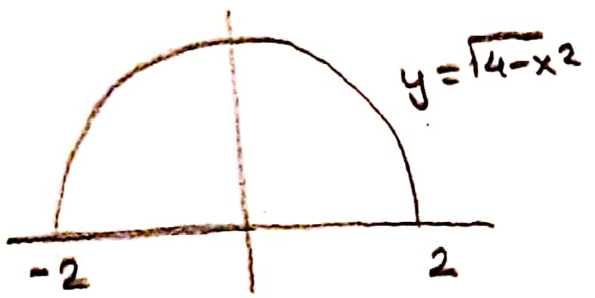
# 14.1) Double Integrals

$\boxed{13-14}$ Evaluate the given double integral by inspection

Q13) $\iint_R dA$, where $R$ is the rectangle $-1 \leq x \leq 3$, $-4 \leq y \leq 1$

$$\iint_R dA = \text{the area of } R = [3-(-1)][1-(-4)] = 20$$

Q14) $\iint_D (x+3)\,dA$, where $D$ is the half disk $0 \leq y \leq \sqrt{4-x^2}$




Notice that the half disk $D$ is symmetric about $x=0$

$$\iint_D (x+3)\,dA = \underbrace{\iint_D x\,dA}_{I_1} + \underbrace{\iint_D 3\,dA}_{I_2}$$

$I_1 = \iint_D x\,dA = 0$ since $x$ is an odd function and $D$ is symmetric about $x=0$

$$I_2 = \iint_D 3\,dA = 3\iint_D dA = 3(\text{area of } D) = 3 \cdot \frac{1}{2}\pi 2^2 = 6\pi$$

## 14.2) Iteration of Double Integrals in Cartesian Coordinates

$\boxed{3-4}$ Calculate the given iterated integrals

Q3) $\int_0^{\pi} \int_{-x}^{x} \cos y \, dy \, dx$ :

$$= \int_0^{\pi} \sin y \Big|_{y=-x}^{y=x} dx = \int_0^{\pi} (\sin x - \sin(-x))\, dx$$

$$= 2\int_0^{\pi} \sin x \, dx = -2\cos x \Big|_0^{\pi} = -2(\overbrace{\cos\pi}^{-1} - \overbrace{\cos 0}^{1}) = 4$$

Q4) $\int_0^2 dy \int_0^y y^2 e^{xy} dx$ :

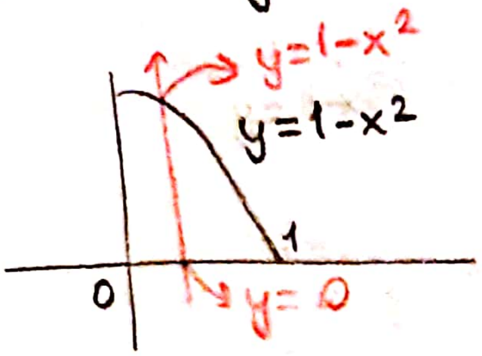
$$= \int_0^2 dy \; y^2 \int_0^y e^{xy} dx = \int_0^2 dy \, y^2 \left(\frac{1}{y} e^{xy}\right)\Big|_{x=0}^{x=y}$$

$$= \int_0^2 y(e^{y^2} - 1)\, dy = \int_0^2 y e^{y^2} dy + \int_0^2 y \, dy$$

$$= \frac{e^{y^2} - y^2}{2}\Big|_0^2 = \frac{e^4 - 4}{2} - \frac{1}{2} = \frac{e^4 - 5}{2}$$

10-12-13-14 Evaluate the double integrals by iteration

Q10) $\iint_D x\cos y \, dA$, where D is the finite region in the $1^{st}$ quadrant bounded by the coordinate axes and the curve $y=1-x^2$




y is from 0 to $1-x^2$

x is from 0 to 1.

