MATH 101.2 PS-2

Q1) Evoluate the limit $\lim_{h\to 2}\frac{1}{4-h^2}$ or explain why it does not exist.

Sol: **Remember:** The limit of the function f at a point c exists and equal to L if and only if $\lim_{x\to c^+} f(x) = \lim_{c\to c^-} f(x) = L$.

Equivalently, $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$

$$\to \lim_{h\to 2^+}\frac{1}{4-h^2} = \lim_{h\to 2^+}\left(\frac{1}{2+h}\cdot\frac{1}{2-h}\right) = \lim_{h\to 2^+}\left(\frac{1}{2+h}\right)\cdot\lim_{h\to 2^+}\left(\frac{1}{\underbrace{2-h}_{0^-}}\right) = 4.(-\infty) = -\infty$$

$$\lim_{h\to 2^-}\frac{1}{4-h^2} = \lim_{h\to 2^-}\left(\frac{1}{2+h}\cdot\frac{1}{2-h}\right) = \lim_{h\to 2^-}\left(\frac{1}{2+h}\right)\cdot\lim_{h\to 2^-}\left(\frac{1}{\underbrace{2-h}_{0^+}}\right) = 4\cdot\infty = \infty$$

So, $\lim_{h\to 2^+}\frac{1}{4-h^2} \neq \lim_{h\to 2^-}\frac{1}{4-h^2}$. Thus, $\lim_{h\to 2}\frac{1}{4-h^2}$ does not exist.

Q2) $\lim_{x\to 0}\frac{|x-2|}{x-2} = ?$

Sol:
$$\left.\begin{aligned} \lim_{x\to 0^+}\frac{|x-2|}{(x-2)} &= \frac{|0^+-2|}{(0^+-2)} = \frac{2}{-2} = -1 \\ \lim_{x\to 0^-}\frac{|x-2|}{(x-2)} &= \frac{|0^--2|}{(0^--2)} = \frac{2}{-2} = -1 \end{aligned}\right\} \lim_{x\to 0}\frac{|x-2|}{x-2} = -1 \;//$$

Q3) $\lim_{x\to 2}\frac{|x-2|}{x-2} = ?$

Sol:
$$\lim_{x\to 2^+}\frac{|x-2|}{x-2} = \frac{|2^+-2|}{2^+-2} = \frac{|0^+|}{0^+} = \frac{0^+}{0^+} = 1$$

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2} = \frac{|2^--2|}{2^--2} = \frac{|0^-|}{0^-} = \frac{0^+}{0^-} = -1$$

OR $\lim_{x\to 2^+}\frac{\overbrace{|x-2|}^{\to 0}}{(x-2)} = \lim_{x\to 2^+}\frac{x-2}{x-2} = 1$ $\quad \lim_{x\to 2^-}\frac{\overbrace{|x-2|}^{<0}}{x-2} = \lim_{x\to 2^-}\frac{2-x}{x-2} = -1$

Therefore, $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exists.

Q4) $\lim\limits_{s\to 0} \dfrac{(s+1)^2-(s-1)^2}{s} = ?$

Sol: $\lim\limits_{s\to 0} \dfrac{(s+1)^2-(s-1)^2}{s} = \dfrac{(0+1)^2-(0-1)^2}{0} = \dfrac{0}{0}$ ⚡

$$\lim_{s\to 0} \frac{(s+1)^2-(s-1)^2}{s} = \lim_{s\to 0} \frac{\cancel{s^2}+2s+\cancel{1}-\cancel{s^2}+2s-\cancel{1}}{s} = \lim_{s\to 0}\frac{4s}{s} = 4 \,//$$

Q5) $\lim\limits_{y\to 1} \dfrac{y-4\sqrt{y}+3}{y^2-1} = ?$

Sol: $\lim\limits_{y\to 1} \dfrac{y-4\sqrt{y}+3}{y^2-1} = \dfrac{1-4\sqrt{1}+3}{1^2-1} = \dfrac{0}{0}$ ⚡

We know that $y=(\sqrt{y})^2$ and $a^2-b^2=(a+b)(a-b)$.

