What are the solutions to system (8)?" is equivalent to the question, "What coefficients allow $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}2\\1\\-4\end{bmatrix}$, and $\begin{bmatrix}-2\\-5\\1\end{bmatrix}$ to form $\begin{bmatrix}-15\\-21\\18\end{bmatrix}$ as a linear combination?" Here, $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}2\\1\\-4\end{bmatrix}$, and $\begin{bmatrix}-2\\-5\\1\end{bmatrix}$ are the columns of the coefficient matrix.

The *Row* Picture

The row interpretation gives us another perspective. Let $\vec r_1$, $\vec r_2$, and $\vec r_3$ be the rows of the coefficient matrix for system (8). Then the system is equivalent to

$$\begin{bmatrix}1&2&-2\\2&1&-5\\1&-4&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\vec r_1\\ \hline \vec r_2\\ \hline \vec r_3\end{bmatrix}\vec x=\begin{bmatrix}\vec r_1\cdot\vec x\\\vec r_2\cdot\vec x\\\vec r_3\cdot\vec x\end{bmatrix}=\begin{bmatrix}-15\\-21\\18\end{bmatrix}.$$

In other words, we can interpret solutions to system (8) as vectors whose dot product with $\vec r_1$ is -15, whose dot product with $\vec r_2$ is -12, and whose dot product with $\vec r_3$ is 18. Given that the dot product has a geometric interpretation, this perspective will sometimes be enlightening.

Interpreting Homogeneous Systems

Consider the homogeneous system/matrix equation

$$\begin{aligned}x+2y-2z&=0\\2x+\;\,y-5z&=0\\x-4y+\;\,z&=0\end{aligned}\qquad\iff\qquad\underbrace{\begin{bmatrix}1&2&-2\\2&1&-5\\1&-4&1\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.\tag{9}$$

Now, the column interpretation of system (9) is: what linear combinations of the columns vectors of A give $\vec 0$? This directly relates to the question of whether the column vectors of A are linearly independent.

Let $\vec r_1$, $\vec r_2$, and $\vec r_3$ be the rows of A. The row interpretation of system (9) asks what vectors are simultaneously orthogonal to $\vec r_1$, $\vec r_2$, and $\vec r_3$.

> **Takeaway.** There are three ways to interpret solutions to a matrix equation $A\vec x=\vec b$: (i) the intersection of hyperplanes specified by the rows; (ii) what linear combinations of the columns of A give $\vec b$; (iii) what vectors yield the entries of $\vec b$ when dot producted with the rows of A.

Example. Find all vectors orthogonal to $\vec a=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\vec b=\begin{bmatrix}1\\2\\1\end{bmatrix}$.

XXX Finish

The row picture is particularly applicable when trying to find normal vectors.

Example. Let $\mathcal{Q}$ be the hyperplane specified in vector form by

$$\vec x=t\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}+s\begin{bmatrix}0\\1\\0\\1\end{bmatrix}+r\begin{bmatrix}2\\0\\0\\0\end{bmatrix}+\begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

Find a normal vector for $\mathcal{Q}$ and write $\mathcal{Q}$ in normal form.

XXX Finish

33 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

33.1 Compute the product $A\vec{x}$.

33.2 Write down a system of equations that corresponds to the matrix equation $A\vec{x} = \vec{b}$.

33.3 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

33.4 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

34 Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

34.1 How could you determine if $\{\vec{u}, \vec{v}, \vec{w}\}$ was a linearly independent set?

34.2 Can your method be rephrased in terms of a matrix equation? Explain.

35 Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

35.1 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

35.2 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

36 Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Let $\mathcal{P}$ be the plane given in vector form by $\vec{x} = t\vec{d}_1 + s\vec{d}_2$. Further, suppose M is a matrix so that $M\vec{r} \in \mathcal{P}$ for any $\vec{r} \in \mathbb{R}^2$.

36.1 How many rows does M have?

36.2 Find such an M.

36.3 Find necessary and sufficient conditions (phrased as equations) for $\vec{n}$ to be a normal vector for $\mathcal{P}$.

36.4 Find a matrix K so that non-zero solutions to $K\vec{x} = \vec{0}$ are normal vectors for $\mathcal{P}$. How do K and M relate?

Coordinates & Change of Basis I

In this module you will learn

- Notation for representing a vector in multiple bases.
- The distinction between a vector and its representation.
- How to compute multiple representation of a vector.
- The definition of an *oriented* basis.

Recall that when we write $\vec{x} = \begin{bmatrix}2\\3\end{bmatrix}$, what we actually mean is $\vec{x} = 2\vec{e}_1 + 3\vec{e}_2$. The numbers 2 and 3 are called the coordinates of the vector $\vec{x}$ with respect to the standard basis. However, in general, subspaces have many bases, and so it is possible to represent a single vector in many ways as coordinates with respect to *many* bases.

Let $\vec{x} = \begin{bmatrix}2\\3\end{bmatrix}$; let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ be another basis for $\mathbb{R}^2$. Now, the coordinates of $\vec{x}$ with respect to $\mathcal{E}$ are $(2, 3)$, but the coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $(1, 1)$.

XXX Figure

The coordinates $(2, 3)$ and $(1, 1)$ represent $\vec{x}$ equally well, and when solving problems, we should pick the coordinates that make our problem the easiest.[21] However, now that we are representing vectors in multiple bases, we need a way to keep track of what coordinates correspond to which basis.

Representation in a Basis.
Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notated $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix}.$$

where $\alpha_1, \ldots, \alpha_n$ uniquely satisfy $\vec{v} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$.
Conversely,

$$\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix}_{\mathcal{B}} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \ldots, \vec{b}_n$ with coefficients $\alpha_1, \ldots, \alpha_n$.

