12.1) Functions of Several Variables

$\boxed{6,7}$ Specify the domains of the given functions.

Q6) $f(x,y) = \dfrac{1}{\sqrt{x^2-y^2}}$

* The denominator must be nonzero.
* The number in squareroot must be ≥ 0.

So f is defined if $x^2-y^2 > 0 \Rightarrow x^2 > y^2 \Rightarrow |x| > |y|$

$$\mathrm{Dom}(f) = \{ (x,y) \in \mathbb{R}^2 : |x| > |y| \}$$

Q7) $f(x,y) = \ln(1+xy)$

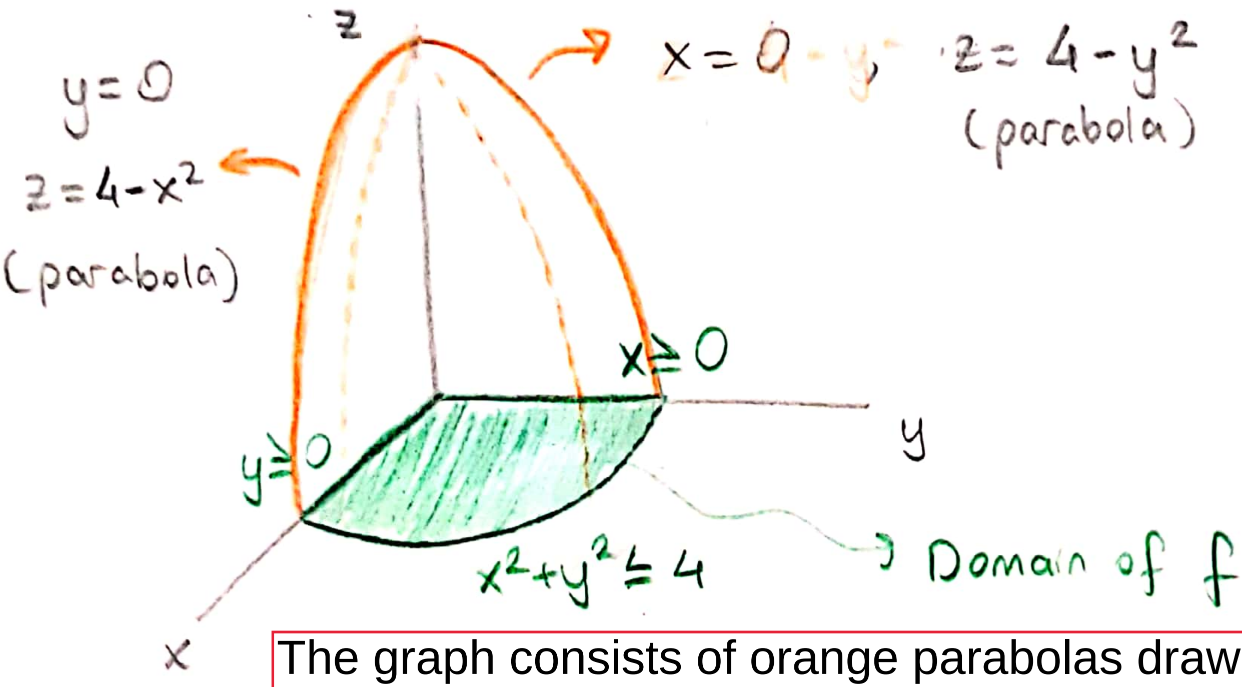
* $\ln t$ is defined for $t > 0$.

So, $1 + xy > 0 \Rightarrow xy > -1$

$$\mathrm{Dom}(f) = \{ (x,y) \in \mathbb{R}^2 : xy > -1 \}$$

$\boxed{14}$ Sketch the graph of

$$f(x,y) = 4 - x^2 - y^2 \quad (x^2+y^2 \leq 4,\ x \geq 0,\ y \geq 0)$$



The graph consists of orange parabolas drawn from z=4 to the green quarter circle.

