Inquiry Based Linear Algebra

About the Document

This document was originally designed in the spring of 2015 to guide students through an eleven week Linear Algebra course (Math 211, Linear Algebra for Scientists) at the University of Victoria. The order of topics closely follows that in *Linear Algebra for Science and Engineering* second edition by Daniel Norman and Dan Wolczuk.

A typical class day using the problem-sets:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

These problem-sets strike a balance between concepts and computation, leaning towards the conceptual side. Two algorithms, *row reduction* and *determinant of a matrix by cofactors*, are not introduced in these problem-sets. Instead, students are expected to learn them on their own and be prepared to apply them on problems in the problem-set (these topics were left up to the students because of time constraints). Further, not ever Linear Algebra definition is given in these problem-sets, since these notes are not intended to replace a Linear Algebra textbook, but most definitions are given to expedite the transition to new topics in the middle of class time.

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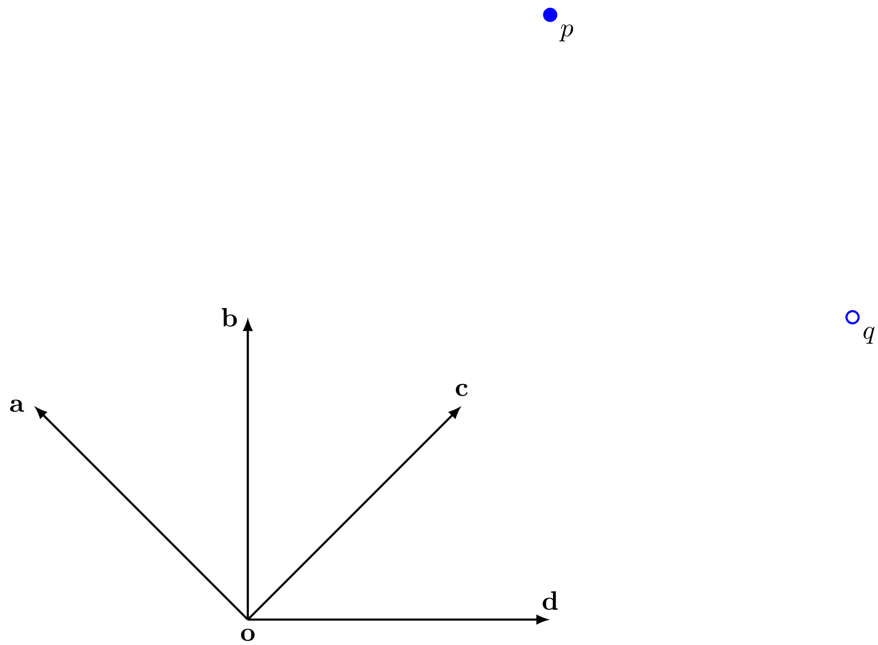
Vectors

1



Notice that all arrows in this diagram are the same length. We will call this length a *unit*.

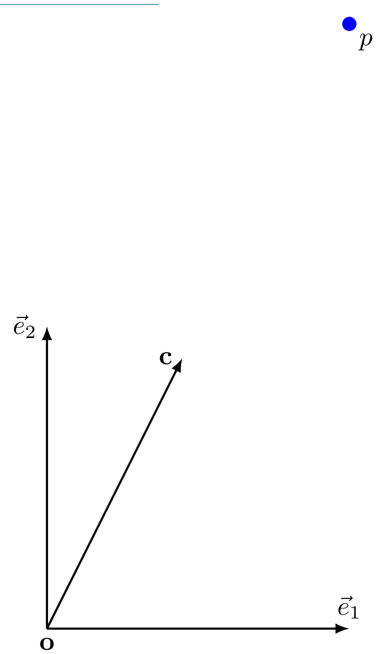
1.1 Give directions from **o** to p of the form "Walk ___units in the direction of arrow ___, then walk ___units in the direction of arrow ___."

1.2 Can you give directions with the two arrows you haven't used? Give such directions, or explain why it cannot be done.

1.3 Give directions from **o** to q.

1.4 Can you give directions from **o** to q using **c** and **a**? Give such directions, or explain why it cannot be done.

2



We are going to start using a more mathematical notation for giving directions. Our directions will now look like

$$p = \underline{\hspace{2em}}\,\vec{e}_1 + \underline{\hspace{2em}}\,\vec{e}_2$$

which is read as "To get to p (=) go ___units in the direction $\vec{e}_1$ then (+) go ___units in the direction $\vec{e}_2$."

2.1 What is the difference between $p = \underline{\hspace{2em}}\,\vec{e}_1 + \underline{\hspace{2em}}\,\vec{e}_2$ and $p = \underline{\hspace{2em}}\,\vec{e}_2 + \underline{\hspace{2em}}\,\vec{e}_1$? Can they both give valid directions?

2.2 (a) Give directions to p using the new notation.
(b) Give directions to p using **c**.
(c) What is the distance from **o** to p in units?

2.3 (a) $r = 1\mathbf{c}$. Give directions from **o** to r using $\vec{e}_1$ and $\vec{e}_2$.
(b) What is the distance from **o** to r?

2.4 (a) $q = -2\vec{e}_1 + 3\vec{e}_2$; find the exact distance from **o** to q.
(b) $s = 2\vec{e}_1 + \mathbf{c}$; find the exact distance from **o** to s.

The vectors $\vec{e}_1$ and $\vec{e}_2$ are called the *standard basis vectors* for $\mathbb{R}^2$ (the plane).

Column Vector Notation

We previously wrote $q = -2\vec{e}_1 + 3\vec{e}_2$. In column vector notation we write

$$q = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We may call q either a *vector* or a *point*. If we call q a vector, we are emphasizing that q gives direction of some sort. If we call q a point, we emphasize that q is some absolute location in space. (What's the philosophical difference between a location in space and directions from the origin to said location?)

3

$r = 1\mathbf{c}$ and $s = 2\vec{e}_1 + \mathbf{c}$ where $\mathbf{c}$ is the vector from before.

3.1 Write r and s in column vector form.

Sets and Set Notation

Def A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1, 2, 3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X.

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \,|\, \text{some rule involving } a\}$.

Def Some common sets are

$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}$.

$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}$.

$\mathbb{R} = \{\text{real numbers}\}$.

$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}$.

