


# Inquiry Based Linear Algebra

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## About the Document

This document was originally designed in the spring of 2015 to guide students through an eleven week Linear Algebra course (Math 211, Linear Algebra for Scientists) at the University of Victoria. The order of topics closely follows that in *Linear Algebra for Science and Engineering* second edition by Daniel Norman and Dan Wolczuk.

A typical class day using the problem-sets:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.
2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).  
  
If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.
4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

These problem-sets strike a balance between concepts and computation, leaning towards the conceptual side. Two algorithms, *row reduction* and *determinant of a matrix by cofactors*, are not introduced in these problem-sets. Instead, students are expected to learn them on their own and be prepared to apply them on problems in the problem-set (these topics were left up to the students because of time constraints). Further, not every Linear Algebra definition is given in these problem-sets, since these notes are not intended to replace a Linear Algebra textbook, but most definitions are given to expedite the transition to new topics in the middle of class time.

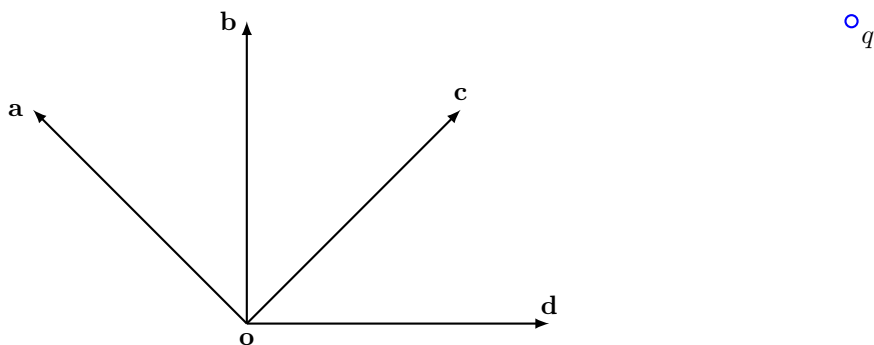
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# Vectors

1

$p$



Notice that all arrows in this diagram are the same length. We will call this length a *unit*.

- 1.1 Give directions from  $o$  to  $p$  of the form “Walk \_\_\_\_units in the direction of arrow \_\_\_\_, then walk \_\_\_\_units in the direction of arrow \_\_\_\_.”
- 1.2 Can you give directions with the two arrows you haven’t used? Give such directions, or explain why it cannot be done.
- 1.3 Give directions from  $o$  to  $q$ .
- 1.4 Can you give directions from  $o$  to  $q$  using  $c$  and  $a$ ? Give such directions, or explain why it cannot be done.

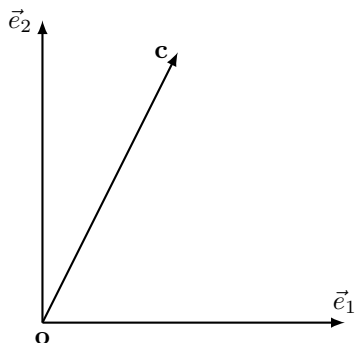
2

$p$

We are going to start using a more mathematical notation for giving directions. Our directions will now look like

$$p = \text{__} \vec{e}_1 + \text{__} \vec{e}_2$$

which is read as “To get to  $p$  ( $=$ ) go \_\_\_\_units in the direction  $\vec{e}_1$  then (+) go \_\_\_\_units in the direction  $\vec{e}_2$ .”



- 2.1 What is the difference between  $p = \text{__} \vec{e}_1 + \text{__} \vec{e}_2$  and  $p = \text{__} \vec{e}_2 + \text{__} \vec{e}_1$ ? Can they both give valid directions?
- 2.2 (a) Give directions to  $p$  using the new notation.  
(b) Give directions to  $p$  using  $c$ .  
(c) What is the distance from  $o$  to  $p$  in units?
- 2.3 (a)  $r = 1c$ . Give directions from  $o$  to  $r$  using  $\vec{e}_1$  and  $\vec{e}_2$ .  
(b) What is the distance from  $o$  to  $r$ ?
- 2.4 (a)  $q = -2\vec{e}_1 + 3\vec{e}_2$ ; find the exact distance from  $o$  to  $q$ .  
(b)  $s = 2\vec{e}_1 + c$ ; find the exact distance from  $o$  to  $s$ .

The vectors  $\vec{e}_1$  and  $\vec{e}_2$  are called the *standard basis vectors* for  $\mathbb{R}^2$  (the plane).

# Column Vector Notation

We previously wrote  $q = -2\vec{e}_1 + 3\vec{e}_2$ . In column vector notation we write

$$q = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We may call  $q$  either a *vector* or a *point*. If we call  $q$  a vector, we are emphasizing that  $q$  gives direction of some sort. If we call  $q$  a point, we emphasize that  $q$  is some absolute location in space. (What's the philosophical difference between a location in space and directions from the origin to said location?)

3

$r = 1\mathbf{c}$  and  $s = 2\vec{e}_1 + \mathbf{c}$  where  $\mathbf{c}$  is the vector from before.

3.1 Write  $r$  and  $s$  in column vector form.

## Sets and Set Notation

Def

A **set** is a (possibly infinite) collection of items and is notated with curly braces (for example,  $\{1, 2, 3\}$  is the set containing the numbers 1, 2, and 3). We call the items in a set **elements**.

If  $X$  is a set and  $a$  is an element  $X$ , we may write  $a \in X$ , which is read “ $a$  is an element of  $X$ .”

If  $X$  is a set, a **subset**  $Y$  of  $X$  (written  $Y \subseteq X$ ) is a set such that every element of  $Y$  is an element of  $X$ .

We can define a subset using **set-builder notation**. That is, if  $X$  is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read “ $Y$  is the set of  $a$  in  $X$  **such that** some rule involving  $a$  is true.” If  $X$  is intuitive, we may omit it and simply write  $Y = \{a : \text{some rule involving } a\}$ . You may equivalently use “ $|$ ” instead of “ $:$ ”, writing  $Y = \{a \mid \text{some rule involving } a\}$ .

Def

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

4

4.1 Which of the following are true?

- (a)  $3 \in \{1, 2, 3\}$ .
- (b)  $4 \in \{1, 2, 3\}$ .
- (c) “b”  $\in \{x : x \text{ is an English letter}\}$ .
- (d) “ö”  $\in \{x : x \text{ is an English letter}\}$ .
- (e)  $\{1, 2\} \subseteq \{1, 2, 3\}$ .
- (f) For some  $a \in \{1, 2, 3\}$ ,  $a \geq 3$ .
- (g) For any  $a \in \{1, 2, 3\}$ ,  $a \geq 3$ .
- (h)  $1 \subseteq \{1, 2, 3\}$ .
- (i)  $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$ .
- (j)  $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$ .

---

5 Write the following in set-builder notation

5.1 The subset  $A \subset \mathbb{R}$  of real numbers larger than  $\sqrt{2}$ .

5.2 The subset  $B \subset \mathbb{R}^2$  of vectors whose first coordinate is twice the second.

**Def** Two common set operations are *unions* and *intersections*. Let  $X$  and  $Y$  be sets.

(union)  $X \cup Y = \{a : a \in X \text{ or } a \in Y\}$ .

(intersection)  $X \cap Y = \{a : a \in X \text{ and } a \in Y\}$ .

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6 Let  $X = \{1, 2, 3\}$  and  $Y = \{2, 3, 4, 5\}$  and  $Z = \{4, 5, 6\}$ . Compute

6.1  $X \cup Y$

6.2  $X \cap Y$

6.3  $X \cup Y \cup Z$

6.4  $X \cap Y \cap Z$

---

7 Draw the following subsets of  $\mathbb{R}^2$ .

7.1  $V = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .

7.2  $H = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .

7.3  $J = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$ .

7.4  $V \cup H$ .

7.5  $V \cap H$ .

7.6 Does  $V \cup H = \mathbb{R}^2$ ?

## Linear Combinations, Span, and Linear Independence

**Def** A *linear combination* of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  is a vector

$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n$$

where  $\alpha_1, \alpha_2, \dots, \alpha_n$  are scalars.

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8 Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , and  $\vec{w} = 2\vec{v}_1 + \vec{v}_2$ .

