GTU, Fall 2020, MATH 101

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* The numbers $a_n, a_{n-1}, ..., a_1, a_0$ are called the *coefficients* of $P(x)$.
* If $a_n \neq 0$, the degree of $P(x)$ is said to be n and we simply write $\deg(P) = n$.

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* A polynomial of degree one is also called a linear function. A polynomial of degree two is called a quadratic polynomial.

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- (!) But $f(x)$ is not equal to the polynomial $x + 1$ (Why?).

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- To multiply $A(x)$ and $B(x)$, use the rule $x^i x^j = x^{i+j}$ and distribution rule.
- $A(x)B(x) = (3x^3 - x + 5)(-x^2 + x + 4) =$ $3x^3(-x^2+x+4) - x(-x^2+x+4) + 5(-x^2+x+4) = (-3x^5+3x^4+12x^3) - (-x^3 + x^2 + 4x) + (-5x^2 + 5x + 20) = -3x^5 + 3x^4 + 13x^3 - 6x^2x + 20$

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* Exercise: Compute $A(x)B(x)$ in an easier way (Try to figure out how you can obtain x^5, x^4, ..., x and the constant term in this product.

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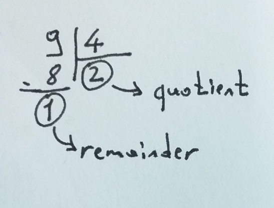

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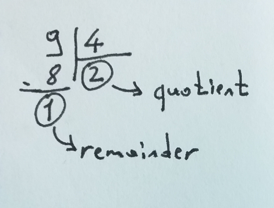

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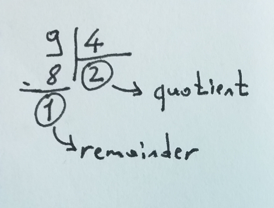

If you divide 9 by 4 then the quotient and remainder are found as 2 and 1 respectively, and there is not any other choice. Even in this simple fact you used an important theorem of mathematics ☺: For any natural numbers A and B, there are unique natural numbers q (quotient) and r (remainder) such that

$$A = Bq + r \text{ and } 0 \leq r < B$$

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* To find the quotient and remainder we use the usual division.

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\end{array}
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to eliminate $4x^3$

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to eliminate $-x^2$

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* $\deg(8x) < \deg(x^2 - 2)$, so the division is completed.

$\Longrightarrow$

$$4x^3 - x^2 + 2 = (x^2 - 2)(4x - 1) + 8x \quad \text{or} \quad \frac{4x^3 - x^2 + 2}{x^2 - 2} = 4x - 1 + \frac{8x}{x^2 - 2}$$

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* Example: The polynomial $P(x) = x^2 + 1$ has no real root (The complex roots of $P(x)$ are i and $-i$).

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* Also the roots of $P(x)$ are $x = 2$ and $x = 3$.

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  So $P(x) = Q(x)(x - r) + c$, evaluate both side at $x = r$

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Preliminaries, Roots and Factors of Polynomials

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  Exercise: Complete the proof.

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* Observe that the multiplicities are just the powers of the corresponding linear factors x and $(x-1)$.

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Preliminaries, Linear Quadratic Polynomials

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* Degree two polynomials are also called quadratic polynomials.
* A quadratic polynomial can be written as $P(x) = ax^2 + bx + c$, $a \neq 0$. The graph of such a polynomial is a parabola.

* We can factorize quadratic polynomials easily:

$$\begin{aligned}
\frac{P(x)}{a} &= x^2 + \frac{bx}{a} + \frac{c}{a} = x^2 + \frac{bx}{a} + \frac{c}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} \\
&= \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) \\
&= \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)
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* Put $\Delta = b^2 - 4ac$, so we have $P(x) = a\left(x + \frac{b + \sqrt{\Delta}}{2a}\right)\left(x + \frac{b - \sqrt{\Delta}}{2a}\right)$

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* Equivalently the roots of $P(x)$ are $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$

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* Exercise: Factorize $x^4 + 3x^2 - 4$ (Hint: First put $x^2 = t$ and factorize $t^2 + 3t - 4$).