$\boxed{19,20}$ Sketch some of the level curves of the given functions

Q19) $f(x,y) = x - y.$

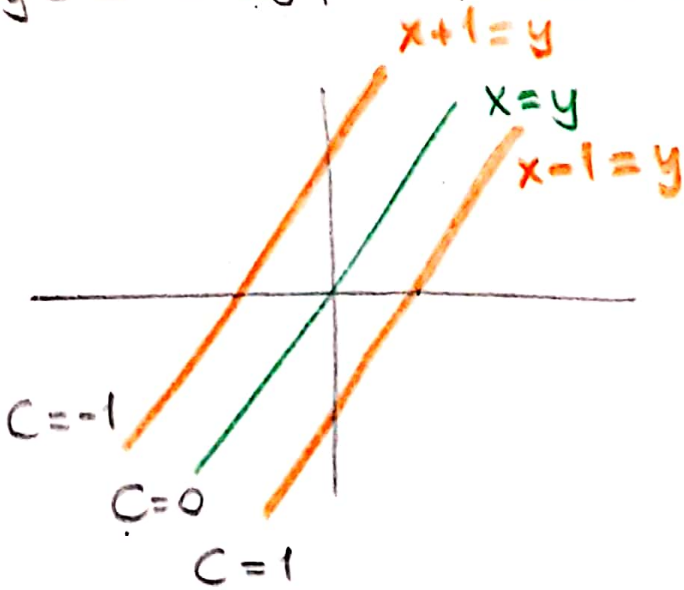
Consider the curves $x - y = C$ in xy-plane for some $C \in \mathbb{R}$:

$x - y = 0 \Rightarrow y = x$

$x - y = 1 \Rightarrow y = x - 1$

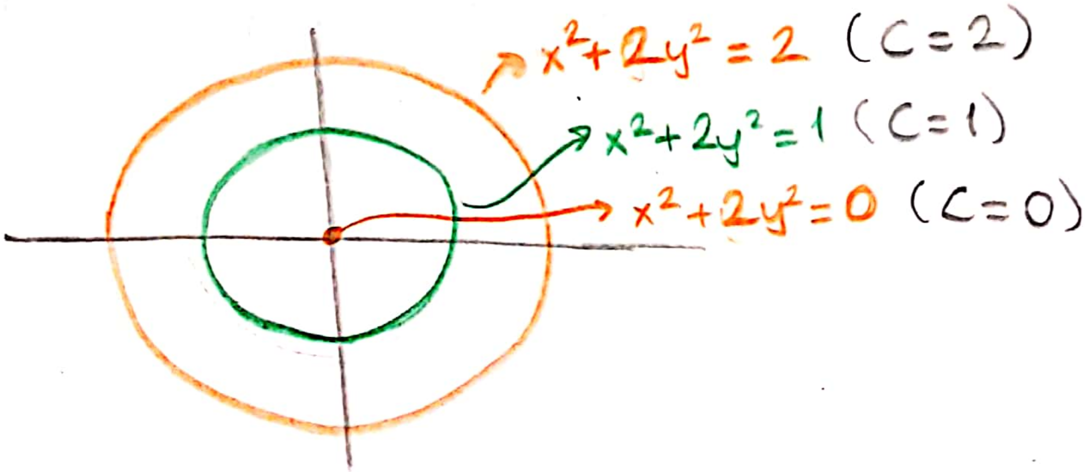
$x - y = 2 \Rightarrow y = x - 2$

$x - y = -1 \Rightarrow y = x + 1$



Q20) $f(x,y) = x^2 + 2y^2$

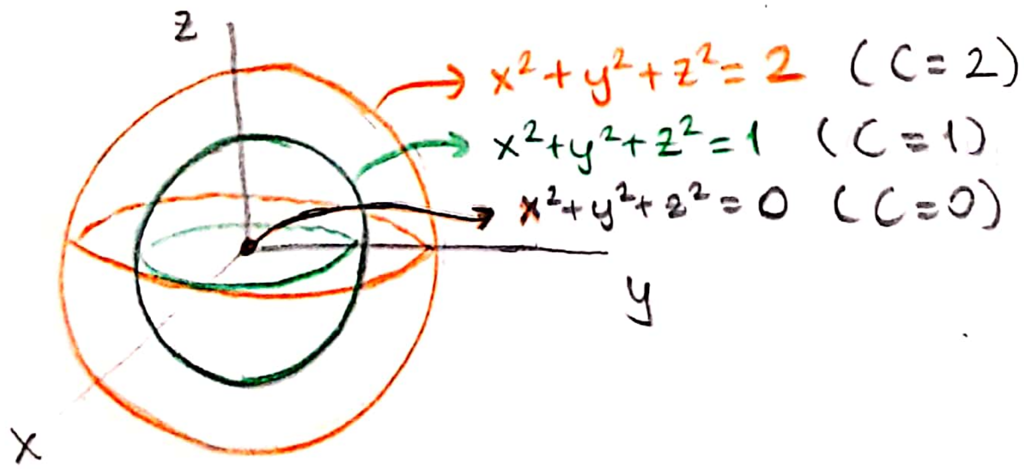
* $x^2 + 2y^2 = 0$ (origin)
* $x^2 + 2y^2 = 1$ (ellipse) $x = \mp 1$, $y = \mp \frac{1}{\sqrt{2}}$ intercepts
* $x^2 + 2y^2 = 2$ (ellipse) $x = \mp\sqrt{2}$ $y = \mp 1$ intercepts