8.1 Write the coordinates of  $\vec{w}$ .

8.2 Draw a picture with  $\vec{w}$ ,  $\vec{v}_1$ , and  $\vec{v}_2$ .

8.3 Is  $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$  a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?

8.4 Is  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?

8.5 Is  $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$  a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?

8.6 Can you find a vector in  $\mathbb{R}^2$  that isn't a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ ?

8.7 Can you find a vector in  $\mathbb{R}^2$  that isn't a linear combination of  $\vec{v}_1$ ?

**Def** The *span* of a set of vectors  $V$  is the set of all linear combinations of vectors in  $V$ . That is,  
 $\text{span } V = \{ \vec{v} : \vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \dots, \alpha_n \}.$

9 Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , and  $\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ .

9.1 Draw  $\text{span} \{ \vec{v}_1 \}$ .

9.2 Draw  $\text{span} \{ \vec{v}_2 \}$ .

9.3 Describe  $\text{span} \{ \vec{v}_1, \vec{v}_2 \}$ .

9.4 Describe  $\text{span} \{ \vec{v}_1, \vec{v}_3 \}$ .

9.5 Describe  $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ .

10 Give an example of:

10.1 two vectors in  $\mathbb{R}^3$  that span a plane;

10.2 two vectors in  $\mathbb{R}^3$  that span a line;

10.3 four vectors in  $\mathbb{R}^3$  that span a plane;

10.4 a set of 50 vectors in  $\mathbb{R}^3$  whose span is the line through the origin and the point  $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ .

In some sets, every vector is essential for computing a span. In others, there are “excess” vectors. This leads us to the concept of linear independence.

**Def** We say  $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$  is *linearly dependent* if for at least one  $i$ ,

$$\vec{v}_i \in \text{span} \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n \},$$

and a set is *linearly independent* otherwise.

11 Let  $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , and  $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

11.1 Describe  $\text{span} \{ \vec{u}, \vec{v}, \vec{w} \}$ .

11.2 Is  $\{ \vec{u}, \vec{v}, \vec{w} \}$  linearly independent? Why or why not?

Let  $X = \{ \vec{u}, \vec{v}, \vec{w} \}$ .

11.3 Give a subset  $Y \subseteq X$  so that  $\text{span } Y = \text{span } X$  and  $Y$  is linearly independent.

11.4 Give a subset  $Z \subseteq X$  so that  $\text{span } Z = \text{span } X$  and  $Z$  is linearly independent and  $Z \neq Y$ .

**Def** We say a linear combination  $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$  is **trivial** if  $a_1 = a_2 = \cdots = a_n = 0$ .

12

Recall  $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , and  $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ .

12.1 Consider the linearly dependent set  $\{\vec{u}, \vec{v}, \vec{w}\}$  (where  $\vec{u}, \vec{v}, \vec{w}$  are defined as above). Can you write  $\vec{0}$  as a non-trivial linear combination of vectors in this set?

12.2 Consider the linearly independent set  $\{\vec{u}, \vec{v}\}$ . Can you write  $\vec{0}$  as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

**Def**  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  is **linearly dependent** if there is a non-trivial linear combination of  $\vec{v}_1, \dots, \vec{v}_n$  that equals the zero vector.

13

13.1 Explain how this new definition implies the old one.

13.2 Explain how the old definition implies this new one.

Since have old def  $\implies$  new def, and new def  $\implies$  old def ( $\implies$  should be read aloud as ‘implies’), the two definitions are *equivalent* (which we write as new def  $\iff$  old def).

14

Suppose for some unknown  $\vec{u}, \vec{v}, \vec{w}$ , and  $\vec{a}$ ,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \quad \text{and} \quad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

14.1 Could the set  $\{\vec{u}, \vec{v}, \vec{w}\}$  be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write  $\vec{a}$  using  $\vec{u}, \vec{r}, \vec{s}$ .

14.2 Is  $\{\vec{u}, \vec{r}, \vec{s}\}$  linearly independent?

14.3 Is  $\{\vec{u}, \vec{r}\}$  linearly independent?

14.4 Is  $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$  linearly independent?

## Subspaces and Bases

**Def** A **subspace**  $V \subseteq \mathbb{R}^n$  is a subset such that

(i)  $\vec{u}, \vec{v} \in V$  implies  $\vec{u} + \vec{v} \in V$ .

(ii)  $\vec{u} \in V$  implies  $k\vec{u} \in V$  for all scalars  $k$ .

Subspaces give a mathematically precise definition of a “flat space through the origin.”

15

For each set, draw it and explain whether or not it is a subspace of  $\mathbb{R}^2$ .

15.1  $A = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for some } a \in \mathbb{Z}\}.$

15.2  $B = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}.$

15.3  $C = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\}.$

15.4  $D = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\}.$

15.5  $E = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\}.$

15.6  $F = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\}.$

15.7  $G = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$

15.8  $H = \text{span} \{\vec{u}, \vec{v}\}$  for some unknown vectors  $\vec{u}, \vec{v} \in \mathbb{R}^2$ .

**Def** A **basis** for a subspace  $V$  is a linearly independent set of vectors,  $\mathcal{B}$ , so that  $\text{span } \mathcal{B} = V$ .

16

Let  $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ , and  $V = \text{span} \{\vec{u}, \vec{v}, \vec{w}\}.$

16.1 Describe  $V$ .

16.2 Is  $\{\vec{u}, \vec{v}, \vec{w}\}$  a basis for  $V$ ? Why or why not?

16.3 Give a basis for  $V$ .

16.4 Give another basis for  $V$ .

16.5 Is  $\text{span} \{\vec{u}, \vec{v}\}$  a basis for  $V$ ? Why or why not?

**Def** The **dimension** of a subspace  $V$  is the number of elements in a basis for  $V$ .

16.6 What is the dimension of  $V$ ?

17

Let  $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ ,  $\vec{c} = \begin{bmatrix} 7 \\ 8 \\ 8 \end{bmatrix}$  and let  $P = \text{span} \{\vec{a}, \vec{b}\}$  and  $Q = \text{span} \{\vec{b}, \vec{c}\}.$

17.1 Give a basis for and the dimension of  $P$ .

17.2 Give a basis for and the dimension of  $Q$ .

17.3 Is  $P \cap Q$  a subspace? If so, give a basis for it and its dimension.

17.4 Is  $P \cup Q$  a subspace? If so, give a basis for it and its dimension.

## Dot Product

**Def**

If  $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$  are two vectors in  $n$ -dimensional space, then the **dot product** of  $\vec{a}$  and  $\vec{b}$  is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ .

Let  $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ , and  $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ .

- 18.1 (a) Draw a picture of  $\vec{a}$  and  $\vec{b}$ .  
 (b) Compute  $\vec{a} \cdot \vec{b}$ .  
 (c) Find  $\|\vec{a}\|$  and  $\|\vec{b}\|$  and use your knowledge of the multiple ways to compute the dot product to find  $\theta$ , the angle between  $\vec{a}$  and  $\vec{b}$ . Label  $\theta$  on your picture.
- 18.2 Draw the graph of  $\cos$  and identify which angles make  $\cos$  negative, zero, or positive.
- 18.3 Draw a new picture of  $\vec{a}$  and  $\vec{b}$  and on that picture draw
- (a) a vector  $\vec{c}$  where  $\vec{c} \cdot \vec{a}$  is negative.
  - (b) a vector  $\vec{d}$  where  $\vec{d} \cdot \vec{a} = 0$  and  $\vec{d} \cdot \vec{b} < 0$ .
  - (c) a vector  $\vec{e}$  where  $\vec{e} \cdot \vec{a} = 0$  and  $\vec{e} \cdot \vec{b} > 0$ .
  - (d) Could you find a vector  $\vec{f}$  where  $\vec{f} \cdot \vec{a} = 0$  and  $\vec{f} \cdot \vec{b} = 0$ ? Explain why or why not.
- 18.4 Recall the vector  $\vec{u}$  whose coordinates are given at the beginning of this problem.
- (a) Write down a vector  $\vec{v}$  so that the angle between  $\vec{u}$  and  $\vec{v}$  is  $\pi/2$ . (Hint, how does this relate to the dot product?)
  - (b) Write down another vector  $\vec{w}$  (in a different direction from  $\vec{v}$ ) so that the angle between  $\vec{w}$  and  $\vec{u}$  is  $\pi/2$ .
  - (c) Can you write down other vectors different than both  $\vec{v}$  and  $\vec{w}$  that still form an angle of  $\pi/2$  with  $\vec{u}$ ? How many such vectors are there?

