10.1) Analytic Geometry in 3 Dimensions

$\boxed{4}$ Find the distance between points $(3,8,-1)$ and $(-2,3,-6)$:

* Distance between $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$:

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_1 - z_2)}$$

$$\Rightarrow d = \sqrt{(3-(-2))^2 + (8-3)^2 + (-1-(-6))^2}$$

$$= \sqrt{25 + 25 + 25}$$

$$= 5\sqrt{3}$$

$\boxed{18-29}$ Describe the set of points in $\mathbb{R}^3$ that satisfy the given equation or inequality

Q18) $x^2 + y^2 + z^2 = 2z$:

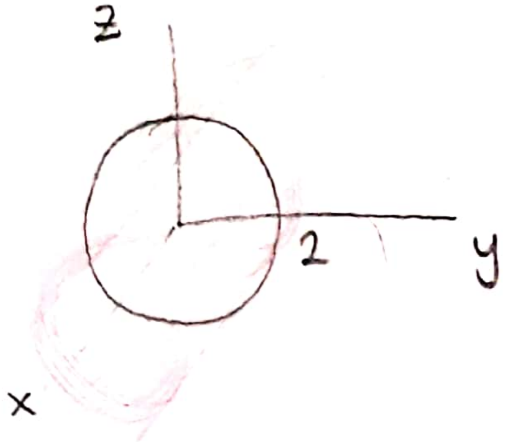
We can write this equation as:

$$x^2 + y^2 + z^2 - 2z = 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 - 2z + 1 = 1$$

$$\Rightarrow x^2 + y^2 + (z-1)^2 = 1$$

This equation gives the sphere with center $(0,0,1)$ and radius 1.

Q19) $y^2+z^2 \leq 4$:



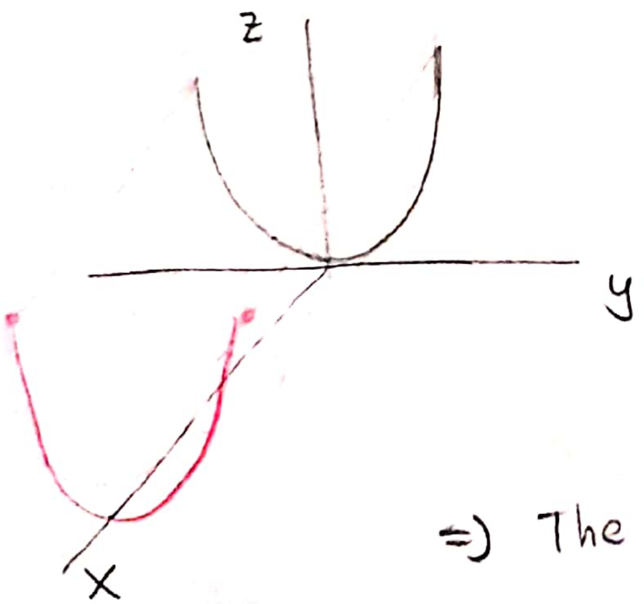
In $\mathbb{R}^2$ (with axes y and z) $y^2+z^2 \leq 4$ gives a disk with center (0,0) and radius 2.

In $\mathbb{R}^3$, this equations means that x takes any value. So; we can consider all disks with radius 2 along the x-axis.

$\Rightarrow$ This inequality gives the circular cylinder of radius 2 with central axis along the x-axis.

a solid cylinder, not just the surface of the cylinder

Q21) $z = y^2$



In $\mathbb{R}^2$ (with axes y and z) $z=y^2$ gives a parabola.