Example. Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \vec{e}_1 + \vec{e}_2$, and $\vec{b}_2 = 3\vec{e}_2$ be another basis for $\mathbb{R}^2$. Given that $\vec{v} = 2\vec{e}_1 - \vec{e}_2$, find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{B}}$.

XXX Finish

Notation Conventions

In light of this notation, we need to revisit some past notation. Again, we have been writing $\vec{x} = \begin{bmatrix}1\\3\end{bmatrix}$ to mean $\vec{x} = 2\vec{e}_1 + 3\vec{e}_2$. However, given the representation-in-a-basis notation, we should be writing

$$\vec{x} = \begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^2$. We should write $\begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{E}}$ because the coordinates $(2, 3)$ refer to *different* vectors for *different* bases. However, most of the time we are only thinking about the standard basis. So, the convention we will follow is:

- If a problem involves only one basis, we may write $\begin{bmatrix}x\\y\end{bmatrix}$ to mean $\begin{bmatrix}x\\y\end{bmatrix}_{\mathcal{E}}$ where $\mathcal{E}$ is the standard basis.

[21] For example, maybe in one choice of coordinates, we can avoid all fractions in our calculations—this could be good if you're programming a computer that rounds it.

- If there are multiple bases in a problem, we will always write $\begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{X}}$ to specify a vector in coordinates relative to a particular basis $\mathcal{X}$.

Takeaway. If a problem only involves the standard basis, we may use the notation we always have. If a problem involves multiple bases, we must ***always*** use representation-in-a-basis notation.

True Vectors vs. Representations

[22]

The Belgian surrealist René Magritte painted the work above, which is subtitled, "This is not a pipe". Why? Because, of course, it is not a pipe. It is a painting of a pipe! In this work, Magritte points out a distinction that will soon become very important to us—the distinction between an object and a representation of that object.

Let $\vec{x} = 2\vec{e}_1 + 3\vec{e}_2 \in \mathbb{R}^2$. The vector $\vec{x}$ is a *real-life geometrical thing*, and to emphasize this, we will call $\vec{x}$ a *true* vector. In contrast, when we write the column matrix $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we are writing a *list of numbers*. The list of numbers $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has no meaning until we give it a meaning by assigning it a basis. For example, by writing $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{E}}$ we declare that the numbers 2 and 3 are coefficients of $\vec{e}_1$ and $\vec{e}_2$. By writing $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, we declare that the numbers 2 and 3 are coefficients of $\vec{b}_1$ and $\vec{b}_2$. Since a list of numbers without a basis has no meaning, we must write

$$\vec{x} \neq [\vec{x}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

since the left side of the equation is a *true vector* and the right side is a *list of numbers*. Similarly, we must write

$$[\vec{x}]_{\mathcal{E}} \neq \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{E}} = \vec{x},$$

since the left side is a *list of numbers* and the right side is a *true vector*.

To help keep the notation straight in your head, for a basis $\mathcal{X}$, remember the rule

$$[\text{true vector}]_{\mathcal{X}} = \text{list of numbers} \qquad \text{and} \qquad [\text{list of numbers}]_{\mathcal{X}} = \text{true vector}.$$

It's easy to get confused when answering questions that involve multiple bases; precision will make these problems much easier.

Orientation of a Basis

How can you tell the difference between a hand and a foot? They're similar in structure[23]—a hand has five fingers and a foot has five toes—but they're different in shape—fingers are much longer than toes and the thumb sticks off the hand at a different angle than the big toe sticks off the foot.

[22] Image take from Wikipedia: `https://en.wikipedia.org/wiki/File:MagrittePipe.jpg`
[23] We might say hands and feet are *topologically* equivalent.

How about a harder question: how can you tell the difference between a left hand and a right hand? Any length or angle measurement you make on an (idealized) left hand or right hand will be identical. But, we know they're different because they differ in *orientation*.[24]

We'll build up to the definition of orientation in stages. Consider the ordered bases $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$ shown below.

XXX Figure

The $\mathcal{A}$ basis can be rotated to get the $\mathcal{E}$ basis, but it is impossible to rotate the $\mathcal{B}$ basis to get the $\mathcal{E}$ basis. In this case, we say that $\mathcal{E}$ and $\mathcal{A}$ have the same orientation and $\mathcal{E}$ and $\mathcal{B}$ have opposite orientations. In this way, even though the lengths and angles between all vectors in the $\mathcal{A}$ basis and the $\mathcal{B}$ basis are the same, we can distinguish the $\mathcal{A}$ and $\mathcal{B}$ bases because they have different *orientations*.

Orientations of bases come in exactly two flavors: *right-handed* (or *positively oriented*) and *left-handed* (or *negatively oriented*). By convention, the standard basis is called right-handed.

Orthonormal bases—bases consisting of unit vectors that are orthogonal to each other—are called right-handed if they can be rotated to align with the standard basis, otherwise they are called left-handed. In this way, the right-hand–left-hand analogy should be clear: two right hands or two left hands can be rotated to align with each other, but a left hand and a right can never be rotated to alignment.

However, not all bases are orthonormal! Consider the bases $\mathcal{E}$, $\mathcal{A}'$, $\mathcal{B}'$.

XXX Figure where A' and B' differ only slightly from A and B

The bases $\mathcal{A}'$ and $\mathcal{B}'$ differ only slightly from $\mathcal{A}$ and $\mathcal{B}$. Neither can be *rotated* to obtain $\mathcal{E}$, however we'd still like to say $\mathcal{A}'$ is right-handed and $\mathcal{B}'$ is left-handed. The following, fully general definition, allows us to do so.