**Def** The **norm** of a vector  $\vec{v} \in \mathbb{R}^n$ , denoted  $\|\vec{v}\|$  is its length and is given by the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

- 19.1 Let  $\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ . Find  $\|\vec{a}\|$  using the Pythagorean theorem and using the formula from the definition of the norm. How do these quantities relate?
- 19.2 Let  $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}$ , and find  $\|\vec{b}\|$ . Did you know how to find 4-d lengths before?
- 19.3 Suppose  $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$  for some  $x, y \in \mathbb{R}$ . Could  $\vec{u} \cdot \vec{u}$  be negative? Compute  $\vec{u} \cdot \vec{u}$  algebraically and use this to justify your answer.

**Def** The **distance** between two vectors  $\vec{u}$  and  $\vec{v}$  is  $\|\vec{u} - \vec{v}\|$ .

**Def** A vector  $\vec{v}$  is called a **unit vector** if  $\|\vec{v}\| = 1$ .

Let  $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  and  $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ .

- 20.1 Find the distance between  $\vec{u}$  and  $\vec{v}$ .
- 20.2 Find a unit vector in the direction of  $\vec{u}$ .
- 20.3 Does there exist a *unit vector*  $\vec{x}$  that is distance 1 from  $\vec{u}$ ?
- 20.4 Suppose  $\vec{y}$  is a unit vector and the distance between  $\vec{y}$  and  $\vec{u}$  is 2. What is the angle between  $\vec{y}$  and  $\vec{u}$ ?



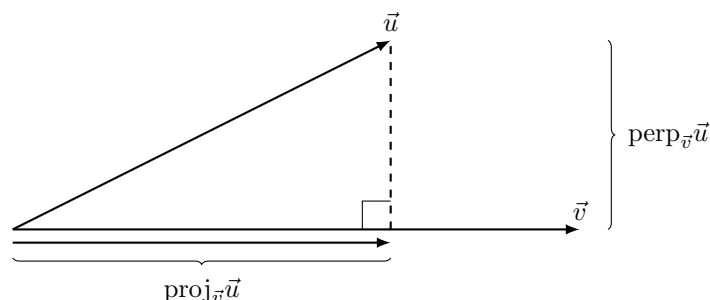
**Def** Two vectors  $\vec{u}$  and  $\vec{v}$  are **orthogonal** to each other if  $\vec{u} \cdot \vec{v} = 0$ . The word orthogonal is synonymous with the word perpendicular.

- 21
- 21.1 Find two vectors orthogonal to  $\vec{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ . Could you find two linearly independent vectors that are orthogonal to  $\vec{a}$ ?
- 21.2 Find two vectors orthogonal to  $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ . Could you find two linearly independent vectors that are orthogonal to  $\vec{b}$ ?
- 21.3 Suppose  $\vec{x}$  and  $\vec{y}$  are orthogonal to each other and  $\|\vec{x}\| = 5$  and  $\|\vec{y}\| = 3$ . What is the distance between  $\vec{x}$  and  $\vec{y}$ ?

## Projections

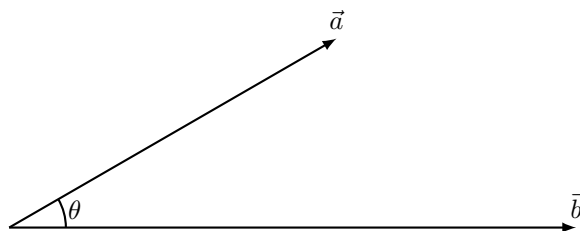
Projections (sometimes called orthogonal projections) are a way to measure how much one vector points in the direction of another.

**Def**



The **projection** of  $\vec{u}$  onto  $\vec{v}$  is written  $\text{proj}_{\vec{v}} \vec{u}$  and is the vector in the direction of  $\vec{v}$  such that  $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$  is orthogonal to  $\vec{v}$ . The vector  $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$  is called the **perpendicular component** of  $\vec{u}$  with respect to  $\vec{v}$  and is notated as  $\text{perp}_{\vec{v}} \vec{u}$ .

22



- 22.1 In this picture  $\|\vec{a}\| = 4$  and  $\theta = \pi/6$ . Find  $\|\text{proj}_{\vec{b}} \vec{a}\|$  and  $\|\text{perp}_{\vec{b}} \vec{a}\|$ .
- 22.2 If  $\vec{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ , write down  $\text{proj}_{\vec{b}} \vec{a}$  and  $\text{perp}_{\vec{b}} \vec{a}$  in column vector form.
- 22.3 If  $\vec{b} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$ , write down  $\text{proj}_{\vec{b}} \vec{a}$  and  $\text{perp}_{\vec{b}} \vec{a}$  in column vector form.
- 22.4 If  $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , write down  $\text{proj}_{\vec{b}} \vec{a}$  and  $\text{perp}_{\vec{b}} \vec{a}$  in column vector form. (You may need to use your knowledge of how dot products and angles relate to answer this one.)
- 22.5 Consider  $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . Compute  $\text{proj}_{\vec{e}_1} \vec{u}$  and  $\text{proj}_{\vec{e}_2} \vec{u}$ . How do these projections relate to the coordinates of  $\vec{u}$ ? What can you say in general about projections onto  $\vec{e}_1$  and  $\vec{e}_2$ ?

# Lines, Planes, Normals, Equations

23

23.1 Draw  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and *all* vectors perpendicular to it.

23.2 If  $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $\vec{x}$  is perpendicular to  $\vec{u}$ , what is  $\vec{x} \cdot \vec{u}$ ?

23.3 Expand the dot product  $\vec{u} \cdot \vec{x}$  to get an equation for a line. This equation is called the *scalar equation* representing the line.

Def

A **normal vector** to a line (or plane or hyperplane) is one that is orthogonal to it.

23.4 Rewrite the line  $\vec{u} \cdot \vec{x} = 0$  in  $y = mx + b$  form and verify it matches the line you drew above.

24

We can also write a line in *parametric form* by introducing a parameter that traces out the line as the parameter runs over all real numbers.

24.1 Draw the line  $L$  with  $x, y$  coordinates given by

$$\begin{aligned} x &= t \\ y &= 2t \end{aligned}$$

as  $t$  ranges over  $\mathbb{R}$ .

24.2 Write the line  $\vec{u} \cdot \vec{x} = 0$  (where  $\vec{u}$  is the same as before) in parametric form.

25

*Vector form* is the same as parametric form but written in vector notation. For example, the line  $L$  from earlier could be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

25.1 Write the line  $\vec{u} \cdot \vec{x} = 0$  in vector form. That is, find a vector  $\vec{v}$  so the line  $\vec{u} \cdot \vec{x} = 0$  can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\vec{v}$$

as  $t$  ranges over  $\mathbb{R}$ .

25.2 What is  $\vec{v} \cdot \vec{u}$ ? Why? Will this always happen?

## Moving to Planes

When solving equations, sometimes we get to make choices. For example, if  $x + 2y = 0$ , we can find solutions by fixing either  $x$  or  $y$  and solving for the other. e.g., if  $x = 2$ , then  $y = -1$  and if  $y = 3$  then  $x = -6$ .

26

26.1 Write down three solutions  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$  to

$$2x + y - z = 0. \tag{1}$$

26.2 Is  $2\vec{a} - \vec{b}$  a solution? Is any linear combination of solutions a solution? Justify why or why not.

26.3 Find  $\vec{n} \in \mathbb{R}^3$  so that equation (1) is equivalent to  $\vec{n} \cdot \vec{x} = 0$  where  $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

26.4 What do you notice about the angle between solutions to equation (1) and  $\vec{n}$ ?

26.5 Prove that any plane of the form  $\vec{n} \cdot \vec{x} = 0$  is a subspace.

When writing down solutions to equation (1), you got to choose two coordinates before the remaining coordinate became determined. This means the solutions have two parameters (and consequently form a two dimensional space).

26.6 Write down parametric form of a line of solutions to equation (1).

26.7 Write down parametric form of a different line of solutions to equation (1).

26.8 Write down all solutions to equation (1) in parametric form. That is, find  $a_x, a_y, a_z, b_x, b_y, b_z$  so that

$$\begin{aligned}x &= a_x t + b_x s \\y &= a_y t + b_y s \\z &= a_z t + b_z s\end{aligned}$$

gives all solutions as  $t, s$  vary over all of  $\mathbb{R}$ .

26.9 Write all solutions to equation (1) in vector form.

## Arbitrary Lines and Planes

So far, all of our lines and planes have passed through the origin. To produce the equation of an arbitrary line/plane, we first make one of same “slope” that passes through the origin, then we translate it to the appropriate place.

