12.8) Implicit Functions.

$\boxed{4-11-12}$ Calculate the indicated derivatives from the given equations. What condition on the variables will guarantee the existence of a solution that has the indicated derivative?

Q4) $\dfrac{\partial y}{\partial z}$ if $e^{yz} - x^2 z \ln y = \pi$

Consider $y = y(x, z)$ and use implicit differentiation.

$$\frac{\partial}{\partial z}\left(e^{yz} - x^2 z \ln y\right) = \frac{\partial}{\partial z}\pi$$

$$\Rightarrow e^{yz}\left(\frac{\partial y}{\partial z} z + y\right) - x^2\left(\ln y + z\frac{1}{y}\frac{\partial y}{\partial z}\right) = 0$$

$$\frac{\partial y}{\partial z}\left(z e^{yz} - \frac{x^2 z}{y}\right) + e^{yz} y - x^2 \ln y = 0$$

$$\Rightarrow \frac{\partial y}{\partial z} = \frac{-e^{yz} y + x^2 \ln y}{z e^{yz} - \frac{x^2 z}{y}} = \frac{x^2 y \ln y - e^{yz} y^2}{z e^{yz} y - x^2 z}$$

The given equation has a solution $y = y(x,y)$ with this derivative near any point where $y > 0$, $z \neq 0$ and $y e^{yz} \neq x^2$

Q11) $\left(\frac{\partial x}{\partial y}\right)_z$ if $x^2+y^2+z^2+w^2=1$ and $x+2y+3z+4w=2$

Consider $x=x(y,z)$, $w=w(y,z)$.

$\frac{\partial}{\partial y}(x^2+y^2+z^2+w^2)=0 \Rightarrow 2x\frac{\partial x}{\partial y}+2y+2w\frac{\partial w}{\partial y}=0$ (1)

$\frac{\partial}{\partial y}(x+2y+3z+4w)=0 \Rightarrow 1.\frac{\partial x}{\partial y}+2+4\frac{\partial w}{\partial y}=0$ (2)

From (1) and (2), we get

$$\begin{cases} 2x\frac{\partial x}{\partial y}+2w\frac{\partial w}{\partial y}=-2y & \rightarrow \text{multiply by } 2 \\ \frac{\partial x}{\partial y}+4\frac{\partial w}{\partial y}=-2 & \rightarrow \text{multiply by } -w \end{cases}$$

$\Rightarrow (4x-w)\frac{\partial x}{\partial y}=-4+2w \Rightarrow \left(\frac{\partial x}{\partial y}\right)_z=\frac{2w-4}{4x-w}$

The given equations have a solution $\left.\begin{matrix} x=x(y,z) \\ w=w(y,z) \end{matrix}\right\}$

with this derivative near any point

where $w \neq 4x$

Q12) $\frac{du}{dx}$ if $x^2y + y^2u - u^3 = 0$ and $x^2 + yu = 1$

Consider $u = u(x)$, $y = y(x)$.

$$\frac{d}{dx}(x^2y + y^2u - u^3) = 0$$

$$\Rightarrow (1) \quad 2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx}u + y^2\frac{du}{dx} - 3u^2\frac{du}{dx} = 0$$

$$\frac{d}{dx}(x^2 + yu) = 0$$

$$\Rightarrow (2) \quad 2x + \frac{dy}{dx}u + y\frac{du}{dx} = 0$$

From (1) & (2), we have

$$(x^2 + 2yu)\frac{dy}{dx} + (y^2 - 3u^2)\frac{du}{dx} = -2xy \rightarrow \text{multiply by } -u$$

$$u\frac{dy}{dx} + y\frac{du}{dx} = -2x \rightarrow \text{Multiply by } x^2 + 2yu$$

$$\Rightarrow [-u(y^2 - 3u^2) + y(x^2 + 2yu)]\frac{du}{dx} = 2xyu - 2x(x^2 + 2yu)$$

$$\Rightarrow \frac{du}{dx} = \frac{-2xyu - 2x^3}{3u^3 + x^2y + y^2u} = \frac{-2x(x^2 + yu)}{3u^3 + (x^2y + y^2u)} = \frac{-x}{2u^3}$$

The equations have a solution $u = u(x)$, $y = y(x)$ with this derivative near any point where $u \neq 0$.

