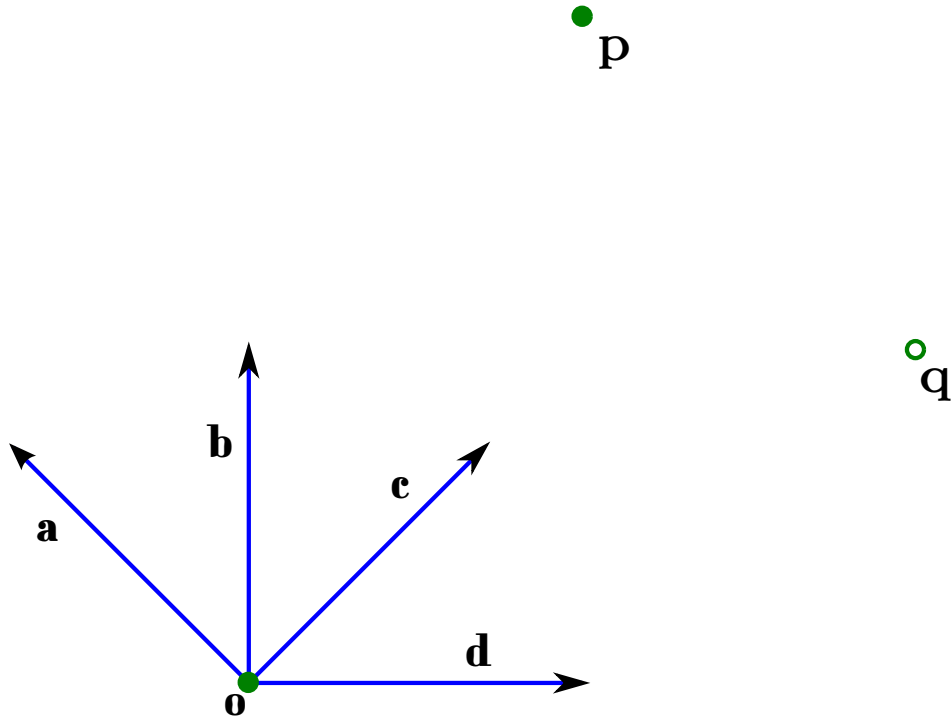# Vectors


Notice that all arrows in this diagram are the same length. We will call this length a *unit*.

1.1 Give directions from **o** to **p** of the form "Walk ____units in the direction of arrow ____, then walk ____units in the direction of arrow ____."

1.2 Can you give directions with the two arrows you haven't used? Give such directions, or explain why it cannot be done.

1.3 Give directions from **o** to q.

1.4 Can you give directions from **o** to q using **c** and **a**? Give such directions, or explain why it cannot be done.

Unit Vectors

$\bullet p$
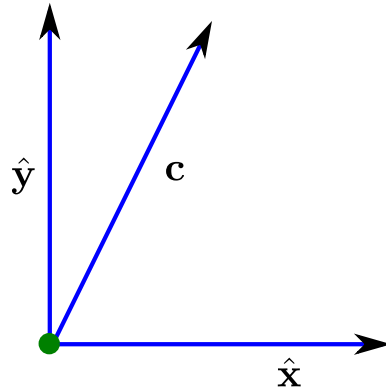
We are going to start using a more mathematical notation for giving directions. Our directions will now look like

$$p = \underline{\quad} \, \hat{\mathbf{x}} + \underline{\quad} \, \hat{\mathbf{y}}$$

which is read as "To get to p (=) go ____units in the direction $\hat{\mathbf{x}}$ then (+) go ____units in the direction $\hat{\mathbf{y}}$."

2.1 What is the difference between $p = \underline{\quad} \, \hat{\mathbf{x}} + \underline{\quad} \, \hat{\mathbf{y}}$ and $p = \underline{\quad} \, \hat{\mathbf{y}} + \underline{\quad} \, \hat{\mathbf{x}}$? Can they both give valid directions?

2.2 (a) Give directions to p using the new notation.

(b) Give directions to p using $\mathbf{c}$.

(c) What is the distance from $\mathbf{o}$ to p in units?

2.3 (a) $r = 1\mathbf{c}$. Give directions from $\mathbf{o}$ to r using $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

(b) What is the distance from $\mathbf{o}$ to r?

2.4 (a) $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$; find the exact distance from $\mathbf{o}$ to q.

(b) $s = 2\hat{\mathbf{x}} + \mathbf{c}$; find the exact distance from $\mathbf{o}$ to s.