4

4.1 Which of the following are true?

(a) $3 \in \{1, 2, 3\}$.
(b) $4 \in \{1, 2, 3\}$.
(c) "b"$\in \{x : x \text{ is an English letter}\}$.
(d) "ò"$\in \{x : x \text{ is an English letter}\}$.
(e) $\{1, 2\} \subseteq \{1, 2, 3\}$.
(f) For some $a \in \{1, 2, 3\}$, $a \geq 3$.
(g) For any $a \in \{1, 2, 3\}$, $a \geq 3$.
(h) $1 \subseteq \{1, 2, 3\}$.
(i) $\{1, 2, 3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.
(j) $\{1, 2, 3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$.

5 Write the following in set-builder notation

5.1 The subset $A \subset \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

5.2 The subset $B \subset \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Def Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)} \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

6 Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4, 5\}$ and $Z = \{4, 5, 6\}$. Compute

6.1 $X \cup Y$

6.2 $X \cap Y$

6.3 $X \cup Y \cup Z$

6.4 $X \cap Y \cap Z$

7 Draw the following subsets of $\mathbb{R}^2$.

7.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.3 $J = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.4 $V \cup H$.

7.5 $V \cap H$.

7.6 Does $V \cup H = \mathbb{R}^2$?

Linear Combinations, Span, and Linear Independence

Def A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

8 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

8.1 Write the coordinates of $\vec{w}$.

8.2 Draw a picture with $\vec{w}$, $\vec{v}_1$, and $\vec{v}_2$.

8.3 Is $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

8.4 Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

8.5 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

8.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

8.7 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Def The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ for some $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

9

Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

9.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

9.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

9.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

9.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

9.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

10

Give an example of:

10.1 two vectors in $\mathbb{R}^3$ that span a plane;

10.2 two vectors in $\mathbb{R}^3$ that span a line;

10.3 four vectors in $\mathbb{R}^3$ that span a plane;

10.4 a set of 50 vectors in $\mathbb{R}^3$ whose span is the line through the origin and the point $\begin{bmatrix}1\\2\\-3\end{bmatrix}$.

In some sets, every vector is essential for computing a span. In others, there are "excess" vectors. This leads us to the concept of linear independence.

Def We say $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\},$$

and a set is ***linearly independent*** otherwise.

11

Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

11.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

11.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

11.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

11.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

Def We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

12 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

12.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

12.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

Def $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector.

13 13.1 Explain how this new definition implies the old one.

13.2 Explain how the old definition implies this new one.

Since have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

14 Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

14.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

14.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

14.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

14.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Subspaces and Bases

Def A ***subspace*** $V \subseteq \mathbb{R}^n$ is a subset such that

(i) $\vec{u}, \vec{v} \in V$ implies $\vec{u} + \vec{v} \in V$.

(ii) $\vec{u} \in V$ implies $k\vec{u} \in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

15 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

15.1 $A = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}a\\0\end{bmatrix}$ for some $a \in \mathbb{Z}\}$.

15.2 $B = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix}0\\0\end{bmatrix}\}$.

15.3 $C = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}0\\t\end{bmatrix}$ for some $t \in \mathbb{R}\}$.

15.4 $D = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}0\\t\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$ for some $t \in \mathbb{R}\}$.

15.5 $E = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix}0\\t\end{bmatrix}$ or $\vec{x} = \begin{bmatrix}t\\0\end{bmatrix}$ for some $t \in \mathbb{R}\}$.

15.6 $F = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix}3\\1\end{bmatrix}$ for some $t \in \mathbb{R}\}$.

15.7 $G = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

15.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Def A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

16

Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

16.1 Describe V.

16.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

16.3 Give a basis for V.

16.4 Give another basis for V.

16.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

Def The ***dimension*** of a subspace V is the number of elements in a basis for V.

16.6 What is the dimension of V?

17

Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}7\\8\\8\end{bmatrix}$ and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

17.1 Give a basis for and the dimension of P.

17.2 Give a basis for and the dimension of Q.

17.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

17.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Dot Product

Def If $\vec{a} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

18

Let $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}3\\2\end{bmatrix}$, and $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$.

18.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

18.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

18.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

18.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

Def The ***norm*** of a vector $\vec{v} \in \mathbb{R}^n$, denoted $\|\vec{v}\|$ is its length and is given by the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

19

19.1 Let $\vec{a} = \begin{bmatrix}3\\2\end{bmatrix}$. Find $\|\vec{a}\|$ using the Pythagorean theorem and using the formula from the definition of the norm. How do these quantities relate?

19.2 Let $\vec{b} = \begin{bmatrix}1\\1\\-2\\2\end{bmatrix}$, and find $\|\vec{b}\|$. Did you know how to find 4-d lengths before?

19.3 Suppose $\vec{u} = \begin{bmatrix}x\\y\end{bmatrix}$ for some $x, y \in \mathbb{R}$. Could $\vec{u} \cdot \vec{u}$ be negative? Compute $\vec{u} \cdot \vec{u}$ algebraically and use this to justify your answer.

Def The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

Def A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

20

Let $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

20.1 Find the distance between $\vec{u}$ and $\vec{v}$.

20.2 Find a unit vector in the direction of $\vec{u}$.

20.3 Does there exists a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

20.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Def Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v}=0$. The word orthogonal is synonymous with the word perpendicular.

21

21.1 Find two vectors orthogonal to $\vec{a}=\begin{bmatrix}1\\-3\end{bmatrix}$. Could you find two linearly independent vectors that are orthogonal to $\vec{a}$?

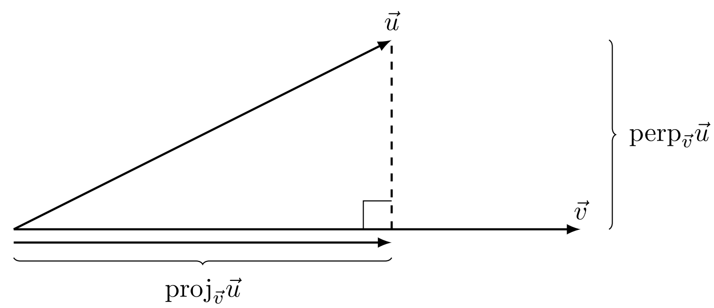
21.2 Find two vectors orthogonal to $\vec{b}=\begin{bmatrix}1\\-3\\4\end{bmatrix}$. Could you find two linearly independent vectors that are orthogonal to $\vec{b}$?

21.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\|=5$ and $\|\vec{y}\|=3$. What is the distance between $\vec{x}$ and $\vec{y}$?