> **Orientation of a Basis.** The ordered basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i \mapsto \vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

The term *continuously transformed* can be given a precise definition,[25] but it will be enough for us to imagine that there is a continuous transform of one basis to another if there is a "movie" where one basis smoothly and without jumps transforms into another basis.

Let's consider some examples. Let $\mathcal{X} = \{\vec{x}_1, \vec{x}_2\}$. We could imagine $\vec{x}_1, \vec{x}_2$ continuously transforming to $\vec{e}_1, \vec{e}_2$ by $\vec{x}_1$ staying in place and $\vec{x}_2$ smoothly moving along the dotted line.

XXX Figure

Because at every step along this motion, the set of $\vec{x}_1$ and the transformed $\vec{x}_2$ stayed linearly independent, $\mathcal{X}$ is *positively* oriented.

Let $\mathcal{Y} = \{\vec{y}_1, \vec{y}_2\}$. We are in a similar situation, except this time, somewhere along $\vec{y}_2$'s path, the set of $\vec{y}_1$ and the transformed $\vec{y}_2$ becomes linearly dependent.

XXX Figure

Maybe that was just bad luck and we might be able to transform along a different path and stay linearly independent? It turns out, we are doomed to fail, because $\mathcal{Y}$ is *negatively* oriented.

Using the definition of the orientation of a basis to answer questions is difficult because to determine that a basis is negatively oriented, you need to make a determination about *every possible* way to continuously transform a basis to the standard basis. This is hard enough in $\mathbb{R}^2$ and gets much harder in $\mathbb{R}^3$. Fortunately, we will encounter computational tools that will allow us to numerically determine the orientation of a basis, but, for now, the idea is what's important.

Reversing Orientation

Reflections reverse orientation and can manifest in two ways.[26] Consider the reflection of $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ across the line $y = x$.

XXX Figure

[24] Other words for orientation include *chirality* and *handedness*.

[25] Because you crave precision, here it is: the basis $\vec{a}_1, \ldots, \vec{a}_n$ can be *continuously transformed* to the basis $\vec{b}_1, \ldots, \vec{b}_n$ if there exists a continuous function $\Phi : [0,1] \to \{n\text{-tuples of vectors}\}$ so that $\Phi(0) = (\vec{a}_1, \ldots, \vec{a}_n)$ and $\Phi(1) = (\vec{b}_1, \ldots, \vec{b}_n)$. Here, continuity is defined in the multi-variable calculus sense.

[26] Think back to hands. The left and right hands *are* reflections of each other.

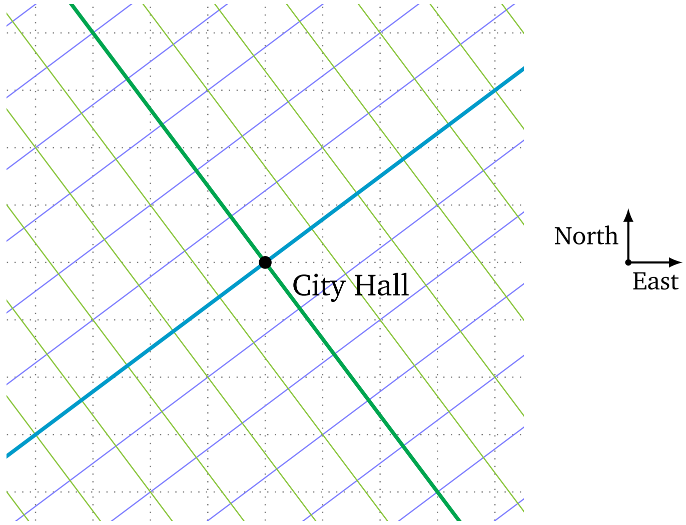
This reflection sends $\{\vec{e}_1, \vec{e}_2\} \mapsto \{\vec{e}_2, \vec{e}_1\}$. Alternatively, reflection across the line $x = 0$ sends $\{\vec{e}_1, \vec{e}_2\} \mapsto \{-\vec{e}_1, \vec{e}_2\}$.

XXX Figure

Both $\{\vec{e}_2, \vec{e}_1\}$ and $\{-\vec{e}_1, \vec{e}_2\}$, as ordered bases, are negatively oriented. This is indicative of a general theorem.

Theorem. Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be an ordered basis. The ordered basis obtained from $\mathcal{B}$ by replacing $\vec{b}_i$ with $-\vec{b}_i$ and the ordered basis obtained from $\mathcal{B}$ by swapping the order of $\vec{b}_i$ and $\vec{b}_{i+1}$ have the opposite orientation as $\mathcal{B}$.

37 The fictional town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \frac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

37.1 The points $A = (2, 1)$ and $B = (3, -1)$ are given in street coordinates. Find their east-north coordinates.

37.2 The points $X = (4, 3)$ and $Y = (1, 7)$ are given in east-north coordinates. Find their street coordinates.

37.3 Define $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$. Does $\operatorname{span}\{\vec{e}_1, \vec{e}_2\} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$?

37.4 Notice that $Y = 5\vec{d}_1 + 5\vec{d}_2 = \vec{e}_1 + 7\vec{e}_2$. Is the point Y better represented by the pair $(5, 5)$ or by the pair $(1, 7)$? Explain.

Representation in a Basis

DEFINITION

Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notated $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

where $\alpha_1, \dots, \alpha_n$ uniquely satisfy $\vec{v} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$.

Conversely,

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{B}} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \dots, \vec{b}_n$ with coefficients $\alpha_1, \dots, \alpha_n$.

38 Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

38.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$.

38.2 Express $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$.

38.3 Let $\vec{v} = 2\vec{e}_1 + 2\vec{e}_2$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

38.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}} = [\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

38.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}} = [\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

38.6 What is YX?

Orientation of a Basis

DEF

The ordered basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i \mapsto \vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

---

39 Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\vec{e}_1$ measured counter-clockwise starting at $\vec{e}_1$.

