GTU, Fall 2021, MATH 101

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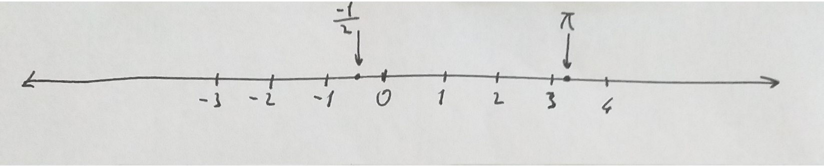

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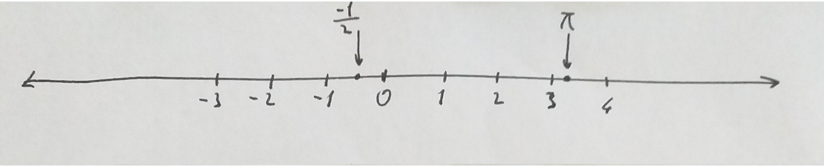

* There is no hole on the real line, so there is no *gap* between real numbers.

Preliminaries, Real Numbers

$*$ Recall some important subsets of real numbers.

i) Natural numbers, $\mathbb{N}$: The numbers 0,1,2,3,...

ii) Integers, $\mathbb{Z}$: ..,-2,-1,0,1,2,... (or $0, \pm 1, \pm 2$, ...)

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* But the set of real numbers, $\mathbb{R}$ is much more larger (!). If a real number is not rational then it is called irrational. For example, $\sqrt{2} = 1.41...$ and $\pi = 3.14...$ are irrational.
* Exercise: What is the difference between the decimal expansion of a rational number and of an irrational number?

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∗ $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

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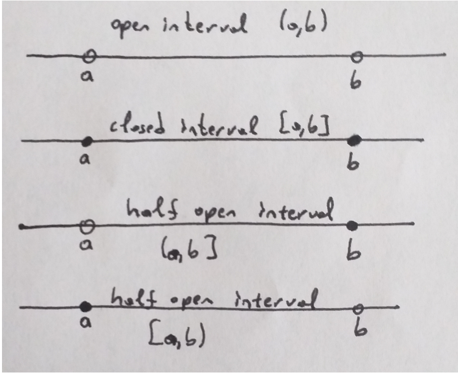
∗ These are all *finite* intervals.



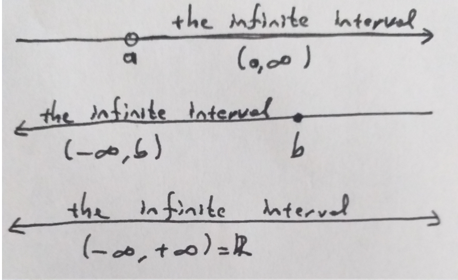
* We may also talk about *infinite* intervals.

* $(a, +\infty) = \{x \in \mathbb{R} \mid a < x\}$

* $(-\infty, \mathrm{b}] = \{x \in \mathbb{R} \mid x \leq \mathrm{b}\}$

* $(-\infty, +\infty) = \mathbb{R}$

* $+\infty$: plus infinity,
* $-\infty$: minus infinity.



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- For example; $|-3| = 3$, $|5.1| = 5.1$, $|0| = 0$
- Absolute value can not be negative.
- Exercise: Solve the following equalities/inequalities (Find all values of x which satisfy the given equality/inequality).
  i) $x|x-2| \leq 0$ (Answer: $x \leq 0$ or $x = 2$).
  ii) $x^2 - 3x + 2 > 0$ (Answer: $x > 2$ or $x < 1$).
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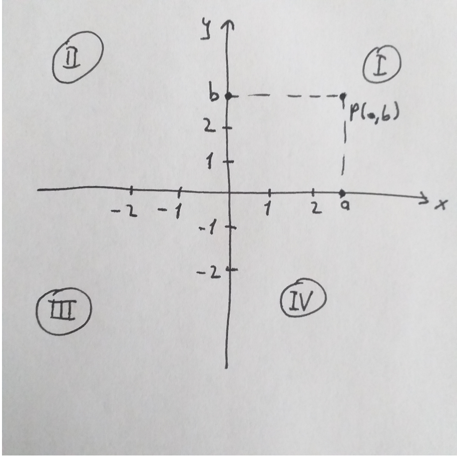
  (Note: These are of high school level, you can use sign table to solve such inequlities. Similar problems will be solved in detail in problem sessions).

