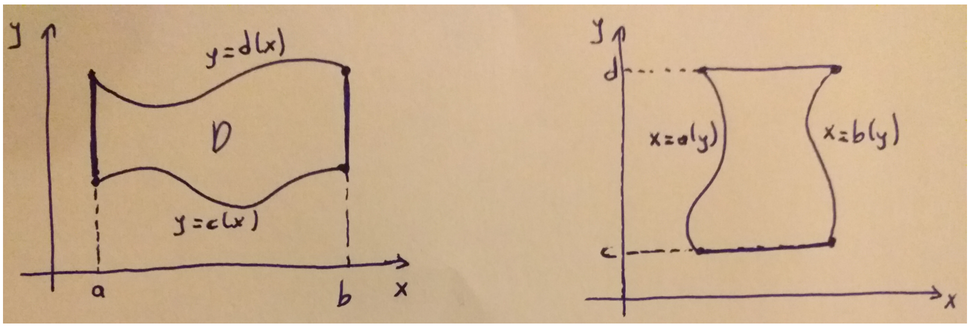
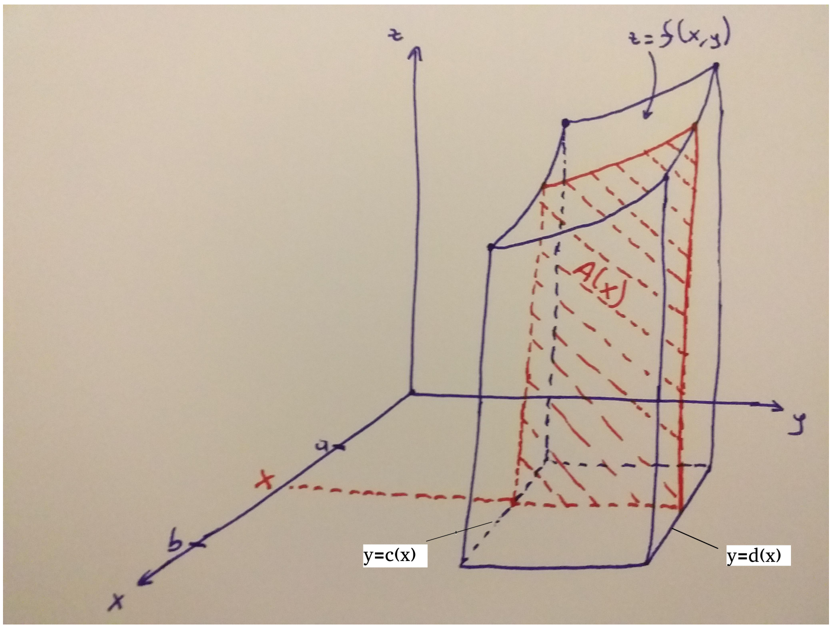
§14.2 Iteration of Double Integrals in Cartesian Coordinates

We will see evaluation of double integrals by iteration of single definite integrals. The difficulty of evaluation of double integrals depend on the function as well as the domain itself. We start with *simple* domains figured below.



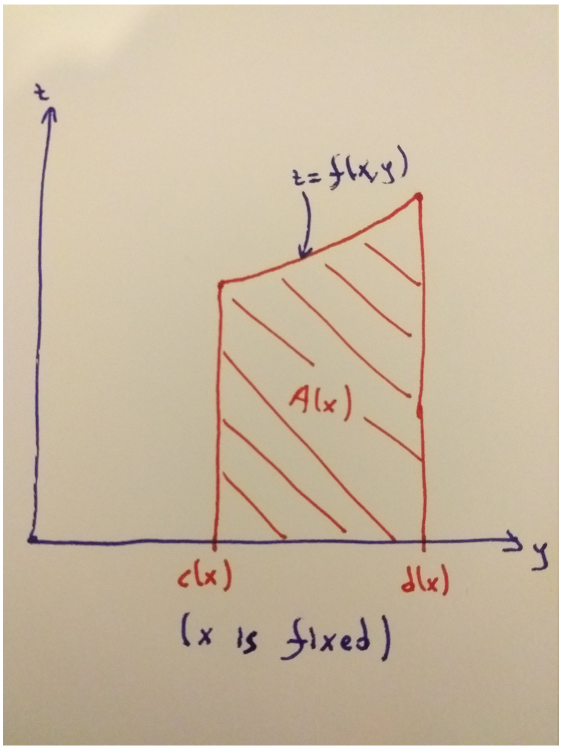
$a \leq x \leq b$, $c(x) \leq y \leq d(x)$ $\qquad$ $c \leq y \leq d$, $a(y) \leq x \leq b(y)$

If we have a simple domain then we can express the double integral as an *iterated* integral. Recall computation of volume of solids by slicing.