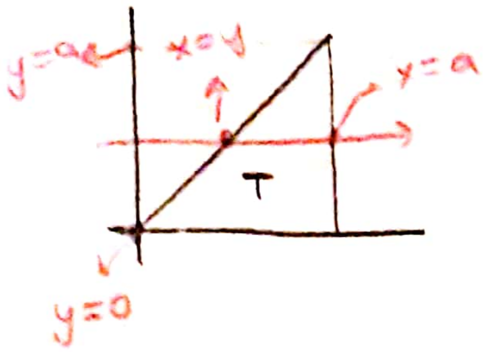
$$\int_0^1 \int_0^{1-x^2} x\cos y \, dy\, dx = \int_0^1 x \int_0^{1-x^2} \cos y \, dy \, dx.$$

$$= \int_0^1 x \left( \sin y \right)_{y=0}^{y=1-x^2} dx = \int_0^1 x \sin(1-x^2) dx$$

Let $u=1-x^2 \Rightarrow du=-2x\,dx$

$$= -\frac{1}{2}\int_1^0 \sin u \, du = \frac{1}{2}\cos u \Big|_1^0 = \frac{1}{2}(1-\cos 1)$$

Q12) $\iint_T \sqrt{a^2-y^2}\, dA$ , where T is the triangle with vertices $(0,0)$, $(a,0)$, $(a,a)$.




x is from y to a

y is from 0 to a

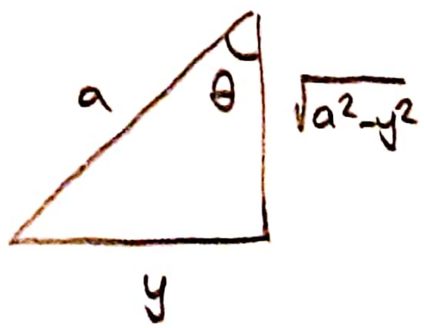
$$\iint_T \sqrt{a^2-y^2}\, dA = \int_0^a \int_y^a \sqrt{a^2-y^2}\, dx\, dy = \int_0^a \sqrt{a^2-y^2}\left(x\Big|_{x=y}^{x=a}\right) dy$$

$$= \int_0^a \sqrt{a^2-y^2}\,(a-y)\, dy = a\underbrace{\int_0^a \sqrt{a^2-y^2}\, dy}_{I_1} - \underbrace{\int_0^a y\sqrt{a^2-y^2}\, dy}_{I_2}$$

$$I_1 = \int_0^a \sqrt{a^2-y^2}\, dy.$$

Let $y = a\sin\theta \Rightarrow dy = a\cos\theta\, d\theta.$

$\sqrt{a^2-y^2} = a\cos\theta$




$y = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{y}{a}\right)$

$y = a \Rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}$ , $y = 0 \Rightarrow \theta = \sin^{-1}(0) = 0$

$$I_1 = \int_0^a \sqrt{a^2-y^2}\,dy = \int_0^{\pi/2} (a\cos\theta)(a\cos\theta\,d\theta)$$

$$= a^2\int_0^{\pi/2} \cos^2\theta\,d\theta = a^2\int_0^{\pi/2} (\cos 2\theta + 1)\,d\theta$$

$$= a^2\left(\frac{\sin 2\theta}{2} + \theta\right)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{a^2\pi}{4}$$

$$I_2 = \int_0^a y\sqrt{a^2-y^2}\,dy$$

Let $u = a^2 - y^2 \Rightarrow du = -2y\,dy$

$y = a \Rightarrow u = 0, \quad y = 0 \Rightarrow u = a^2$

$$I_2 = \int_0^a y\sqrt{a^2-y^2}\,dy = -\frac{1}{2}\int_{a^2}^{0} \sqrt{u}\,du$$

$$= -\frac{1}{2}\,\frac{u^{3/2}}{3/2}\Big|_{u=a^2}^{u=0} = -\frac{1}{3}a^3$$

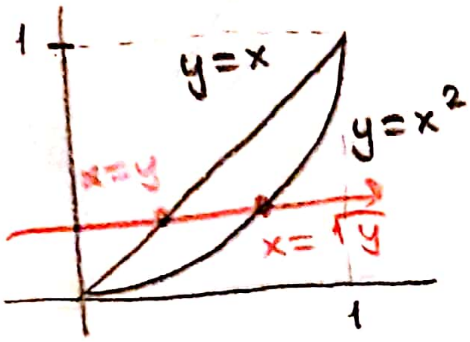
$$\Rightarrow \iint_T \sqrt{a^2-y^2}\,dA = aI_1 - I_2 = \frac{a^3\pi}{4} + \frac{a^3}{3}$$