Projections

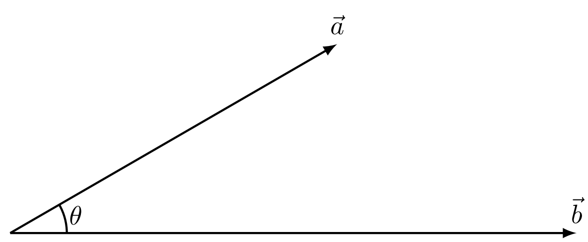
Projections (sometimes called orthogonal projections) are a way to measure how much one vector points in the direction of another.

Def



The ***projection*** of $\vec{u}$ onto $\vec{v}$ is written $\mathrm{proj}_{\vec{v}}\vec{u}$ and is the vector in the direction of $\vec{v}$ such that $\vec{u}-\mathrm{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$. The vector $\vec{u}-\mathrm{proj}_{\vec{v}}$ is called the ***perpendicular component*** of $\vec{u}$ with respect to $\vec{v}$ and is notated as $\mathrm{perp}_{\vec{v}}\vec{u}$.

22



22.1 In this picture $\|\vec{a}\|=4$ and $\theta=\pi/6$. Find $\|\mathrm{proj}_{\vec{b}}\vec{a}\|$ and $\|\mathrm{perp}_{\vec{b}}\vec{a}\|$.

22.2 If $\vec{b}=\begin{bmatrix}6\\0\end{bmatrix}$, write down $\mathrm{proj}_{\vec{b}}\vec{a}$ and $\mathrm{perp}_{\vec{b}}\vec{a}$ in column vector form.

22.3 If $\vec{b}=\begin{bmatrix}-4\\0\end{bmatrix}$, write down $\mathrm{proj}_{\vec{b}}\vec{a}$ and $\mathrm{perp}_{\vec{b}}\vec{a}$ in column vector form.

22.4 If $\vec{b}=\begin{bmatrix}3\\1\end{bmatrix}$, write down $\mathrm{proj}_{\vec{b}}\vec{a}$ and $\mathrm{perp}_{\vec{b}}\vec{a}$ in column vector form. (You may need to use your knowledge of how dot products and angles relate to answer this one.)

22.5 Consider $\vec{u}=\begin{bmatrix}3\\1\end{bmatrix}$. Compute $\mathrm{proj}_{\vec{e}_1}\vec{u}$ and $\mathrm{proj}_{\vec{e}_2}\vec{u}$. How do these projections relate to the coordinates of $\vec{u}$? What can you say in general about projections onto $\vec{e}_1$ and $\vec{e}_2$?

Lines, Planes, Normals, Equations

23

23.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors perpendicular to it.

23.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is perpendicular to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

23.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for a line. This equation is called the *scalar equation* representing the line.

Def A ***normal vector*** to a line (or plane or hyperplane) is one that is orthogonal to it.

23.4 Rewrite the line $\vec{u} \cdot \vec{x} = 0$ in $y = mx + b$ form and verify it matches the line you drew above.

24

We can also write a line in *parametric form* by introducing a parameter that traces out the line as the parameter runs over all real numbers.

24.1 Draw the line L with x, y coordinates given by

$$\begin{aligned} x &= t \\ y &= 2t \end{aligned}$$

as t ranges over $\mathbb{R}$.

24.2 Write the line $\vec{u} \cdot \vec{x} = 0$ (where $\vec{u}$ is the same as before) in parametric form.

25

Vector form is the same as parametric form but written in vector notation. For example, the line L from earlier could be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

25.1 Write the line $\vec{u} \cdot \vec{x} = 0$ in vector form. That is, find a vector $\vec{v}$ so the line $\vec{u} \cdot \vec{x} = 0$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\vec{v}$$

as t ranges over $\mathbb{R}$.

25.2 What is $\vec{v} \cdot \vec{u}$? Why? Will this always happen?

Moving to Planes

When solving equations, sometimes we get to make choices. For example, if $x + 2y = 0$, we can find solutions by fixing either x or y and solving for the other. e.g., if $x = 2$, then $y = -1$ and if $y = 3$ then $x = -6$.

26

26.1 Write down three solutions $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ to

$$2x + y - z = 0. \tag{1}$$

26.2 Is $2\vec{a} - \vec{b}$ a solution? Is any linear combination of solutions a solution? Justify why or why not.

26.3 Find $\vec{n} \in \mathbb{R}^3$ so that equation (1) in is equivalent to $\vec{n} \cdot \vec{x} = 0$ where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

26.4 What do you notice about the angle between solutions to equation (1) and $\vec{n}$?

26.5 Prove that any plane of the form $\vec{n} \cdot \vec{x} = 0$ is a subspace.

When writing down solutions to equation (1), you got to choose two coordinates before the remaining coordinate became determined. This means the solutions have two parameters (and consequently form a two dimensional space).

26.6 Write down parametric form of a line of solutions to equation (1).

26.7 Write down parametric form of a different line of solutions to equation (1).

26.8 Write down all solutions to equation (1) in parametric form. That is, find $a_x, a_y, a_z, b_x, b_y, b_z$ so that

$$\begin{aligned} x &= a_x t + b_x s \\ y &= a_y t + b_y s \\ z &= a_z t + b_z s \end{aligned}$$

gives all solutions as t, s vary over all of $\mathbb{R}$.

26.9 Write all solutions to equation (1) in vector form.

Arbitrary Lines and Planes

So far, all of our lines and planes have passed through the origin. To produce the equation of an arbitrary line/plane, we first make one of same "slope" that passes through the origin, then we translate it to the appropriate place.

27

We'd like to write the equation of a line L with normal vector $\vec{n} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ that passes through the point $p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

27.1 Give a scalar equation of the line L_2 which is parallel to L but passes through the origin.

27.2 Draw a picture of L and L_2, and find two points that lie on L. Call these points p_1 and p_2.

27.3 Verify the vector $\overline{p_1 p_2}$ is perpendicular to $\vec{n}$.

27.4 What is $\vec{n} \cdot p_1$, $\vec{n} \cdot p_2$, $\vec{n} \cdot p$? Should these values be zero, equal, or different? Explain (think about projections).

27.5 How does the equation $\vec{n} \cdot (\vec{x} - p) = 0$ relate to L?

28

W is the plane with normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and passes through the point $p = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

28.1 Write normal form of W.

28.2 Write vector form of W.

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.