2

We've been learning vector addition. $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are called the *standard basis vectors* for $\mathbb{R}^2$ (the plane). Everyone has agreed that if we give directions from the origin to some point and we don't specify otherwise, we will give directions in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

Column Vector Notation

We previously wrote $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$. In column vector notation we write

$$q = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We may call q either a *vector* or a *point*. If we call q a vector, we are emphasizing that q gives direction of some sort. If we call q a point, we emphasize that q is some absolute location in space. (What's the philosophical difference between a location in space and directions from the origin to said location?)

$r = 1\mathbf{c}$; $s = 2\hat{\mathbf{x}} + \mathbf{c}$.

3.1 Write r and s in column vector form.

Vector Length

The *length* or *norm* of a vector $\vec{w}$ is denoted $\|\vec{w}\|$ and is the distance from $\mathbf{o}$ to the point you end up at if you follow $\vec{w}$'s instructions.

4.1 Find $\|\vec{a}\|$, $\|\vec{b}\|$, $\|\vec{c}\|$ where

  (a) $\vec{a} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$
  (b) $\vec{b} = 2\vec{a}$
  (c) $\vec{c} = -\vec{a}/2$

4.2 $\hat{\mathbf{z}}$ points perpendicular to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ into the 3rd dimension.
  Let $\vec{v} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$ and $\vec{w} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}}$.

  (a) Write $\vec{v}$ in terms of $\vec{w}$ and $\hat{\mathbf{z}}$ and draw a picture showing the relationship between the three vectors (3-d pictures are a hard but essential skill in this course).
  (b) Find $\|\vec{w}\|$ and $\|\vec{v}\|$. (Hint, look at your picture and see if there are any right triangles to exploit).

4.3 Let $\vec{u} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$.

  (a) Find $\|\vec{u}\|$.
  (b) Find $\|k\vec{u}\|$ where k is some unknown constant.
  (c) What value(s) of k makes $\|k\vec{u}\| = 1$?
  (d) Write down a vector in column form that points in the same direciton as $\vec{u}$ and has length 1.

Unit Vectors

Vectors that have length 1 are called *unit vectors*.

5.1 $\vec{a} = -\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$. Find a unit vector in the direction of $\vec{a}$, and call this vector $\vec{u}$ (u for *unit*, get it?).

5.2 Write $\vec{a}$ in terms of $\vec{u}$. Does $\|\vec{a}\|$ show up in your formula at all?

5.3 Write $3\vec{u}$ in column vector form and find its length.

5.4 Write $7.5\vec{u}$ in column vector form and find its length.

5.5 $\vec{v}$ is a different unit vector (I won't tell you its exact form). Find $\|9\vec{v}\|$ Why do we like unit vectors so much?

Dot Product

The *dot product* is incredible because it is easy to compute and has a useful geometric meaning.

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the dot product of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

We also have a geometry-related formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

6.1 Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

   (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

   (b) Compute $\vec{a} \cdot \vec{b}$.

   (c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

6.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

6.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

   (a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

   (b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

   (c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

   (d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

6.4 $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   (a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

   (b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

   (c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

We've explored how dot products relate to angles, but how do they relate to lengths?

7.1 Let $\vec{a} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$

    (a) Find $\|\vec{a}\|$ and $\vec{a} \cdot \vec{a}$. How do the two quantities relate?

    (b) Write down an equation for the length of a vector $\vec{v}$ in terms of dot products.

7.2 Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}$, and find $\|\vec{b}\|$. Did you know how to find 4-d lengths before?
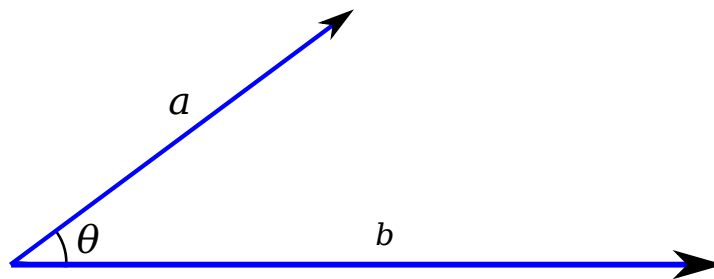
7.3 Suppose $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ for $x, y \in \mathbb{R}$. Could $\vec{u} \cdot \vec{u}$ be negative? Compute $\vec{u} \cdot \vec{u}$ algebraically and use this to justify your answer.

