Learning Objectives

In this tutorial you will be working with inverses and examining how they connect algebra and geometry.

These problems relate to the following course learning objectives: *Translate between algebraic and geometric viewpoints to solve problems*, and *clearly and correctly express the mathematical ideas of linear algebra to others, and understand and apply logical arguments and definitions that have been written by others*.

Problems

1. Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a function and let A be a matrix. Write down the definition of (i) what it means for f to be *invertible* and (ii) what it means for A to be invertible.

2. Below are several samples from student answers to a MAT223 exam question relating to solving the matrix equation $A\vec{x}=\vec{b}$. For each sample, decide whether it is totally correct, mostly correct, mostly incorrect, or totally incorrect. If it is not totally correct, explain what is wrong, and if possible, how to fix it.

   You may assume the student can correctly compute A^{-1}.

   (a) $A\vec{x}=\vec{b} \quad\Longrightarrow\quad \vec{x}=\vec{b}A^{-1}$.

   (b) $A\vec{x}=\vec{b} \quad\Longrightarrow\quad \vec{x}=A^{-1}\vec{b}$.

   (c) $A\vec{x}=\vec{b} \quad\Longrightarrow\quad \vec{x}=\vec{b}/A$.

   (d) $A\vec{x}=\vec{b} \quad\Longrightarrow\quad \vec{x}=\frac{1}{A}\vec{b}$.

   (e) $A\vec{x}=\vec{b} \quad\Longrightarrow\quad A/\vec{b}=1/\vec{x}. \quad\Longrightarrow\quad 1/(A/\vec{b})=\vec{x}$.

3. Let $R:\mathbb{R}^2\to\mathbb{R}^2$ be rotation counter-clockwise by 30°; let $D:\mathbb{R}^2\to\mathbb{R}^2$ be reflection across the line $y=4x$; let $P:\mathbb{R}^2\to\mathbb{R}^2$ be projection onto the line $y=4x$; and let $S:\mathbb{R}^2\to\mathbb{R}^2$ be the transformation that doubles the length of every vector.

   For each transformation, (i) decide if it is invertible, and (ii) describe the inverse-transformation in words.

4. Let R, D, P, and S be defined as before.

   (a) Determine the rank of R, D, P, and S.

   (b) Does rank relate to invertibility? Write down an argument to support your conjecture.

   (c) Find matrices for R, D, P, and S. How does the invertibility of these matrices relate to the invertibility of the original transformations? Is this relationship affected by which basis you choose to represent the transformation in?

1. $f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ so that $f \circ g = g \circ f =$id, the identity function. The matrix A is invertible if there exists a matrix B so that $AB = I$ and $BA = I$.

2. (a) Totally incorrect. If $\vec{b}$ is a column vector $\vec{b}A^{-1}$ won't be defined. It should be $\vec{x} = A^{-1}\vec{b}$.
   (b) Totally correct.
   (c) Totally incorrect. There is no division operation for matrices. Instead, you must multiply both sides by A^{-1}.
   (d) Mostly incorrect. The notation $\frac{1}{A}$ is not defined. We use the notation A^{-1} for the matrix such that $A^{-1}A = AA^{-1} = I$.
   (e) Totally incorrect. You cannot divide by a vector.

3. R This transformation is invertible and its inverse is clockwise rotation by 30°.
   D This transformation is invertible and its inverse is itself.
   P This transformation is not invertible. Since $P(\vec{0}) = P\left(\begin{bmatrix}-4\\1\end{bmatrix}\right) = \vec{0}$, there is no way to "undo" P. Formally, P is not one-to-one, so it cannot be invertible.
   S This transformation is invertible and its inverse is halving the length of each vector.

4. (a) R, D, S are rank 2 and P is rank 1.
   (b) Yes. If a transformation from $\mathbb{R}^n \to \mathbb{R}^n$ is invertible its rank must be n. We can argue as follows:

   Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ has rank $< n$. Then the range of T cannot be all of $\mathbb{R}^n$. Thus, there cannot exist a transformation S so that $T \circ S =$id, where id is the identity function on all of $\mathbb{R}^n$.
   (c)
$$[R]_{\mathcal{E}} = \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix} \qquad [D]_{\mathcal{E}} = \tfrac{1}{17}\begin{bmatrix}-15 & 8\\ 8 & 15\end{bmatrix}$$
$$[P]_{\mathcal{E}} = \tfrac{1}{17}\begin{bmatrix}1 & 4\\ 4 & 16\end{bmatrix} \qquad [S]_{\mathcal{E}} = \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

   The matrix for a transformation is invertible if and only if the transformation is invertible. This is not affected by the choice of basis.

Learning Objectives

Students need to be able to...

- Read others' mathematical arguments and identify whether they are correct
- Multiply matrices in the correct order when solving matrix equations
- Apply the meaning of invertibility (that a transformation can be "undone")
- Formally define invertibility
- Recognize the identity function as a legitimate function

Context

In lecture, students have seen rank of a matrix and a linear transformation; null space of a matrix/linear transformation; column space, row space of a matrix; and elementary matrices. However, most of them will be shaky on what an inverse actually "means" and even more shaky on how to define the inverse of a function (even though this is grade 11 material). Many lectures, but not all, have used the terms "one-to-one" and "onto" when talking about inverses. You may use these terms (the students should know them from high school). Prefer "one-to-one" and "onto" over "injective" and "surjective".

What to Do

Start the tutorial by stating the day's learning objectives and that today has two focuses: *inverses* and *reading others' work*. You may tell them that the idea of inverse *functions* is used in general contexts, not just in linear algebra, but that we only deal with inverse *matrices* in linear algebra. This is why they are asked for two definitions to start.

After everyone has written down a definition for inverse functions and inverse matrices, get a correct definition on the board and have a short (≤ 3 minute) discussion on why inverses are useful for solving equations/matrix equations. Then, have them start #2.

#2 shouldn't take long, but should allow a for a good discussion of the fact that when we write out matrix equations, they look the same as equations with numbers, but if we don't remember that they're actually linear algebra objects, we'll do nonsensical things. *The power of algebra (in the symbolic sense) is that it allows us to shut off our brains and follow symbolic steps, but we cannot shut off our brains too much!*

#3 asks students to connect geometric transformations with the formal definition of a function being invertible. This will be hard for some and easy for others. If they get stuck and frustrated, you can ask them to try 4(c) and then come back to 3.

6 minutes before the end of class, pick a suitable problem to do as a wrap-up.

Notes

- Students will be uncomfortable with function composition. They would prefer to write $h(x) = f(g(x))$ over $h = f \circ g$. We want them to get comfortable with $f \circ g$ notation, but you may use $f(g(x))$ notation if they're confused.

- #2 is trivial in some sense. And, we could extend notational definitions to make many of those correct. However, in this class we're using the standard North-American notations, so only (b) is acceptable.

- For #3 encourage them to draw some "before and after" pictures of vectors subjected to each transformation. This will aid their thinking (and should be their natural inclination).

- For #4(b), it is easier to argue invertible $\implies$ full rank than full rank $\implies$ invertible. Focus on invertible $\implies$ full rank and leave the converse as a challenge for the stronger students.

  Also, you may use matrix theory when arguing #4, but make sure that a distinction between *matrices* and *transformations* is made. Just like we can tell things about vectors by looking at their representation in a basis, so can we tell things about transformations by looking at their matrix representation. However these aren't the same thing!