The volume of the solid under $z = f(x, y)$ above D is equal to $\int_a^b A(x)\,dx$ where $A(x)$ is the area of the section at x.

Now we consider a fixed x and express $A(x)$ as an integral with respect to y. Imagine that this section lies on the yz-plane, i.e. project this section onto the yz-plane, so the area $A(x)$ remains the same. The projection of this section onto the yz-plane is the region bounded by the y-axis, the vertical (!) lines



$y = c(x)$, $y = d(x)$ and the curve

$z = f(x, y)$ (Recall that we fixed x). So we must have that

$$A(x) = \int_{c(x)}^{d(x)} f(x, y)\, dy$$

(A single definite integral !). Combining these integrals, we obtain

$$\iint_D f(x, y)\, dA = \text{Volume} = \int_a^b A(x) dx = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\, dy \right) dx.$$

Hence we expressed the double integral as iterated single integrals. We shall omit the parantheses and write the above integral as

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx.$$

We may extend to not necessarily positive functions and to the other type of simple domains in the next theorem.

Theorem 1. i) Let $f(x, y)$ be a continous function on the domain D given by $a \leq x \leq b$, $c(x) \leq y \leq d(x)$. Then

$$\iint\limits_D f(x,y)\,dA = \int\limits_a^b \int\limits_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

ii) Similarly let $g(x,y)$ be a continous function on the domain D given by $c \leq y \leq d$, $a(y) \leq x \leq b(y)$. Then

$$\iint\limits_D g(x,y)\,dA = \int\limits_c^d \int\limits_{a(y)}^{b(y)} g(x,y)\,dx\,dy$$

Note that the evaluation order is as from *the inner integral* to *the outer integral*

Example 1: Evaluate $\iint\limits_D (1-x+y)\,dA$ where D is the rectangle $-1 \leq x \leq 2$, $2 \leq y \leq 4$.

Solution: The domian is a rectangle, so it's a simple domain of both types. So we can write

$$\iint\limits_D (1-x+y)\,dA = \int\limits_{-1}^{2} \int\limits_{2}^{4} (1-x+y)\,dy\,dx$$

(!) Note that we first take integral with respect to y, so the limits of the inner integral are the endpoints of y. Now we have that

$$\int_{-1}^{2}\int_{2}^{4}(1-x+y)\,dy\,dx=\int_{-1}^{2}\left[(y-xy+y^2/2)\Big|_{y=2}^{4}\right]\,dx$$

$$=\int_{-1}^{2}\left((12-4x)-(4-2x)\right)\,dx=\int_{-1}^{2}(8-2x)\,dx=21$$

In the first equality above we evaluated the integral of $1-x+y$ with respect to y. Since x and y are independent, the integral (or antiderivative) of x turns out to be xy.

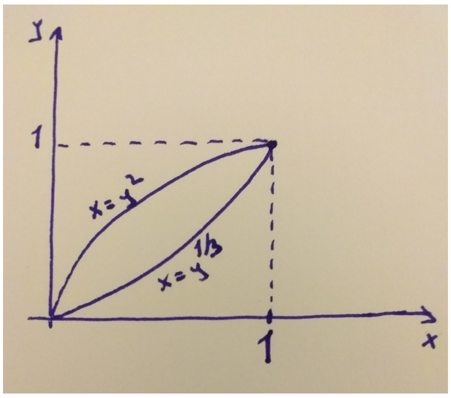
We could also swap dx and dy and write the double integral as

$$\iint_D(1-x+y)\,dA=\int_{2}^{4}\int_{-1}^{2}(1-x+y)\,dx\,dy.$$

But we shall not forget to also adopt the limits of the single definite integrals. As an exercise compute the integral in the order $dx\,dy$ (Answer must be 21).

Example 2: Find the volume of the solid under the graph of $z=3x^2y$ and above the region D on the xy-plane bounded by the curves $x=y^2$ and $x=y^{1/3}$.