Projections

Projections (sometimes called orthogonal projections) are a way to measure how much one vector points in the direction of another.
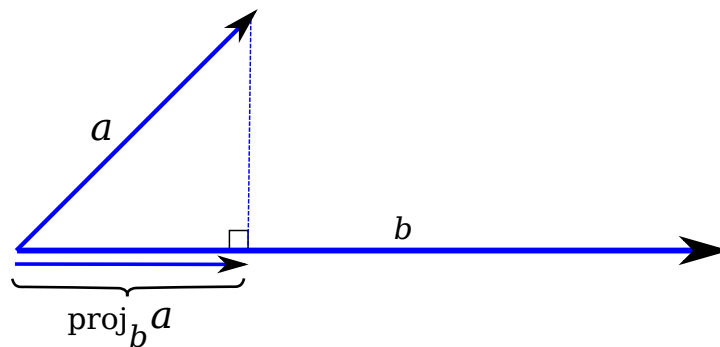
The projection of $\vec{a}$ onto $\vec{b}$ is written $\text{proj}_{\vec{b}}\vec{a}$ and is a vector in the direction of $\vec{b}$.

8.1 In this picture $\|\vec{a}\| = 4$ and $\theta = \pi/6$. Find $\|\text{proj}_{\vec{b}}\vec{a}\|$.

8.2 If $\vec{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, write down $\text{proj}_{\vec{b}}\vec{a}$ in column vector form. How do the coordinates relate to $\|\text{proj}_{\vec{b}}\vec{a}\|$?

8.3 Consider $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Compute $\text{proj}_{\hat{\mathbf{x}}}\vec{u}$ and $\text{proj}_{\hat{\mathbf{y}}}\vec{u}$. How do these projections relate to the coordinates of $\vec{u}$? What can you say in general about projections onto $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$?

$$\vec{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

9.1 Find θ, the angle between $\vec{w}$ and $\vec{v}$.

9.2 Use θ to compute $\text{proj}_{\vec{v}}\vec{w}$ and $\text{proj}_{\vec{w}}\vec{v}$.

9.3 Write down a formula for $\text{proj}_{\vec{b}}\vec{a}$ where $\vec{a}$ and $\vec{b}$ are arbitrary vectors.

10.1 For the arbitrary vector $\vec{a}$, what is $\text{proj}_{3\vec{a}}\vec{a}$?

10.2 If $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) vectors, what is $\text{proj}_{\vec{b}}\vec{a}$? $\text{proj}_{\vec{a}}\vec{b}$?

Lines, Planes, Normals, Equations

11.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors perpendicular to it.

11.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is perpendicular to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

11.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for a line. This is called *normal form*

A *normal vector* to a line is one that is orthogonal to it.

11.4 Rewrite the line $\vec{u} \cdot \vec{x} = 0$ in $y = mx + b$ form and verify it matches the line you drew above.

We can also write a line in *parametric form* by introducing a parameter that traces out the line as the parameter runs over all real numbers.

12.1 Draw the line L with x, y coordinates given by

$$x = t$$
$$y = 2t$$

as t ranges over $\mathbb{R}$.

12.2 Write the line $\vec{u} \cdot \vec{x} = 0$ (where $\vec{u}$ is the same as before) in parametric form.

Vector form is the same as parametric form but written in vector notation. For example, the line L from earlier could be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

13.1 Write $\vec{u} \cdot \vec{x} = 0$ in vector form. That is, find a vector $\vec{v}$ so the line $\vec{u} \cdot \vec{x} = 0$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\vec{v}$$

as t ranges over $\mathbb{R}$.

13.2 What is $\vec{v} \cdot \vec{u}$? Why? Will this always happen?

Moving to Planes

When solving equations, sometimes we get to make choices. For example, if $x + 2y = 0$, we can find solutions by fixing either x or y and solving for the other. e.g., if $x = 2$, then $y = -1$ and if $y = 3$ then $x = -6$.

14.1 Write down three solutions $\vec{a}$, $\vec{b}$, $\vec{c}$ to

$$2x + y - z = 0. \tag{1}$$

14.2 Is $2\vec{a} - \vec{b}$ a solution? Is any linear combination of solutions a solution? Justify why or why not.

14.3 Rewrite equation (1) in normal form $\vec{n} \cdot \vec{x} = 0$ where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

14.4 What do you notice about the angle between solutions to equation (1) and $\vec{n}$?

14.5 You've already seen that scalars come out of dot products (e.g., $\vec{a} \cdot (3\vec{b}) = 3(\vec{a} \cdot \vec{b})$). Use this combined with normal form to prove a linear combination of solutions is still a solution.

When writing down solutions to equation (1), you got to choose two coordinates before the remaining coordinate became determined. This means the solutions have two parameters (and consequently form a two dimensional space).

