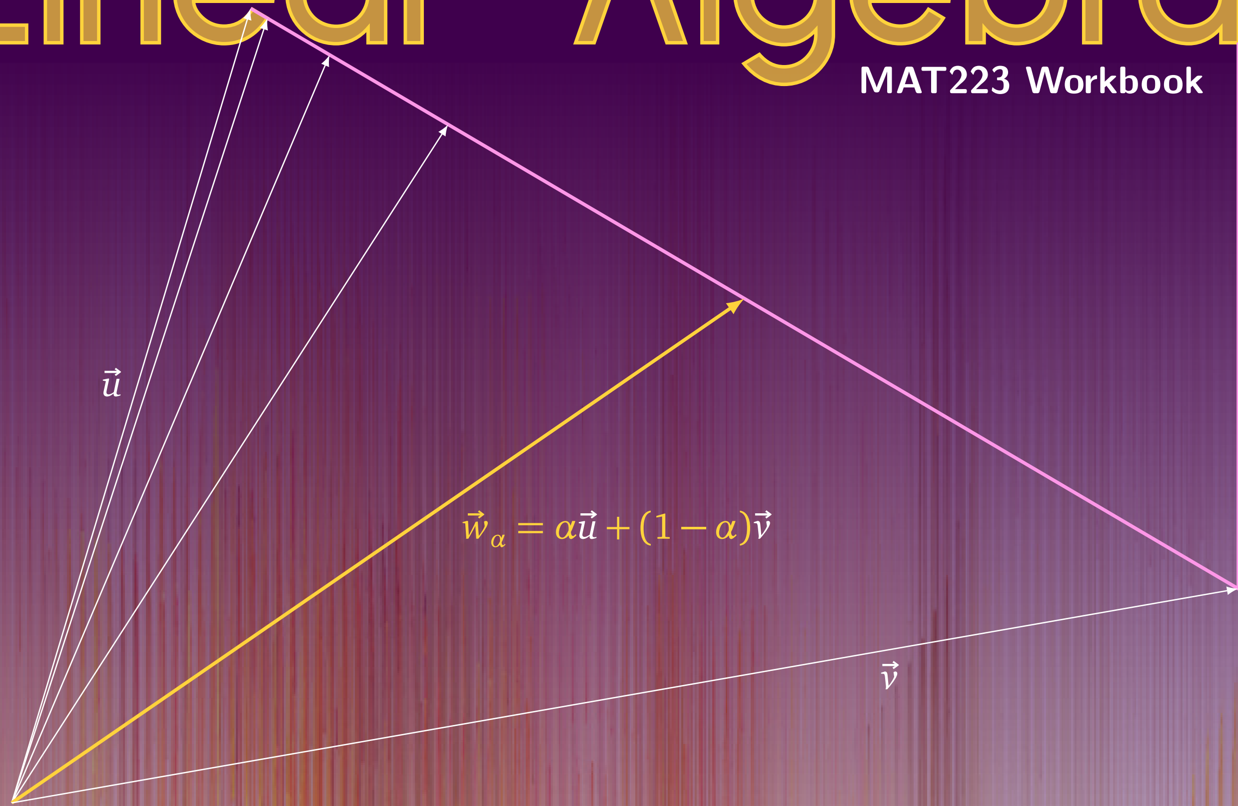
Linear Algebra

MAT223 Workbook



Jason Siefken

Inquiry Based Linear Algebra

About the Document

This document is a hybrid of many linear algebra resources, including those of the IOLA (Inquiry Oriented Linear Algebra) project and Jason Siefken's IBL Linear Algebra project.

This document is a mix of short problems and more involved exploratory question. A typical class day looks like:

1. **Introduction by instructor.** This may involve giving a definition, a broader context for the day's topics, or answering questions.

2. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

3. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.

4. **Repeat step 2.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question especially primed to hear the insights of the instructor.

These problems are geared towards concepts instead of computation, though some problems focus on simple computation. The questions also have a geometric lean. Vectors are initially introduced with familiar coordinate notation, but eventually, coordinates are understood to be *representations* of vectors rather than "true" geometric vectors, and objects like the determinant are defined via oriented volumes rather than formulas involving matrix entries.

Contributors

This book is a collaborative effort. The following people have contributed to its creation:

◦ Jesse Frohlich ◦ Sameul Khan ◦ Avery King ◦ Dan Le ◦ Tianhao (Patrick) Wang ◦ Zack Wolske ◦

Sets, Vectors & Notation

In this module you will learn

- The basics of sets and set-builder notation.
- The definition of vectors and how they relate to points.
- Column vector notation and to how represent vectors in drawings.
- How to compute linear combinations of vectors and use systems of linear equations to answer questions about linear combinations of vectors.

Module 1

Sets

Modern mathematics makes heavy use of *sets*. A set is an unordered collection of distinct objects. We won't try and pin it down more than this—our intuition about collections of objects will suffice.[1] We write a set with curly-braces { and } and list the objects inside. For instance

$$\{1,2,3\}.$$

This would be read aloud as "the set containing the elements 1, 2, and 3". Things in a set are called *elements*, and the symbol $\in$ is used to specify that something is an element of a set. In contrast, $\notin$ is used to specify something is not an element of a set. For example,

$$3\in\{1,2,3\}\qquad 4\notin\{1,2,3\}.$$

Sets can contain mixtures of objects, including other sets. For example,

$$\{1,2,a,\{-70,\infty\},x\}$$

is a perfectly valid set.

It is tradition to use capital letters to name sets. So we might say $A=\{6,7,12\}$ or $X=\{7\}$. However there are some special sets which already have names/symbols associated with them. The *empty set* is the set containing no elements and is written $\emptyset$ or $\{\}$. Note that $\{\emptyset\}$ is *not* the empty set—it is the set containing the empty set! It is also traditional to call elements of a set *points* regardless of whether you consider them "point-like".

Operations on Sets

If the set A contains all the elements that the set B does, we call B a *subset* of A. Conversely, we call A a *superset* of B.

> **Subset & Superset.** The set B is a ***subset*** of the set A, written $B\subseteq A$, if for all $b\in B$ we also have $b\in A$. In this case, A is called a ***superset*** of B.[a]
>
> [a] Some mathematicians use the symbol $\subset$ instead of $\subseteq$.

Some simple examples are $\{1,2,3\}\subseteq\{1,2,3,4\}$ and $\{1,2,3\}\subseteq\{1,2,3\}$. There's something funny about that last example, though. Those two sets are not only subsets/supersets of each other, they're *equal*. As surprising as it seems, we actually need to define what it means for two sets to be equal.

> **Set Equality.** The sets A and B are ***equal***, written $A=B$, if $A\subseteq B$ and $B\subseteq A$.

Having a definition of equality to lean on will help us when we need to prove things about sets.

> **Example.** Let A be the set of numbers that can be expressed as $2n$ for some whole number n, and let B be the set of numbers that can be expressed as $m+1$ where m is an odd whole number. We will show $A=B$.
>
> First, let us show $A\subseteq B$. If $x\in A$, then $x=2n$ for some whole number n. Therefore
>
> $$x=2n=2(n-1)+1+1=m+1$$
>
> where $m=2(n-1)+1$ is, by definition, an odd number. Thus $x\in B$, which proves $A\subseteq B$.

[1] When you pursue more rigorous math, you rely on definitions to get yourself out of philosophical jams. For instance, with our definition of set, consider "the set of all sets that don't contain themselves". Such a set cannot exist! This is called *Russel's Paradox*, and shows that if we start talking about sets of sets, we may need more than intuition.

Now we will show $B \subseteq A$. Let $x \in B$. By definition, $x = m+1$ for some odd m. By the definition of oddness, $m = 2k+1$ for some whole number k. Thus

$$\begin{aligned} x = m+1 = (2k+1)+1 &= 2k+2 \\ &= 2(k+1) = 2n, \end{aligned}$$

where $n = k+1$, and so $x \in A$. Since $A \subseteq B$ and $B \subseteq A$, by definition $A = B$.

Set-builder Notation

Specifying sets by listing all their elements can be a hassle, and if there are an infinite number of elements, it's impossible! Fortunately, *set-builder notation* solves these problems. If X is a set, we can define a subset

$$Y = \{a \in X \,:\, \text{some rule involving } a\},$$

which is read "Y is the set of a in X *such that* some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a \,:\, \text{some rule involving } a\}$.[2] You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

There are also some common operations we can do with two sets.

Unions & Intersections. Let X and Y be sets. The ***union*** of X and Y and the ***intersection*** of X and Y are defined as follows.

$$\text{(union)} \quad X \cup Y = \{a \,:\, a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a \,:\, a \in X \text{ and } a \in Y\}.$$

For example, if $A = \{1,2,3\}$ and $B = \{-1,0,1,2\}$, then $A \cap B = \{1,2\}$ and $A \cup B = \{-1,0,1,2,3\}$. Set unions and intersections are *associative*, which means it doesn't matter how you apply parentheses to an expression involving just unions or just intersections. For example $(A \cup B) \cup C = A \cup (B \cup C)$, which means we can give an unambiguous meaning to an expression like $A \cup B \cup C$ (just put the parentheses wherever you like). But watch out, $(A \cup B) \cap C$ means something different than $A \cup (B \cap C)$!

Some common sets have special notation:

$$\begin{aligned} \emptyset &= \{\}, \text{ the empty set} \\ \mathbb{N} &= \{0,1,2,3,\ldots\} = \{\text{natural numbers}\} \\ \mathbb{Z} &= \{\ldots,-3,-2,-1,0,1,2,3,\ldots\} = \{\text{integers}\} \\ \mathbb{Q} &= \{\text{rational numbers}\} \\ \mathbb{R} &= \{\text{real numbers}\} \\ \mathbb{R}^n &= \{\text{vectors in } n\text{-dimensional Euclidean space}\} \end{aligned}$$

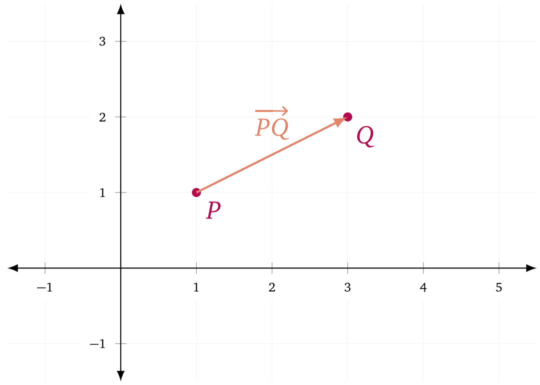
Vectors & Scalars

A *scalar* number (also referred to as a *scalar* or just an ordinary *number*) models a relationship between quantities. For example, a recipe might call for *six* times as much flour as sugar. In contrast, a *vector* models a relationship between points. For example, the store might be *2km East and 4km North* from your house. In this way, a vector may be thought of as a *displacement* with a *direction* and a *magnitude*.[3]

Given points $P = (1,1)$ and $Q = (3,2)$, we specify the *displacement* from P to Q as a vector $\overrightarrow{PQ}$ whose magnitude is $\sqrt{5}$ (as given by the Pythagorean theorem) and whose direction is specified by a directed line segment from P to Q.

[2] If you want to get technical, to make this notation unambiguous, you define a *universe of discourse*. That is, a set $\mathcal{U}$ containing every object you might want to talk about. Then $\{a \,:\, \text{some rule involving } a\}$ is short for $\{a \in \mathcal{U} \,:\, \text{some rule involving } a\}$

[3] Though in this book we will treat vectors as geometric objects relating to Euclidean space, they are much more general. For instance, someone's internet browsing habits could be described by a vector—the topics they find most interesting might be the "direction" and the amount of time they browse might be the "magnitude."

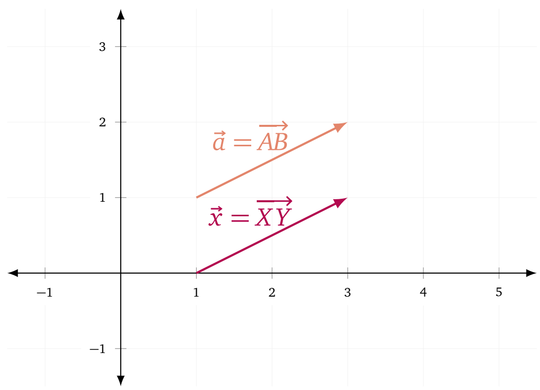


Vector Notation

There are many ways to represent vector quantities in writing. If we have two points, P and Q, we write $\overrightarrow{PQ}$ to represent the vector from P to Q. Absent points, a bold-faced letter (like **a**) or an arrow over a letter (like $\vec{a}$) are the most common vector notations. In this text, we will use $\vec{a}$ to represent a vector. The notation $\|\vec{a}\|$ represents the magnitude of the vector $\vec{a}$, which is sometimes called the *norm* or *length* of $\vec{a}$.

Graphically, we may represent vectors as directed line segments (a line segment with an arrow at one end), however we must take care to distinguish between the picture we draw and the “true” vector. For example, directed line segments always *start* somewhere, but a vector models a displacement and has no sense of “origin”.

Consider the following: for the points $A=(1,1)$, $B=(3,2)$, $X=(1,0)$, and $Y=(3,1)$, define the vectors $\vec{a}=\overrightarrow{AB}$ and $\vec{x}=\overrightarrow{XY}$.



Are these the same or different vectors? As directed line segments, they are different because they are at different locations in space. However, both $\vec{a}$ and $\vec{x}$ have the same magnitude and direction. Thus, $\vec{a}=\vec{x}$ despite the fact that $A\neq X$.[4]

Takeaway. A vector is not the same as a line segment and a vector by itself has no “origin”.

Vectors and Points

The distinction between vectors and points is sometimes nebulous because the two are so closely related. A *point* in Euclidean space specifies an absolute position whereas a vector specifies a displacement (i.e., a magnitude and direction). However, given a point P, one associates P with the vector $\vec{p}=\overrightarrow{OP}$, where O is the origin. Similarly, we associate the vector $\vec{v}$ with the point V so that $\overrightarrow{OV}=\vec{v}$. Thus, we have a way to unambiguously go back and forth between vectors and points.[5] As such, *we will treat vectors and points interchangeably*.

Takeaway. Vectors and points can and will be treated interchangeably.

[4] Some theories use *rooted vectors* instead of vectors as the fundamental object of study. A rooted vector represents a magnitude, direction, *and* a starting point. And, as rooted vectors, $\vec{a}\neq\vec{x}$ (from the example above). But for us, vectors will always be unrooted, even though our graphical representations of vectors might appear rooted.

[5] Mathematically, we say there is an *isomorphism* between vectors and points (once an origin is fixed, of course!).

Vector Arithmetic

Vectors provide a natural way to give directions. For example, suppose $\vec{e}_1$ points one kilometer eastwards and $\vec{e}_2$ points one kilometer northwards. Now, if you were standing at the origin and wanted to move to a location 3 kilometers east and 2 kilometers north, you might say: "Walk 3 times the length of $\vec{e}_1$ in the $\vec{e}_1$ direction and 2 times the length of $\vec{e}_2$ in the $\vec{e}_2$ direction." Mathematically, we express this as

$$3\vec{e}_1 + 2\vec{e}_2.$$

Of course, we've incidentally described a new vector. Let P be the point at 3-east and 2-north. Then

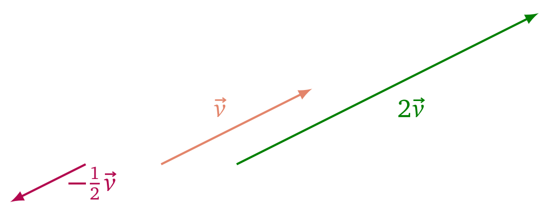
$$\overrightarrow{OP} = 3\vec{e}_1 + 2\vec{e}_2.$$

If the vector $\vec{r}$ points north but has a length of 10 kilometers, we have a similar formula:

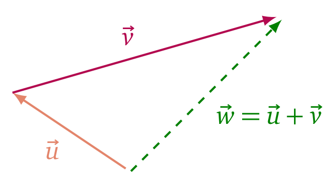
$$\overrightarrow{OP} = 3\vec{e}_1 + \tfrac{1}{5}\vec{r},$$

and we have the relationship $\vec{r} = 10\vec{e}_2$. Our notation here is very suggestive. Indeed, if we could make sense of "$\alpha\vec{v}$" (scalar multiplication) and "$\vec{v}+\vec{w}$" (vector addition) for any scalar α and any vectors $\vec{v}$ and $\vec{w}$, we could do algebra with vectors.

We will define scalar multiplication and vector addition intuitively: For a vector $\vec{v}$ and a scalar $\alpha > 0$, the vector $\vec{w} = \alpha\vec{v}$ is the vector pointing in the same direction as $\vec{v}$ but with length scaled up by α. That is, $\|\vec{w}\| = \alpha\|\vec{v}\|$. Similarly, $-\vec{v}$ is the vector of the same length as $\vec{v}$ but pointing in the exact opposite direction.



For two vectors $\vec{u}$ and $\vec{v}$, the sum $\vec{w} = \vec{u} + \vec{v}$ represents the displacement vector created by first displacing along $\vec{u}$ and then displacing along $\vec{v}$.



Takeaway. You add vectors tip to tail and you scale vectors by changing their length.

Now, there is one snag. What should $\vec{v} + (-\vec{v})$ be? Well, first we displace along $\vec{v}$ and then we displace in the exact opposite direction by the same amount. So, we have gone nowhere. This corresponds to a displacement with zero magnitude. But, what direction did we displace? Here we make a philosophical stand.

Zero Vector. The ***zero vector***, notated as $\vec{0}$, is the vector with no magnitude.

We will be pragmatic about the direction of the zero vector and say, *the zero vector does not have a well-defined direction*.[6] That means sometimes we consider the zero vector to point in every direction and sometimes we consider it to point in no directions. It depends on our mood—but we must never talk about *the* direction of the zero vector, since it's not defined.

Formalizing, for vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and scalars α and β are scalars, the following are laws are always satisfied:

$$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w}) \qquad \text{(Associativity)}$$
$$\vec{u}+\vec{v} = \vec{v}+\vec{u} \qquad \text{(Commutativity)}$$
$$\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v} \qquad \text{(Distributivity)}$$

[6] In the mathematically precise definition of vector, the idea of "magnitude" and "direction" are dropped. Instead, a set of vectors is defined to be a set over which you can reasonably define addition and scalar multiplication.

and

$$(\alpha\beta)\vec{v} = \alpha(\beta\vec{v}) \qquad \text{(Associativity II)}$$
$$(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v} \qquad \text{(Distributivity II)}$$

Indeed, if we intuitively think about vectors in flat (Euclidean) space, all of these properties are satisfied.[7] From now on, these properties of vector operations will be considered the *laws (or axioms) of vector arithmetic*.

We group scalar multiplication and vector addition under one name: *linear combinations*.

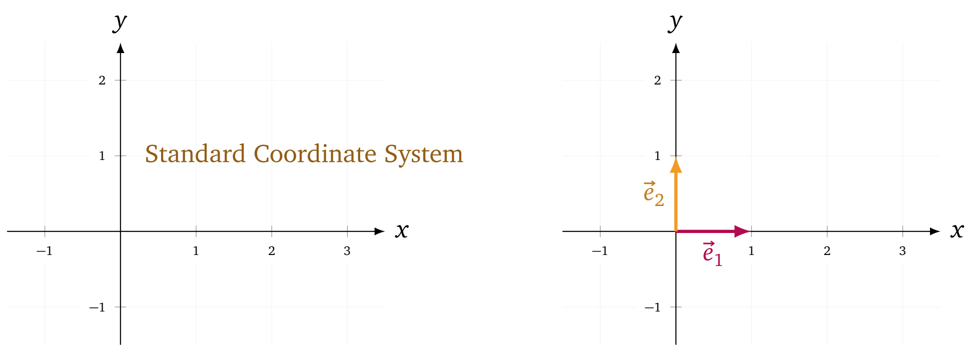
Linear Combination. A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called the ***coefficients*** of the linear combination.

Coordinates and the Standard Basis

Consider the standard, flat, Euclidean plane (which is notated by $\mathbb{R}^2$). A coordinate system for $\mathbb{R}^2$ is a way to assign a unique pair of numbers to every point in $\mathbb{R}^2$. Though there are infinitely many coordinate systems we could choose for the plane, there is one standard one: the xy-coordinate system depicted below (which you're already familiar with).



In conjunction with the standard coordinate system, there are also *standard basis vectors*. The vector $\vec{e}_1$ always points one unit in the direction of the positive x-axis and $\vec{e}_2$ always points one unit in the direction of the positive y-axis.

Using the standard basis, we can represent every point (or vector) in the plane as a linear combination. If the point P has xy-coordinates (α, β), then $\overrightarrow{OP} = \alpha\vec{e}_1 + \beta\vec{e}_2$. Not only that, but this is the *only* way to represent the vector $\overrightarrow{OP}$ as a linear combination of α and β.

Takeaway. Every vector in $\mathbb{R}^2$ can be written uniquely as a linear combination of the standard basis vectors.

For a vector $\vec{w} = \alpha\vec{e}_1 + \beta\vec{e}_2$, we call the pair (α, β) the *standard coordinates* of the vector $\vec{w}$. There are many equivalent notations used to represent a vector in coordinates.

(α, β)	parentheses
$\langle \alpha, \beta \rangle$	angle brackets
$\begin{bmatrix} \alpha & \beta \end{bmatrix}$	square brackets in a row (a row matrix)
$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$	square brackets in a column (a column matrix)

Coordinates and vectors go hand in hand, and we will often write

$$\vec{v} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

as a shorthand for "$\vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2$".

[7] If we deviate from flat space, some of these rules are no longer respected. Consider moving 100 kilometers north then 100 kilometers east on a sphere. Is this the same as moving 100 kilometers east and then 100 kilometers north?

Solving Problems with Coordinates

Coordinates allow for vector arithmetic to be carried out in a mechanical way. Suppose $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then,

$$\vec{u} = \vec{v} \qquad \iff \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \iff \qquad a = x \text{ and } b = y.$$

Further,

$$\vec{u} + \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+x \\ b+y \end{bmatrix} \qquad \text{and} \qquad t\vec{v} = t\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ta \\ tb \end{bmatrix}$$

for any scalar t.

Using these rules, otherwise complicated questions about vectors can be reduced to simple algebra questions.[8]

> **Example.** Let $\vec{x} = \vec{e}_1 - \vec{e}_2$, $\vec{y} = 3\vec{e}_1 - \vec{e}_2$, and $\vec{r} = 2\vec{e}_1 + 2\vec{e}_2$. Is $\vec{r}$ a linear combination of $\vec{x}$ and $\vec{y}$?
>
> By definition, $\vec{r}$ is a linear combination of $\vec{x}$ and $\vec{y}$ if there exist scalars a and b such that
>
> $$\vec{r} = a\vec{x} + b\vec{y}.$$
>
> Rewriting everything in coordinates, we see this is equivalent to the equation
>
> $$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} a+3b \\ -a-b \end{bmatrix}.$$
>
> Therefore, we need to determine if the system of equations
>
> $$\begin{cases} a + 3b = 2 \\ -a - b = 2 \end{cases}$$
>
> has a solution. After solving, we find $a = -4$ and $b = 2$ is the only solution. Thus, $\vec{r}$ is a linear combination of $\vec{x}$ and $\vec{y}$. More specifically,
>
> $$\vec{r} = -4\vec{x} + 2\vec{y}.$$

Higher Dimensions

We coordinatize three dimensional space (notated by $\mathbb{R}^3$) by constructing x, y, and z axes. Again, $\mathbb{R}^3$ has standard basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ which each point one unit along the x, y, and z axes, respectively.

Since we live in three dimensional space, its study has a long history, and many notations for the standard basis of three dimensional space are in use. This text will use $\vec{e}_1, \vec{e}_2, \vec{e}_3$, but other common notations include:

$$\begin{array}{ccc} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \hat{\imath} & \hat{\jmath} & \hat{k} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}$$

Beyond three dimensions, drawing pictures becomes hard, but we can still use vectors. We use $\mathbb{R}^n$ to notate n-dimensional Euclidean space. The standard basis for $\mathbb{R}^n$ is $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. Again, every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of the standard basis, and a coordinate representation of a vector in $\mathbb{R}^n$ is a list of n scalars.

> **Example.** Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be given by $\vec{x} = 2\vec{e}_1 - \vec{e}_3$ and $\vec{y} = 6\vec{e}_2 + 3\vec{e}_3$. Compute $\vec{z} = \vec{x} + 2\vec{y}$.
>
> $$\vec{z} = \vec{x} + 2\vec{y} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 5 \end{bmatrix} = 2\vec{e}_1 + 12\vec{e}_2 + 6\vec{e}_3$$

[8] So simple, that computers are able to answer billions of such questions a second as you play your favorite video game!

Practice Problems

1[B] Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2,4,8)$

(b) $(15,6,-21)$

Task 1.1: The Magic Carpet Ride

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Task 1.2: The Magic Carpet Ride, Hide and Seek

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X \,:\, \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \,|\, \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

1

1.1 Which of the following statements are true?

(a) $3 \in \{1,2,3\}$.
(b) $1.5 \in \{1,2,3\}$.
(c) $4 \in \{1,2,3\}$.
(d) "b"$\in \{x \,:\, x \text{ is an English letter}\}$.
(e) "ò"$\in \{x \,:\, x \text{ is an English letter}\}$.
(f) $\{1,2\} \subseteq \{1,2,3\}$.
(g) For some $a \in \{1,2,3\}$, $a \geq 3$.
(h) For any $a \in \{1,2,3\}$, $a \geq 3$.
(i) $1 \subseteq \{1,2,3\}$.
(j) $\{1,2,3\} = \{x \in \mathbb{R} \,:\, 1 \leq x \leq 3\}$.
(k) $\{1,2,3\} = \{x \in \mathbb{Z} \,:\, 1 \leq x \leq 3\}$.

2

Write the following in set-builder notation

2.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

2.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

DEFINITION

Unions & Intersections

Let X and Y be sets. The ***union*** of X and Y and the ***intersection*** of X and Y are defined as follows.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

---

3 Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

3.1 $X \cup Y$

3.2 $X \cap Y$

3.3 $X \cup Y \cup Z$

3.4 $X \cap Y \cap Z$

4 Draw the following subsets of $\mathbb{R}^2$.

4.1 $V = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.2 $H = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.3 $D = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.4 $N = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R} \right\}.$

4.5 $V \cup H$.

4.6 $V \cap H$.

4.7 Does $V \cup H = \mathbb{R}^2$?

Vector Combinations

Linear Combination

DEFINITION

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the ***coefficients*** of the linear combination.

---

5 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

5.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

5.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

5.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

6 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

6.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

6.3 Rephrase the sentence "$\mathbb{R}^2$ *is the set of all linear combinations of* $\vec{h}$ *and* $\vec{m}$" using formal mathematical language.

Sets of Vectors, Lines & Planes

In this module you will learn

- How to draw a set of vectors making an appropriate choice of when to use line segments and when to use dots to represent vectors.
- The *vector form* of lines and planes, including how to determine the intersection of lines and planes in vector form.
- Restricted linear combinations and how to use them to represent common geometric objects (like line segments or polygons).

With a handle on vectors, we can now use them to describe some common geometric objects: lines and planes.

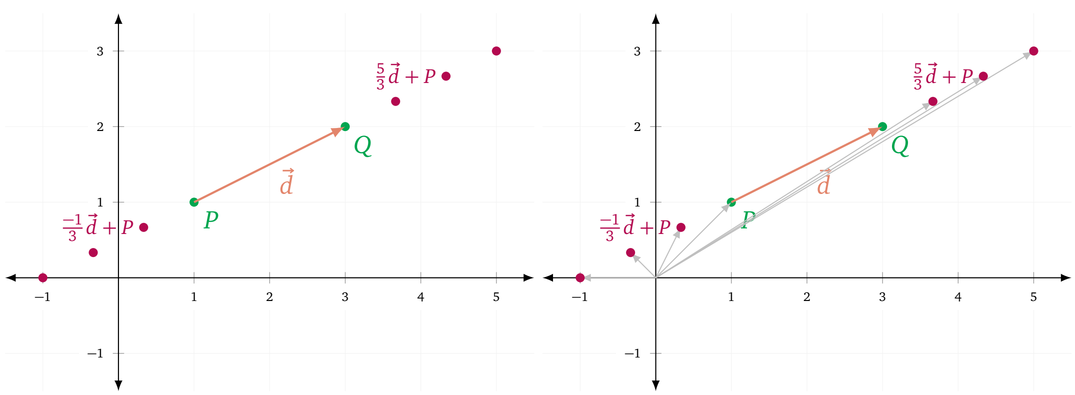
Lines

Consider for a moment the line ℓ through the points P and Q. When $P,Q \in \mathbb{R}^2$, we can describe ℓ with an equation of the form $y = mx + b$ (provided it isn't a vertical line), but if $P,Q \in \mathbb{R}^3$, it's much harder to describe ℓ with an equation. We can solve this problem by using vectors.

Let $\vec{d} = \overrightarrow{PQ}$ and consider the set of points (or vectors) $\vec{x}$ that can be expressed as

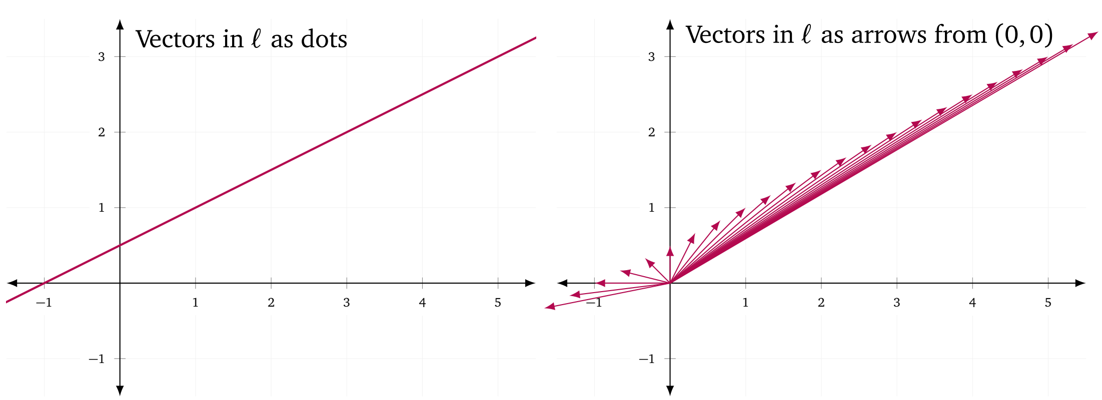
$$\vec{x} = t\vec{d} + P$$

for $t \in \mathbb{R}$. Geometrically, this is the set of all points we get by starting at P and displacing by some multiple of $\vec{d}$. This is a line!



We simultaneously interpret this line as a set of points (the points that make up the line) and as a set of vectors rooted at the origin (the vectors pointing from the origin to the line). Note that sometimes we draw vectors as directed line segments. Other times, we draw each vector by marking only its ending point because drawing each vector as line segment would make it hard to see what is going on.

Which picture below do you think best represents ℓ?



Takeaway. When drawing a picture depicting several vectors, make an appropriate choice (arrows, dots, or a mix) so that the picture is clear.

The line ℓ described above can be written in set-builder notation as:

$$\ell = \{\vec{x} : \vec{x} = t\vec{d} + P \text{ for some } t \in \mathbb{R}\}.$$

Notice that in set-builder notation, we write "for some $t \in \mathbb{R}$." Make sure you understand why replacing "for some $t \in \mathbb{R}$" with "for all $t \in \mathbb{R}$" would be incorrect.

Writing lines with set-builder notation all the time can be overkill, so we will allow ourselves to describe lines in a shorthand called *vector form*.[9]

Vector Form of a Line. Let ℓ be a line and let $\vec{d}$ and $\vec{p}$ be vectors. If $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$, we say the vector equation

$$\vec{x} = t\vec{d} + \vec{p}$$

is ℓ expressed in ***vector form***. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

We can also use coordinates when writing a line in vector form. For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

corresponds to the line passing through $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ with $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ as a direction vector.

The "t" that appears in a vector form is called the *parameter variable*, and for this reason, some textbooks use the term *parametric form* in place of "vector form".

Writing a line in vector form requires only a point on the line and a direction for the line,[10] which makes converting from another form into vector form straightforward.

Example. Find vector form of the line $\ell \subseteq \mathbb{R}^2$ with equation $y = 2x + 3$.

First, we find two points on ℓ. By guess-and-check, we see $P = (0, 3)$ and $Q = (1, 5)$ are on ℓ. Thus, a direction vector for ℓ is given by

$$\vec{d} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We may now express ℓ in vector form as

$$\vec{x} = t\vec{d} + P$$

or, using coordinates, as

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

It's important to note that when we write a line in vector form, it is a *specific shorthand* notation. If we augment the notation, we no longer have written a line in "vector form".

Example. Let ℓ be a line, let $\vec{d}$ be a direction vector for ℓ, and let $\vec{p} \in \ell$ be a point on ℓ. Writing

$$\vec{x} = t\vec{d} + \vec{p}$$

or

$$\vec{x} = t\vec{d} + \vec{p} \quad \text{where} \quad t \in \mathbb{R}$$

specifies ℓ in vector form; both are shorthands for $\{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$. But,

$$\vec{x} = t\vec{d} + \vec{p} \quad \text{for some} \quad t \in \mathbb{R}$$

and

$$\vec{x} = t\vec{d} + \vec{p} \quad \text{for all} \quad t \in \mathbb{R}$$

[9] $y = mx + b$ form of a line is also shorthand. The line ℓ described by the equation $y = mx + b$ is actually the set $\{(x, y) \in \mathbb{R}^2 : y = mx + b\}$.

[10] Notice that a direction vector *for* a line ℓ is different than a vector *in* a line ℓ.

are logical statements about the vectors $\vec{x}$, $\vec{d}$, and $\vec{p}$. These statements are either true or false; they do *not* specify ℓ in vector form.

Similarly, the statement

$$\ell = t\vec{d} + \vec{p}$$

is mathematically nonsensical and does not specify ℓ in vector form. (On the left is a *set* and on the right is a *vector*!)

Takeaway. Vector form is a specific shorthand for a set. If "extra" words or symbols are added to the vector form, it stops being a shorthand.

But, why is vector form useful? For starters, every line can be expressed in vector form (you cannot write a vertical line in $y = mx + b$ form, and in $\mathbb{R}^3$, you would need two linear equations to represent a line). But, the most useful thing about expressing a line in vector form is that you can easily generate points on that line.

Suppose ℓ can be represented in vector form as $\vec{x} = t\vec{d} + \vec{p}$. Then, for every $t \in \mathbb{R}$, the vector $t\vec{d} + \vec{p} \in \ell$. Not only that, but as t ranges over $\mathbb{R}$, all points on ℓ are "traced out". Thus, we can find points on ℓ without having to "solve" any equations.

The downside to using vector form is that it is not unique. There are multiple direction vectors and multiple points for every line. Thus, merely by looking at the vector equation for two lines, it can be hard to tell if they're equal.

For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

all represent the same line. In the second equation, the direction vector is parallel but scaled, and in the third equation, a different point on the line was chosen.

Recall that in vector form, the variable t is called the *parameter variable*. It is an instance of a *dummy variable*. In other words, t is a placeholder—just because "t" appears in two different vector forms, doesn't mean it's the same quantity.

To drive this point home, let's think about vector form in terms of the sets it specifies. Let $\vec{d}_1, \vec{d}_2 \neq \vec{0}$ and $\vec{p}_1, \vec{p}_2$ be vectors and define the lines

$$\ell_1 = \{\vec{x} \,:\, \vec{x} = t\vec{d}_1 + \vec{p}_1 \text{ for some } t \in \mathbb{R}\}$$

$$\ell_2 = \{\vec{x} \,:\, \vec{x} = t\vec{d}_2 + \vec{p}_2 \text{ for some } t \in \mathbb{R}\}.$$

These lines have vector forms $\vec{x} = t\vec{d}_1 + \vec{p}_1$ and $\vec{x} = t\vec{d}_2 + \vec{p}_2$. However, declaring that $\ell_1 = \ell_2$ if and only if $t\vec{d}_1 + \vec{p}_1 = t\vec{d}_2 + \vec{p}_2$ does *not* make sense. Instead, as per the definition, $\ell_1 = \ell_2$ if $\ell_1 \subseteq \ell_2$ and $\ell_2 \subseteq \ell_1$. If $\vec{x} \in \ell_1$ then $\vec{x} = t\vec{d}_1 + \vec{p}_1$ for some $t \in \mathbb{R}$. If $\vec{x} \in \ell_2$ then $\vec{x} = t\vec{d}_2 + \vec{p}_2$ for some *possibly different* $t \in \mathbb{R}$. This can get confusing really quickly. The easiest way to avoid confusion is to use different parameter variables when comparing different vector forms.

Example. Determine if the lines ℓ_1 and ℓ_2, given in vector form as

$$\vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \vec{x} = t\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

are the same line.

To determine this, we need to figure out if $\vec{x} \in \ell_1$ implies $\vec{x} \in \ell_2$ and if $\vec{x} \in \ell_2$ implies $\vec{x} \in \ell_1$.

If $\vec{x} \in \ell_1$, then $\vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}$. If $\vec{x} \in \ell_2$, then $\vec{x} = s\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ for some $s \in \mathbb{R}$. Thus if

$$t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{x} = s\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

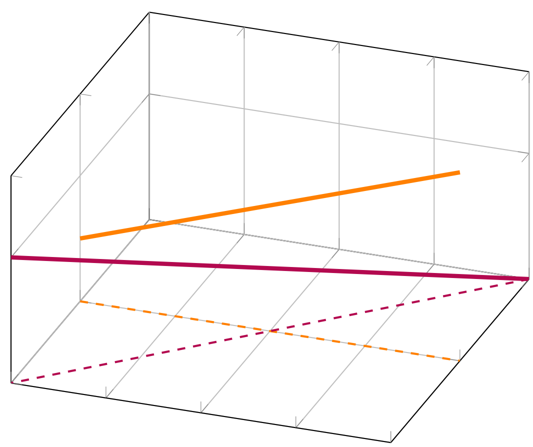
always has a solution, $\ell_1 = \ell_2$. Moving everything to one side, we see

$$\begin{aligned}\vec{0} = \begin{bmatrix}4\\3\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} + s\begin{bmatrix}2\\2\end{bmatrix} - t\begin{bmatrix}1\\1\end{bmatrix} &= \begin{bmatrix}2\\2\end{bmatrix} + s\begin{bmatrix}2\\2\end{bmatrix} - t\begin{bmatrix}1\\1\end{bmatrix}\\ &= (s+1)\begin{bmatrix}2\\2\end{bmatrix} - \tfrac{t}{2}\begin{bmatrix}2\\2\end{bmatrix}\\ &= (s+1-\tfrac{t}{2})\begin{bmatrix}2\\2\end{bmatrix}.\end{aligned}$$

This equation has a solution whenever $s+1-t/2 = 0$ has a solution. Since for every s, the equation $s+1-t/2=0$ has a solution, and for every t, the equation $s+1-t/2=0$ has a solution, we know $\ell_1 = \ell_2$.

Vector Form in Higher Dimensions

The geometry of lines in space ($\mathbb{R}^3$ and above) is a bit more complicated than that of lines in the plane. Lines in the plane either intersect or are parallel. In space, we have to be a more careful about what we mean by "parallel lines," since lines with entirely different directions can still fail to intersect.[11]



Example. Consider the lines described by

$$\begin{aligned}\vec{x} &= t(1,3,-2) + (1,2,1)\\ \vec{x} &= t(-2,-6,4) + (3,1,0).\end{aligned}$$

They have parallel directions since $(-2,-6,4) = -2(1,3,-2)$. Hence, in this case, we say the lines are *parallel*. (How can we be sure the lines are not the same?)

Example. Consider the lines described by

$$\begin{aligned}\vec{x} &= t(1,3,-2) + (1,2,1)\\ \vec{x} &= t(0,2,3) + (0,3,9).\end{aligned}$$

They are not parallel because neither of the direction vectors is a multiple of the other. They may or may not intersect. (If they don't, we say the lines are *skew*.) How can we find out? Mirroring our earlier approach, we can set their equations equal and see if we can solve for a point of intersection *after ensuring we give their parametric variables different names*. We'll keep one parametric variable named t and name the other one s. Thus, we want

$$\vec{x} = t(1,3,-2) + (1,2,1) = s(0,2,3) + (0,3,9),$$

which after collecting terms yields

$$(t+1, 3t+2, -2t+1) = (0, 2s+3, 3s+9).$$

[11] Recall that in Euclidean geometry two lines are defined to be parallel if they coincide or never intersect.

Reading coordinate by coordinate, we get three equations

$$\begin{aligned} t+1 &= 0 \\ 3t+2 &= 2s+3 \\ -2t+1 &= 3s+9 \end{aligned}$$

in two unknowns s and t. This is an *overdetermined* system, and it may or may not have a solution. The first two equations yield $t=-1$ and $s=-2$. Putting these values in the last equation yields $(-2)(-1)+1 = 3(-2)+9$, which is indeed true. Hence, the equations are consistent, and the lines intersect. To find the point of intersection, put $t=-1$ in the equation for the vector equation of the first line (or $s=-2$ in that for the second) to obtain $(0,-1,3)$.

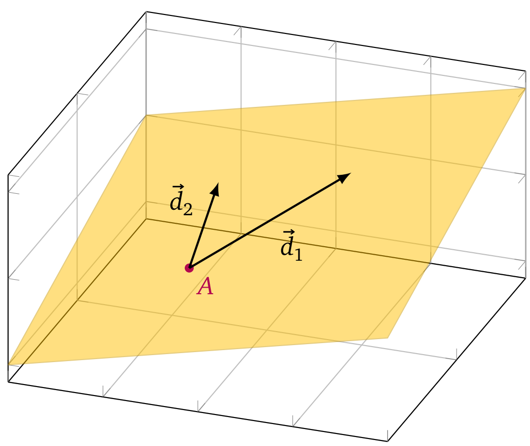
Planes

Any two distinct points define a line. To define a plane, we need three points. But there's a caveat: the three points cannot be on the same line, otherwise they'd define a line and not a plane. Let $A, B, C \in \mathbb{R}^3$ be three points that are not collinear and let $\mathcal{P}$ be the plane that passes through A, B, and C.

Just like lines, planes have direction vectors. For $\mathcal{P}$, both $\vec{d}_1 = \overrightarrow{AB}$ and $\vec{d}_2 = \overrightarrow{AC}$ are direction vectors. Of course, $\vec{d}_1$, $\vec{d}_2$ and their multiples are not the only direction vectors for $\mathcal{P}$. There are infinitely many more, including $\vec{d}_1 + \vec{d}_2$, and $\vec{d}_1 - 7\vec{d}_2$, and so on. However, since a plane is a *two*-dimensional object, we only need two different direction vectors to describe it.

Like lines, planes have a vector form. Using the direction vectors $\vec{d}_1 = \overrightarrow{AB}$ and $\vec{d}_2 = \overrightarrow{AC}$, the plane $\mathcal{P}$ can be written in vector form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\vec{d}_1 + s\vec{d}_2 + A.$$



Vector Form of a Plane. A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

Example. Describe the plane $\mathcal{P} \subseteq \mathbb{R}^3$ with equation $z = 2x + y + 3$ in vector form.

