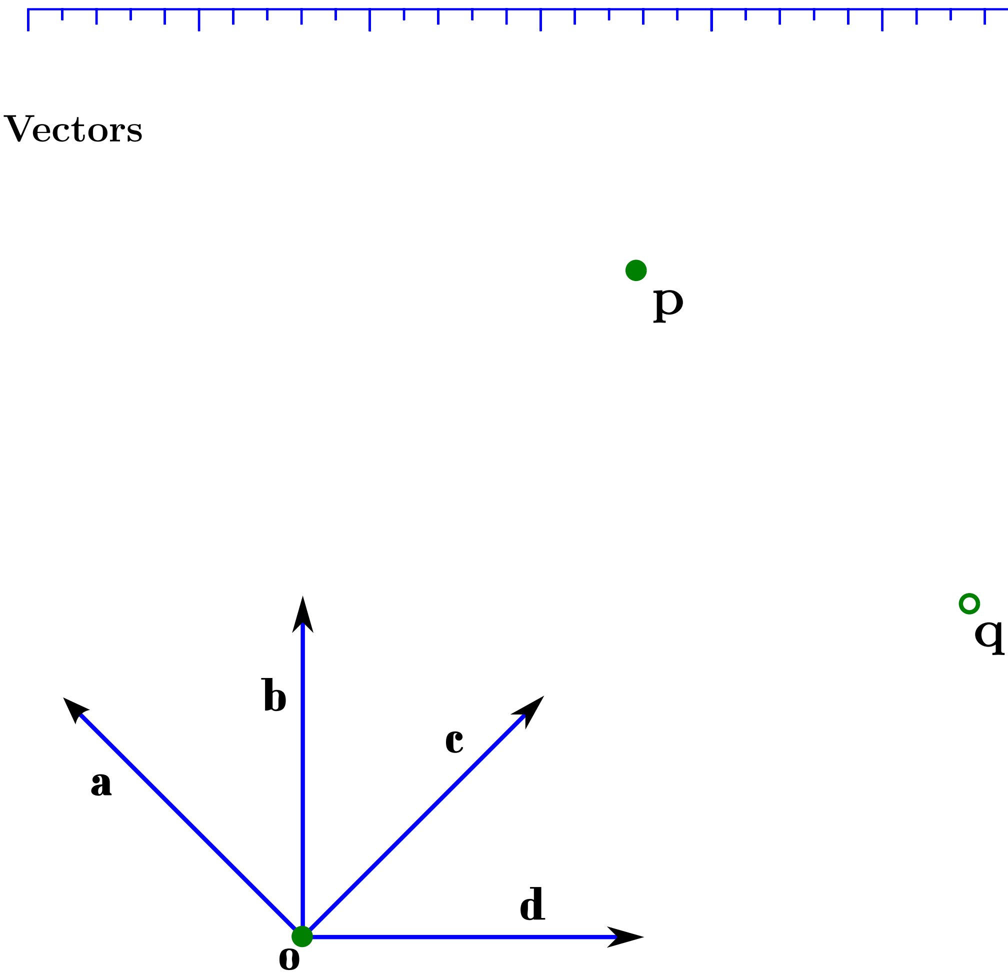
Vectors



Notice that all arrows in this diagram are the same length. We will call this length a *unit*.

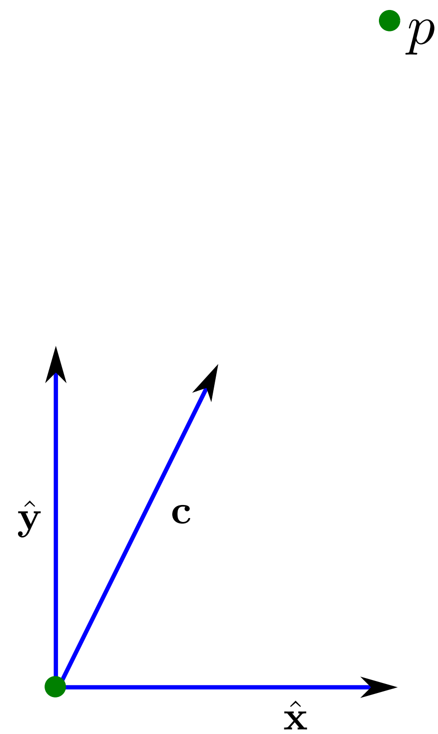
1.1 Give directions from **o** to **p** of the form "Walk ___units in the direction of arrow ___, then walk ___units in the direction of arrow ___."

1.2 Can you give directions with the two arrows you haven't used? Give such directions, or explain why it cannot be done.

