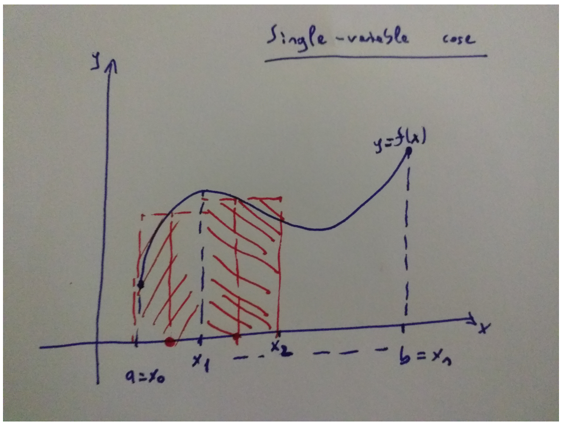
Chapter 14 Multiple Integration

§14.1 Double Integral

We will extend the theory of definite integral to functions in several variables (two or three variables). We start with integral of functions in two variables. Recall the main motivation of definite integrals of functions in one variable; if f is a positive function then $\int_a^b f(x)dx$ represents the area of the region determined by $y = f(x)$, the x-axis, $x = a$ and $x = b$. Similarly a motivation for the double integral is the *volume* of the region *under the graph* of a positive function $f(x, y)$ lying above the $xy - plane$.

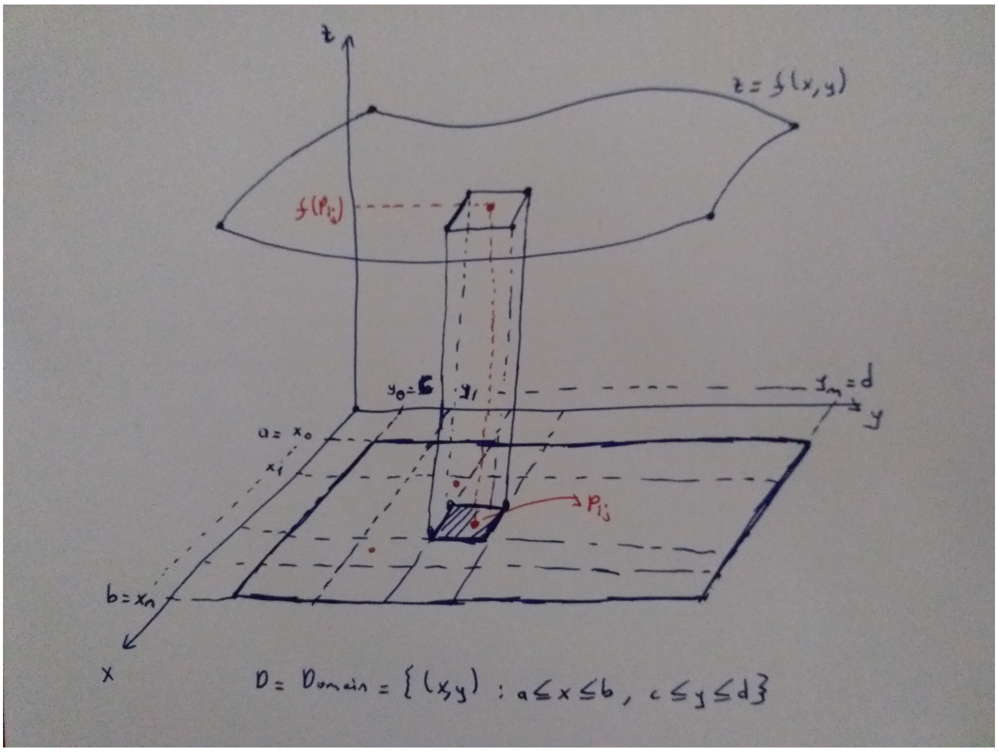
Recall the definition of definite integral of $y = f(x)$ on $[a, b]$.



We divide the interval $[a, b]$ into n subintervals, i.e. we take a partition of $[a, b]$ as $a = x_0 \leq x_1 \leq \ldots \leq x_n = b$. If we set $\Delta x_i = x_{i+1} - x_i$, and choose any arbitrary *sample points* $x_i^* \in [x_i, x_{i+1}]$ then

$$\int_a^b f(x)dx = \lim_{n\to\infty, \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Now we may extend this idea to define double integrals. Let $z = f(x, y)$ be a continuous function on the rectangle $D = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$.



Divide D into mn-rectangles. Take a sample point P_{ij} from each of these rectangles. The total volume of the rectangular boxes

$$\sum_{i,j} f(P_{ij})\Delta A_{ij}$$

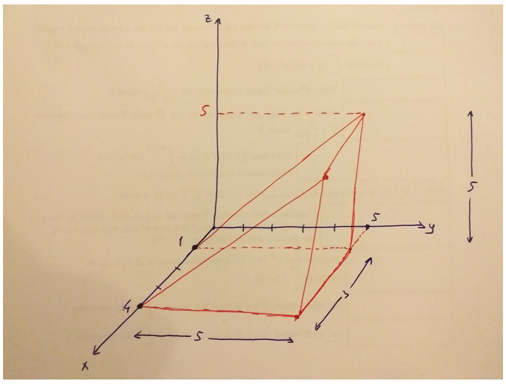
is an *approximation* of the volume under the graph of $z = f(x, y)$ above D. So the *limit* as the number of rectangular boxes goes to infinity and as the length of the sides of rectangles goes to 0 is the volume under the graph of $z = f(x, y)$ above D.