To describe $\mathcal{P}$ in vector form, we need a point on $\mathcal{P}$ and two direction vectors for $\mathcal{P}$. By guess-and-check, we see the points

$$A = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \qquad C = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

are all in $\mathcal{P}$. Thus

$$\vec{d}_1 = B - A = \begin{bmatrix}1\\0\\2\end{bmatrix} \qquad \text{and} \qquad \vec{d}_2 = C - A = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

are both direction vectors for $\mathcal{P}$. Since these vectors are not parallel, we can express $\mathcal{P}$ in vector form as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + A = t\begin{bmatrix}1\\0\\2\end{bmatrix} + s\begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}.$$

Example. Find the line of intersection between $\mathcal{P}_1$ and $\mathcal{P}_2$ where the planes are given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\1\\0\end{bmatrix} + s\begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix}}^{\mathcal{P}_1} \qquad \text{and} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\0\\2\end{bmatrix} + s\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}}^{\mathcal{P}_2}.$$

Just like in the example for lines, we are looking for points $\vec{x}$ that are in both planes. To keep from getting mixed, we'll use a, b, c, and d as parameter variables. Therefore, we are looking for solutions to

$$a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix} = \vec{x} = c\begin{bmatrix}-1\\0\\2\end{bmatrix} + d\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix}.$$

Collecting terms, this is equivalent to the system of equations

$$\begin{cases} a - b + c - d = -1 \\ a \phantom{{}-b+2c} - 2d = -2 \\ \phantom{a-{}} b - 2c - d = 0 \end{cases}.$$

This system is underdetermined (there are four variables and three equations). If $\mathcal{P}_1$ and $\mathcal{P}_2$ indeed intersect in a line, we know there must an infinite number of solutions to this system. After row reducing, we see

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}r\\r/2-1\\-1\\r/2+1\end{bmatrix}$$

is a solution for every $r \in \mathbb{R}$. We can substitute these parameters into either of the original equations to get an equation for the line of intersection. Picking the second one, we see

$$\vec{x} = c\begin{bmatrix}-1\\0\\2\end{bmatrix} + d\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix} = -\begin{bmatrix}-1\\0\\2\end{bmatrix} + (\tfrac{r}{2}+1)\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}0\\0\\3\end{bmatrix} = \tfrac{r}{2}\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}2\\2\\2\end{bmatrix}$$

is in both planes for every $r \in \mathbb{R}$. Therefore, we may express $\mathcal{P}_1 \cap \mathcal{P}_2$ in vector form as

$$\vec{x} = r\begin{bmatrix}1/2\\1\\1/2\end{bmatrix} + \begin{bmatrix}2\\2\\2\end{bmatrix}.$$

Restricted Linear Combinations

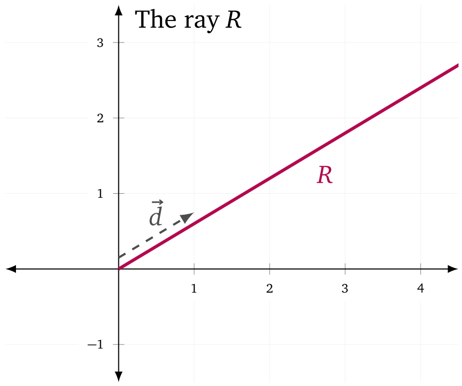
Using vectors, we can describe more than just lines and planes—we can describe all sorts of geometric objects.

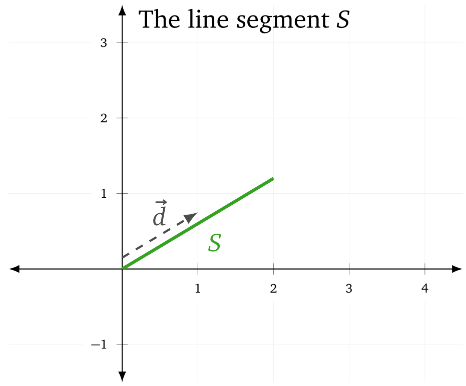
Recall that when we write $\vec{x} = t\vec{d} + \vec{p}$ to describe the line ℓ, what we mean is

$$\ell = \{\vec{x} \,:\, \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}.$$

The line ℓ stretches off infinitely in both directions. But, what if we wanted to describe just a part of ℓ? We can do this by placing additional restrictions on t. For example, consider the ray R and the line segment S:

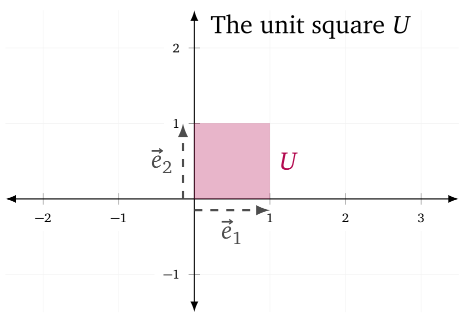
$$R = \{\vec{x} \,:\, \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \geq 0\}$$
$$S = \{\vec{x} \,:\, \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in [0, 2]\}$$

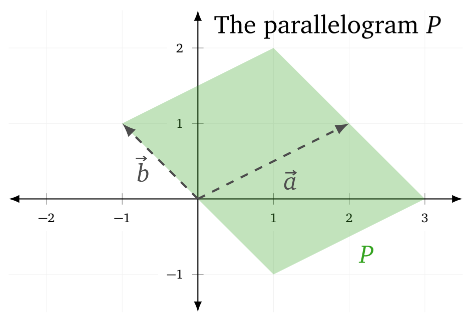




We can also make polygons by adding restrictions to the vector form of a plane. Let $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and consider the unit square U and the parallelogram P defined by

$$U = \{\vec{x} \,:\, \vec{x} = t\vec{e}_1 + s\vec{e}_2 \text{ for some } t, s \in [0, 1]\}$$
$$P = \{\vec{x} \,:\, \vec{x} = t\vec{a} + s\vec{b} \text{ for some } t \in [0, 1] \text{ and } s \in [-1, 1]\}$$





Each set so far is a set of linear combinations, and we have made different shapes by restricting the coefficients of those linear combinations. There are two ways of restricting linear combinations that arise often enough to get their own names.

Non-negative & Convex Linear Combinations.
Let $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$. The vector $\vec{w}$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0.$$

The vector $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1.$$

You can think of a non-negative linear combinations as vector you can arrive at by only displacing "forward". Convex linear combinations can be thought of as weighted averages of vectors (the average of $\vec{v}_1, \ldots, \vec{v}_n$ would be the convex linear combination with coefficients $\alpha_i = \frac{1}{n}$). A convex linear combination of two vectors gives a point on the line segment connecting them.

Example. Let $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and define

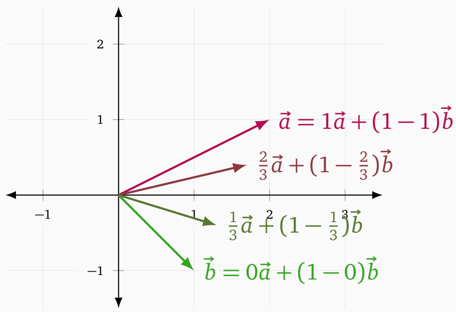
$$\begin{aligned} A &= \{\vec{x} \,:\, \vec{x} \text{ is a convex linear combination of } \vec{a} \text{ and } \vec{b}\} \\ &= \{\vec{x} \,:\, \vec{x} = \alpha\vec{a} + (1-\alpha)\vec{b} \text{ for some } \alpha \in [0, 1]\}. \end{aligned}$$

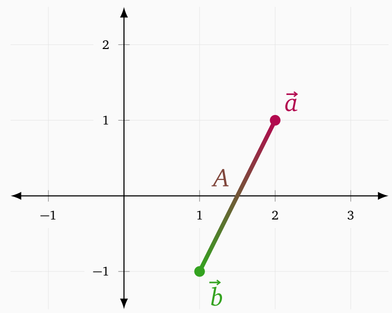
Draw A.

We know $\vec{x} = \alpha\vec{a} + (1-\alpha)\vec{b} \in A$ whenever $\alpha \in [0,1]$. If we rearrange the equation $\vec{x} = \alpha\vec{a} + (1-\alpha)\vec{b}$, we see

$$\vec{x} = \alpha\vec{a} - \alpha\vec{b} + \vec{b} = \alpha(\vec{a} - \vec{b}) + \vec{b},$$

which looks like the vector form of a line which passes through $\vec{b}$ with direction $\vec{a} - \vec{b}$. However, we have the additional restriction $\alpha \in [0,1]$, so A is only the part of that line which connects $\vec{a}$ and $\vec{b}$.





Since A is an infinite collection of vectors, it's better to draw vectors in A as dots rather than lines from the origin.

Practice Problems

1 Express the following lines in vector form.

(a) $\ell \subseteq \mathbb{R}^2$ with equation $4x - 3y = -10$.

(b) $\ell \subseteq \mathbb{R}^2$ which passes through the points $A = (1,1)$ and $B = (2,7)$.

(c) $\ell \subseteq \mathbb{R}^2$ which passes through $\vec{0}$ and is parallel to the line with equation $4x - 3y = -10$.

(d) $\ell \subseteq \mathbb{R}^3$ which passes through the points $A = (-1,-1,0)$ and $B = (2,3,5)$.

(e) $\ell \subseteq \mathbb{R}^3$ which is contained in the yz-plane and whose coordinates satisfy $x + 2y - 3z = 5$.

2 Express the following planes in vector form

(a) $\mathcal{P} \subseteq \mathbb{R}^3$ with equation $4x - 3y + z = -10$.

(b) $\mathcal{P} \subseteq \mathbb{R}^3$ with equation $4x - z = 0$.

(c) $\mathcal{P} \subseteq \mathbb{R}^3$ which passes through the points $A = (-1,-1,0)$, $B = (2,3,5)$, and $C = (3,3,3)$.

(d) $\mathcal{P} \subseteq \mathbb{R}^3$ which is parallel to the yz-plane but passes through the point $X = (1,-1,1)$.

(e) $\mathbb{R}^2$.

(f) $\mathcal{P} \subseteq \mathbb{R}^4$ which passes through the points $A = (1,1,-1,-1)$, $B = (1,-1,1,-1)$, and whose coordinates satisfy the equation $x + y + 2z - w = 3$.

3 Let ℓ_1, ℓ_2, and ℓ_3 be described in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix}}^{\ell_1} \quad \overbrace{\vec{x} = t\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}}^{\ell_2} \quad \overbrace{\vec{x} = t\begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}2\\4\end{bmatrix}}^{\ell_3}.$$

(a) Determine which pairs of the lines ℓ_1, ℓ_2, and ℓ_3 intersect, coincide, and are parallel.

(b) What is $\ell_1 \cap \ell_2 \cap \ell_3$?

4 Let $\mathcal{P}_1 \subseteq \mathbb{R}^3$ be the plane with equation $x + 2y - z = 3$. Let $\mathcal{P}_2, \ell \subseteq \mathbb{R}^3$ be described in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\0\\2\end{bmatrix} + \begin{bmatrix}1\\3\\1\end{bmatrix}}^{\mathcal{P}_2} \quad \overbrace{\vec{x} = t\begin{bmatrix}1\\3\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix}}^{\ell}.$$

(a) Find $\mathcal{P}_1 \cap \ell$.

(b) Find $\mathcal{P}_1 \cap \mathcal{P}_2$.

(c) Find $\mathcal{P}_2 \cap \ell$.

(d) Give an example of a plane $\mathcal{P}_3$ so that $\mathcal{P}_3 \cap \ell$ is empty.

(e) Does there exist a plane $\mathcal{P}_2'$ that is parallel to $\mathcal{P}_2$, but which does not intersect ℓ? Why or why not?

5 Let $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\-1\end{bmatrix}$. The goal of this question is to give and justify a drawing of the set of convex linear combinations of $\vec{a}$ and $\vec{b}$.

(a) Let A be the set of all non-negative linear combinations of $\vec{a}$ and $\vec{b}$. Draw A.

(b) Let ℓ be the set

$$\{\alpha\vec{a} + \beta\vec{b} \,:\, \alpha, \beta \in \mathbb{R} \text{ and } \alpha + \beta = 1\}$$

Rewrite ℓ in set-builder notation using only a single variable t. (Hint: Let t be α.)

(c) Justify why ℓ is a line, and write ℓ in vector form.

(d) Draw both A and ℓ on the same grid. On a separate grid, draw $A \cap \ell$.

(e) Write the $A \cap \ell$ in set builder notation. How does $A \cap \ell$ relate to convex linear combinations?

(f) Determine the endpoints of $A \cap \ell$.

(a) The lines $y = x$ and $y = -x$ divide $\mathbb{R}^2$ into four regions. A is the right-most region.

(b) ℓ is given by the set

$$\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\vec{a} + (1-t)\vec{b} \text{ for some } t \in \mathbb{R}\}.$$

(c) The above set can be rewritten as

$$\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t(\vec{a}+\vec{b}) + \vec{b} \text{ for some } t \in \mathbb{R}\}.$$

This is exactly the line given in vector form by

$$\vec{x} = t(\vec{a} - \vec{b}) + \vec{b}.$$

Since $\vec{a} - \vec{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, ℓ is the vertical line containing $\vec{b}$ (and $\vec{a}$).

(d) $A \cap \ell$ is the set

$$\{\alpha\vec{a} + \beta\vec{b} \;:\; \alpha, \beta \geq 0 \text{ and } \alpha + \beta = 1\}.$$

This is the set of convex linear combinations of $\vec{a}$ and $\vec{b}$.

(e) $A \cap \ell$ is the line segment with endpoints $\vec{a}$ and $\vec{b}$.

6 Let $\vec{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. The goal of this question is to give and justify a drawing of the set of convex linear combinations of $\vec{a}$, $\vec{b}$, and $\vec{c}$. This requires an understanding of the previous question.

(a) Let $\vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Write $\vec{d}$ as a convex linear combination of $\vec{a}$ and $\vec{b}$.

(b) Let $\vec{e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Write $\vec{e}$ as a convex linear combination of $\vec{c}$ and $\vec{d}$.

(c) Substituting the answer to (6a) into the answer to part (6b), write $\vec{e}$ as a convex linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

(d) Draw and label $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$ on the same grid.

(e) Draw the set of convex linear combinations of $\vec{a}$, $\vec{b}$, and $\vec{c}$. Justify your answer.

(a) $\vec{d} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$.

(b) $\vec{e} = \frac{1}{2}\vec{c} + \frac{1}{2}\vec{d}$.

(c)

$$\begin{aligned} \vec{e} &= \tfrac{1}{2}\vec{c} + \tfrac{1}{2}\left(\tfrac{1}{2}\vec{a} + \tfrac{1}{2}\vec{b}\right) \\ &= \tfrac{1}{2}\vec{c} + \tfrac{1}{4}\vec{a} + \tfrac{1}{4}\vec{b}. \end{aligned}$$

(d)

(e) The set is the filled-in triangle with vertices given by $\vec{a}$, $\vec{b}$, and $\vec{c}$. To see this, notice that the set of convex linear combinations of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the set of convex linear combinations of $\vec{c}$ and any convex linear combination of $\vec{a}$ and $\vec{b}$. Indeed,

$$\begin{aligned} &\alpha_1\vec{a} + \alpha_2\vec{b} + \alpha_3\vec{c} \\ &\quad = (1-\alpha_3)\left(\tfrac{\alpha_1}{1-\alpha_3}\vec{a} + \tfrac{\alpha_2}{1-\alpha_3}\vec{b}\right) + \alpha_3\vec{c}, \end{aligned}$$

and $\alpha_1\vec{a} + \alpha_2\vec{b} + \alpha_3\vec{c}$ is a convex linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$ exactly when

$$\frac{\alpha_1}{1-\alpha_3}\vec{a} + \frac{\alpha_2}{1-\alpha_3}\vec{b}$$

is a convex linear combination of $\vec{a}$ and $\vec{b}$ (verify this!).

By the previous part, the set of convex linear combinations of $\vec{a}$ and $\vec{b}$ is the line segment between $\vec{a}$ and $\vec{b}$. Call this line segment S. Now we know the set of convex linear combinations of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the union of every line segment from $\vec{c}$ and a vector in S. This is the solid triangle with vertices given by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

7 Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$. Draw the following subsets of $\mathbb{R}^2$.

(a) All non-negative linear combinations of $\vec{x}$ and $\vec{y}$.

(b) All non-negative linear combinations of $\vec{x}$ and $\vec{z}$.

(c) All convex linear combinations of $\vec{y}$ and $\vec{z}$.

(d) All convex linear combinations of $\vec{x}$ and $\vec{z}$.

(e) All convex linear combinations of $\vec{x}$, $\vec{y}$ and $\vec{z}$.

(a) The acute angled section of $\mathbb{R}^2$ between the ray from the origin through $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the ray from the origin through $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

(b) The line through the origin and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) The line segment between $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

(d) The line segment between $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

(e) The filled-in triangle with vertices $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

8 Describe the sets in (7c) and (7d) in set builder notation.

(7c)

$$\left\{\vec{v} \in \mathbb{R}^2 \;:\; \vec{v} = t\begin{bmatrix} -5 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ for some } 0 \leq t \leq$$

(7d)

$$\left\{\vec{w} \in \mathbb{R}^2 \;:\; \vec{w} = t\begin{bmatrix} -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } 0 \leq t \leq 1$$

9 Determine if the points $P = (-2, 0)$ and $Q = (0, -2)$ are convex linear combinations of the vectors $\vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$. First solve this question by drawing a picture. Then justify algebraically.

Geometrically, the set of convex linear combinations of $\vec{u}$, $\vec{v}$, and $\vec{w}$ is a filled-in triangle with vertices at $\vec{u}, \vec{v}$ and $\vec{w}$. The point P lies inside this triangle, while Q does not.

To argue algebraically, suppose $P = t_1\vec{u} + t_2\vec{v} + t_3\vec{w}$ and $t_1 + t_2 + t_3 = 1$. From these assumptions, we can set up a system of equations which has a unique solution

$$P = \tfrac{1}{4}\vec{u} + \tfrac{1}{4}\vec{v} + \tfrac{1}{2}\vec{w},$$

and so P is a convex linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$. The same procedure with Q gives a unique solution with coefficients $t_1 = \frac{5}{9}, t_2 = -\frac{1}{9}, t_3 = \frac{5}{9}$, and so Q is not a convex linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Non-negative & Convex Linear Combinations

DEFINITION

Let $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$. The vector $\vec{w}$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0.$$

The vector $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1.$$

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7 Let

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \vec{e} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

7.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$?

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$?

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$?

7.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that

(a) $\vec{a}$ and $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not.

(b) $\vec{a}$ and $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$.

(c) $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not.

(d) $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$.

Otherwise, explain why it's not possible.

8 Let L be the set of points $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 1$.

8.1 Describe L using set-builder notation.

8.2 Draw L as a subset of $\mathbb{R}^2$.

8.3 Add the vectors $\vec{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{d} = \vec{b} - \vec{a}$ to your drawing.

8.4 Is $\vec{d} \in L$? Explain.

8.5 For which $t \in \mathbb{R}$ is it true that $\vec{a} + t\vec{d} \in L$? Explain using your picture.

Vector Form of a Line

DEFINITION

Let ℓ be a line and let $\vec{d}$ and $\vec{p}$ be vectors. If $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$, we say the vector equation

$$\vec{x} = t\vec{d} + \vec{p}$$

is ℓ expressed in ***vector form***. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

---

9 Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

9.1 Write ℓ in vector form. Is vector form of ℓ unique?

9.2 Write L in vector form.

9.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

10 Let A, B, and C be given in vector form by

$$\overbrace{\vec{x}=t\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}}^{A}\qquad\overbrace{\vec{x}=t\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}\qquad\overbrace{\vec{x}=t\begin{bmatrix}2\\-1\\1\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

10.1 Do the lines A and B intersect? Justify your conclusion.

10.2 Do the lines A and C intersect? Justify your conclusion.

10.3 Let $\vec{p}\neq\vec{q}$ and suppose X has vector form $\vec{x}=t\vec{d}+\vec{p}$ and Y has vector form $\vec{x}=t\vec{d}+\vec{q}$. Is it possible that X and Y intersect?

Vector Form of a Plane

DEFINITION

A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

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11 Recall the intersecting lines A and B given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

11.1 Find two direction vectors for $\mathcal{P}$.

11.2 Write $\mathcal{P}$ in vector form.

11.3 Describe how vector form of a plane relates to linear combinations.

11.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

12 Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be a plane with equation $x + y + z = 1$.

12.1 Find three points in $\mathcal{Q}$.

12.2 Find two direction vectors for $\mathcal{Q}$.

12.3 Write $\mathcal{Q}$ in vector form.

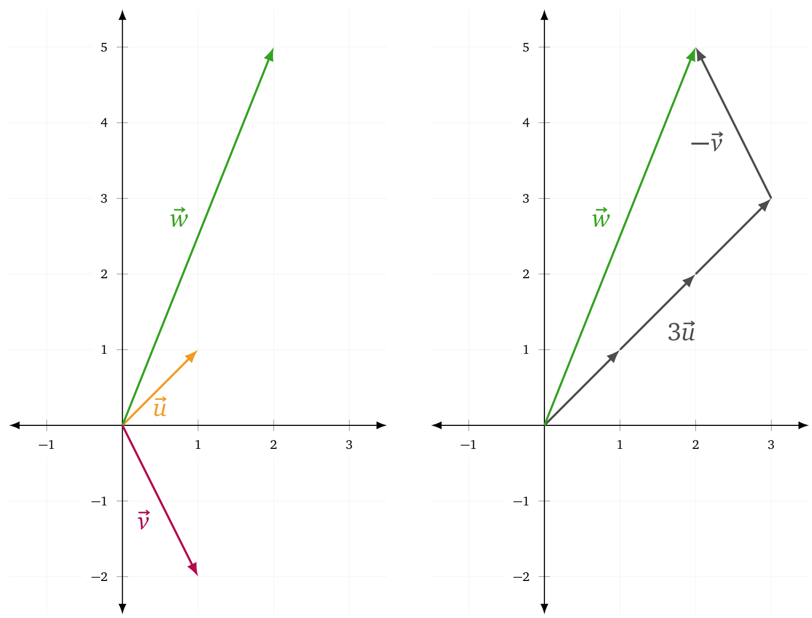
Spans, Translated Spans, and Linear Independence/Dependence

In this module you will learn

- The definition of span and how to visualize spans.
- How to express lines/planes/volumes through the origin as spans.
- How to express lines/planes/volumes *not* through the origin as *translated* spans using set addition.
- Geometric and algebraic definitions of linear independence and linear dependence.
- How to find linearly independent subsets.

Let $\vec{u} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\-2\end{bmatrix}$. Can the vectors $\vec{w} = \begin{bmatrix}2\\5\end{bmatrix}$ be obtained as a linear combination of $\vec{u}$ and $\vec{v}$?

By drawing a picture, the answer appears to be *yes*.



Algebraically, we can use the definition of a *linear combination* to set up a system of equations. We know $\vec{w}$ can be expressed as a linear combination of $\vec{u}$ and $\vec{v}$ if and only if the vector equation

$$\vec{w} = \begin{bmatrix}2\\5\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-2\end{bmatrix} = \alpha\vec{u} + \beta\vec{v}$$

has a solution. By inspection, we see $\alpha = 3$ and $\beta = -1$ solve this equation.

After initial success, we might ask the following: *what are all the locations in $\mathbb{R}^2$ that can be obtained as a linear combination of $\vec{u}$ and $\vec{v}$?* Geometrically, it appears any location can be reached. To verify this algebraically, consider the vector equation

$$\vec{x} = \begin{bmatrix}x\\y\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-2\end{bmatrix} = \alpha\vec{u} + \beta\vec{v}. \tag{1}$$

Here $\vec{x}$ represents an arbitrary point in $\mathbb{R}^2$. If equation (1) always has a solution,[12] any vector in $\mathbb{R}^2$ can be obtained as a linear combination of $\vec{u}$ and $\vec{v}$.

We can solve this equation for α and β by considering the equations arising from the first and second coordinates. Namely,

$$\begin{aligned} x &= \alpha + \beta \\ y &= \alpha - 2\beta \end{aligned}$$

[12] The official terminology would be to say that the equations is always *consistent*.

Subtracting the second equation from the first, we get $x - y = 3\beta$ and so $\beta = (x-y)/3$. Plugging β into the first equation and solving, we get $\alpha = (2x+y)/3$. Thus, equation (1) *always* has the solution

$$\begin{aligned} \alpha &= \tfrac{1}{3}(2x+y) \\ \beta &= \tfrac{1}{3}(x-y). \end{aligned}$$

There is a formal term for the set of vectors that can be obtained as linear combinations of others: *span*.

Span. The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} \,:\, \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Additionally, we define $\operatorname{span}\{\} = \{\vec{0}\}$.

We just showed above that $\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\} = \mathbb{R}^2$.

Example. Let $\vec{u} = \begin{bmatrix}-1\\2\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\-2\end{bmatrix}$. Find $\operatorname{span}\{\vec{u}, \vec{v}\}$.

By the definition of span,

$$\operatorname{span}\{\vec{u}, \vec{v}\} = \{\vec{x} \,:\, \vec{x} = \alpha\vec{u} + \beta\vec{v} \text{ for some } \alpha, \beta \in \mathbb{R}\}.$$

We need to determine for which x and y the vector equation $\begin{bmatrix}x\\y\end{bmatrix} = \alpha\begin{bmatrix}-1\\2\end{bmatrix} + \beta\begin{bmatrix}1\\-2\end{bmatrix}$ is consistent. From the first and second coordinates, we get the system

$$\begin{aligned} x &= -\alpha + \beta \\ y &= 2\alpha - 2\beta \end{aligned}.$$

Adding 2 times the first equation to the second, we get $2x + y = 0$ and so $y = -2x$. Therefore, if $\begin{bmatrix}x\\y\end{bmatrix}$ make the above system consistent, we must have

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}t\\-2t\end{bmatrix} = t\vec{v}$$

for some t. Thus,

$$\operatorname{span}\{\vec{u}, \vec{v}\} = \{\vec{x} \,:\, \vec{x} = t\vec{v} \text{ for some } t\} = \operatorname{span}\{\vec{v}\},$$

which is a line through the origin with direction $\vec{v}$.

Example. Let $\vec{a} = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{c} = \begin{bmatrix}1\\1\\2\end{bmatrix}$. Show that $\mathbb{R}^3 = \operatorname{span}\{\vec{a}, \vec{b}, \vec{c}\}$.

If the equation

$$\vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha_1\begin{bmatrix}1\\2\\1\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}1\\1\\2\end{bmatrix} = \alpha_1\vec{a} + \alpha_2\vec{b} + \alpha_3\vec{c}$$

is always consistent, then any vector in $\mathbb{R}^3$ can be obtained as a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$. Reading off the coordinates, we get the system

$$\begin{aligned} x &= \alpha_1 + \alpha_3 \\ y &= 2\alpha_1 + \alpha_2 + \alpha_3 \\ z &= \alpha_1 + 2\alpha_3 \end{aligned}.$$

Solving this system, we see

$$\begin{aligned} \alpha_1 &= 2x - z \\ \alpha_2 &= -3x + y + z \\ \alpha_3 &= -x + z \end{aligned}$$

is always a solution (no matter the values of x, y, and z). Therefore, $\operatorname{span}\{\vec{a}, \vec{b}, \vec{c}\} = \mathbb{R}^3$.

Representing Lines & Planes as Spans

If spans remind you of vector forms of lines and planes, your intuition is keen. Consider the line ℓ given in vector form by

$$\vec{x} = t\vec{d} + \vec{0}.$$

The line ℓ passes through the origin, and if we unravel its definition, we see

$$\ell = \{\vec{x} \,:\, \vec{x} = t\vec{d} + \vec{0} \text{ for some } t \in \mathbb{R}\} = \{\vec{x} \,:\, \vec{x} = t\vec{d} \text{ for some } t \in \mathbb{R}\} = \operatorname{span}\{\vec{d}\}.$$

Similarly, if $\mathcal{P}$ is a plane given in vector form by

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{0},$$

then

$$\mathcal{P} = \{\vec{x} \,:\, \vec{x} = t\vec{d}_1 + s\vec{d}_2 \text{ for some } t, s \in \mathbb{R}\} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}.$$

If the "$\vec{p}$" in our vector form is $\vec{0}$, then that vector form actually defines a span. This means (if you accept that every line/plane through the origin has a vector form) that every line/plane through the origin can be written as a span. Conversely, if $X = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a span, we know $\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_n \in X$, and so every span passes through the origin.

As it turns out, spans exactly describe points, lines, planes, and volumes[13] through the origin.

Example. The line $\ell_1 \subseteq \mathbb{R}^2$ is described by the equation $x + 2y = 0$ and the line $\ell_2 \subseteq \mathbb{R}^2$ is described by the equation $4x - 2y = 6$. If possible, describe ℓ_1 and ℓ_2 using spans.

We can express ℓ_1 in vector form by

$$\vec{x} = t\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \vec{0},$$

and so

$$\ell_1 = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}.$$

However, ℓ_2 does not pass through $\vec{0}$, and so ℓ_2 cannot be written as a span.

Takeaway. Lines and planes through the origin, and only lines and planes through the origin, can be expressed as spans.

Set Addition

We're going to work around the fact that only objects which pass through the origin can be written as spans, but first let's take a detour and learn about *set addition*.

Set Addition. If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A + B$, is

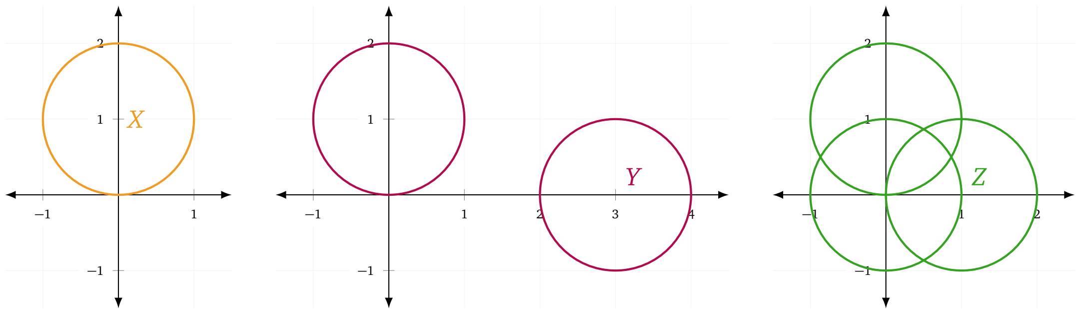
$$A + B = \{\vec{x} \,:\, \vec{x} = \vec{a} + \vec{b} \text{ for some } \vec{a} \in A \text{ and } \vec{b} \in B\}.$$

Set sums are very different than regular sums despite using the same symbol, "+".[14] However, they are very useful. Let $C = \{\vec{x} \in \mathbb{R}^2 \,:\, \|\vec{x}\| = 1\}$ be the unit circle centered at the origin, and consider the sets

$$X = C + \{\vec{e}_2\} \qquad Y = C + \{3\vec{e}_1, \vec{e}_2\} \qquad Z = C + \{\vec{0}, \vec{e}_1, \vec{e}_2\}.$$

Rewriting, we see $X = \{\vec{x} + \vec{e}_2 \,:\, \|\vec{x}\| = 1\}$ is just C translated by $\vec{e}_2$. Similarly, $Y = \{\vec{x} + \vec{v} \,:\, \|\vec{x}\| = 1 \text{ and } \vec{v} = 3\vec{e}_1 \text{ or } \vec{v} = \vec{e}_2\} = (C + \{3\vec{e}_1\}) \cup (C + \{\vec{e}_1\})$, and so Y is the union of two translated copies of C.[15]

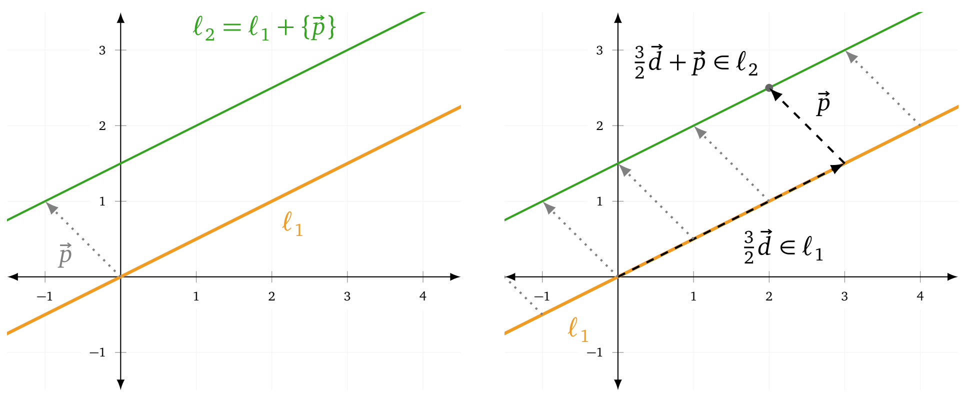
[13] We use the word *volume* to indicate the higher-dimensional analogue of a plane.
[14] For example, $A + \{\} = \{\}$, which might seem counterintuitive for an "addition" operation.
[15] If you want to stretch your mind, consider what $C + C$ is as a set.



Translated Spans

Set addition allows us to easily create parallel lines and planes by translation. For example, consider the lines ℓ_1 and ℓ_2 given in vector form as $\vec{x} = t\vec{d}$ and $\vec{x} = t\vec{d} + \vec{p}$, respectively, where $\vec{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. These lines differ from each other by a translation. That is, every point in ℓ_2 can be obtained by adding $\vec{p}$ to a corresponding point in ℓ_1. Using the idea of set addition, we can express this relationship by the equation

$$\ell_2 = \ell_1 + \{\vec{p}\}.$$



Note: it would be incorrect to write "$\ell_2 = \ell_1 + \vec{p}$". Because ℓ_1 is a set and $\vec{p}$ is not a set, "$\ell_1 + \vec{p}$" does not make mathematical sense.

Example. Recall $\ell_2 \subseteq \mathbb{R}^2$ is the line described by the equation $4x - 2y = 6$. Describe ℓ_2 as a translated span.

We can express ℓ_2 in vector form with the equation

$$\vec{x} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$\ell_2 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} + \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

We can now see translated spans provide an alternative notation to vector form for specifying lines and planes. If Q is described in vector form by

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p},$$

then

$$Q = \operatorname{span}\{\vec{d}_1, \vec{d}_2\} + \{\vec{p}\}.$$

Takeaway. All lines and planes, whether through the origin or not, can be expressed as translated spans.

Linear Independence & Linear Dependence

Let

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Since $\vec{w} = \vec{u} + \vec{v}$, we know that $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$. Geometrically, this is also clear because $\operatorname{span}\{\vec{u}, \vec{v}\}$ is the xy-plane in $\mathbb{R}^3$ and $\vec{w}$ lies on that plane.

What about $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$? Intuitively, since $\vec{w}$ is already a linear combination of $\vec{u}$ and $\vec{v}$, we can't get anywhere *new* by taking linear combinations of $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ compared to linear combinations of just $\vec{u}$ and $\vec{v}$. So $\operatorname{span}\{\vec{u}, \vec{v}\} = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

Can we prove this from the definitions? Yes! Suppose $\vec{r} \in \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$. By definition,

$$\vec{r} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. Since $\vec{w} = \vec{u} + \vec{v}$, we see

$$\vec{r} = \alpha\vec{u} + \beta\vec{v} + \gamma(\vec{u} + \vec{v}) = (\alpha + \gamma)\vec{u} + (\beta + \gamma)\vec{v} \in \operatorname{span}\{\vec{u}, \vec{v}\}.$$

Thus, $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\} \subseteq \operatorname{span}\{\vec{u}, \vec{v}\}$. Conversely, if $\vec{s} \in \operatorname{span}\{\vec{u}, \vec{v}\}$, by definition,

$$\vec{s} = a\vec{u} + b\vec{v} = a\vec{u} + b\vec{v} + 0\vec{w}$$

for some $a, b \in \mathbb{R}$, and so $\vec{s} \in \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$. Thus $\operatorname{span}\{\vec{u}, \vec{v}\} \subseteq \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$. We conclude $\operatorname{span}\{\vec{u}, \vec{v}\} = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

In this case, $\vec{w}$ was a redundant vector—it wasn't needed for the span. When a set contains a redundant vector, we call the set *linearly dependent*.

Linearly Dependent & Independent (Geometric).
We say the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

We will also refer to sets of vectors (for example $\{\vec{v}_1, \dots, \vec{v}_n\}$) as being linearly independent or linearly dependent. For technical reasons, we didn't state the definition in terms of sets.[16]

The geometric definition of linear dependence says that the vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent if you can remove at least one vector without changing the span. In other words, $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent *if there is a redundant vector*.

Example. Let $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{b} = \begin{bmatrix}2\\3\end{bmatrix}$, $\vec{c} = \begin{bmatrix}4\\6\end{bmatrix}$, and $\vec{d} = \begin{bmatrix}4\\5\end{bmatrix}$. Determine whether $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ is linearly independent or linearly dependent.

By inspection, we see $\vec{c} = 2\vec{a}$. Therefore,

$$\operatorname{span}\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\} = \operatorname{span}\{\vec{a}, \vec{b}, \vec{d}\},$$

and so $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ is linearly dependent.

Example. The planes $\mathcal{P}$ and $\mathcal{Q}$ are given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\1\end{bmatrix} + s\begin{bmatrix}2\\2\\1\end{bmatrix}}^{\mathcal{P}} \qquad \text{and} \qquad \overbrace{\vec{x} = t\begin{bmatrix}3\\4\\2\end{bmatrix} + s\begin{bmatrix}2\\2\\1\end{bmatrix}}^{\mathcal{Q}}.$$

Determine if $\mathcal{P}$ and $\mathcal{Q}$ are the same plane.

Let $\vec{a}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix}2\\2\\1\end{bmatrix}$ be the direction vectors for $\mathcal{P}$ and let Let $\vec{b}_1 = \begin{bmatrix}3\\4\\2\end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix}2\\2\\1\end{bmatrix}$ be the direction vectors for $\mathcal{Q}$. We know that $\mathcal{P} = \mathcal{Q}$ if every direction vector for $\mathcal{Q}$ is a linear combination of the direction vectors for $\mathcal{P}$. In other words, $\mathcal{P} = \mathcal{Q}$ if

$$\{\vec{a}_1, \vec{a}_2, \vec{b}_1\} \qquad \text{and} \qquad \{\vec{a}_1, \vec{a}_2, \vec{b}_2\}$$

[16] The issue is, every element of a set is unique. Clearly, the vectors $\vec{v}$ and $\vec{v}$ are linearly dependent, but $\{\vec{v}, \vec{v}\} = \{\vec{v}\}$, and so $\{\vec{v}, \vec{v}\}$ is technically a linearly independent set. This issue would be resolved by talking about *multisets* instead of sets, but it isn't worth the hassle.

are both linearly dependent sets. Since $\vec{a}_2 = \vec{b}_2$, clearly $\{\vec{a}_1, \vec{a}_2, \vec{b}_2\}$ is linearly dependent.
We will now check whether $\{\vec{a}_1, \vec{a}_2, \vec{b}_1\}$ is linearly dependent. Since $\{\vec{a}_1, \vec{a}_2\}$ is a linearly independent set, we only need to check if $\vec{b}_1$ can be written as a linear combination of $\vec{a}_1$ and $\vec{a}_2$. After solving the system

$$\begin{bmatrix}3\\4\\2\end{bmatrix} = x\begin{bmatrix}1\\2\\1\end{bmatrix} + y\begin{bmatrix}2\\2\\1\end{bmatrix},$$

we see that $\vec{b}_1 = \vec{a}_1 + \vec{a}_2$, and so $\{\vec{a}_1, \vec{a}_2, \vec{b}_1\}$ is linearly dependent. Therefore, $\mathcal{P} = \mathcal{Q}$.

We can also think of linear independence/dependence from an algebraic perspective. Suppose the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ satisfy

$$\vec{w} = \vec{u} + \vec{v}. \tag{2}$$

The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent since $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$, but equation (2) can be rearranged to get

$$\vec{0} = \vec{u} + \vec{v} - \vec{w}. \tag{3}$$

Here we have expressed $\vec{0}$ as a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$. By itself, this is nothing special. After all, we know $\vec{0} = 0\vec{u} + 0\vec{v} + 0\vec{w}$ is a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$. However, the right side of equation (3) has non-zero coefficients, which makes the linear combination *non-trivial*.

Trivial Linear Combination.
The linear combination $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ is called ***trivial*** if $\alpha_1 = \cdots = \alpha_n = 0$. If at least one $\alpha_i \neq 0$, the linear combination is called ***non-trivial***.

We can always write $\vec{0}$ as a linear combination of vectors if we let all the coefficients be zero, but it turns out we can only write $\vec{0}$ as a *non-trivial* linear combination of vectors if those vectors are linearly dependent. This is the inspiration for another definition of linear independence/dependence.

Linearly Dependent & Independent (Algebraic).
The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals the zero vector. Otherwise they are linearly independent.

The idea of a "redundant vector" coming from the geometric definition of linear dependence is easy to visualize, but it can be hard to prove things with—checking for linear independence with the geometric definition involves verifying for every vector that it is not in the span of the others. The algebraic definition on the other hand is less obvious, but the reasoning is easier. You only need to analyze solutions to one equation!

Example. Let $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}4\\5\end{bmatrix}$. Use the algebraic definition of linear independence to determine whether $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent or dependent.

We need to determine if there is a non-trivial solution to

$$x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}.$$

This vector equation is equivalent to the system of equations

$$\begin{cases} x + 2y + 4z = 0 \\ 2x + 3y + 5z = 0 \end{cases}.$$

Solving this system using row reduction, we see the complete solution set can be expressed as

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}2\\-3\\1\end{bmatrix}.$$

In particular, $(x, y, z) = (2, -3, 1)$ is a non-trivial solution to this system. Therefore $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

Theorem. The geometric and algebraic definitions of linear independence are equivalent.

Proof. To show the two definitions are equivalent, we need to show that geometric $\implies$ algebraic and algebraic $\implies$ geometric.

(geometric $\implies$ algebraic) Suppose $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent by the geometric definition. That means that for some i, we have

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Fix such an i. Then, by the definition of span we know

$$\vec{v}_i = \alpha_1\vec{v}_1 + \cdots \alpha_{i-1}\vec{v}_{i-1} + \alpha_{i+1}\vec{v}_{i+1} + \cdots + \alpha_n\vec{v}_n,$$

and so

$$\vec{0} = \alpha_1\vec{v}_1 + \cdots \alpha_{i-1}\vec{v}_{i-1} - \vec{v}_i + \alpha_{i+1}\vec{v}_{i+1} + \cdots + \alpha_n\vec{v}_n.$$

This must be a non-trivial linear combination because the coefficient of $\vec{v}_i$ is $-1 \neq 0$. Therefore, $\vec{v}_1, \dots, \vec{v}_n$ is linearly dependent by the algebraic definition.

(geometric $\implies$ algebraic) Suppose $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent by the algebraic definition. That means there exist $\alpha_1, \dots, \alpha_n$, not all zero, so that

$$\vec{0} = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n.$$

Fix i so that $\alpha_i \neq 0$ (why do we know there is such an i?). Rearranging we get

$$-\alpha_i\vec{v}_i = \alpha_1\vec{v}_1 + \cdots \alpha_{i-1}\vec{v}_{i-1} + \alpha_{i+1}\vec{v}_{i+1} + \cdots + \alpha_n\vec{v}_n,$$

and since $\alpha_i \neq 0$, we can multiply both sides by $\frac{-1}{\alpha_i}$ to get

$$\vec{v}_i = \tfrac{-\alpha_1}{\alpha_i}\vec{v}_1 + \cdots + \tfrac{-\alpha_{i-1}}{\alpha_i}\vec{v}_{i-1} + \tfrac{-\alpha_{i+1}}{\alpha_i}\vec{v}_{i+1} + \cdots + \tfrac{-\alpha_n}{\alpha_i}\vec{v}_n.$$

This shows that

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\},$$

and so $\vec{v}_1, \dots, \vec{v}_n$ is linearly dependent by the geometric definition. ■

Linear Independence and Unique Solutions

The algebraic definition of linear independence can teach us something about solutions to systems of equations.