---

27

We'd like to write the equation of a line  $L$  with normal vector  $\vec{n} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$  that passes through the point  $p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

27.1 Give a scalar equation of the line  $L_2$  which is parallel to  $L$  but passes through the origin.

27.2 Draw a picture of  $L$  and  $L_2$ , and find two points that lie on  $L$ . Call these points  $p_1$  and  $p_2$ .

27.3 Verify the vector  $\overline{p_1 p_2}$  is perpendicular to  $\vec{n}$ .

27.4 What is  $\vec{n} \cdot p_1$ ,  $\vec{n} \cdot p_2$ ,  $\vec{n} \cdot p$ ? Should these values be zero, equal, or different? Explain (think about projections).

27.5 How does the equation  $\vec{n} \cdot (\vec{x} - p) = 0$  relate to  $L$ ?

---

28

$W$  is the plane with normal vector  $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and passes through the point  $p = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ .

28.1 Write normal form of  $W$ .

28.2 Write vector form of  $W$ .

# Systems of Linear Equations

*Linear equations* are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

---

29

Consider the system

$$\begin{aligned}x - y &= 2 \\ 2x + y &= 1\end{aligned}\tag{2}$$

29.1 Draw the lines in (2) on the same coordinate plane.

29.2 Algebraically solve the system (2). What does this solution represent on your graph?

---

30

Let  $L$  be the line given by  $x - y = 2$ .

30.1 Write an equation of a line that doesn't intersect  $L$ .

30.2 Write an equation of a line that intersects  $L$  in

- (a) one place.
- (b) infinitely many places
- (c) exactly two places

or explain why no such equation exists.

30.3 For each equation you came up with, solve the system algebraically. How can you tell algebraically how many solutions there are?

## The Row Reduction Algorithm

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31

31.1 Solve the system

$$\begin{aligned}x - y - 2z &= -5 \\ 2x + 3y + z &= 5 \\ 0x + 2y + 3z &= 8\end{aligned}\tag{3}$$

any way you like.

31.2 Use an augmented matrix to solve the system (3).

The system (3) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (3) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

31.3 Rewrite (3) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where  $x, y, z$  are interpreted as scalar quantities.

31.4 If  $(x, y, z)$  is a solution to (3), explain how to get from the origin to  $\vec{p}$  using only  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ .

31.5 If  $(x, y, z)$  is a solution to (3), is  $\vec{p} \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ ?

---

32

Consider the augmented matrix

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

32.1 Write the system of equations corresponding to  $A$ .

32.2 Solve the system of equations corresponding to  $A$ .

## Infinite Solutions

33

Consider the system

$$\begin{aligned}x + 2y &= 3 \\ 2x + 4y &= 6\end{aligned}\tag{4}$$

33.1 How many solutions does (4) have?

33.2 Write the solutions to (4) in vector form.

33.3 What happens when you use an augmented matrix to solve (4)?

## Free Variables

34

Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array} \right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make  $2 - 1 = 1$  choices when writing a solution. Let's make the choice  $y = t$ .

34.1 With the added equation  $y = t$ , solve the system represented by  $B$ .

35

Consider the system given by the augmented matrix

$$C = \left[ \begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right].$$

and call the variables in this system  $x_1, x_2, x_3, x_4, x_5$ .

35.1 Write the system of equations represented by  $C$ .

35.2 Identify how many choices you can make when writing down a solution corresponding to  $C$ .

35.3 Add one equation (of the form  $x_i = t$  or  $x_j = s$ , etc.) for each choice you must make when solving the system.

35.4 Write in vector form all solutions to  $C$ .

36

36.1 An unknown system  $U$  is represented by an augmented matrix with 4 rows and 6 columns. What is the minimum number of free variables solutions to  $U$  will have?

36.2 An unknown system  $V$  is represented by an augmented matrix with 6 rows and 4 columns. What is the minimum number of free variables solutions to  $V$  will have?

37

**Def** A system is called *homogeneous* if all equations equal 0.

Let  $A$  be an unknown system of 3 equations and 3 variables and suppose  $(x, y, z) = (1, 2, 1)$  and  $(x, y, z) = (-1, 1, 1)$  are solutions to  $A$ .

37.1 Can you produce another solution to the system?

37.2 Can you produce a solution to the homogeneous version of  $A$  (the version of  $A$  where every equation equals 0)?

37.3 Suppose when you use an augmented matrix to solve the system  $A$ , you only have one free variable. Could  $A$  be homogeneous? Can you produce all solutions to the system  $A$ ?

# Rank

**Def** The *rank* of the matrix  $A$  is the number of leading ones in the reduced row echelon form of  $A$ .

38

38.1 Determine the rank of (a)  $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  (c)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  (d)  $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$  (e)  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

39

Consider the homogeneous system

$$\begin{array}{rrcr} x & +2y & +z & = 0 \\ x & +2y & +3z & = 0 \\ -x & -2y & +z & = 0 \end{array} \quad (5)$$

and the non-augmented matrix of coefficients  $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$ .

39.1 What is  $\text{rank}(A)$ ?

39.2 Give the general solution to (5).

39.3 Are the column vectors of  $A$  linearly independent?

39.4 Give a non-homogeneous system with the same coefficients as (5) that has

- (a) infinitely many solutions
- (b) no solutions.

40

40.1 The rank of a  $3 \times 4$  matrix  $A$  is 3. Are the column vectors of  $A$  linearly independent?

40.2 The rank of a  $4 \times 3$  matrix  $B$  is 3. Are the column vectors of  $B$  linearly independent?

## Span Again

41

Consider the system

$$\begin{array}{rrcr} x & -y & -z & = 0 \\ 0x & +1y & +2z & = 0 \\ 3x & -3y & +3z & = 0 \end{array} \quad (6)$$

which has the unique solution  $(x, y, z) = (0, 0, 0)$ .

41.1 Give vectors  $\vec{u}, \vec{v}, \vec{w}$  so that the system (6) corresponds to the vector equation  $x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}$ .

41.2 Is  $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$ ? If so, write it as a linear combination of  $\vec{u}$  and  $\vec{v}$ .

The matrix  $M$  is the non-augmented matrix corresponding to a homogeneous system of linear equations.  $M$  also corresponds to the vector equation  $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ . Further, we know

$$\text{rref}(M) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

41.3 Give a solution to the vector equation  $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ .

41.4 Is  $\vec{c} \in \text{span}\{\vec{a}, \vec{b}\}$ ? If so, write it as a linear combination of  $\vec{a}$  and  $\vec{b}$ .

41.5 Do you have enough information to tell if  $\{\vec{a}, \vec{b}\}$  is linearly independent? Why or why not?

## Finding Linearly Independent Subsets

42

Suppose when you use an augmented matrix to solve  $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$  you have no free variables.

42.1 Is  $\{\vec{u}, \vec{v}, \vec{w}\}$  linearly independent?

Suppose when you use an augmented matrix to solve  $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ , the second column corresponds to a free variable.

42.2 Is  $\{\vec{u}, \vec{v}, \vec{w}\}$  linearly independent?

42.3 Is  $\{\vec{u}, \vec{w}\}$  linearly independent?

42.4 Is  $\{\vec{u}, \vec{v}\}$  linearly independent?

**Def**

Given a set of vectors  $X$ , a **maximal linearly independent subset** of  $X$  is a linearly independent subset  $V \subseteq X$  with the most possible vectors in it (i.e., if you took any subset of  $X$  with more vectors, it would be linearly dependent).

43

43.1 Give a maximal linearly independent subset,  $T$ , of  $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ .

43.2 What is the size of  $T$ ?

44

Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \vec{v}_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and the matrices

$$A = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 2 & -1 & 1 & 2 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(Notice that the columns of  $A$  are the vectors  $\vec{v}_1, \dots, \vec{v}_5$ )

44.1 Is  $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$  linearly independent?

44.2 Pick a maximal linearly independent subset of  $V$ .

44.3 Pick another (different) maximal linearly independent subset of  $V$ .

44.4 Give a basis for  $\text{span}(V)$ .

44.5 What is the dimension of  $\text{span}(V)$ ?

# Matrices

---

45

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

- 45.1 Write the shape of the matrices  $A, B, C$  (i.e., for each one, write the dimensions in  $m \times n$  form).
- 45.2 List *all* products between the matrices  $A, B, C$  that are defined. (Your list will be some subset of  $AB, AC, BA, CA, BC, CB$ .)
- 45.3 Compute  $AC$  and  $CA$ .

---

46

- 46.1 If the matrices  $X$  and  $Y$  are both square  $n \times n$  matrices, does  $XY = YX$ ? Explain.
- 46.2 If the matrices  $X$  and  $Y$  are both square  $n \times n$  matrices, does  $X + Y = Y + X$ ? Explain.