In general for any function $f(x, y)$ (which is not necessarily positive) for which this limit exists we say that $f(x, y)$ is integrable over D. We call this limit as the double integral of $f(x, y)$ over D denoted by

$$\iint_D f(x, y)\, dA.$$

For example if f is continuous on a rectangular region D then f is integrable over D. Note that if $f(x, y)$ is positive on D then $\iint_D f(x, y)\, dA$ gives the area under $z = f(x, y)$ above D.

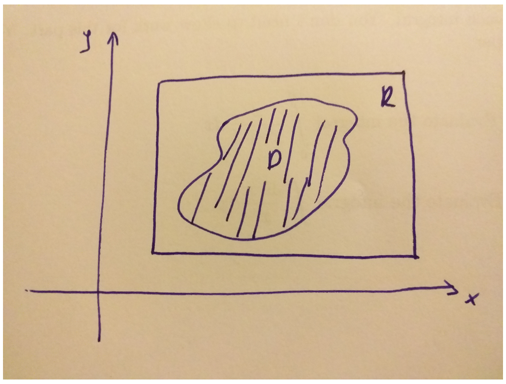
Example: Let $f(x, y) = y$ and $D = \{(x, y) : 1 \le x \le 4, 0 \le y \le 5\}$. Then the double integral



$$\iint_D f(x, y)\, dA = \iint_D y\, dA$$

represents the volume of the half rectangular box given in the skecth. So

$$\iint_D f(x, y)\, dA = \iint_D y\, dA = 75/2$$



Note: What if the domain is not a rectangle? For example consider a bounded domain D given in the skecth. Since D is bounded we can put inside a rectangle, say R. We can extend $f(x, y)$ as 0 outside of D. Then we may write

$$\iint_D f(x,y)\,dA = \iint_R f(x,y)\,dA$$

where $f(x, y) = 0$ if (x, y) does not lie in D. In general we have the following theorem.

Theorem 1. Let D be a bounded and closed region whose boundary is a union of finitely many curves (eg; circle, triangle, square,...). If $f(x, y)$ is continuous on D then f is integrable on D, i.e.

$$\iint_D f(x,y)\,dA$$

exists.

Properties of the Double Integral

Let f and g be integrable on a region D. Then the following holds.

i) If the area of D is zero then $\displaystyle\iint\limits_D f(x,y)\,dA = 0$.

ii) $\displaystyle\iint\limits_D 1\,dA = 1\times$ (area of D) $=$ area of D.

iii) If $f(x,y) \geq 0$ on D, then $\displaystyle\iint\limits_D f(x,y)\,dA$ represents the volume of the solid under the graph of $z = f(x,y)$ above D.

iv) For any $\alpha, \beta \in \mathbb{R}$,

$$\iint\limits_D (\alpha f(x,y)+\beta g(x,y))\,dA = \alpha\iint\limits_D f(x,y)\,dA+\beta\iint\limits_D g(x,y)\,dA$$

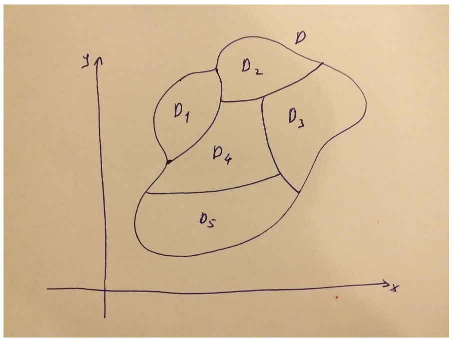
v) If $f(x,y) \leq g(x,y)$ on D then $\displaystyle\iint\limits_D f(x,y)\,dA \leq \iint\limits_D g(x,y)\,dA$

vi) $\displaystyle\left|\iint\limits_D f(x,y)\,dA\right| \leq \iint\limits_D |f(x,y)|\,dA$

vii) Let $D_1, D_2, ..., D_n$ be nonoverlapping domains, and $f(x,y)$ be continuous on each D_i. Put D be the union of all D_i. Then f is integrable on D and

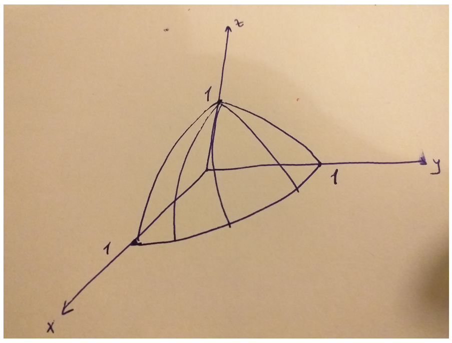
$$\iint\limits_D f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + ... + \iint\limits_{D_n} f(x,y)\,dA$$

As an example for vii) consider the domain D in the skecth. The boundaries of D_i's may overlap but this is not a problem (the area of the boundary is 0). But D_i shall not overlap on a region with positive area.



Example: Let $D = \{(x, y) : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$. Then

$$\iint_D \sqrt{1 - x^2 - y^2}\, dA$$



represents the volume of unit sphere in the first octant, so it equals to $\pi/6$.