29 Consider the system

$$\begin{aligned} x - y &= 2 \\ 2x + y &= 1 \end{aligned} \tag{2}$$

29.1 Draw the lines in (2) on the same coordinate plane.

29.2 Algebraically solve the system (2). What does this solution represent on your graph?

30 Let L be the line given by $x - y = 2$.

30.1 Write an equation of a line that doesn't intersect L.

30.2 Write an equation of a line that intersects L in

(a) one place.
(b) infinitely many places
(c) exactly two places

or explain why no such equation exists.

30.3 For each equation you came up with, solve the system algebraically. How can you tell algebraically how many solutions there are?

The Row Reduction Algorithm

31 31.1 Solve the system

$$\begin{aligned} x - y - 2z &= -5 \\ 2x + 3y + z &= 5 \\ 0x + 2y + 3z &= 8 \end{aligned} \tag{3}$$

any way you like.

31.2 Use an augmented matrix to solve the system (3).

The system (3) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (3) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

31.3 Rewrite (3) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

31.4 If (x, y, z) is a solution to (3), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

31.5 If (x, y, z) is a solution to (3), is $\vec{p} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

32 Consider the augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -7 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

32.1 Write the system of equations corresponding to A.

32.2 Solve the system of equations corresponding to A.

Infinite Solutions

33 Consider the system

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \end{aligned} \tag{4}$$

33.1 How many solutions does (4) have?

33.2 Write the solutions to (4) in vector form.

33.3 What happens when you use an augmented matrix to solve (4)?

Free Variables

34 Suppose the row-reduced augmented matrix corresponding to a system is

$$B = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

After reducing, we have 1 equation and 2 unknowns, so we can make $2 - 1 = 1$ choices when writing a solution. Let's make the choice $y = t$.

34.1 With the added equation $y = t$, solve the system represented by B.

35 Consider the system given by the augmented matrix

$$C = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

and call the variables in this system x_1, x_2, x_3, x_4, x_5.

35.1 Write the system of equations represented by C.

35.2 Identify how many choices you can make when writing down a solution corresponding to C.

35.3 Add one equation (of the form $x_i = t$ or $x_j = s$, etc.) for each choice you must make when solving the system.

35.4 Write in vector form all solutions to C.

36

36.1 An unknown system U is represented by an augmented matrix with 4 rows and 6 columns. What is the minimum number of free variables solutions to U will have?

36.2 An unknown system V is represented by an augmented matrix with 6 rows and 4 columns. What is the minimum number of free variables solutions to V will have?

37

Def A system is called ***homogeneous*** if all equations equal 0.

Let A be an unknown system of 3 equations and 3 variables and suppose $(x, y, z) = (1, 2, 1)$ and $(x, y, z) = (-1, 1, 1)$ are solutions to A.

37.1 Can you produce another solution to the system?

37.2 Can you produce a solution to the homogeneous version of A (the version of A where every equation equals 0)?

37.3 Suppose when you use an augmented matrix to solve the system A, you only have one free variable. Could A be homogeneous? Can you produce all solutions to the system A?

Rank

Def The ***rank*** of the matrix A is the number of leading ones in the reduced row echelon form of A.

38

38.1 Determine the rank of (a) $\begin{bmatrix}1&1\\2&2\end{bmatrix}$ (b) $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ (c) $\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$ (d) $\begin{bmatrix}3\\3\\2\end{bmatrix}$ (e) $\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

39

Consider the homogeneous system

$$\begin{array}{llll} x & +2y & +z & =0 \\ x & +2y & +3z & =0 \\ -x & -2y & +z & =0 \end{array} \tag{5}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix}1&2&1\\1&2&3\\-1&-2&1\end{bmatrix}$.

39.1 What is $\mathrm{rank}(A)$?

39.2 Give the general solution to (5).

39.3 Are the column vectors of A linearly independent?

39.4 Give a non-homogeneous system with the same coefficients as (5) that has

(a) infinitely many solutions

(b) no solutions.

40

40.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

40.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Span Again

41

Consider the system

$$\begin{array}{llll} x & -y & -z & =0 \\ 0x & +1y & +2z & =0 \\ 3x & -3y & +3z & =0 \end{array} \tag{6}$$

which has the unique solution $(x,y,z)=(0,0,0)$.

41.1 Give vectors $\vec{u},\vec{v},\vec{w}$ so that the system (6) corresponds to the vector equation $x\vec{u}+y\vec{v}+z\vec{w}=\vec{0}$.

41.2 Is $\vec{w}\in\mathrm{span}\,\{\vec{u},\vec{v}\}$? If so, write it as a linear combination of $\vec{u}$ and $\vec{v}$.

The matrix M is the non-augmented matrix corresponding to a homogeneous system of linear equations. M also corresponds to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$. Further, we know

$$\mathrm{rref}(M) = \begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\end{bmatrix}.$$

41.3 Give a solution to the vector equation $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$.

41.4 Is $\vec{c}\in\mathrm{span}\,\{\vec{a},\vec{b}\}$? If so, write it as a linear combination of $\vec{a}$ and $\vec{b}$.

41.5 Do you have enough information to tell if $\{\vec{a},\vec{b}\}$ is linearly independent? Why or why not?

Finding Linearly Independent Subsets

42 Suppose when you use an augmented matrix to solve $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ you have no free variables.

42.1 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

Suppose when you use an augmented matrix to solve $a\vec{u}+b\vec{v}+c\vec{w} = \vec{0}$, the second column corresponds to a free variable.

42.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

42.3 Is $\{\vec{u}, \vec{w}\}$ linearly independent?

42.4 Is $\{\vec{u}, \vec{v}\}$ linearly independent?

Def Given a set of vectors X, a ***maximal linearly independent subset*** of X is a linearly independent subset $V \subseteq X$ with the most possible vectors in it (i.e., if you took any subset of X with more vectors, it would be linearly dependent).

43

43.1 Give a maximal linearly independent subset, T, of $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

43.2 What is the size of T?

44 Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \vec{v}_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and the matrices

$$A = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 2 & -1 & 1 & 2 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(Notice that the columns of A are the vectors $\vec{v}_1, \ldots, \vec{v}_5$)

44.1 Is $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ linearly independent?

44.2 Pick a maximal linearly independent subset of V.

44.3 Pick another (different) maximal linearly independent subset of V.

44.4 Give a basis for $\mathrm{span}(V)$.

44.5 What is the dimension of $\mathrm{span}(V)$?

Matrices

45

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

45.1 Write the shape of the matrices A, B, C (i.e., for each one, write the dimensions in $m \times n$ form).

45.2 List *all* products between the matrices A, B, C that are defined. (Your list will be some subset of AB, AC, BA, CA, BC, CB.)