39.1 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ a linearly independent set?

39.2 For which θ can $\{\vec{e}_1, \vec{u}_\theta\}$ be continuously transformed into $\{\vec{e}_1, \vec{e}_2\}$ and remain linearly independent the whole time?

39.3 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ right-handed? Left-handed?

39.4 For which θ is $\{\vec{u}_\theta, \vec{e}_1\}$ (in that order) right-handed? Left-handed?

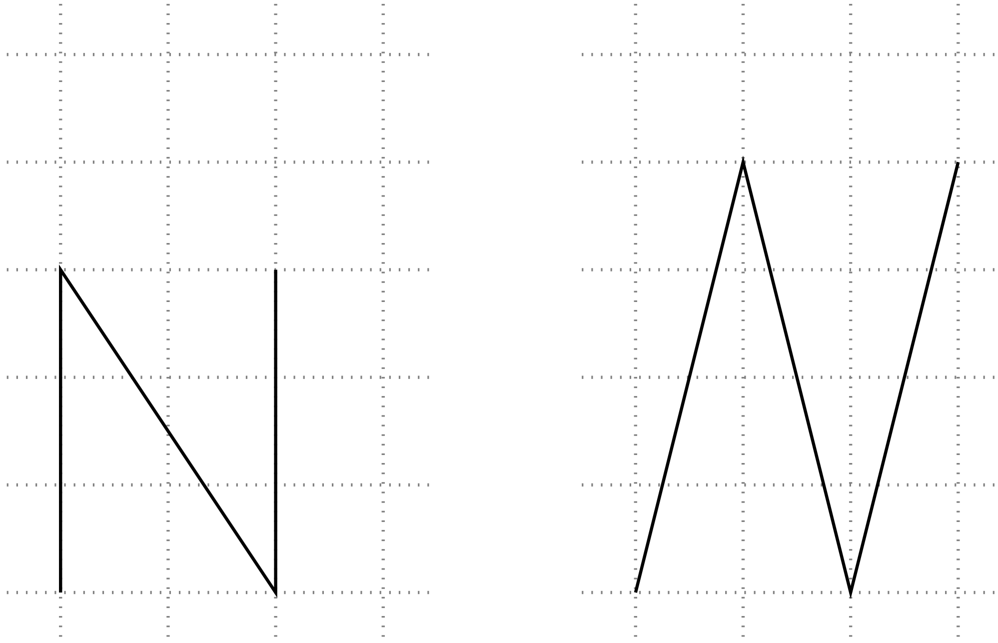
39.5 Is $\{2\vec{e}_1, 3\vec{e}_2\}$ right-handed or left-handed? What about $\{2\vec{e}_1, -3\vec{e}_2\}$?

Linear Transformations

In this module you will learn

- The definition of a linear transformation.
- How to prove whether a transformation is linear or not.
- How to find a matrix for a linear transformation.
- The difference between a matrix and a linear transformation.