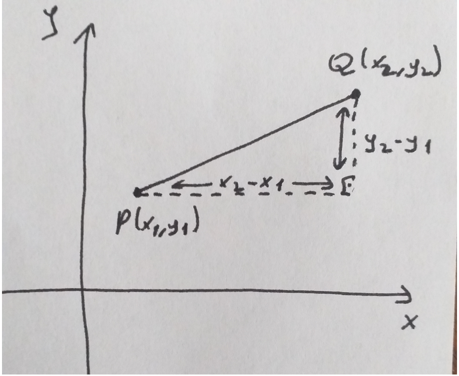
Preliminaries, Cartesian Coordinate System



$*$ xy-plane is divided into four regions called *quadrants* numbered by I, II, III and IV.

$*$ O: The origin
$*$ P: a point on the xy-plane.
$*$ a: the x-coordinate of P
$*$ b: the y-coordinate of P

Preliminaries, Cartesian Coordinate System - Distance Formula



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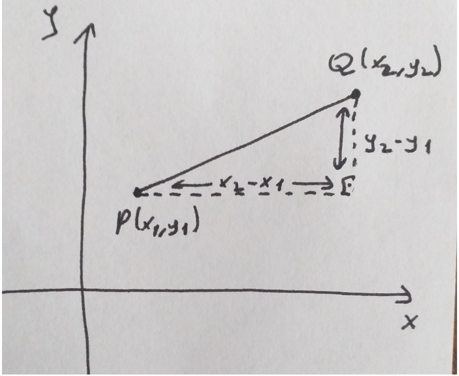

The distance between two points $P(x_1, x_2)$ and $Q(x_2, y_2)$ is given by

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 \text{ - } y_2)^2}$$

(This
follows by Pythagorean theorem).

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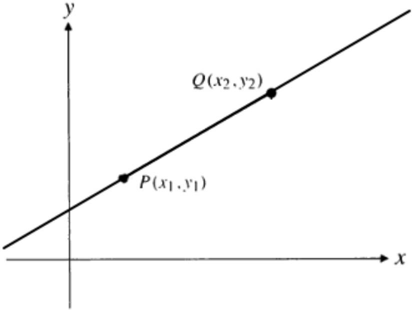

* The slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
* The line is given by the equation

$$L : \frac{y - y_1}{x - x_1} = m$$

(Why?)

Preliminaries, Cartesian Coordinate System - Line Equations

* Example; the slope of the line L passing through $P(1,2)$ and $Q(2,5)$ is

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- The x-intercept (the point at which the line cuts the x-axis) can be found by setting $y = 0$, so it is $x = 1/3$.

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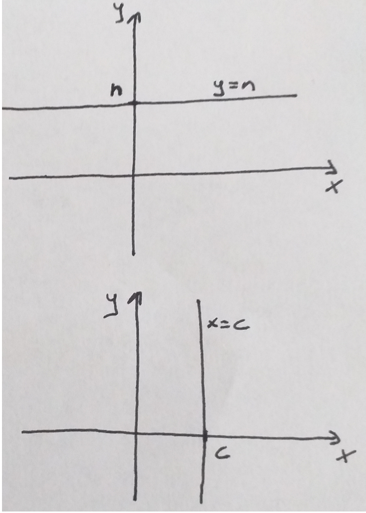

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is $y = n$, so that it is a *horizontal* line.

* We may also have *vertical*
lines with equation $x = c$. Vertical
lines do not have a slope (!) (or
do they have? what would you choose
for the 'slope' of a vertical line?).