14.6 Write down parametric form of a line of solutions to equation (1).

14.7 Write down parametric form of a different line of solutions to equation (1).

7

14.8 Write down all solutions to equation (1) in parametric form. That is, find $a_x, a_y, a_z, b_x, b_y, b_z$ so that

$$x = a_x t + b_x s$$
$$y = a_y t + b_y s$$
$$z = a_z t + b_z s$$

gives all solutions as t, s vary over all of $\mathbb{R}$.

14.9 Write all solutions to equation (1) in vector form.

Arbitrary Lines and Planes

So far, all of our lines and planes have passed through the origin. To produce the equation of an arbitrary line/plane, we first make one of same "slope" that passes through the origin, then we translate it to the appropriate place.

We'd like to write the equation of a line L with normal vector $\vec{n} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ that passes through the point $p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

15.1 Write normal form of the line L_2 which is parallel to L, but passes through the origin.

15.2 Draw a picture of L and L_2, and find two points that lie on L. Call these points p_1 and p_2.

15.3 Verify the vector $\vec{p_1 p_2}$ is perpendicular to $\vec{n}$.

15.4 What is $\vec{n} \cdot p_1$, $\vec{n} \cdot p_2$, $\vec{n} \cdot p$? Should these values be zero, equal, or different? Explain (think about projections).

15.5 How does the equation $\vec{n} \cdot (\vec{x} - p) = 0$ relate to L?

W is the plane with normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and passes through the point $p = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

16.1 Write normal form of W.

16.2 Write vector form of W.

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

Consider the system

$$\begin{aligned} x - y &= 2 \\ 2x + y &= 1 \end{aligned} \tag{2}$$

17.1 Draw the lines in (2) on the same coordinate plane.

17.2 Algebraically solve the system (2). What does this solution represent on your graph?

Let L be the line given by $x - y = 2$.

18.1 Write an equation of a line that doesn't intersect L.

18.2 Write an equation of a line that intersects L in

   (a) one place.
   (b) infinitely many places
   (c) exactly two places

   or explain why no such equation exists.

18.3 For each equation you came up with solve the system algebraically. How can you tell algebraically how many solutions there are?

The Row Reduction Algorithm

19.1 Solve the system

$$\begin{aligned} x - y - 2z &= -5 \\ 2x + 3y + z &= 5 \\ 0x + 2y + 3z &= 8 \end{aligned} \tag{3}$$

any way you like.

19.2 Use an augmented matrix to solve the system (3).

The system (3) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (3) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

19.3 Rewrite (3) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

19.4 If (x, y, z) is a solution to (3), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

20.1 Write the system of equations corresponding to A.

20.2 Solve the system of equations corresponding to A.

Infinite Solutions

Consider the system

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \end{aligned} \qquad (4)$$

21.1 How many solutions does (4) have?

21.2 Write the solutions to (4) in vector form.

21.3 What happens when you use an augmented matrix to solve (4)?

Free Variables

Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array} \right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make $2 - 1 = 1$ choices when writing a solution. Let's make the choice $y = t$.

22.1 With the added equation $y = t$, solve the system represented by B.

Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right].$$

and call the variables in this system x_1, x_2, x_3, x_4, x_5.

23.1 Write the system of equations represented by C.

23.2 Identify how many choices you can make when writing down a solution corresponding to C.

23.3 Add one equation (of the form $x_i = t$ or $x_j = s$, etc.) for each choice you must make when solving the system.

23.4 Write in vector form all solutions to C.

24.1 An unknown system U is represented by an augmented matrix with 4 rows and 6 columns. What is the minimum number of free variables solutions to U will have?

24.2 An unknown system V is represented by an augmented matrix with 6 rows and 4 columns. What is the minimum number of free variables solutions to V will have?

Span

Let

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{r} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

25.1 Is $\vec{w}$ a linear combination of $\vec{u}$ and $\vec{v}$?

25.2 Is $\vec{r}$ a linear combination of $\vec{u}$ and $\vec{v}$?

25.3 What does the space of all linear combinations of $\vec{u}$ and $\vec{v}$ look like? (Do you expect a randomly chosen vector to be in this space?)

The set of all linear combinations of a set of vectors V is called the *span* of V and is denoted "span V."

26.1 Describe span $\{\vec{u}\}$.

26.2 Describe span $\{\vec{u}, \vec{v}\}$

26.3 Describe span $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$.

26.4 Describe span $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$.

26.5 Describe span $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\} \cap$ span $\{\vec{u}, \vec{v}\}$.

27.1 How do span $\{\vec{u}, \vec{v}\}$ and span $\{\vec{u}, \vec{v}, \vec{w}\}$ relate?

Linear Independence and Dependence

We've seen sometimes adding a vector to a set doesn't make its span any larger. This is because the vector was already in the span in the first place!