Task 2.1: Italicizing N

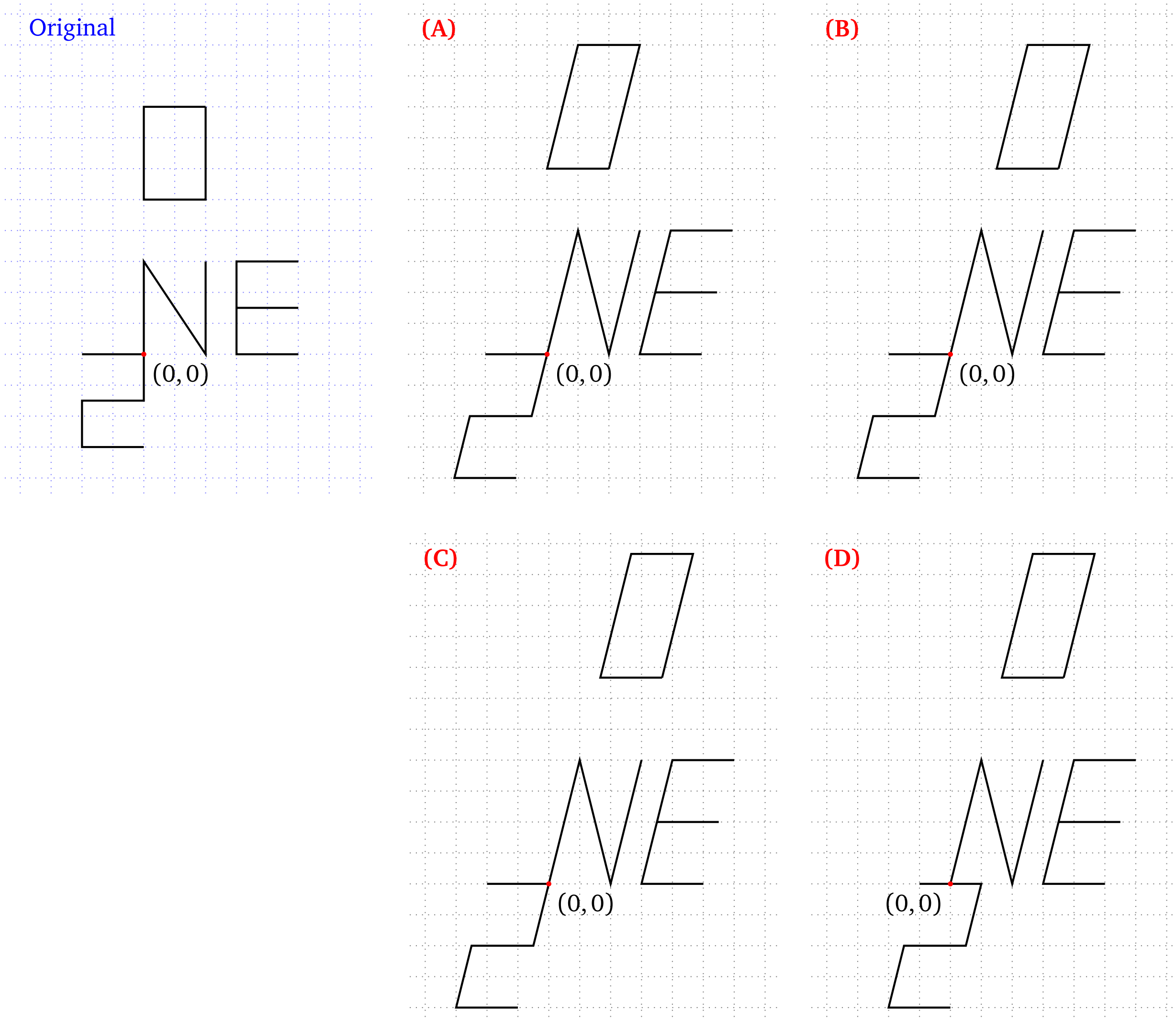


Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

Task 2.2: Beyond the N

A few students were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If other letters are placed around the "N," the students argued over four different possible results for the transformed letters. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?



40 $\mathcal{R}:\mathbb{R}^2\to\mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

40.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

40.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

40.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

40.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter-clockwise by 90°.

Linear Transformation

DEFINITION

Let V and W be subspaces. A function $T:V\to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v})=T\vec{u}+T\vec{v}\qquad\text{and}\qquad T(\alpha\vec{v})=\alpha T\vec{v}$$

for all vectors $\vec{u},\vec{v}\in V$ and all scalars α.

41 41.1 Classify the following as linear transformations or not.

(a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

(b) $W:\mathbb{R}^2\to\mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x^2\\y\end{bmatrix}$.

(c) $T:\mathbb{R}^2\to\mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2\\y\end{bmatrix}$.

(d) $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix}=\mathrm{vcomp}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

Image of a Set

DEF

Let $L:V\to W$ be a transformation and let $X\subseteq V$ be a set. The ***image of the set** X **under** L*, denoted $L(X)$, is the set

$$L(X)=\{\vec{x}\in W\ :\ \vec{x}=L(\vec{y})\text{ for some }\vec{y}\in X\}.$$

42 Let $S=\left\{\begin{bmatrix}x\\y\end{bmatrix}\ :\ 0\leq x,y\leq 1\right\}\subseteq\mathbb{R}^2$ be the filled-in unit square and let $C=\{\vec{0},\vec{e}_1,\vec{e}_2,\vec{e}_1+\vec{e}_2\}\subseteq\mathbb{R}^2$ be the corners of the unit square.

42.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

42.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).

42.3 Let $\ell=\{\text{all convex combinations of }\vec{a}\text{ and }\vec{b}\}$ be a line segment with endpoints $\vec{a}$ and $\vec{b}$ and let A be a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

42.4 Explain how images of sets relate to the *Italicizing N* task.

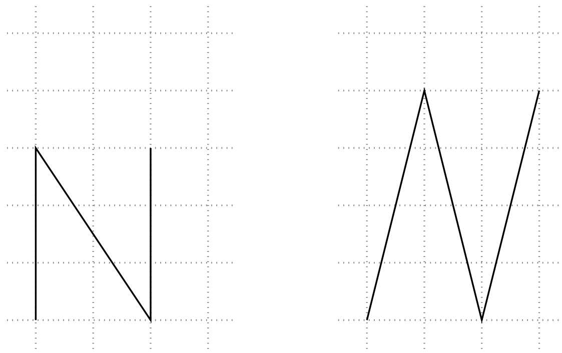
The Composition of Linear Transformations

Module 10

In this module you will learn

- How to break a complicated transformation into the composition of simpler ones.
- How the composition of linear transformations relates to matrix multiplication.

Task 2.3: Pat and Jamie



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

1. Do you think Pat and Jamie's approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?
2. Try Pat and Jamie's approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

43 Define $\mathcal{P}$ to be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

43.1 Find a matrix P so that $P\vec{x} = \mathcal{P}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

43.2 Find a matrix R so that $R\vec{x} = \mathcal{R}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

43.3 Write down matrices A and B for $\mathcal{P} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{P}$.

43.4 How do the matrices A and B relate to the matrices P and R?

Range & Nullspace of a Linear Transformation

In this module you will learn

- The definition of the range and null space of a linear transformation.
- The fundamental subspaces corresponding to a matrix (row space, column space, null space) and how they relate to the range and null space of a linear transformation.
- How to find a basis for the fundamental subspaces of a matrix.
- The definition of rank and the rank-nullity theorem.

DEF

Range

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\mathrm{range}(T) = \{\vec{y} \in W \;:\; \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

DEF

Null Space

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$\mathrm{null}(T) = \{\vec{x} \in V \;:\; T\vec{x} = \vec{0}\}.$$

---

44 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto $\mathrm{span}\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

44.1 What is the range of $\mathcal{P}$?

44.2 What is the null space of $\mathcal{P}$?

---

45 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

45.1 Show that the null space of T is a subspace.

45.2 Show that the range of T is a subspace.

DEFINITION

Induced Transformation

Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $T_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[T_M\vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

---

46 Let M be a 2×2 matrix and let $\vec{v} \in \mathbb{R}^2$. Further, let T_M be the transformation induced by M.

46.1 What is the difference between "$M\vec{v}$" and "$M[\vec{v}]_{\mathcal{E}}$"?

46.2 What is $[T_M\vec{e}_1]_{\mathcal{E}}$?

46.3 Can you relate the columns of M to the range of T_M?

DEF

Fundamental Subspaces

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\mathrm{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\mathrm{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\mathrm{null}(M)$, is the set of solutions to $M\vec{x} = \vec{0}$.

