Inquiry Based Vector Calculus

About the Document

This document was originally designed in the spring of 2016 to guide students through an ten week Linear Algebra course (Math 281-3) at Northwestern University.

A typical class day using the problem-sets:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

4. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

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Sets of Vectors

1 Write the following sets in set-builder notation

1.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

1.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Unions & Intersections (DEFINITION)

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

2 Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

2.1 $X \cup Y$

2.2 $X \cap Y$

2.3 $X \cup Y \cup Z$

2.4 $X \cap Y \cap Z$

3 Draw the following subsets of $\mathbb{R}^2$.

3.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.3 $J = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

3.4 $V \cup H$.

3.5 $V \cap H$.

3.6 Does $V \cup H = \mathbb{R}^2$?

Linear Combinations, Span, and Linear Independence

Linear Combination (DEFINITION)

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.

4 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

4.1 Write the coordinates of $\vec{w}$.

4.2 Draw a picture with $\vec{w}$, $\vec{v}_1$, and $\vec{v}_2$.

4.3 Is $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.4 Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.5 Is $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

4.7 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Span

DEF

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

5 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

5.1 Draw span $\{\vec{v}_1\}$.

5.2 Draw span $\{\vec{v}_2\}$.

5.3 Describe span $\{\vec{v}_1, \vec{v}_2\}$.

5.4 Describe span $\{\vec{v}_1, \vec{v}_3\}$.

5.5 Describe span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

6 Give an example of:

6.1 two vectors in $\mathbb{R}^3$ that span a plane;

6.2 two vectors in $\mathbb{R}^3$ that span a line;

6.3 four vectors in $\mathbb{R}^3$ that span a plane;

6.4 a set of 50 vectors in $\mathbb{R}^3$ whose span is the line through the origin and the point $\begin{bmatrix}1\\2\\-3\end{bmatrix}$.

In some sets, every vector is essential for computing a span. In others, there are "excess" vectors. This leads us to the concept of linear independence.

DEFINITION

We say $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\},$$

and a set is ***linearly independent*** otherwise.

7 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

7.1 Describe span $\{\vec{u}, \vec{v}, \vec{w}\}$.

7.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

7.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

7.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

Trivial Linear Combination

DEF

We say a linear combination $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ is ***trivial*** if $a_1 = a_2 = \cdots = a_n = 0$.

8 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

8.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

8.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

DEF $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector.

9

9.1 Explain how this new definition implies the old one.

9.2 Explain how the old definition implies this new one.

Since have old def $\implies$ new def, and new def $\implies$ old def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as new def $\iff$ old def).

10 Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

10.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

10.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

10.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

10.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Subspaces and Bases

DEFINITION A ***subspace*** $V \subseteq \mathbb{R}^n$ is a subset such that

(i) $\vec{u}, \vec{v} \in V$ implies $\vec{u} + \vec{v} \in V$.

(ii) $\vec{u} \in V$ implies $k\vec{u} \in V$ for all scalars k.

Subspaces give a mathematically precise definition of a "flat space through the origin."

11 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

11.1 $A = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ for some $a \in \mathbb{Z}\}$.

11.2 $B = \{\vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$.

11.3 $C = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.4 $D = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.5 $E = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.6 $F = \{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}\}$.

11.7 $G = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

11.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

DEF A ***basis*** for a subspace V is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = V$.

12 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec{u},\vec{v},\vec{w}\}$.

12.1 Describe V.

12.2 Is $\{\vec{u},\vec{v},\vec{w}\}$ a basis for V? Why or why not?

12.3 Give a basis for V.

12.4 Give another basis for V.

12.5 Is $\operatorname{span}\{\vec{u},\vec{v}\}$ a basis for V? Why or why not?

DEF The ***dimension*** of a subspace V is the number of elements in a basis for V.

12.6 What is the dimension of V?

13 Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}7\\8\\8\end{bmatrix}$ and let $P = \operatorname{span}\{\vec{a},\vec{b}\}$ and $Q = \operatorname{span}\{\vec{b},\vec{c}\}$.

13.1 Give a basis for and the dimension of P.

13.2 Give a basis for and the dimension of Q.

13.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

13.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.