$y-4\sqrt{y}+3 = (\sqrt{y})^2 - 4\sqrt{y}+3 = (\sqrt{y}-3)(\sqrt{y}-1)$

$y^2-1 = (y+1)(y-1) = (y+1)(\sqrt{y}^2-1^2) = (y+1)(\sqrt{y}+1)(\sqrt{y}-1)$

$$\rightarrow \lim_{y\to 1}\frac{y-4\sqrt{y}+3}{y^2-1} = \lim_{y\to 1}\frac{(\sqrt{y}-3)\cancel{(\sqrt{y}-1)}}{(y+1)(\sqrt{y}+1)\cancel{(\sqrt{y}-1)}} = \lim_{y\to 1}\frac{\sqrt{y}-3}{(y+1)(\sqrt{y}+1)} = \frac{1-3}{2.2} = -\frac{1}{2} \,//$$

Q6) $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2} - \dfrac{4}{x^2-4}\right) = ?$

Sol: $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2} - \dfrac{4}{x^2-4}\right) = \dfrac{1}{0} - \dfrac{4}{0} = \infty - \infty$ ⚡

$$\lim_{x\to 2}\left(\underset{(x+2)}{\frac{1}{x-2}} - \frac{4}{x^2-4}\right) = \lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{4} \checkmark$$

Q7) $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2} = ?$

Sol: $\lim\limits_{x\to 0}\dfrac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2} = \dfrac{\sqrt{2}-\sqrt{2}}{0^2} = \dfrac{0}{0}$ ⚡

$$\lim_{x\to 0}\frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\underset{(\sqrt{2+x^2}+\sqrt{2-x^2})}{x^2}} = \lim_{x\to 0}\frac{(\sqrt{2+x^2}-\sqrt{2-x^2})(\sqrt{2+x^2}+\sqrt{2-x^2})}{x^2(\sqrt{2+x^2}+\sqrt{2-x^2})}$$

$$= \lim_{x\to 0}\frac{(2+x^2)-(2-x^2)}{x^2(\sqrt{2+x^2}+\sqrt{2-x^2})} = \lim_{x\to 0}\frac{2\cancel{x^2}}{\cancel{x^2}(\sqrt{2+x^2}+\sqrt{2-x^2})}$$

$$= \lim_{x\to 0}\frac{2}{\sqrt{2+x^2}+\sqrt{2-x^2}} = \frac{2}{\sqrt{2}+\sqrt{2}} = \frac{1}{\sqrt{2}} \,//$$

Q8) If $2-x^2 \leq g(x) \leq 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

Sol: Squeeze Theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for all x in an open interval about a (except possibly at a itself). Further, suppose $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then, $\lim_{x\to a} g(x) = L$.

$\rightarrow \lim_{x\to 0}(2-x^2) = 2$ and $\lim_{x\to 0} 2\cos x = 2.\cos 0 = 2.1 = 2$.

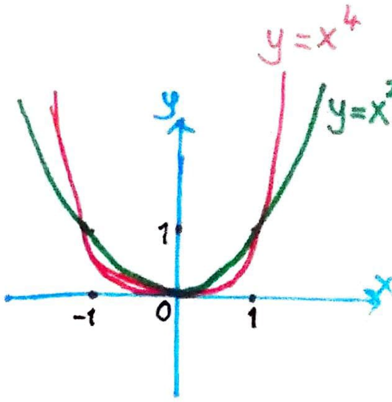
By the Squeeze theorem, $\lim_{x\to 0} g(x) = 2$.

Q~~10~~ 9) a) Sketch the curves $y=x^2$ ~~any~~ and $y=x^4$ on the same graph. Where do they intersect?

b) The function $f(x)$ satisfies:

$$\begin{cases} x^2 < f(x) \leq x^4 & \text{if } x<-1 \text{ or } x>1 \\ x^4 \leq f(x) \leq x^2 & \text{if } -1 \leq x \leq 1 \end{cases}$$

Find $\lim_{x\to -1} f(x)$, $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$.



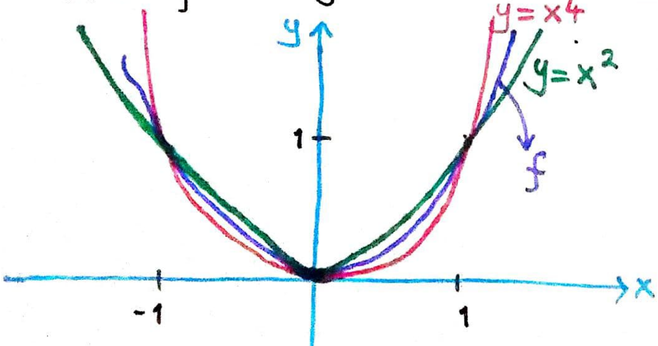
Sol: a) Let us find the intersection points.

$x^4 = x^2 \Rightarrow x^4 - x^2 = 0 \Rightarrow x^2(x^2-1) = 0$.

Then, $x^2(x+1)(x-1) = 0$ and, $x = 0, -1, 1$.