---

47 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

47.1 Describe the row space of A.

47.2 Describe the column space of A.

47.3 Is the row space of A the same as the column space of A?

47.4 Describe the set of all vectors perpendicular to the rows of A.

47.5 Describe the null space of A.

47.6 Describe the range and null space of T_A, the transformation induced by A.

48

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

48.1 How does the row space of B relate to the row space of C?

48.2 How does the null space of B relate to the null space of C?

48.3 Compute the null space of B.

49

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \mathrm{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

49.1 How does the column space of P relate to the column space of Q?

49.2 Describe the column space of P and the column space of Q.

Rank

DEF

For a linear transformation $T : V \to W$, the ***rank*** of T, denoted $\mathrm{rank}(T)$, is the dimension of the range of T.

For an $n \times m$ matrix M, the ***rank*** of M, denoted $\mathrm{rank}(M)$, is the number of pivots in $\mathrm{rref}(M)$.

50 Let $\mathcal{P}$ be projection onto $\mathrm{span}\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

50.1 Describe $\mathrm{range}(\mathcal{P})$ and $\mathrm{range}(\mathcal{R})$.

50.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

50.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

50.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

51

51.1 Determine the rank of (a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

52 Consider the homogeneous system

$$\begin{array}{rrrl} x & +2y & +z & = 0 \\ x & +2y & +3z & = 0 \\ -x & -2y & +z & = 0 \end{array} \tag{10}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$.

52.1 What is $\mathrm{rank}(A)$?

52.2 Give the general solution to system (10).

52.3 Are the column vectors of A linearly independent?

52.4 Give a non-homogeneous system with the same coefficients as (10) that has

(a) infinitely many solutions

(b) no solutions.

53

53.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

53.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = \#\text{ of columns in } A.$$

54

54.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

55 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

55.1 What is the rank and nullity of A^T?

55.2 What is the rank and nullity of A?

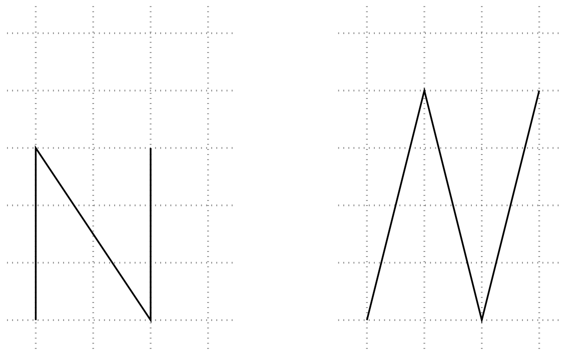
Module 12

Inverse Functions & Inverse Matrices

In this module you will learn

- The definition of an inverse function and an inverse matrix.
- How to decompose a matrix into the product of elementary matrices and how to use elementary matrices to compute inverses.
- How the order of matrix multiplication matters.
- How row-reduction and matrix inverses relate.

Task 2.4: Getting back N



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> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

Consider the new task: find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.
2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

56

56.1 Apply the row operation $R_3 \mapsto R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

56.2 Apply the row operation $R_3 \mapsto R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

ElementaryMatrix

DEF An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

56.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

56.4 Without computing, what should the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to E_1 be? Compute and verify.

56.5 Without computing, what should E_2E_1 be? What about E_1E_2? Now compute and verify.

Matrix Inverse

DEF The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated by A^{-1}.

57 Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

57.1 Which pairs of matrices above are inverses of each other?

58

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

58.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

58.2 What is E_2E_1B?

58.3 Find B^{-1}.

58.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

59

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

59.1 What is $A^{-1}A$?

59.2 What is $\text{rref}(A)$?

59.3 What is $\text{rref}(C)$? (Hint, there is no need to actually do row reduction!)

59.4 Solve the system $A\vec{x} = \vec{b}$.

60

60.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

60.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

Change of Basis II

In this module you will learn

- How to create *change-of-basis* matrices.
- How to write a linear transformation in multiple bases.

Module 13

61 Let $\mathcal{B}=\{\vec{b}_1,\vec{b}_2\}$ where $\vec{b}_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2=\begin{bmatrix}1\\-1\end{bmatrix}$ and let $X=[\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

61.1 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

61.2 Compute $X[\vec{e}_1]_{\mathcal{B}}$ and $X[\vec{e}_2]_{\mathcal{B}}$. What do you notice?

61.3 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

62 Let $\mathcal{E}=\{\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B}=\{\vec{b}_1,\vec{b}_2,\ldots,\vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X=[\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}}=[\vec{v}]_{\mathcal{E}}.$$

62.1 Should X^{-1} exist? Explain.

62.2 Consider the equation

$$X^{-1}[\vec{v}]_?=[\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

62.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

63 Let $\vec{c}_1=\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$, $\vec{c}_2=\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$, $\mathcal{C}=\{\vec{c}_1,\vec{c}_2\}$, and $A=\begin{bmatrix}2&5\\1&3\end{bmatrix}$. Note that $A^{-1}=\begin{bmatrix}3&-5\\-1&2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

63.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

63.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and $T\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$.

63.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

63.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

63.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

63.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

DEFINITION

Similar Matrices

A matrices A and B are called ***similar matrices***, denoted $A\sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A\sim B$ if there is an invertible matrix X so that

$$A=XBX^{-1}.$$

Determinants

In this module you will learn

- The definition of the determinant of a linear transformation and of a matrix.
- How to interpret the determinant as a change-of-volume factor.
- How to relate the determinant of $S \circ T$ to the determinant of S and of T.
- How to compute the determinants of elementary matrices and how to compute determinants of large matrices using row reduction.