Q14) Near what points (r, s) can the transformation $x = r^2 + 2s$, $y = s^2 - 2r$ be solved for r and s as functions of x and y? Calculate the values of the 1st partial derivatives of the solution at the origin.

$$\frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} 2r & 2 \\ -2 & 2s \end{vmatrix}$$

(Jacobian of $x = x(r,s)$ and $y = y(r,s)$)

$$= 4(rs+1) \neq 0 \text{ if } rs \neq -1$$

By the Implicit function theorem the system $\left.\begin{matrix} x = r^2 + 2s \\ y = s^2 - 2r \end{matrix}\right\}$ can be solved for r and s as functions of x & y near any point (r,s) where

$rs \neq -1$

To find the 1st partial derivatives of the solution, take the partial derivatives of the equations wrt x & y:

$$\frac{\partial}{\partial x} x = \frac{\partial}{\partial x}(r^2+2s) \Rightarrow 1 = 2r\frac{\partial r}{\partial x} + 2\frac{\partial s}{\partial x}$$

$$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}(s^2-2r) \Rightarrow 0 = -2\frac{\partial r}{\partial x} + 2s\frac{\partial s}{\partial x}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{s}{2(rs+1)}, \quad \frac{\partial s}{\partial x} = \frac{1}{2(rs+1)}$$

$$\frac{\partial}{\partial y} x = \frac{\partial}{\partial y}(r^2+2s) \Rightarrow 0 = 2r\frac{\partial r}{\partial y} + 2\frac{\partial s}{\partial y}$$

$$\frac{\partial}{\partial y} y = \frac{\partial}{\partial y}(s^2-2r) \Rightarrow 1 = -2\frac{\partial r}{\partial y} + 2s\frac{\partial s}{\partial y}$$

$$\Rightarrow \frac{\partial r}{\partial y} = -\frac{1}{2(rs+1)}, \quad \frac{\partial s}{\partial y} = \frac{r}{2(rs+1)}$$

$$\frac{\partial r}{\partial x}(0,0) = \frac{\partial s}{\partial y} = 0, \quad \frac{\partial r}{\partial y} = -\frac{1}{2}, \quad \frac{\partial s}{\partial x} = \frac{1}{2}$$

13.1) Extreme Values

$\boxed{4-5-8-11}$ Find and classify the critical points of the given functions.

Q4) $f(x,y) = x^4 + y^4 - 4xy$

⊛ f has a critical point at (a,b) if $\nabla f(a,b) = 0$

$$\Rightarrow \begin{cases} f_x = 4x^3 - 4y = 0 & \Rightarrow x^3 = y \\ f_y = 4y^3 - 4x = 0 & \Rightarrow y^3 = x \end{cases}$$

$\Rightarrow x^9 = x \quad \Rightarrow x^9 - x = 0 \quad \Rightarrow x = 0$ or $x^8 - 1 = 0$

$\Rightarrow x = 0$ or $x = \mp 1$ ($y = 0$, $y = \mp 1$, respectively)

So, the critical points are

$(0,0)$, $(1,1)$, $(-1,-1)$

Hessian matrix:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

⊛ Let f has a critical point at (a,b).

(I) $f_{xx} f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0 \Rightarrow f$ has local min

(II) $f_{xx} f_{yy} - f_{xy}^2 > 0$, $f_{xx} < 0 \Rightarrow f$ has local max.

(III) $f_{xx} f_{yy} - f_{xy}^2 < 0 \Rightarrow f$ has a saddle point.

$f_{xx} = 12x^2$, $f_{xy} = f_{yx} = -4$, $f_{yy} = 12y^2$

So at $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16 = -16 < 0$

$\Rightarrow$ Saddle point at $(0,0)$

At $(1,1)$, $f_{xx}f_{yy} - f_{xy}^2 = 144 - 16 = 128 > 0$,

$f_{xx} = 12 > 0$

$\Rightarrow$ Local min at $(1,1)$

At $(-1,-1)$, $f_{xx}f_{yy} - f_{xy}^2 = 144 - 16 = 128 > 0$

$f_{xx} = 12 > 0$

$\Rightarrow$ Local min at $(-1,-1)$.

Q5) $f(x,y) = \frac{x}{y} + \frac{8}{x} - y$

$f_x = \frac{1}{y} - \frac{8}{x^2} = 0 \Rightarrow 8y = x^2$

$f_y = -\frac{x}{y^2} - 1 = 0 \Rightarrow y^2 = -x$

$\Rightarrow \frac{x^4}{64} = -x \Rightarrow x^4 - 64x = 0 \Rightarrow x = 0 \text{ or } x = -4$
$\qquad (y=0) \qquad (y=2)$

The function is not defined at $(0,0)$.

The only critical point is $(-4, 2)$.

$f_{xx} = \frac{16}{x^3}, \quad f_{xy} = f_{yx} = -\frac{1}{y^2}, \quad f_{yy} = \frac{2x}{y^3}$

At $(-4, 2)$;

$f_{xx} f_{yy} - f_{xy}^2 = \left(-\frac{1}{4}\right)(-1) - \left(-\frac{1}{4}\right)^2 = \frac{3}{16} > 0$

$f_{xx} = -\frac{1}{4} < 0$

$\Rightarrow f$ has a local max at $(-4, 2)$.

Q8) $f(x,y) = \cos x + \cos y$

$f_x = -\sin x = 0 \Rightarrow x = k\pi, \ k \in \mathbb{Z}$

$f_y = -\sin y = 0 \Rightarrow y = l\pi, \ l \in \mathbb{Z}$

Critical points: $(k\pi, l\pi)$, $k, l \in \mathbb{Z}$

$f_{xx} = -\cos x$, $f_{xy} = f_{yx} = 0$, $f_{yy} = -\cos y$

$\Rightarrow f_{xx} f_{yy} - f_{xy}^2 = \cos x \cos y$.