1.3 Give directions from **o** to q.

1.4 Can you give directions from **o** to q using **c** and **a**? Give such directions, or explain why it cannot be done.

Unit Vectors



We are going to start using a more mathematical notation for giving directions. Our directions will now look like

$$p = \underline{\hspace{1.5em}}\,\hat{\mathbf{x}} + \underline{\hspace{1.5em}}\,\hat{\mathbf{y}}$$

which is read as "To get to p (=) go ___units in the direction $\hat{\mathbf{x}}$ then (+) go ___units in the direction $\hat{\mathbf{y}}$."

2.1 What is the difference between $p = \underline{\hspace{1.5em}}\,\hat{\mathbf{x}} + \underline{\hspace{1.5em}}\,\hat{\mathbf{y}}$ and $p = \underline{\hspace{1.5em}}\,\hat{\mathbf{y}} + \underline{\hspace{1.5em}}\,\hat{\mathbf{x}}$? Can they both give valid directions?

2.2 (a) Give directions to p using the new notation.
 (b) Give directions to p using $\mathbf{c}$.
 (c) What is the distance from $\mathbf{o}$ to p in units?

2.3 (a) $r = 1\mathbf{c}$. Give directions from $\mathbf{o}$ to r using $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.
 (b) What is the distance from $\mathbf{o}$ to r?

2.4 (a) $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$; find the exact distance from $\mathbf{o}$ to q.
 (b) $s = 2\hat{\mathbf{x}} + \mathbf{c}$; find the exact distance from $\mathbf{o}$ to s.

We've been learning vector addition. $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are called the *standard basis vectors* for $\mathbb{R}^2$ (the plane). Everyone has agreed that if we give directions from the origin to some point and we don't specify otherwise, we will give directions in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

Column Vector Notation

We previously wrote $q = -2\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$. In column vector notation we write

$$q = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

We may call q either a *vector* or a *point*. If we call q a vector, we are emphasizing that q gives direction of some sort. If we call q a point, we emphasize that q is some absolute location in space. (What's the philosophical difference between a location in space and directions from the origin to said location?)
$r = 1\mathbf{c}$; $s = 2\hat{\mathbf{x}} + \mathbf{c}$.

3.1 Write r and s in column vector form.

Vector Length

The *length* or *norm* of a vector $\vec{w}$ is denoted $\|\vec{w}\|$ and is the distance from $\mathbf{o}$ to the point you end up at if you follow $\vec{w}$'s instructions.

4.1 Find $\|\vec{a}\|$, $\|\vec{b}\|$, $\|\vec{c}\|$ where

- (a) $\vec{a} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$
- (b) $\vec{b} = 2\vec{a}$
- (c) $\vec{c} = -\vec{a}/2$

4.2 $\hat{\mathbf{z}}$ points perpendicular to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ into the 3rd dimension.
Let $\vec{v} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$ and $\vec{w} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}}$.

- (a) Write $\vec{v}$ in terms of $\vec{w}$ and $\hat{\mathbf{z}}$ and draw a picture showing the relationship between the three vectors (3-d pictures are a hard but essential skill in this course).
- (b) Find $\|\vec{w}\|$ and $\|\vec{v}\|$. (Hint, look at your picture and see if there are any right triangles to exploit).

4.3 Let $\vec{u} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$.

- (a) Find $\|\vec{u}\|$.
- (b) Find $\|k\vec{u}\|$ where k is some unknown constant.
- (c) What value(s) of k makes $\|k\vec{u}\| = 1$?
- (d) Write down a vector in column form that points in the same direciton as $\vec{u}$ and has length 1.

Unit Vectors

Vectors that have length 1 are called *unit vectors*.

5.1 $\vec{a} = -\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$. Find a unit vector in the direction of $\vec{a}$, and call this vector $\vec{u}$ (u for $u$nit, get it?).

5.2 Write $\vec{a}$ in terms of $\vec{u}$. Does $\|\vec{a}\|$ show up in your formula at all?

5.3 Write $3\vec{u}$ in column vector form and find its length.

5.4 Write $7.5\vec{u}$ in column vector form and find its length.

5.5 $\vec{v}$ is a different unit vector (I won't tell you its exact form). Find $\|9\vec{v}\|$ Why do we like unit vectors so much?

Dot Product

The *dot product* is incredible because it is easy to compute and has a useful geometric meaning.

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the dot product of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

We also have a geometry-related formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

6.1 Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

6.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

6.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

6.4 $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

We've explored how dot products relate to angles, but how do they relate to lengths?

7.1 Let $\vec{a} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$

(a) Find $\|\vec{a}\|$ and $\vec{a} \cdot \vec{a}$. How do the two quantities relate?