Recall the linearly dependent vectors

$$\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

which satisfy the non-trivial relationship $\vec{u} + \vec{v} - \vec{w} = \vec{0}$. Since $\vec{u} + \vec{v} - \vec{w} = \vec{0}$ is a non-trivial relationship giving $\vec{0}$, we can use it to generate others. For example,

$$\begin{aligned} 17(\vec{u} + \vec{v} - \vec{w}) &= 17\vec{u} + 17\vec{v} - 17\vec{w} = 17\vec{0} = \vec{0} \\ -3(\vec{u} + \vec{v} - \vec{w}) &= -3\vec{u} - 3\vec{v} + 3\vec{w} = -3\vec{0} = \vec{0} \\ &\vdots \end{aligned}$$

are all different non-trivial linear combinations that give $\vec{0}$. In other words, if the equation $\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0}$ has a non-trivial solution, it has *infinitely many* non-trivial solutions. Conversely, if the equation $\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0}$ has infinitely many solutions, one of them has to be non-trivial!

Equations where one side is $\vec{0}$ show up often and are called *homogeneous* equations.

Homogeneous System.
A system of linear equations or a vector equation in the variables $\alpha_1, \dots, \alpha_n$ is called ***homogeneous*** if it takes the form

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0},$$

where the right side of the equation is $\vec{0}$.

This insight links linear independence and homogeneous systems together, and is encapsulated in the following theorem.

Theorem. The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if and only if the homogeneous equation

$$\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = \vec{0}$$

has a unique solution.

This theorem has a practical application: suppose you wanted to decide if the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ were linearly independent. You could (i) find a non-trivial solution to $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, or (ii) merely show that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ has more than one solution. Sometimes one is easier than the other.

Linear Independence and Vector Form

The equation

$$\vec{x} = t_1\vec{d}_1 + t_2\vec{d}_2$$

represents a plane in vector form whenever $\vec{d}_1$ and $\vec{d}_2$ are non-zero, non-parallel vectors. In other words, $\vec{x} = t_1\vec{d}_1 + t_2\vec{d}_2$ represents a plane whenever $\{\vec{d}_1, \vec{d}_2\}$ is linearly independent.

Does this reasoning work for lines too? The equation

$$\vec{x} = t\vec{d}$$

represents a line in vector form precisely when $\vec{d} \neq \vec{0}$. And $\{\vec{d}\}$ is linearly independent exactly when $\vec{d} \neq 0$.

This reasoning generalizes to volumes. The equation

$$\vec{x} = t_1\vec{d}_1 + t_2\vec{d}_2 + t_3\vec{d}_3$$

represents a *volume* in vector form exactly when $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ is linearly independent. To see this, suppose $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ were linearly dependent. That means one or more vectors could be removed from $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ without changing its span. Therefore, if $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ is linearly dependent $\vec{x} = t_1\vec{d}_1 + t_2\vec{d}_2 + t_3\vec{d}_3$ at best represents a plane (though it could be a line or a point).

We now have a way of testing the validity of a vector-form representation of a line/plane/volume. Just check whether the chosen direction vectors are linearly independent!

Takeaway. When writing an object in vector form, the direction vectors must always be linearly independent.

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

Span

Span

DEFINITION

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Additionally, we define $\operatorname{span}\{\} = \{\vec{0}\}$.

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13 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

13.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

13.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

13.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

13.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

13.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

14 Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

14.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

14.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

14.3 Does the expression span(ℓ_1) make sense? If so, what is it? How about span(ℓ_2)?

Set Addition

DEF

If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A+B$, is

$$A+B=\{\vec{x}\,:\,\vec{x}=\vec{a}+\vec{b}\text{ for some }\vec{a}\in A\text{ and }\vec{b}\in B\}.$$

---

15 Let $A=\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell=\operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

15.1 Draw A, B, and $A+B$ in the same picture.

15.2 Is $A+B$ the same as $B+A$?

15.3 Draw $\ell+A$.

15.4 Consider the line ℓ_2 given in vector form by $\vec{x}=t\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

Task 1.3: The Magic Carpet, Getting Back Home

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$).

1. Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.
2. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?
3. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?
4. What is $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}$?

Linearly Dependent & Independent (Geometric)

DEFINITION

We say the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

---

16 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

16.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

16.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

16.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

16.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

Trivial Linear Combination

DEF

The linear combination $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ is called ***trivial*** if $\alpha_1 = \cdots = \alpha_n = 0$. If at least one $\alpha_i \neq 0$, the linear combination is called ***non-trivial***.

---

17 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

17.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

17.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

Linearly Dependent & Independent (Algebraic)

DEF The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector. Otherwise they are linearly independent.

18 18.1 Explain how this algebraic definition (new) implies the geometric one (original).

18.2 Explain how the geometric definition (original) implies this algebraic one (new).

Since we have geometric def $\implies$ algebraic def, and algebraic def $\implies$ geometric def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as algebraic def $\iff$ geometric def).

19 Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

19.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

19.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

19.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

19.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

Task 1.4: Linear Independence and Dependence, Creating Examples

1. Fill in the following chart keeping track of the strategies you used to generate examples.

	Linearly independent	Linearly dependent
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

2. Write at least two generalizations that can be made from these examples and the strategies you used to create them.

Dot Products & Normal Forms

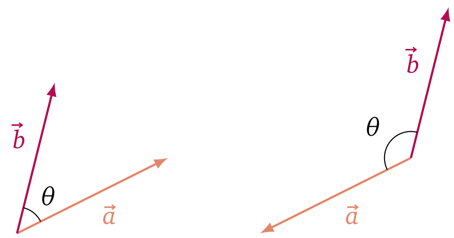
In this module you will learn

- Geometric an algebraic definitions of the dot product.
- How dot products relate to the length of a vector and the angle between two vectors.
- The *normal form* of lines, planes, and hyperplanes.

Let $\vec{a}$ and $\vec{b}$ be vectors rooted at the same point and let θ denote the *smaller* of the two angles between them (so $0 \leq \theta \leq \pi$). The *dot product* of $\vec{a}$ and $\vec{b}$ is defined to be

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos \theta .$$

We will call this the *geometric definition of the dot product*.



The dot product is also sometimes called the *scalar product* because the result is a scalar.

Algebraically, we can define the dot product in terms of coordinates:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n .$$

We will call this the *algebraic definition of the dot product*.

By switching between algebraic and geometric definitions, we can use the dot product to find quantities that are otherwise difficult to find.

Example. Find the angle between the vectors $\vec{v} = (1, 2, 3)$ and $\vec{w} = (1, 1, -2)$.

From the algebraic definition of the dot product, we know

$$\vec{v} \cdot \vec{w} = 1(1) + 2(1) + 3(-2) = -3.$$

From the geometric definition, we know

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos \theta = \sqrt{14}\sqrt{6} \cos \theta = 2\sqrt{21} \cos \theta .$$

Equating the two definitions of $\vec{v} \cdot \vec{w}$, we see

$$\cos \theta = \frac{-3}{2\sqrt{21}}$$

and so $\theta = \arccos\left(\frac{-3}{2\sqrt{21}}\right)$.

The dot product has several interesting properties. Since the angle between $\vec{a}$ and itself is 0, the geometric definition of the dot product tells us

$$\vec{a} \cdot \vec{a} = \|\vec{a}\|\|\vec{a}\| \cos 0 = \|\vec{a}\|^2 .$$

In other words,

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}},$$

and so dot products can be used to compute the length of vectors.[17]

From the algebraic definition of the dot product, we can deduce several distributive laws. Namely, for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ and any scalar k we have

$$(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} \qquad \vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

$$(k\vec{a})\cdot\vec{b}=k(\vec{a}\cdot\vec{b})=\vec{a}\cdot(k\vec{b})$$

and

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}.$$

Orthogonality

Recall that for vectors $\vec{a}$ and $\vec{b}$, the relationship $\vec{a}\cdot\vec{b}=0$ can hold for two reasons: (i) either $\vec{a}=\vec{0}$, $\vec{b}=\vec{0}$, or both or (ii) $\vec{a}$ and $\vec{b}$ meet at 90°. Thus, the dot product can be used to tell if two vectors are perpendicular. There is some strangeness with the zero vector here, but it turns out this strangeness simplifies our lives mathematically.

Orthogonal. Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v}=0$. The word orthogonal is synonymous with the word perpendicular.

The definition of orthogonal encapsulates both the idea of two vectors forming a right angle and the idea of one of them being $\vec{0}$.

Before we continue, let's pin down exactly what we mean by the *direction* of a vector. There are many ways we could define this term, but we'll go with the following.

Direction. The vector $\vec{u}$ points in the ***direction*** of the vector $\vec{v}$ if $k\vec{u}=\vec{v}$ for some scalar k. The vector $\vec{u}$ points in the ***positive direction*** of $\vec{v}$ if $k\vec{u}=\vec{v}$ for some positive scalar k.

The vector $2\vec{e}_1$ points in the direction of $\vec{e}_1$ since $\frac{1}{2}(2\vec{e}_1)=\vec{e}_1$. Since $\frac{1}{2}>0$, $2\vec{e}_1$ also points in the positive direction of $\vec{e}_1$. In contrast, $-\vec{e}_1$ points in the direction $\vec{e}_1$ but not the positive direction of $\vec{e}_1$.

When it comes to the relationship between two vectors, there are two extremes: they point in the same direction, or they are orthogonal. The dot product can be used to tell you which of these cases you're in, and more than that, it can tell you to what extent one vector points in the direction of another (even if they don't point in the same direction).

Example. Let $\vec{a}=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec{b}=\begin{bmatrix}3\\3\end{bmatrix}$, $\vec{c}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{v}=\begin{bmatrix}3\\4\end{bmatrix}$. Which vector out of $\vec{a}$, $\vec{b}$, and $\vec{c}$ has a direction closest to the direction of $\vec{v}$?

We would like to know when θ, the angle between a pair of the given vectors, is smallest. This is equivalent to finding when $\cos\theta$ is closest to 1 (since $\cos 0=1$). By equating the geometric and algebraic definitions of the dot product, we know

$$\cos\theta=\frac{\vec{p}\cdot\vec{q}}{\|\vec{p}\|\|\vec{q}\|}.$$

Let α, β and γ between the vector $\vec{v}$ and the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, respectively. Computing, we find

$$\cos\alpha=\frac{3+8}{5\sqrt{5}}=\frac{11\sqrt{5}}{25}\approx 0.9838699101$$

$$\cos\beta=\frac{9+12}{5\cdot 3\sqrt{2}}=\frac{7\sqrt{2}}{10}\approx 0.989949437$$

$$\cos\gamma=\frac{6+4}{5\sqrt{5}}=\frac{2\sqrt{5}}{5}\approx 0.894427191.$$

Since $\cos\beta$ is the closest to 1, we know $\vec{b}$ has a direction closest to that of $\vec{v}$.

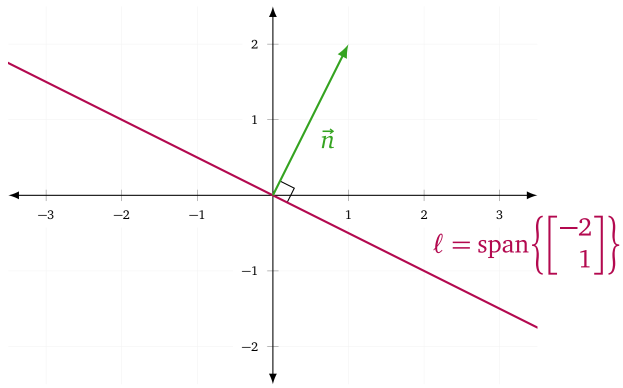
Normal Form of Lines and Planes

[17] Oftentimes in non-geometric settings, the dot product between two vectors is defined first and then the length of $\vec{a}$ is actually *defined* to be $\sqrt{\vec{a}\cdot\vec{a}}$.

Let $\vec{n} = \begin{bmatrix}1\\2\end{bmatrix}$. If a vector $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ is orthogonal to $\vec{n}$, then

$$\vec{n} \cdot \vec{v} = v_1 + 2v_2 = 0,$$

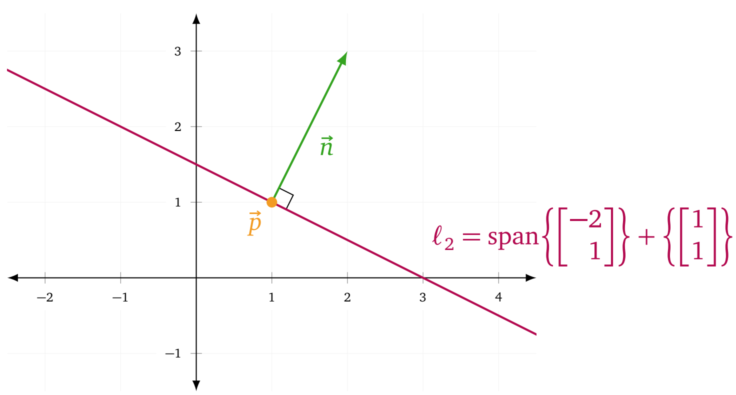
and so $v_1 = -2v_2$. In other words, $\vec{v}$ is orthogonal to $\vec{n}$ exactly when $\vec{v} \in \operatorname{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$. What have we learned? The set of all vectors orthogonal to $\vec{n}$ forms a line $\ell = \operatorname{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$. In this case, we call $\vec{n}$ a *normal vector* for ℓ.



Normal Vector. A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

In $\mathbb{R}^2$, normal vectors provide yet another way to describe lines, including lines which don't pass through the origin.

Let $n = \begin{bmatrix}1\\2\end{bmatrix}$ as before, and fix $\vec{p} = \begin{bmatrix}1\\1\end{bmatrix}$. If we draw the set of all vectors orthogonal to $\vec{n}$ but *root all the vectors at* $\vec{p}$, again we get a line, but this time the line passes through $\vec{p}$.



In fact, the line we get is $\ell_2 = \operatorname{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\} + \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} = \ell + \{\vec{p}\}$, which is just ℓ (the parallel line through the origin) translated by $\vec{p}$.

Let's relate this to dot products and normal vectors. By definition, for every $\vec{v} \in \ell$, we have $\vec{n} \cdot \vec{v} = 0$. Since ℓ_2 is a translate of ℓ by $\vec{p}$, we deduce the relationship that for every $\vec{v} \in \ell_2$,

$$\vec{n} \cdot (\vec{v} - \vec{p}) = 0.$$

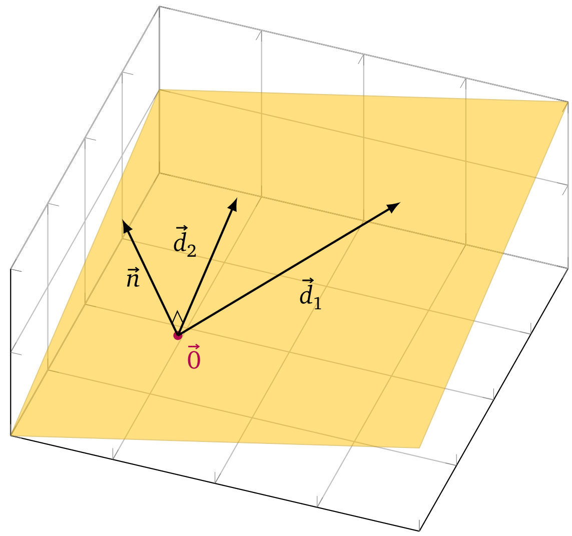
When a line is expressed as above, we say it is expressed in *normal form*.

Normal Form of a Line. A line $\ell \subseteq \mathbb{R}^2$ is expressed in ***normal form*** if ℓ is the solution set to the equation

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

where $\vec{n}$ and $\vec{p}$ are fixed vectors.

What about in $\mathbb{R}^3$? Fix a non-zero vector $\vec{n} \in \mathbb{R}^3$ and let $\mathcal{Q} \subseteq \mathbb{R}^3$ be the set of vectors orthogonal to $\vec{n}$. $\mathcal{Q}$ is a plane through the origin, and again, we call $\vec{n}$ a *normal vector* of the plane $\mathcal{Q}$.



In a similar way to the line, $\mathcal{Q}$ is the set of solutions to $\vec{n} \cdot \vec{x} = 0$. And, for any $\vec{p} \in \mathbb{R}^3$, the translated plane $\mathcal{Q} + \{\vec{p}\}$ is the solution set to

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0.$$

Thus, we see planes in $\mathbb{R}^3$ have a normal form just like lines in $\mathbb{R}^2$ do.

Example. Find vector form and normal form of the plane $\mathcal{P}$ passing through the points $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$.

To find vector form of $\mathcal{P}$, we need a point on the plane and two direction vectors. We have three points on the plane, so we can obtain two direction vectors by subtracting these points in different ways. Let

$$\vec{d}_1 = \overrightarrow{AB} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{d}_2 = \overrightarrow{AC} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Using the point A, we may now express $\mathcal{P}$ in vector form by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

To write $\mathcal{P}$ in normal, form we need to find a normal vector for $\mathcal{P}$. By inspection, we see that $\vec{n} = (1, 1, 1)$ is a normal vector to $\mathcal{P}$. (If we weren't so insightful, we could also solve the system $\vec{n} \cdot \vec{d}_1 = 0$ and $\vec{n} \cdot \vec{d}_2 = 0$ to find a normal vector.) Now, we may express $\mathcal{P}$ in normal form as

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = 0.$$

In $\mathbb{R}^2$, only lines have a normal form, and in $\mathbb{R}^3$ only planes have a normal form. In general, we call objects in $\mathbb{R}^n$ which have a normal form *hyperplanes*.

Hyperplane. The set $X \subseteq \mathbb{R}^n$ is called a ***hyperplane*** if there exists $\vec{n} \neq \vec{0}$ and $\vec{p}$ so that X is the set of solutions to the equation

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0.$$

Hyperplanes always have dimension one less than the space they're contained in. So, hyperplanes in $\mathbb{R}^2$ are (one-dimensional) lines, hyperplanes in $\mathbb{R}^3$ are regular (two-dimensional) planes, and hyperplanes in $\mathbb{R}^4$ are (three-dimensional) volumes.

Hyperplanes and Linear Equations

Suppose $\vec{n},\vec{p}\in\mathbb{R}^3$ and $\vec{n}\neq\vec{0}$. Then, solutions to

$$\vec{n}\cdot(\vec{x}-\vec{p})=0$$

define a plane $\mathcal{P}$. But, $\vec{n}\cdot(\vec{x}-\vec{p})=0$ if and only if

$$\vec{n}\cdot\vec{x}=\vec{n}\cdot\vec{p}=\alpha.$$

Since $\vec{n}$ and $\vec{p}$ are fixed, α is a constant. Expanding using coordinates, we see

$$\vec{n}\cdot(\vec{x}-\vec{p})=\vec{n}\cdot\vec{x}-\alpha=n_xx+n_yy+n_zz-\alpha=0$$

and so $\mathcal{P}$ is the set of solutions to

$$n_xx+n_yy+n_zz=\alpha. \tag{4}$$

Equation (4) is sometimes called *scalar form* of a plane. For us, it will not be important to distinguish between scalar and normal form, but what is important is that we can use the row reduction algorithm to write the complete solution to (4), and this complete solution will necessarily be written in vector form.

Example. Let $\mathcal{Q}\subseteq\mathbb{R}^3$ be the plane passing through $\vec{p}$ and with normal vector $\vec{n}$ where

$$\vec{p}=\begin{bmatrix}1\\1\\0\end{bmatrix}\qquad\text{and}\qquad\vec{n}=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Write $\mathcal{Q}$ in vector form.

We know $\mathcal{Q}$ is the set of solutions to $\vec{n}\cdot(\vec{x}-\vec{p})=0$. In scalar form, this equation becomes

$$\vec{n}\cdot(\vec{x}-\vec{p})=\vec{n}\cdot\vec{x}-\vec{n}\cdot\vec{p}=x+y+z-2=0.$$

Rearranging, we see $\mathcal{Q}$ is the set of all solutions to

$$x+y+z=2.$$

Using the row reduction algorithm to write the complete solution,[a] we get

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}-1\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\end{bmatrix}+\begin{bmatrix}2\\0\\0\end{bmatrix}.$$

[a] In some sense, this is overkill because the equation corresponds to the augmented matrix $\left[\begin{array}{ccc|c}1&1&1&2\end{array}\right]$, which is already row reduced.

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2,4,8)$

(b) $(15,6,-21)$

Dot Product

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

---

20 Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

20.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

20.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

20.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

20.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

THM

For a vector $\vec{v} \in \mathbb{R}^n$, the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

always holds.

Distance

DEF

The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

Unit Vector

DEF

A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

21

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

21.1 Find the distance between $\vec{u}$ and $\vec{v}$.

21.2 Find a unit vector in the direction of $\vec{u}$.

21.3 Does there exist a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

21.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Orthogonal

DEF Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u} \cdot \vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

22

22.1 Find two vectors orthogonal to $\vec{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Can you find two such vectors that are not parallel?

22.2 Find two vectors orthogonal to $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Can you find two such vectors that are not parallel?

22.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\| = 5$ and $\|\vec{y}\| = 3$. What is the distance between $\vec{x}$ and $\vec{y}$?

23

23.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.

23.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

23.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for A.

23.4 If possible, express A as a span.

Normal Vector

DEF

A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

24 Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}1\\-1\end{bmatrix}$, and define the lines

$$\ell_1 = \mathrm{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \mathrm{span}\{\vec{d}\} + \{\vec{p}\}.$$

24.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

24.2 Let $\vec{v} \in \ell_1$ and $\vec{u} \in \ell_2$. What is $\vec{n} \cdot \vec{v}$? What about $\vec{n} \cdot (\vec{u} - \vec{p})$? Explain using a picture.

24.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n} \cdot (\vec{x} - \vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

24.4 Some textbooks would claim that ℓ_2 could be expressed in normal form as $\begin{bmatrix}2\\-1\end{bmatrix} \cdot \vec{x} = 3$. How does this relate to the $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ normal form? Where does the 3 come from?

25 Let $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

25.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

25.2 Describe X using an equation.

25.3 Describe X as a span.

Projections & Vector Components

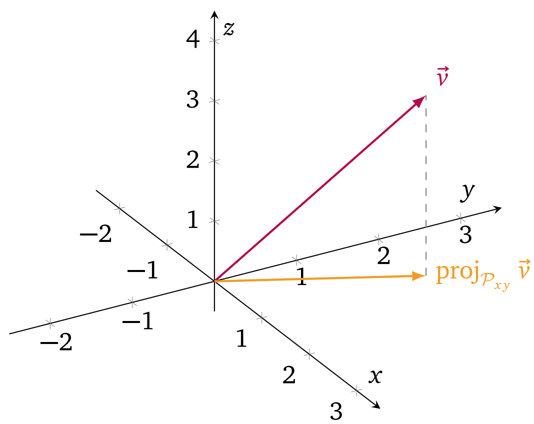
In this module you will learn

- The definition of the projection of a vector onto a set and the definition of the vector component of one vector in the direction of another.
- The relationship between projection, orthogonality, and vector components.
- How to project a vector onto a line.

Consider the following situation: you're designing a 3d video game, but your users only have 2d screens. Or, you have a 900-dimensional dataset, but you want to visualize it on a continuum (i.e., as a line). Each of these is an example of finding the best approximation to particular points given restrictions. In general, this operation is called a *projection*,[18] and in the world of linear algebra, it has a very particular meaning.

Projection. Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\mathrm{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

Let $\mathcal{P}_{xy} \subseteq \mathbb{R}^3$ be the xy-plane in $\mathbb{R}^3$ and let $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Intuitively, $\mathrm{proj}_{\mathcal{P}_{xy}} \vec{v}$ is the "shadow" that $\vec{v}$ would casts on $\mathcal{P}_{xy}$ if the sun were directly overhead. Upon drawing a picture, we conclude $\mathrm{proj}_{\mathcal{P}_{xy}} \vec{v} = \begin{bmatrix}1\\2\\0\end{bmatrix}$.



Continuing, let $\ell_y \subseteq \mathbb{R}^3$ be the y-axis in $\mathbb{R}^3$. It's a little bit harder to visualize what $\mathrm{proj}_{\ell_y} \vec{v}$ is, so let's appeal to some definitions.

By definition, every vector in ℓ_y takes the form $\vec{u}_t = \begin{bmatrix}0\\t\\0\end{bmatrix}$ for some $t \in \mathbb{R}$. The distance between $\vec{u}_t$ and $\vec{v}$ is

$$\|\vec{u}_t - \vec{v}\| = \left\| \begin{bmatrix}0\\t\\0\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} \right\| = \sqrt{1^2 + (t-2)^2 + 3^2}.$$

Since $(t-2)^2$ is always positive, the quantity $\sqrt{1^2 + (t-2)^2 + 3^2}$ is minimized when $(t-2)^2 = 0$; that is, when $t = 2$. Thus, we see $\vec{u}_2$ is the closest vector in ℓ_y to $\vec{v}$ and so,

$$\mathrm{proj}_{\ell_y} \vec{v} = \vec{u}_2 = \begin{bmatrix}0\\2\\0\end{bmatrix}.$$

[18] What we define as a *projection* is sometimes called the *orthogonal projection* to distinguish it from other types of projections.

Example. Let $\ell \subseteq \mathbb{R}^2$ be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix}$, and let $\vec{v} = \begin{bmatrix}-1\\-1\end{bmatrix}$. Use the definition of projection to find $\operatorname{proj}_\ell \vec{v}$.

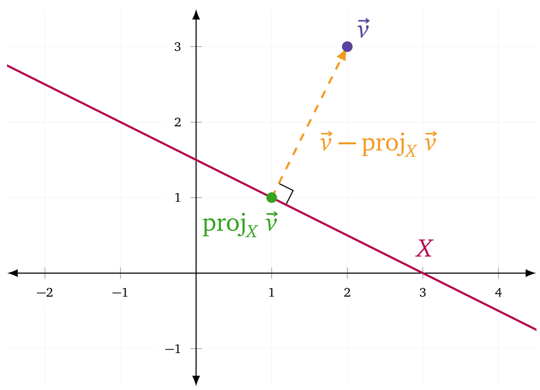
Let $\vec{u}_t = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix} \in \ell$. By definition, the distance between $\vec{v}$ and $\vec{u}_t$ is given by

$$\|\vec{u}_t - \vec{v}\| = \left\| \left(t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix}\right) - \begin{bmatrix}-1\\-1\end{bmatrix} \right\| = \left\| \begin{bmatrix}t+2\\t-3\end{bmatrix} \right\| = \sqrt{2t^2+6t+17}.$$

The quantity $2t^2+6t+17$ is minimized when $t = -\frac{3}{2}$, and so the closest point in ℓ to $\vec{v}$ is $\vec{u}_{-3/2}$. Thus,

$$\operatorname{proj}_\ell \vec{v} = -\tfrac{3}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}3/2\\-7/2\end{bmatrix}.$$

Every example of a projection so far shares a geometric property. In the case of lines and planes, the vector from the projection to the original point is a normal vector for the line or plane (provided it's non-zero).



Stated precisely, if X is a line or plane and $\vec{v} \notin X$ is a vector, then $\vec{v} - \operatorname{proj}_X \vec{v}$ is a normal vector for X. Using this fact, we can find projections onto lines and planes without needing to compute any distances!

Example. Let $\ell \subseteq \mathbb{R}^2$ be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix}$, and let $\vec{v} = \begin{bmatrix}-1\\-1\end{bmatrix}$. Use the fact that $\vec{v} - \operatorname{proj}_\ell \vec{v}$ is a normal vector to ℓ to find Find $\operatorname{proj}_\ell \vec{v}$.

Since $\vec{v} - \operatorname{proj}_\ell \vec{v}$ is a normal vector to ℓ, we know $\vec{v} - \operatorname{proj}_\ell \vec{v}$ is orthogonal to $\vec{d} = \begin{bmatrix}1\\1\end{bmatrix}$. Let $\begin{bmatrix}x\\y\end{bmatrix} = \operatorname{proj}_\ell \vec{v}$ for some unknown $x, y \in \mathbb{R}$. We now know

$$(\vec{v} - \operatorname{proj}_\ell \vec{v}) \cdot \vec{d} = \left(\begin{bmatrix}-1\\-1\end{bmatrix} - \begin{bmatrix}x\\y\end{bmatrix} \right) \cdot \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1-x\\-1-y\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = -2 - x - y = 0.$$

That is,

$$x + y = -2. \tag{5}$$

Also, since $\operatorname{proj}_\ell \vec{v} \in \ell$, we know $\operatorname{proj}_\ell \vec{v}$ know

$$\begin{bmatrix}x\\y\end{bmatrix} = t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}t+3\\t-2\end{bmatrix}.$$

From this, we have that $x - t = 3$ and $y - t = -3$. Combined with Equation (5), we have three equations and three unknowns which produce the following system of linear equations.

$$\begin{cases} \phantom{-t+{}} x + y = -2 \\ -t + x \phantom{{}+y} = 3 \\ -t \phantom{{}+x} + y = -2 \end{cases}$$

Solving this system, we conclude that $x = 3/2$ and $y = -7/2$ (we don't care about the value of t). Therefore $\operatorname{proj}_\ell \vec{v} = \begin{bmatrix}3/2\\-7/2\end{bmatrix}$.

Takeaway. When projecting onto lines an planes, right angles appear in key places.

Projections Onto Other Sets

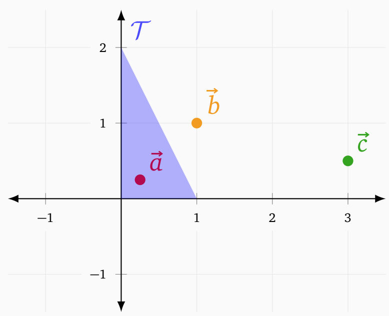
For projections onto lines and planes, we can use what we know about normal vectors to simplify our life. The same is true when projecting onto other sets, but we must always keep the definition in mind.

Example. Let $\mathcal{T} \subseteq \mathbb{R}^2$ be the filled in triangle with vertices $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\2\end{bmatrix}$, and let

$$\vec{a} = \begin{bmatrix}1/4\\1/4\end{bmatrix} \qquad \vec{b} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \vec{c} = \begin{bmatrix}3\\1/2\end{bmatrix}.$$

Find $\text{proj}_{\mathcal{T}}\,\vec{a}$, $\text{proj}_{\mathcal{T}}\,\vec{b}$, and $\text{proj}_{\mathcal{T}}\,\vec{c}$.

We'll start by drawing a picture.



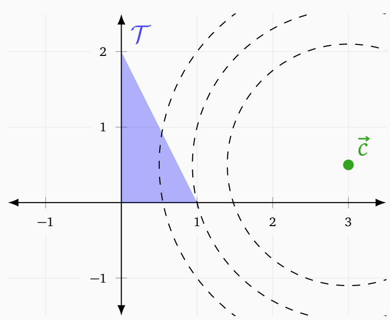
From the picture, we see that $\vec{a} \in \mathcal{T}$ and so

$$\text{proj}_{\mathcal{T}}\,\vec{a} = \vec{a}.$$

We also see that $\vec{b}$ is closest to the hypotenuse of $\mathcal{T}$, and so $\text{proj}_{\mathcal{T}}\,\vec{b}$ is the same as the projection of $\vec{b}$ onto the line $y = -2x + 2$. Computing, we find

$$\text{proj}_{\mathcal{T}}\,\vec{b} = \begin{bmatrix}3/5\\4/5\end{bmatrix}.$$

Finally, drawing concentric circles centered at $\vec{c}$, we see that the lower-right corner of $\mathcal{T}$ is the closest point in $\mathcal{T}$ to $\vec{c}$.



And so,

$$\text{proj}_{\mathcal{T}}\,\vec{c} = \begin{bmatrix}1\\0\end{bmatrix}.$$

Subtleties of Projections

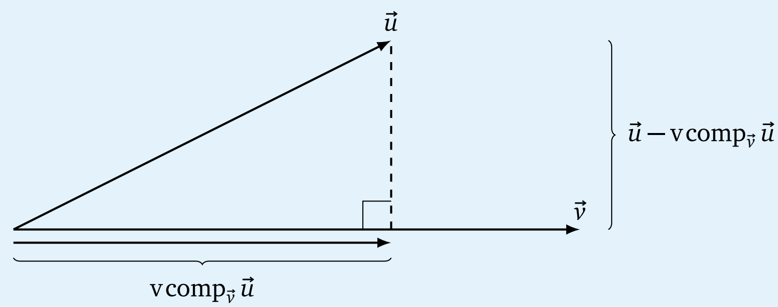
You might be wondering, what is $\text{proj}_X\,\vec{v}$ if $\vec{v}$ is equidistant from *two* closest points in X? Or, what if X is an *open* set (for example, an open interval in $\mathbb{R}^1$)? Then there might not be a closest point in X to $\vec{v}$. In both these cases, we say $\text{proj}_X\,\vec{v}$ is *undefined*.

Formally, for a fixed set X, we consider $P(\vec{v}) = \text{proj}_X\,\vec{v}$ as a function that inputs and outputs vectors. And, as a function, P has a domain consisting of exactly the vectors $\vec{v}$ for which $P(\vec{v})$ is defined. As it happens, if X is a line or a plane in $\mathbb{R}^n$, the domain of P is all of $\mathbb{R}^n$, and in this text, we will be sensible and only ask about projections that exist.

Vector Components

We've seen before that dot products can be used to measure how much one vector points in the direction of another. But, we can go further. Suppose $\vec{v} \neq \vec{0}$ and $\vec{u}$ are vectors. We might want to *decompose* $\vec{u}$ into the sum of two vectors, one which is in the direction of $\vec{v}$ and the other which is orthogonal to $\vec{v}$. The tool to that does this is the *vector component*.

Vector Components. Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***vector component of $\vec{u}$ in the $\vec{v}$ direction***, written $\operatorname{vcomp}_{\vec{v}} \vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.



From the definition, it's obvious that

$$\vec{u} = \operatorname{vcomp}_{\vec{v}} \vec{u} + (\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u})$$

is a decomposition of $\vec{u}$ into the sum of two vectors, one ($\operatorname{vcomp}_{\vec{v}} \vec{u}$) is in the direction of $\vec{v}$, and the other ($\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$) is orthogonal to $\vec{v}$.

Example. Find the vector component of $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$ in the direction of $\vec{b} = \begin{bmatrix}1\\1\end{bmatrix}$.

Since $\operatorname{vcomp}_{\vec{b}} \vec{a}$ is a vector in the direction of $\vec{b}$, we know

$$\operatorname{vcomp}_{\vec{b}} \vec{a} = k\vec{b}$$

for some $k \in \mathbb{R}$. Since $\vec{a} - \operatorname{vcomp}_{\vec{b}} \vec{a}$ is orthogonal to $\vec{b}$, we know

$$(\vec{a} - \operatorname{vcomp}_{\vec{b}} \vec{a}) \cdot \vec{b} = 0.$$

Combining these facts, we see

$$(\vec{a} - \operatorname{vcomp}_{\vec{b}} \vec{a}) \cdot \vec{b} = \left(\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\vec{a}} - \underbrace{\begin{bmatrix}k\\k\end{bmatrix}}_{k\vec{b}} \right) \cdot \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{b}} = (1-k) + (2-k) = 3 - 2k = 0,$$

and so $k = 3/2$. Therefore

$$\operatorname{vcomp}_{\vec{b}} \vec{a} = k\vec{b} = \begin{bmatrix}3/2\\3/2\end{bmatrix}.$$

Since we'll be computing vector components often, let's try to find a formula for $\operatorname{vcomp}_{\vec{v}} \vec{u}$.

By definition, $\operatorname{vcomp}_{\vec{v}} \vec{u}$ is a vector in the direction of $\vec{v}$, so

$$\operatorname{vcomp}_{\vec{v}} \vec{u} = k\vec{v}.$$

Further, from the definition $\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$, and so

$$\vec{v} \cdot (\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}) = \vec{v} \cdot (\vec{u} - k\vec{v}) = \vec{v} \cdot \vec{u} - k\vec{v} \cdot \vec{v} = 0,$$

Because $\vec{v} \neq \vec{0}$, we know $\vec{v} \cdot \vec{v} \neq 0$. Therefore, we may rearrange and solve for k to find

$$k = \frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}},$$

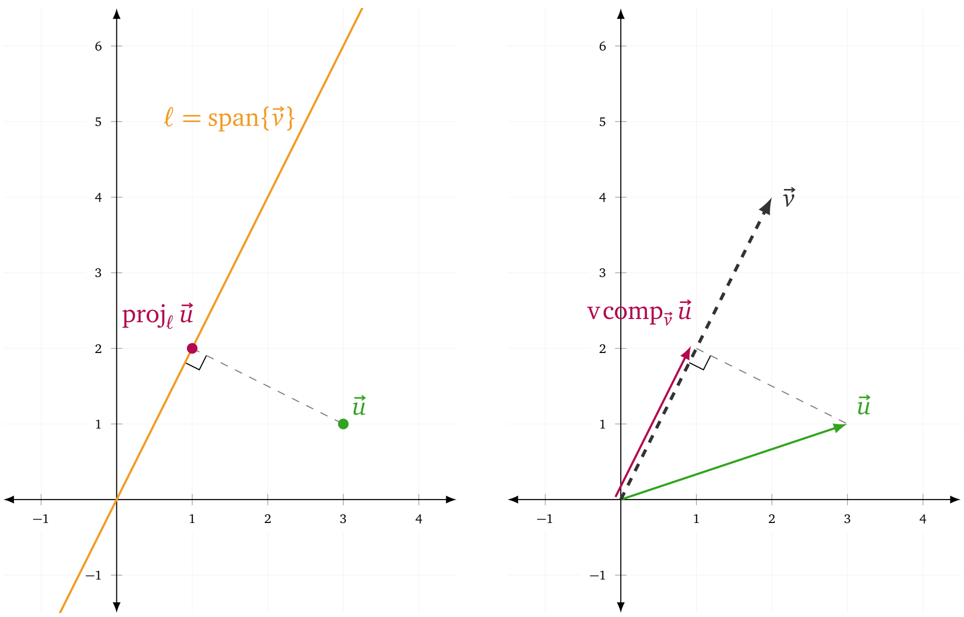
which means

$$\text{vcomp}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}}\right)\vec{v}.$$

The Relationship Between Vector Components and Projections

Vector components and projections onto lines are closely related. So closely related that many textbooks use the single word *projection* to talk about both vector components and projections.[19] Let's take a moment to explore this relationship.

Let $\vec{v} = \begin{bmatrix}2\\4\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}3\\1\end{bmatrix}$ and let $\ell = \text{span}\{\vec{v}\}$. Drawing a picture of ℓ, $\vec{u}$, and $\text{proj}_\ell\,\vec{u}$, we see that $\text{proj}_\ell\,\vec{u}$ satisfies all the properties of $\text{vcomp}_{\vec{v}}\,\vec{u}$.



Since $\ell = \text{span}\{\vec{v}\}$ and $\text{proj}_\ell\,\vec{u} \in \ell$, we know that $\text{proj}_\ell\,\vec{u}$ is in the direction of $\vec{v}$. Further, using geometric arguments, we know $\vec{u} - \text{proj}_\ell\,\vec{u}$ is a normal vector for ℓ and is therefore orthogonal to its direction vector $\vec{v}$! What's more, we didn't use anything in particular about $\vec{u}$ and $\vec{v}$ when making this argument (other than $\vec{v} \neq \vec{0}$). This means, we have established a general fact.

Theorem. For vectors $\vec{u}$ and $\vec{v} \neq 0$, we have

$$\text{proj}_{\text{span}\{\vec{v}\}}\,\vec{u} = \text{vcomp}_{\vec{v}}\,\vec{u}.$$

This is great news because vector components are easy to compute using dot products while projections are usually hard to compute.

Example. Compute the projection of $\vec{a} = \begin{bmatrix}3\\7\end{bmatrix}$ onto $\ell = \text{span}\left\{\begin{bmatrix}1\\-4\end{bmatrix}\right\}$.

Let $\vec{b} = \begin{bmatrix}1\\-4\end{bmatrix}$. Since $\ell = \text{span}\{\vec{b}\}$ and $\vec{b} \neq \vec{0}$, by the theorem above, we have

$$\text{proj}_\ell\,\vec{a} = \text{vcomp}_{\vec{b}}\,\vec{a} = \left(\frac{\vec{d}\cdot\vec{a}}{\vec{b}\cdot\vec{b}}\right)\vec{b} = -\tfrac{3-28}{1+16}\begin{bmatrix}1\\-4\end{bmatrix} = \begin{bmatrix}-25/17\\100/17\end{bmatrix}.$$

It's worth noting, however, that vector components are equal to projections *only in the case when you're projecting onto a span*. In general, projections and vector components are unrelated.

Example. Let $\vec{a} = \begin{bmatrix}3\\7\end{bmatrix}$, $\vec{b} = \begin{bmatrix}1\\-4\end{bmatrix}$, and let ℓ be the line given in vector form by $\vec{x} = t\vec{b} + \vec{a}$. Show that $\text{proj}_\ell\,\vec{a} \neq \text{vcomp}_{\vec{b}}\,\vec{a}$.

[19] We will not.

By definition, $\operatorname{proj}_\ell \vec a$ is the closest point in ℓ to $\vec a$. Since $\vec a \in \ell$, we must have

$$\operatorname{proj}_\ell \vec a = \vec a = \begin{bmatrix}3\\7\end{bmatrix}.$$

We already computed $\operatorname{vcomp}_{\vec b} \vec a = \begin{bmatrix}-25/17\\100/17\end{bmatrix}$ in the previous example, and so we see

$$\operatorname{vcomp}_{\vec b} \vec a = \begin{bmatrix}-25/17\\100/17\end{bmatrix} \neq \begin{bmatrix}3\\7\end{bmatrix} = \operatorname{proj}_\ell \vec b.$$

Takeaway. When projecting onto the span of a single vector, you can use vector components as a computational shortcut, but if the set isn't a span, you cannot.

Practice Problems

1 Let $T = \left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right\}$. Find $\operatorname{proj}_T \begin{bmatrix}3\\1\end{bmatrix}$.

The distances from $\begin{bmatrix}3\\1\end{bmatrix}$ to $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-1\\2\end{bmatrix}$, and $\begin{bmatrix}1\\-2\end{bmatrix}$ are $\sqrt{10}$, $\sqrt{17}$, and $\sqrt{13}$, respectively. So $\operatorname{proj}_T \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

2 Let $C = \left\{\vec v \in \mathbb{R}^2 : \|\vec v\| = 1\right\}$ be the unit circle in $\mathbb{R}^2$. Find $\operatorname{proj}_C \begin{bmatrix}2\\0\end{bmatrix}$. Justify your answer.

Suppose $\vec v = \begin{bmatrix}x\\y\end{bmatrix} \in C$. We would like to minimize $\left\|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix}\right\|$, or equivalently $\left\|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix}\right\|^2$. This expression can be rewritten as

$$\left\|\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix}\right\|^2 = (x-2)^2 + y^2 = (x^2+y^2) - 4x + 4$$

$$= \|\vec v\|^2 - 4x + 4 = 5 - 4x.$$

Since $x \leq 1$, the above expression is minimized when $x = 1$ (and thus $y = 0$). That is,

$$\operatorname{proj}_C \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}.$$

3 Let $\ell = \operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$, $L = \operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\} + \left\{\begin{bmatrix}4\\0\end{bmatrix}\right\}$, and let S be the set of convex linear combinations of $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}4\\2\end{bmatrix}$. For $\vec v = \begin{bmatrix}1\\0\end{bmatrix}$, find

(a) $\operatorname{proj}_\ell \vec v$.

(b) $\operatorname{proj}_L \vec v$.

(c) $\operatorname{proj}_S \vec v$.