Thus, intersection points are $(0,0)$, $(-1,1)$ and $(1,1)$.

b) Let us redraw the graph we just drew and take an ~~a~~ arbitrary function f on the graph which satisfies given conditions.



It is clear by the squeeze theorem that

$\lim_{x\to -1} f(x) = \lim_{x\to 1} f(x) = 1$ and

$\lim_{x\to 0} f(x) = 0$.

Q10) $\lim\limits_{x\to-\infty} \dfrac{x^2+3}{x^3+2} = ?$

Sol: $\lim\limits_{x\to-\infty} \dfrac{x^2+3}{x^3+2} = \lim\limits_{x\to-\infty} \dfrac{x^3\left(\frac{1}{x}+\frac{3}{x^3}\right)}{x^3\left(1+\frac{2}{x^3}\right)} = \lim\limits_{x\to-\infty} \dfrac{\frac{1}{x}+\frac{3}{x^3}}{1+\frac{2}{x^3}} = \dfrac{0+0}{1+0} = 0 //$

Q11) $\lim\limits_{x\to\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}} = ?$

Sol: $\lim\limits_{x\to\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}} = \lim\limits_{x\to\infty} \dfrac{2x-1}{\sqrt{x^2\left(3+\frac{1}{x}+\frac{1}{x^2}\right)}} = \lim\limits_{x\to\infty} \dfrac{2x-1}{|x|\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}$

$= \lim\limits_{x\to\infty} \dfrac{x\left(2-\frac{1}{x}\right)}{x\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = \lim\limits_{x\to\infty} \dfrac{2-\frac{1}{x}}{\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}$

$= \dfrac{2-0}{\sqrt{3+0+0}} = \dfrac{2}{\sqrt{3}} //$

Q12) $\lim\limits_{x\to-\infty} \dfrac{2x-5}{|3x+2|} = ?$

Sol: $\lim\limits_{x\to-\infty} \dfrac{2x-5}{\underbrace{|3x+2|}_{<0}} = \lim\limits_{x\to-\infty} \dfrac{2x-5}{-3x-2} = -\dfrac{2}{3} //$

Q13) $\lim\limits_{x\to\left(-\frac{2}{5}\right)^-} \dfrac{2x+5}{5x+2} = ?$

Sol: $\lim\limits_{x\to\left(-\frac{2}{5}\right)^-} \dfrac{2x+5}{5x+2} = \dfrac{2.\left(-\frac{2}{5}\right)^- + 5}{5.\left(-\frac{2}{5}\right)^- + 2} = \dfrac{\frac{21}{5}}{0^-} = -\infty .$

Q14) $\lim\limits_{x\to1^-} \dfrac{1}{|x-1|} = ?$

Sol: $\lim\limits_{x\to1^-} \dfrac{1}{|x-1|} = \dfrac{1}{|1^- - 1|} = \dfrac{1}{|0^-|} = \dfrac{1}{0^+} = \infty .$

Q15) $\lim_{x\to 1^+} \frac{\sqrt{x^2-x}}{x-x^2} = ?$

if $x>1$, $x^2>x$
and $x^2-x>0$
$x^2-x\to 0^+$ as $x\to 1^+$

Sol: $\lim_{x\to 1^+} \frac{\sqrt{x^2-x}}{x-x^2} = \lim_{x\to 1^+} -\frac{\sqrt{x^2-x}}{x^2-x} = \lim_{x\to 1^+} \frac{-1}{\sqrt{x^2-x}} = \frac{-1}{\sqrt{0^+}} = \frac{-1}{0^+} = -\infty$

Q16) $\lim_{x\to -\infty} \frac{1}{\sqrt{x^2+2x}-x} = ?$

Sol: $\lim_{x\to -\infty} \frac{1}{\sqrt{x^2+2x}-x} = \lim_{x\to -\infty} \frac{1}{\sqrt{x^2(1+\frac{2}{x})}-x} = \lim_{x\to -\infty} \frac{1}{|x|\sqrt{1+\frac{2}{x}}-x}$

$= \lim_{x\to -\infty} \frac{1}{-x\sqrt{1+\frac{2}{x}}-x} = \lim_{x\to -\infty} \frac{1}{x\left(-\sqrt{1+\frac{2}{x}}-1\right)}$

$= \frac{1}{(-\infty).(-2)} = 0.$

Q17) Let $f(x) = \begin{cases} x & \text{if } x<0 \\ x^2 & \text{if } x\geqslant 0 \end{cases}$. State where in its domain the given function is continuous, and where it is just discontinuous.

Sol: Clearly, f is continuous for all $x>0$ and $x<0$. We must examine only $0\in\mathbb{R}$.