$$= \left(\frac{\pi}{4} + \frac{1}{3}\right)a^3$$

Q13) $\iint_R \frac{x}{y} e^y \, dA$ where $R$ is the region

$0 \le x \le 1, \; x^2 \le y \le x.$

It is hard to calculate $\int_0^1 x \int_{x^2}^{x} \frac{e^y}{y} \, dy \, dx.$




So, calculate

$$\int_0^1 \frac{e^y}{y} \int_y^{\sqrt{y}} x \, dx \, dy:$$

$$= \int_0^1 \frac{e^y}{y} \left( \frac{x^2}{2} \Big|_{x=y}^{x=\sqrt{y}} \right) dy = \frac{1}{2} \int_0^1 \frac{e^y}{y} (y - y^2) \, dy$$

$$= \frac{1}{2} \int_0^1 (1-y) e^y \, dy$$

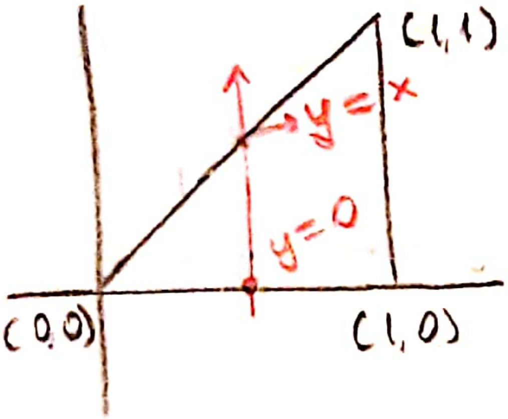
Let $u = 1-y \Rightarrow du = -dy$

$dv = e^y dy \Rightarrow v = e^y$

$$= \frac{1}{2} \left[ e^y (1-y) \Big|_0^1 + \int_0^1 e^y \, dy \right]$$

$$= -\frac{1}{2} + \frac{1}{2}(e-1) = \frac{e}{2} - 1$$

Q14) $\iint_T \frac{xy}{1+x^4}\, dA$, where $T$ is the triangle with vertices $(0,0), (1,0), (1,1)$.




$y$ is from $0$ to $x$

$x$ is from $0$ to $1$.

$$\int_0^1 \int_0^x \frac{xy}{1+x^4}\, dy\, dx = \int_0^1 \frac{x}{1+x^4} \int_0^x y\, dy\, dx$$

$$= \int_0^1 \frac{x}{1+x^4} \left( \frac{y^2}{2} \Big|_{y=0}^{y=x} \right) dx = \frac{1}{2} \int_0^1 \frac{x^3}{1+x^4}\, dx$$

$$= \frac{1}{2} \int_1^2 \frac{1}{4} \frac{du}{u} = \frac{1}{8} \ln u \Big|_1^2 = \frac{1}{8} \ln 2.$$

↑

Let $u = 1+x^4 \Rightarrow du = 4x^3\, dx$

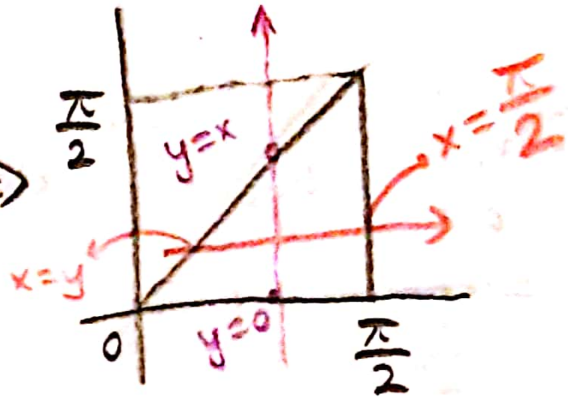
$x=1 \Rightarrow u=2$, $x=0 \Rightarrow u=1$

Q16) Sketch the domain of integration and evaluate the given iterated integral :

$$\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x}\, dx \quad \text{(PINK)}$$

$y$ is from $0$ to $\frac{\pi}{2}$.