45.3 Compute AC and CA.

46

46.1 If the matrices X and Y are both square $n \times n$ matrices, does $XY = YX$? Explain.

46.2 If the matrices X and Y are both square $n \times n$ matrices, does $X + Y = Y + X$? Explain.

47

Consider the system

$$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \tag{7}$$

47.1 Find values of a, b, c, d, e, f so that the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

represents the same system as (7).

Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

47.2 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

47.3 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

48

The entries of a matrix are specified by (row,column) pairs of integers. If a_{ij} is the (i, j) entry of a matrix A, we may write $A = [a_{ij}]$.

48.1 Write the 2×2 matrix A with entries $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 7$ and $a_{22} = 9$.

48.2 Let $B = [b_{ij}]$ be the 3×3 matrix where $b_{ij} = i + j$. Write B.

48.3 Let $C = [c_{ij}]$ be the 3×4 matrix where $c_{ij} = 0$ if $i = j$ and $c_{ij} = 1$ if $i \neq j$.

49

Def The ***transpose*** of a matrix $A = [a_{ij}]$ is the matrix $A^T = [a_{ji}]$.

Visually, the transpose of a matrix swaps rows and columns.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

49.1 What is the shape of A and A^T?

49.2 Write down A^T.

B and D are 4×6 matrices and C is a 6×4 matrix.

49.3 Does $(BC)^T = B^T C^T$? Explain.

49.4 Does $(B + D)^T = B^T + D^T$? Explain.

49.5 Compute AA^T and $A^T A$ (where A is the matrix defined earlier). What do you notice?

50

Def A matrix X is called ***symmetric*** if $X = X^T$.

Symmetric matrices have many useful properties, and have deep connections with orthogonality and eigenvectors (which we will get to later on).

50.1 Prove that if W is a square matrix, then $V = W^T W + W + W^T$ is a symmetric matrix.

51

Def A ***zero matrix*** is a matrix whose entries are all zeros. An ***identity matrix*** is a square matrix whose diagonal entries are 1 and non-diagonal entries are 0.

We write the $m \times n$ zero matrix as $0_{m \times n}$ or just 0 if the shape is determined by context. The $n \times n$ identity matrix is notated $I_{n \times n}$ or just I if the shape is determined by context.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

51.1 Write down the 3×3 identity matrix and the 3×3 zero matrix.

51.2 Compute $I_{3\times3}A$, $AI_{3\times3}$, $0_{3\times3}A$, and $A0_{3\times3}$.

51.3 If we were to think of matrices as numbers, what numbers would the zero matrix and the identity matrix correspond to?

52

52.1 Solve the matrix equation

$$I_{4\times4} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

Linear Transformations

53 $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

53.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

53.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

53.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

53.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter clockwise by 90°.

54 $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^3$ stretches in the $\vec{e}_3$ direction by a factor of 2 and contracts in the $\vec{e}_2$ direction by a factor of 3.

54.1 Write a matrix representation of $\mathcal{S}$.

Def A ***Linear Transformation*** is a transformation of vectors that distributes with respect to vector addition and scalar multiplication. That is T is a linear transformation if

$$T(\vec{u}+\vec{v}) = T\vec{u}+T\vec{v} \qquad \text{and} \qquad T(a\vec{v}) = aT\vec{v}$$

for all scalars a.

55

55.1 Classify the following as linear transformation or not

(a) $\mathcal{R}$ from above.

(b) $\mathcal{S}$ from above.

(c) $W : \mathbb{R}^2 \to \mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x^2\\y\end{bmatrix}$.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+2\\y\end{bmatrix}$.

(e) $P : \mathbb{R}^2 \to \mathbb{R}^2$ where $P\begin{bmatrix}x\\y\end{bmatrix} = \operatorname{proj}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$.

It turns out every linear transformation can be written as a matrix (in fact this is why matrix multiplication was invented).

56 Define $\mathcal{P}$ to be projection onto $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$.

56.1 Write down a matrix for $\mathcal{P}$.

56.2 What is the rank of the matrix corresponding to $\mathcal{P}$?

Matrix multiplication was designed to exactly model composition of linear transformations.

56.3 Write down a matrix for $\mathcal{P}$ and for $\mathcal{R}$, the counter-clockwise rotation by 90°.

56.4 Write down matrices for $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{R}\circ\mathcal{P}$.

Def The ***range*** (or ***image***) of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of vectors that T can output. That is,

$$\operatorname{range}(T) = \{\vec{y} \in \mathbb{R}^m : \vec{y} = T\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}.$$

Def The ***null space*** (or ***kernel***) of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of vectors that get mapped to zero under T. That is,

$$\operatorname{null}(T) = \{\vec{x} \in \mathbb{R}^n : T\vec{x} = \vec{0}\}.$$

57 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the vector $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

57.1 What is the range of $\mathcal{P}$?

57.2 What is the null space of $\mathcal{P}$?

58 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

58.1 Show that the null space of T is a subspace.

58.2 Show that the range of T is a subspace.

Matrix Inverses

59

59.1 Apply the row operation $R_3 \to R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

59.2 Apply the row operation $R_3 \to R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

Def An ***elementary matrix*** is the identity matrix with a single row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

59.3 Compute $E_1 A$ and $E_2 A$. How do the resulting matrices relate to row operations?

59.4 Without computing, what should the result of applying the row operation $R_3 \to R_3 - 2R_1$ to E_1 be? Compute and verify.

59.5 Without computing, what should $E_1 E_2$ be? What about $E_2 E_1$? Now compute and verify.

If two square matrices A, B satisfy $AB = I = BA$, we call A and B *inverses*. We notate the inverse of A as A^{-1}.

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

60.1 Which pairs of matrices above are inverses of each other?

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

61.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

61.2 What is $E_2 E_1 B$?

61.3 Find B^{-1}.

61.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

62.1 What is $A^{-1}A$?

62.2 What is $\mathrm{rref}(A)$?

62.3 What is $\mathrm{rref}(C)$?

62.4 Solve the system $A\vec{x} = \vec{b}$.

63.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

$$A^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1/3 & -2/3 \\ 0 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix}$$

64.1 Find $(AB)^{-1}$.

64.2 Solve $AB\vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Algorithms for Computing Inverses

65.1 What is $A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$? (Where A is the matrix from earlier).

If A is invertible (which it happens to be) we could solve the system $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as $\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

65.2 Solve $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$D = [A|I_{3\times 3}]$$

65.3 What is $\mathrm{rref}(D)$?

When dealing with matrices, there are several subspaces we often refer to

- The *row space* of A is the span of the row vectors in A.
- The *column space* of A is the span of the column vectors in A.
- The *null space* of A is the set of vectors $\vec{x}$ so that $A\vec{x} = \vec{0}$.

