Complex Numbers

A large part of mathematics is the process of abstraction. For instance, numbers are an abstraction of the concept of length. If we had two numbers a, b representing length, their product would be the area of the rectangle whose sides were a, b. Strictly speaking, with this interpretation, ab is an area and a, b are lengths and they cannot be compared (this is what the Greeks thought), but we have come to know numbers as a more abstract concept. After all, we can write both a, b and ab using decimals, so why shouldn't we compare them?

Taking this idea to solving polynomials gives us complex numbers. The solutions to $x^2 = a$ are $\pm\sqrt{a}$ formally, but if I said $a = -4$, we might claim that $\sqrt{-4}$ doesn't make any sense. However, with a simple definition, we can make a larger class of numbers where $\sqrt{-4}$ fits in perfectly.

Definition 1 *The imaginary unit i is defined as a solution to $x^2 = -1$. That is $i^2 = -1$.*

Definition 2 *A* complex number *is a number $a + bi$ where a is the* real part *and b is the* imaginary part.

You can think of complex numbers as vectors with two components, a real one and an imaginary one. You add complex numbers as you would vectors.

Compute the following:

1.1 $(1 + 2i)+(2 - 7i)$

1.2 $i - (6 + i)$

1.3 $24(3 + i) - \frac{1}{2}(2 + 2i)$

Unlike vectors however, complex numbers can be multiplied. Multiplication follows all the same rules as for real numbers, except $(i)(i) = -1$. So, for example, $i(2 - 3i) = 2i - 3i^2 = 3 + 2i$.

Compute the following products:

2.1 $i(7i)$

2.2 $(3 + i)(2 + 4i)$

2.3 $(7 + i)(7 - i)$

Definition 3 *The* conjugate *of a complex number $a + bi$ is written with a bar over top the number and is*

$$\overline{a + bi} = a - bi.$$

If c is a complex number then $c\bar{c}$ is always a real number. This means that while $1/c$ has complex numbers in the denominator of a fraction, $1/c = \bar{c}/(c\bar{c})$ does not. Now you know how to divide by a complex number!

Compute the following:

3.1 $\overline{2 - 7i}$

3.2 $\bar{i}$

3.3 $1/(2 - 7i)$

3.4 $(4 + 3i)/(-5 + i)$

Recall that for the square-root function $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Thus for example $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$. Use this and your knowledge of the quadratic formula to solve the following:

4.1 $x^2 + 3x + 9 = 0$

4.2 $-x^2 - 2 = 1$

4.3 $-x^2 + 5x - 12 = 7$

The amazing thing about complex numbers is that just by introducing the solution to the equation $x^2 = -1$ we can now write the solutions to any n-degree polynomial as complex numbers!

Theorem 1 *If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then we can factor p as*

$$p(x) = (x - c_1)(x - c_2)\cdots(x - c_n)$$

where $c_1, \ldots, c_n$ are complex numbers.

Theorem 1 will be important as we start solving equations involving determinants, but you can just think of Theorem 1 as telling us that we always have n solutions to an n-degree polynomial (though the solutions may not be all distinct).

Complex Numbers and Matrices

Using complex numbers as entries in a matrix allows us to represent things we couldn't before.

Compute the following:

5.1 $\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$

5.2 $\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

5.3 $\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

5.4 Compute $i \begin{bmatrix} i \\ 1 \end{bmatrix}$ and compare it with $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix}$. What do you notice? (How does this relate to eigenvalues and eigenvectors?)

Geometry of Complex Numbers

When we think of complex numbers as vectors with two components, we say they are vectors in the *complex plane*. The complex plane has a *real* axis and an *imaginary* axis.

Being vectors in a plane, we can represent a complex number in terms of a pair of rectangular coordinates or as a pair of polar coordinates.

Definition 4 *The* rectangular *coordinates of a complex number c are (a, b) where $c = a + bi$. The* rectangular form *of c is $a + bi$.*

Definition 5 *The* polar *coordinates of a complex number c are (r, θ) where $c = r(\cos\theta + i\sin\theta)$. The* polar form *(sometimes called the exponential form) of c is $re^{i\theta}$.*

For the moment, look past the funny use of e. The complex number $a = \cos\theta + i\sin\theta$ is the rectangular form of a unit vector in the complex plane that forms an angle θ with the real axis. (Draw a picture if this doesn't make sense.)

Let $a = \cos\theta + i\sin\theta$

6.1 Compute $a\bar{a}$ and $|a|$.

6.2 Compute $|16a|$.

Let $c = 1 + \sqrt{3}i$.

6.3 Find the angle c makes with the real axis.

6.4 Find r and θ so $c = r(\cos\theta + i\sin\theta)$.

The exponential in polar form comes from the following incredible identity.

Definition 6 Euler's formula *(sometimes called Euler's identity) states*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus, $r(\cos\theta + i\sin\theta) = re^{i\theta}$ is an application of Euler's formula.

7.1 Write $1 + \sqrt{3}i$ in polar form.

7.2 Write $-2i$ in polar form.

7.3 Write $\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$ in polar form.

7.4 Write $e^{-\frac{4}{3}\pi i}$ in rectangular form.

7.5 Write $-2e^{-\frac{\pi}{2}i}$ in rectangular form.

Polar form and rectangular form each have their advantages. Rectangular form makes adding complex numbers easy but multiplying them hard. Polar form makes adding hard but multiplying easy.

8.1 Compute xy where $x = 2e^{\frac{3}{4}\pi i}$ and $y = 3e^{\frac{\pi}{2}i}$.

8.2 Let $x = 2e^{\frac{3}{4}\pi i}$ and compute $\bar{x}$ in polar form. What do you notice about the exponent?

When you multiply complex number in polar form, you see their angles add and their lengths multiply. This means that multiplication by a complex number is a rotation combined with a scaling (and you can bet this is a linear transformation!).

9.1 Compute z^7 where $z = \frac{1}{2}e^{\frac{2}{3}\pi i}$.

Finally, we observe that polar form also makes it easy to find roots. For example $\sqrt{z} = (re^{i\theta})^{1/2} = r^{1/2}e^{i\theta/2}$. However, we are by now quite familiar with the fact that $z^2 = k$ has more solutions than just $\sqrt{k}$ (in this case it also has $-\sqrt{k}$). Polar form helps us find them all!

10.1 If $z = re^{i\theta}$, find all r and all θ such that $z^2 = 4e^{\pi i}$ (you need to use Euler's formula and the periodicity of sin and cos to figure this out).

10.2 Find all unique solutions to $z^2 = 4e^{\pi i}$. Write them in rectangular form (there should only be two).

10.3 Find all solutions to $z^3 = 1$ (Hint, $1 = e^{0i}$).

10.4 Compute all solutions to $z^4 = i$.