GTU, Fall 2020, MATH 101

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* if f is not continuous then f is not differentiable

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since g is continuous.

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$$f'(a) = 1 - \frac{1}{a^2} = -3 \implies -\frac{1}{a^2} = -4 \implies a = \pm\frac{1}{2}$$

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