9.2) Infinite Series

① Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots \quad , \quad r: \text{common ratio.}$$

- converges to $\frac{a}{1-r}$ if $|r|<1$
- diverges otherwise.

② Telescoping Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$$

Use partial fraction: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\Rightarrow S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

(partial sum)

$$= 1 - \underbrace{\frac{1}{n+1}}_{\to 0 \text{ as } n \to \infty}$$

$\Rightarrow$ converges to 1.

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③ Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

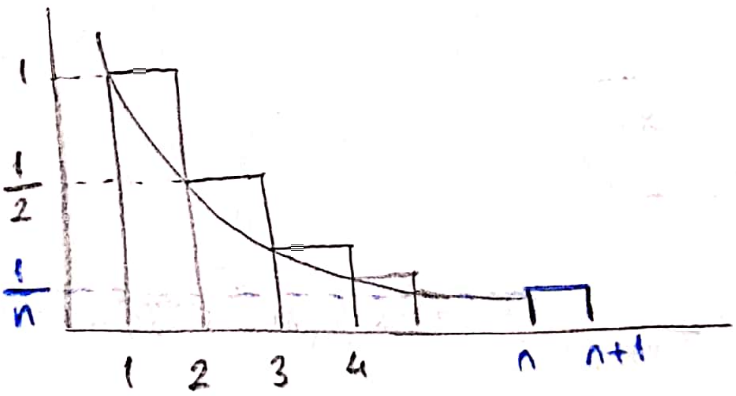
$$\Rightarrow s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \left[\begin{array}{l}\text{sum of the areas of the} \\ \text{rectangles in } (*)\end{array}\right]$$

$> \text{area under } y = \frac{1}{x} \text{ from } x = 1 \text{ to } x = n+1$

$$= \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

$\Rightarrow$ diverges to infinity.

(*) :



n^{th} Term Test for divergence of a series

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

(So, if $a_n \not\to 0$ as $n \to \infty$, then the series diverges)

$\boxed{2-16}$ Find the sum of the given series, or show that the series diverges (possibly to infinity or negative infinity)

Q2) $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \cdots = \sum_{n=1}^{\infty} 3\left(-\frac{1}{4}\right)^{n-1}$

Geometric series with common ratio $r = -\frac{1}{4}$, $|r| < 1$

and $a = 3$.

$\Rightarrow \sum_{n=1}^{\infty} 3\left(-\frac{1}{4}\right)^{n-1} = \frac{3}{1-\left(-\frac{1}{4}\right)} = \frac{12}{5}$

Q7) $\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}}$

$$\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}} = \sum_{k=0}^{\infty} 8e^3\left(\frac{2}{e}\right)^k$$
$$= 8e^3\left(1 + \frac{2}{e} + \frac{4}{e^2} + \frac{8}{e^3} + \frac{16}{e^4} + \cdots\right)$$
$$= \sum_{k=1}^{\infty} 8e^3\left(\frac{2}{e}\right)^{k-1}$$

Geometric series, $r = \frac{2}{e}$, $|r| < 1$. $a = 8e^3$

$\Rightarrow$ The series converges to $\frac{8e^3}{1-\frac{2}{e}} = \frac{8e^4}{e-2}$

Q10) $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}}$

$$\sum_{n=0}^{\infty} \frac{3}{3^{n+2}} + \sum_{n=0}^{\infty} \frac{2^n}{3^{n+2}} = \underbrace{\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^n}_{(\text{I})} + \underbrace{\sum_{n=0}^{\infty} \frac{1}{9}\left(\frac{2}{3}\right)^n}_{(\text{II})}$$

(I) is a geometric series with $r=\frac{1}{3}$, $a=\frac{1}{3}$

(II) is a geometric series with $r=\frac{2}{3}$, $a=\frac{1}{9}$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}} = \frac{1/3}{1-\frac{1}{3}} + \frac{1/9}{1-\frac{2}{3}} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Q12) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots$

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{2n(A+B)+(A-B)}{(2n-1)(2n+1)} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=\frac{-1}{2} \end{cases}$$

$$\Rightarrow S_n = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2}\left[\left(1-\cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \underbrace{\frac{1}{2n+1}}_{\to 0 \text{ as } n\to\infty}\right]$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{n\to\infty} S_n = \frac{1}{2}$$

Q16) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

$\frac{n}{n+2} \to 1$ as $n \to \infty$. So, by n^{th} term test,

this series diverges.

