Learning Objectives

In this tutorial you will practice using the determinant to answer geometric and algebraic questions.

These problems relate to the following course learning objectives: *Translate between algebraic and geometric viewpoints to solve problems*, and *use determinants to solve problems*.

Problems

1. Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{e} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

   A gem expert has found several crystals in the shape of parallelepipeds (the three-dimensional analogue of a parallelogram). Carefully measuring, she discovers: crystal X has edges described by $\vec{a}$, $\vec{b}$, $\vec{c}$; crystal Y has edges described by $\vec{c}$, $\vec{d}$, $\vec{e}$; and crystal Z has edges described by $\vec{a}$, $\vec{d}$, $\vec{e}$.

   (a) Which crystal has the largest volume?

   (b) Which crystal is the "pointiest"? Justify your conclusion.

2. Use row reduction to solve the system

$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned}$$

   recording all your steps. If $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0$, which row-reduction step fails? Why?

3. Let $\vec{a}$ and $\vec{d}$ be as in problem 1, and let $\vec{f}$ be a unit vector. What is the largest possible volume of the parallelepiped with edges $\vec{a}$, $\vec{d}$, $\vec{f}$?

4. For $\vec{u}, \vec{v} \in \mathbb{R}^3$, the *cross product* of $\vec{u}$ and $\vec{v}$, written $\vec{u} \times \vec{v}$, is a vector orthogonal to $\vec{u}$ and $\vec{v}$ whose length is the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$ and such that $\det([\vec{u}|\vec{v}|\vec{u} \times \vec{v}]) \geq 0$.

   Use the definition[1] of the cross product to find the cross product of the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. *Hint: you already know how to use dot products to find the angle between vectors.*

[1] If you know a formula for the cross product, feel free to use it to check your work, but the definition given above is not stated in terms of a formula.

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1. (a) We can use the determinant to compute the volume of each gem.

$$\mathrm{vol}(X) = |\det([\vec{a}|\vec{b}|\vec{c}])| = 5 \qquad \mathrm{vol}(Y) = |\det([\vec{c}|\vec{d}|\vec{e}])| = 2 \qquad \mathrm{vol}(Z) = |\det([\vec{a}|\vec{d}|\vec{e}])| = 1$$

So gem X has the largest volume.

(b) One way to define "pointiness" of a parallelepiped is as the ratio between the volume of the parallelepiped if it were a rectangular prism and its true volume. A sharp "point" of the crystal will cause it to have less volume than a point that is close to 90°.

Computing,

$$\|\vec{a}\| = \sqrt{2} \qquad \|\vec{b}\| = \sqrt{6} \qquad \|\vec{c}\| = \sqrt{11} \qquad \|\vec{d}\| = \sqrt{3} \qquad \|\vec{e}\| = 1,$$

so the pointiness ratios are

$$\mathrm{rat}(X) = \frac{5}{\sqrt{2}\sqrt{6}\sqrt{11}} \approx 0.435 \qquad \mathrm{rat}(Y) = \frac{2}{\sqrt{11}\sqrt{3}(1)} \approx 0.348 \qquad \mathrm{rat}(Z) = \frac{1}{\sqrt{2}\sqrt{3}(1)} \approx 0.408.$$

Using this measure, gem Y would be the pointiest.

2. After row reduction we get

$$x = \frac{d}{ad - bc} \qquad y = \frac{-c}{ad - bc}.$$

In both cases, we're dividing by $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0$, which is not allowed.

3. The largest possible volume occurs when $\vec{f}$ is orthogonal to $\vec{a}$ and $\vec{d}$. By inspection we see that $\vec{f} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Since $\vec{f}$ is a unit vector, we know $\vec{f} = \pm\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$. Computing,

$$\det([\vec{a}|\vec{d}|\vec{f}]) = \pm\frac{2}{\sqrt{2}},$$

and so $2/\sqrt{2}$ is the largest possible volume.

4. Let $\vec{C} = \vec{u} \times \vec{v}$. Since $\vec{u} \perp \vec{v}$, we know $\|\vec{C}\| = \|\vec{u}\|\|\vec{v}\| = \sqrt{42}$. Further, $\vec{C}$ is orthogonal to $\vec{u}$ and $\vec{v}$, so $\vec{C} \in \mathrm{null}(M)$ where M is the matrix with rows $\vec{u}$ and $\vec{v}$. Row reducing,

$$\mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix},$$

and so $\vec{C} = t\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$. Since $\|\vec{C}\| = \sqrt{42}$, we in fact see $\vec{C} = \pm\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$. Finally, computing

$$\det([\vec{u}|\vec{v}|\vec{C}]) = +42$$

when $\vec{C} = -\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$ and so $\vec{u} \times \vec{v} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}$.

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Students need to be able to...

- Apply determinants to solve geometric questions
- Use linear algebra tools to answer open-ended questions (i.e., define "pointiness")
- Work with new definitions (cross product)

Context

At this point, all students have seen determinants in class; additionally, they have had online homework where they had to compute 2×2 and 3×3 determinants.

In this class, determinants are defined as the oriented volume of the image of the unit cube (for a transformation) and the oriented volume of the parallelepiped given by the column/row vectors (for a matrix). We do *not* define the determinant in terms of cofactors, so don't expect students to know this or use it. However, they must know know to compute 2×2 and 3×3 determinants.

What to Do

Start the tutorial by stating the day's learning objectives. Re-emphasize that the goal of tutorial is not to get through many problems, but to spend quality time thinking and applying math tools we've learned.

Have students start in groups on #1. When most groups have answers for part (a), have a class discussion about how they found the volumes (this should be straightforward). Then, have groups resume working on part (b). Part (b) will be very hard. There isn't a "right" way to do this, and they'll need to be creative. After students have adequately struggled, you might choose to have a group discussion about ideas for (b). Some might come up with ideas involving volumes, others might want to use dot products to find angles. Get two different ideas on the board and then let students pick which one to implement.

It is likely that #1 takes the whole classtime. If you do have extra time, continue as usual, letting students work in small groups on #2 and then having a mini-discussion when half the groups have figured it out.

6 minutes before the end of class, pick a suitable problem to do as a wrap-up.

Notes

- Students should know how to compute 2×2 and 3×3 determinants via an algorithm, but they will be slow at it. Let them take their time—today is their practice day!
- For #1 the idea of "pointiness" isn't precise and there are several valid metrics. First, have them use pens/pencils as vectors and actually have them orient them the way the crystal is oriented. Can they judge based just on their "3d diagram" which is pointiest?

  After they have an estimate, try to get them to formalize their intuition. If they need a hint, have them think about angles, and how could they tell whether an angle is more acute or not.

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- For #3, the big barrier will be figuring out that $\vec{f}$ should be orthogonal to $\vec{a}$ and $\vec{d}$. Once they figure that out, they might be stuck on how to find such a vector. Remind them that they can set up a system of linear equations if they aren't able to guess a solution.

- #4 would normally be a nightmare, but the problem is set up to make the numbers easy. They might not notice that $\vec{u}$ and $\vec{v}$ form the sides of a rectangle, so they won't need trigonometry to compute its area.

  The second stumbling block will be how to get a hold of a vector orthogonal to both $\vec{u}$ and $\vec{v}$. This one is hard to get by guessing. Have them write equations for what they need.

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