(a) Let $\vec u = \begin{bmatrix}2\\1\end{bmatrix}$. Then $\operatorname{proj}_\ell \vec v = t\vec u$ for some $t \in \mathbb{R}$ which minimizes

$$\|\vec v - t\vec u\|^2 = \|\vec u\|^2 t^2 - (2\vec u \cdot \vec v)t + \|\vec v\|^2.$$

This quantity is minimized when $t = \frac{\vec u \cdot \vec v}{\|\vec u\|^2} = \frac{2}{5}$, so

$$\operatorname{proj}_\ell \vec v = \tfrac{2}{5}\vec u = \tfrac{1}{5}\begin{bmatrix}4\\2\end{bmatrix}.$$

(b) Let $\vec u = \begin{bmatrix}2\\1\end{bmatrix}$. Then $\operatorname{proj}_L \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix} + t\vec u$ for some $t \in \mathbb{R}$ which minimizes

$$\left\|\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}4\\0\end{bmatrix} - t\vec u\right\|^2 = (-3-2t)^2 + (0-t)^2$$

$$= 9 + 12t + 5t^2.$$

The quantity $9 + 12t + 5t^2 = 5(t + \frac{6}{5})^2 + \frac{9}{5}$ is minimized when $t = -\frac{6}{5}$, so

$$\operatorname{proj}_L \vec v = \begin{bmatrix}4\\0\end{bmatrix} - \tfrac{6}{5}\vec u = \tfrac{1}{5}\begin{bmatrix}8\\-6\end{bmatrix}$$

(c) The set S is equal to $\{t\vec u : 1 \leq t \leq 2\}$ (check this), and so $S \subseteq \ell$. We found $\operatorname{proj}_\ell \vec v = \begin{bmatrix}4/5\\2/5\end{bmatrix}$ which is not in S. Therefore, $\operatorname{proj}_S \vec v$ must be one of the endpoints of S. Checking both endpoints, we conclude $\operatorname{proj}_S \vec v = \begin{bmatrix}2\\1\end{bmatrix}$.

4 Let T be the set of convex linear combinations of $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\end{bmatrix}\right\}$. Find $\operatorname{proj}_T(\vec v)$, for

(a) $\vec v = \begin{bmatrix}3\\3\end{bmatrix}$

(b) $\vec v = \begin{bmatrix}0\\0\end{bmatrix}$

(c) $\vec v = \begin{bmatrix}1\\-2\end{bmatrix}$

(d) $\vec v = \begin{bmatrix}0\\-4\end{bmatrix}$

Geometrically, T is a filled in triangle with vertices $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\-2\end{bmatrix}$. So for points outside the triangle, the closest point in T will be on the nearest side of T, and so we can project onto T by projecting onto line segments.

(a) $\text{proj}_T \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$; since $\begin{bmatrix}3\\3\end{bmatrix}$ is above T, the closest point will be on the line segment $y = 1, -1 \le x \le 1$.

(b) $\text{proj}_T \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, since $\vec{0} \in T$.

(c) $\text{proj}_T \begin{bmatrix}1\\-2\end{bmatrix} = \frac{1}{13}\begin{bmatrix}-5\\-14\end{bmatrix}$; let ℓ be the line segment $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix} + t\begin{bmatrix}-2\\-3\end{bmatrix}, 0 \le t \le 1$. Then $\text{proj}_T \begin{bmatrix}1\\-2\end{bmatrix} = \text{proj}_\ell \begin{bmatrix}1\\-2\end{bmatrix}$, and then either by minimizing the length function, or drawing a perpendicular line to ℓ, we find the closest point is when $t = \frac{9}{13}$.

(d) $\text{proj}_T \begin{bmatrix}0\\-4\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$, as in ($c$), but now the minimizer is at $t > 1$, so the constraints $0 \le t \le 1$ force us to take the closest point on the line segment.

5 Explain in your own words how to find $\text{proj}_\ell(\vec{v})$ when $\ell = \text{span}\{\vec{d}\}$ for some $\vec{d} \ne \vec{0}$.

6 Let $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, and $\vec{u} = \begin{bmatrix}2\\3\end{bmatrix}$.

(a) Draw $\vec{e}_1, \vec{e}_2, \vec{u}, \text{vcomp}_{\vec{e}_1}\vec{u}$, and $\text{vcomp}_{\vec{e}_2}\vec{u}$ on the same grid.

(b) Write down two characterizing properties for $\text{vcomp}_{\vec{e}_2}\vec{u}$.

(c) Check that $\vec{u} - \text{vcomp}_{\vec{e}_1}\vec{u}$ satisfies the above properties.

(d) $\text{vcomp}_{\vec{e}_1}\vec{u} + \text{vcomp}_{\vec{e}_2}\vec{u} = \vec{u}$. Does this always happen? Explain.

7 In this problem, we will find the projection of a vector onto a plane in $\mathbb{R}^3$. Let $\vec{u} = \begin{bmatrix}1\\2\\-2\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $\vec{a} = \begin{bmatrix}6\\4\\-2\end{bmatrix}$, and let $P = \text{span}\{\vec{u}, \vec{v}\}$.

(a) Find $\text{vcomp}_{\vec{u}}(\vec{a})$ and $\text{vcomp}_{\vec{v}}(\vec{a})$.

(b) Show that $\vec{a} - \text{vcomp}_{\vec{u}}(\vec{a}) - \text{vcomp}_{\vec{v}}(\vec{a})$ is a normal vector for P.

(c) Use (7b) to find $\text{proj}_P(\vec{a})$.

(a) Using the formula for vector components, we have

$$\text{vcomp}_{\vec{u}}(\vec{a}) = \frac{\vec{a}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{18}{9}\vec{u} = \begin{bmatrix}2\\4\\-4\end{bmatrix}$$

$$\text{vcomp}_{\vec{v}}(\vec{a}) = \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{u} = \frac{2}{2}\vec{v} = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

(b) Let $\vec{b} = \vec{a} - \text{vcomp}_{\vec{u}}(\vec{a}) - \text{vcomp}_{\vec{v}}(\vec{a})$. Directly computing, we have $\vec{b} = \begin{bmatrix}4\\-1\\1\end{bmatrix}$.

To show $\vec{b}$ is orthogonal to P, we need to check that

$$\vec{b}\cdot\vec{u} = 4 - 2 - 2 = 0$$

and

$$\vec{b}\cdot\vec{v} = 0 - 1 + 1 = 0.$$

Hence $\vec{b}$ is a normal vector to P. (Note that this only worked because $\vec{u}\cdot\vec{v} = 0$. Subtracting each vector component from $\vec{a}$ will not produce a normal vector in general.)

(c) Since $\vec{b}$ is orthogonal to P and $\vec{a} - \vec{b}$ is a linear combination of $\vec{u}$ and $\vec{v}$, the vector $\vec{a} - \vec{b} = \begin{bmatrix}2\\5\\-3\end{bmatrix}$ is the closest point to $\vec{a}$ on P.

Projections

Projection

DEF Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\operatorname{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

26 Let $\vec{a}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b}=\begin{bmatrix}4\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}2\\2\end{bmatrix}$ and $\ell=\operatorname{span}\{\vec{a}\}$.

26.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.

26.2 Find $\operatorname{proj}_{\{\vec{b}\}} \vec{v}$, $\operatorname{proj}_{\{\vec{a},\vec{b}\}} \vec{v}$.

26.3 Find $\operatorname{proj}_\ell \vec{v}$. (Recall that a quadratic at^2+bt+c has a minimum at $t=-\frac{b}{2a}$).

26.4 Is $\vec{v}-\operatorname{proj}_\ell \vec{v}$ a normal vector for ℓ? Why or why not?

27 Let K be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}$ and let $\vec{c} = \begin{bmatrix}1\\3\end{bmatrix}$.

27.1 Make a sketch with $\vec{c}$, K, and $\mathrm{proj}_K \vec{c}$ (you don't need to compute $\mathrm{proj}_K \vec{c}$ exactly).

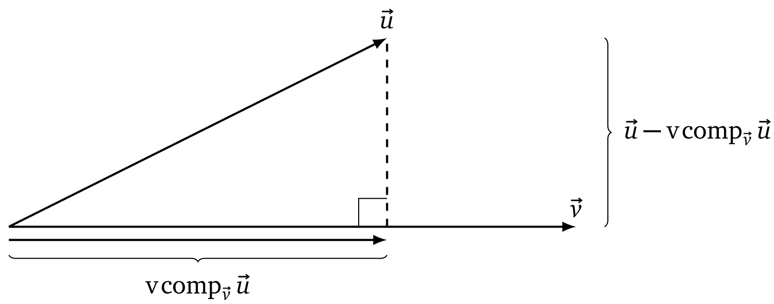
27.2 What should $(\vec{c} - \mathrm{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix}$ be? Explain.

27.3 Use your formula from the previous part to find $\mathrm{proj}_K \vec{c}$ *without* computing any distances.

Vector Components

DEFINITION

Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***vector component of $\vec{u}$ in the $\vec{v}$ direction***, written $\operatorname{vcomp}_{\vec{v}} \vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.



28 Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

28.1 List two conditions that $\operatorname{vcomp}_{\vec{b}} \vec{a}$ must satisfy.

28.2 Find a formula for $\operatorname{vcomp}_{\vec{b}} \vec{a}$.

29 Let $\vec d = \begin{bmatrix}3\\3\end{bmatrix}$ and $\vec u = \begin{bmatrix}1\\2\end{bmatrix}$.

29.1 Draw $\vec d$, $\vec u$, $\operatorname{span}\{\vec d\}$, and $\operatorname{proj}_{\operatorname{span}\{\vec d\}} \vec u$ in the same picture.

29.2 How do $\operatorname{proj}_{\operatorname{span}\{\vec d\}} \vec u$ and $\operatorname{vcomp}_{\vec d} \vec u$ relate?

29.3 Compute $\operatorname{proj}_{\operatorname{span}\{\vec d\}} \vec u$ and $\operatorname{vcomp}_{\vec d} \vec u$.

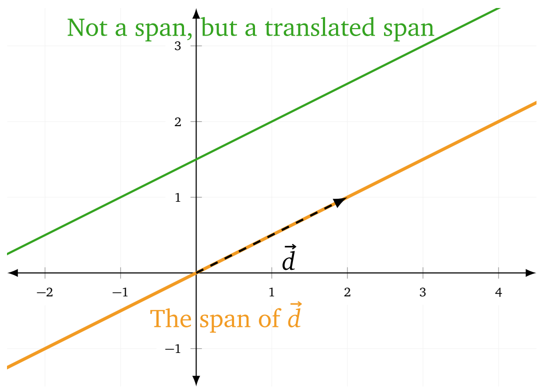
29.4 Compute $\operatorname{vcomp}_{-\vec d} \vec u$. Is this the same as or different from $\operatorname{vcomp}_{\vec d} \vec u$? Explain.

Subspaces & Bases

In this module you will learn

- Formal and intuitive definitions of subspaces.
- The relationship between subspaces and spans.
- How to prove whether or not a set is a subspace.
- How to find a basis for and the dimension of a subspace.

Lines or planes through the origin can be written as spans of their direction vectors. However, a line or plane that doesn't pass through the origin cannot be written as a span—it must be expressed as a *translated* span.



There's something special about sets that can be expressed as (untranslated) spans. In particular, since a linear combination of linear combinations is still a linear combination, a span is *closed* with respect to linear combinations. That is, by taking linear combinations of vectors in a span, you cannot escape the span. In general, sets that have this property are called *subspaces*.

Subspace. A non-empty subset $V \subseteq \mathbb{R}^n$ is called a ***subspace*** if for all $\vec{u}, \vec{v} \in V$ and all scalars k we have

(i) $\vec{u} + \vec{v} \in V$; and

(ii) $k\vec{u} \in V$.

In the definition of a subspace, property (i) is called begin *closed with respect to vector addition* and property (ii) is called being *closed with respect to scalar multiplication*.

Subspaces generalize the idea of *flat spaces through the origin*. They include lines, planes, volumes and more.

Example. Let $\mathcal{V} \subseteq \mathbb{R}^2$ be the complete solution to $x + 2y = 0$. Show that $\mathcal{V}$ is a subspace.

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be in $\mathcal{V}$, and let k be a scalar.
By definition, we have

$$\begin{aligned} u_1 + 2u_2 &= 0 \\ v_1 + 2v_2 &= 0 \end{aligned}.$$

We will show that $\mathcal{V}$ is nonempty and that (i) $\vec{u} + \vec{v} \in \mathcal{V}$; and (ii) $k\vec{u} \in \mathcal{V}$.
First we will show (i). Observe that

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

and the coordinates of $\vec{u} + \vec{v}$ satisfy

$$(u_1 + v_1) + 2(u_2 + v_2) = (u_1 + 2u_2) + (v_1 + 2v_2) = 0 + 0 = 0.$$

Since the coordinates of $\vec{u} + \vec{v}$ satisfy the equation $x + 2y = 0$, we have shown that $\vec{u} + \vec{v} \in \mathcal{V}$.
Next we will show (ii). Observe that

$$k\vec{u} = \begin{bmatrix} ku_1 \\ ku_2 \end{bmatrix}$$

and the coordinates of $k\vec{u}$ satisfy

$$(ku_1)+2(ku_2)=k(u_1+2u_2)=k0=0.$$

And so, we have shown that $k\vec{u}\in\mathcal{V}$.

Finally, since $\vec{0}=\begin{bmatrix}0\\0\end{bmatrix}$ satisfies $x+2y=0$, we conclude that $\vec{0}\in\mathcal{V}$ and so $\mathcal{V}$ is non-empty.

Thus, by the definition, we have shown that $\mathcal{V}$ is a subspace.

Example. Let $\mathcal{W}\subseteq\mathbb{R}^2$ be the line expressed in vector form as

$$\vec{x}=t\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}.$$

Determine whether $\mathcal{W}$ is a subspace.

$\mathcal{W}$ is *not* a subspace. To see this, notice that $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}\in\mathcal{W}$, but $0\vec{v}=\vec{0}\notin\mathcal{W}$. Therefore, $\mathcal{W}$ is not closed under scalar multiplication and so cannot be a subspace.

As mentioned earlier, subspaces and spans are deeply connected. This connection is given by the following theorem.

Theorem. Every subspace is a span and every span is a subspace. More precisely, $\mathcal{V}\subseteq\mathbb{R}^n$ is a subspace if and only if $\mathcal{V}=\operatorname{span}\mathcal{X}$ for some set $\mathcal{X}$.

Proof. We will start by showing every span is a subspace. Fix $\mathcal{X}\subseteq\mathbb{R}^2$ and let $\mathcal{V}=\operatorname{span}\mathcal{X}$. First note that if $\mathcal{X}\neq\{\}$, then $\mathcal{V}$ is non-empty because $\mathcal{X}\subseteq\mathcal{V}$ and if $\mathcal{X}=\{\}$, then $\mathcal{V}=\{\vec{0}\}$, and so is still non-empty.

Fix $\vec{v},\vec{u}\in\mathcal{V}$. By definition there are $\vec{x}_1,\vec{x}_2,\ldots,\vec{y}_1,\vec{y}_2,\ldots\in\mathcal{X}$ and scalars $\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2,\ldots$ so that

$$\vec{v}=\sum\alpha_i\vec{x}_i\qquad\vec{u}=\sum\beta_i\vec{y}_i.$$

To verify property (i), observe that

$$\vec{u}+\vec{v}=\sum\alpha_i\vec{x}_i+\sum\beta_i\vec{y}_i$$

is also a linear combination of vectors in $\mathcal{X}$ (because all $\vec{x}_i$ and $\vec{y}_i$ are in $\mathcal{X}$), and so $\vec{u}+\vec{v}\in\operatorname{span}\mathcal{X}=\mathcal{V}$.

To verify property (ii), observe that for any scalar α,

$$\alpha\vec{v}=\alpha\sum\alpha_i\vec{x}_i=\sum(\alpha\alpha_i)\vec{x}_i\in\operatorname{span}\mathcal{X}=\mathcal{V}.$$

Since $\mathcal{V}$ is non-empty and satisfies both properties (i) and (ii), it is a subspace.

Now we will prove that every subspace is a span. Let $\mathcal{V}$ be a subspace and consider $\mathcal{V}'=\operatorname{span}\mathcal{V}$. Since taking a span may only enlarge a set, we know $\mathcal{V}\subseteq\mathcal{V}'$. If we establish that $\mathcal{V}'\subseteq\mathcal{V}$, then $\mathcal{V}=\mathcal{V}'=\operatorname{span}\mathcal{V}$, which would complete the proof.

Fix $\vec{x}\in\mathcal{V}'$. By definition, there are $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\in\mathcal{V}$ and scalars $\alpha_1,\alpha_2,\ldots,\alpha_n$ so that

$$\vec{x}=\sum\alpha_i\vec{v}_i.$$

Observe that $\alpha_i\vec{v}_i\in\mathcal{V}$ for all i, since $\mathcal{V}$ is closed under scalar multiplication. It follows that $\alpha_1\vec{v}_1+\alpha_2\vec{v}_2\in\mathcal{V}$, because $\mathcal{V}$ is closed under sums. Continuing, $(\alpha_1\vec{v}_1+\alpha_2\vec{v}_2)+\alpha_3\vec{v}_3\in\mathcal{V}$ because $\mathcal{V}$ is closed under sums. Applying the principle finite induction, we see

$$\vec{x}=\sum\alpha_i\vec{v}_i=\Big(\big((\alpha_1\vec{v}_1+\alpha_2\vec{v}_2)+\alpha_3\vec{v}_3\big)+\cdots+\alpha_{n-1}\vec{v}_{n-1}\Big)+\alpha_n\vec{v}_n\in\mathcal{V}.$$

Thus $\mathcal{V}'\subseteq\mathcal{V}$, which completes the proof. ■

The previous theorem is saying that spans and subspaces are two ways of talking about the same thing. Spans provide a *constructive* definition of lines/planes/volumes/etc. through the origin. That is, when you describe a line/plane/etc. through the origin as a span, you're saying "this is a line/plane/etc. through the origin because every point in it is a linear combination of *these specific vectors*". In contrast, subspaces provide a *categorical* definition of lines/planes/etc. through the origin. When you describe a line/plane/etc. through the origin as a subspace, you're saying "this is a line/plane/etc. through the origin because these *properties* are satisfied".[20]

[20] Categorical definitions are useful when working with objects where it's hard to pin down exactly what the elements inside are.

Takeaway. Spans and subspaces are two different ways of talking about the same objects: points/lines/planes/etc. through the origin.

Special Subspaces

When thinking about $\mathbb{R}^n$, there are two special subspaces that are always available. The first is $\mathbb{R}^n$ itself. $\mathbb{R}^n$ is obviously non-empty, and linear combinations of vectors in $\mathbb{R}^n$ remain in $\mathbb{R}^n$. The second is the *trivial subspace*, $\{\vec{0}\}$.

Trivial Subspace. The subset $\{\vec{0}\} \subseteq \mathbb{R}^n$ is called the ***trivial subspace***.

Theorem. The trivial subspace is a subspace.

Proof. First note that $\{\vec{0}\}$ is non-empty since $\vec{0} \in \{\vec{0}\}$. Now, since $\vec{0}$ is the only vector in $\{\vec{0}\}$, properties (i) and (ii) follow quickly:

$$\vec{0} + \vec{0} = \vec{0} \in \{\vec{0}\}$$

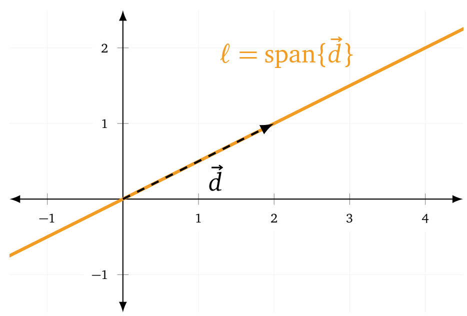
and

$$\alpha\vec{0} = \vec{0} \in \{\vec{0}\}.$$

■

Bases

Let $\vec{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and consider $\ell = \operatorname{span}\{\vec{d}\}$.



We know that ℓ is a subspace, and we defined ℓ as the span of $\{\vec{d}\}$, but we didn't have to define ℓ that way. We could have, for instance, defined $\ell = \operatorname{span}\{\vec{d}, -2\vec{d}, \frac{1}{2}\vec{d}\}$. However, $\operatorname{span}\{\vec{d}\}$ is a simpler way to describe ℓ than $\operatorname{span}\{\vec{d}, -2\vec{d}, \frac{1}{2}\vec{d}\}$. This property is general: the simplest description of a line involve the span of only one vector.

Analogously, let $\mathcal{P} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$ be the plane through the origin with direction vectors $\vec{d}_1$ and $\vec{d}_2$. There are many ways to write $\mathcal{P}$ as a span, but the simplest ones involve exactly two vectors. The idea of a *basis* comes from trying to find the simplest description of a subspace.

Basis. A ***basis*** for a subspace $\mathcal{V}$ is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = \mathcal{V}$.

In short, a basis for a subspace is a linearly independent set that spans that subspace.

Example. Let $\ell = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$. Find two different bases for ℓ.

We are looking for a set of linearly independent vectors that spans ℓ. Notice that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -4 \end{bmatrix} = 2\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. Therefore,

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\} = \ell.$$

Because $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ is linearly independent and spans ℓ, we have that $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ is a basis for ℓ. Similarly, $\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$ is another basis for ℓ.

Unpacking the definition of basis a bit more, we can see that a basis for a subspace is a set of vectors that is *just the right size* to describe everything in the subspace. It's not too big—because it is linearly independent, there are no redundancies. It's not too small—because we require it to span the subspace.[21]

There are several facts everyone should know about bases:

1. Bases are not unique. Every subspace (except the trivial subspace) has multiple bases.
2. Given a basis for a subspace, every vector in the subspace can be written as a *unique* linear combination of vectors in that subspace.
3. Any two bases for the same subspace have the same number of elements.

You can prove the first fact by observing that if $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots\}$ is a basis with at least one element,[22] then $\{2\vec{b}_1, 2\vec{b}_2, \ldots\}$ is a different basis. The second fact is a consequence of all bases being linearly independent. The third fact is less obvious and takes some legwork to prove, so we will accept it as is.

Dimension

Let $\mathcal{V}$ be a subspace. Though there are many bases for $\mathcal{V}$, they all have the same number of vectors in them. And, this number says something fundamental about $\mathcal{V}$: it tells us the maximum number of linearly independent vectors that can simultaneously exist in $\mathcal{V}$. We call this number the *dimension* of $\mathcal{V}$.

Dimension. The ***dimension*** of a subspace V is the number of elements in a basis for V.

This definition agrees with our intuition about lines and planes: the dimension of a line through $\vec{0}$ is 1, and the dimension of a plane through $\vec{0}$ is 2. It even tells us the dimension of the single point $\{\vec{0}\}$ is 0.[23]

Example. Find the dimension of $\mathbb{R}^2$.

Since $\{\vec{e}_1, \vec{e}_2\}$ is a basis for $\mathbb{R}^2$, we know $\mathbb{R}^2$ is two dimensional.

Example. Let $\ell = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\-4\end{bmatrix}, \begin{bmatrix}1/2\\1\end{bmatrix}\right\}$. Find the dimension of the subspace ℓ.

This is the same subspace from the earlier example where we found $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}1/2\\1\end{bmatrix}\right\}$ were bases for ℓ. Both these bases contain one element, and so ℓ is a one dimensional subspace.

Example. Let $A = \{(x_1, x_2, x_3, x_4) : x_1 + 2x_2 - x_3 = 0 \text{ and } x_1 + 6x_4 = 0\}$. Find a basis for and the dimension of A.

A is the complete solution to the system

$$\begin{cases} x_1 + 2x_2 - x_3 \qquad\quad = 0 \\ x_1 \qquad\qquad\quad + 6x_4 = 0 \end{cases},$$

which can be expressed in vector form as

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = t\begin{bmatrix}0\\1/2\\1\\0\end{bmatrix} + s\begin{bmatrix}-6\\3\\0\\1\end{bmatrix}.$$

Therefore $A = \operatorname{span}\left\{\begin{bmatrix}0\\1/2\\1\\0\end{bmatrix}, \begin{bmatrix}-6\\3\\0\\1\end{bmatrix}\right\}$. Since $\left\{\begin{bmatrix}0\\1/2\\1\\0\end{bmatrix}, \begin{bmatrix}-6\\3\\0\\1\end{bmatrix}\right\}$ is a linearly independent spanning set with two elements, A is two dimensional.

Like $\mathbb{R}^2$ and $\mathbb{R}^3$, whenever we discuss $\mathbb{R}^n$, we always have a standard basis that comes along for the ride.

[21] If you're into British fairy tales, you might call a basis a *Goldilocks set*.

[22] The empty set is a basis for the trivial subspace.

[23] The dimension of a line, plane, or point not through the origin is defined to be the dimension of the subspace obtained when it is translated to the origin.

Standard Basis. The ***standard basis*** for $\mathbb{R}^n$ is the set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \qquad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \qquad \cdots .$$

That is $\vec{e}_i$ is the vector with a 1 in its ith coordinate and zeros elsewhere.

Note: the notation $\vec{e}_i$ is context specific. If we say $\vec{e}_i \in \mathbb{R}^2$, then $\vec{e}_i$ must have exactly two components. If we say $\vec{e}_i \in \mathbb{R}^{45}$, then $\vec{e}_i$ must have 45 components.

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

Subspaces and Bases

Subspace

DEFINITION

A non-empty subset $V \subseteq \mathbb{R}^n$ is called a ***subspace*** if for all $\vec{u}, \vec{v} \in V$ and all scalars k we have

(i) $\vec{u} + \vec{v} \in V$; and

(ii) $k\vec{u} \in V$.

Subspaces give a mathematically precise definition of a "flat space through the origin."

30 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

30.1 $A = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for some } a \in \mathbb{Z} \right\}$.

30.2 $B = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

30.3 $C = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.4 $D = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.5 $E = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.6 $F = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

30.7 $G = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

30.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Basis

DEF A ***basis*** for a subspace $\mathcal{V}$ is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = \mathcal{V}$.

Dimension

DEF The ***dimension*** of a subspace V is the number of elements in a basis for V.

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31 Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

31.1 Describe V.

31.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

31.3 Give a basis for V.

31.4 Give another basis for V.

31.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

31.6 What is the dimension of V?

32 Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}7\\8\\8\end{bmatrix}$ (notice these vectors are linearly independent) and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

32.1 Give a basis for and the dimension of P.

32.2 Give a basis for and the dimension of Q.

32.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

32.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Matrix Representations of Systems of Linear Equations

Module 7

In this module you will learn

- How to represent a system of linear equations as a matrix equation.
- Multiple ways to interpret solutions of systems of linear equations.
- How linear independence/dependence relates to solutions to matrix equations.
- How to use matrix equations to find normal vectors to lines or planes.

Matrix-vector multiplication gives a compact way to represent systems of linear equations.

Consider the system

$$\begin{cases} x+2y-2z=-15 \\ 2x+\ \ y-5z=-21 \\ x-4y+\ \ z=18 \end{cases}, \tag{6}$$

which is equivalent to the vector equation

$$\begin{bmatrix} x+2y-2z \\ 2x+y-5z \\ x-4y+z \end{bmatrix} = \begin{bmatrix} -15 \\ -21 \\ 18 \end{bmatrix}.$$

We can rewrite (6) using matrix-vector multiplication:

$$\underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -5 \\ 1 & -4 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -15 \\ -21 \\ 18 \end{bmatrix}.$$

The matrix A on the left is called the *coefficient matrix* because it is made up of the coefficients from equation (6).

By using coefficient matrices, every system of linear equations can be rewritten as a single matrix equation of the form

$$A\vec{x}=\vec{b}$$

where A is a coefficient matrix, $\vec{x}$ is a column vector of variables, and $\vec{b}$ is a column vector of constants.

Example. Consider the one equation system

$$\{x-4y+z=5 \tag{7}$$

and the two-equation system

$$\begin{cases} x-4y+z=5 \\ \qquad y-z=9 \end{cases}. \tag{8}$$

Rewrite each system as a single matrix equation.

We can rewrite (7) as

$$\begin{bmatrix} 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}.$$

Multiplying out to verify, we see,

$$\begin{bmatrix} 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-4y+z \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix},$$

which is indeed equivalent to (7).

Similarly, we can rewrite (8) as

$$\begin{bmatrix}1 & -4 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}5\\ 9\end{bmatrix}.$$

Multiplying out to verify, we see,

$$\begin{bmatrix}1 & -4 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}x-4y+z\\ 0x+y-z\end{bmatrix}=\begin{bmatrix}5\\ 9\end{bmatrix},$$

which is equivalent to (8).

Interpretations of Matrix Equations

The solution set to a system of linear equations, like

$$\begin{cases} x+2y-2z=-15\\ 2x+\ \ y-5z=-21\\ x-4y+\ \ z=18\end{cases}, \tag{9}$$

can be interpreted as the intersection of three planes (or hyperplanes if there were more variables). Each equation (each row) specifies a plane, and the solution set is the intersection of all of these planes. Rewriting a system of equations in matrix form gives two additional ways to interpret the solution set.

The *Column* Picture

Using the column interpretation of matrix-vector multiplication, we see that system (9) is equivalent to

$$\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & -5\\ 1 & -4 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=x\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}+y\begin{bmatrix}2\\ 1\\ -4\end{bmatrix}+z\begin{bmatrix}-2\\ -5\\ 1\end{bmatrix}=\begin{bmatrix}-15\\ -21\\ 18\end{bmatrix}.$$

We now see that asking, "*What coefficients allow* $\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$, $\begin{bmatrix}2\\ 1\\ -4\end{bmatrix}$, *and* $\begin{bmatrix}-2\\ -5\\ 1\end{bmatrix}$ *to form* $\begin{bmatrix}-15\\ -21\\ 18\end{bmatrix}$ *as a linear combination?*" is equivalent to asking, "*What are the solutions to system* (9)*?*" Here, $\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$, $\begin{bmatrix}2\\ 1\\ -4\end{bmatrix}$, and $\begin{bmatrix}-2\\ -5\\ 1\end{bmatrix}$ are the columns of the coefficient matrix.

The *Row* Picture

The row interpretation gives another perspective. Let $\vec r_1$, $\vec r_2$, and $\vec r_3$ be the rows of the coefficient matrix for system (9). Then, system (9) is equivalent to

$$\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & -5\\ 1 & -4 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\vec r_1\\ \hline \vec r_2\\ \hline \vec r_3\end{bmatrix}\vec x=\begin{bmatrix}\vec r_1\cdot\vec x\\ \vec r_2\cdot\vec x\\ \vec r_3\cdot\vec x\end{bmatrix}=\begin{bmatrix}-15\\ -21\\ 18\end{bmatrix}.$$

In other words, we can interpret solutions to system (9) as vectors whose dot product with $\vec r_1$ is -15, whose dot product with $\vec r_2$ is -12, and whose dot product with $\vec r_3$ is 18. Given that the dot product has a geometric interpretation, this perspective is powerful (especially when the right side of the equation is all zeros!).

Interpreting Homogeneous Systems

Consider the homogeneous system/matrix equation

$$\begin{aligned} x+2y-2z&=0\\ 2x+\ \ y-5z&=0\\ x-4y+\ \ z&=0\end{aligned}\qquad\iff\qquad \underbrace{\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & -5\\ 1 & -4 & 1\end{bmatrix}}_{A}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}. \tag{10}$$

Now, the column interpretation of system (10) is: *what linear combinations of the column vectors of A give $\vec 0$?* This directly relates to the question of whether the column vectors of A are linearly independent.

Let $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ be the rows of A. The row interpretation of system (10) asks: *what vectors are simultaneously orthogonal to $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$?*

Takeaway. There are three ways to interpret solutions to a matrix equation $A\vec{x} = \vec{b}$: (i) the intersection of hyperplanes specified by the rows; (ii) what linear combinations of the columns of A give $\vec{b}$; (iii) what vectors yield the entries of $\vec{b}$ when dot producted with the rows of A.

Example. Find all vectors orthogonal to $\vec{a} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}1\\2\\1\end{bmatrix}$.

To find all vectors orthogonal to $\vec{a}$ and $\vec{b}$ we need to find vectors $\vec{x}$ satisfying $\vec{a} \cdot \vec{x} = 0$ and $\vec{b} \cdot \vec{x} = 0$. This is is equivalent to solving the matrix equation

$$\begin{bmatrix}\vec{a}\\ \hline \vec{b}\end{bmatrix}\vec{x} = \begin{bmatrix}\vec{a}\cdot\vec{x}\\ \vec{b}\cdot\vec{x}\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

By row reducing A, we get

$$\mathrm{rref}(A) = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix},$$

and so the complete solution expressed in vector form is

$$\vec{x} = t\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

The row picture is particularly applicable when trying to find normal vectors.

Example. Let Q be the hyperplane specified in vector form by

$$\vec{x} = t\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + r\begin{bmatrix}2\\0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

Find a normal vector for Q and write Q in normal form.

Like the above example, since normal vectors for Q need to be orthogonal to $\vec{d}_1 = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$, $\vec{d}_2 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$, and $\vec{d}_3 = \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$, we can find the normal vectors by solving

$$\underbrace{\begin{bmatrix}1 & 1 & -1 & 1\\ 0 & 1 & 0 & 1\\ 2 & 0 & 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

By row reducing A, we get

$$\mathrm{rref}(A) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix},$$

and so we get that the complete solution expressed in vector form is

$$\vec{x} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Therefore, any non-zero multiple of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ is a normal vector for $\mathcal{Q}$. For example, $\vec{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ is a normal vector for $\mathcal{Q}$, and $\mathcal{Q}$ can be written in normal form as

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right) = 0.$$

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

33 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

33.1 Compute the product $A\vec{x}$.

33.2 Write down a system of equations that corresponds to the matrix equation $A\vec{x} = \vec{b}$.

33.3 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

33.4 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

34 Let $\vec{u}=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{v}=\begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{w}=\begin{bmatrix}7\\8\\9\end{bmatrix}$.

34.1 How could you determine if $\{\vec{u}, \vec{v}, \vec{w}\}$ was a linearly independent set?

34.2 Can your method be rephrased in terms of a matrix equation? Explain.

35 Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

35.1 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

35.2 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

36 Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Let $\mathcal{P}$ be the plane given in vector form by $\vec{x} = t\vec{d}_1 + s\vec{d}_2$. Further, suppose M is a matrix so that $M\vec{r} \in \mathcal{P}$ for any $\vec{r} \in \mathbb{R}^2$.

36.1 How many rows does M have?

36.2 Find such an M.

36.3 Find necessary and sufficient conditions (phrased as equations) for $\vec{n}$ to be a normal vector for $\mathcal{P}$.

36.4 Find a matrix K so that non-zero solutions to $K\vec{x} = \vec{0}$ are normal vectors for $\mathcal{P}$. How do K and M relate?

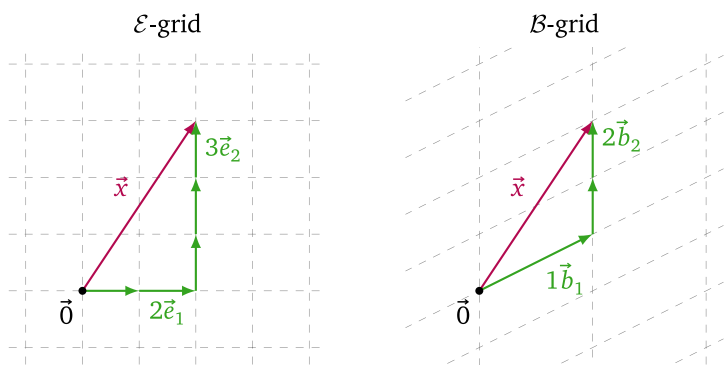
Coordinates & Change of Basis I

In this module you will learn

- Notation for representing a vector in multiple bases.
- The distinction between a vector and its representation.
- How to compute multiple representation of a vector.
- The definition of an *oriented* basis.

Recall that when we write $\vec{x}=\begin{bmatrix}2\\3\end{bmatrix}$, what we actually mean is $\vec{x}=2\vec{e}_1+3\vec{e}_2$. The numbers 2 and 3 are called the coordinates of the vector $\vec{x}$ with respect to the standard basis. However, in general, subspaces have many bases, and so it is possible to represent a single vector in *many different ways* as coordinates with respect to different bases.

Let $\vec{x}=\begin{bmatrix}2\\3\end{bmatrix}$, let $\mathcal{E}=\{\vec{e}_1,\vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $\mathcal{B}=\{\vec{b}_1,\vec{b}_2\}$ where $\vec{b}_1=\begin{bmatrix}2\\1\end{bmatrix}$ and $\vec{b}_2=\begin{bmatrix}0\\1\end{bmatrix}$ be another basis for $\mathbb{R}^2$. The coordinates of $\vec{x}$ with respect to $\mathcal{E}$ are $(2,3)$, but the coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $(1,2)$.



The coordinates $(2,3)$ and $(1,1)$ represent $\vec{x}$ equally well, and when solving problems, we should pick the coordinates that make our problem the easiest.[24] However, now that we are representing vectors in multiple bases, we need a way to keep track of what coordinates correspond to which basis.

Representation in a Basis.
Let $\mathcal{B}=\{\vec{b}_1,\ldots,\vec{b}_n\}$ be a basis for a subspace V and let $\vec{v}\in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notated $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}$$

where $\alpha_1,\ldots,\alpha_n$ uniquely satisfy $\vec{v}=\alpha_1\vec{b}_1+\cdots+\alpha_n\vec{b}_n$.
Conversely,

$$\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}_{\mathcal{B}}=\alpha_1\vec{b}_1+\cdots+\alpha_n\vec{b}_n$$

is notation for the linear combination of $\vec{b}_1,\ldots,\vec{b}_n$ with coefficients $\alpha_1,\ldots,\alpha_n$.

Example. Let $\mathcal{E}=\{\vec{e}_1,\vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C}=\{\vec{c}_1,\vec{c}_2\}$ where $\vec{c}_1=\vec{e}_1+\vec{e}_2$, and $\vec{c}_2=3\vec{e}_2$ be another basis for $\mathbb{R}^2$. Given that $\vec{v}=2\vec{e}_1-\vec{e}_2$, find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

[24] For example, maybe in one choice of coordinates, we can avoid all fractions in our calculations—this could be good if you're programming a computer that rounds decimals.

Since $\vec{v} = 2\vec{e}_1 - \vec{e}_2$, we know

$$[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

To find $[\vec{v}]_{\mathcal{C}}$, we need to write $\vec{v}$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$.
Suppose

$$\vec{v} = x\vec{c}_1 + y\vec{c}_2$$

for some unknown scalars x and y. On the one hand,

$$\vec{v} = 2\vec{e}_1 - \vec{e}_2,$$

and on the other hand,

$$\vec{v} = x\vec{c}_1 + y\vec{c}_2 = x(\vec{e}_1 + \vec{e}_2) + 3y\vec{e}_2 = x\vec{e}_1 + (x + 3y)\vec{e}_2.$$

Combining these two equations, we have

$$2\vec{e}_1 - \vec{e}_2 = x\vec{e}_1 + (x + 3y)\vec{e}_2,$$

and so

$$(x - 2)\vec{e}_1 + (x + 3y + 1)\vec{e}_2 = \vec{0}.$$

Since $\vec{e}_1$ and $\vec{e}_2$ are linearly independent, the only way for the above equation to be satisfied is if $x - 2 = 0$ *and* $x + 3y + 1 = 0$. Thus, we need to solve the system

$$\begin{cases} x \phantom{{}+3y} = 2 \\ x + 3y = -1 \end{cases}.$$

After solving, we see $\vec{v} = 2\vec{c}_1 - \vec{c}_2$, and so

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Notation Conventions

We need to revisit some past notation. Up to this point, we have been writing $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to mean $\vec{x} = 2\vec{e}_1 + 3\vec{e}_2$. However, given the representation-in-a-basis notation, we should be writing

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^2$. We should write $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{E}}$ because the coordinates $(2, 3)$ refer to *different* vectors for *different* bases. However, most of the time we are only thinking about the standard basis. So, the convention we will follow is:

- If a problem involves only one basis, we may write $\begin{bmatrix} x \\ y \end{bmatrix}$ to mean $\begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{E}}$ where $\mathcal{E}$ is the standard basis.
- If there are multiple bases in a problem, we will always write $\begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{X}}$ to specify a vector in coordinates relative to a particular basis $\mathcal{X}$.

Takeaway. If a problem only involves the standard basis, we may use the notation we always have. If a problem involves multiple bases, we must ***always*** use representation-in-a-basis notation.

True Vectors vs. Representations

[25]

The Belgian surrealist René Magritte painted the work above, which is subtitled, “This is not a pipe”. Why? Because, of course, it is not a pipe. It is a painting of a pipe! In this work, Magritte points out a distinction that will soon become very important to us—the distinction between an object and a representation of that object.

Let $\vec{x} = 2\vec{e}_1 + 3\vec{e}_2 \in \mathbb{R}^2$. The vector $\vec{x}$ is a *real-life geometrical thing*, and to emphasize this, we will call $\vec{x}$ a *true* vector. In contrast, when we write the column matrix $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix}2\\3\end{bmatrix}$, we are writing a *list of numbers*. The list of numbers $\begin{bmatrix}2\\3\end{bmatrix}$ has no meaning until we give it a meaning by assigning it a basis. For example, by writing $\begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{E}}$, we declare that the numbers 2 and 3 are the coefficients of $\vec{e}_1$ and $\vec{e}_2$. By writing $\begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, we declare that the numbers 2 and 3 are the coefficients of $\vec{b}_1$ and $\vec{b}_2$. Since a list of numbers without a basis has no meaning, we must acknowledge

$$\vec{x} \neq [\vec{x}]_{\mathcal{E}} = \begin{bmatrix}2\\3\end{bmatrix},$$

since the left side of the equation is a *true vector* and the right side is a *list of numbers*. Similarly, we must acknowledge

$$[\vec{x}]_{\mathcal{E}} \neq \begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{E}} = \vec{x},$$

since the left side is a *list of numbers* and the right side is a *true vector*.

To help keep the notation straight in your head, for a basis $\mathcal{X}$, remember the rule

$$[\text{true vector}]_{\mathcal{X}} = \text{list of numbers} \qquad \text{and} \qquad [\text{list of numbers}]_{\mathcal{X}} = \text{true vector}.$$

It’s easy to get confused when answering questions that involve multiple bases; precision will make these problems much easier.

Orientation of a Basis

How can you tell the difference between a hand and a foot? They’re similar in structure[26]—a hand has five fingers and a foot has five toes—but they’re different in shape—fingers are much longer than toes and the thumb sticks off the hand at a different angle than the big toe sticks off the foot.

How about a harder question: how can you tell the difference between a left hand and a right hand? Any length or angle measurement you make on an (idealized) left hand or right hand will be identical. But, we know they’re different because they differ in *orientation*.[27]

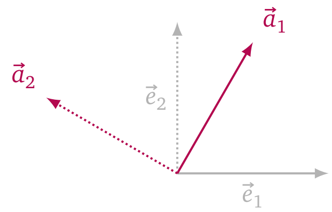
We’ll build up to the definition of orientation in stages. Consider the ordered bases $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$ shown below.

[25]Image taken from Wikipedia: `https://en.wikipedia.org/wiki/File:MagrittePipe.jpg`

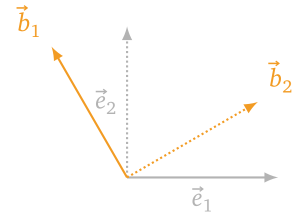
[26] We might say hands and feet are *topologically* equivalent.

[27] Other words for orientation include *chirality* and *handedness*.

$\mathcal{A} = \{\vec{a}_1, \vec{a}_2\}$



$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$



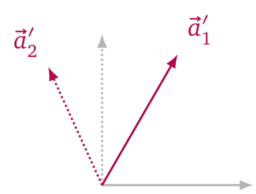
The $\mathcal{A}$ basis can be rotated to get the $\mathcal{E}$ basis while maintaining the proper order of the basis vectors (i.e., $\vec{a}_1 \mapsto \vec{e}_1$ and $\vec{a}_2 \mapsto \vec{e}_2$), but it is impossible to rotate the $\mathcal{B}$ basis to get the $\mathcal{E}$ basis while maintaining the proper order. In this case, we say that $\mathcal{E}$ and $\mathcal{A}$ have the same orientation and $\mathcal{E}$ and $\mathcal{B}$ have opposite orientations. Even though the lengths and angles between all vectors in the $\mathcal{A}$ basis and the $\mathcal{B}$ basis are the same, we can distinguish the $\mathcal{A}$ and $\mathcal{B}$ bases because they have different *orientations*.