$\cos(k\pi)\cos(l\pi) = (-1)^{k-1}(-1)^{l-1} = (-1)^{k+l}$

So, if $k+l$ is odd, $f_{xx} f_{yy} - f_{xy}^2 < 0$

$\Rightarrow$ f has a saddle point at $(k\pi, l\pi)$

If $k+l$ is even, then $f_{xx} f_{yy} - f_{xy}^2 > 0$

If k is even, $f_{xx} = -\cos(k\pi) < 0$

$\Rightarrow$ f has a local max at $(k\pi, l\pi)$

If k is odd, $f_{xx} = -\cos(k\pi) > 0$

$\Rightarrow$ f has a local min at $(k\pi, l\pi)$

Q11) $f(x,y) = x e^{-x^3+y^3}$

$f_x = e^{-x^3+y^3} - 3x^3 e^{-x^3+y^3} = (1-3x^3) e^{-x^3+y^3} = 0$

$f_y = 3xy^2 e^{-x^3+y^3} = 0$

$\Rightarrow \begin{cases} 1-3x^3 = 0 & \Rightarrow \quad x = 3^{-1/3} \\ x=0 \text{ or } y=0 \end{cases}$

So, $(3^{-1/3}, 0)$ is the only critical point.

$f_{xx} = -9x^2 e^{-x^3+y^3} - 3x^2(1-3x^3) e^{-x^3+y^3}$

$= (-12x^2 + 9x^5) e^{-x^3+y^3} = 3x^2(3x^3-4) e^{-x^3+y^3}$

$f_{xy} = f_{yx} = 3y^2 e^{-x^3+y^3} - 9x^3y^2 e^{-x^3+y^3}$

$= 3y^2(1-3x^3) e^{-x^3+y^3}$

$f_{yy} = 6xy e^{-x^3+y^3} + 9xy^4 e^{-x^3+y^3}$

$= 3xy(2+3y^3) e^{-x^3+y^3}$

$f_{xy} = f_{yy} = 0$ at $(3^{-1/3}, 0)$. So

$f_{xx} f_{yy} - f_{xy}^2 = 0 \Rightarrow$ Test is inconclusive.

Fix x.

$$\frac{d}{dy}\left(xe^{-x^3+y^3}\right) = \underbrace{xe^{-x^3}}_{\text{constant}}\underbrace{(3y^2e^{y^3})}_{\text{greater than 0 for } y\neq 0}$$

So, the sign of this derivative remains the same (with the constant part) for $y>0$ and $y<0$.

By 1st derivative test for single variable functions, there exists no max or min at $y=0$.

So, $f(x,y)$ has a saddle point at $(3^{-1/3}, 0)$

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Q27) Find the critical points of the function $z=g(x,y)$ that satisfies the equation $e^{2zx-x^2}-3e^{2zy+y^2}=2$.

To find the critical points of $z=g(x,y)$, we need to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using implicit differentiation

$$\frac{\partial}{\partial x}\left(e^{2zx-x^2}-3e^{2zy+y^2}\right)=0$$

$$\Rightarrow \left(2x\frac{\partial z}{\partial x}+2z-2x\right)e^{2zx-x^2}-3\left(2y\frac{\partial z}{\partial x}\right)e^{2zx-x^2}=0$$

If $\frac{\partial z}{\partial x}=0$, the equation becomes

$$(2z-2x)e^{2zx-x^2}=0 \Rightarrow z=x \quad (*)$$

$$\frac{\partial}{\partial y}\left(e^{2zx-x^2}-3e^{2zy+y^2}\right)=0$$

$$\Rightarrow \left(2x\frac{\partial z}{\partial y}\right)e^{2zx-x^2}-3\left(2y\frac{\partial z}{\partial y}+2z+2y\right)e^{2zy+y^2}=0$$

If $\frac{\partial z}{\partial y}=0$, the equation becomes

$$-3(2z+2y)e^{2zy+y^2}=0 \Rightarrow z=-y. \quad (**)$$

Using (*) and (**) in the equation, we get

$$e^{2z^2-z^2} - 3e^{-2z^2+z^2} = 2$$

$$\Rightarrow e^{z^2} - 3e^{-z^2} = 2$$

$$\Rightarrow e^{2z^2} - 2e^{z^2} - 3 = 0$$

$$\Rightarrow (e^{z^2} - 3)(e^{z^2} + 1) = 0$$

$$\Rightarrow e^{z^2} = 3 \quad \text{or} \quad \underbrace{e^{z^2} = -1}_{\text{not possible}}$$

$$\Rightarrow z = \mp\sqrt{\ln 3}$$

So, the critical points are

$(\sqrt{\ln 3}, -\sqrt{\ln 3})$ and $(-\sqrt{\ln 3}, \sqrt{\ln 3})$