Learning Objectives

In this tutorial you will examine the pros and cons of different bases.

These problems relate to the following course learning objectives: *Write vectors in different bases and pick an appropriate basis when working on problems*.

Problems

1. Write down the definition of what it means for a set $\mathcal{B}$ to be a *basis* for the subspace V.

2. Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$, and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \right\}$ be ordered bases, and let $\vec{v} = 4\vec{e}_1 - 4\vec{e}_2 + 2\vec{e}_3$.

   (a) Are the vectors in $\mathcal{B}$ and $\mathcal{C}$ written in a basis? If so, which one(s)? What about the vectors in $\mathcal{E}$?

   (b) Compute $[\vec{v}]_{\mathcal{E}}$, $[\vec{v}]_{\mathcal{B}}$, and $[\vec{v}]_{\mathcal{C}}$.

   (c) Compute $[7\vec{v}]_{\mathcal{E}}$, $[7\vec{v}]_{\mathcal{B}}$, and $[7\vec{v}]_{\mathcal{C}}$. *Hint: look at what you've already done; you might not have to do much more work.*

   (d) Suppose, during a scientific experiment, you repeatedly measure multiples of the vector $\vec{v}$. Which basis would you prefer to write down your measurements in? Why?

3. Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$ and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter-clockwise by $90°$.

   (a) Find a matrix, P, for R in the basis $\mathcal{E}$. That is $P[\vec{x}]_{\mathcal{E}} = [R\vec{x}]_{\mathcal{E}}$.

   (b) Find a matrix, Q, for R in the basis $\mathcal{B}$. That is $Q[\vec{x}]_{\mathcal{B}} = [R\vec{x}]_{\mathcal{B}}$.

4. In math, a real number can have infinitely many digits. In a computer, however, there is limited space, so a computer will *truncate* numbers it stores.[1] Consider the bases

$$\mathcal{E}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{E}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ .1 \end{bmatrix} \right\} \qquad \mathcal{E}_3 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ .01 \end{bmatrix} \right\} \qquad \mathcal{E}_4 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ .001 \end{bmatrix} \right\} \qquad \cdots$$

   and the vector $\vec{v} = \begin{bmatrix} 1 \\ 0.1234567\ldots \end{bmatrix}$, written in the standard basis. Suppose your computer can only store two decimal places (after the decimal point) when it stores a number. Which basis should you use to represent $\vec{v}$? Why?

[1] *Truncate* means erasing digits after a certain point. So $3.14159\ldots$ might become 3.141.

1. $\mathcal{B}$ is a basis for V if it is a linearly independent set of vectors that spans V.

2. (a) Yes. The vectors in $\mathcal{B}$ and $\mathcal{C}$ are written in the standard basis. The vectors in $\mathcal{E}$ are not written in any basis—they are referred to directly by name.

   (b) $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$, $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, and $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$.

   (c) $[7\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 28 \\ -28 \\ 14 \end{bmatrix}$, $[7\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 14 \end{bmatrix}$, and $[7\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 56 \\ 0 \\ -14 \end{bmatrix}$.

   (d) I would pick basis $\mathcal{B}$ since every multiple of $\vec{v}$ written in the basis $\mathcal{B}$ takes the form $\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$, so there are fewer numbers to write down.

3. (a) $[R]_{\mathcal{E}} = P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (b) $[R]_{\mathcal{B}} = Q = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$.

4. Computing, we see

$$[\vec{v}]_{\mathcal{E}_1} = \begin{bmatrix} 0.8765\ldots \\ 0.1234\ldots \end{bmatrix} \quad [\vec{v}]_{\mathcal{E}_2} = \begin{bmatrix} -0.2345\ldots \\ 1.2345\ldots \end{bmatrix} \quad [\vec{v}]_{\mathcal{E}_3} = \begin{bmatrix} -11.3456\ldots \\ 12.3456\ldots \end{bmatrix} \quad [\vec{v}]_{\mathcal{E}_4} = \begin{bmatrix} -122.4567\ldots \\ 123.4567\ldots \end{bmatrix}$$

   Given the pattern, it might be best to pick $\mathcal{E}_i$ for i as large as possible. However, if the computer were to also store the basis in standard form, $\mathcal{E}_i$ for $i \geq 4$ couldn't be stored. Thus, the most accuracy we could get is by using $\mathcal{E}_3$.

Learning Objectives

Students need to be able to...

- Write vectors in multiple bases
- Write matrices for linear transformations in multiple bases
- Explain the pros and cons for one basis over another in a particular problem

Context

In class, students have gone over bases, subspaces, dimension, linear transformations, and matrix transformations. In this class, we only work with $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$, so we won't be talking about bases for abstract spaces.

Bases are especially hard for students—they're used to thinking of lists of numbers as the actual vectors instead of representations of the vectors. They need a lot of hand-holding to make this transition.

What to Do

Start by explaining to students that bases and linear transformations are *the* two big ideas from linear algebra and that today we're going to focus on bases. Further tell them that up till now we've been considering vectors and lists of numbers as interchangeable, but now we think of lists of numbers as a *representation* of a vector instead of a true vector (kind of like the symbol "4" is not literally four, it is a graphical representation of the abstract idea of the number four).

Proceed as usual, asking students to form small groups and start working. Again, this tutorial starts with a definition question, which they all need to actually write out. The meat of this tutorial is problem #2. When most groups are on #2(c), have a class discussion on 2(a) and 2(b), to make sure everyone is on the same page.

Remember, the point of this tutorial (and all tutorials) is not to get through all the problems. It is to get practice with difficult and new mathematics. Don't cut thinking time short trying to get through more problems!

8 minutes before the end of class, pick a problem that most groups have started, or a problem that they've finished but you think needs more attention, and do that problem as a wrap-up.

Notes

1. To talk about representations in a basis we technically need *ordered* sets. In this class, when writing a set down and declaring it a basis, we imply that the order of the basis vectors is the left-to-right order that they're written in. The student's won't think about regular sets lacking order, so don't bring it up unless they ask.

2. We use the notation $[\vec{v}]_{\mathcal{X}}$ to mean the list of numbers representing $\vec{v}$ in the basis $\mathcal{X} = \{\vec{x}_1, \vec{x}_2, \ldots\}$. We use $\begin{bmatrix}1\\2\\\vdots\end{bmatrix}_{\mathcal{X}}$ to mean the linear combination $1\vec{x}_1 + 2\vec{x}_2 + \cdots$. Thus $[[\vec{v}]_{\mathcal{X}}]_{\mathcal{X}} =$

$\vec{v}$.

3. Lists of number that are not subscripted and are treated as vectors have an implicit subscript of $\mathcal{E}$, the standard basis. Students may be uncomfortable that sometimes we demand subscripts and sometimes we don't. It should always be clear from context whether we are talking about a list of numbers or a vector written in the standard basis, and on tests we will be very clear.

4. For 2(c), make sure that the shortcut comes out that $[7\vec{v}]_{\mathcal{X}} = 7[\vec{v}]_{\mathcal{X}}$ and, in fact, changing basis is a linear operation.

5. Problems 2(d) and 4 ask for value judgments. There's no "right" answer to these questions, but some answers are better reasoned than others. Hold the students accountable for coming up with good reasons.

6. No one will get to #4, but if they do, encourage them to use a calculator to speed things along.