We say $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is *linearly dependent* if for at least one i,

$$\vec{v}_i \in \text{span}\, \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\},$$

and a set is *linearly independent* otherwise.

28.1 Can you state linear independence in terms of linear combinations?

28.2 Is the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$ linearly independent? What is its span?

28.3 In $\mathbb{R}^2$ what is the largest linearly independent set you could have?

28.4 In $\mathbb{R}^2$, is every set of two or fewer vectors linearly independent?

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is *trivial* if $a_1 = a_2 = \cdots = a_n = 0$.

29.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

29.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence. Namely, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is *linearly dependent* if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that forms the zero vector.

30.1 Explain how this new definition implies the old one.

30.2 Explain how the old definition implies this new one.

We now have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'). This means the two definitions are *equivalent* (which we write as new def $\iff$ old def).

Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

31.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

31.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

31.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

31.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Finding Linearly Independent Subsets

Suppose when you use an augmented matrix to solve $a\vec{u} + b\vec{v} + c\vec{w} = \vec{y}$ you have no free variables.

32.1 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

Suppose when you use an augmented matrix to solve $a\vec{u} + b\vec{v} + c\vec{w} = \vec{y}$ the second column corresponds to a free variable.

32.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

32.3 Is $\{\vec{u}, \vec{w}\}$ linearly independent?

32.4 Is $\{\vec{u}, \vec{v}\}$ linearly independent?

A *maximal linearly independent subset* X of a set of vectors V is a linearly independent subset of V with the most possible vectors in it (i.e., if you too any subset of V with more vectors, it would be linearly dependent).

33.1 Give a maximal linearly independent subset, T, of $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

33.2 What is the size of T?

Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \vec{v}_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and the matrices

$$A = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 2 & -1 & 1 & 2 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(Notice that the columns of A are the vectors $\vec{v}_1, \dots \vec{v}_5$)

34.1 Is $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ linearly independent?

34.2 Pick a maximal linearly independent subset of V.

34.3 Pick another (different) maximal linearly independent subset of V.

Matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

35.1 Write the shape of the matrices A, B, C (i.e., for each one, write the dimensions in $m \times n$ form).

35.2 List *all* products between the matrices A, B, C that are defined. (Your list will be some subset of AB, AC, BA, CA, BC, CB.)

35.3 Compute AC and CA.

36.1 If the matrices X and Y are both square $n \times n$ matrices, does $XY = YX$? Explain.

36.2 If the matrices X and Y are both square $n \times n$ matrices, does $X + Y = Y + X$? Explain.

Consider the system
$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \tag{5}$$

37.1 Find values of a, b, c, d, e, f so that the matrix equation
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$
represents the same system as (5).

Consider the system represented by
$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

37.2 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the solution set to this system a point, line, plane, or other?

37.3 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the solution set to this system a point, line, plane, or other?

The transpose of a matrix (written with a superscript T, e.g. A^T) swaps the rows an columns of a matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

38.1 What is the shape of A and A^T?

38.2 Write down A^T.

B and D are 4×6 matrices and C is a 6×4 matrix.

38.3 Does $(BC)^T = B^T C^T$? Explain.

38.4 Does $(B + D)^T = B^T + D^T$? Explain?

38.5 Compute AA^T and $A^T A$ (where A is the matrix defined earlier). What do you notice?

A matrix X is called symmetric if $X = X^T$. Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

39.1 Prove that if W is a square matrix, then $V = W^T W + W + W^T$ is a symmetric matrix.

There are two very special matrices that have special names. The *zero* matrix is a square matrix consisting of only zeros. We sometimes write $0_{n \times n}$ to signify the $n \times n$ zero matrix. Sometimes we leave off the $n \times n$ when it is obvious what size it should be. The *identity* matrix is a square matrix with ones on the diagonal and zeros everywhere else. Again, we may write $I_{n \times n}$ to specify the $n \times n$ identity matrix, or we may just write I and assume the dimensions are clear from context. (In Matlab, the command `eye(n)` will create an $n \times n$ identity matrix and `zeros(n)` will create an $n \times n$ zero matrix).

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

40.1 Write down the 3×3 identity matrix and the 3×3 zero matrix.

40.2 Compute $I_{3 \times 3} A$, $A I_{3 \times 3}$, $0_{3 \times 3} A$, and $A 0_{3 \times 3}$.

40.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

41.1 Solve the matrix equation

$$I_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$