$x$ is from $y$ to $\frac{\pi}{2}$ $\Rightarrow$




So the region is the triangle with vertices

$(0,0)$, $(\frac{\pi}{2},0)$, $(\frac{\pi}{2},\frac{\pi}{2})$

But it is hard to calculate this integral.

Instead, calculate $\int_0^{\pi/2} \frac{\sin x}{x}\, dx \int_0^{x} dy$ : (PURPLE)

$$= \int_0^{\pi/2} \frac{\sin x}{x}\, y \Big|_{y=0}^{y=x} dx = \int_0^{\pi/2} \sin x\, dx$$

$$= -\cos x \Big|_0^{\pi/2} = 1$$

Q20) Find the volume under $z = 1 - x^2$ and above the region $0 \leq y \leq 1$, $0 \leq x \leq y$.

We need to calculate $\displaystyle\int_0^1 \int_0^y (1-x^2)\,dx\,dy$:

$$= \int_0^1 \left(x - \frac{x^3}{3}\right)\Big|_{x=0}^{x=y} dy = \int_0^1 \left(y - \frac{y^3}{3}\right) dy$$

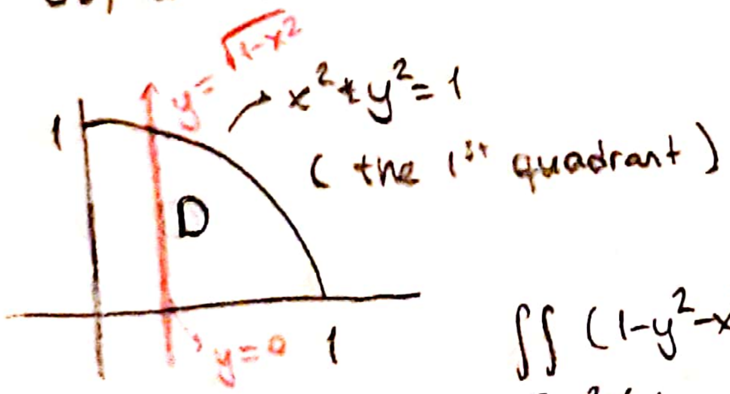
$$= \left(\frac{y^2}{2} - \frac{y^4}{12}\right)\Big|_{y=0}^{y=1} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

Q22) Find the volume under $z = 1 - y^2$ and above $z = x^2$.

$$1 - y^2 = x^2 \implies x^2 + y^2 = 1$$

These two surfaces intersect on the cylinder $x^2 + y^2 = 1$.

So, we need to calculate $\displaystyle\iint_{x^2+y^2 \leq 1} (1 - y^2 - x^2)\,dA$:




Note that,

$$\iint_{x^2+y^2 \leq 1} (1-y^2-x^2)\,dA = 4 \cdot \iint_D (1-y^2-x^2)\,dA$$

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$$\iint_D (1-y^2-x^2)\,dA = \int_0^1\int_0^{\sqrt{1-x^2}} (1-y^2-x^2)\,dy\,dx$$

$$= \int_0^1 \left( y - \frac{y^3}{3} - x^2 y \right) \Big|_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left( \sqrt{1-x^2} - \frac{(1-x^2)^{3/2}}{3} - x^2\sqrt{1-x^2} \right) dx$$

$$= \frac{2}{3}\int_0^1 (1-x^2)^{3/2}\,dx \qquad \text{Let } x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{1}{6}\int_0^{\pi/2} (\cos 2\theta + 1)^2\, d\theta$$

$$= \frac{1}{6}\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{6}\left( \theta + \sin 2\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8} \right) \Big|_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{1}{6}\left( \frac{\pi}{2} + \frac{\pi}{4} \right) = \frac{\pi}{8}$$

So, the required volume is $\frac{\pi}{2}$.