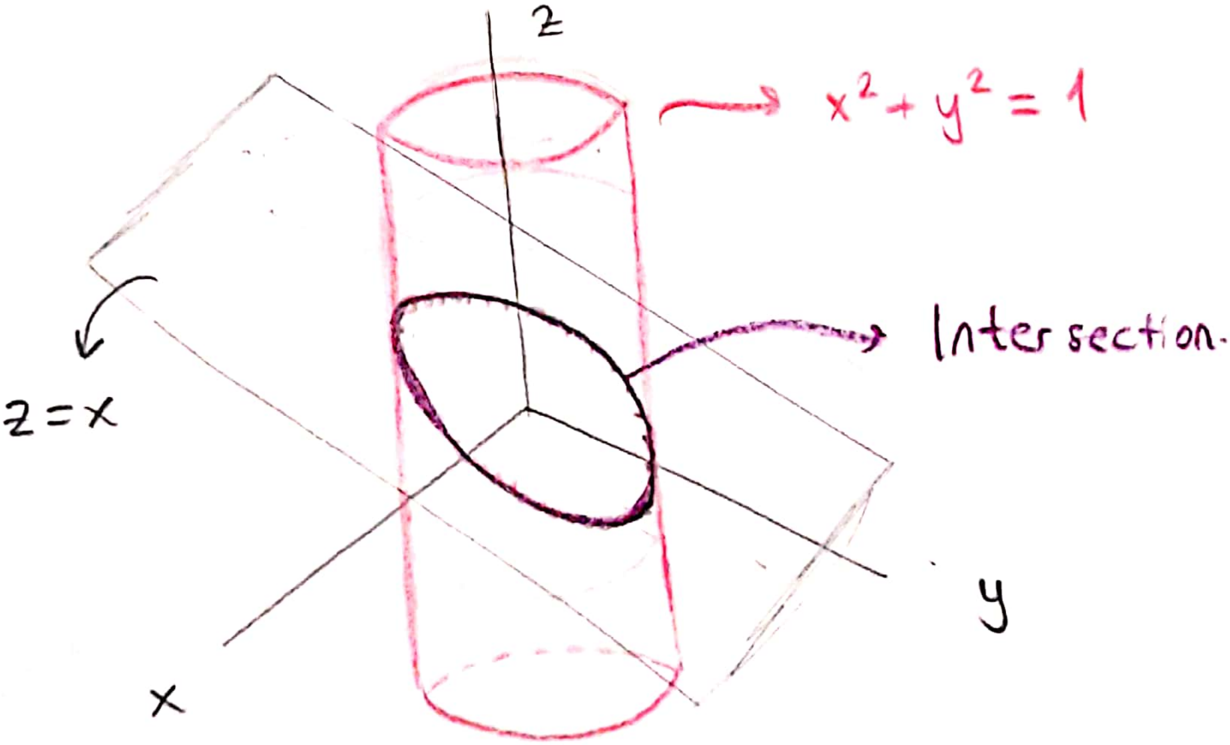
In $\mathbb{R}^3$, consider all parabolas along x.

$\Rightarrow$ The equation gives a parabolic cylinder

$\boxed{29}$ Describe the set of points in $\mathbb{R}^3$ that satisfy the given pair of equations:
$$\begin{cases} x^2+y^2=1 \\ z=x \end{cases}$$

$x^2+y^2=1$: Circular cylinder of radius 1 with central axis along the z-axis.

$z=x$: Slanted plane.



The pair of equations gives an ellipse.

10.2) Vectors

$\boxed{3}$ Calculate the following for the vectors

$u = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $v = 3\hat{i} - 4\hat{j} - 5\hat{k}$

(a) $u + v = (3\hat{i} + 4\hat{j} - 5\hat{k}) + (3\hat{i} - 4\hat{j} - 5\hat{k}) = 6\hat{i} - 10\hat{k}$

$u - v = (3\hat{i} + 4\hat{j} - 5\hat{k}) - (3\hat{i} - 4\hat{j} - 5\hat{k}) = 8\hat{j}$

$$
\begin{aligned}
2u - 3v &= 2(3\hat{i} + 4\hat{j} - 5\hat{k}) - 3(3\hat{i} - 4\hat{j} - 5\hat{k}) \\
&= (6\hat{i} + 8\hat{j} - 10\hat{k}) + (-9\hat{i} + 12\hat{j} + 15\hat{k}) \\
&= -3\hat{i} + 20\hat{j} + 5\hat{k}
\end{aligned}
$$

(b) the lengths $|u|$ and $|v|$.

Note that if $u = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$, then the length of u is $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

$$\Rightarrow \quad |u| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$$

$$|v| = \sqrt{3^2 + (-4)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$$

(c) unit vectors $\hat{u}$ and $\hat{v}$ in the direction of u and v, respectively.

Consider $\frac{u}{|u|}$. Since $|u|$ is a scalar, $\frac{u}{|u|}$ is in the direction of u. Furthermore, the length of $\frac{u}{|u|}$ is 1. So,

$$\hat{u} = \frac{u}{|u|} = \frac{1}{5\sqrt{2}}(3\hat{i}+4\hat{j}-5\hat{k})$$

$$\hat{v} = \frac{v}{|v|} = \frac{1}{5\sqrt{2}}(3\hat{i}-4\hat{j}-5\hat{k})$$

(d) the dot product $u \cdot v$:

$u = u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$, $v = v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

$\Rightarrow$ dot product of u & v : $u \cdot v = u_1v_1+u_2v_2+u_3v_3$

So, for given u and v,

$$u \cdot v = (3\hat{i}+4\hat{j}-5\hat{k}) \cdot (3\hat{i}-4\hat{j}-5\hat{k})$$

$$= 3 \cdot 3 + 4(-4) + (-5)(-5)$$

$$= 18$$

(e) the angle between u and v:

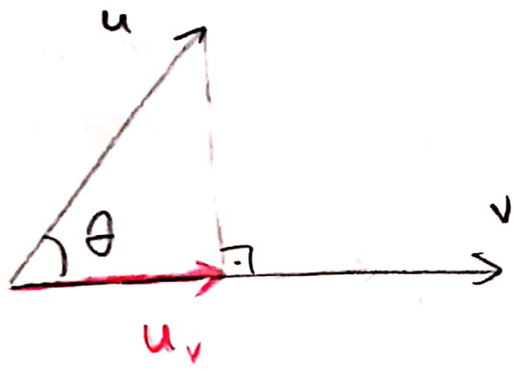
Note that $u \cdot v = |u||v| \cos\theta$ where $\theta \subset [0, \pi]$

is the angle between u and v.

