§12.1. Functions of Several Variables

The notation $y = f(x)$ is used to indicate that the variable y depends on the single real variable x, that is, that y is a function of x. This chapter we will study on functions of more than one variable.

For example, the volume of a circular cylinder of radius r and height h is given by $V = \pi r^2 h$; we say that V is a function of two variables r and h and we write it $V = f(r, h)$ where $r \geq 0$ and $h \geq 0$. Thus, f is a function of two variables having as domain the set of points in the rh-plane with coordinates (r, h) satisfying $r \geq 0$ and $h \geq 0$. In general we define a function of n variables as follows:

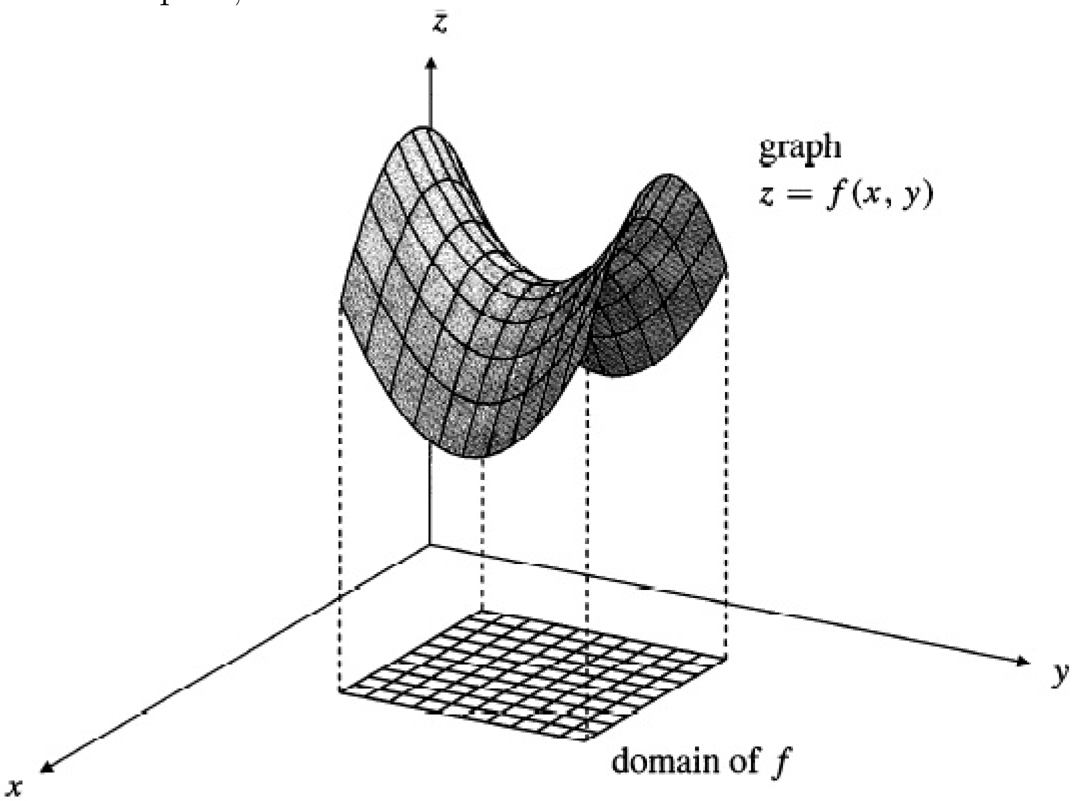
Definition. A function f of n real variables is a rule that assigns a unique real number $f(x_1, x_2, ..., x_n)$ to each $(x_1, x_2, ..., x_n)$ in some subset $\mathcal{D}(f)$ of $\mathbb{R}^n$. $\mathcal{D}(f)$ is called the **domain** of f. The set of real numbers $f(x_1, x_2, ..., x_n)$ obtained from points in the domain is called the **range** of f.

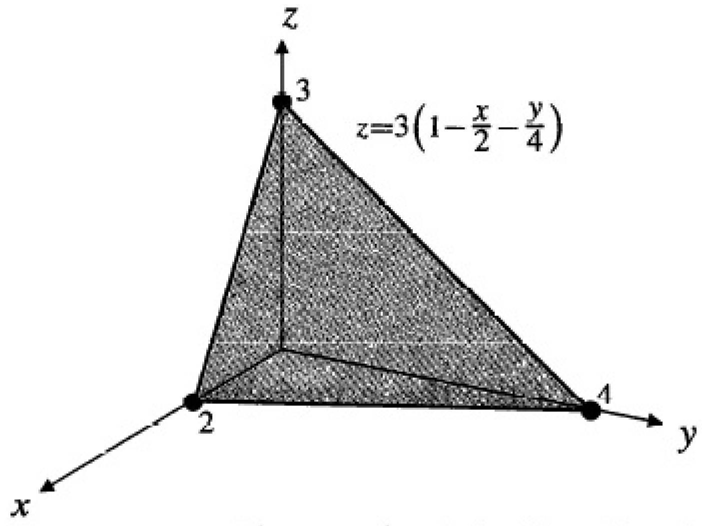
We will generally deal with functions of two or three independent variables. When a function f depends on two variables, we will usually call these independent variables x and y, and we will use z to denote the dependent variable that represents the value of the function; that is, $z = f(x, y)$.