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} x = 0 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x^2 = 0 \end{aligned}\right\} \lim_{x\to 0} f(x) = 0.$$

Also, $\lim_{x\to 0} f(x) = f(0) = 0$. It means f is continuous at $0\in\mathbb{R}$. So, f is continuous everywhere.

Q18) How should the function $\frac{x^2-4}{x-2}$ be defined at $x=2$ to be continuous there. Give a formula for the continuous extention.

Sol: $\frac{x^2-4}{x-2}$ is not defined at $x=2$. (undefined)

$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} (x+2) = 4$. If we choose f such that $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x=2 \end{cases}$

Then, f is continuous function. (Actually, $f(x)=x+2$).

Q19) Find k so that $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ k-x^2 & \text{if } x>2 \end{cases}$ is a continuous function.

Sol: If f is continuous at 2, then $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (k-x^2) = k-4$

$4 = k-4 \Rightarrow k=8$ //

Q20) Find m so that $g(x) = \begin{cases} x-m & \text{if } x<3 \\ 1-mx & \text{if } x \geq 3 \end{cases}$ is continuous for all x.

Sol: $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} (x-m) = 3-m$

$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} (1-mx) = 1-3m$

$\Rightarrow 3-m = 1-3m \Rightarrow m=-1$ //

Q21) Find the intervals which the function $f(x) = \frac{x^2-1}{x^2-4}$ is positive or negative.

Sol: $f(x) = \frac{x^2-1}{x^2-4} = \frac{(x+1)(x-1)}{(x+2)(x-2)}$ → roots of neme. are $1, -1$ → roots of denom. are $2, -2$

We can make a table by using this information.

	$-\infty$ to -2	-2 to -1	-1 to 1	1 to 2	2 to ∞
$(x+1)(x-1)$	+	+	−	+	+
$(x+2)(x-2)$	+	−	−	−	+
$f(x)$	+	−	+	−	+

Hence, $f(x) > 0$ on $(-\infty, -2) \cup (-1, 1) \cup (2, \infty)$
$f(x) < 0$ on $(-2, -1) \cup (1, 2)$

Q22) Show that $f(x) = x^3 + x - 1$ has has a zero between $x = 0$ and $x = 1$.

Sol: Intermediate Value Theorem (IVT): If f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval.

→ So, we can see that f is continuous function on $[0, 1]$. Also $-1 = f(0) < 0 < f(1) = 1$. Then, there exists a point $c \in [0, 1]$ such that $f(c) = 0$ by the IVT.

Q23) Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

Sol: Clearly, $f(x) = x^3 - 15x + 1$ is continuous on $[-4, 4]$. We can write $[-4, 4]$ as $[-4, 4] = [-4, -3] \cup [-3, 1] \cup [1, 4]$.
Also, $f(-4) = -3$, $f(-3) = 19$, $f(1) = -13$ and $f(4) = 5$
By the IVT, there exists $a \in [-4, -3]$, $b \in [-3, 1]$ and $c \in [1, 4]$ such that $f(a) = f(b) = f(c) = 0$.

Q24) Show that $\lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$ by using formal definition of limit.

Sol: Def: Let $f(x)$ be a function defined on an interval that contains $x=a$, except possibly $x=a$. Then, we say that, $\lim_{x \to a} f(x) = L$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $0 < |x-a| < \delta$.

→ Let $\varepsilon > 0$ be given. Then,

$$\left| \frac{1}{x+1} - \frac{1}{2} \right| = \left| \frac{1-x}{2(x+1)} \right| = \frac{|x-1|}{2|x+1|}$$

If $|x-1| < 1$, then $-1 < x-1 < 1 \Rightarrow 0 < x < 2$ and $1 < x+1 < 3$ so that $|x+1| > 1$. Let $\delta = \min(1, 2\varepsilon)$. If $|x-1| < \delta$, Then,

$$\left| \frac{1}{x+1} - \frac{1}{2} \right| = \frac{|x-1|}{2|x+1|} < \frac{2\varepsilon}{2} = \varepsilon.$$

Q25) Show that $\lim_{x \to 1^+} \frac{1}{x-1} = \infty$ by using formal definition of limit.

Sol: Let $M > 0$ be given. (Here M is arbirary positive number)

Then, $\frac{1}{x-1} > B$ if $0 < x-1 < \frac{1}{B}$. It means

$\frac{1}{x-1} > B$ if $1 < x < 1 + \frac{1}{B}$. If we choose $\delta = \frac{1}{B}$,

then $\frac{1}{x-1} > B$ for all $1 < x < 1+\delta$. This completes the proof.