Orientations of bases come in exactly two flavors: *right-handed* (or *positively oriented*) and *left-handed* (or *negatively oriented*). By convention, the standard basis is called right-handed.

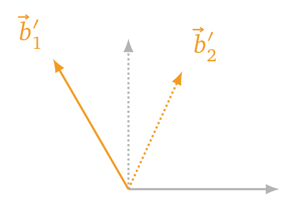
Orthonormal bases—bases consisting of unit vectors that are orthogonal to each other—are called right-handed if they can be rotated to align with the standard basis, otherwise they are called left-handed. In this way, the right-hand–left-hand analogy should be clear: two right hands or two left hands can be rotated to align with each other, but a left hand and a right can never be rotated to alignment.

However, not all bases are orthonormal! Consider the bases $\mathcal{E}$, $\mathcal{A}'$, $\mathcal{B}'$.

$\mathcal{A}' = \{\vec{a}_1', \vec{a}_2'\}$



$\mathcal{B}' = \{\vec{b}_1', \vec{b}_2'\}$



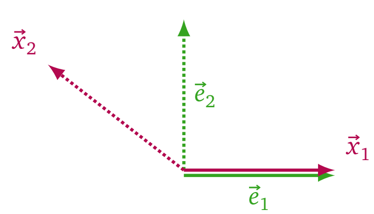
The bases $\mathcal{A}'$ and $\mathcal{B}'$ differ only slightly from $\mathcal{A}$ and $\mathcal{B}$. Neither can be *rotated* to obtain $\mathcal{E}$, however we'd still like to say $\mathcal{A}'$ is right-handed and $\mathcal{B}'$ is left-handed. The following, fully general definition, allows us to do so.

Orientation of a Basis. The ordered basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i \mapsto \vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

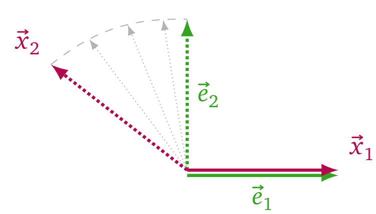
The term *continuously transformed* can be given a precise definition,[28] but it will be enough for us to imagine that a continuous transform between two bases is equivalent to a "movie" where one basis smoothly and without jumps transforms into the other.

Let's consider some examples. Let $\mathcal{X} = \{\vec{x}_1, \vec{x}_2\}$ as depicted below. We could imagine $\vec{x}_1, \vec{x}_2$ continuously transforming to $\vec{e}_1, \vec{e}_2$ by $\vec{x}_1$ staying in place and $\vec{x}_2$ smoothly moving along the dotted line.

$\mathcal{X} = \{\vec{x}_1, \vec{x}_2\}$



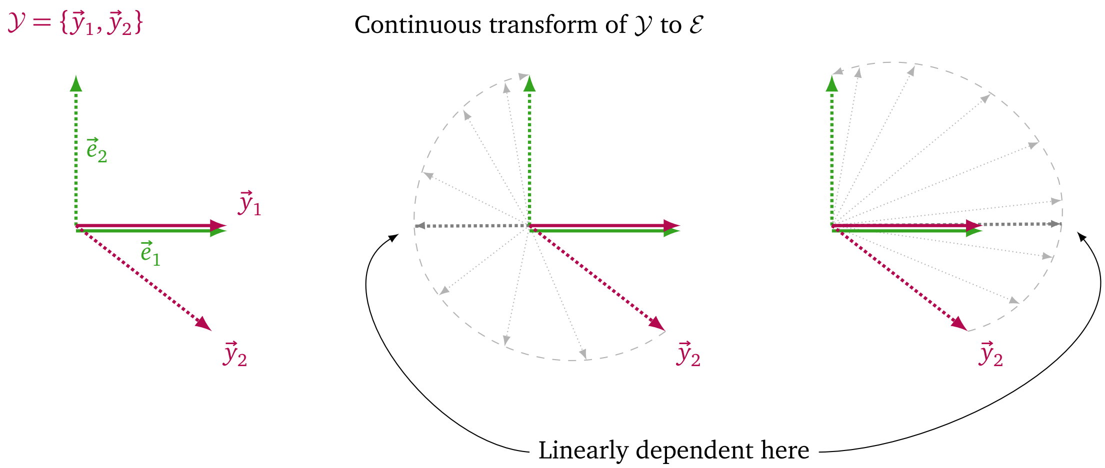
Continuous transform of $\mathcal{X}$ to $\mathcal{E}$



Because at every step along this motion, the set of $\vec{x}_1$ and the transformed $\vec{x}_2$ is linearly independent, $\mathcal{X}$ is *positively* oriented.

[28] Because you crave precision, here it is: the basis $\vec{a}_1, \dots, \vec{a}_n$ can be *continuously transformed* to the basis $\vec{b}_1, \dots, \vec{b}_n$ if there exists a continuous function $\Phi : [0,1] \to \{n\text{-tuples of vectors}\}$ so that $\Phi(0) = (\vec{a}_1, \dots, \vec{a}_n)$ and $\Phi(1) = (\vec{b}_1, \dots, \vec{b}_n)$. Here, continuity is defined in the multi-variable calculus sense.

Let $\mathcal{Y} = \{\vec{y}_1, \vec{y}_2\}$ as depicted below. We are in a similar situation, except this time, somewhere along $\vec{y}_2$'s path, the set of $\vec{y}_1$ and the transformed $\vec{y}_2$ becomes linearly dependent.

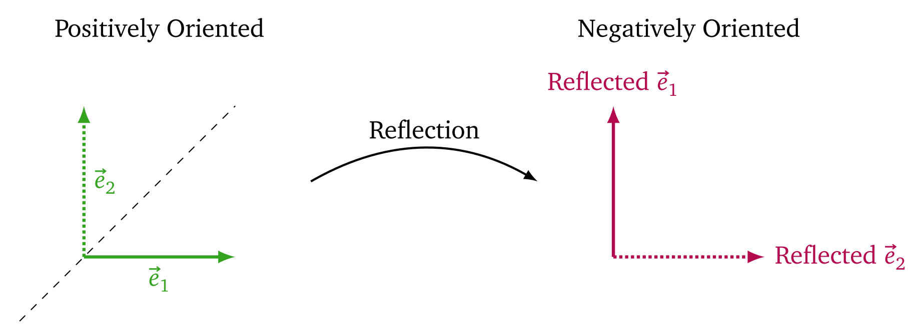


Maybe that was just bad luck and we might be able to transform along a different path and stay linearly independent. It turns out, we are doomed to fail, because $\mathcal{Y}$ is *negatively* oriented.

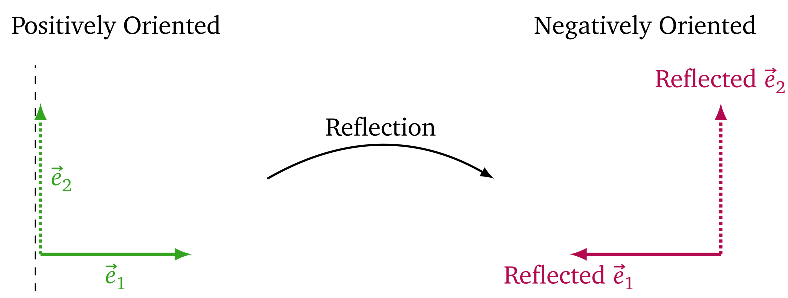
Using the definition of the orientation of a basis to answer questions is difficult because to determine that a basis is negatively oriented, you need to make a determination about *every possible* way to continuously transform a basis to the standard basis. This is hard enough in $\mathbb{R}^2$ and gets much harder in $\mathbb{R}^3$. Fortunately, we will encounter computational tools that will allow us to numerically determine the orientation of a basis, but, for now, the idea is what's important.

Reversing Orientation

Reflections reverse orientation and can manifest in two ways.[29] Consider the reflection of $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ across the line $y = x$.



This reflection sends $\{\vec{e}_1, \vec{e}_2\} \mapsto \{\vec{e}_2, \vec{e}_1\}$. Alternatively, reflection across the line $x = 0$ sends $\{\vec{e}_1, \vec{e}_2\} \mapsto \{-\vec{e}_1, \vec{e}_2\}$.



Both $\{\vec{e}_2, \vec{e}_1\}$ and $\{-\vec{e}_1, \vec{e}_2\}$, as ordered bases, are negatively oriented. This is indicative of a general theorem.

[29] Think back to hands. The left and right hands *are* reflections of each other.

Theorem. Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be an ordered basis. The ordered basis obtained from $\mathcal{B}$ by replacing $\vec{b}_i$ with $-\vec{b}_i$ and the ordered basis obtained from $\mathcal{B}$ by swapping the order of $\vec{b}_i$ and $\vec{b}_j$ (with $i \neq j$) have the opposite orientation as $\mathcal{B}$.

Practice Problems

1 Example problem with parts

(a) The first part

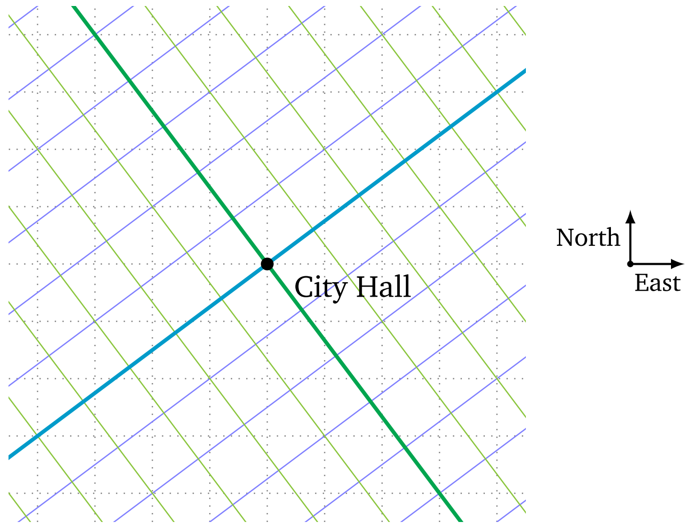
(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

37 The fictional town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \frac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

37.1 The points $A = (2, 1)$ and $B = (3, -1)$ are given in street coordinates. Find their east-north coordinates.

37.2 The points $X = (4, 3)$ and $Y = (1, 7)$ are given in east-north coordinates. Find their street coordinates.

37.3 Define $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$. Does $\text{span}\{\vec{e}_1, \vec{e}_2\} = \text{span}\{\vec{d}_1, \vec{d}_2\}$?

37.4 Notice that $Y = 5\vec{d}_1 + 5\vec{d}_2 = \vec{e}_1 + 7\vec{e}_2$. Is the point Y better represented by the pair $(5, 5)$ or by the pair $(1, 7)$? Explain.

Representation in a Basis

DEFINITION

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notated $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix}$$

where $\alpha_1, \ldots, \alpha_n$ uniquely satisfy $\vec{v} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$.

Conversely,

$$\begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix}_{\mathcal{B}} = \alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \ldots, \vec{b}_n$ with coefficients $\alpha_1, \ldots, \alpha_n$.

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38 Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

38.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$.

38.2 Express $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$.

38.3 Let $\vec{v} = 2\vec{e}_1 + 2\vec{e}_2$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

38.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}} = [\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

38.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}} = [\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

38.6 What is YX?

Orientation of a Basis

DEF

The ordered basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i \mapsto \vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

---

39 Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\vec{e}_1$ measured counter-clockwise starting at $\vec{e}_1$.

39.1 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ a linearly independent set?

39.2 For which θ can $\{\vec{e}_1, \vec{u}_\theta\}$ be continuously transformed into $\{\vec{e}_1, \vec{e}_2\}$ and remain linearly independent the whole time?

39.3 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ right-handed? Left-handed?

39.4 For which θ is $\{\vec{u}_\theta, \vec{e}_1\}$ (in that order) right-handed? Left-handed?

39.5 Is $\{2\vec{e}_1, 3\vec{e}_2\}$ right-handed or left-handed? What about $\{2\vec{e}_1, -3\vec{e}_2\}$?

Module 9

Linear Transformations

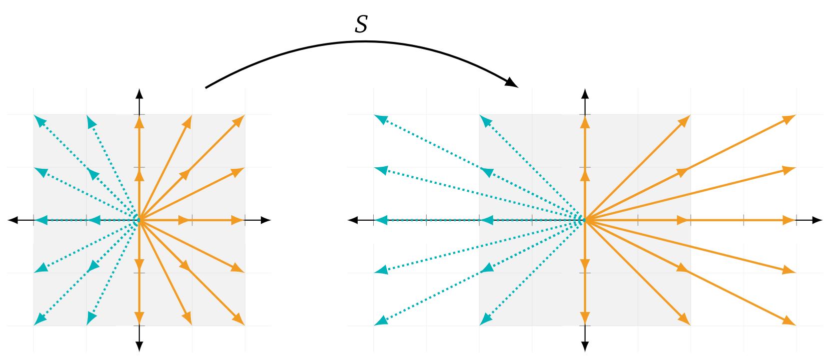
In this module you will learn

- The definition of a linear transformation.
- The definition of the image of a set under a transformation.
- How to prove whether a transformation is linear or not.
- How to find a matrix for a linear transformation.
- The difference between a matrix and a linear transformation.

Now that we have a handle on the basics of vectors, we can start thinking about *transformations*. Transformation (or map) is another word for a function, and transformations show up any time you need to describe vectors changing. For example, the transformation

$$S:\mathbb{R}^2\to\mathbb{R}^2 \qquad \text{defined by} \qquad \begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}2x\\y\end{bmatrix}$$

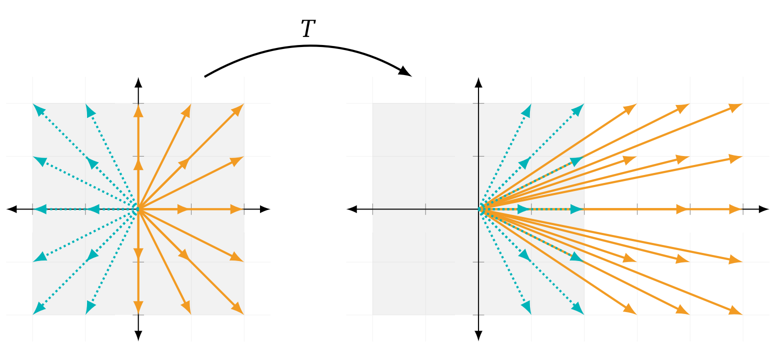
stretches all vectors in the $\vec e_1$ direction by a factor of 2.



The transformation

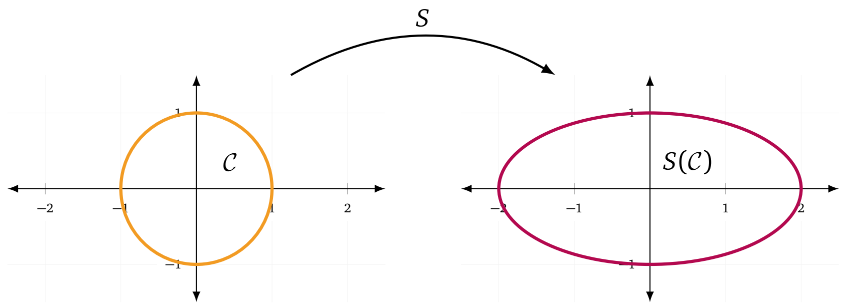
$$T:\mathbb{R}^2\to\mathbb{R}^2 \qquad \text{defined by} \qquad \begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}x+3\\y\end{bmatrix}$$

translates all vectors 3 units in the $\vec e_1$ direction.



Images of Sets

Recall the transformation $S:\mathbb{R}^2\to\mathbb{R}^2$ defined by $\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}2x\\y\end{bmatrix}$. If we had a bunch of vectors in the plane, applying S would stretch those vectors in the $\vec e_1$ direction by a factor of 2. For example, let $\mathcal{C}$ be the circle of radius 1 centered at $\vec 0$. Applying S to all the vectors that make up $\mathcal{C}$ produces an ellipse.



The operation of applying a transformation to a specific set of vectors and seeing what results is called taking the *image* of a set.

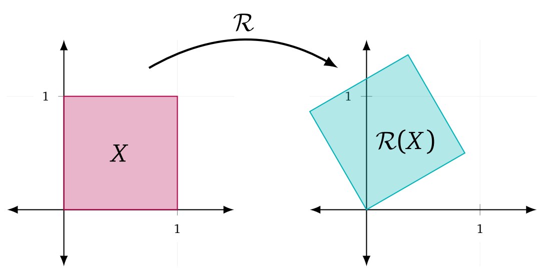
Image of a Set. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation and let $X \subseteq \mathbb{R}^n$ be a set. The ***image of the set** X **under** L*, denoted $L(X)$, is the set

$$L(X) = \{\vec{x} \in W \ : \ \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\}.$$

In plain language, the image of a set X under a transformation L is the set of all outputs of L when the inputs come from X.

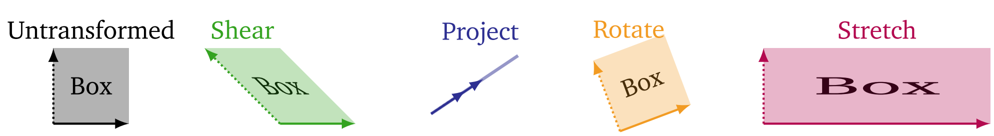
If you think of sets in $\mathbb{R}^n$ as black-and-white "pictures" (a point is black if it's in the set and white if it's not), then the image of a set under a transformation is the output after applying the transformation to the "picture".

Images allow one to describe complicated geometric figures in terms of an original figure and a transformation. For example, let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter clockwise by 30° and let $X = \{x\vec{e}_1 + y\vec{e}_2 \ : \ x, y \in [0, 1]\}$ be the filled-in unit square. Then, $\mathcal{R}(X)$ is the filled-in unit square that meets the x-axis at an angle of 30°. Try describing that using set builder notation!



Linear Transformations

Linear algebra's main focus is the study of a special category of transformations: the *linear* transformations. Linear transformations include rotations, dilations (stretches), shears, and more.



Linear transformations are an important type of transformation because (i) we have a complete theory of linear transformations (non-linear transformations are notoriously difficult to understand), and (ii) many non-linear transformations can be approximated by linear ones.[30] All this is to say that linear transformations are worthy of our study.

Without further ado, let's define what it means for a transformation to be linear.

Linear Transformation. Let V and W be subspaces. A function $T : V \to W$ is called a ***linear transformation*** if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \text{and} \qquad T(\alpha\vec{v}) = \alpha T(\vec{v})$$

for all vectors $\vec{u}, \vec{v} \in V$ and all scalars α.

[30] Just like in one-variable calculus where if you zoom into a function at a point its graph looks like a line, if you zoom into a (non-linear) transformation at a point, it looks like a linear one.

In plain language, the transformation T is linear if it distributes over addition and scalar multiplication. In other words, T *distributes over linear combinations*.

Example. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{S}{\mapsto} \begin{bmatrix} 2x \\ y \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x \\ y \end{bmatrix} \overset{T}{\mapsto} \begin{bmatrix} x \\ y+4 \end{bmatrix}.$$

For each of S and T, determine whether the transformation is linear.

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors, and let α be a scalar.
We first consider $\mathcal{S}$. We need to verify that $\mathcal{S}(\vec{u}+\vec{v}) = \mathcal{S}(\vec{u}) + \mathcal{S}(\vec{v})$ and $\mathcal{S}(\alpha\vec{u}) = \alpha\mathcal{S}(\vec{u})$.
Computing, we see

$$\mathcal{S}(\vec{u}+\vec{v}) = \mathcal{S}\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} 2u_1+2v_1 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 \\ v_2 \end{bmatrix} = \mathcal{S}(\vec{u}) + \mathcal{S}(\vec{v})$$

and

$$\mathcal{S}(\alpha\vec{u}) = \begin{bmatrix} 2\alpha u_1 \\ a u_2 \end{bmatrix} = \mathcal{S}\begin{bmatrix} \alpha u_1 \\ a u_2 \end{bmatrix} = \alpha\begin{bmatrix} 2u_1 \\ u_2 \end{bmatrix} = \alpha\mathcal{S}(\vec{u}),$$

and so $\mathcal{S}$ satisfies all the properties of a linear transformation.

Next we consider $\mathcal{T}$. Notice that $\mathcal{T}(\vec{u}+\vec{v}) = \begin{bmatrix} u_1+v_1 \\ u_2+v_2+4 \end{bmatrix}$ doesn't look like $\mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v}) = \begin{bmatrix} u_1+v_1 \\ u_2+v_2+8 \end{bmatrix}$.
Therefore, we will guess that $\mathcal{T}$ is not linear and look for a counter example.
Using $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we see

$$\mathcal{T}(\vec{e}_1+\vec{e}_2) = \mathcal{T}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \mathcal{T}(\vec{e}_1) + \mathcal{T}(\vec{e}_2).$$

Since at least one required property of a linear transformation is violated, $\mathcal{T}$ cannot be a linear transformation.

Function Notation vs. Linear Transformation Notation

Linear transformations are just special types of functions. In calculus, it is traditional to use lower case letters for a function and parenthesis "(" and ")" around the input to the function.

$$\underbrace{f : \mathbb{R} \to \mathbb{R}}_{\text{a function}} \qquad \underbrace{f(x)}_{f \text{ evaluated at } x}$$

For (linear) transformations, it is traditional to use capital letters to describe the function/transformation and parenthesis around the input are optional.

$$\underbrace{T : \mathbb{R}^n \to \mathbb{R}^m}_{\text{a transformation}} \qquad \underbrace{T(\vec{x})}_{T \text{ evaluated at } \vec{x}} \qquad \underbrace{T\vec{x}}_{\text{also } T \text{ evaluated at } \vec{x}}$$

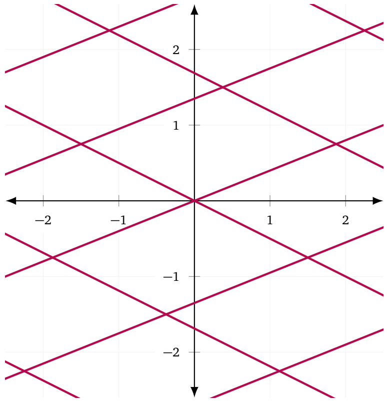
Since sets are also traditionally written using capital letters, sometimes a font variant is used to when writing the transformation or the set. For example, we might use a regular X to denote a set and a calligraphic $\mathcal{T}$ to describe a transformation.

Another difference you might not be used to is that, in linear algebra, we make a careful distinction between a function and its output. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. In calculus, you might consider the phrases "the function f" and "the function $f(x)$" to both make sense. In linear algebra, the first phrase is valid and the second is *not*. By writing $f(x)$, we are indicating "the output of the function f when x is input". So, properly we should say "the *number* $f(x)$".

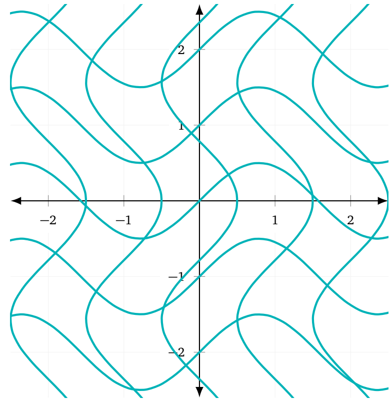
This distinction might seem pedantic now, but by keeping our functions as functions and our numbers/vectors as numbers/vectors, we can avoid some major confusion in the future.

The "look" of a Linear Transformation

Images under linear transformations have a certain look to them. Based just on the word *linear* you can probably guess which figure below represents the image of a grid under a linear transformation.



Linear



Non-linear

Let's prove some basic facts about linear transformations.

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(\vec{0}) = \vec{0}$.

Proof. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $\vec{v} \in \mathbb{R}^n$. We know that $0\vec{v} = \vec{0}$, so by linearity we have

$$T(\vec{0}) = T(0\vec{v}) = 0T(\vec{v}) = \vec{0}.$$

■

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T takes lines to lines (or points).

Proof. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and let $\ell \subseteq \mathbb{R}^n$ be the line given in vector form by $\vec{x} = t\vec{d} + \vec{p}$. We want to prove that $T(\ell)$, the image of ℓ under the transformation T, is a line or a point.

By definition, every point in ℓ takes the form $t\vec{d} + \vec{p}$ for some scalar t. Therefore, every point in $T(\ell)$ take the form $T(t\vec{d} + \vec{p})$ for some scalar t. But, T is a linear transformation, so

$$T(t\vec{d} + \vec{p}) = tT(\vec{d}) + T(\vec{p}).$$

If $T(\vec{d}) \neq \vec{0}$, then $\vec{x} = tT(\vec{d}) + T(\vec{p})$ describes a line in vector form and so $T(\ell)$ is a line. If $T(\vec{d}) = \vec{0}$, then $T(\ell) = \{t\vec{0} + T(\vec{p}) : t \text{ is a scalar}\} = \{T(\vec{p})\}$ is a point. ■

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T takes parallel lines to parallel lines (or points).

Proof. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and let ℓ_1 and ℓ_2 be parallel lines. Then, we may describe ℓ_1 in vector form as $\vec{x} = t\vec{d} + \vec{p}_1$ and we may describe ℓ_2 in vector form as $\vec{x} = t\vec{d} + \vec{p}_2$. Note that since the lines are parallel, the direction vectors are the same.

Now, $T(\ell_1)$ can be described in vector form by

$$\vec{x} = tT(\vec{d}) + T(\vec{p}_1)$$

and $T(\ell_2)$ can be described in vector form by

$$\vec{x} = tT(\vec{d}) + T(\vec{p}_2).$$

Written this way and provided $T(\ell_1)$ and $T(\ell_2)$ are actually lines, we immediately see that $T(\ell_1)$ and $T(\ell_2)$ have the same direction vectors and so are parallel.

If $T(\ell_1)$ is instead a point, then we must have $T(\vec{d}) = \vec{0}$, and so $T(\ell_2)$ must also be a point. ■

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T takes subspaces to subspaces.

Proof. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $V \subseteq \mathbb{R}^n$ be a subspace. We need to show that $T(V)$ satisfies the properties of a subspace.

Since V is non-empty, we know $T(V)$ is non-empty.

Let $\vec{x}, \vec{y} \in T(V)$. By definition, there are vectors $\vec{u}, \vec{v} \in V$ so that

$$\vec{x} = T(\vec{u}) \qquad \text{and} \qquad \vec{y} = T(\vec{v}).$$

Since T is linear, we know

$$\vec{x}+\vec{y}=T(\vec{u})+T(\vec{v})=T(\vec{u}+\vec{v}).$$

Because V is a subspace, we know $\vec{u}+\vec{v}\in V$ and so we conclude $\vec{x}+\vec{y}=T(\vec{u}+\vec{v})\in T(V)$.

Similarly, for any scalar α we have

$$\alpha\vec{x}=\alpha T(\vec{u})=T(\alpha\vec{u}).$$

Since V is a subspace, $\alpha\vec{u}\in V$ and so $\alpha\vec{x}=T(\alpha\vec{u})\in T(V)$. ■

Linear Transformations and Proofs

When proving things in math, you have all of logic at your disposal, and that freedom can be combined with creativity to show some truly amazing things. But, for better or for worse, proving whether or not a transformation is linear usually doesn't require substantial creativity.

Let $T:\mathbb{R}^n\to\mathbb{R}^n$ defined by $T(\vec{v})=2\vec{v}$. To show that T is linear, we need to show that for *all* inputs $\vec{x}$ and $\vec{y}$ and for *all* scalars α we have

$$T(\vec{x}+\vec{y})=T(\vec{x})+T(\vec{y}) \qquad \text{and} \qquad T(\alpha\vec{x})=\alpha T(\vec{x}).$$

But, there are an infinite number of choices for $\vec{x}$, $\vec{y}$, and α. How can we argue about all of them at once?

Consider the following proof that T is linear.

Proof. Let $\vec{x},\vec{y}\in\mathbb{R}^n$ and let α be a scalar. By applying the definition of T, we see

$$T(\vec{x}+\vec{y})=2(\vec{x}+\vec{y})=2\vec{x}+2\vec{y}=T(\vec{x})+T(\vec{y}).$$

Similarly,

$$T(\alpha\vec{x})=2(\alpha\vec{x})=\alpha(2\vec{x})=\alpha T(\vec{x}).$$

Since T satisfies the two properties of a linear transformation, T is a linear transformation. ■

This proof starts out with "let $\vec{x},\vec{y}\in\mathbb{R}^n$ and let α be a scalar". In what follows, the only properties of $\vec{x}$ and $\vec{y}$ we use come from the fact that they're in $\mathbb{R}^n$ (the domain of T) and the only fact about α we use is that it's a scalar. Because of this, $\vec{x}$, and $\vec{y}$ are considered *arbitrary* vectors and α is an *arbitrary* scalar. Put another way, the argument that followed would work for every single pair of vectors $\vec{x},\vec{y}\in\mathbb{R}^n$ and for every scalar α. Thus, by fixing arbitrary vectors at the start of our proof, we are (i) able to argue about all vectors at once while (ii) having named vectors that we can actually use in equations.

Takeaway. Starting a linearity proof with "***let*** $\vec{x},\vec{y}\in\mathbb{R}^n$ ***and let*** α ***be a scalar***" allows you to argue about all vectors and scalars simultaneously.

The proof given above is very typical, and almost every proof of the linearity of a function $T:\mathbb{R}^n\to\mathbb{R}^m$ will look something like

Proof. Let $\vec{x},\vec{y}\in\mathbb{R}^n$ and let α be a scalar. By applying the definition of T, we see

$$T(\vec{x}+\vec{y})=\text{application(s) of the definition}=T(\vec{x})+T(\vec{y}).$$

Similarly,

$$T(\alpha\vec{x})=\text{application(s) of the definition}=\alpha T(\vec{x}).$$

Since T satisfies the two properties of a linear transformation, T is a linear transformation. ■

This isn't to say that proving whether or not a transformation is linear is *easy*, but all the cleverness and insight required appears in the "application(s) of the definition" parts.

What about showing a transformation is *not* linear? Here we don't need to show something true for all vectors and all scalars. We only need to show something is false for *one* pair of vectors or *one* pair of a vector and a scalar.

When proving a transformation is not linear, we can pick one of the properties of linearity (distribution over vector addition or distribution over scalar multiplication) and a *single example* where that property fails.[31]

[31] It's often tempting to argue that the properties of linearity fail for all inputs, but this is a dangerous path! For instance, if $T(\vec{0})=\vec{0}$, then $T(\vec{a})=T(\vec{a}+\vec{0})=T(\vec{a})+T(\vec{0})=T(\vec{a})$ *regardless* of whether T is linear or not.

Example. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = \vec{x} + \vec{e}_1$. Show that T is *not* linear.

Proof. We will show that T does not distribute with respect to scalar multiplication.
Observe that

$$T(2\vec{0}) = T(\vec{0}) = \vec{e}_1 \neq 2\vec{e}_1 = 2T(\vec{0}).$$

Therefore, T cannot be a linear transformation. ∎

Matrix Transformations

We already know two ways to interpret matrix multiplication—linear combinations of the columns and dot products with the rows—and we're about to have a third.

Let $M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. For a vector $\vec{v} \in \mathbb{R}^2$, $M\vec{v}$ is another vector in $\mathbb{R}^2$. In this way, we can think of multiplication by M as a transformation on $\mathbb{R}^2$. Define

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{by} \qquad T(\vec{x}) = M\vec{x}.$$

Because T is defined by a matrix, we call T a *matrix transformation*. It turns out all matrix transformations are linear transformations and most linear transformations are matrix transformations.[32]

When it comes to specifying linear transformations, matrices are heroes, providing a compact notation (just like they did for systems of linear equations). For example, we could say, "The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that doubles the x-coordinate and triples the y-coordinate", or we could say, "The matrix transformation given by $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$".

When talking about matrices and linear transformations, we must keep in mind that they are not the same thing. A matrix is a box of numbers and has no meaning until we give it meaning. A linear transformation is a function that inputs vectors and outputs vectors. We can *specify* a linear transformation using a matrix, but a matrix by itself is *not* a linear transformation.[33]

Takeaway. Matrices and linear transformations are closely related, but they aren't the same thing.

So what are some correct ways to specify a linear transformation using a matrix? For a matrix M, the following are correct.

- The transformation T defined by $T(\vec{x}) = M\vec{x}$.
- The transformation given by multiplication by M.
- The transformation induced by M.
- The matrix transformation given by M.
- The linear transformation whose matrix is M.

Finding a Matrix for a Linear Transformation

Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ has a matrix, and we can use basic algebra to find an appropriate matrix.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Since T inputs vectors with n coordinates and outputs vectors with m coordinates, we know any matrix for T must be $m \times n$. The process of finding a matrix for T can now be summarized as follows: (i) create an $m \times n$ matrix of variables, (ii) use known input-output pairs for T to set up a system of equations involving the unknown variables, (iii) solve for the variables.

Example. Let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathcal{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x \end{bmatrix}$. Find a matrix, M, for $\mathcal{T}$.

Because $\mathcal{T}$ is a transformation for $\mathbb{R}^2 \to \mathbb{R}^2$, M will be a 2×2 matrix. Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[32] If you believe in the axiom of choice and you allow infinitely sized matrices, every linear transformation can be expressed as a matrix transformation.

[33] Consider the function defined by $f(x) = 2x$. You would never say that the function f is 2!

We now need to use input-output pairs to "calibrate" M. We know

$$\mathcal{T}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix} \qquad \text{and} \qquad \mathcal{T}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

Since M is a matrix for $\mathcal{T}$, we know $\mathcal{T}\vec{x}=M\vec{x}$ for all $\vec{x}$, and so

$$M\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}a+b\\c+d\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$$

and

$$M\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}b\\d\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

This gives us the system of equations

$$\begin{cases} a+b = 3 \\ c+d = 1 \\ b = 1 \\ d = 0 \end{cases},$$

and solving this system tells us

$$M=\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}2&1\\1&0\end{bmatrix}.$$

Practice Problems

1 Example problem with parts

(a) The first part

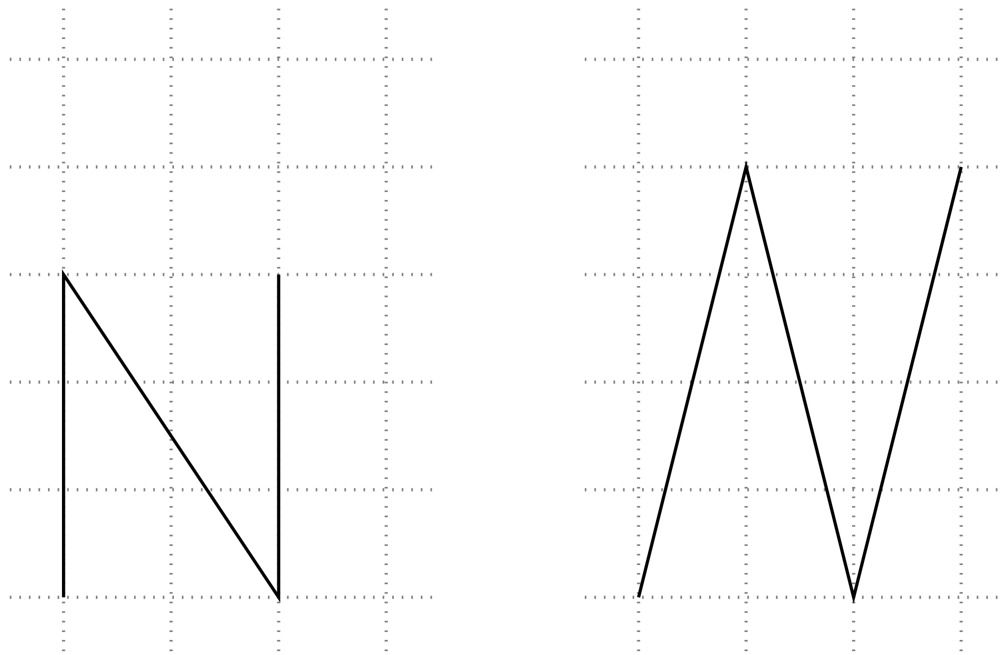
(b) The second part

Solutions to the parts

(a) $(2,4,8)$

(b) $(15,6,-21)$

Task 2.1: Italicizing N

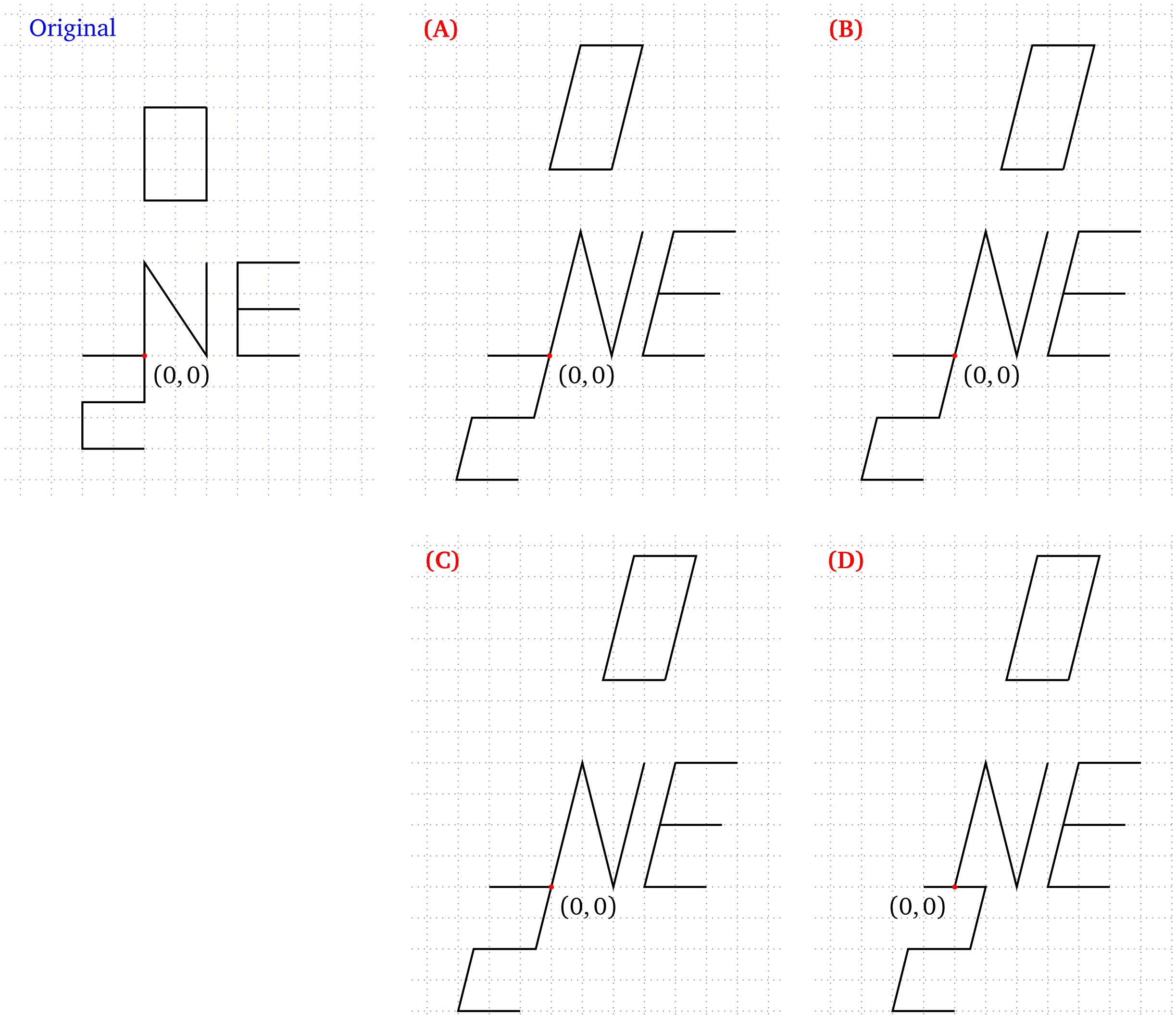


Suppose that the “N” on the left is written in regular 12-point font. Find a matrix A that will transform the “N” into the letter on the right which is written in an *italic* 16-point font.

Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

Task 2.2: Beyond the N

A few students were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If other letters are placed around the "N," the students argued over four different possible results for the transformed letters. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?



40 $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

40.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

40.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

40.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

40.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter-clockwise by 90°.

DEFINITION

Linear Transformation

Let V and W be subspaces. A function $T : V \to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \text{and} \qquad T(\alpha\vec{v}) = \alpha T(\vec{v})$$

for all vectors $\vec{u}, \vec{v} \in V$ and all scalars α.

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41.1 Classify the following as linear transformations or not.

(a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

(b) $W : \mathbb{R}^2 \to \mathbb{R}^2$ where $W\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$.

(d) $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix} x \\ y \end{bmatrix} = \operatorname{vcomp}_{\vec{u}}\begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

DEFINITION

Image of a Set

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation and let $X \subseteq \mathbb{R}^n$ be a set. The ***image of the set X under L***, denoted $L(X)$, is the set

$$L(X) = \{\vec{x} \in W \ : \ \vec{x} = L(\vec{y}) \text{ for some } \vec{y} \in X\}.$$

42 Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x, y \leq 1 \right\} \subseteq \mathbb{R}^2$ be the filled-in unit square and let $C = \{\vec{0}, \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2\} \subseteq \mathbb{R}^2$ be the corners of the unit square.

42.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

42.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).

42.3 Let $\ell = \{\text{all convex combinations of } \vec{a} \text{ and } \vec{b}\}$ be a line segment with endpoints $\vec{a}$ and $\vec{b}$ and let A be a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

42.4 Explain how images of sets relate to the *Italicizing N* task.

The Composition of Linear Transformations

Module 10

In this module you will learn

- How to break a complicated transformation into the composition of simpler ones.
- How the composition of linear transformations relates to matrix multiplication.

In life, we encounter situations where we do one thing after another. For example, you put on your socks and then your shoes. We might call this whole operation (of first putting on your socks and then your shoes) "getting dressed", and it is an example of *function composition*.

Composition of Functions. Let $f : A \to B$ and $g : B \to C$. The ***composition*** of g and f, notated $g \circ f$, is the function $h : A \to C$ defined by

$$h(x) = g \circ f(x) = g\big(f(x)\big).$$

We can formalize the shoes-socks example with mathematical notation.

$$\text{getting dressed} = (\text{putting on shoes}) \circ (\text{putting on socks})$$

Or, if X represented putting on socks, S represented putting on shoes, and D represented getting dressed, $D = S \circ X$.

This real-life example has utility when talking to children. Getting dressed is a complicated operation, but by breaking it up into simpler operations, even a young child can understand the process. In this vein, we can understand complicated linear transformations by breaking them up into the composition of simpler ones.

For example, define

$$\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{by} \qquad \mathcal{T}(\vec{x}) = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\vec{x}.$$

It's hard to understand what $\mathcal{T}$ does just by looking at inputs and outputs. However, if we were keen enough to notice that

$$\mathcal{T} = \mathcal{S} \circ \mathcal{R}$$

where $\mathcal{R}$ was rotation counter-clockwise by 45° and $\mathcal{S}$ was a stretch in the $\vec{e}_1$ direction by a factor of 2, suddenly $\mathcal{T}$ wouldn't be such a mystery.

How does one figure out the "simple transformations" that when composed give the full transformation? In truth, there are many, many methods and there are whole books written about how to find these decompositions efficiently. We will encounter two algorithms for this,[34] but for now our methods will be ad hoc.

Example. Let $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by $M = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 \end{bmatrix}$, let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter clockwise by 45°, and let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the x-axis. Write $\mathcal{U}$ as the composition (in some order) of $\mathcal{R}$ and $\mathcal{P}$.