Module 14

DEFINITION

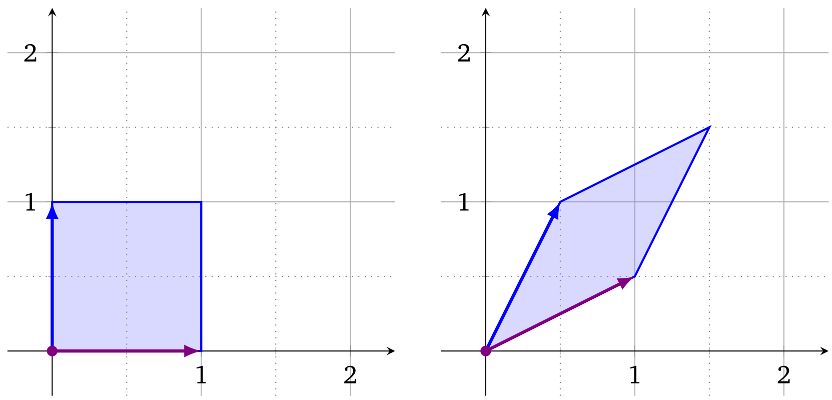
Unit n-cube

The ***unit n-cube*** is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The sides of the unit n-cube are always length 1 and its volume is always 1.

64 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



64.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

64.2 Write down a matrix for T.

64.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may use trigonometry.

DEF

Determinant

The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the determinant of its induced transformation.

65 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

65.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

65.2 Compute the area of $A(C_2)$.

65.3 Compute $\det(A)$.

66 Suppose R is a rotation counter-clockwise by 30°.

66.1 Draw C_2 and $R(C_2)$.

66.2 Compute the area of $R(C_2)$.

66.3 Compute $\det(R)$.

67 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad F\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}.$$

67.1 What is $\det(F)$?

Volume Theorem I

THM For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors of M.

Volume Theorem II

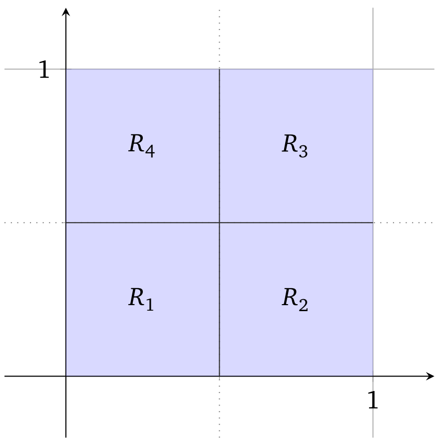
THM For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the row vectors of M.

68

68.1 Explain Volume Theorem I using the definition of determinant.

68.2 Based on Volume Theorems I and II, how should $\det(M)$ and $\det(M^T)$ relate for a square matrix M?

69



Let $R = R_1 \cup R_2 \cup R_3 \cup R_4$. You know the following about the linear transformations M, T, and S.

$$M\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\y\end{bmatrix}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ has determinant } 2$$

$$S : \mathbb{R}^2 \to \mathbb{R}^2 \text{ has determinant } 3$$

69.1 Find the volumes (areas) of R_1, R_2, R_3, R_4, and R.

69.2 Compute the oriented volume of $M(R_1)$, $M(R_2)$, and $M(R)$.

69.3 Do you have enough information to compute the oriented volume of $T(R_2)$? What about the oriented volume of $T(R + \{\vec{e}_2\})$?

69.4 What is the oriented volume of $S \circ T(R)$? What is $\det(S \circ T)$?

70

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

70.1 What is $\det(E_f)$?

70.2 What is $\det(E_m)$?

70.3 What is $\det(E_a)$?

70.4 What is $\det(E_f E_m)$?

70.5 What is $\det(4I_{3\times 3})$?

70.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

71

$$U = \begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 3 & -2 & 4\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 4\end{bmatrix}$$

71.1 What is $\det(U)$?

71.2 V is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

72 72.1 V is a square matrix whose columns are linearly dependent. What is $\det(V)$?