---

47

Consider the system

$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \tag{7}$$

- 47.1 Find values of  $a, b, c, d, e, f$  so that the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

represents the same system as (7).

Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

- 47.2 If  $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , is the set of solutions to this system a point, line, plane, or other?

- 47.3 If  $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ , is the set of solutions to this system a point, line, plane, or other?

---

48

The entries of a matrix are specified by (row,column) pairs of integers. If  $a_{ij}$  is the  $(i, j)$  entry of a matrix  $A$ , we may write  $A = [a_{ij}]$ .

- 48.1 Write the  $2 \times 2$  matrix  $A$  with entries  $a_{11} = 4$ ,  $a_{12} = 3$ ,  $a_{21} = 7$  and  $a_{22} = 9$ .
- 48.2 Let  $B = [b_{ij}]$  be the  $3 \times 3$  matrix where  $b_{ij} = i + j$ . Write  $B$ .
- 48.3 Let  $C = [c_{ij}]$  be the  $3 \times 4$  matrix where  $c_{ij} = 0$  if  $i = j$  and  $c_{ij} = 1$  if  $i \neq j$ .



**Def** The *transpose* of a matrix  $A = [a_{ij}]$  is the matrix  $A^T = [a_{ji}]$ .

Visually, the transpose of a matrix swaps rows and columns.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

49.1 What is the shape of  $A$  and  $A^T$ ?

49.2 Write down  $A^T$ .

$B$  and  $D$  are  $4 \times 6$  matrices and  $C$  is a  $6 \times 4$  matrix.

49.3 Does  $(BC)^T = B^T C^T$ ? Explain.

49.4 Does  $(B + D)^T = B^T + D^T$ ? Explain.

49.5 Compute  $AA^T$  and  $A^T A$  (where  $A$  is the matrix defined earlier). What do you notice?

**Def** A matrix  $X$  is called *symmetric* if  $X = X^T$ .

Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

50.1 Prove that if  $W$  is a square matrix, then  $V = W^T W + W + W^T$  is a symmetric matrix.

**Def** A *zero matrix* is a matrix whose entries are all zeros. An *identity matrix* is a square matrix whose diagonal entries are 1 and non-diagonal entries are 0.

We write the  $m \times n$  zero matrix as  $0_{m \times n}$  or just 0 if the shape is determined by context. The  $n \times n$  identity matrix is notated  $I_{n \times n}$  or just  $I$  if the shape is determined by context.

Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ .

51.1 Write down the  $3 \times 3$  identity matrix and the  $3 \times 3$  zero matrix.

51.2 Compute  $I_{3 \times 3} A$ ,  $A I_{3 \times 3}$ ,  $0_{3 \times 3} A$ , and  $A 0_{3 \times 3}$ .

51.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

52.1 Solve the matrix equation

$$I_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

# Linear Transformations

53

$\mathcal{R} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the transformation that rotates vectors counter-clockwise by  $90^\circ$ .

53.1 Compute  $\mathcal{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\mathcal{R} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

53.2 Compute  $\mathcal{R} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . How does this relate to  $\mathcal{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\mathcal{R} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ?

53.3 What is  $\mathcal{R} \left( a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ ?

53.4 Write down a matrix  $R$  so that  $R\vec{v}$  is  $\vec{v}$  rotated counter clockwise by  $90^\circ$ .

54

$\mathcal{S} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  stretches in the  $\vec{e}_3$  direction by a factor of 2 and contracts in the  $\vec{e}_2$  direction by a factor of 3.

54.1 Write a matrix representation of  $\mathcal{S}$ .

Def

A **linear transformation** is a function that takes in vectors and outputs vectors and which distributes with respect to vector addition and scalar multiplication. That is  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation if

$$T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v} \quad \text{and} \quad T(a\vec{v}) = aT\vec{v}$$

for all scalars  $a$ .

55

55.1 Classify the following as linear transformation or not

(a)  $\mathcal{R}$  from above.

(b)  $\mathcal{S}$  from above.

(c)  $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $W \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ .

(d)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$ .

(e)  $\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where  $\mathcal{P} \begin{bmatrix} x \\ y \end{bmatrix} = \text{proj}_{\vec{u}} \begin{bmatrix} x \\ y \end{bmatrix}$  and  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

It turns out every linear transformation can be written as a matrix (in fact this is why matrix multiplication was invented).

56

Define  $\mathcal{P}$  to be projection onto  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

56.1 Write down a matrix for  $\mathcal{P}$ .

56.2 What is the rank of the matrix corresponding to  $\mathcal{P}$ ?

Matrix multiplication was designed to exactly model composition of linear transformations.

56.3 Write down a matrix for  $\mathcal{P}$  and for  $\mathcal{R}$ , the counter-clockwise rotation by  $90^\circ$ .

56.4 Write down matrices for  $\mathcal{P} \circ \mathcal{R}$  and  $\mathcal{R} \circ \mathcal{P}$ .

Def

The **range** (or **image**) of a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is the set of vectors that  $T$  can output. That is,

$$\text{range}(T) = \{\vec{y} \in \mathbb{R}^m : \vec{y} = T\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}.$$

Def

The **null space** (or **kernel**) of a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is the set of vectors that get mapped to zero under  $T$ . That is,

$$\text{null}(T) = \{\vec{x} \in \mathbb{R}^n : T\vec{x} = \vec{0}\}.$$

57

Let  $\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be projection onto the vector  $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  (like before).

57.1 What is the range of  $\mathcal{P}$ ?

57.2 What is the null space of  $\mathcal{P}$ ?

58

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be an arbitrary linear transformation.

58.1 Show that the null space of  $T$  is a subspace.

58.2 Show that the range of  $T$  is a subspace.

**Def**

Associated with any matrix  $M$  are three fundamental subspaces: the **row space** of  $M$  is the span of the rows of  $M$ ; the **column space** of  $M$  is the span of the columns of  $M$ ; and the **null space** of  $M$  is the set of solutions to  $M\vec{x} = \vec{0}$ .

The **nullity** of  $M$  is the dimension of the null space of  $M$ .

59

Consider  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

59.1 Describe the row space of  $A$ .

59.2 Describe the column space of  $A$ .

59.3 Is the row space of  $A$  the same as the column space of  $A$ ?

59.4 Describe the set of all vectors perpendicular to the rows of  $A$ .

59.5 Describe the null space of  $A$ .

60

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad C = \text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

60.1 How does the row space of  $B$  relate to the row space of  $C$ ?

60.2 How does the null space of  $B$  relate to the null space of  $C$ ?

60.3 Compute the null space of  $B$ .

61

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \quad Q = \text{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

61.1 How does the column space of  $P$  relate to the column space of  $Q$ ?

61.2 Describe the columns space of  $P$  and the column space of  $Q$ .

**Def**

The **nullity** of a matrix is the dimension of the null space.

The rank-nullity theorem states

$$\text{rank}(A) + \text{nullity}(A) = \# \text{ of columns in } A.$$

62

The vectors  $\vec{u}, \vec{v} \in \mathbb{R}^9$  are linearly independent and  $\vec{w} = 2\vec{u} - \vec{v}$ . Define  $A = [\vec{u} | \vec{v} | \vec{w}]$ .

62.1 What is the rank and nullity of  $A^T$ ?

62.2 What is the rank and nullity of  $A$ ?

## Matrix Inverses

63

63.1 Apply the row operation  $R_3 \rightarrow R_3 + 2R_1$  to the  $3 \times 3$  identity matrix and call the result  $E_1$ .

63.2 Apply the row operation  $R_3 \rightarrow R_3 - 2R_1$  to the  $3 \times 3$  identity matrix and call the result  $E_2$ .

Def

An *elementary matrix* is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

63.3 Compute  $E_1A$  and  $E_2A$ . How do the resulting matrices relate to row operations?

63.4 Without computing, what should the result of applying the row operation  $R_3 \rightarrow R_3 - 2R_1$  to  $E_1$  be? Compute and verify.

63.5 Without computing, what should  $E_1E_2$  be? What about  $E_2E_1$ ? Now compute and verify.

Def

The *inverse* of an  $n \times n$  matrix  $A$  is an  $n \times n$  matrix  $B$  such that  $AB = I_{n \times n} = BA$ . In this case,  $B$  is called the inverse of  $A$  and is notated as  $A^{-1}$ .