$\boxed{37\ 39}$ Describe the level surfaces of the given functions.

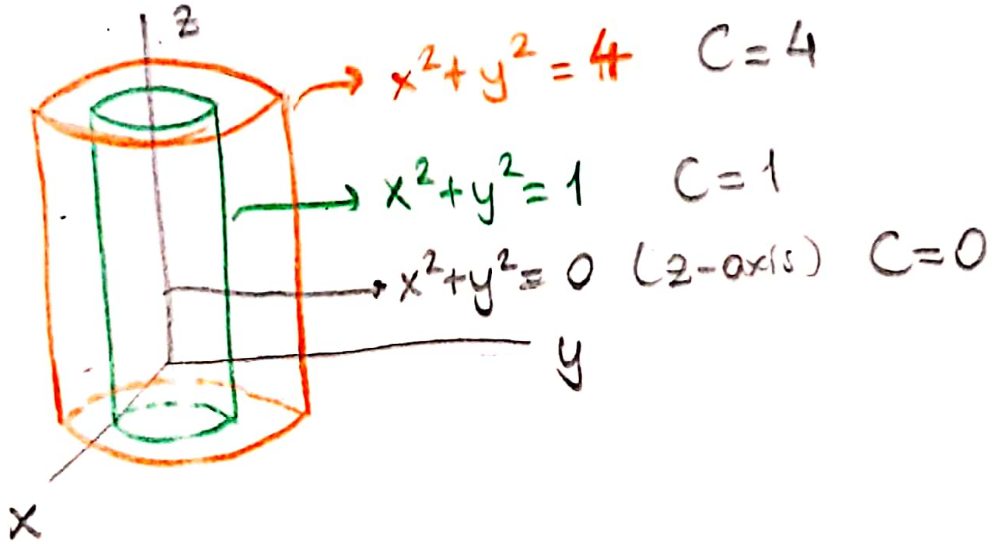
Q37) $f(x,y,z) = x^2 + y^2 + z^2$

Consider the surfaces $x^2+y^2+z^2 = C$ in xyz-space for some $C \in \mathbb{R}$ (a sphere of radius $\sqrt{C}$ centered at $(0,0,0)$)



Q39) $f(x,y,z) = x^2 + y^2$

z arbitrary, $x^2+y^2=C$ gives a circular cylinder of radius $\sqrt{C}$ along the z-axis.



12.2) Limits and Continuity

$\boxed{9-12}$ Evaluate the indicated limit or explain why it does not exist.

Q9) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

* Path: $y=x$

$$\lim_{x\to 0} \frac{\sin(x^2)}{2x^2} = \frac{1}{2} \lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \frac{1}{2}$$

* Path: $x=0$

$$\lim_{y\to 0} \frac{\sin(0\cdot y)}{0+y^2} = 0$$

From different paths, $f(x,y)$ approaches different numbers.

So, the limit does not exist.

Q11) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4}$

$$0 \le \frac{x^2y^2}{x^2+y^4} \le \frac{x^2y^2}{x^2} = y^2 \to 0 \text{ as } y\to 0$$

By squeeze theorem, $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4} = 0$

Q12) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{2x^4+y^4}$

* Path: $y=x$

$\lim_{x\to 0} \dfrac{x^4}{3x^4} = \dfrac{1}{3}$

From different paths $f(x,y)$ approaches different numbers.

* Path: $y=0$

$\lim_{x\to 0} \dfrac{x^4 \cdot 0}{2x^4+0} = 0$

So, the limit does not exist.