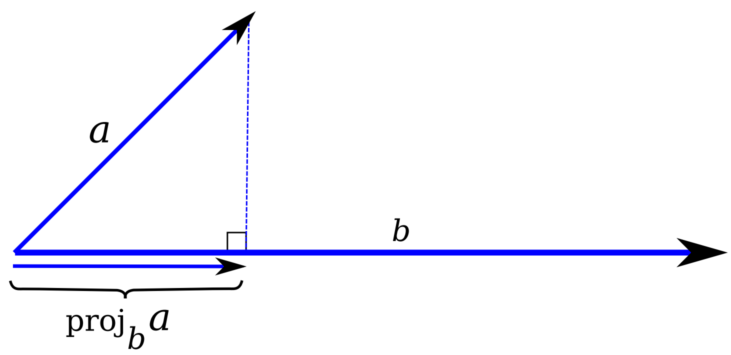
(b) Write down an equation for the length of a vector $\vec{v}$ in terms of dot products.

7.2 Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}$, and find $\|\vec{b}\|$. Did you know how to find 4-d lengths before?

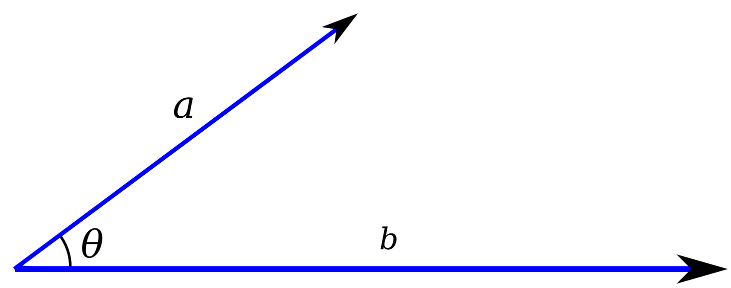
7.3 Suppose $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ for $x, y \in \mathbb{R}$. Could $\vec{u} \cdot \vec{u}$ be negative? Compute $\vec{u} \cdot \vec{u}$ algebraically and use this to justify your answer.

Projections

Projections (sometimes called orthogonal projections) are a way to measure how much one vector points in the direction of another.



The projection of $\vec{a}$ onto $\vec{b}$ is written $\text{proj}_{\vec{b}}\vec{a}$ and is a vector in the direction of $\vec{b}$.



8.1 In this picture $\|\vec{a}\| = 4$ and $\theta = \pi/6$. Find $\|\text{proj}_{\vec{b}}\vec{a}\|$.

8.2 If $\vec{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, write down $\text{proj}_{\vec{b}}\vec{a}$ in column vector form. How do the coordinates relate to $\|\text{proj}_{\vec{b}}\vec{a}\|$?

8.3 Consider $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Compute $\text{proj}_{\hat{\mathbf{x}}}\vec{u}$ and $\text{proj}_{\hat{\mathbf{y}}}\vec{u}$. How do these projections relate to the coordinates of $\vec{u}$? What can you say in general about projections onto $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$?

$$\vec{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

9.1 Find θ, the angle between $\vec{w}$ and $\vec{v}$.

9.2 Use θ to compute $\text{proj}_{\vec{v}}\vec{w}$ and $\text{proj}_{\vec{w}}\vec{v}$.

9.3 Write down a formula for $\text{proj}_{\vec{b}}\vec{a}$ where $\vec{a}$ and $\vec{b}$ are arbitrary vectors.

10.1 For the arbitrary vector $\vec{a}$, what is $\text{proj}_{3\vec{a}}\vec{a}$?

10.2 If $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) vectors, what is $\text{proj}_{\vec{b}}\vec{a}$ $\text{proj}_{\vec{a}}\vec{b}$?

Lines, Planes, Normals, Equations

11.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors perpendicular to it.

11.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is perpendicular to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

11.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for a line. This is called *normal form*

A *normal vector* to a line is one that is orthogonal to it.

11.4 Rewrite the line $\vec{u} \cdot \vec{x} = 0$ in $y = mx + b$ form and verify it matches the line you drew above.

We can also write a line in *parametric form* by introducing a parameter that traces out the line as the parameter runs over all real numbers.

12.1 Draw the line L with x, y coordinates given by

$$\begin{aligned} x &= t \\ y &= 2t \end{aligned}$$

as t ranges over $\mathbb{R}$.

12.2 Write the line $\vec{u} \cdot \vec{x} = 0$ (where $\vec{u}$ is the same as before) in parametric form.

Vector form is the same as parametric form but written in vector notation. For example, the line L from earlier could be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

13.1 Write $\vec{u} \cdot \vec{x} = 0$ in vector form. That is, find a vector $\vec{v}$ so the line $\vec{u} \cdot \vec{x} = 0$ can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\vec{v}$$

as t ranges over $\mathbb{R}$.