Graphs

The graph of a function of two variables is the set of all points in 3-space having coordinates $(x, y, f(x, y))$, where (x, y) belongs to domain of $f(x, y)$. This graph is a surface in $\mathbb{R}^3$ lyings above (if $f(x, y) > 0$) or below (if $f(x, y) < 0$) the domain of f in the xy-plane.

The graph of a function of three variables is a three-dimensional hyperspace in 4-space,$\mathbb{R}^4$.

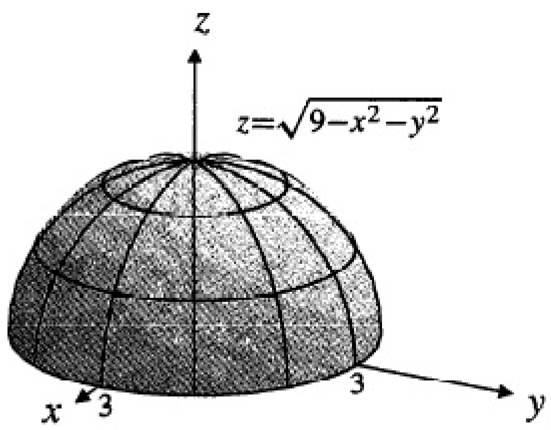




Example 1.

Consider the function $z = f(x, y) = 3(1 - \frac{x}{2} - \frac{y}{4})$ for $0 \le x \le 2$ and $0 \le y \le 4 - 2x$.

The graph of f is the plane triangular surface with vertices $(2, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$. If the domain of f had not been explicitly stated to be a particular set in xy-plane, the graph would have been the whole plane through these three points!



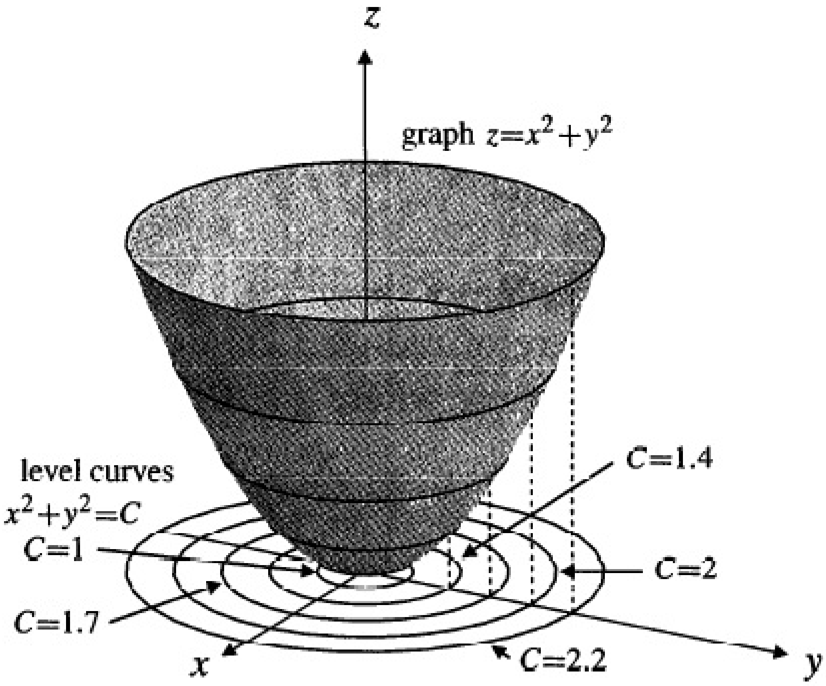
Example 2.

Consider $f(x, y) = \sqrt{9 - x^2 - y^2}$. The expression under the square root cannot be negative, so the domain is the disk $x^2 + y^2 \leq 9$.

If we square the equation $z = \sqrt{9 - x^2 - y^2}$, we can rewrite the result in the form $x^2+y^2+z^2 = 9$ which is a sphere of radius 3 centered at the origin. But the graph of f is only the upper hemisphere where $z \geq 0$.

