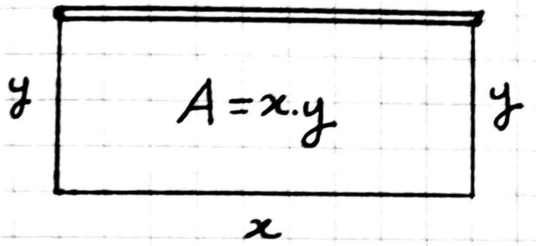
EXTREME-VALUE PROBLEMS

Example: A rectangular animal enclosure is to be contructed having one side along an existing long wall and the other three sides fenced. If 100 m of fence are available, what is the largest possible area for the enclosure?



Since the total length of the fence is 100 m, we must have $x+2y=100$.

The area A can be written as a function of only one variable;

$$x = 100-2y$$

$$A = A(y) = (100-2y)y = 100y-2y^2$$

Evidently, we require $y \geq 0$ and $y \leq 50$, in order that the area make sense. Thus, we must maximize the function $A(y)$ on the interval $[0,50]$. Being continuous on this closed, finite interval, A must have a maximum value.

Clearly, $A(0) = A(50) = 0$ and $A(y) > 0$ for $0<y<50$.

Hence, the maximum cannot occur at an endpoint. Since A has no singular points, the maximum must occur at a critical point.

$$A'(y) = 100 - 4y = 0$$

Therefore, $y = 25$ is the critical point.

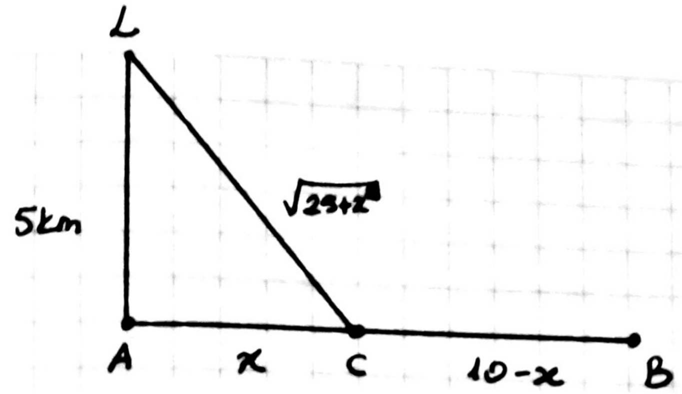
Thus, $A(25) = 1.250 \ m^2$ is the greatest possible area for the enclosure.

Example: A lighthouse L is located on a small island 5 km north of a point A on a straight east-west shoreline. A cable is to be laid from L to point B on the shoreline 10 km east of A. The cable will be laid through the water in a straight line from L to a point C on the shoreline between A and B, and from there to B along the shoreline. The part of the cable lying in the water costs \$ 5.000 / km, and the part along the shoreline costs \$ 3.000 / km.

a) Where should C be chosen to minimize the total cost of the cable?

b) Where should C be chosen if B is only 3 km from A?

a)



$T = T(x) = 5000\sqrt{25+x^2} + 3000(10-x)$; $(0 \le x \le 10)$.

T is continuous on the closed, finite interval $[0,10]$, so it has a minimum value that may occur at one of the endpoints $x=0$ or $x=10$ or at critical points in the interval $(0,10)$. T has no singular points.

$$\frac{dT}{dx} = \frac{5000(2x)}{2\sqrt{25+x^2}} - 3000 = 0$$

$$5000x = 3000\sqrt{25+x^2}$$

$$25x^2 = 9(25+x^2)$$

$$16x^2 = 225$$

$$x^2 = \frac{15^2}{4^2} \Rightarrow x = \pm\frac{15}{4} \Big\} \text{are critical points.}$$

Only one critical point $x = \frac{15}{4} = 3{,}75 \in (0,10)$.

Since $T(0) = 55.000$

$T(3{,}75) = T(15/4) = 50.000$

$T(10) \approx 55{,}902$

Thus, for minimum cost C should be 3,75 km from A.

b) If B is 3 km from A, the corresponding total cost function is

$$T(x) = 5000\sqrt{25+x^2} + 3000(3-x) \quad ; \quad (0 \leq x \leq 3).$$

The critical points, $x = \pm 15/4$ neither of which lies in the interval $(0,3)$.

Since $T(0) = 34.000$ and $T(3) \approx 29,155$ in this case we should choose $x = 3$. To minimize the total cost, the cable should go straight from A to B.