66 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

66.1 Describe the row space of A.

66.2 Describe the column space of A.

66.3 Is the row space of A the same as the column space of A?

66.4 Describe the set of all vectors perpendicular to the rows of A.

66.5 Describe the null space of A.

67

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

67.1 How does the row space of B relate to the row space of C?

67.2 How does the null space of B relate to the null space of C?

67.3 Compute the null space of B.

68

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \text{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

68.1 How does the column space of P relate to the column space of Q?

68.2 Describe the columns space of P and the column space of Q.

Def The ***nullity*** of a matrix is the dimension of the null space.
The rank-nullity theorem states

$$\text{rank}(A) + \text{nullity}(A) = \#\text{of rows in } A.$$

69 The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

69.1 What is the rank and nullity of A^T?

69.2 What is the rank and nullity of A?

70 Not only is every linear transformation modeled by a matrix, but every matrix models a linear transformation.

70.1 Describe in words the transformation that each matrix represents.

(a) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$

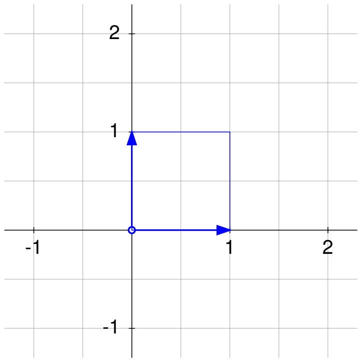
(e) $E = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$

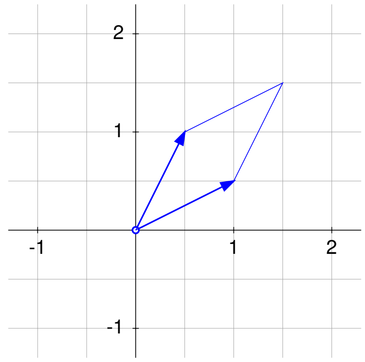
70.2 Which transformations listed above are invertible?

70.3 For each transformation, describe its image as a point, line, plane, or other. How does this relate to the column space?

Eigenvectors

71 The picture shows what the transformation T does to the unit square.





71.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

71.2 Write down a matrix for T.

71.3 Are there any vectors $\vec{v}$ so that $T\vec{v}$ points in the same direction as $\vec{v}$?

Def For a transformation X, an ***eigenvector*** for X is a vector that doesn't change directions when X is applied. That is,

$$X\vec{v} = \lambda\vec{v}$$

for some $\lambda \in \mathbb{R}$. λ is called the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

71.4 Give an eigenvector for T. What is the eigenvalue?

71.5 Can you give another?

We will now develop tools to allow us to compute eigenvalues and eigenvectors.

Def The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$ is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors or the row vectors.

72 Let $\mathcal{S}$ be the unit square in $\mathbb{R}^2$ (The 1×1 square with lower left corner at the origin).

We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}.$$

72.1 Draw $\mathcal{S}$ and $A\mathcal{S}$, the image of the unit square under A.

72.2 Compute the area of $A\mathcal{S}$.

72.3 Compute $\det(A)$.

73 Suppose R is a rotation counterclockwise by $30°$.

73.1 Draw $\mathcal{S}$ and $R\mathcal{S}$.

73.2 Compute the area of $R\mathcal{S}$.

73.3 Compute $\det(R)$.

74 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

74.1 What is $\det(F)$?

75

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \to R_1 + 2R_2$ applied.

75.1 What is $\det(E_f)$?

75.2 What is $\det(E_m)$?

75.3 What is $\det(E_a)$?

75.4 What is $\det(E_f E_m)$?

75.5 What is $\det(4I_{3\times 3})$?

75.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

76

$$U = \begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 3 & -2 & 4\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 4\end{bmatrix}$$

76.1 What is $\det(U)$?

When you row reduce the square matrix V, there is a row of zeros.

76.2 What is $\det(V)$?

P is projection onto the vector $\begin{bmatrix}-1\\-1\end{bmatrix}$.

76.3 What is $\det(P)$?

77 Suppose you know $\det(X) = 4$.

77.1 What is $\det(X^{-1})$?

77.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

77.3 Suppose Y is not invertible. What is $\det(Y)$?

After all this work with determinants, we see that (like dot products) there is a geometric and an algebraic way of thinking about them, and they *determine* if a matrix is invertible.

Eigenvectors and Eigenvalues Cont.

78 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2/3\end{bmatrix}.$$

78.1 Give an eigenvector and a corresponding eigenvalue for A.

79 $B = A - \frac{2}{3}I$.

79.1 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

79.2 What is the dimension of $\mathrm{null}(B)$?

79.3 What is $\det(B)$?

80 $C=\begin{bmatrix}-1&2\\1&0\end{bmatrix}$ and $E_\lambda = C-\lambda I$

80.1 For what values of λ does E_λ have a non-trivial null space?

80.2 What are the eigenvalues of C?

80.3 Find the eigenvectors of C.

81

Def For a matrix A, the ***characteristic polynomial*** of A is

$$\mathrm{char}(A)=\det(A-\lambda I).$$

Let $D=\begin{bmatrix}1&2\\3&0\end{bmatrix}$.

81.1 Compute $\mathrm{char}(D)$.

81.2 Find the eigenvalues of D.

82 Suppose $\mathrm{char}(E)=\lambda(\lambda-2)(\lambda+3)$ for some unknown 3×3 matrix E.

82.1 What are the eigenvalues of E?

82.2 Is E invertible?

82.3 What is $\mathrm{nullity}(E)$, $\mathrm{nullity}(E-3I)$, $\mathrm{nullity}(E+3I)$?

83 Define

$$A=\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}\qquad \vec v_1=\begin{bmatrix}1\\1\\1\end{bmatrix}\qquad \vec v_2=\begin{bmatrix}1\\1\\-2\end{bmatrix}\qquad \vec v_3=\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

and notice that $\vec v_1, \vec v_2, \vec v_3$ are eigenvectors for A.

83.1 Find the eigenvalues of A.

83.2 Find the characteristic polynomial of A.

83.3 Compute $A\vec{w}$ where $w = 2\vec{v}_1 - \vec{v}_2$.

83.4 Compute $A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

83.5 If $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}_V$ is $\vec{x}$ written in the V basis, compute $A\vec{x}$ in the V basis.