13.2 What is $\vec{v} \cdot \vec{u}$? Why? Will this always happen?

Moving to Planes

When solving equations, sometimes we get to make choices. For example, if $x + 2y = 0$, we can find solutions by fixing either x or y and solving for the other. e.g., if $x = 2$, then $y = -1$ nad if $y = 3$ then $x = -6$.

14.1 Write down three solutions $\vec{a}$, $\vec{b}$, $\vec{c}$ to

$$2x + y - z = 0. \tag{1}$$

14.2 Is $2\vec{a} - \vec{b}$ a solution? Is any linear combination of solutions a solution? Justify why or why not.

14.3 Rewrite equation (1) in normal form $\vec{n} \cdot \vec{x} = 0$ where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

14.4 What do you notice about the angle between solutions to equation (1) and $\vec{n}$?

14.5 You've already seen that scalars come out of dot products (e.g., $\vec{a} \cdot (3\vec{b}) = 3(\vec{a} \cdot \vec{b})$. Use this combined with normal form to prove a linear combination of solutions is still a solution.

When writing down solutions to equation (1), you got to choose two coordinates before the remaining coordinate became determined. This means the solutions have two parameters (and consequently form a two dimensional space).

14.6 Write down parametric form of a line of solutions to equation (1).

14.7 Write down parametric form of a different line of solutions to equation (1).

14.8 Write down all solutions to equation (1) in parametric form. That is, find $a_x, a_y, a_z, b_x, b_y, b_z$ so that

$$\begin{aligned} x &= a_x t + b_x s \\ y &= a_y t + b_y s \\ z &= a_z t + b_z s \end{aligned}$$

gives all solutions as t, s vary over all of $\mathbb{R}$.

14.9 Write all solutions to equation (1) in vector form.

Arbitrary Lines and Planes

So far, all of our lines and planes have passed through the origin. To produce the equation of an arbitrary line/plane, we first make one of same "slope" that passes through the origin, then we translate it to the appropriate place.

We'd like to write the equation of a line L with normal vector $\vec{n} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ that passes through the point $p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

15.1 Write normal form of the line L_2 which is parallel to L, but passes through the origin.

15.2 Draw a picture of L and L_2, and find two points that lie on L. Call these points p_1 and p_2.

15.3 Verify the vector $\overrightarrow{p_1 p_2}$ is perpendicular to $\vec{n}$.

15.4 What is $\vec{n} \cdot p_1$, $\vec{n} \cdot p_2$, $\vec{n} \cdot p$? Should these values be zero, equal, or different? Explain (think about projections).

15.5 How does the equation $\vec{n} \cdot (\vec{x} - p) = 0$ relate to L?

W is the plane with normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and passes through the point $p = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

16.1 Write normal form of W.

16.2 Write vector form of W.

Systems of Linear Equations

Linear equations are equations only involving variables, multiplication by constants, and addition/subtraction. *Systems* of equations are sets of equations that share common variables.
Consider the system

$$\begin{aligned} x - y &= 2 \\ 2x + y &= 1 \end{aligned} \tag{2}$$

17.1 Draw the lines in (2) on the same coordinate plane.

17.2 Algebraically solve the system (2). What does this solution represent on your graph?

Let L be the line given by $x - y = 2$.

18.1 Write an equation of a line that doesn't intersect L.

18.2 Write an equation of a line that intersects L in

(a) one place.

(b) infinitely many places

(c) exactly two places

or explain why no such equation exists.

18.3 For each equation you came up with solve the system algebraically. How can you tell algebraically how many solutions there are?

The Row Reduction Algorithm

19.1 Solve the system

$$\begin{aligned} x - y - 2z &= -5 \\ 2x + 3y + z &= 5 \\ 0x + 2y + 3z &= 8 \end{aligned} \tag{3}$$

any way you like.

19.2 Use an augmented matrix to solve the system (3).

The system (3) can be interpreted in two ways (and switching between these interpretations when appropriate is one of the most powerful tools of Linear Algebra). We can think of solutions to (3) as the intersection of three planes, or we can interpret the solution as coefficients of a linear combination.

19.3 Rewrite (3) as a vector equation of the form

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{p}$$

where x, y, z are interpreted as scalar quantities.

19.4 If (x, y, z) is a solution to (3), explain how to get from the origin to $\vec{p}$ using only $\vec{v}_1, \vec{v}_2, \vec{v}_3$.