72.2 P is projection onto $\operatorname{span}\left\{\begin{bmatrix}-1\\-1\end{bmatrix}\right\}$. What is $\det(P)$?

73 Suppose you know $\det(X) = 4$.

73.1 What is $\det(X^{-1})$?

73.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

73.3 Suppose Y is not invertible. What is $\det(Y)$?

Eigenvalues and Eigenvectors

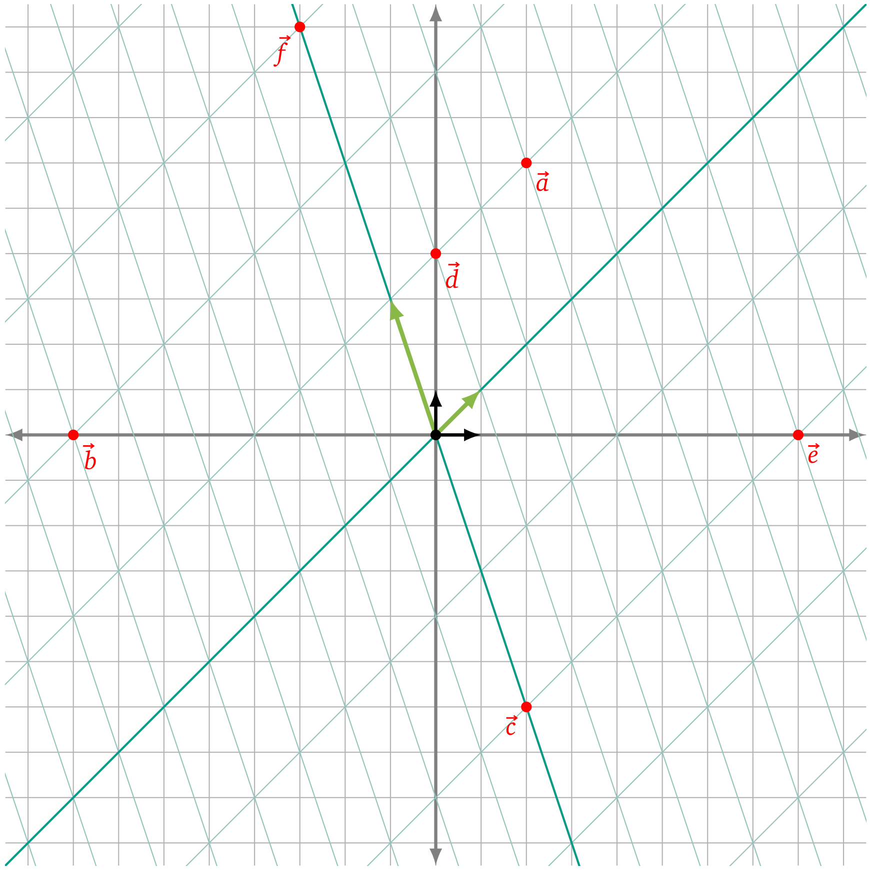
In this module you will learn

- The definition of eigenvalues and eigenvectors.
- That eigenvectors give a particularly nice basis in which to study a linear transformation.
- How the characteristic polynomial relates to eigenvalues.

Module 15

Task 3.1: The Green and the Black

Consider the following two bases for $\mathbb{R}^2$: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$ and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.



1. Write each point above in both the green and the black bases.
2. Find a change-of-basis matrix X that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

4. When working with the transformation T, which basis do you prefer vectors be represented in?

Eigenvectors

Eigenvector

DEFINITION

Let X be a linear transformation. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

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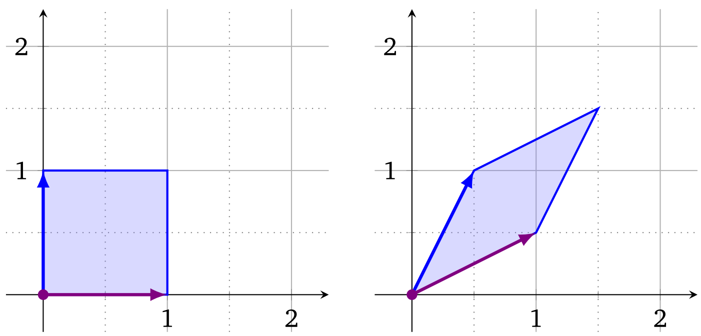
74 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



74.1 Give an eigenvector for T. What is the eigenvalue?

74.2 Can you give another?

---

75 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

75.1 Give an eigenvector and a corresponding eigenvalue for A.

75.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

75.3 What is the dimension of null(B)?

75.4 What is det(B)?

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76 Let $C = \begin{bmatrix}-1 & 2\\ 1 & 0\end{bmatrix}$ and $E_\lambda = C - \lambda I$.

76.1 For what values of λ does E_λ have a non-trivial null space?

76.2 What are the eigenvalues of C?

76.3 Find the eigenvectors of C.

Characteristic Polynomial

DEF

For a matrix A, the ***characteristic polynomial*** of A is

$$\text{char}(A) = \det(A - \lambda I).$$

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77 Let $D = \begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$.

77.1 Compute char(D).

77.2 Find the eigenvalues of D.

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78 Suppose $\text{char}(E) = -\lambda(2-\lambda)(-3-\lambda)$ for some unknown 3×3 matrix E.

78.1 What are the eigenvalues of E?

78.2 Is E invertible?

78.3 What can you say about nullity(E), nullity$(E - 3I)$, nullity$(E + 3I)$?

Diagonalization

Module 16

In this module you will learn

- How to diagonalize a matrix.
- When a matrix can and cannot be diagonalized.

79 Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for A. Let T_A be the transformation induced by A.

79.1 Find the eigenvalues of T_A.

79.2 Find the characteristic polynomial of T_A.

79.3 Compute $T_A\vec{w}$ where $w = 2\vec{v}_1 - \vec{v}_2$.

79.4 Compute $T_A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

79.5 If $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is $\vec{x}$ written in the $\mathcal{V}$ basis, compute $T_A\vec{x}$ in the $\mathcal{V}$ basis.

80 The matrix P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation. Here, A, T_A, and $\mathcal{V}$ come from Problem 79.

80.1 Describe in words what P does.

80.2 Describe how you can use P and P^{-1} to compute $T_A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

80.3 Can you find a matrix D so that

$$PDP^{-1} = A?$$

80.4 $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $T_A^{100}\vec{x}$. Express your answer in both the $\mathcal{V}$ bais and the standard basis.

Diagonalizable

DEF A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

81 Let B be an $n \times n$ matrix and let T_B be the induced transformation. Suppose T_B has eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ which form a basis for $\mathbb{R}^n$, and let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues.

81.1 How do the eigenvalues and eigenvectors of B and T_B relate?

81.2 Is B diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

81.3 What if one of the eigenvalues of T_B is zero? Would B be diagonalizable?

81.4 What if the eigenvectors of T_B did not form a basis for $\mathbb{R}^n$. Would B be diagonalizable?

Eigenspace

DEFINITION

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

82 Let $F = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

82.1 Is F diagonalizable? Why or why not?

82.2 Is G diagonalizable? Why or why not?

82.3 What are the geometric and algebraic multiplicities of each eigenvalue of F? What about the multiplicities for each eigenvalue of G?

82.4 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?