Solution: The region D can be given by $0 \le y \le 1$, $y^2 \le x \le y^{1/3}$. So the volume of the prescribed solid is



$$\iint_D 3x^2y\,dA = \int_0^1 \int_{y^2}^{y^{1/3}} 3x^2y\,dx\,dy$$

$$= \int_0^1 (x^3y)\Big|_{x=y^2}^{y^{1/3}}\,dy$$

$$= \int_0^1 (y^2 - y^7)\,dy = 5/24$$

Observe that D can also be given as

$0 \le x \le 1$, $x^3 \le y \le \sqrt{x}$, and so we can evaluate the volume as

$$\iint_D 3x^2y\,dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 3x^2y\,dy\,dx = \ldots \text{ Exercise } \ldots = 5/24.$$

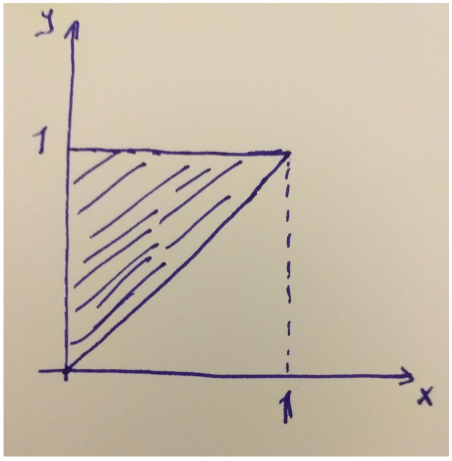
Remark: *Note that if change the order of integration,i.e. if we swap dx and dy then we can not simply swap the limits of integration directly, i.e.* $\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx \neq \int_{c(x)}^{d(x)} \int_a^b f(x,y)\,dx\,dy$ *if $c(x)$ and $d(x)$ are non-constant.*

In the above examples it is a 'matter of taste' to evaluate the integrals in the order $dx\,dy$ or $dy\,dx$. But in some cases we are forced to use a fixed integration order.

Example 3: Evaluate the interated integral $\int\limits_0^1 \int\limits_x^1 e^{-y^2}\,dy\,dx$.

Solution: Observe that the inner integral $\int\limits_x^1 e^{-y^2}\,dy$ can not be evaluated. We need to write the integral in the order $dx\,dy$, so firstly we need to skecth the region.

The domain of integration is given by $D : 0 \leq x \leq 1, x \leq y \leq 1$, so it the triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$. As we want to evaluate the integral in the order $dx\,dy$ we need to write the domain in the form $c \leq y \leq d$, $a(y) \leq x \leq b(y)$. First observe that y can take any value between 0 and 1, so $0 \leq y \leq 1$. For the bounds of x, consider the *barriers* as our points move to right (or left), i.e. in the positive (or negative) x-direction. The left (respectively right) barrier for all points is the y-axis (respectively the line $y = x$), so $0 \leq x \leq y$. So our region is

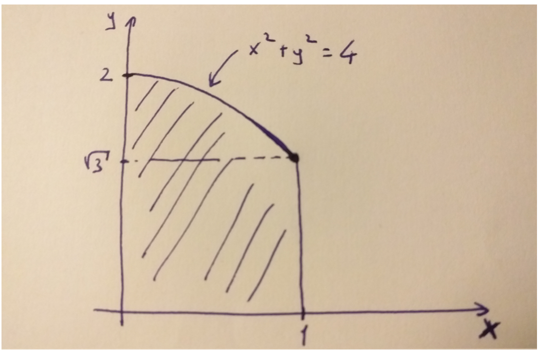


given by $0 \leq y \leq 1, 0 \leq x \leq y$. Then we have that

$$\int_0^1 \int_x^1 e^{-y^2}\,dy\,dx = \int_0^1 \int_0^y e^{-y^2}\,dx\,dy = \int_0^1 \left(xe^{-y^2}\right)\Big|_{x=0}^{y} dy$$
$$= \int_0^1 ye^{-y^2}\,dy = \frac{1}{2}\left(1 - \frac{1}{e}\right)$$

(In the last step use the substitution $u = y^2$).

Example 4: Express $\iint_D f(x,y)\,dA$ as interated integral(s) in the order **a)** $dy\,dx$ **b)** $dx\,dy$ where D be the region figured below.



a) The limits of x are apperantly 0 and 1. Also the lower (respectively upper) limit of y coordinates of the points in the region is the x-axis (respectively) the arc $x^2 + y^2 = 4$. So

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

b) The limits of y are 0 and 2. Also the left *barrier* of the points in our region is the y-axis. But the right barrier may be the arc or the line $x = 1$. So we need to divide our region into two regions corresponding to the rectangle and disc. So

$$\iint_D f(x,y)\,dA = \int_0^{\sqrt{3}} \int_0^1 f(x,y)\,dx\,dy + \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$