Level Curves

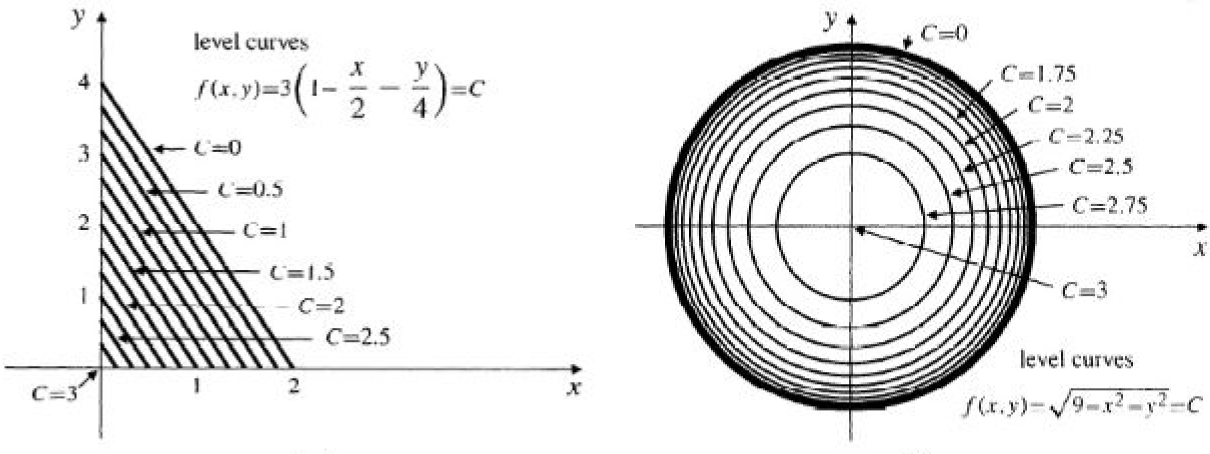
Another way to represent the function $f(x, y)$ graphically is to produce a two dimensional topographic map of the surface $z = f(x, y)$. In the xy-plane we sketch the curves $f(x, y) = C$ for various of the constant C. These curves are called level curves of f because they are the vertical projections onto the xy-plane of the curves in which the graph $z = f(x, y)$ intersects the horizontal (level) planes $z = C$.



The graph of $f(x, y) = x^2 + y^2$
and some level curves of f

Example 3. a) The level curves of the function $f(x, y) = 3(1 - \frac{x}{2} - \frac{y}{4})$ of the Example 1 are the segments of the straight lines $3(1 - \frac{x}{2} - \frac{y}{4}) = C$ or $\frac{x}{2} + \frac{y}{4} = 1 - \frac{C}{3}$ where $0 \leq C \leq 3$, which lie in the first quadrant.
b) The level curves of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$ of the Example 2 are the concentric circles. $\sqrt{9 - x^2 - y^2} = C$ or $x^2 + y^2 =$

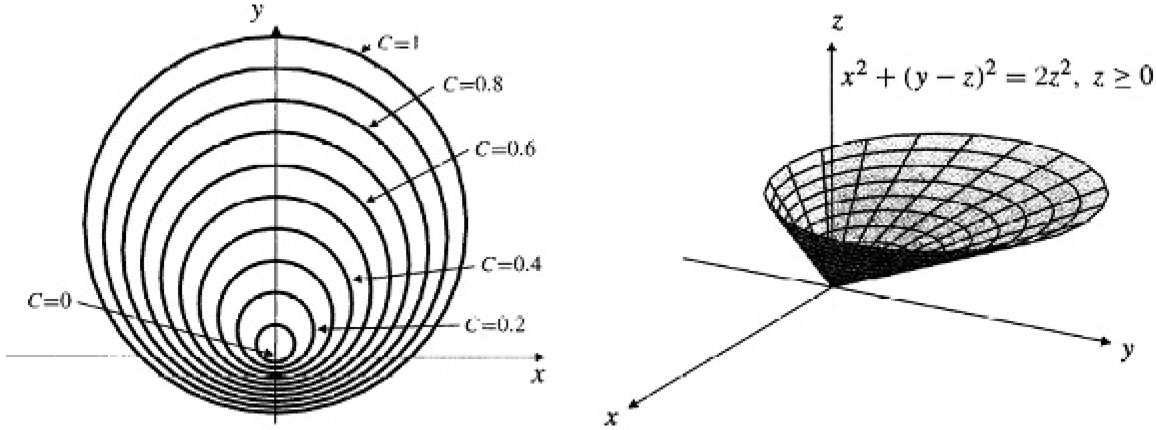
$9 - C^2$ where $0 \leq C \leq 3$.



(a) (b)

A function determines its level curves with any given spacing between consecutive values of C. However, level curves only determine the function if **all of them** are known.

Example 4. Describe and sketch the graph and some level curves of the function $z = g(x, y)$ defined by $z \geq 0$, and $x^2 + (y - z)^2 = 2z^2$. The level curve $z = g(x, y) = C$ where C is a positive constant, has equation $x^2 + (y - C)^2 = 2C^2$ and is, therefore, a circle of radius $\sqrt{2}C$ centered at $(0, C)$. The surface $z = g(x, y)$ is an oblique cone.



(a) (b)