$\boxed{27-28}$ Decide whether the given statement is TRUE or FALSE. If it is true, prove it. If it is false, give a counter example.

Q27) If $\sum a_n$ converges, then $\sum \frac{1}{a_n}$ diverges to ∞.

FALSE. Choose $a_n = \left(-\frac{1}{2}\right)^n$. The series $\sum_{n=0} \left(-\frac{1}{2}\right)^n$,

converges. Now consider $\sum_{n=0}^{\infty} \frac{1}{a_n} = \sum_{n=0}^{\infty} (-2)^n$

Partial sum: $S_n = 1 - 2 + 4 - 8 + \dots + (-1)^n 2^n$

As n gets larger, S_n becomes larger if n is even and becomes smaller if n is odd.

So, the partial sum diverges (but neither to ∞ nor to $-\infty$)

Q28) If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum (a_n + b_n)$.

FALSE. Let $a_n = (-1)^n$, $b_n = (-1)^{n+1}$

Then both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ diverge. However

$$\sum_{n=0}^{\infty} (a_n + b_n) = (1-1) + (-1+1) + (1-1) + (-1+1) + \cdots$$
$$= 0$$

$\Rightarrow \sum_{n=0}^{\infty} (a_n + b_n)$ converges to 0.

9.3) Convergence Tests for Positive Series

① The Integral Test

Suppose that $a_n = f(n)$ where f is positive, continuous and nonincreasing on an interval $[N, \infty)$ for some positive integer N. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(t)\,dt$$

either both converge or both diverge to infinity.

* p-test $\displaystyle\sum_{n=1}^{\infty} n^{-p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges to } \infty & \text{if } p \leq 1 \end{cases}$

② Comparison Tests

I) A comparison test

Let $\{a_n\}$ and $\{b_n\}$ be sequences for which there exists a positive constant K such that ultimately

$0 \leq a_n \leq K b_n$.

(a) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

(b) $\sum a_n$ diverges to ∞ $\Rightarrow$ $\sum b_n$ diverges to ∞.

II) A limit comparison test

Suppose that $\{a_n\}$ and $\{b_n\}$ are positive sequences and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \quad \text{where} \quad L \geq 0 \quad \text{or} \quad L = \infty$$

(a) $L < \infty$ and $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

(b) $L > 0$ and $\sum b_n$ diverges to ∞ $\Rightarrow$ $\sum a_n$ diverges to ∞.

③ The Ratio Test

Suppose that $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists or is ∞.

(a) $0 \leq \rho < 1 \Rightarrow \sum a_n$ converges

(b) $1 < \rho \leq \infty \Rightarrow \sum a_n$ diverges to ∞.

(c) $\rho = 1 \Rightarrow$ the test is inconclusive

④ The Root Test

Suppose that $\sigma = \lim_{n\to\infty} (a_n)^{1/n}$ exists or is ∞

(a) $0 \leq \sigma < 1 \Rightarrow \sum a_n$ converges

(b) $1 < \sigma \leq \infty \Rightarrow \sum a_n$ diverges to ∞.

(c) $\sigma = 1 \Rightarrow$ the test is inconclusive.

$\boxed{4\text{-}24}$ Determine whether the given series converges or diverges by using any appropriate test.

Q4) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$: $\sim \frac{1}{n^{3/2}}$ (So, try to compare with $\frac{1}{n^{3/2}}$)

$$\frac{\sqrt{n}}{n^2+n+1} = \frac{1}{n^{3/2}+n^{1/2}+n^{-1/2}} \leq \frac{1}{n^{3/2}} \quad (1)$$

By p-test, $\sum \frac{1}{n^{3/2}}$ converges. (2)

Since we have (2) and (1), by comparison test,

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$ converges.