$\Rightarrow \quad 18 = (5\sqrt{2})(5\sqrt{2}) \cos\theta$

$\Rightarrow \quad \cos\theta = \frac{18}{50} = \frac{9}{25} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{9}{25}\right)$

⊛ Scalar and Vector Projections



We obtain a vector u_v if we project u along v.

u_v is the vector projection of u along v.

The scalar projection s of a vector u in the direction of v:

$$s = u \cdot \hat{v} = \frac{u \cdot v}{|v|} = |u| \cos\theta$$

The vector projection u_v of a vector u in the direction of v:

$$u_v = \frac{u \cdot v}{|v|} \hat{v} = \frac{u \cdot v}{|v|^2} v = s \frac{v}{|v|}$$

(f) the scalar projection of u in the direction of v:

$$s_1 = \frac{u \cdot v}{|v|} = \frac{18}{5\sqrt{2}}$$

(g) the vector projection of v along u:

$$s_2 = \frac{u \cdot v}{|u|} = \frac{18}{5\sqrt{2}}$$ (scalar projection of v in the direction of u)

$$v_u = \frac{u \cdot v}{|u|} \hat{u} = \frac{18}{5\sqrt{2}} \left(\frac{1}{5\sqrt{2}} (3\hat{i} + 4\hat{j} - 5\hat{k}) \right)$$

$$= \frac{9}{25} (3\hat{i} + 4\hat{j} - 5\hat{k})$$ (vector projection of v along u)

$\boxed{13}$ For what value of t is the vector $2t\,\hat{i} + 4\hat{j} - (10+t)\hat{k}$ perpendicular to the vector $\hat{i} + t\hat{j} + \hat{k}$?

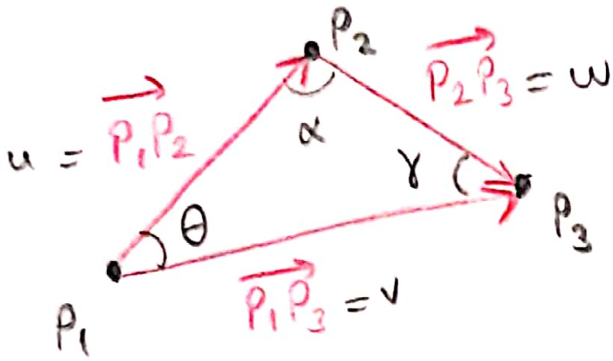
u and v are perpendicular (the angle between u & v is $\frac{\pi}{2}$) $\iff u \cdot v = 0$

$\Rightarrow (2t\hat{i} + 4\hat{j} - (10+t)\hat{k}) \cdot (\hat{i} + t\hat{j} + \hat{k}) = 0$

$\Rightarrow 2t + 4t - (10+t) = 0 \Rightarrow 5t - 10 = 0 \Rightarrow t = 2$

perpendicular if $t = 2$.

18 Find the three angles of the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,3)$:



$P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$

$$\Rightarrow \overrightarrow{P_1P_2} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

So, $u = \overrightarrow{P_1P_2} = -\hat{\imath} + 2\hat{\jmath} \Rightarrow |u| = \sqrt{5}$

$v = \overrightarrow{P_1P_3} = -\hat{\imath} + 3\hat{k} \Rightarrow |v| = \sqrt{10}$

$w = \overrightarrow{P_2P_3} = -2\hat{\jmath} + 3\hat{k} \Rightarrow |w| = \sqrt{13}$

$$\circledast \quad u \cdot v = |u|\,|v| \cos\theta \Rightarrow \cos\theta = \frac{u \cdot v}{|u|\,|v|}$$

$$\Rightarrow \cos\theta = \frac{u \cdot v}{|u||v|} = \frac{1}{\sqrt{5}\sqrt{10}} = \frac{1}{5\sqrt{2}}$$

$$\cos\alpha = \frac{-u \cdot w}{|u||w|} = \frac{(\hat{\imath} - 2\hat{\jmath}) \cdot (-2\hat{\jmath} + 3\hat{k})}{\sqrt{5}\ \sqrt{13}} = \frac{4}{\sqrt{65}}$$

$$\cos\gamma = \frac{-v \cdot (-w)}{|v|\,|w|} = \frac{9}{\sqrt{10}\sqrt{13}} = \frac{9}{\sqrt{130}}$$

So, $\theta = \cos^{-1}\left(\frac{1}{5\sqrt{2}}\right)$

$\alpha = \cos^{-1}\left(\frac{4}{\sqrt{165}}\right)$

$\gamma = \cos^{-1}\left(\frac{9}{\sqrt{130}}\right)$

Note on Projections

If the angle between u and v is less than pi/2, then scalar projection of u gives the length of the vector projection of u and means that the vector projection of u is in the same direction with v.

If the angle between u and v is greater than pi/2, then the scalar projection of u gives the negative of the length of the vector projection of u. It means that the vector projection of u is in the opposite direction of v