We will use $\vec{e}_1$ and $\vec{e}_2$ to determine whether $\mathcal{U}$ is $\mathcal{R} \circ \mathcal{P}$ or $\mathcal{P} \circ \mathcal{R}$.
Computing,

$$\mathcal{U}(\vec{e}_1) = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{U}(\vec{e}_2) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}.$$

For $\mathcal{R} \circ \mathcal{P}$:

$$\mathcal{R} \circ \mathcal{P}(\vec{e}_1) = \mathcal{R}(\mathcal{P}(\vec{e}_1)) = \mathcal{R}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\mathcal{R} \circ \mathcal{P}(\vec{e}_2) = \mathcal{R}(\mathcal{P}(\vec{e}_2)) = \mathcal{R}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[34] Phrased in terms of matrices instead of linear transformations, the decompositions we will study are called: (i) decomposition into *elementary* matrices, and (ii) *diagonalization*.

For $\mathcal{P}\circ\mathcal{R}$:

$$\mathcal{P}\circ\mathcal{R}(\vec{e}_1)=\mathcal{P}(\mathcal{R}(\vec{e}_1))=\mathcal{P}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{2}}{2}\\ 0\end{bmatrix}$$
$$\mathcal{P}\circ\mathcal{R}(\vec{e}_2)=\mathcal{P}(\mathcal{R}(\vec{e}_2))=\mathcal{P}\begin{bmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}=\begin{bmatrix}-\frac{\sqrt{2}}{2}\\ 0\end{bmatrix}.$$

Since $\mathcal{P}\circ\mathcal{R}$ agrees with $\mathcal{U}$ on the standard basis (i.e., $\mathcal{P}\circ\mathcal{R}$ and $\mathcal{U}$ output the same vectors when $\vec{e}_1$ and $\vec{e}_2$ are input), they must agree for all vectors. Therefore $\mathcal{U}=\mathcal{P}\circ\mathcal{R}$.

Compositions and Matrix Products

Let $\mathcal{A}:\mathbb{R}^2\to\mathbb{R}^2$ and $\mathcal{B}:\mathbb{R}^2\to\mathbb{R}^2$ be matrix transformations with matrices

$$M_A=\begin{bmatrix}1&2\\0&2\end{bmatrix}\qquad\text{and}\qquad M_B=\begin{bmatrix}-1&-1\\-2&0\end{bmatrix}.$$

(Make sure you understand why $\mathcal{A}\neq M_A$ before continuing!)

Define $\mathcal{T}=\mathcal{A}\circ\mathcal{B}$. Since $\mathcal{T}:\mathbb{R}^2\to\mathbb{R}^2$, we know $\mathcal{T}$ has a matrix M_T. We can find M_T by the usual methods. First, compute some input-output pairs for $\mathcal{T}$.

$$\mathcal{T}\begin{bmatrix}1\\0\end{bmatrix}=\mathcal{A}\left(\mathcal{B}\begin{bmatrix}1\\0\end{bmatrix}\right)=\mathcal{A}\begin{bmatrix}-1\\-2\end{bmatrix}=\begin{bmatrix}-5\\-4\end{bmatrix}\qquad\text{and}\qquad\mathcal{T}\begin{bmatrix}0\\1\end{bmatrix}=\mathcal{A}\left(\mathcal{B}\begin{bmatrix}0\\1\end{bmatrix}\right)=\mathcal{A}\begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$$

Letting $M_T=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, we use the input-output pairs to see

$$\begin{bmatrix}a\\c\end{bmatrix}=M_T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-5\\-4\end{bmatrix}\qquad\text{and}\qquad\begin{bmatrix}b\\d\end{bmatrix}=M_T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix},$$

and so

$$M_T=\begin{bmatrix}-5&-1\\-4&0\end{bmatrix}.$$

We found M_T, the matrix for $\mathcal{T}$, using traditional techniques, but could we have used M_A and M_B to somehow find M_T? As it turns out, yes, we could have!

By definition,

$$\mathcal{A}\vec{x}=M_A\vec{x}\qquad\text{and}\qquad\mathcal{B}\vec{x}=M_B\vec{x},$$

since $\mathcal{A}$ and $\mathcal{B}$ are matrix transformations. Therefore,

$$\mathcal{A}(\mathcal{B}\vec{x})=M_A(M_B\vec{x}).$$

But, matrix multiplication is *associative*,[35] and so

$$M_A(M_B\vec{x})=(M_AM_B)\vec{x}.$$

Thus M_AM_B must be a matrix for $\mathcal{A}\circ\mathcal{B}=\mathcal{T}$. Indeed, computing the matrix product, we see

$$M_AM_B=\begin{bmatrix}1&2\\0&2\end{bmatrix}\begin{bmatrix}-1&-1\\-2&0\end{bmatrix}=\begin{bmatrix}-5&-1\\-4&0\end{bmatrix}=M_T.$$

The fact that matrix multiplication corresponds to function composition is no coincidence. It is the very reason matrix multiplication is defined the way it is. This is reiterated in the following theorem.

Theorem. If $\mathcal{P}:\mathbb{R}^a\to\mathbb{R}^b$ and $\mathcal{Q}:\mathbb{R}^c\to\mathbb{R}^a$ are matrix transformations with matrices M_P and M_Q, then $\mathcal{P}\circ\mathcal{Q}$ is a matrix transformation whose matrix is given by the matrix product M_PM_Q.

It should now be clear why the order of matrix multiplication matters. The order of function composition matters (you must put on your socks before your shoes!), and since matrix multiplication corresponds to function composition, the order of matrix multiplication must matter.

[35] If an operation is associative, it means that where you put the parenthesis doesn't matter.

Practice Problems

1 Example problem with parts

(a) The first part

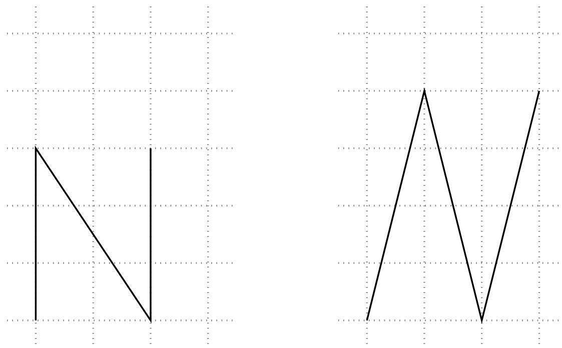
(b) The second part

Solutions to the parts

(a) $(2,4,8)$

(b) $(15,6,-21)$

Task 2.3: Pat and Jamie



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

1. Do you think Pat and Jamie's approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?
2. Try Pat and Jamie's approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

43 Define $\mathcal{P}$ to be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

43.1 Find a matrix P so that $P\vec{x} = \mathcal{P}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

43.2 Find a matrix R so that $R\vec{x} = \mathcal{R}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

43.3 Write down matrices A and B for $\mathcal{P} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{P}$.

43.4 How do the matrices A and B relate to the matrices P and R?

Range & Nullspace of a Linear Transformation

In this module you will learn

- The definition of the range and null space of a linear transformation.
- How to precisely notate the matrix for a linear transformation.
- The fundamental subspaces corresponding to a matrix (row space, column space, null space) and how they relate to the range and null space of a linear transformation.
- How to find a basis for the fundamental subspaces of a matrix.
- The definition of rank and the rank-nullity theorem.

Associated with every linear transformation are two specially named subspaces: the range and the null space.

Range

Range. The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\operatorname{range}(T) = \{\vec{y} \in W \,:\, \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

The range of a linear transformation has the exact same definition as the range of a function—it's the set of all outputs. In other words, the range of a linear transformation is the *image* of the entire domain with respect to that linear transformation.[36] However, unlike the range of an arbitrary function, the range of a linear transformation is always a subspace.

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\operatorname{range}(T) \subseteq \mathbb{R}^m$ is a subspace.

Proof. Since $\operatorname{range}(T) = T(\mathbb{R}^n)$ and $\mathbb{R}^n$ is non-empty, we know that $\operatorname{range}(T)$ is non-empty. Therefore, to show that $\operatorname{range}(T)$ is a subspace, what remains to be shown is (i) that it's closed under vector addition, and (ii) that it is closed under scalar multiplication.

(i) Let $\vec{x}, \vec{y} \in \operatorname{range}(T)$. By definition, there exist $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $\vec{x} = T(\vec{u})$ and $\vec{y} = T(\vec{v})$. Since T is linear,

$$\vec{x} + \vec{y} = T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}),$$

and so $\vec{x} + \vec{y} \in \operatorname{range}(T)$.

(ii) Let $\vec{x} \in \operatorname{range}(T)$ and let α be a scalar. By definition, there exists $\vec{u} \in \mathbb{R}^n$ such that $\vec{x} = T(\vec{u})$, and so by the linearity of T,

$$\alpha\vec{x} = \alpha T(\vec{u}) = T(\alpha\vec{u}).$$

Therefore $\alpha\vec{x} \in \operatorname{range}(T)$.

∎

When analyzing subspaces, we are often interested in how big they are. That information is captured by a numbers—the *dimension* of the subspace. For transformations, we also have a notion of how "big" they are, which is captured in a number called the *rank*.

Rank of a Linear Transformation. For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the ***rank*** of T, denoted $\operatorname{rank}(T)$, is the dimension of the range of T.

The rank of a linear transformation can be used to measure its complexity or compressibility. A rank 0 transformation must send all vectors to $\vec{0}$. A rank 1 transformation must send all vectors to a line, etc.. So, by knowing just a single number—the rank—you can judge how complicated the set of outputs of a linear transformation will be.

Example. Let $\mathcal{P}$ be the plane given by $x + y + z = 0$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be projection onto $\mathcal{P}$. Find $\operatorname{range}(T)$ and $\operatorname{rank}(T)$.

[36] Some people say "the *image* of T" as a short way of saying "the image of the entire domain of T under T". Used in this sense $\operatorname{Image}(T) = \operatorname{range}(T)$.

First we will find range(T). Since T is a projection onto $\mathcal{P}$, we know range$(T) \subseteq \mathcal{P}$. Because $T(\vec{p}) = \vec{p}$ for all $\vec{p} \in \mathcal{P}$, we know $\mathcal{P} \subseteq$ range(T), and so

$$\text{range}(T) = \mathcal{P}.$$

Since $\mathcal{P}$ is a plane, we know $\dim(\mathcal{P}) = 2 = \dim(\text{range}(T)) = \text{rank}(T)$.

Null Space

The second special subspace is called the *null space*.

Null Space. The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$\text{null}(T) = \{\vec{x} \in V \;:\; T\vec{x} = \vec{0}\}.$$

We've seen null spaces before. In the context of matrices when we asked questions like, "Are these column vectors linearly independent?" Now that we understand linear transformation and subspaces, we can consider this question anew.

Just like the range of a linear transformation, the null space of a linear transformation is always a subspace.

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then null$(T) \subseteq \mathbb{R}^n$ is a subspace.

Proof. Since T is linear, $T(\vec{0}) = \vec{0}$ and so $\vec{0} \in$ null(T) which shows that null(T) is non-empty. Therefore, to show that null(T) is a subspace, we only need to show (i) that it's closed under vector addition, and (ii) that it is closed under scalar multiplication.

(i) Let $\vec{x}, \vec{y} \in$ null(T). By definition, $T(\vec{x}) = T(\vec{y}) = \vec{0}$. By linearity we see

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{0} + \vec{0} = \vec{0},$$

and so $\vec{x} + \vec{y} \in$ null(T).

(ii) Let $\vec{x} \in$ null(T) and let α be a scalar. By definition, $T(\vec{x}) = \vec{0}$, and so by the linearity of T,

$$T(\alpha\vec{x}) = \alpha T(\vec{x}) = \alpha\vec{0} = \vec{0}.$$

Therefore $\alpha\vec{x} \in$ null(T).

■

Akin to the rank–range connection, there is a special number call the *nullity* which specifies the dimension of the null space.

Nullity. For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the ***nullity*** of T, denoted nullity(T), is the dimension of the null space of T.

Example. Let $\mathcal{P}$ be the plane given by $x + y + z = 0$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be projection onto $\mathcal{P}$. Find null(T) and nullity(T).

First we will find null(T). Since T is a projection onto $\mathcal{P}$ (and because $\mathcal{P}$ passes through $\vec{0}$), we know every normal vector for $\mathcal{P}$ will get sent to $\vec{0}$ when T is applied. And, besides $\vec{0}$ itself, these are the only vectors that get sent to $\vec{0}$. Therefore

$$\text{null}(T) = \{\text{normal vectors}\} \cup \{\vec{0}\} = \text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

Since null(T) is a line, we know nullity$(T) = 1$.

Fundamental Subspaces of a Matrix

Every linear transformation has a range and a null space. Analogously, every matrix is associated with three fundamental subspaces.

Fundamental Subspaces. Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\mathrm{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\mathrm{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\mathrm{null}(M)$, is the set of solutions to $M\vec{x} = \vec{0}$.

Computationally, it's much easier to find the row space/column space/null space of a matrix than it is to find the range/null space of a linear transformation because we can turn matrix questions into systems of linear equations.

Example. Find the null space of $M = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}$.

To find the null space of M, we need to solve the homogeneous matrix equation $M\vec{x} = \vec{0}$. Row reducing, we see

$$\mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

and so the z column is a free variable column. Therefore, the complete solution can be expressed in vector form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix},$$

and so

$$\mathrm{null}(M) = \mathrm{span}\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}.$$

The column space and row space are just as easy to compute, since it just involves picking a basis from the existing row or column vectors.

Example. Let $M = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}$. Find a basis for the row space and the column space of M.

First the column space. We need to pick a basis for $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \end{bmatrix}\right\}$, which is the same thing as picking a maximal linearly independent subset of $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \end{bmatrix}\right\}$.
Putting these vectors as columns in a matrix and row reducing, we see

$$\mathrm{rref}\left(\begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The first and second columns are the only pivot columns and so the first and second *original* vectors form a maximal linearly independent subset. Thus,

$$\mathrm{col}(M) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\} = \mathbb{R}^2 \qquad \text{and a basis is} \qquad \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}.$$

To find the column space, we need to pick a basis for $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}\right\}$. Repeating a similar procedure, we see

$$\mathrm{rref}\left(\begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 5 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and so $\left\{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}\right\}$ is linearly independent. Therefore

$$\mathrm{row}(M) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}\right\} \qquad \text{and a basis is} \qquad \left\{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}\right\}.$$

When talking about fundamental subspaces, we often switch between talking about column vectors and row vectors belonging to a matrix. The operation of swapping rows for columns is called the *transpose*.

Transpose.
Let M be an $n \times m$ matrix defined by

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{3n} & \cdots & a_{nm} \end{bmatrix}.$$

The ***transpose*** of M, notated M^T, is the $m \times n$ matrix produced by swapping the rows and columns of M. That is

$$M^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Using the transpose, we can make statements like

$$\mathrm{col}(M) = \mathrm{row}(M^T) \qquad \text{and} \qquad \mathrm{row}(M) = \mathrm{col}(M^T).$$

In addition, it helps us state the following theorem.

Theorem. For a matrix A, the dimension of the row space equals the dimension of the column space. That is, $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Proof. To prove $\mathrm{rank}(A) = \mathrm{rank}(A^T)$, we will rely on what we know about the row reduction algorithm and what the reduced row echelon form of a matrix tells us.

Claim 1: $\mathrm{row}(\mathrm{rref}(A)) \subseteq \mathrm{row}(A)$. To see this, observe that to get $\mathrm{rref}(A)$, we take linear combinations of the rows of A. Therefore, it must be that the span of the rows of $\mathrm{rref}(A)$ is contained in the span of the rows of A.

Claim 2: $\mathrm{row}(\mathrm{rref}(A)) = \mathrm{row}(A)$. To see this, observe that every elementary row operation is reversible. Therefore every row in A can be obtained as a linear combination of rows in $\mathrm{rref}(A)$ (by just reversing the steps). Thus the row vectors of $\mathrm{rref}(A)$ and the row vectors of A must have the same span.

Claim 3: The non-zero rows of $\mathrm{rref}(A)$ form a basis for $\mathrm{row}(A)$. We already know that the non-zero rows of $\mathrm{rref}(A)$ span $\mathrm{row}(A)$, so we only need to argue that they are linearly independent. However, this follows immediately from the fact that $\mathrm{rref}(A)$ is in reduced row echelon form. Above and below every pivot in $\mathrm{rref}(A)$ are zeros. Therefore, a row in $\mathrm{rref}(A)$ with a pivot cannot be written as a linear combination of any other row. Since every non-zero row has a pivot, this proves the claim.

Now, note the following two facts.

1. The columns of A corresponding to pivot columns of $\mathrm{rref}(A)$ form a basis for $\mathrm{col}(A)$.

2. The non-zero rows of $\mathrm{rref}(A)$ form a basis for $\mathrm{row}(A)$.

To complete the proof, note that every pivot of $\mathrm{rref}(A)$ lies in exactly one row and one column. Therefore, the number of basis vectors in $\mathrm{row}(A)$ is the same as the number of basis vectors in $\mathrm{col}(A)$. Thus $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.
■

Equations, Null Spaces, and Geometry

Let $M = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}$. Using the typical row-reduction steps, we know that the complete solution to $M\vec{x} = \vec{0}$ (i.e., the null space of M) can be expressed in vector form as

$$\vec{x} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

Similarly, the complete solution to $M\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix}1\\2\end{bmatrix}$ can be expressed in vector form as

$$\vec{x} = t\begin{bmatrix}-1\\-2\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

The set of solutions to $M\vec{x} = \vec{0}$ and $M\vec{x} = \vec{b}$ look very similar. In fact,

$$\{\text{solutions to } M\vec{x} = \vec{b}\} = \{\text{solutions to } M\vec{x} = \vec{0}\} + \{\vec{p}\} \qquad \text{where} \qquad \vec{p} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Or, phrased another way, the solution set to $M\vec{x} = \vec{b}$ is

$$\text{null}(M) + \{\vec{p}\}.$$

In the context of what we already know about lines and translated subspaces, this makes perfect sense. We know that the solution set to $M\vec{x} = \vec{b}$ is a line (which doesn't pass through the origin) and may therefore be written as a translated span $\text{span}\{\vec{d}\} + \{\vec{p}\}$. Here $\vec{d}$ is a direction vector for the line and $\vec{p}$ is a point on the line.

Because $\vec{p} \in \text{span}\{\vec{d}\} + \{\vec{p}\}$, we call $\vec{p}$ a *particular solution* to $M\vec{x} = \vec{b}$. Using a similar argument, we can show that for any matrix A, and any vector $\vec{b}$, the set of all solutions to $A\vec{x} = \vec{b}$ (provided there are any) can be expressed as

$$V + \{\vec{p}\}$$

where V is a subspace and $\vec{p}$ is a particular solution. In fact, we can do better. We can say $V = \text{null}(A)$.

Theorem. Let A be a matrix, $\vec{b}$ be a vector, and let $\vec{p}$ be a particular solution to $A\vec{x} = \vec{b}$. Then, the set of all solutions to $A\vec{x} = \vec{b}$ is

$$\text{null}(A) + \{\vec{p}\}.$$

Proof. Let $S = \{\text{all solutions to } A\vec{x} = \vec{b}\}$ and assume $\vec{p} \in S$. We will show $S = \text{null}(A) + \{\vec{p}\}$.

First we will show $\text{null}(A) + \{\vec{p}\} \subseteq S$. Let $\vec{v} \in \text{null}(A) + \{\vec{p}\}$. By definition, $\vec{v} = \vec{n} + \vec{p}$ for some $\vec{n} \in \text{null}(A)$. Now, by linearity of matrix multiplication and the definition of the null space,

$$A\vec{v} = A(\vec{n} + \vec{p}) = A\vec{n} + A\vec{p} = \vec{0} + \vec{b} = \vec{b},$$

and so $\vec{v} \in S$.

Next we will show $S \subseteq \text{null}(A) + \{\vec{p}\}$. First observe that for any $\vec{u}, \vec{v} \in S$ we have

$$A(\vec{u} - \vec{v}) = A\vec{u} - A\vec{v} = \vec{b} - \vec{b} = \vec{0},$$

and so $\vec{u} - \vec{v} \in \text{null}(A)$.

Fix $\vec{w} \in S$. By our previous observation, $\vec{w} - \vec{p} \in \text{null}(A)$. Therefore

$$\vec{w} = (\vec{w} - \vec{p}) + \vec{p} \in \text{null}(A) + \{\vec{p}\},$$

which completes the proof. ■

Takeaway. To write the complete solution to $A\vec{x} = \vec{b}$, all you need is the null space of A and a particular solution to $A\vec{x} = \vec{b}$.

Null spaces are also closely connected with row spaces. Let $\mathcal{P} \subseteq \mathbb{R}^3$ be the plane with equation $x + 2y + 2z = 0$. We can rewrite this equation as a matrix equation and as the equation of a plane in normal form.

$$\underbrace{\begin{bmatrix}1 & 2 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{0}}_{\text{a matrix equation}} \qquad \underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = 0}_{\text{normal form}}$$

Now we see that $\mathcal{P} = \text{null}\left(\begin{bmatrix}1 & 2 & 2\end{bmatrix}\right)$ and that every non-zero vector in $\text{row}\left(\begin{bmatrix}1 & 2 & 2\end{bmatrix}\right)$ is a normal vector for $\mathcal{P}$. In other words, $\text{null}\left(\begin{bmatrix}1 & 2 & 2\end{bmatrix}\right)$ is orthogonal to $\text{row}\left(\begin{bmatrix}1 & 2 & 2\end{bmatrix}\right)$.

This is no coincidence. Let M be a matrix and let $\vec{r}_1, \ldots, \vec{r}_n$ be the rows of M. By definition,

$$M\vec{x} = \begin{bmatrix}\vec{r}_1 \cdot \vec{x} \\ \vdots \\ \vec{r}_n \cdot \vec{x}\end{bmatrix},$$

and so solutions to $M\vec{x} = \vec{0}$ are precisely the vectors which are orthogonal to every row of M. In other words, $\text{null}(M)$ consists of all vectors orthogonal to the rows of M. Conversely, $\text{row}(M)$ consists of all vectors orthogonal to everything in $\text{null}(M)$. We can use this fact to approach questions in a new way.

> **Example.** Let $\vec{a} = \begin{bmatrix}1\\2\\5\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}2\\-2\\-2\end{bmatrix}$. Find the set of all vectors orthogonal to both $\vec{a}$ and $\vec{b}$.
>
> Let $M = \begin{bmatrix}1 & 2 & 5\\ 2 & -2 & -2\end{bmatrix}$ be the matrix whose rows are $\vec{a}$ and $\vec{b}$. Since $\text{null}(M)$ consists of all vectors orthogonal to $\text{row}(M)$, the set we are looking for is $\text{null}(M)$. Computing via row reduction, we find
>
> $$\text{null}(M) = \text{span}\left\{\begin{bmatrix}-1\\-2\\1\end{bmatrix}\right\}.$$

Transformations and Matrices

Matrices are connected to systems of linear equations via matrix equations (like $A\vec{x} = \vec{b}$) and to linear transformations through matrix transformations (like $\mathcal{T}(\vec{x}) = M\vec{x}$). This means that we can think about systems of equations in terms of linear transformations and we can gain insight about linear transformations by looking at systems of equations!

In preparation for this, let's reconsider matrix transformations and be pedantic about our notation.

Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let M be its corresponding matrix. $\mathcal{T}$ is a function that inputs and outputs *vectors*. M is a box of numbers, which has no meaning by itself, but we know how to multiply M by lists of numbers (or other boxes of numbers). Therefore, strictly speaking, the expression "$M\vec{x}$" doesn't make sense. The quantity "$\vec{x}$" is a vectors, but we only know how to multiply M by lists of numbers.

Ah! But we know how to turn $\vec{x}$ into a list of numbers. Just pick a basis! The expression

$$M[\vec{x}]_{\mathcal{E}}$$

makes perfect sense since $[\vec{x}]_{\mathcal{E}}$ is a list of numbers. Continuing to be pedantic, we know $\mathcal{T}(\vec{x}) \neq M[\vec{x}]_{\mathcal{E}}$ since the left side is a vector and the right side is a list of numbers. We can fix this by either turning the right side into a vector or the left side into a list of numbers. Doing this, we see the precise relationship between a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ and its matrix M is

$$[\mathcal{T}(\vec{x})]_{\mathcal{E}} = M[\vec{x}]_{\mathcal{E}}.$$

If we have a matrix M, by picking a basis (usually the standard basis), we can define a linear transformation by first taking the input vector and rewriting it in the basis, next multiplying by the matrix, and finally taking the list of numbers and using them as coefficients for a linear combination involving the basis vectors. This is what we actually mean when we say that a matrix *induces* a linear transformation.

> **Induced Transformation.**
> Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $\mathcal{T}_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by
>
> $$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$
>
> where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

Before we would write "$\mathcal{T}(\vec{x}) = M\vec{x}$" which hides the fact that when we relate a matrix and a linear transformation, there is a basis hidden in the background. And, like before, when we're only considering a single basis, we can be sloppy with our notation and write things like "$M\vec{x}$", but when there are multiple bases or when we're trying to be extra precise, we must make sure our boxes/lists of numbers and our transformations/vectors stay separate.

Example. Let $\mathcal{T}$ be the transformation induced by the matrix $M = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}$, and let $\vec{v} = 3\vec{e}_1 - 3\vec{e}_3$. Compute $\mathcal{T}(\vec{v})$.

Since $\mathcal{T}$ is induced by $M = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}$, by definition,

$$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}[\vec{v}]_{\mathcal{E}}.$$

Further, since $\vec{v} = 3\vec{e}_1 - 3\vec{e}_3$, by definition we have $[\vec{v}]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$. Therefore,

$$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -2 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix}.$$

In other words, $\mathcal{T}(\vec{v}) = \begin{bmatrix} -12 \\ 12 \end{bmatrix}_{\mathcal{E}'} = -12\vec{e}_1 + 12\vec{e}_2$.

Using induced transformations, we can extend linear-transformation definitions to matrix definitions. In particular, we can define the rank and nullity of a matrix.

Rank of a Matrix. Let M be a matrix. The ***rank*** of M, denoted $\operatorname{rank}(M)$, is the rank of the linear transformation induced by M.

Nullity of a Matrix. Let M be a matrix. The ***nullity*** of M, denoted $\operatorname{null}(M)$, is the nullity of the linear transformation induced by M.

Range vs. Column Space & Null Space vs. Null Space

Let $M = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix}$ be an $m \times m$ matrix with columns $C_1, \dots, C_m$, and let $\mathcal{T}$ be the transformation induced by M. The column space of M is the set of all linear combinations of the columns of M. But, let's be precise. The columns of M are lists of numbers, so to talk about the column space of M, we need to turn them into vectors. Fortunately, we have a nice notation for that. Since C_i is a list of numbers, $[C_i]_{\mathcal{E}}$ is a (true) vector, and

$$\operatorname{col}(M) = \operatorname{span}\{[C_1]_{\mathcal{E}}, [C_2]_{\mathcal{E}}, \dots, [C_m]_{\mathcal{E}}\}.$$

Can we connect this to the range of $\mathcal{T}$? Well, by the definition of matrix multiplication, we know that

$$M\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = M[\vec{e}_1]_{\mathcal{E}} = C_1$$

and in general $M[\vec{e}_i]_{\mathcal{E}} = C_i$. By the definition of induced transformation, we know

$$[\mathcal{T}(\vec{e}_i)]_{\mathcal{E}} = M[\vec{e}_i]_{\mathcal{E}} = C_i,$$

and so

$$\mathcal{T}(\vec{e}_i) = [C_i]_{\mathcal{E}}.$$

Every input to $\mathcal{T}$ can be written as a linear combination of $\vec{e}_i$'s (because $\mathcal{E}$ is a basis) and so, because $\mathcal{T}$ is linear, every output of $\mathcal{T}$ can be written as a linear combination of $[C_i]_{\mathcal{E}}$'s. In other words,

$$\operatorname{range}(\mathcal{T}) = \operatorname{col}(M).$$

This means that when trying to answer a question about the range of a linear transformation, we could think about the column space of its matrix instead (or vice verse).

Example. Let $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$\mathcal{T}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - z \\ 4x - 2z \end{bmatrix}.$$

Find range($\mathcal{T}$) and rank($\mathcal{T}$).

Let M be a matrix for $\mathcal{T}$. We know $\operatorname{range}(\mathcal{T}) = \operatorname{col}(M)$ and $\operatorname{rank}(\mathcal{T}) = \dim(\operatorname{range}(\mathcal{T})) = \dim(\operatorname{col}(M))$. By inspection, we see that

$$M = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix}.$$

Again, by inspection, we see that $\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ is a basis for col(M) and col(M) is one dimensional. Therefore,

$$\operatorname{range}(\mathcal{T}) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\} \qquad \text{and} \qquad \operatorname{rank}(\mathcal{T}) = 1.$$

There is an alternative definition of the rank of a matrix which commonly appears. We'll state it as a theorem.

Theorem. Let M be a matrix. The rank of M is equal to the number of pivots in rref(M).

Proof. We know that $\operatorname{rank}(M) = \dim(\operatorname{range}(\mathcal{T}_M)) = \dim(\operatorname{col}(M))$ where $\mathcal{T}_M$ is the transformation induced by M. Further, a basis for col(M) consists of a maximal linearly independent subset of the columns of M. To find such a subset, we row reduce M and look at the columns of M that correspond to pivot columns of rref(M).

When all is said and done, the number of elements in a basis for col(M) will be the number of pivots in rref(M), which is the same as rank(M). ■

Takeaway. If $\mathcal{T}$ is a linear transformation and M is a corresponding matrix, $\operatorname{range}(\mathcal{T}) = \operatorname{col}(M)$, and answering questions about M answers questions about $\mathcal{T}$.

Just like the range–column-space relationship, we also have a null-space–null-space relationship. More specifically, if $\mathcal{T}$ is a linear transformation with matrix M, then $\operatorname{null}(\mathcal{T}) = \operatorname{null}(M)$. From this fact, we deduce the following theorem.

Theorem. Let $\mathcal{T}$ be a linear transformation and let M be a matrix for $\mathcal{T}$. Then nullity($\mathcal{T}$) is equal to the number of free variable columns in rref(M).

Proof. We know $\operatorname{nullity}(\mathcal{T}) = \dim(\operatorname{null}(\mathcal{T})) = \dim(\operatorname{null}(M))$. Further, we know that the complete solution to $M\vec{x} = \vec{0}$ will take the form

$$\vec{x} = t_1\vec{d}_1 + \cdots + t_k\vec{d}_k$$

where k is the number of free variable columns in rref(M). The algorithm for writing the complete solution to $M\vec{x} = \vec{0}$ ensures that $\{\vec{d}_1, \ldots, \vec{d}_k\}$ is a basis for null(M), and so $\operatorname{nullity}(\mathcal{T}) = k$, which completes the proof. ■

The Rank-Nullity Theorem

The rank and the nullity of a linear transformation/matrix are connected by a powerful theorem.

Theorem. (Rank-nullity Theorem for Matrices) For a matrix A,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \text{ \# of columns in } A.$$

The Rank-nullity Theorem's statement is simple, but it is surprisingly useful. Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}.$$

We already know that null(M) is the plane with equation $x + 2y + 2z = 0$ and therefore is two dimensional. Since M has three columns, $\operatorname{rank}(M) = 1$, and so $\dim(\operatorname{col}(M)) = \dim(\operatorname{row}(M)) = 1$, which means that M has a one-dimensional set of normal vectors (if we include $\vec{0}$).

By contrast, let $\mathcal{P} \subseteq \mathbb{R}^4$ be the plane in $\mathbb{R}^4$ given in vector form by

$$\vec{x} = t\begin{bmatrix}1\\2\\2\\2\end{bmatrix} + s\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}.$$

How many normal vectors does $\mathcal{P}$ have? Well, the matrix $A = \begin{bmatrix}1 & 2 & 2 & 2\\ -1 & 1 & -1 & 1\end{bmatrix}$ is rank 2, and so has nullity 2. Therefore, there exists two linearly independent normal directions for $\mathcal{P}$.

There is an equivalent Rank-nullity Theorem for linear transformations.

Theorem. (Rank-nullity Theorem for Linear Transformations) Let $\mathcal{T}$ be a linear transformation. Then

$$\operatorname{rank}(\mathcal{T}) + \operatorname{nullity}(\mathcal{T}) = \dim(\text{domain of } \mathcal{T}).$$

Just like the Rank-nullity Theorem for matrices, the Rank-nullity Theorem for linear transformations can give insights about linear transformations that would be otherwise hard to see.

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

DEF

Range

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\operatorname{range}(T) = \{\vec{y} \in W \;:\; \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

DEF

Null Space

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to zero under T. That is,

$$\operatorname{null}(T) = \{\vec{x} \in V \;:\; T\vec{x} = \vec{0}\}.$$

---

44 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

44.1 What is the range of $\mathcal{P}$?

44.2 What is the null space of $\mathcal{P}$?

45 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

45.1 Show that the null space of T is a subspace.

45.2 Show that the range of T is a subspace.

DEFINITION

Induced Transformation

Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $\mathcal{T}_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

---

46 Let M be a 2×2 matrix and let $\vec{v} \in \mathbb{R}^2$. Further, let T_M be the transformation induced by M.

46.1 What is the difference between "$M\vec{v}$" and "$M[\vec{v}]_{\mathcal{E}}$"?

46.2 What is $[T_M \vec{e}_1]_{\mathcal{E}}$?

46.3 Can you relate the columns of M to the range of T_M?

DEF

Fundamental Subspaces

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\text{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\text{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\text{null}(M)$, is the set of solutions to $M\vec{x} = \vec{0}$.

---

47 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

47.1 Describe the row space of A.

47.2 Describe the column space of A.

47.3 Is the row space of A the same as the column space of A?

47.4 Describe the set of all vectors perpendicular to the rows of A.

47.5 Describe the null space of A.

47.6 Describe the range and null space of T_A, the transformation induced by A.

48

$$B=\begin{bmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{bmatrix} \qquad C=\mathrm{rref}(B)=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}$$

48.1 How does the row space of B relate to the row space of C?

48.2 How does the null space of B relate to the null space of C?

48.3 Compute the null space of B.

49

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \operatorname{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

49.1 How does the column space of P relate to the column space of Q?

49.2 Describe the column space of P and the column space of Q.

Rank

DEF

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the ***rank*** of T, denoted rank(T), is the dimension of the range of T.

For an $n \times m$ matrix M, the ***rank*** of M, denoted rank(M), is the number of pivots in rref(M).

---

50 Let $\mathcal{P}$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

50.1 Describe range$(\mathcal{P})$ and range$(\mathcal{R})$.

50.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

50.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

50.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

51

51.1 Determine the rank of (a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

52 Consider the homogeneous system

$$\begin{array}{llll} x & +2y & +z & =0 \\ x & +2y & +3z & =0 \\ -x & -2y & +z & =0 \end{array} \tag{11}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$.

52.1 What is rank(A)?

52.2 Give the general solution to system (11).

52.3 Are the column vectors of A linearly independent?

52.4 Give a non-homogeneous system with the same coefficients as (11) that has

(a) infinitely many solutions

(b) no solutions.

53 53.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

53.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\operatorname{rank}(A)+\operatorname{nullity}(A) = \#\text{ of columns in }A.$$

54

54.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

55

The vectors $\vec{u},\vec{v}\in\mathbb{R}^9$ are linearly independent and $\vec{w}=2\vec{u}-\vec{v}$. Define $A=[\vec{u}|\vec{v}|\vec{w}]$.

55.1 What is the rank and nullity of A^T?

55.2 What is the rank and nullity of A?

Inverse Functions & Inverse Matrices

Module 12

In this module you will learn

- The definition of an inverse function and an inverse matrix.
- How to decompose a matrix into the product of elementary matrices and how to use elementary matrices to compute inverses.
- How the order of matrix multiplication matters.
- How row-reduction and matrix inverses relate.

We should think of transformations or functions as machines that perform some manipulation of their input and then give an output. This perspective allows us to divide functions into two natural categories: those that can be undone and those that cannot. The official term for a function that can be undone is an *invertible* function.

Invertible Functions

The simplest function is the *identity function*.

> **Identity Function.**
> Let X be a set. The ***identity function*** with domain and codomain X, notated $\mathrm{id}: X \to X$, is the function satisfying
> $$\mathrm{id}(x) = x$$
> for all $x \in X$.

The identity function is the function that does nothing to its input.[37] When doing precise mathematics, we often prove a function or composition of functions does nothing to its input by showing it is *equal* to the identity function.[38]

In plain terms, a function is invertible if it can be undone. More precisely a function is invertible if there exists an inverse function that when composed with the original function produces the identity function and vice verse.

> **Inverse Function.**
> Let $f : X \to Y$ be a function. We say f is ***invertible*** if there exists a function $g : Y \to X$ so that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. In this case, we call g an ***inverse*** of f and write
> $$f^{-1} = g.$$

Let's consider an example. You have some money in your pockets. Let $l : \{\text{nickels in left pocket}\} \to \mathbb{N}$ be the function that adds up the value of all the nickels in your left pocket. Let $r : \{\text{nickels in either pocket}\} \to \mathbb{N}$ be the function that adds up the value of all the nickels in both of your pockets. In this case, l would be invertible—if you know that $l(\#\text{ nickels}) = 25$, you must have had 5 nickels in your left pocket. We can write down a formula for l^{-1} as
$$l^{-1}(n) = \frac{n}{5}.$$

However, r is not invertible. If $r(\#\text{ nickels}) = 25$, you *might* have had 5 nickels in your left pocket, but you might have 3 nickels in your left pocket and 2 in your right. We just don't know, so no inverse to r can exist.

What we've just learned is that for a function to be invertible, it must be *one-to-one*.

> **One-to-one.**
> Let $f : X \to Y$ be a function. We say f is ***one-to-one*** (or ***surjective***) if distinct inputs to f produce distinct outputs. That is $f(x) = f(y)$ implies $x = y$.

Whenever a function f is one-to-one, there exists a function g so that $g \circ f = \mathrm{id}$. However, this is not enough to declare that f is invertible[39] because we also need $f \circ g = \mathrm{id}$. To ensure this, we need f to be *onto*.

---

[37] Technically, for every set there exists a unique identity function with that set as the domain/codomain, but we won't belabor this point.
[38] This is similar to saying that we know $\vec{x} = \vec{y}$ if and only if $\vec{x} - \vec{y} = \vec{0}$.
[39] In this situation, we say that f is *left*-invertible.

Onto.
Let $f : X \to Y$ be a function. We say f is ***onto*** (or ***injective***) if every point in the codomain of f gets mapped to. That is $\text{range}(f) = Y$.

Every invertible function is both one-to-one and onto, and every one-to-one and onto function is invertible. And, as we will learn, this has implications for the rank and nullity of linear transformations.

Invertibility and Linear Transformations

Let's now focus on linear transformations. We know that a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if it is one-to-one and onto.

If $\mathcal{T}$ is one-to-one, that means that distinct inputs to $\mathcal{T}$ yield distinct outputs. In other words, the solution to $\mathcal{T}(\vec{x}) = \vec{b}$ is always unique. But, the set of all solutions to $\mathcal{T}(\vec{x}) = \vec{b}$ can be expressed as

$$\text{null}(\mathcal{T}) + \{\vec{p}\}.$$

Therefore, $\mathcal{T}$ is one-to-one if and only if $\text{nullity}(\mathcal{T}) = 0$. If $\mathcal{T}$ is onto, then $\text{range}(\mathcal{T}) = \mathbb{R}^m$ and so $\text{rank}(\mathcal{T}) = m$.

Now, suppose $\mathcal{T}$ is one-to-one and onto. By the Rank-nullity Theorem,

$$\text{rank}(\mathcal{T}) + \text{nullity}(\mathcal{T}) = 0 + m = m = n = \dim(\text{domain of } \mathcal{T}),$$

and so $\mathcal{T}$ has the same domain and codomain (at least a domain and codomain of the same dimension).

Using the rank-nullity theorem, we can start developing a list of properties that are equivalent to invertibility of a linear transformation.

- $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if $\text{nullity}(\mathcal{T}) = 0$ and $\text{rank}(\mathcal{T}) = m$.
- $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if $m = n$ and $\text{nullity}(\mathcal{T}) = 0$.
- $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if $m = n$ and $\text{rank}(\mathcal{T}) = n$.

Example. Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the x-axis and let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter-clockwise by $15°$. Classify each of $\mathcal{P}$ and $\mathcal{R}$ as invertible or not.

Notice that $\mathcal{P}(\vec{e}_2) = \mathcal{P}(2\vec{e}_2) = \vec{0}$, therefore $\mathcal{P}$ is not one-to-one and so is not invertible.

Let $\mathcal{Q} : R^2 \to \mathbb{R}^2$ be rotation clockwise by $15°$. $\mathcal{R}$ and $\mathcal{Q}$ will undo each other. Phrased mathematically,

$$\mathcal{R} \circ \mathcal{Q} = \text{id} \qquad \text{and} \qquad \mathcal{Q} \circ \mathcal{R} = \text{id}.$$

Therefore, $\mathcal{Q}$ is an inverse of $\mathcal{R}$, and so $\mathcal{R}$ is invertible.

One important fact about linear transformations is that if a linear transformation is invertible, it's inverse is also a linear transformation.

Theorem. Let $\mathcal{T}$ be an invertible linear transformation. Then $\mathcal{T}^{-1}$ is also a linear transformation.

Proof. Let $\mathcal{T}$ be an invertible linear transformation and let $\mathcal{T}^{-1}$ be its inverse. We need to show that (i) $\mathcal{T}^{-1}$ distributes over addition and (ii) $\mathcal{T}^{-1}$ distributes over scalar multiplication.

(i) First observe that since $\mathcal{T} \circ \mathcal{T}^{-1} = \text{id}$ and because $\mathcal{T}$ is linear, we have

$$\vec{a} + \vec{b} = \mathcal{T} \circ \mathcal{T}^{-1}\vec{a} + \mathcal{T} \circ \mathcal{T}^{-1}\vec{b} = \mathcal{T}(\mathcal{T}^{-1}\vec{a} + \mathcal{T}^{-1}\vec{b}).$$

Since $\mathcal{T}^{-1} \circ \mathcal{T} = \text{id}$, by using the fact that $\vec{a} + \vec{b} = \mathcal{T}(\mathcal{T}^{-1}\vec{a} + \mathcal{T}^{-1}\vec{b})$ we know

$$\mathcal{T}^{-1}(\vec{a} + \vec{b}) = \mathcal{T}^{-1}\Big(\mathcal{T}(\mathcal{T}^{-1}\vec{a} + \mathcal{T}^{-1}\vec{b})\Big) = \mathcal{T}^{-1}\vec{a} + \mathcal{T}^{-1}\vec{b}.$$

(ii) Similar to the proof of (i), we see

$$\mathcal{T}^{-1}(\alpha\vec{a}) = \mathcal{T}^{-1}\Big(\alpha\big(\mathcal{T} \circ \mathcal{T}^{-1}\vec{a}\big)\Big) = \mathcal{T}^{-1} \circ \mathcal{T}(\alpha\mathcal{T}^{-1}\vec{a}) = \alpha\mathcal{T}^{-1}\vec{a}.$$

■

Invertibility and Matrices

In the world of matrices, the *identity matrix* takes the place of the identity function.

Identity Matrix.
An ***identity matrix*** is a square matrix with ones on the diagonal and zeros everywhere else. The $n \times n$ identity matrix is denoted $I_{n\times n}$, or just I when its size is implied.

We can now define what it means for a matrix to be invertible.[40]

Matrix Inverse.
The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated A^{-1}.

Example. Determine whether the matrices $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ are inverses of each other.

$$AB = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$
$$BA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Therefore, A and B are inverses of each other.