Q6) $\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$: $\sim \frac{1}{\pi^n}$

$$\frac{1}{\pi^n+5} \leq \frac{1}{\pi^n} \quad (3)$$

since $\left|\frac{1}{\pi}\right| < 1$

$\sum_{n=8}^{\infty} \left(\frac{1}{\pi}\right)^n$ is a convergent geometric series (4)

Since we have (4) and (3), by comparison test

$\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

Q8) $\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$:

$\ln x < x$ for $x>0 \Rightarrow \frac{1}{\ln(3n)} \geq \frac{1}{3n}$ (5)

$\sum^{\infty} \frac{1}{3n}$ is harmonic series, i.e, it is divergent. (6)

By comparison test, $\sum \frac{1}{\ln(3n)}$ diverges.

Q12) $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

$\frac{n^2}{1+n\sqrt{n}} \to \infty$ as $n \to \infty$ $\left(\deg(n^2) > \deg(1+n\sqrt{n}) \right)$

So, by nth term test, the series diverges.

Q14) $\sum_{n=2}^{\infty} \frac{1}{n\ln n(\ln\ln n)}$:

Let $f(x) = \frac{1}{n\ln n(\ln\ln n)^2}$ and consider the following integral.

$$\int_3^{\infty} \frac{dx}{x\ln x(\ln\ln x)^2} = \lim_{R\to\infty} \int_3^R \frac{dx}{x\ln x(\ln\ln x)^2}$$

Substitution

$$\boxed{u = \ln\ln x \Rightarrow du = \frac{dx}{x\ln x}}$$

$\lim_{R\to\infty} u(R) = \infty$

$u(3) = \ln\ln 3$

$$= \int_{\ln\ln 3}^{\infty} \frac{du}{u^2} = -\frac{1}{u}\Big|_{\ln\ln 3}^{\infty} = \frac{1}{\ln\ln 3}$$

The integral converges.

So by the Integral test, the series also converges.

NOTE. The lower limit of the integral needs to be greater than 2, since $\ln 2 < 1 \Rightarrow \ln\ln 2 < 0$ (2 is not an appropriate limit point) but $\ln 3 > 1$ $\Rightarrow \ln\ln 3 > 0$.

Q20) $\sum_{n=1}^{\infty} \frac{(2n)!\,6^n}{(3n)!}$:

$$\rho = \lim_{n\to\infty}\left(\frac{(2(n+1))!\,6^{n+1}}{(3(n+1))!} \Big/ \frac{(2n)!\,6^n}{(3n)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{(2n)!}(2n+1)(2n+2)\,6^{n+1}}{\cancel{(3n)!}(3n+1)(3n+2)(3n+3)} \cdot \frac{\cancel{(3n)!}}{\cancel{(2n)!}\,\cancel{6^n}}\right)$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)\cdot 6}{(3n+1)(3n+2)(3n+3)} = 0$$

(degree of numerator is less than degree of denominator)

$\Rightarrow \rho < 1 \Rightarrow$ the series converges by the ratio test.

Q24) $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

$$\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{1+n} \quad (7)$$

$$\sum_{n=1}^{\infty} \frac{1}{1+n} = \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) - 1 \Rightarrow \text{harmonic series} \Rightarrow \text{diverges.} \quad (8)$$

So, by comparison test, $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$ diverges.

$\boxed{39}$ Use the root test to test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

Let $\sigma = \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$

Indeterminate form: 1^{∞}. Use ln.

$$\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) \quad [\infty \cdot 0]$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} \quad \left[\frac{0}{0}\right]. \text{ Use L'Hopital:}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{n+1}{\cancel{n}}\right)\frac{(n+1-n)}{(n+1)^{\cancel{2}}}}{-\frac{1}{n^{\cancel{2}}}} = \lim_{n\to\infty} \left(-\frac{n}{n+1}\right) = -1$$

$\Rightarrow \sigma = e^{-1} < 1$. By root test, the series converges.