84 The transformation P takes vectors in the standard basis and outputs vectors in the V basis.

84.1 Describe in words what P^{-1} does.

84.2 Describe how you can use P and P^{-1} to easily compute $A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

84.3 Can you find a matrix D so that

$$P^{-1}DP = A?$$

84.4 $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}_V$. Compute $A^{100}\vec{x}$.

Def Two matrices A, B are called ***similar*** if there is a matrix P so that

$$A = PBP^{-1}.$$

Similar matrices represent the same transformation but in different bases.

A matrix A is called ***diagonalizable*** if A is similar to a diagonal matrix D.

85 For an $n \times n$ matrix T, suppose its eigenvectors $\{\vec{v}_1, \dots \vec{v}_n\}$ form a basis for $\mathbb{R}^n$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues.

85.1 Is T diagonalizable? If so, explain how to obtain its diagonalized form.

85.2 What if one of the eigenvalues of T is zero? Is T diagonalizable?

85.3 What if the eigenvectors of T did not form a basis for $\mathbb{R}^n$. Would T be diagonalizable?

Def Let A be a matrix with eigenvalues $\{\lambda_1, \dots, \lambda_m\}$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the eigenspace corresponding to λ_i. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

86 Define $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

86.1 Is F diagonalizable? Why or why not?

86.2 What is the geometric and algebraic multiplicity of each eigenvalue of F?

86.3 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?

Orthogonality

Def A set of vectors is ***orthogonal*** if every pair of vectors in the set is orthogonal.

Def A set of vectors is ***orthonormal*** if the set is orthogonal and every vector is a unit vector.

87

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} \qquad \vec{b}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

The matrix $A = [\vec{b}_1|\vec{b}_2]$ takes vectors in the $\mathcal{B}$ basis and rewrites them in the standard basis.

87.1 What does A^{-1} do?

87.2 Find a matrix B that takes vectors in the standard basis and rewrites them in the $\mathcal{B}$ basis.

87.3 Write $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_S$ in the $\mathcal{B}$ basis.

87.4 What is the relationship between A and B?

Def An ***orthogonal*** matrix is a square matrix whose columns are orthonormal (Yes, a better name would be orthonormal matrix, but that is not the term the rest of the world uses).

88 Suppose $X = [\vec{x}_1|\vec{x}_2|\vec{x}_3|\vec{x}_4]$ is an orthogonal matrix.

88.1 What is the shape of X (i.e., it is a what×what matrix)?

88.2 Compute $X^T X$.

88.3 What is X^{-1}?

89

$$Y = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

89.1 Is Y an orthogonal matrix?

89.2 Fix Y so it is an orthogonal matrix. Call the new matrix X.

89.3 Compute X^{-1}.

89.4 Compute Y^{-1}.

89.5 Compute $|\det(X)|$ and $|\det(Y)|$ (the absolute value of the determinant of X and Y).

Matrix equations involving orthogonal matrices are easy to solve because the inverse of an orthogonal matrix is so easy to compute!

90 Let $A = [\vec{a}_1|\vec{a}_2|\vec{a}_3|\vec{a}_4]$ be an orthogonal matrix.

90.1 Explain why $\vec{x} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b} \\ \vec{a}_2 \cdot \vec{b} \\ \vec{a}_3 \cdot \vec{b} \\ \vec{a}_4 \cdot \vec{b} \end{bmatrix}$ is a solution to $A\vec{x} = \vec{b}$.

90.2 Find scalars a, b, c, d so $\vec{b} = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3 + d\vec{a}_4$ (your answers will have variables in them).

Orthogonal matrices also allow us to compute projections quite easily.

Def If V is a subspace of $\mathbb{R}^n$, the ***projection*** (sometimes called the orthogonal projection) of $\vec{x}$ onto V is the closest point in V to $\vec{x}$. We notate the projection of $\vec{x}$ onto V as $\text{proj}_V \vec{x}$.

Projections are normally hard to compute and a priori might require some sort of calculus-style optimization to find. However, from geometry we know that if we travel from $\text{proj}_V \vec{x}$ to $\vec{x}$, we should always trace out a path perpendicular to V. Otherwise, we could find a point in V that was slightly closer to $\vec{x}$, violating the definition of $\text{proj}_V \vec{x}$. Thus, orthogonality will be our savior.

91 Let $\mathcal{S} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis.

91.1 If $\vec{x} = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$, find the projection of $\vec{x}$ onto the xy-plane.

Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

91.2 If $\vec{y} = 3\vec{b}_1 - 2\vec{b}_2 + 2\vec{b}_3$, find the projection of $\vec{y}$ onto span $\{\vec{b}_1, \vec{b}_3\}$.

Suppose $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is a basis for $\mathbb{R}^3$ with

$$\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1 \qquad \vec{c}_1 \cdot \vec{c}_2 = 0 \qquad \vec{c}_1 \cdot \vec{c}_3 = 0 \qquad \vec{c}_2 \cdot \vec{c}_3 = \sqrt{2}/2.$$

91.3 If $\vec{z} = 5\vec{c}_1 + 2\vec{c}_2 - \vec{c}_3$, find the projection of $\vec{z}$ onto span $\{\vec{c}_1, \vec{c}_2\}$.

92 Let's put this all together. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Let $\mathcal{P}$ be the plane defined by

$$0x + y - z = 0.$$

92.1 Write $\mathcal{P}$ in vector form (Hint: think about the vectors listed in the $\mathcal{B}$ basis).

92.2 Find an orthonormal basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ for $\mathbb{R}^3$ so $\mathcal{P} = \text{span}\{\vec{c}_1, \vec{c}_2\}$.

92.3 Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\text{proj}_{\mathcal{P}} \vec{x}$.

Gram-Schmidt Orthogonalization

We've seen how useful orthonormal bases are. The incredible thing is that we can turn any basis into an orthonormal basis through a process called Gram-Schmidt orthogonalization.

93 Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

93.1 Draw $\vec{a}$ and $\vec{b}$ and find $\vec{w} = \text{proj}_{\vec{b}} \vec{a}$.

93.2 Add $\vec{c} = \vec{a} - \vec{w}$ to your drawing. What is the angle between $\vec{c}$ and $\vec{b}$.

93.3 Can you write $\vec{a}$ as the sum of two vectors, one in the direction of $\vec{b}$ and one orthogonal to $\vec{b}$? If so, do it.