Example. Determine whether the matrices $A = \begin{bmatrix} 2 & 5 & 0 \\ -3 & -7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$ are inverses of each other.

$$AB = \begin{bmatrix} 2 & 5 & 0 \\ -3 & -7 & 0 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

but

$$BA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & 0 \\ -3 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix} \neq I.$$

Therefore, A and B are not inverses of each other.

Since every matrix induces a linear transformation, we can use the facts we know about invertible linear transformations to produce facts about invertible matrices. In particular:

- An $n \times m$ matrix A is invertible if and only if $\operatorname{nullity}(A) = 0$ and $\operatorname{rank}(A) = n$.
- An $n \times n$ matrix A is invertible if and only if $\operatorname{nullity}(A) = 0$.
- An $n \times n$ matrix A is invertible if and only if $\operatorname{rank}(A) = n$.

Matrix Algebra

The linear equation $ax = b$ has solution $x = \frac{b}{a}$ whenever $a \neq 0$. We arrive at this solution by dividing both sides of the equation by a. Does a similar process exist for solving the matrix equation $A\vec{x} = \vec{b}$? It sure does!

Unfortunately, we cannot divide by a matrix, but to solve $A\vec{x} = \vec{b}$, we don't need to "divide" by a matrix, we just need to eliminate A from the left side. This could be accomplished by using an inverse.

Suppose A is invertible, then

$$A\vec{x} = \vec{b} \qquad \implies \qquad A^{-1}A\vec{x} = A^{-1}\vec{b} \qquad \implies \qquad \vec{x} = A^{-1}\vec{b}.$$

Thus, if we have the inverse of a matrix handy, we can use it to solve a system of equations.

Example. Use the fact that $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ to solve the system $\begin{cases} 2x + 5y = 2 \\ -3x - 7y = 1 \end{cases}$.

[40] This should look very similar to what it means for a function to be invertible.

The system can be rewritten as

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Multiplying both sides by $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1}$ gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -19 \\ 8 \end{bmatrix}.$$

It's important to note that, unlike in the case with regular scalars, the order of matrix multiplication matters. So, whereas with scalars you could get away with something like

$$ax = b \qquad \implies \qquad \tfrac{1}{a}ax = b\tfrac{1}{a} \qquad \implies \qquad x = \tfrac{b}{a},$$

with matrices $A\vec{x} = \vec{b}$ *does not imply* $A^{-1}A\vec{x} = \vec{b}A^{-1}$. In fact, if $\vec{b}$ is a column vector, the expression $\vec{b}A^{-1}$ is almost always undefined!

Finding a Matrix Inverse

Whereas before we only knew how to solve a matrix equation $A\vec{x} = \vec{b}$ using row reduction, we now know how to use A^{-1} to solve the same system. In fact, A^{-1} is the exact matrix so that $\vec{x} = A^{-1}\vec{b}$ is the solution to $A\vec{x} = \vec{b}$. Therefore, by picking different $\vec{b}$'s and solving for $\vec{x}$, we can find A^{-1}.

Example. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. Find A^{-1}.

We know $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ will be a 2×2 matrix, and we know $\vec{x} = A^{-1}\vec{b}$ will always be the unique solution to $A\vec{x} = \vec{b}$. Therefore, we can find A^{-1} by finding $\vec{x}, \vec{b}$ pairs that satisfy $A\vec{x} = \vec{b}$.
Using row reduction, we see

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{has solution} \quad \vec{x} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}, \qquad \text{and} \qquad A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{has solution} \quad \vec{x} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}.$$

Therefore

$$A^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix} \qquad \text{and} \qquad A^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix},$$

and so

$$A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}.$$

Elementary Matrices

Finding the inverse of a matrix can be a lot of work. However if you already know how to undo what the matrix does, finding the inverse might not be so hard. For example, if R_{30} is the matrix that rotates vectors in $\mathbb{R}^2$ counter-clockwise by 30°, its inverse must be R_{-30}, the matrix that rotates vectors in $\mathbb{R}^2$ *clockwise* by 30°.

Like before when we analyzed linear transformations by breaking them up into compositions of simpler linear transformations, another strategy for find an inverse matrix is to break a matrix into simpler ones whose inverses we can just write down.

Some of the simplest matrices around are the *elementary matrices*.

Elementary Matrix.
A matrix is called an ***elementary matrix*** if it is an identity matrix with a single elementary row operation applied.

Examples of elementary matrices include

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These matrices are obtained from the row operations "multiply the last row by −5", "add 7 times the last row to the first", and "swap the first two rows".

Elementary matrices are useful because multiplying by an elementary matrix performs the corresponding elementary row operation! See for yourself:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&-5\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\-5g&-5h&-5i\end{bmatrix}$$

$$\begin{bmatrix}1&0&7\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a+7g&b+7h&c+7i\\d&e&f\\g&h&i\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}d&e&f\\a&b&c\\g&h&i\end{bmatrix}$$

As a refresher, the elementary row operations are:

- multiply a row by a non-zero constant;
- add a multiple of one row to another; and
- swap to rows.

Each one of these operations can be undone, and so every elementary matrix is invertible. What's more, the inverse is another elementary matrix that is easy to write down.

Example. Find the inverse of $E=\begin{bmatrix}1&0&7\\0&1&0\\0&0&1\end{bmatrix}$.

Since E corresponds to the row operation "add 7 times the last row to the first", E^{-1} must correspond to the row operation "subtract 7 times the last row from the first". Therefore,

$$E^{-1}=\begin{bmatrix}1&0&-7\\0&1&0\\0&0&1\end{bmatrix}.$$

Elementary Matrices and Inverses

For a matrix M to be invertible, we know that M must be square and $\mathrm{nullity}(M)=0$. That means, M is invertible if and only if $\mathrm{rref}(M)=I$. In other words, M is invertible if there is a sequence of elementary row operations that turn M into I. Each one of these row operations can represented by an elementary matrix, which gives us the following theorem.

Theorem. A matrix M is invertible if and only if there are elementary matrices $E_1,\dots,E_k$ so that

$$E_k\cdots E_2E_1M=I.$$

Now, suppose M is invertible and let $E_1,\dots,E_k$ be elementary matrices so that $E_k\cdots E_2E_1M=I$. We now know

$$E_k\cdots E_2E_1M=\underbrace{(E_k\cdots E_2E_1)}_{Q}M=QM=I.$$

If we can argue that $MQ=I$, then Q will be the inverse of M!

Theorem. If A is a square matrix and $AB=I$ for some matrix B, then $BA=I$.

Proof. Suppose A is a square matrix and that $AB=I$. Since $AB=I$, B must also be square. Since $\mathrm{null}(B)\subseteq\mathrm{null}(AB)$, we know $\mathrm{nullity}(B)\le\mathrm{nullity}(AB)=\mathrm{nullity}(I)=0$, and so B is invertible (since it's a square matrix whose nullity is 0). Let B^{-1} be the inverse of B. Observe now that

$$A=AI=A(BB^{-1})=(AB)B^{-1}=IB^{-1}=B^{-1},$$

and so $A=B^{-1}$. Finally, substituting B^{-1} for A shows

$$BA=BB^{-1}=I.$$

■

In light of this theorem, we now have a new algorithm for finding the inverse of a matrix—find elementary matrices that turn the matrix into the identity matrix and multiply those elementary matrices together to find the inverse.

Example. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Find A^{-1} using elementary matrices.

We can row-reduce A with the following steps:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The elementary matrices corresponding to these steps are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We now have

$$E_3E_2E_1A = I,$$

and so

$$A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 1 \end{bmatrix}.$$

Decomposition into Elementary Matrices

If A is an invertible matrix, then the double-inverse of A (i.e., $(A^{-1})^{-1}$) is A itself.[41] This is easily proved. By definition, $(A^{-1})^{-1}$ is a matrix B so that $BA^{-1} = I$ and $A^{-1}B = I$. But $B = A$ satisfies this condition!

Now, suppose M is an invertible matrix. Then, there exists a sequence of elementary matrices $E_1, \dots, E_k$ so that $E_k \cdots E_2E_1M = I$ and

$$M^{-1} = E_k \cdots E_2E_1.$$

Therefore

$$M = (M^{-1})^{-1} = (E_k \cdots E_2E_1)^{-1}.$$

Thinking carefully about what $(E_k \cdots E_2E_1)^{-1}$ should be, we see that

$$(E_1^{-1}E_2^{-1} \cdots E_k^{-1})E_k \cdots E_2E_1 = I \qquad \text{and} \qquad E_k \cdots E_2E_1(E_1^{-1}E_2^{-1} \cdots E_k^{-1}) = I,$$

and so

$$M = (E_k \cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \cdots E_k^{-1}.$$

(Notice the order of matrix multiplication reversed!) Each E_i^{-1} is also an elementary matrix, and so we have just shown that every invertible matrix can be written as the product of elementary matrices. This is actually an if and only if.

Theorem. A matrix M is invertible if and only if it can be written as the product of elementary matrices.

Proof. Suppose M is invertible. Then, there exists a sequence of elementary matrices $E_1, \dots, E_k$ so that $E_k \cdots E_1M = I$. It follows that

$$M = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$$

is the product of elementary matrices. Conversely, since the product of invertible matrices is invertible and every elementary matrix is invertible, the product of elementary matrices must be invertible. Therefore, if M is not invertible, it *cannot* be written as the product of elementary matrices. ■

[41] Formally we say that the operation of taking a matrix inverse is an *involution*.

Practice Problems

1 Example problem with parts

(a) The first part

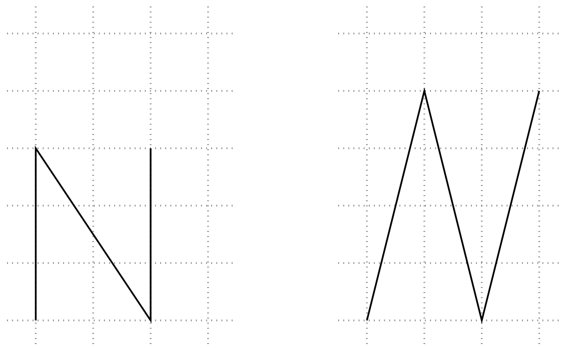
(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

Task 2.4: Getting back N



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

Consider the new task: find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.
2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

56 56.1 Apply the row operation $R_3 \mapsto R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

56.2 Apply the row operation $R_3 \mapsto R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

Elementary Matrix

DEF A matrix is called an ***elementary matrix*** if it is an identity matrix with a single elementary row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

56.3 Compute $E_1 A$ and $E_2 A$. How do the resulting matrices relate to row operations?

56.4 Without computing, what should the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to E_1 be? Compute and verify.

56.5 Without computing, what should $E_2 E_1$ be? What about $E_1 E_2$? Now compute and verify.

DEF

Matrix Inverse

The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated A^{-1}.

---

57 Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

57.1 Which pairs of matrices above are inverses of each other?

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

58.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

58.2 What is E_2E_1B?

58.3 Find B^{-1}.

58.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

59

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

59.1 What is $A^{-1}A$?

59.2 What is $\operatorname{rref}(A)$?

59.3 What is $\operatorname{rref}(C)$? (Hint, there is no need to actually do row reduction!)

59.4 Solve the system $A\vec{x} = \vec{b}$.

60 60.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

60.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

Change of Basis II

In this module you will learn

- How to create *change-of-basis* matrices.
- How to write a linear transformation in multiple bases.

Given a basis $\mathcal{A}$ for $\mathbb{R}^n$, every vector $\vec{x} \in \mathbb{R}^n$ uniquely corresponds to the list of numbers $[\vec{x}]_\mathcal{A}$ (its coordinates with respect to $\mathcal{A}$), and the operation of writing a vector in a basis is *invertible*.

If we have two bases, $\mathcal{A}$ and $\mathcal{B}$, for $\mathbb{R}^n$, we have two equally valid ways of representing a vector in coordinates.

$$\begin{tikzcd}
 & \vec{x} \in \mathbb{R}^n \arrow[dl, leftrightarrow, "\text{Represent in } \mathcal{A} \text{ basis}"'] \arrow[dr, leftrightarrow, "\text{Represent in } \mathcal{B} \text{ basis}"] & \\
[\vec{x}]_\mathcal{A} & & [\vec{x}]_\mathcal{B}
\end{tikzcd}$$

Not only that, but there must be a function that converts between $[\vec{x}]_\mathcal{A}$ and $[\vec{x}]_\mathcal{B}$. The function works as follows: input the list of numbers $[\vec{x}]_\mathcal{A}$, use those numbers as coefficients of the $\mathcal{A}$ basis vectors to get the true vector $\vec{x}$, and then find the coordinates of that vector with respect to the $\mathcal{B}$ basis.

Example. Let $\mathcal{A} = \{\vec{a}_1, \vec{a}_2\}$ where $\vec{a}_1 = \begin{bmatrix}1\\1\end{bmatrix}_\mathcal{E}$ and $\vec{a}_2 = \begin{bmatrix}1\\-1\end{bmatrix}_\mathcal{E}$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}2\\1\end{bmatrix}_\mathcal{E}$ and $\vec{b}_2 = \begin{bmatrix}5\\3\end{bmatrix}_\mathcal{E}$ be bases for $\mathbb{R}^2$. Given that $[\vec{x}]_\mathcal{A} = \begin{bmatrix}2\\-3\end{bmatrix}$, find $[\vec{x}]_\mathcal{B}$.

By definition,

$$\vec{x} = 2\vec{a}_1 - 3\vec{a}_2 = 2(\vec{e}_1 + \vec{e}_2) - 3(\vec{e}_1 - \vec{e}_2) = -\vec{e}_1 + 5\vec{e}_2.$$

We need to rewrite $\vec{x}$ as a linear combination of $\vec{b}_1 = 2\vec{e}_1 + \vec{e}_2$ and $\vec{b}_2 = 5\vec{e}_1 + 3\vec{e}_2$. That is, we need to solve the equation

$$\vec{x} = -\vec{e}_1 + 5\vec{e}_2 = \alpha(2\vec{e}_1 + \vec{e}_2) + \beta(5\vec{e}_1 + 3\vec{e}_2) = (2\alpha + 5\beta)\vec{e}_1 + (\alpha + 3\beta)\vec{e}_2.$$

Equating the coefficients of $\vec{e}_1$ and $\vec{e}_2$, we get

$$\begin{cases} 2\alpha + 5\beta = -1 \\ \alpha + 3\beta = 5 \end{cases},$$

which has a unique solution $(\alpha, \beta) = (-28, 11)$. We conclude

$$[\vec{x}]_\mathcal{B} = \begin{bmatrix}-28\\11\end{bmatrix}.$$

For a basis $\mathcal{A}$, the invertible function that takes a vector $\vec{x}$ and generates the coordinates $[\vec{x}]_\mathcal{A}$ is a linear function. Therefore, for bases $\mathcal{A}$ and $\mathcal{B}$, the function that converts $[\vec{x}]_\mathcal{A}$ to $[\vec{x}]_\mathcal{B}$ must have a matrix. This matrix is called the *change of basis* matrix.

Change of Basis Matrix. Let $\mathcal{A}$ and $\mathcal{B}$ be bases for $\mathbb{R}^n$. The matrix M is called a ***change of basis*** matrix (which converts from $\mathcal{A}$ to $\mathcal{B}$) if for all $\vec{x} \in \mathbb{R}^n$

$$M[\vec{x}]_\mathcal{A} = [\vec{x}]_\mathcal{B}.$$

Notationally, $[\mathcal{B} \leftarrow \mathcal{A}]$ stands for the change of basis matrix converting from $\mathcal{A}$ to $\mathcal{B}$, and we may write $M = [\mathcal{B} \leftarrow \mathcal{A}]$.

Example. Let $\mathcal{A} = \{\vec{a}_1, \vec{a}_2\}$ where $\vec{a}_1 = \begin{bmatrix}1\\1\end{bmatrix}_\mathcal{E}$ and $\vec{a}_2 = \begin{bmatrix}1\\-1\end{bmatrix}_\mathcal{E}$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}2\\1\end{bmatrix}_\mathcal{E}$ and

$\vec{b}_2 = \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$ be bases for $\mathbb{R}^2$. Find the change of basis matrix $[\mathcal{B}\leftarrow\mathcal{A}]$.

We know $[\mathcal{B}\leftarrow\mathcal{A}]$ will be a 2×2 matrix and that

$$[\mathcal{B}\leftarrow\mathcal{A}][\vec{a}_1]_{\mathcal{A}} = [\vec{a}_1]_{\mathcal{B}} \qquad \text{and} \qquad [\mathcal{B}\leftarrow\mathcal{A}][\vec{a}_2]_{\mathcal{A}} = [\vec{a}_2]_{\mathcal{B}}.$$

Therefore, we need to compute $[\vec{a}_1]_{\mathcal{B}}$ and $[\vec{a}_2]_{\mathcal{B}}$. Repeating the procedure from the previous example, we find

$$[\vec{a}_1]_{\mathcal{B}} = \begin{bmatrix}-2\\1\end{bmatrix} \qquad \text{and} \qquad [\vec{a}_2]_{\mathcal{B}} = \begin{bmatrix}8\\-3\end{bmatrix},$$

and so

$$[\mathcal{B}\leftarrow\mathcal{A}] = \begin{bmatrix}-2 & 8\\1 & -3\end{bmatrix}.$$

We can now enhance our diagram from earlier.

$$\begin{tikzcd}
 & \vec{x}\in\mathbb{R}^n \arrow[dl, "\text{Represent in } \mathcal{A} \text{ basis}"'] \arrow[dr, "\text{Represent in } \mathcal{B} \text{ basis}"] & \\
{[\vec{x}]_{\mathcal{A}}} \arrow[rr, dashed, shift right, "{[\mathcal{B}\leftarrow\mathcal{A}]}"'] & & {[\vec{x}]_{\mathcal{B}}} \arrow[ll, dashed, shift right, "{[\mathcal{A}\leftarrow\mathcal{B}]}"']
\end{tikzcd}$$

The notation $[\mathcal{B}\leftarrow\mathcal{A}]$ for the matrix that changes from the $\mathcal{A}$ basis to the $\mathcal{B}$ basis is suggestive. Suppose we have another basis $\mathcal{C}$. We can obtain $[\mathcal{C}\leftarrow\mathcal{A}]$ by multiplying $[\mathcal{B}\leftarrow\mathcal{A}]$ on the left by $[\mathcal{C}\leftarrow\mathcal{B}]$. That is,

$$[\mathcal{C}\leftarrow\mathcal{A}] = [\mathcal{C}\leftarrow\mathcal{B}][\mathcal{B}\leftarrow\mathcal{A}].$$

The backwards arrow "$\leftarrow$" in the change-of-basis matrix notation comes because we multiply vectors on the *left* by matrices. So,

$$[\vec{x}]_{\mathcal{C}} = [\mathcal{C}\leftarrow\mathcal{A}][\vec{x}]_{\mathcal{A}} = [\mathcal{C}\leftarrow\mathcal{B}][\mathcal{B}\leftarrow\mathcal{A}][\vec{x}]_{\mathcal{A}}.$$

As such, the notation for the change of basis matrix chains, allowing you to figure out what's going on without too much trouble.

Change of Basis Matrix in Detail

Let $\mathcal{A}$ and $\mathcal{B}$ be bases for $\mathbb{R}^n$ and $M = [\mathcal{B}\leftarrow\mathcal{A}]$ be the matrix that changes from the $\mathcal{A}$ to the $\mathcal{B}$ basis. Since we can change vectors back from $\mathcal{B}$ to $\mathcal{A}$, we know M is invertible and

$$M^{-1} = [\mathcal{A}\leftarrow\mathcal{B}].$$

Just playing with notation, we see

$$M^{-1}M = [\mathcal{A}\leftarrow\mathcal{B}][\mathcal{B}\leftarrow\mathcal{A}] = [\mathcal{A}\leftarrow\mathcal{A}] = I \qquad MM^{-1} = [\mathcal{B}\leftarrow\mathcal{A}][\mathcal{A}\leftarrow\mathcal{B}] = [\mathcal{B}\leftarrow\mathcal{B}] = I,$$

which makes sense. The matrices $[\mathcal{A}\leftarrow\mathcal{A}]$ and $[\mathcal{B}\leftarrow\mathcal{B}]$ take vectors and rewrite them in the same basis, which is to say, they do nothing to the vectors.

The argument above shows that every change of basis matrix is invertible. The converse is also true.

Theorem. An $n\times n$ matrix is invertible if and only if it is a change of basis matrix.

Proof. Suppose $M = [\mathcal{B}\leftarrow\mathcal{A}]$ is a change-of-basis matrix. Then

$$M^{-1} = [\mathcal{A}\leftarrow\mathcal{B}].$$

Alternatively, suppose $M = [C_1|C_2|\cdots|C_n]$ is an invertible $n\times n$ matrix with columns $C_1, \ldots, C_n$. Let $\vec{c}_i = [C_i]_{\mathcal{E}}$. That is, $\vec{c}_i$ is the vector which comes from interpreting C_i as coordinates with respect to the standard basis.

Since M is invertible, $\mathrm{rref}(M) = I$, and so $\{\vec{c}_1, \ldots, \vec{c}_n\}$ is a linearly independent set of n vectors. Therefore $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$ is a basis for $\mathbb{R}^n$. Now, observe

$$M[\vec{c}_i]_{\mathcal{C}} = C_i = [\vec{c}_i]_{\mathcal{E}}$$

for $i = 1, \ldots, n$, and so $M = [\mathcal{E} \leftarrow \mathcal{C}]$ is a change-of-basis matrix. ■

The proof of the above theorem highlights something interesting. Let $\mathcal{A} = \{\vec{a}_1, \ldots, \vec{a}_n\}$ be a basis for $\mathbb{R}^n$. It is always the case that

$$[\vec{a}_i]_{\mathcal{A}} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

has a 1 in the ith position and zeros elsewhere. Now, let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be another basis for $\mathbb{R}^n$ and define the matrix $M = \begin{bmatrix} [\vec{a}_1]_{\mathcal{B}} & [\vec{a}_2]_{\mathcal{B}} & \cdots & [\vec{a}_n]_{\mathcal{B}} \end{bmatrix}$ to be the matrix with columns $[\vec{a}_1]_{\mathcal{B}}, \ldots, [\vec{a}_n]_{\mathcal{B}}$. Since multiplying a matrix by $[\vec{a}_i]_{\mathcal{A}}$ will pick out the ith column, we have that

$$M[\vec{a}_i]_{\mathcal{A}} = [\vec{a}_i]_{\mathcal{B}}.$$

In other words,

$$M = [\mathcal{B} \leftarrow \mathcal{A}].$$

Transformations and Bases

A linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ always has a matrix associated with it. This matrix is defined as the matrix M so that

$$[\mathcal{T}\vec{x}]_{\mathcal{E}} = M[\vec{x}]_{\mathcal{E}}.$$

But, what if we swapped out $\mathcal{E}$ for a different basis?

Linear Transformation in a Basis. Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $\mathcal{B}$ be a basis for $\mathbb{R}^n$. The ***matrix for $\mathcal{T}$ with respect to $\mathcal{B}$***, notated $[\mathcal{T}]_{\mathcal{B}}$, is the $n \times n$ matrix satisfying

$$[\mathcal{T}\vec{x}]_{\mathcal{B}} = [\mathcal{T}]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

In this case, we say the matrix $[\mathcal{T}]_{\mathcal{B}}$ is the representation of $\mathcal{T}$ in the $\mathcal{B}$ basis.

Just like there are many ways to write down coordinates for a vector—one per choice of basis—there are many ways to write down a matrix for a linear transformation. Up to this point, when we've said "M is a matrix for $\mathcal{T}$", what we meant is "$M = [\mathcal{T}]_{\mathcal{E}}$". And, like with vectors, if we talk about a matrix for a linear transformation without specifying the basis, we mean the matrix for the transformation with respect to the standard basis.

Example. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}_{\mathcal{E}}$ and $\vec{b}_2 = \begin{bmatrix} 5 \\ -7 \end{bmatrix}_{\mathcal{E}}$ be a basis for $\mathbb{R}^2$ and let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that stretches in the $\vec{e}_1$ direction by a factor of 2. Find $[\mathcal{T}]_{\mathcal{E}}$ and $[\mathcal{T}]_{\mathcal{B}}$.

Since $\mathcal{T}\vec{e}_1 = 2\vec{e}_1$ and $\mathcal{T}\vec{e}_2 = \vec{e}_2$, We know

$$[\mathcal{T}]_{\mathcal{E}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \text{and} \qquad [\mathcal{T}]_{\mathcal{E}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and so

$$[\mathcal{T}]_{\mathcal{E}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can find $[\mathcal{T}]_{\mathcal{B}}$ in two ways: directly from the definition, or by using change of basis matrices. First, we will work directly from the definition.

To find $[\mathcal{T}]_{\mathcal{B}}$, we need to figure out what $\mathcal{T}$ does to $\vec{b}_1$ and $\vec{b}_2$. However, since $\mathcal{T}$ is described in term of $\vec{e}_1$ and $\vec{e}_2$, it might be easier to express $\vec{e}_1$ and $\vec{e}_2$ in the $\mathcal{B}$ basis, and then analyze $\mathcal{T}$.

Computing,

$$[\vec{e}_1]_{\mathcal{B}} = [\mathcal{B} \leftarrow \mathcal{E}]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$
$$[\vec{e}_2]_{\mathcal{B}} = [\mathcal{B} \leftarrow \mathcal{E}]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}.$$

Now we know

$$\begin{aligned}[\mathcal{T}]_{\mathcal{B}}[\vec{e}_1]_{\mathcal{B}} &= [\mathcal{T}\vec{e}_1]_{\mathcal{B}} = [2\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix}-14\\6\end{bmatrix} \\ [\mathcal{T}]_{\mathcal{B}}[\vec{e}_2]_{\mathcal{B}} &= [\mathcal{T}\vec{e}_2]_{\mathcal{B}} = [\vec{e}_2]_{\mathcal{B}} = \begin{bmatrix}-5\\2\end{bmatrix}\end{aligned}.$$

Since $[\mathcal{T}]_{\mathcal{B}}$ is a 2×2 matrix, we can use what we know to solve for its entries, finding

$$[\mathcal{T}]_{\mathcal{B}} = \begin{bmatrix}-13 & -35\\ 6 & 16\end{bmatrix}.$$

Let's try finding $[\mathcal{T}]_{\mathcal{B}}$ using change of basis matrices. We already know $[\mathcal{T}]_{\mathcal{E}}$, and so

$$[\mathcal{T}]_{\mathcal{B}} = [\mathcal{B}\leftarrow\mathcal{E}][\mathcal{T}]_{\mathcal{E}}[\mathcal{E}\leftarrow\mathcal{B}].$$

Further, we know

$$[\mathcal{E}\leftarrow\mathcal{B}] = \begin{bmatrix}2 & 5\\ -3 & -7\end{bmatrix} \qquad\text{and}\qquad [\mathcal{B}\leftarrow\mathcal{E}] = \begin{bmatrix}2 & 5\\ -3 & -7\end{bmatrix}^{-1} = \begin{bmatrix}-7 & -5\\ 3 & 2\end{bmatrix}.$$

Putting it all together,

$$[\mathcal{T}]_{\mathcal{E}} = \begin{bmatrix}-7 & -5\\ 3 & 2\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 5\\ -3 & -7\end{bmatrix} = \begin{bmatrix}-13 & -35\\ 6 & 16\end{bmatrix}.$$

Similar Matrices

Just like some bases are better than others to represent particular vectors, some bases are better than others to represent a particular linear transformation.

Example. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}2\\-3\end{bmatrix}_{\mathcal{E}}$ and $\vec{b}_2 = \begin{bmatrix}5\\-7\end{bmatrix}_{\mathcal{E}}$ be a basis for $\mathbb{R}^2$ and let $\mathcal{S}:\mathbb{R}^2\to\mathbb{R}^2$ be the transformation that stretches in the $\vec{b}_1 = 2\vec{e}_1 - 3\vec{e}_3$ direction by a factor of 3 and reflects vectors in the $\vec{b}_2 = 5\vec{e}_1 - 7\vec{e}_2$ direction. Find $[\mathcal{S}]_{\mathcal{E}}$ and $[\mathcal{S}]_{\mathcal{B}}$.

In this example, $\mathcal{S}$ is described in terms of the $\mathcal{B}$ basis. That is, we know

$$\mathcal{S}\vec{b}_1 = 2\vec{b}_1 \qquad\text{and}\qquad \mathcal{S}\vec{b}_2 = -\vec{b}_2.$$

Therefore,

$$[\mathcal{S}]_{\mathcal{B}} = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}.$$

To find $[\mathcal{S}]_{\mathcal{E}}$, we will use change of basis matrices. Notice that

$$[\mathcal{S}]_{\mathcal{B}} = [\mathcal{E}\leftarrow\mathcal{B}][\mathcal{S}]_{\mathcal{B}}[\mathcal{B}\leftarrow\mathcal{E}],$$

and that

$$[\mathcal{E}\leftarrow\mathcal{B}] = \begin{bmatrix}2 & 5\\ -3 & -7\end{bmatrix} \qquad\text{and}\qquad [\mathcal{B}\leftarrow\mathcal{E}] = \begin{bmatrix}2 & 5\\ -3 & -7\end{bmatrix}^{-1} = \begin{bmatrix}-7 & -5\\ 3 & 2\end{bmatrix}.$$

Therefore

$$[\mathcal{S}]_{\mathcal{B}} = \begin{bmatrix}2 & 5\\ -3 & -7\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-7 & -5\\ 3 & 2\end{bmatrix} = \begin{bmatrix}-34 & -30\\ 63 & 44\end{bmatrix}.$$

In the example above, $[\mathcal{S}]_{\mathcal{B}}$ is a much nicer matrix than $[\mathcal{S}]_{\mathcal{E}}$. However, the two matrices relate to each other. After all,

$$[\mathcal{S}]_{\mathcal{B}} = [\mathcal{B}\leftarrow\mathcal{E}][\mathcal{S}]_{\mathcal{E}}[\mathcal{E}\leftarrow\mathcal{B}].$$

In this case, we call these matrices *similar*.[42]

Similar Matrices. The matrices A and B are called ***similar matrices***, denoted $A\sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A\sim B$ if there is an invertible

[42] Another commonly used term is *conjugate*.

matrix X so that

$$A = XBX^{-1}.$$

The X in the definition of similar matrices is always a change-of-basis matrix.

When studying a linear transformation, you can pick any basis to represent it in and study the resulting matrix. Different choices of basis will give you different perspectives on the linear transformation. In what's to follow, we will work to find the "best" basis in which to study a given linear transformation.[43]

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

[43] If you cannot wait, the "best" basis will turn out to be the *eigen basis* (provided it exists).

61 Let $\mathcal{B}=\{\vec{b}_1,\vec{b}_2\}$ where $\vec{b}_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2=\begin{bmatrix}1\\-1\end{bmatrix}$ and let $X=[\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$.

61.1 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

61.2 Compute $X[\vec{e}_1]_{\mathcal{B}}$ and $X[\vec{e}_2]_{\mathcal{B}}$. What do you notice?

61.3 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

62 Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{E}}.$$

62.1 Should X^{-1} exist? Explain.

62.2 Consider the equation

$$X^{-1}[\vec{v}]_{?} = [\vec{v}]_{?}.$$

Can you fill in the "?" symbols so that the equation makes sense?

62.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

63 Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

63.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

63.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and $T\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$.

63.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

63.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

63.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

63.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

DEFINITION

Similar Matrices

The matrices A and B are called ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

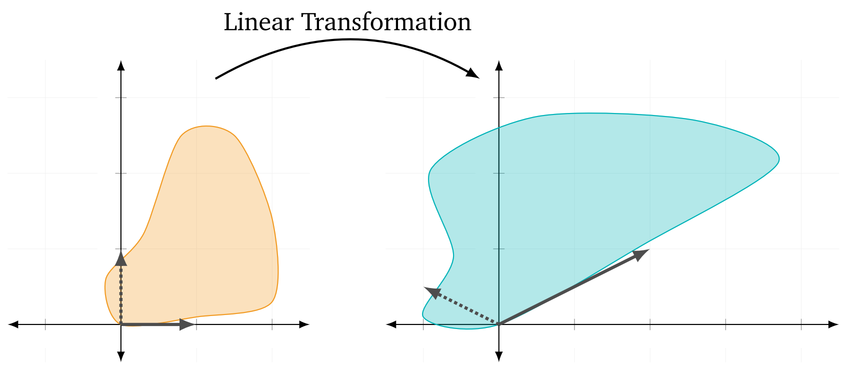
$$A = XBX^{-1}.$$

Determinants

In this module you will learn

- The definition of the determinant of a linear transformation and of a matrix.
- How to interpret the determinant as a change-of-volume factor.
- How to relate the determinant of $S \circ T$ to the determinant of S and of T.
- How to compute the determinants of elementary matrices and how to compute determinants of large matrices using row reduction.

Linear transformations transform vectors, but they also change *sets*.



It turns out to be particularly useful to track by how much a linear transformation changes area/volume. This number (which is associated with a linear transformation with the same domain and codomain) is called the *determinant*.[44]

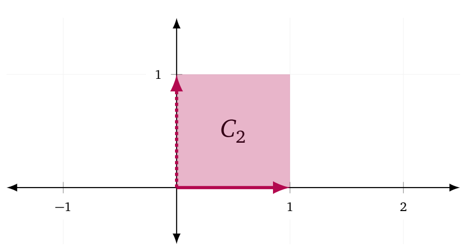
Volumes

In this module, most examples will be in $\mathbb{R}^2$ because they're easier to draw. The definitions given will extend to $\mathbb{R}^n$ for any n, however we need to establish some conventions to properly express these ideas in English. In English, we say that a two-dimensional figure has an *area* and a three-and-up dimensional figure has a *volume*. In this section, *we will use the term volume to also mean area* where appropriate.

To measure how volume changes, we need to compare input volumes and output volumes. The easiest volume to compute is that of the *unit n-cube*, which has a special notation.

> **Unit n-cube.** The ***unit n-cube*** is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is
>
> $$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

C_2 should look familiar as the unit square in $\mathbb{R}^2$ with lower-left corner at the origin.



C_n always has volume 1,[45] and by analyzing the image of C_n under a linear transformation, we can see by how much a given transformation changes volume.

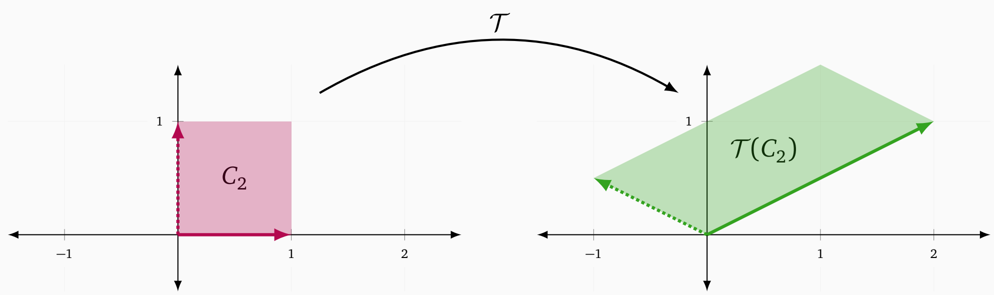
[44] This number is *almost* the determinant. The only difference is that the determinant might have a ± in front.
[45] The fact that the volume of C_n is 1 is actually by definition.

Example. Let $\mathcal{T}:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $\mathcal{T}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x+y\\-x+\frac{1}{2}y\end{bmatrix}$. Find the volume of $\mathcal{T}(C_2)$.

Recall that C_2 is the unit square in $\mathbb{R}^2$ with sides given by $\vec{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$. Applying the linear transformation $\mathcal{T}$ to $\vec{e}_1$ and $\vec{e}_2$, we obtain

$$\mathcal{T}(\vec{e}_1)=\begin{bmatrix}2\\-1\end{bmatrix}\qquad\text{and}\qquad\mathcal{T}(\vec{e}_2)=\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}.$$

Plotting $\begin{bmatrix}2\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}$, we see $\mathcal{T}(C_2)$ is a parallelogram with base $\sqrt{5}$ and height $\frac{2\sqrt{5}}{5}$.



Therefore, the volume of $\mathcal{T}(C_2)$ is 2.

Let $\mathrm{Vol}(X)$ stand for the volume of the set X. Given a linear transformation $\mathcal{S}:\mathbb{R}^n\to\mathbb{R}^n$, we can define a number

$$\mathrm{Vol\,Change}(\mathcal{S})=\frac{\mathrm{Vol}(\mathcal{S}(C_n))}{\mathrm{Vol}(C_n)}=\frac{\mathrm{Vol}(\mathcal{S}(C_n))}{1}=\mathrm{Vol}(\mathcal{S}(C_n)).$$

A priori, $\mathrm{Vol\,Change}(\mathcal{S})$ only describes how $\mathcal{S}$ changes the volume of C_n. However, because $\mathcal{S}$ is a linear transformation, $\mathrm{Vol\,Change}(\mathcal{S})$ actually describes how $\mathcal{S}$ changes the volume of any figure.

Theorem. Let $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation and let $X\subseteq\mathbb{R}^n$ be a subset with volume α. Then the volume of $\mathcal{T}(X)$ is $\alpha\cdot\mathrm{Vol\,Change}(\mathcal{T})$.

A full proof of the above theorem requires calculus and limits, but the linear algebra ideas are based on the following theorems.

Theorem. Suppose $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$ is a linear transformation, $X\subseteq\mathbb{R}^n$ is a subset, and the volume of $\mathcal{T}(X)$ is α. Then for any $\vec{p}\in\mathbb{R}^n$, the volume of $\mathcal{T}(X+\{\vec{p}\})$ is α.

Proof. Fix $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$, $X\subseteq\mathbb{R}^n$, and $\vec{p}\in\mathbb{R}^n$. Combining linearity with the definition of set addition, we see

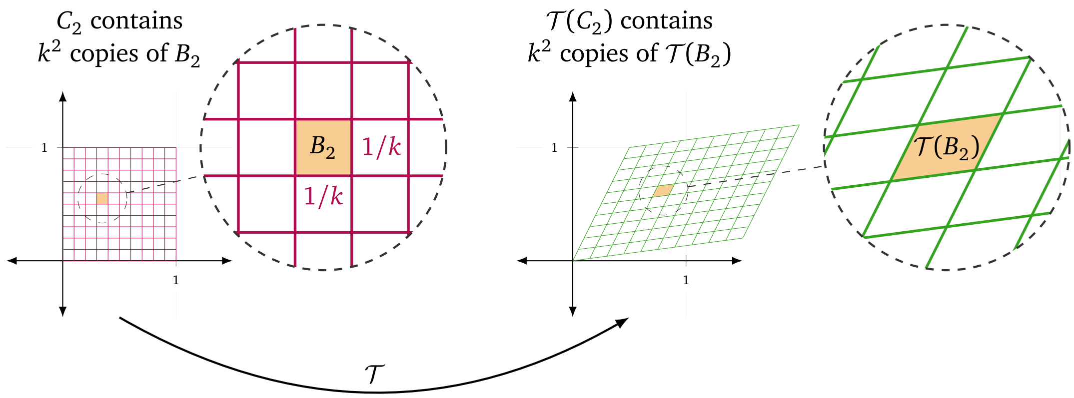
$$\mathcal{T}(X+\{\vec{p}\})=\mathcal{T}(X)+\mathcal{T}(\{\vec{p}\})=\mathcal{T}(X)+\{\mathcal{T}(\vec{p})\}$$

and so $\mathcal{T}(X+\{\vec{p}\})$ is just a translation of $\mathcal{T}(X)$. Since translations don't change volume, $\mathcal{T}(X+\{\vec{p}\})$ and $\mathcal{T}(X)$ must have the same volume. ■

Theorem. Fix k and let B_n be C_n scaled to have side lengths $\frac{1}{k}$ and let $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation. Then

$$\mathrm{Vol\,Change}(\mathcal{T})=\frac{\mathrm{Vol}(\mathcal{T}(B_n))}{\mathrm{Vol}(B_n)}.$$

Rather than giving a formal proof of the above theorem, let's make a motivating picture.

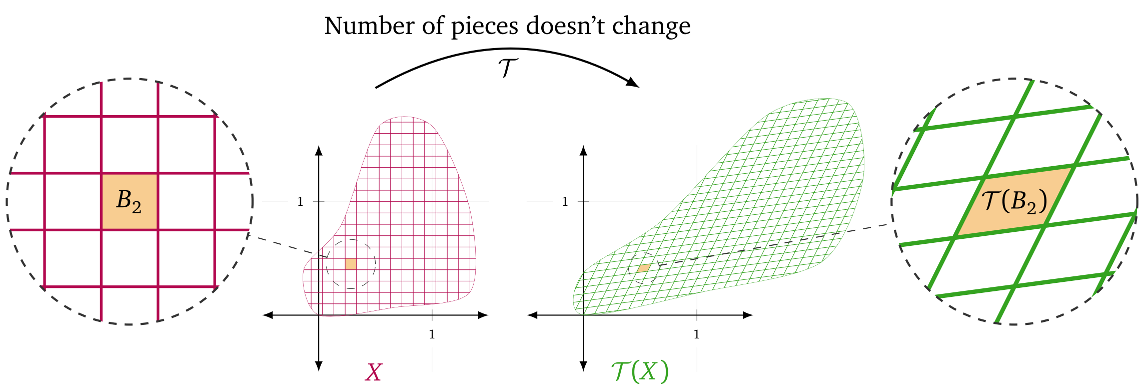


The argument now goes: there are k^n copies of B_n in C_n and k^n copies of $\mathcal{T}(B_n)$ in $T(C_n)$. Thus,

$$\operatorname{Vol Change}(\mathcal{T}) = \frac{\operatorname{Vol}(\mathcal{T}(C_n))}{\operatorname{Vol}(C_n)} = \frac{k^n \operatorname{Vol}(\mathcal{T}(B_n))}{k^n \operatorname{Vol}(B_n)} = \frac{\operatorname{Vol}(\mathcal{T}(B_n))}{\operatorname{Vol}(B_n)}.$$

Now we can finally show that for a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$, the number "Vol Change($\mathcal{T}$)" actually corresponds to how much $\mathcal{T}$ changes the volume of any figure by.

The argument goes as follows: for a figure $X \subseteq \mathbb{R}^n$, we can fill it with shrunken and translated copies, B_n, of C_n. The same number of copies of $\mathcal{T}(B_n)$ fit inside $\mathcal{T}(X)$ as do B_n's fit inside X. Therefore, the change in volume between $\mathcal{T}(X)$ and X must be the same as the change in volume between $\mathcal{T}(B_n)$ and B_n, which is Vol Change($\mathcal{T}$).



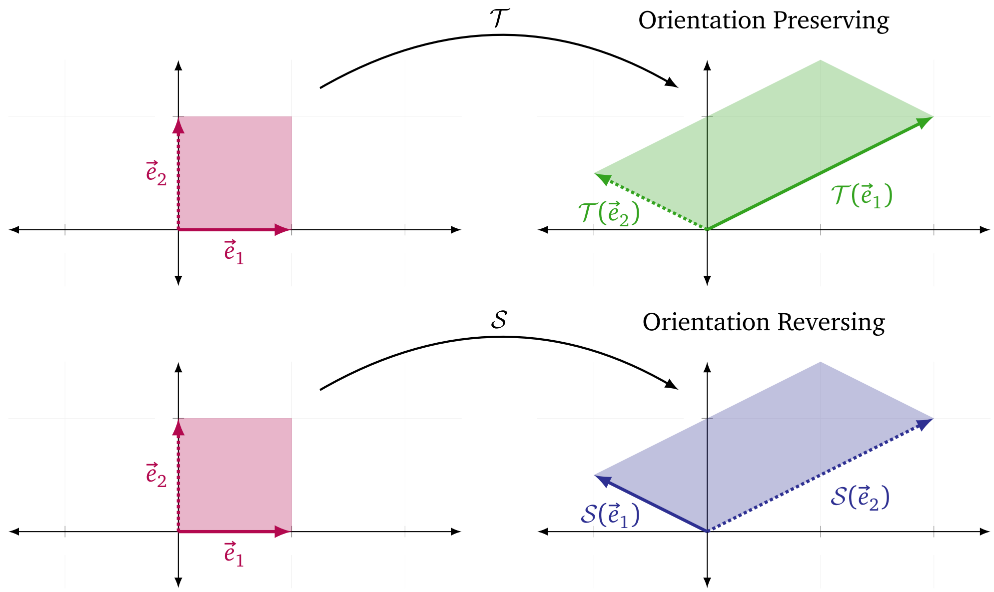
The Determinant

The determinant of a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ is *almost* the same as Vol Change($\mathcal{T}$), but with one twist: *orientation*.