64

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

64.1 Which pairs of matrices above are inverses of each other?

65

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

65.1 Use two row operations to reduce  $B$  to  $I_{2 \times 2}$  and write an elementary matrix  $E_1$  corresponding to the first operation and  $E_2$  corresponding to the second.

65.2 What is  $E_2E_1B$ ?

65.3 Find  $B^{-1}$ .

65.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

66

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad C = [A|\vec{b}] \quad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

66.1 What is  $A^{-1}A$ ?

66.2 What is  $\text{rref}(A)$ ?

66.3 What is  $\text{rref}(C)$ ? (Hint, there is no need to actually do row reduction!)

66.4 Solve the system  $A\vec{x} = \vec{b}$ .

67

67.1 For two square matrices  $X, Y$ , should  $(XY)^{-1} = X^{-1}Y^{-1}$ ?

67.2 If  $M$  is a matrix corresponding to a non-invertible linear transformation  $T$ , could  $M$  be invertible?

# Change of Basis

68

Let  $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ,  $\vec{c} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ , and  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ .

68.1 Is  $\mathcal{B}$  a basis for  $\mathbb{R}^2$ ?

68.2 Find coefficients  $\alpha_1$  and  $\alpha_2$  so that  $\vec{c} = \alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2$ .

We call the vector  $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$  the representation of  $\vec{c}$  in the  $\mathcal{B}$  basis and notate this by  $[\vec{c}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ .

68.3 Compute  $[\vec{e}_1]_{\mathcal{B}}$  and  $[\vec{e}_2]_{\mathcal{B}}$ .

Let  $X = [\vec{b}_1 | \vec{b}_2]$  be the matrix whose columns are  $\vec{b}_1$  and  $\vec{b}_2$ .

68.4 Compute  $X[\vec{c}]_{\mathcal{B}}$ . What do you notice?

69

Let  $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  be the standard basis for  $\mathbb{R}^n$ . Given a basis  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$  for  $\mathbb{R}^n$ , the matrix  $X = [\vec{b}_1 | \vec{b}_2 | \dots | \vec{b}_n]$  converts vectors from the  $\mathcal{B}$  basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}.$$

69.1 Should  $X^{-1}$  exist? Explain.

69.2 Consider the equation

$$X^{-1}[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}.$$

Can you fill in the “?” symbols so that the equation makes sense?

69.3 What is  $[\vec{b}_1]_{\mathcal{B}}$ ? How about  $[\vec{b}_2]_{\mathcal{B}}$ ? Can you generalize to  $[\vec{b}_i]_{\mathcal{B}}$ ?

70

Let  $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ ,  $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ , and  $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ . Note that  $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$  and that  $A$  changes vectors from the  $\mathcal{C}$  basis to standard basis and  $A^{-1}$  changes vectors from the standard basis to the  $\mathcal{C}$  basis.

70.1 Compute  $[\vec{c}_1]_{\mathcal{C}}$  and  $[\vec{c}_2]_{\mathcal{C}}$ .

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation that stretches in the  $\vec{c}_1$  direction by a factor of 2 and doesn't stretch in the  $\vec{c}_2$  direction at all.

70.2 Compute  $T \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $T \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ .

70.3 Compute  $[T\vec{c}_1]_{\mathcal{C}}$  and  $[T\vec{c}_2]_{\mathcal{C}}$ .

70.4 Compute the result of  $T \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathcal{C}}$  and express the result in the  $\mathcal{C}$  basis (i.e., as a vector of the form  $\begin{bmatrix} ? \\ ? \end{bmatrix}_{\mathcal{C}}$ ).

70.5 Find a matrix for  $T$  in the  $\mathcal{C}$  basis.

70.6 Find a matrix for  $T$  in the standard basis.

**Def**

A matrix  $A$  and a matrix  $B$  are *similar matrices*, denoted  $A \sim B$ , if  $A$  and  $B$  represent the same linear transformation but in possibly different bases. Equivalently,  $A \sim B$  if there is an invertible matrix  $X$  so that

$$A = XBX^{-1}.$$

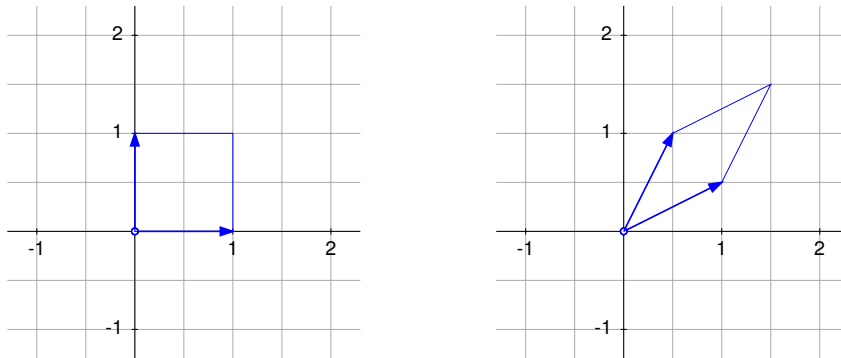
# Determinants

**Def** The unit  $n$ -cube is the  $n$ -dimensional cube with side length 1 and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^n \alpha_i \vec{e}_i \text{ for some } \alpha_1, \dots, \alpha_n \in [0, 1] \right\} = [0, 1]^n.$$

The volume of the unit  $n$ -cube is always 1.

71 The picture shows what the linear transformation  $T$  does to the unit square (i.e., the unit 2-cube).



71.1 What is  $T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ?

71.2 Write down a matrix for  $T$ .

71.3 What is the volume of the image of the unit square (i.e., the volume of  $T(C_2)$ )? You may need to use trigonometry.

**Def** The *determinant* of a linear transformation  $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the oriented volume of the image of the unit  $n$ -cube. The determinant of a square matrix is the oriented volume of the parallelepiped ( $n$ -dimensional parallelogram) given by the column vectors or the row vectors.

72 We know the following about the transformation  $A$ :

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

72.1 Draw  $C_2$  and  $A(C_2)$ , the image of the unit square under  $A$ .

72.2 Compute the area of  $A(C_2)$ .

72.3 Compute  $\det(A)$ .

73 Suppose  $R$  is a rotation counterclockwise by  $30^\circ$ .

73.1 Draw  $C_2$  and  $R(C_2)$ .

73.2 Compute the area of  $R(C_2)$ .

73.3 Compute  $\det(R)$ .

We know the following about the transformation  $F$ :

$$F \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad F \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

74.1 What is  $\det(F)$ ?

- $E_f$  is  $I_{3 \times 3}$  with the first two rows swapped.
- $E_m$  is  $I_{3 \times 3}$  with the third row multiplied by 6.
- $E_a$  is  $I_{3 \times 3}$  with  $R_1 \rightarrow R_1 + 2R_2$  applied.

75.1 What is  $\det(E_f)$ ?

75.2 What is  $\det(E_m)$ ?

75.3 What is  $\det(E_a)$ ?

75.4 What is  $\det(E_f E_m)$ ?

75.5 What is  $\det(4I_{3 \times 3})$ ?

75.6 What is  $\det(W)$  where  $W = E_f E_a E_f E_m E_m$ ?

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

76.1 What is  $\det(U)$ ?

When you row reduce the square matrix  $V$ , there is a row of zeros.

76.2 What is  $\det(V)$ ?

$P$  is projection onto the vector  $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ .

76.3 What is  $\det(P)$ ?

Suppose you know  $\det(X) = 4$ .

77.1 What is  $\det(X^{-1})$ ?

77.2 Derive a relationship between  $\det(Y)$  and  $\det(Y^{-1})$  for an arbitrary matrix  $Y$ .

77.3 Suppose  $Y$  is not invertible. What is  $\det(Y)$ ?

After all this work with determinants, we see that (like dot products) there is a geometric and an algebraic way of thinking about them, and they *determine* if a matrix is invertible.

# Eigenvectors

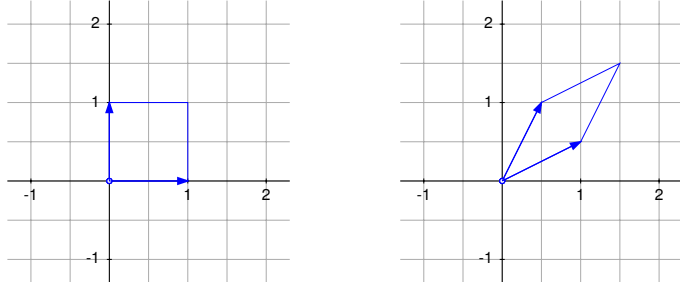
**Def** For a transformation  $X$ , an *eigenvector* for  $X$  is a vector that doesn't change directions when  $X$  is applied. That is,  $\vec{v}$  is an eigenvector for  $X$  if

$$X\vec{v} = \lambda\vec{v}$$

for some  $\lambda \in \mathbb{R}$ . We call  $\lambda$  the *eigenvalue* of  $X$  corresponding to the eigenvector  $\vec{v}$ .