94

Let $\vec{a} = \begin{bmatrix}1\\2\\6\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$.

94.1 Write $\vec{a} = \vec{u} + \vec{v}$ where $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is orthogonal to $\vec{b}$.

94.2 Find an orthonormal basis for $\operatorname{span}\{\vec{a}, \vec{b}\}$.

With two vectors, making an orthonormal set without changing the span is quite easy. With more vectors, it is only slightly harder.

Def The ***Gram-Schmidt*** orthogonalization procedure takes in a set of vectors and outputs a set of orthonormal vectors with the same span. The idea is to iteratively produce a set of vectors where each new vector you produce is orthogonal to the previous vectors.

The algorithm is as follows: Let $\{v_1, \ldots, v_n\}$ be a set of vectors. Produce a set $\{v_2', \ldots, v_n'\}$ that is orthogonal to v_1 by subtracting off the respective projections of $v_2, \ldots, v_n$ onto v_1. Next, produce a set $\{v_3'', \ldots, v_n''\}$ orthogonal to both v_1 and v_2' by subtracting off the respective projections onto v_2'. Continue this process until you have a set $V = \{v_1, v_2', v_3'', v_4''', \ldots\}$ that is orthogonal. Finally, normalize V so all vectors have unit length.

95

Let $\vec{x}_1 = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix}2\\1\\0\\1\end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix}2\\2\\1\\2\end{bmatrix}$.

95.1 Use the Gram-Schmidt procedure to find an orthonormal basis for $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

95.2 Find an orthonormal basis $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ for $\mathbb{R}^4$ so that $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Let $R = \begin{bmatrix}1 & -1 & -1 & 1\\2 & 1 & 0 & 1\\2 & 2 & 1 & 2\end{bmatrix}$.

95.3 Find an orthonormal basis for the row space of R.

95.4 Find the null space of R (Hint, you've already done the work, so there is no need to row reduce).

96

Let

$$\vec{y}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix} \qquad \vec{y}_2 = \begin{bmatrix}-1\\-1\\2\end{bmatrix} \qquad \vec{y}_3 = \begin{bmatrix}1\\1\\6\end{bmatrix}.$$

96.1 Find an orthonormal basis $\mathcal{W}$ so that $\operatorname{span}\mathcal{W} = \operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$.

Def The ***orthogonal complement*** of a subspace V is written $V^\perp$ and defined as

$$V^\perp = \{\vec{x} : \vec{x} \text{ is orthogonal to } V\}.$$

96.2 Find the orthogonal complement of span $\mathcal{W}$.

96.3 Write $\vec{v} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ in the form $\vec{v} = \vec{r} + \vec{n}$ where $\vec{r} \in \operatorname{span}\mathcal{W}$ and $\vec{n} \in (\operatorname{span}\mathcal{W})^{\perp}$.

QR Decomposition

Def For a matrix A, we can rewrite $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Writing A as QR is called the QR ***decomposition*** of A.

97 Suppose A, B, C are square matrices and $C = AB$.

97.1 How do the column spaces of A and C relate?

97.2 How do the column spaces of B and C relate?

98 $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms a basis for $\mathbb{R}^3$. When we apply the Gram-Schmidt process to $\mathcal{V}$, we get

$$\begin{aligned} q_1' &= \vec{v} \\ q_2' &= \vec{v}_2 - \tfrac{1}{2}\vec{v}_2 \\ q_3' &= \vec{v}_3 - \vec{v}_1 + 2\vec{v}_2 \end{aligned}$$

form an orthogonal set. Normalizing we get

$$\begin{aligned} \vec{q}_1 &= 2q_1' \\ \vec{q}_2 &= 3q_2' \\ \vec{q}_3 &= \tfrac{1}{2}q_3' \end{aligned}$$

form an orthonormal set.

98.1 Write $\vec{v}_1$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

98.2 Write $\vec{v}_2$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

98.3 Write $\vec{v}_3$ as a linear combination of $\vec{q}_1, \vec{q}_2, \vec{q}_3$.

Define $A = [\vec{v}_1|\vec{v}_2|\vec{v}_2]$ and $Q = [\vec{q}_1|\vec{q}_2|\vec{q}_3]$.

98.4 Find a matrix R so that $A = QR$.

We've just discovered one process to find the QR decomposition of a matrix. It's really as simple as doing Gram-Schmidt and keeping track of your coefficients. Now, we have another way to the matrix equation $A\vec{x} = \vec{b}$. If we do a QR decomposition and exploit the fact that $Q^{-1} = Q^T$, we have

$$A\vec{x} = QR\vec{x} = \vec{b} \qquad \Longrightarrow \qquad R\vec{x} = Q^T\vec{b}$$

and R is a triangular matrix, so we can just do back substitution! (It turns out that if you solve systems this way, there is less rounding error than if you use row reduction.)

Symmetric Matrices

When you're new to Linear Algebra, learning lots of new concepts and algorithms, it's sometimes hard to grasp the significance of certain properties of a matrix.

Symmetric matrices are easy to forget at first, but they have many profound properties (not to mention they are one of the key concepts of Quantum Mechanics).

99 Let A be a symmetric matrix and let $\vec{v}$ be an eigenvector with eigenvalue 3 and $\vec{w}$ be an eigenvector with eigenvalue 4. Note, for this problem, we are thinking of $\vec{v}$ and $\vec{w}$ as column vectors.

99.1 Write $A\vec{v}$, $\vec{v}^T A^T$, $\vec{v}^T A$, $A\vec{w}$, $\vec{w}^T A^T$, and $\vec{w}^T A$ in terms of $\vec{v}$, $\vec{w}$ and scalars.

99.2 How do $\vec{v}^T \vec{w}$ and $\vec{w}^T \vec{v}$ relate?

99.3 What should $\vec{v}^T A\vec{w}$ be in terms of $\vec{v}^T$ and $\vec{w}$? (Note, you could compute $(\vec{v}^T A)\vec{w}$ or $\vec{v}^T (A\vec{w})$. Better do both to be safe).

99.4 What can you conclude about $\vec{v}^T \vec{w}$? How about $\vec{v} \cdot \vec{w}$?

We've just deduced that all eigenspaces of a symmetric matrix are orthogonal! On top of that, symmetric matrices always have a basis of eigenvectors. That means that not only can you always diagonalize a symmetric matrix, but you can *orthogonally* diagonalize a symmetric matrix. (i.e. if A is symmetric, then $A = QDQ^T$ where Q is orthogonal and D is diagonal). This is like the best of all worlds in one!