Determinant. The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$, denoted $\det(X)$ or $|X|$, is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the determinant of its induced transformation.

We need to understand what the term *oriented volume* means. We've previously defined the orientation of a basis, and we can use the orientation of a basis to define whether a linear transformation is *orientation preserving* or *orientation reversing*.

Orientation Preserving Linear Transformation. Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. We say $\mathcal{T}$ is ***orientation preserving*** if the ordered basis $\{\mathcal{T}(\vec{e}_1), \dots, \mathcal{T}(\vec{e})\}$ is a positively oriented and we say $\mathcal{T}$ is ***orientation reversing*** if the ordered basis $\{\mathcal{T}(\vec{e}_1), \dots, \mathcal{T}(\vec{e})\}$ is a negatively. If $\{\mathcal{T}(\vec{e}_1), \dots, \mathcal{T}(\vec{e})\}$ is not a basis for $\mathbb{R}^n$, then $\mathcal{T}$ is neither orientation preserving nor orientation reversing.



In the figure above, $\mathcal{T}$ is orientation preserving and $\mathcal{S}$ is orientation reversing.

For an arbitrary linear transformation $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n$ and a set $X \subseteq \mathbb{R}^n$, we define the *oriented volume* of $\mathcal{Q}(X)$ to be $+\operatorname{Vol}\mathcal{Q}(X)$ if $\mathcal{Q}$ is orientation preserving and $-\operatorname{Vol}\mathcal{Q}(X)$ if $\mathcal{Q}$ is orientation reversing.

Example. Let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathcal{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ -x + \frac{1}{2}y \end{bmatrix}$. Find $\det(\mathcal{T})$.

This is the same $\mathcal{T}$ as from the previous example where we computed $\operatorname{Vol}\mathcal{T}(C_2) = 2$. Since $\mathcal{T}$ is orientation preserving, therefore know $\det(\mathcal{T}) = 2$.

Example. Let $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathcal{S}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + y \\ x + y \end{bmatrix}$. Find $\det(\mathcal{T})$.

By drawing a picture, we see that $\mathcal{S}(C_2)$ is a square and $\operatorname{Vol}\mathcal{S}(C_2) = 2$. However, $\mathcal{S}(\vec{e}_1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathcal{S}(\vec{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a negatively oriented basis, and so $\mathcal{S}$ is orientation reversing. Therefore, $\det(\mathcal{S}) = -\operatorname{Vol}\mathcal{S}(C_2) = -2$.

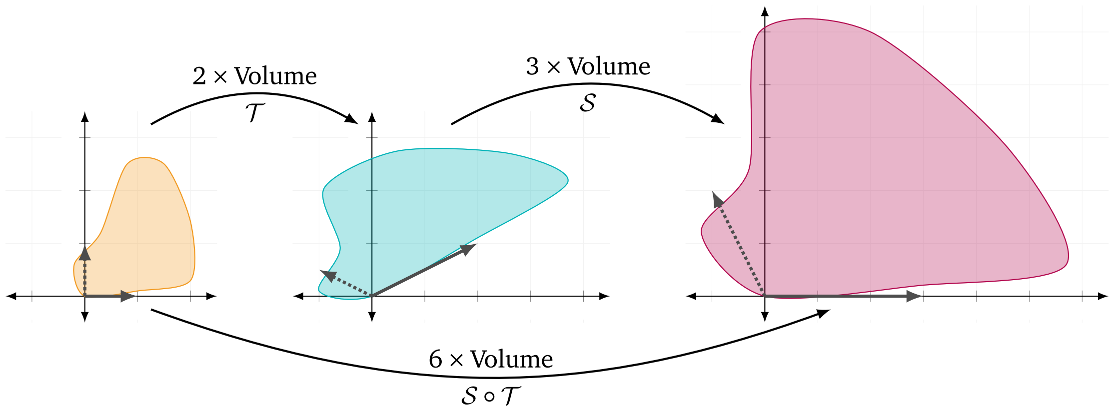
Example. Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the line with equation $x + 2y = 4$. Find $\det(\mathcal{P})$.

Because $\mathcal{P}$ projects everything to a line, we know $\mathcal{P}(C_2)$ must be a line segment and therefore has volume zero. Thus $\det(\mathcal{P}) = 0$.

Determinants of Composition

Volume changes are naturally multiplicative. If a linear transformation $\mathcal{T}$ changes volume by a factor of α and $\mathcal{S}$ changes volume by a factor of β, then $\mathcal{S} \circ \mathcal{T}$ changes volume by a factor of $\beta\alpha$. Thus, determinants must also be multiplicative.[46]

[46] To fully argue this, we need to show that the composition of two orientation-reversing transformations is orientation preserving.



Theorem. Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Then

$$\det(\mathcal{S} \circ \mathcal{T}) = \det(\mathcal{S}) \det(\mathcal{T}).$$

This means that we can compute the determinant of a complicated transformation by breaking it up into simpler ones and computing the determinant of each piece.

Determinants of Matrices

The determinant of a matrix is defined as the determinant of its induced transformation. That means, the determinant is multiplicative with respect to matrix multiplication (because it's multiplicative with respect to function composition).

Theorem. Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B).$$

We will derive an algorithm for finding the determinant of a matrix by considering the determinant of elementary matrices. But first, consider the following theorem.

Theorem. (Volume Theorem I) For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped[a] given by the column vectors.

[a] A parallelepiped is the n-dimensional analog of a parallelogram.

Proof. Let M be an $n \times n$ matrix and let $\mathcal{T}_M$ be its induced transformation. We know the sides of $\mathcal{T}_M(C_n)$ are given by $\{\mathcal{T}_M(\vec{e}_1), \ldots, \mathcal{T}_M(\vec{e}_n)\}$. And, by definition,

$$[\mathcal{T}_M(\vec{e}_i)]_{\mathcal{E}} = M[\vec{e}_i]_{\mathcal{E}} = i\text{th column of } M.$$

Therefore $\mathcal{T}_M(C_n)$ is the parallelepiped whose sides are given by the columns of M. ∎

This means we can think about the determinant of a matrix by considering its columns. Now we are ready to consider the determinants of the elementary matrices!

There are three types of elementary matrices corresponding to the three elementary row operations. For each one, we need to understand how the induced transformation changes volume.

Multiply a row by a non-zero constant α. Let E_m be such an elementary matrix. Scaling one row of I is equivalent to scaling one column of I, and so the columns of E_m specify a parallelepiped that is scaled by α in one direction.

For example, if

$$E_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \qquad \text{then} \qquad \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \mapsto \{\vec{e}_1, \vec{e}_2, \alpha\vec{e}_3\}.$$

Thus $\det(E_m) = \alpha$.

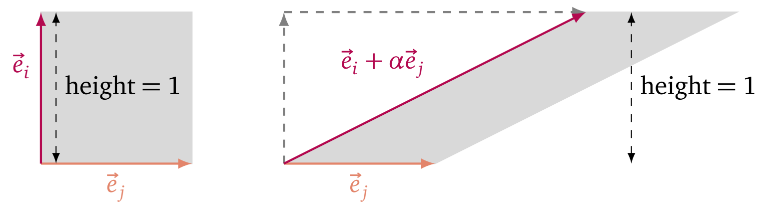
Swap two rows. Let E_s be such an elementary matrix. Swapping two rows of I is equivalent to swapping two columns of I, so E_s is I with two columns swapped. This reverses the orientation of the basis given by the columns.

For example, if

$$E_s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{then} \qquad \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \mapsto \{\vec{e}_2, \vec{e}_1, \vec{e}_3\}.$$

Thus $\det(E_s) = -1$.

Add a multiple of one row to another. Let E_a be such an elementary matrix. The columns of E_a are the same as the columns of I except that one column where $\vec{e}_i$ is replaced with $\vec{e}_i + \alpha\vec{e}_j$. This has the effect of *shearing* C_n in the $\vec{e}_j$ direction.



Since C_n is sheared in a direction parallel to one of its other sides, its volume is not changed. Thus $\det(E_a) = 1$.

Takeaway. The determinants of elementary matrices are all easy to compute and the determinant of the most-used type of elementary matrix is 1.

Now, by decomposing a matrix into the product of elementary matrices, we can use the multiplicative property of the determinant (and the formulas for the determinants of the different types of elementary matrices) to compute the determinant of an invertible matrix.

Example. Use elementary matrices to find the determinant of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

We can row-reduce A with the following steps.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The elementary matrices corresponding to these steps are

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad \text{and} \qquad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

and so $E_3E_2E_1A = I$. Therefore

$$A = E_1^{-1}E_2^{-1}E_3^{-1}I = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Using the fact that the determinant is multiplicative, we get

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\right)\det\left(\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\right)\det\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right) \\ &= (1)(-2)(1) = -2. \end{aligned}$$

Determinants and Invertibility

We can use elementary matrices to compute the determinant of any invertible matrix by decomposing it into the product of elementary matrices. But, what about non-invertible matrices?

Let M be an $n \times n$ matrix that is *not* invertible. Then, we must have $\text{nullity}(M) > 0$ and $\text{col}(M) < n$. Geometrically, this means there is at least one line of vectors, $\text{null}(M)$, that gets collapsed to $\vec{0}$, and the column space of M must be "flattened" (i.e., it has lost a dimension). Therefore, the volume of the parallelepiped given by the columns of M must be zero, and so $\det(M) = 0$.

Based on this argument, we have the following theorem.

Theorem. Let A be an $n \times n$ matrix. A is invertible if and only if $\det(A) \neq 0$.

Proof. If A is invertible, $A = E_1 \cdots E_k$, where $E_1, \ldots, E_k$ are elementary matrices, and so

$$\det(A) = \det(E_1 \cdots E_k) = \det(E_1) \cdots \det(E_k).$$

All elementary matrices have non-zero determinants, and so $\det(A) \neq 0$.

Conversely, if A is not invertible, $\operatorname{rank}(A) < n$, which means the parallelepiped given by the columns of A is "flattened" and has zero volume. ∎

We now have another way to tell if a matrix is invertible! But, for an invertible matrix A, how do $\det(A)$ and $\det(A^{-1})$ relate? Well, by definition

$$AA^{-1} = I,$$

and so

$$\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I) = 1,$$

which gives

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Determinants and Transposes

Somewhat mysteriously, we have the following theorem.

Theorem. (Volume Theorem II) The determinant of a square matrix A is equal to the oriented volume of the parallelepiped given by the rows of A.

Volume Theorem II can be concisely stated as $\det(A) = \det(A^T)$, and joins other strange transpose-related facts (like $\operatorname{rank}(A) = \operatorname{rank}(A^T)$).

We can prove Volume Theorem II using elementary matrices.

Proof. Suppose A is not invertible. Then, neither is A^T and so $\det(A) = \det(A^T) = 0$.

Suppose A is invertible and $A = E_1 \cdots E_k$ where $E_1, \ldots, E_k$ are elementary matrices. We then have

$$A^T = E_k^T \cdots E_1^T.$$

However, for each E_i, we may observe that E_i^T is another elementary matrix of the same type and with the same determinant. Therefore,

$$\begin{aligned}\det(A^T) = \det(E_k^T \cdots E_1^T) &= \det(E_k^T) \cdots \det(E_i^T) \\ &= \det(E_k) \cdots \det(E_1) \\ &= \det(E_1) \cdots \det(E_k) = \det(E_1 \cdots E_k) = \det(A).\end{aligned}$$

The key observations for this proof are that (i) $\det(E_i^T) = \det(E_i)$ and (ii) since the $\det(E_i)$'s are just scalars, the order in which they are multiplied doesn't matter. ∎

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

DEFINITION

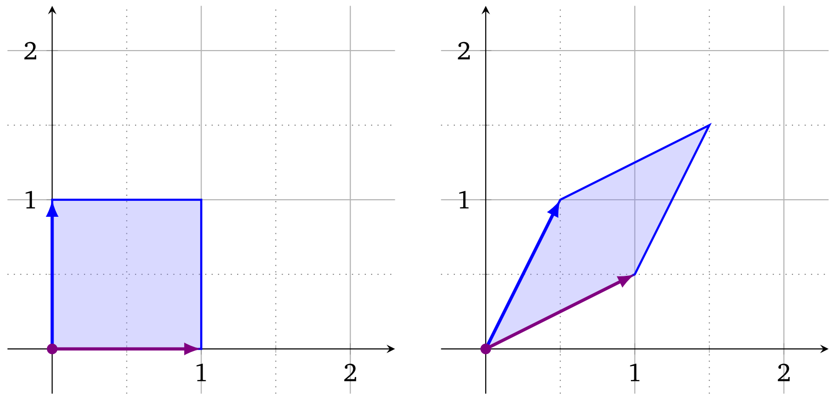
Unit n-cube

The ***unit n-cube*** is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \dots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The sides of the unit n-cube are always length 1 and its volume is always 1.

64 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



64.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

64.2 Write down a matrix for T.

64.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may use trigonometry.

Determinant

DEF The ***determinant*** of a linear transformation $X : \mathbb{R}^n \to \mathbb{R}^n$, denoted $\det(X)$ or $|X|$, is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the determinant of its induced transformation.

---

65 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

65.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

65.2 Compute the area of $A(C_2)$.

65.3 Compute $\det(A)$.

66 Suppose R is a rotation counter-clockwise by 30°.

66.1 Draw C_2 and $R(C_2)$.

66.2 Compute the area of $R(C_2)$.

66.3 Compute $\det(R)$.

67 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

67.1 What is $\det(F)$?

Volume Theorem I

THM

For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors of M.

Volume Theorem II

THM

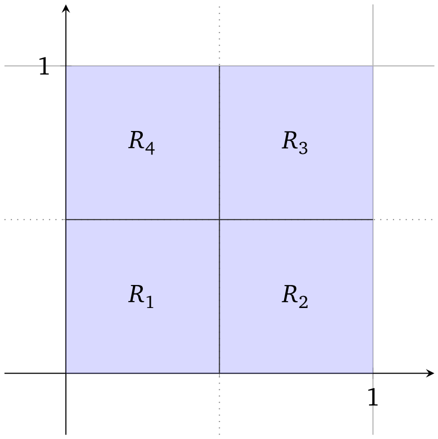
For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the row vectors of M.

68

68.1 Explain Volume Theorem I using the definition of determinant.

68.2 Based on Volume Theorems I and II, how should $\det(M)$ and $\det(M^T)$ relate for a square matrix M?

69



Let $R = R_1 \cup R_2 \cup R_3 \cup R_4$. You know the following about the linear transformations M, T, and S.

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ has determinant } 2$$

$$S : \mathbb{R}^2 \to \mathbb{R}^2 \text{ has determinant } 3$$

69.1 Find the volumes (areas) of R_1, R_2, R_3, R_4, and R.

69.2 Compute the oriented volume of $M(R_1)$, $M(R_2)$, and $M(R)$.

69.3 Do you have enough information to compute the oriented volume of $T(R_2)$? What about the oriented volume of $T(R + \{\vec{e}_2\})$?

69.4 What is the oriented volume of $S \circ T(R)$? What is $\det(S \circ T)$?

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

70.1 What is $\det(E_f)$?

70.2 What is $\det(E_m)$?

70.3 What is $\det(E_a)$?

70.4 What is $\det(E_f E_m)$?

70.5 What is $\det(4I_{3\times 3})$?

70.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

71

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

71.1 What is $\det(U)$?

71.2 V is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

72

72.1 V is a square matrix whose columns are linearly dependent. What is $\det(V)$?

72.2 P is projection onto $\operatorname{span}\left\{\begin{bmatrix}-1\\-1\end{bmatrix}\right\}$. What is $\det(P)$?

73 Suppose you know $\det(X) = 4$.

73.1 What is $\det(X^{-1})$?

73.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

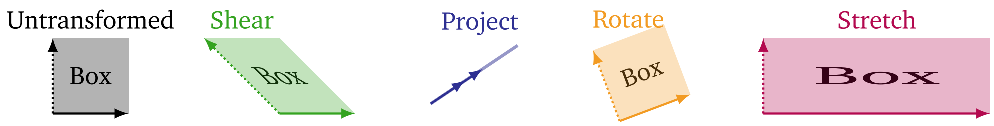
73.3 Suppose Y is not invertible. What is $\det(Y)$?

Eigenvalues and Eigenvectors

In this module you will learn

- The definition of eigenvalues and eigenvectors.
- That eigenvectors give a particularly nice basis in which to study a linear transformation.
- How the characteristic polynomial relates to eigenvalues.

From here on out, we will only be considering linear transformations with the same domain and codomain (i.e., transformations $\mathcal{T}:\mathbb{R}^n \to \mathbb{R}^n$). Why? Because that will allow us to *compare* input and output vectors. By comparing inputs and outputs, we may describe a linear transformation as a stretch, twist, shear, rotation, projection, or some combination of all of these operations.



It's the stretched vectors that we're most interested in now. If $\mathcal{T}$ stretches the vector $\vec{v}$, then $\mathcal{T}$, in that direction, can be described by $\vec{v} \mapsto \alpha\vec{v}$, which is an easy-to-understand linear transformation. The "stretch" directions for a linear transformation have a special name—*eigen directions*—and the vectors that are stretched are called *eigenvectors*.

Eigenvector. Let X be a linear transformation or a matrix. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

The word *eigen* is German for characteristic, representative, or intrinsic, and we will see that eigenvectors provide one of the best contexts in which to understand a linear transformation.

Example. Let $\mathcal{P}:\mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the line ℓ given by $y = x$. Find the eigenvectors and eigenvalues of $\mathcal{P}$.

We are looking for vectors $\vec{v} \neq \vec{0}$ such that $\mathcal{P}\vec{v} = \lambda\vec{v}$ for some λ. Since $\mathcal{P}(\ell) = \ell$, we know for any $\vec{v} \in \ell$

$$\mathcal{P}(\vec{v}) = 1\vec{v} = \vec{v}.$$

Therefore, any non-zero multiple of $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for $\mathcal{P}$ with corresponding eigenvalue 1.
By considering the null space of $\mathcal{P}$, we see, for example,

$$\mathcal{P}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}1\\-1\end{bmatrix},$$

and so $\begin{bmatrix}1\\-1\end{bmatrix}$ and all its non-zero multiples are eigenvectors of $\mathcal{P}$ with corresponding eigenvalue 0.

Finding Eigenvectors

Sometimes you can find the eigenvectors/values of a linear transformation just by thinking about it. For example, for reflections, projections, and dilations, the eigen directions are geometrically clear. However, for an arbitrary matrix transformation, it may not be obvious.

Our goal now will be to see if we can leverage linear algebra knowledge to find eigenvectors/values. So that we don't have to switch back and forth between thinking about linear transformations and thinking about matrices, let just think about matrices for now.

Let M be a square matrix. The vector $\vec{v} \neq \vec{0}$ is an eigenvector for M if and only if there exists a scalar λ so that

$$M\vec{v} = \lambda\vec{v}. \tag{12}$$

Put another way, $\vec{v} \neq \vec{0}$ is an eigenvector for M if and only if

$$M\vec{v} - \lambda\vec{v} = (M - \lambda I)\vec{v} = \vec{0}.$$

The middle equation provides a key insight. The operation $\vec{v} \mapsto M\vec{v} - \lambda\vec{v}$ can be achieved by multiplying $\vec{v}$ by the single matrix $E_\lambda = M - \lambda I$.

Now we have that $\vec{v} \neq \vec{0}$ is an eigenvector for M if and only if

$$E_\lambda\vec{v} = (M - \lambda I)\vec{v} = M\vec{v} - \lambda\vec{v} = \vec{0},$$

or, phrased another way, $\vec{v}$ is a non-zero vector satisfying $\vec{v} \in \mathrm{null}(E_\lambda)$.

We've reduced the problem of finding eigenvectors/values of M to finding the null space of E_λ, a related matrix.

Characteristic Polynomial

Let M be an $n \times n$ matrix and define $E_\lambda = M - \lambda I$. Every eigenvector for M must be in the null space of E_λ for some λ. However, because eigenvectors must be non-zero, the only chance we have of finding an eigenvector is if $\mathrm{null}(E_\lambda) \neq \{\vec{0}\}$. In other words, we would like to know when $\mathrm{null}(E_\lambda)$ is *non-trivial*.

We're well equipped to answer this question. Because E_λ is an $n \times n$ matrix, we know E_λ has a non-trivial null space if and only if E_λ is not invertible which is true if and only if $\det(E_\lambda) = 0$. Every λ defines a different E_λ where eigenvectors could be hiding. By viewing $\det(E_\lambda)$ *as a function of* λ, we can use our mathematical knowledge of single-variable functions to figure out when $\det(E_\lambda) = 0$.

The quantity $\det(E_\lambda)$, view as a function of λ, has a special name—it's called the *characteristic polynomial*.[47]

Characteristic Polynomial.
For a matrix A, the ***characteristic polynomial*** of A is

$$\mathrm{char}(A) = \det(A - \lambda I).$$

Example. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

By the definition of the characteristic polynomial of A, we have

$$\begin{aligned} \mathrm{char}(A) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2. \end{aligned}$$

For an $n \times n$ matrix A, $\mathrm{char}(A)$ has some nice properties.

- $\mathrm{char}(A)$ is a polynomial.[48]
- $\mathrm{char}(A)$ has degree n.
- The coefficient of the λ^n term in $\mathrm{char}(A)$ is ± 1.
- $\mathrm{char}(A)$ evaluated at $\lambda = 0$ is $\det(A)$.
- The roots of $\mathrm{char}(A)$ are precisely the eigenvalues of A.

We will just accept these properties as facts, but each of them can be proved with the tools we've developed.

Using the Characteristic Polynomial to find Eigenvalues

With the characteristic polynomial in hand, finding eigenvectors/values becomes easier.

Example. Find the eigenvectors/values of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

[47] This time the term is traditionally given the English name, rather than being called the *eigenpolynomial*.
[48] A priori, it's not obvious that $\det(A - \lambda I)$ should be a polynomial as opposed to some other type of function.

Like the previous example, we first compute char(A).

$$\begin{aligned}\text{char}(A) &= \det\left(\begin{bmatrix}1-\lambda & 2\\ 3 & 2-\lambda\end{bmatrix}\right)\\ &= (1-\lambda)(2-\lambda)-6 = \lambda^2-3\lambda-4 = (4-\lambda)(-1-\lambda)\end{aligned}$$

Next, we solve for when char$(A)=0$ to find eigenvalues, which are $\lambda_1=-1$ and $\lambda_2=4$.
We know non-zero vectors in null$(A-\lambda_1 I)$ are eigenvectors with eigenvalue -1. Computing,

$$\text{null}(A-\lambda_1 I) = \text{null}\left(\begin{bmatrix}2 & 2\\ 3 & 3\end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix}1\\ -1\end{bmatrix}\right\},$$

And so the eigenvectors of A corresponding to eigenvalue $\lambda_1=-1$ are the non-zero multiples of $\begin{bmatrix}1\\ -1\end{bmatrix}$.
Similarly, for $\lambda_2=4$, we compute

$$\text{null}(A-\lambda_2 I) = \text{null}\left(\begin{bmatrix}-3 & 2\\ 3 & -2\end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix}2\\ 3\end{bmatrix}\right\},$$

and so the eigenvectors for A with eigenvalue 4 are the non-zero multiples of $\begin{bmatrix}2\\ 3\end{bmatrix}$.

Using the characteristic polynomial, we can show that every eigenvalue for a matrix is a root of some polynomial (the characteristic polynomial). In general, finding roots of polynomials is a hard problem,[49] and it's not one we will focus on. However, it's handy to have the quadratic formula in your back pocket for factoring particularly stubborn polynomials.

Example. Find the eigenvectors/values of $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.

First, we find the roots of char(A) by setting it to 0.

$$\begin{aligned}\text{char}(A) &= \det\left(\begin{bmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{bmatrix}\right)\\ &= (1-\lambda)(4-\lambda)-6 = \lambda^2-5\lambda-2 = 0\end{aligned}$$

By the quadratic formula,[a] we find that

$$\lambda_1 = \frac{5-\sqrt{33}}{2} \qquad \lambda_2 = \frac{5+\sqrt{33}}{2}$$

are the roots of char(A).
Following the procedure outlined above, we need to find null$(A-\lambda_1 I)$ and null$(A-\lambda_2 I)$.
We will start by applying row reduction to $A-\lambda_1 I$.

$$\begin{aligned}\begin{bmatrix}1-\frac{5-\sqrt{33}}{2} & 2\\ 3 & 4-\frac{5-\sqrt{33}}{2}\end{bmatrix} &\to \begin{bmatrix}\frac{-3+\sqrt{33}}{2} & 2\\ 3 & \frac{3+\sqrt{33}}{2}\end{bmatrix} \to \begin{bmatrix}1 & \frac{4}{-3+\sqrt{33}}\\ 1 & \frac{3+\sqrt{33}}{6}\end{bmatrix}\\ &\to \begin{bmatrix}1 & \frac{4(3+\sqrt{33})}{(-3+\sqrt{33})(3+\sqrt{33})}\\ 1 & \frac{3+\sqrt{33}}{6}\end{bmatrix} \to \begin{bmatrix}1 & \frac{3+\sqrt{33}}{6}\\ 1 & \frac{3+\sqrt{33}}{6}\end{bmatrix}\\ &\to \begin{bmatrix}1 & \frac{3+\sqrt{33}}{6}\\ 0 & 0\end{bmatrix}\end{aligned}$$

Thus, we conclude that the eigenvectors with eigenvalue $\frac{5-\sqrt{33}}{2}$ are the non-zero multiples of $\begin{bmatrix}\frac{3+\sqrt{33}}{6}\\ -1\end{bmatrix}$.

Similarly, the eigenvectors with eigenvalue $\frac{5\sqrt{33}}{2}$ are the non-zero multiples of $\begin{bmatrix}\frac{3-\sqrt{33}}{6}\\ -1\end{bmatrix}$.

[a] Recall that the roots of ax^2+bx+c are given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

[49] In fact, numerically approximating eigenvalues turns out to be easier than finding roots of a polynomial, so many numerical root finding algorithms actually create a matrix with an appropriate characteristic polynomial and use numerical linear algebra to approximate its roots.

Transformations without Eigenvectors

Are there linear transformations without eigenvectors? Well, it depends on exactly what you mean. Let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter-clockwise by $90°$. Are there any non-zero vectors that don't change direction when $\mathcal{R}$ is applied? Certainly not.

Let's examine further. We know $M_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a matrix for $\mathcal{R}$, and

$$\text{char}(M_R) = \lambda^2 + 1.$$

The polynomial $\lambda^2 + 1$ has no real roots, which means that M_R (and $\mathcal{R}$) have no real eigenvalues. However, $\lambda^2 + 1$ does have *complex* roots of $\pm i$. So far, we've always though of scalars as real numbers, but if we allow complex numbers as scalars and view $\mathcal{R}$ as a transformation from $\mathbb{C}^2 \to \mathbb{C}^2$, it would have eigenvalues and eigenvectors.

Complex numbers play an invaluable role in advanced linear algebra and applications of linear algebra to physics. We will leave the following theorem as food for thought.[50]

> **Theorem.** If A is a square matrix, then A always has an eigenvalue provided complex eigenvalues are permitted.

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

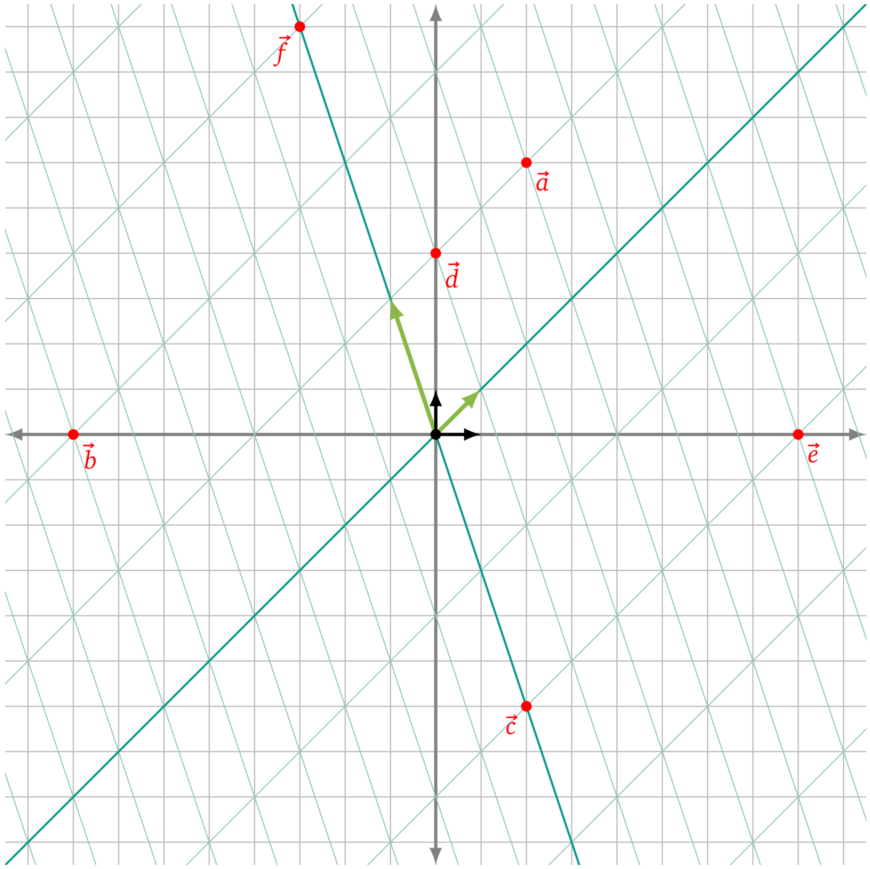
(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

[50] The theorem is a direct corollary of the fundamental theorem of algebra.

Task 3.1: The Green and the Black

Consider the following two bases for $\mathbb{R}^2$: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$ and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.



1. Write each point above in both the green and the black bases.
2. Find a change-of-basis matrix X that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

4. When working with the transformation T, which basis do you prefer vectors be represented in?

Eigenvectors

Eigenvector

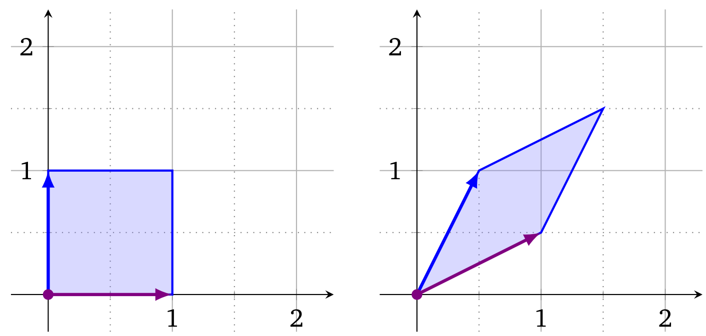
DEFINITION

Let X be a linear transformation or a matrix. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

74 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



74.1 Give an eigenvector for T. What is the eigenvalue?

74.2 Can you give another?

75 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B=A-\tfrac{2}{3}I.$$

75.1 Give an eigenvector and a corresponding eigenvalue for A.

75.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

75.3 What is the dimension of $\text{null}(B)$?

75.4 What is $\det(B)$?

76 Let $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E_\lambda = C - \lambda I$.

76.1 For what values of λ does E_λ have a non-trivial null space?

76.2 What are the eigenvalues of C?

76.3 Find the eigenvectors of C.

DEF

Characteristic Polynomial

For a matrix A, the ***characteristic polynomial*** of A is

$$\operatorname{char}(A) = \det(A - \lambda I).$$

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77 Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

77.1 Compute $\operatorname{char}(D)$.

77.2 Find the eigenvalues of D.

78 Suppose $\operatorname{char}(E) = -\lambda(2-\lambda)(-3-\lambda)$ for some unknown 3×3 matrix E.

78.1 What are the eigenvalues of E?

78.2 Is E invertible?

78.3 What can you say about $\operatorname{nullity}(E)$, $\operatorname{nullity}(E-3I)$, $\operatorname{nullity}(E+3I)$?

Diagonalization

In this module you will learn

- How to diagonalize a matrix.
- When a matrix can and cannot be diagonalized.

Suppose $\mathcal{T}$ is a linear transformation and $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with eigenvalues λ_1 and λ_2. With this setup, for any $\vec{a} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$, we can compute $\mathcal{T}(\vec{a})$ with minimal effort.

Let's get specific. Define $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation with matrix $M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and notice that $\vec{v}_1$ is an eigenvector for $\mathcal{T}$ with eigenvalue -1 and that $\vec{v}_2$ is an eigenvector for $\mathcal{T}$ with eigenvalue 4. Let $\vec{a} = \vec{v}_1 + \vec{v}_2$.

Now,

$$\mathcal{T}(\vec{a}) = \mathcal{T}(\vec{v}_1 + \vec{v}_2) = \mathcal{T}(\vec{v}_1) + \mathcal{T}(\vec{v}_2) = -\vec{v}_1 + 4\vec{v}_2.$$

We didn't need to refer to the entries of M to compute $\mathcal{T}(\vec{a})$.

Let's explore further. Let $\mathcal{V} = \{\vec{v}_1, \vec{v}_2\}$ and notice that $\mathcal{V}$ is a basis for $\mathbb{R}^2$. By definition $[\vec{a}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so we just computed

$$\mathcal{T}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{V}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}_{\mathcal{V}}.$$

When represented in the $\mathcal{V}$ basis, computing $\mathcal{T}$ is easy. In general,

$$\mathcal{T}(\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha\mathcal{T}(\vec{v}_1) + \beta\mathcal{T}(\vec{v}_2) = -\alpha\vec{v}_1 + 4\beta\vec{v}_2,$$

and so

$$\mathcal{T}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathcal{V}} = \begin{bmatrix} -\alpha \\ 4\beta \end{bmatrix}_{\mathcal{V}}.$$

In other words, $\mathcal{T}$, when acting on vectors written in the $\mathcal{V}$ basis, just multiplies each coordinate by an eigenvalue. This is enough information to determine the matrix for $\mathcal{T}$ in the $\mathcal{V}$ basis:

$$[\mathcal{T}]_{\mathcal{V}} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}.$$

The matrix representations $[\mathcal{T}]_{\mathcal{E}} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $[\mathcal{T}]_{\mathcal{V}} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ are equally valid, but writing $\mathcal{T}$ in the $\mathcal{V}$ basis gives a very simple matrix!

Diagonalization

Recall that two matrices are similar if they represent the same transformation but in possibly different bases. The process of *diagonalizing* a matrix A is that of finding a diagonal matrix that is similar to A, and you can bet that this processes is closely related to eigenvectors/values.

Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and suppose that $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis so that

$$[\mathcal{T}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix}$$

is a diagonal matrix. This means that $\vec{b}_1, \ldots, \vec{b}_n$ are eigenvectors for $\mathcal{T}$! The proof goes as follows:

$$[\mathcal{T}]_{\mathcal{B}}[\vec{b}_1]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \alpha_1[\vec{b}_1]_{\mathcal{B}} = [\alpha_1\vec{b}_1]_{\mathcal{B}},$$

and in general

$$[\mathcal{T}]_{\mathcal{B}}[\vec{b}_i]_{\mathcal{B}} = \alpha_i[\vec{b}_i]_{\mathcal{B}} = [\alpha_i\vec{b}_i]_{\mathcal{B}}.$$

Therefore, for $i = 1, \ldots, n$, we have

$$\mathcal{T}\vec{b}_i = \alpha_i\vec{b}_i.$$

Since $\mathcal{B}$ is a basis, $\vec{b}_i \neq \vec{0}$ for any i, and so each $\vec{b}_i$ is an eigenvector for $\mathcal{T}$ with corresponding eigenvalue α_i.

We've just shown that if a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ can be represented by a diagonal matrix, then there must be a basis for $\mathbb{R}^n$ consisting of eigenvectors for $\mathcal{T}$. The converse is also true.

Suppose again that $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and that $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis of eigenvectors for $\mathcal{T}$ with corresponding eigenvalues $\alpha_1, \ldots, \alpha_n$. By definition,

$$\mathcal{T}(k\vec{b}_i) = k\mathcal{T}(\vec{b}_i) = \alpha_i k\vec{b}_i,$$

and so

$$\mathcal{T}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \alpha_1 k_1 \\ \alpha_2 k_2 \\ \vdots \\ \alpha_n k_n \end{bmatrix}_{\mathcal{B}} \qquad \text{which is equivalent to} \qquad [\mathcal{T}]_{\mathcal{B}}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} \alpha_1 k_1 \\ \alpha_2 k_2 \\ \vdots \\ \alpha_n k_n \end{bmatrix}.$$

The only matrix that does this is

$$[\mathcal{T}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix},$$

which is a diagonal matrix.

What we've shown is summarized by the following theorem.

> **Theorem.** A linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ can be represented by a diagonal matrix if and only if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors for $\mathcal{T}$. If $\mathcal{B}$ is such a basis, then $[\mathcal{T}]_{\mathcal{B}}$ is a diagonal matrix.

Now that we have a handle on representing a linear transformation by a diagonal matrix, let's tackle the problem of diagonalizing a matrix itself.

> **Diagonalizable.** A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

Suppose A is an $n \times n$ matrix. A induces some transformation $\mathcal{T}_A : \mathbb{R}^n \to \mathbb{R}^n$. By definition, this means $A = [\mathcal{T}_A]_{\mathcal{E}}$. The matrix B is similar to A if there is some basis $\mathcal{V}$ so that $B = [\mathcal{T}_A]_{\mathcal{V}}$. Using change-of-basis matrices, we see

$$A = [\mathcal{E} \leftarrow \mathcal{V}][\mathcal{T}_A]_{\mathcal{V}}[\mathcal{V} \leftarrow \mathcal{E}] = [\mathcal{E} \leftarrow \mathcal{V}]B[\mathcal{V} \leftarrow \mathcal{E}].$$

In other words, A and B are similar if there is some invertible change-of-basis matrix P so

$$A = PBP^{-1}.$$

Based on our earlier discussion, B will be a diagonal matrix if and only if P is the change-of-basis matrix for a basis of eigenvectors. In this case, we know B will be the diagonal matrix with eigenvalues along the diagonal (in the proper order).

> **Example.** Let $A = \begin{bmatrix} 1 & 2 & 5 \\ -11 & 14 & 5 \\ -3 & 2 & 9 \end{bmatrix}$ be a matrix and notice that $\vec{v}_1 = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ are eigenvectors for A. Diagonalize A.
>
> XXX Finish (eigenvalues are 4,8,12)

Non-diagonalizable Matrices

Is every matrix diagonalizable? Unfortunately the world is not that sweet. But, we have a tool to tell if a matrix is diagonalizable—checking to see if there is a basis of eigenvectors.

Example. Is the matrix $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ diagonalizable?

Computing, $\text{char}(R) = \lambda^2 + 1$ which has no real roots. Therefore, R has no real eigenvalues. Consequently, R has no real eigenvectors, and so R is not diagonalizable.[a]

[a] If we allow complex eigenvalues, then R *is* diagonalizable and is similar to the matrix $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. So, to be more precise, we might say R is not *real* diagonalizable.

Example. Is the matrix $D = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ diagonalizable?

For every vector $\vec{v} \in \mathbb{R}^2$, we have $D\vec{v} = 5\vec{v}$, and so every non-zero vector in $\mathbb{R}^2$ is an eigenvector for D. Thus, $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ is a basis of eigenvectors for $\mathbb{R}^2$, and so D is diagonalizable.[a]

[a] Of course, every square matrix is similar to itself and D is already diagonal, so of course it's diagonalizable.

Example. Is the matrix $J = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ diagonalizable?

XXX Finish

Example. Is the matrix $K = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}$ diagonalizable?

XXX Finish

Takeaway. Not all matrices are diagonalizable, but you can check if an $n \times n$ matrix is diagonalizable by determining whether there is a basis of eigenvectors for $\mathbb{R}^n$.

Geometric and Algebraic Multiplicities

When analyzing linear transformations or matrices, we're often interested in studying the subspaces where vectors are stretched by only one eigenvalue. These are called the *eigenspaces*.

Eigenspace. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.
The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

Now is the time when linear algebra and regular algebra (the solving of non-linear equations) combine. We know, every root of the characteristic polynomial of a matrix gives an eigenvalue for that matrix. Since the degree of the characteristic polynomial of an $n \times n$ matrix is always n, the fundamental theorem of algebra tells us exactly how many roots to expect.

Recall that the *multiplicity* of a root of a polynomial is the power of that root in the factored polynomial. So, for example $p(x) = (4-x)^3(5-x)$ has a root of 4 with multiplicity 3 and a root of 5 with multiplicity 1.

Example. Let $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of R.

XXX Finish

Example. Let $D = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of D.

XXX Finish

Example. Let $J = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of J.

XXX Finish

Example. Let $K = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix}$ and find the geometric and algebraic multiplicity of each eigenvalue of K.

XXX Finish

Consider the following two theorems.

Theorem. (Fundamental Theorem of Algebra) Let p be a polynomial of degree n. Then, if complex roots are allowed, the sum of the multiplicities of the roots of p is n.

Theorem. Let λ be an eigenvalue of the matrix A. Then

$$\text{geometric mult}(\lambda) \leq \text{algebraic mult}(\lambda).$$

We can now deduce the following.

Theorem. An $n \times n$ matrix A is diagonalizable if and only if the sum of its geometric multiplicities is equal to n. Further, provided complex eigenvalues are permitted, A is diagonalizable if and only if all its geometric multiplicities equal its algebraic multiplicities.

Proof. XXX Finish ■

Practice Problems

1 Example problem with parts

(a) The first part

(b) The second part

Solutions to the parts

(a) $(2, 4, 8)$

(b) $(15, 6, -21)$

79 Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are eigenvectors for A. Let T_A be the transformation induced by A.

79.1 Find the eigenvalues of T_A.

79.2 Find the characteristic polynomial of T_A.

79.3 Compute $T_A\vec{w}$ where $w = 2\vec{v}_1 - \vec{v}_2$.

79.4 Compute $T_A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

79.5 If $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is $\vec{x}$ written in the $\mathcal{V}$ basis, compute $T_A\vec{x}$ in the $\mathcal{V}$ basis.

80 The matrix P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation. Here, A, T_A, and $\mathcal{V}$ come from Problem 79.

80.1 Describe in words what P does.

80.2 Describe how you can use P and P^{-1} to compute $T_A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

80.3 Can you find a matrix D so that

$$PDP^{-1} = A?$$

80.4 $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $T_A^{100}\vec{x}$. Express your answer in both the $\mathcal{V}$ basis and the standard basis.

DEF **Diagonalizable**

A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

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81 Let B be an $n \times n$ matrix and let T_B be the induced transformation. Suppose T_B has eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ which form a basis for $\mathbb{R}^n$, and let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues.

81.1 How do the eigenvalues and eigenvectors of B and T_B relate?

81.2 Is B diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

81.3 What if one of the eigenvalues of T_B is zero? Would B be diagonalizable?

81.4 What if the eigenvectors of T_B did not form a basis for $\mathbb{R}^n$. Would B be diagonalizable?

Eigenspace

DEFINITION

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_m$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

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82 Let $F = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

82.1 Is F diagonalizable? Why or why not?

82.2 Is G diagonalizable? Why or why not?

82.3 What are the geometric and algebraic multiplicities of each eigenvalue of F? What about the multiplicities for each eigenvalue of G?

82.4 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?