78

The picture shows what the linear transformation  $T$  does to the unit square (i.e., the unit 2-cube).



78.1 Give an eigenvector for  $T$ . What is the eigenvalue?

78.2 Can you give another?

79

For some matrix  $A$ ,

$$A \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2/3 \end{bmatrix} \quad \text{and} \quad B = A - \frac{2}{3}I.$$

79.1 Give an eigenvector and a corresponding eigenvalue for  $A$ .

79.2 What is  $B \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ ?

79.3 What is the dimension of  $\text{null}(B)$ ?

79.4 What is  $\det(B)$ ?

80

Let  $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$  and  $E_\lambda = C - \lambda I$ .

80.1 For what values of  $\lambda$  does  $E_\lambda$  have a non-trivial null space?

80.2 What are the eigenvalues of  $C$ ?

80.3 Find the eigenvectors of  $C$ .

**Def**

For a matrix  $A$ , the *characteristic polynomial* of  $A$  is

$$\text{char}(A) = \det(A - \lambda I).$$

81

Let  $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ .

81.1 Compute  $\text{char}(D)$ .

81.2 Find the eigenvalues of  $D$ .

82

Suppose  $\text{char}(E) = \lambda(\lambda - 2)(\lambda + 3)$  for some unknown  $3 \times 3$  matrix  $E$ .

82.1 What are the eigenvalues of  $E$ ?

82.2 Is  $E$  invertible?

82.3 What is  $\text{nullity}(E)$ ,  $\text{nullity}(E - 3I)$ ,  $\text{nullity}(E + 3I)$ ?



83

Define

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are eigenvectors for  $A$ .

83.1 Find the eigenvalues of  $A$ .

83.2 Find the characteristic polynomial of  $A$ .

83.3 Compute  $A\vec{w}$  where  $w = 2\vec{v}_1 - \vec{v}_2$ .

83.4 Compute  $A\vec{u}$  where  $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$  for unknown scalar coefficients  $a, b, c$ .

Notice that  $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a basis for  $\mathbb{R}^3$ .

83.5 If  $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}_{\mathcal{V}}$  is  $\vec{x}$  written in the  $\mathcal{V}$  basis, compute  $A\vec{x}$  in the  $\mathcal{V}$  basis.

84

The transformation  $P^{-1}$  takes vectors in the standard basis and outputs vectors in their  $\mathcal{V}$ -basis representation (where  $\mathcal{V}$  is from above).

84.1 Describe in words what  $P$  does.

84.2 Describe how you can use  $P$  and  $P^{-1}$  to easily compute  $A\vec{y}$  for any  $\vec{y} \in \mathbb{R}^3$ .

84.3 Can you find a matrix  $D$  so that

$$P^{-1}DP = A?$$

84.4  $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}_{\mathcal{V}}$ . Compute  $A^{100}\vec{x}$ .

85

For an  $n \times n$  matrix  $T$ , suppose its eigenvectors  $\{\vec{v}_1, \dots, \vec{v}_n\}$  form a basis for  $\mathbb{R}^n$ . Let  $\lambda_1, \dots, \lambda_n$  be the corresponding eigenvalues.

85.1 Is  $T$  diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

85.2 What if one of the eigenvalues of  $T$  is zero? Is  $T$  diagonalizable?

85.3 What if the eigenvectors of  $T$  did not form a basis for  $\mathbb{R}^n$ . Would  $T$  be diagonalizable?

**Def**

Let  $A$  be a matrix with eigenvalues  $\{\lambda_1, \dots, \lambda_m\}$ . The **eigenspace** of  $A$  corresponding to the eigenvalue  $\lambda_i$  is the null space of  $A - \lambda_i I$ . That is, it is the space spanned by all eigenvectors that have the eigenvalue  $\lambda_i$ .

The **geometric multiplicity** of an eigenvalue  $\lambda_i$  is the dimension of the eigenspace corresponding to  $\lambda_i$ . The **algebraic multiplicity** of  $\lambda_i$  is the number of times  $\lambda_i$  occurs as a root of the characteristic polynomial of  $A$  (i.e., the number of times  $x - \lambda_i$  occurs as a factor).

86

Define  $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .

86.1 Is  $F$  diagonalizable? Why or why not?

86.2 What is the geometric and algebraic multiplicity of each eigenvalue of  $F$ ?

86.3 Suppose  $A$  is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is  $A$  diagonalizable? What if all the geometric and algebraic multiplicities match?

# Orthogonality

**Def** A set of vectors is *orthogonal* if every pair of vectors in the set is orthogonal.

**Def** A set of vectors is *orthonormal* if the set is orthogonal and every vector is a unit vector.

---

87

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} \quad \vec{b}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

The matrix  $A = [\vec{b}_1 | \vec{b}_2]$  takes vectors in the  $\mathcal{B}$  basis and rewrites them in the standard basis.

87.1 What does  $A^{-1}$  do?

87.2 Find a matrix  $B$  that takes vectors in the standard basis and rewrites them in the  $\mathcal{B}$  basis.

87.3 Write  $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_S$  in the  $\mathcal{B}$  basis.

87.4 What is the relationship between  $A$  and  $B$ ?

**Def** An *orthogonal* matrix is a square matrix whose columns are orthonormal (Yes, a better name would be orthonormal matrix, but that is not the term the rest of the world uses).

---

88

Suppose  $X = [\vec{x}_1 | \vec{x}_2 | \vec{x}_3 | \vec{x}_4]$  is an orthogonal matrix.

88.1 What is the shape of  $X$  (i.e., it is a what  $\times$  what matrix)?

88.2 Compute  $X^T X$ .

88.3 What is  $X^{-1}$ ?

---

89

$$Y = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

89.1 Is  $Y$  an orthogonal matrix?

89.2 Fix  $Y$  so it is an orthogonal matrix. Call the new matrix  $X$ .

89.3 Compute  $X^{-1}$ .

89.4 Compute  $Y^{-1}$ .

89.5 Compute  $|\det(X)|$  and  $|\det(Y)|$  (the absolute value of the determinant of  $X$  and  $Y$ ).

Matrix equations involving orthogonal matrices are easy to solve because the inverse of an orthogonal matrix is so easy to compute!

---

90

Let  $A = [\vec{a}_1 | \vec{a}_2 | \vec{a}_3 | \vec{a}_4]$  be an orthogonal matrix.

90.1 Explain why  $\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b} \\ \vec{a}_2 \cdot \vec{b} \\ \vec{a}_3 \cdot \vec{b} \\ \vec{a}_4 \cdot \vec{b} \end{bmatrix}$  is a solution to  $A\vec{x} = \vec{b}$ .

90.2 Find scalars  $a, b, c, d$  so  $\vec{b} = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3 + d\vec{a}_4$  (your answers will have variables in them).

Orthogonal matrices also allow us to compute projections quite easily.

**Def** If  $V$  is a subspace of  $\mathbb{R}^n$ , the **projection** (sometimes called the orthogonal projection) of  $\vec{x}$  onto  $V$  is the closest point in  $V$  to  $\vec{x}$ . We notate the projection of  $\vec{x}$  onto  $V$  as  $\text{proj}_V \vec{x}$ .

Projections are normally hard to compute and a priori might require some sort of calculus-style optimization to find. However, from geometry we know that if we travel from  $\text{proj}_V \vec{x}$  to  $\vec{x}$ , we should always trace out a path perpendicular to  $V$ . Otherwise, we could find a point in  $V$  that was slightly closer to  $\vec{x}$ , violating the definition of  $\text{proj}_V \vec{x}$ . Thus, orthogonality will be our savior.

91

Let  $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$  be the standard basis.

91.1 If  $\vec{x} = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$ , find the projection of  $\vec{x}$  onto the  $xy$ -plane.

Suppose  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$  is an orthonormal basis for  $\mathbb{R}^3$ .

91.2 If  $\vec{y} = 3\vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$ , find the projection of  $\vec{y}$  onto  $\text{span}\{\vec{b}_1, \vec{b}_3\}$ .

Suppose  $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$  is a basis for  $\mathbb{R}^3$  with

$$\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1 \quad \vec{c}_1 \cdot \vec{c}_2 = 0 \quad \vec{c}_1 \cdot \vec{c}_3 = 0 \quad \vec{c}_2 \cdot \vec{c}_3 = \sqrt{2}/2.$$

91.3 If  $\vec{z} = 5\vec{c}_1 + 2\vec{c}_2 - \vec{c}_3$ , find the projection of  $\vec{z}$  onto  $\text{span}\{\vec{c}_1, \vec{c}_2\}$ .

92

Let's put this all together.  $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$  is an orthogonal basis for  $\mathbb{R}^3$ . Let  $\mathcal{P}$  be the plane defined by

$$0x + y - z = 0.$$

92.1 Write  $\mathcal{P}$  in vector form (Hint: think about the vectors listed in the  $\mathcal{B}$  basis).

92.2 Find an orthonormal basis  $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$  for  $\mathbb{R}^3$  so  $\mathcal{P} = \text{span}\{\vec{c}_1, \vec{c}_2\}$ .

92.3 Let  $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . Find  $\text{proj}_{\mathcal{P}} \vec{x}$ .

## Gram-Schmidt Orthogonalization

We've seen how useful orthonormal bases are. The incredible thing is that we can turn any basis into an orthonormal basis through a process called Gram-Schmidt orthogonalization.

93

Let  $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

93.1 Draw  $\vec{a}$  and  $\vec{b}$  and find  $\vec{w} = \text{proj}_{\vec{b}} \vec{a}$ .

93.2 Add  $\vec{c} = \vec{a} - \vec{w}$  to your drawing. What is the angle between  $\vec{c}$  and  $\vec{b}$ .

93.3 Can you write  $\vec{a}$  as the sum of two vectors, one in the direction of  $\vec{b}$  and one orthogonal to  $\vec{b}$ ? If so, do it.

94

Let  $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ .

94.1 Write  $\vec{a} = \vec{u} + \vec{v}$  where  $\vec{u}$  is parallel to  $\vec{b}$  and  $\vec{v}$  is orthogonal to  $\vec{b}$ .

94.2 Find an orthonormal basis for  $\text{span}\{\vec{a}, \vec{b}\}$ .

With two vectors, making an orthonormal set without changing the span is quite easy. With more vectors, it is only slightly harder.

**Def**

The **Gram-Schmidt** orthogonalization procedure takes in a set of vectors and outputs a set of orthonormal vectors with the same span. The idea is to iteratively produce a set of vectors where each new vector you produce is orthogonal to the previous vectors.

The algorithm is as follows: Let  $\{v_1, \dots, v_n\}$  be a set of vectors. Produce a set  $\{v'_2, \dots, v'_n\}$  that is orthogonal to  $v_1$  by subtracting off the respective projections of  $v_2, \dots, v_n$  onto  $v_1$ . Next, produce a set  $\{v''_3, \dots, v''_n\}$  orthogonal to both  $v_1$  and  $v'_2$  by subtracting off the respective projections onto  $v'_2$ . Continue this process until you have a set  $V = \{v_1, v'_2, v''_3, v'''_4, \dots\}$  that is orthogonal. Finally, normalize  $V$  so all vectors have unit length.

95

Let  $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ ,  $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\vec{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ .

95.1 Use the Gram-Schmidt procedure to find an orthonormal basis for  $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ .

95.2 Find an orthonormal basis  $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$  for  $\mathbb{R}^4$  so that  $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ .

Let  $R = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$ .

95.3 Find an orthonormal basis for the row space of  $R$ .

95.4 Find the null space of  $R$  (Hint, you've already done the work, so there is no need to row reduce).

96

Let

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{y}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \quad \vec{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}.$$

96.1 Find an orthonormal basis  $\mathcal{W}$  so that  $\text{span } \mathcal{W} = \text{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ .

**Def**

The **orthogonal complement** of a subspace  $V$  is written  $V^\perp$  and defined as

$$V^\perp = \{\vec{x} : \vec{x} \text{ is orthogonal to } V\}.$$

96.2 Find the orthogonal complement of  $\text{span } \mathcal{W}$ .

96.3 Write  $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  in the form  $\vec{v} = \vec{r} + \vec{n}$  where  $\vec{r} \in \text{span } \mathcal{W}$  and  $\vec{n} \in (\text{span } \mathcal{W})^\perp$ .

## QR Decomposition

**Def** For a matrix  $A$ , we can rewrite  $A = QR$  where  $Q$  is an orthogonal matrix and  $R$  is an upper triangular matrix. Writing  $A$  as  $QR$  is called the **QR decomposition** of  $A$ .

97 Suppose  $A, B, C$  are square matrices and  $C = AB$ .

97.1 How do the column spaces of  $A$  and  $C$  relate?

97.2 How do the column spaces of  $B$  and  $C$  relate?

98  $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  forms a basis for  $\mathbb{R}^3$ . When we apply the Gram-Schmidt process to  $\mathcal{V}$ , we get

$$\begin{aligned} q'_1 &= \vec{v} \\ q'_2 &= \vec{v}_2 - \frac{1}{2}\vec{v}_2 \\ q'_3 &= \vec{v}_3 - \vec{v}_1 + 2\vec{v}_2 \end{aligned}$$

form an orthogonal set. Normalizing we get

$$\begin{aligned} \vec{q}_1 &= 2q'_1 \\ \vec{q}_2 &= 3q'_2 \\ \vec{q}_3 &= \frac{1}{2}q'_3 \end{aligned}$$

form an orthonormal set.

98.1 Write  $\vec{v}_1$  as a linear combination of  $\vec{q}_1, \vec{q}_2, \vec{q}_3$ .

98.2 Write  $\vec{v}_2$  as a linear combination of  $\vec{q}_1, \vec{q}_2, \vec{q}_3$ .

98.3 Write  $\vec{v}_3$  as a linear combination of  $\vec{q}_1, \vec{q}_2, \vec{q}_3$ .

Define  $A = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3]$  and  $Q = [\vec{q}_1 | \vec{q}_2 | \vec{q}_3]$ .

98.4 Find a matrix  $R$  so that  $A = QR$ .

We've just discovered one process to find the  $QR$  decomposition of a matrix. It's really as simple as doing Gram-Schmidt and keeping track of your coefficients. Now, we have another way to the matrix equation  $A\vec{x} = \vec{b}$ . If we do a  $QR$  decomposition and exploit the fact that  $Q^{-1} = Q^T$ , we have

$$A\vec{x} = QR\vec{x} = \vec{b} \quad \implies \quad R\vec{x} = Q^T\vec{b}$$

and  $R$  is a triangular matrix, so we can just do back substitution! (It turns out that if you solve systems this way, there is less rounding error than if you use row reduction.)

## Symmetric Matrices

When you're new to Linear Algebra, learning lots of new concepts and algorithms, it's sometimes hard to grasp the significance of certain properties of a matrix.

Symmetric matrices are easy to forget at first, but they have many profound properties (not to mention they are one of the key concepts of Quantum Mechanics).

99 Let  $A$  be a symmetric matrix and let  $\vec{v}$  be an eigenvector with eigenvalue 3 and  $\vec{w}$  be an eigenvector with eigenvalue 4. Note, for this problem, we are thinking of  $\vec{v}$  and  $\vec{w}$  as column vectors.

- 99.1 Write  $A\vec{v}$ ,  $\vec{v}^T A^T$ ,  $\vec{v}^T A$ ,  $A\vec{w}$ ,  $\vec{w}^T A^T$ , and  $\vec{w}^T A$  in terms of  $\vec{v}$ ,  $\vec{w}$  and scalars.
- 99.2 How do  $\vec{v}^T \vec{w}$  and  $\vec{w}^T \vec{v}$  relate?
- 99.3 What should  $\vec{v}^T A \vec{w}$  be in terms of  $\vec{v}^T$  and  $\vec{w}$ ? (Note, you could compute  $(\vec{v}^T A) \vec{w}$  or  $\vec{v}^T (A \vec{w})$ . Better do both to be safe).
- 99.4 What can you conclude about  $\vec{v}^T \vec{w}$ ? How about  $\vec{v} \cdot \vec{w}$ ?

We've just deduced that all eigenspaces of a symmetric matrix are orthogonal! On top of that, symmetric matrices always have a basis of eigenvectors. That means that not only can you always diagonalize a symmetric matrix, but you can *orthogonally* diagonalize a symmetric matrix. (i.e. if  $A$  is symmetric, then  $A = QDQ^T$  where  $Q$  is orthogonal and  $D$  is diagonal). This is like the best of all worlds in one!