12.6) Linear Approximations, Differentiability and Differentials.

$\boxed{4, 6}$ Use suitable linearizations to find approximate values for the given functions at the points indicated.

Q4) $f(x,y) = \dfrac{24}{x^2+xy+y^2}$ at $(2.1, 1.8)$.

> The tangent plane to the graph of $z = f(x,y)$ at (a,b) is $z = L(x,y)$, where
>
> $$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
>
> is the linearization of f at (a,b).
>
> We can use $L(x,y)$ to approximate values of $f(x,y)$ near (a,b).

Take $(a,b) = (2,2)$. $\Rightarrow f(2,2) = 2$

$$f_x(x,y) = \frac{-24(2x+y)}{(x^2+xy+y^2)^2} \Rightarrow f_x(2,2) = -1$$

$$f_y(x,y) = \frac{-24(x+2y)}{(x^2+xy+y^2)^2} \Rightarrow f_y(2,2) = -1$$

So, $L(x,y) = f(2,2) + f_x(2,2)(x-2) + f_y(2,2)(y-2)$

$= 2 - (x-2) - (y-2)$ [linearization of f at $(2,2)$]

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So, $f(2.1, 1.8) \approx L(2.1, 1.8)$

$$\approx 2-(2.1-2)-(1.8-2)$$

$$\approx 2-0.1+0.2 = 2.1$$

Q6) $f(x,y) = xe^{y+x^2}$ at $(2.5, -3.92)$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

(Linearization of $f(x,y)$ at (a,b))

Take $(a,b) = (2,-4) \Rightarrow f(2,-4) = 2$

$f_x(x,y) = e^{y+x^2} + 2x^2 e^{y+x^2} \Rightarrow f_x(2,-4) = 9$

$f_y(x,y) = xe^{y+x^2} \Rightarrow f_y(2,-4) = 2$

So, $L(x,y) = f(2,-4) + f_x(2,-4)(x-2) + f_y(2,-4)(y+4)$

$$= 2 + 9(x-2) + 2(y+4)$$

is the linearization of f at $(2,-4)$

$\Rightarrow f(2.5, -3.92) \approx L(2.5, -3.92)$

$$\approx 2 + 9(0.5) + 2(0.08)$$

$$\approx 2.61$$

12.7) Gradients and Directional Derivative

⊛ The <u>gradient</u> of a function $f(x,y)$ at (a,b):

$$\nabla f(a,b) = f_x(a,b)\,\hat{i} + f_y(a,b)\,\hat{j}$$

Note that $\nabla f(a,b)$ is a normal vector to the level curve of f that passes through (a,b).

⊛ The <u>directional derivative</u> of $f(x,y)$ at (a,b) in the direction of a unit vector u (rate of change of $f(x,y)$ at (a,b) along u):

$$D_u f(a,b) = u \cdot \nabla f(a,b)$$

At (a,b),

- $f(x,y)$ increases most rapidly in the direction of $\nabla f(a,b)$
- $f(x,y)$ decreases most rapidly in the direction of $-\nabla f(a,b)$
- the rate of change is zero in directions tangent to the level curve of f passing through (a,b).

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(*) The gradient of the function $f(x,y,z)$ at (a,b,c):

$$\nabla f(a,b,c) = f_x(a,b,c)\hat{\imath} + f_y(a,b,c)\hat{\jmath} + f_z(a,b,c)\hat{k}$$

The gradient $\nabla f(a,b,c)$ is a normal vector to the level surface of f that passes through (a,b,c).

(*) The directional derivative of $f(x,y,z)$ at (a,b,c) in the direction of the unit vector $\mathbf{v}$:

$$D_{\mathbf{v}} f(a,b,c) = \mathbf{v} \cdot \nabla f(a,b,c)$$

Q5) $f(x,y) = \ln(x^2+y^2)$. Find

(a) the gradient of $f(x,y)$ at $(1,-2)$:

$$\nabla f(x,y) = f_x(x,y)\hat{\imath} + f_y(x,y)\hat{\jmath}$$

$$= \frac{2x}{x^2+y^2}\hat{\imath} + \frac{2y}{x^2+y^2}\hat{\jmath}$$

$$\Rightarrow \nabla f(1,2) = \frac{2}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$$

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(b) the equation of the plane tangent to the graph of $f(x,y)$ at $(1,-2,f(1,-2))$

The tangent plane to $z=f(x,y)$ at $(1,-2)$ is given by

$$z=f(1,-2)+f_x(1,-2)(x-1)+f_y(1,-2)(y+2)$$

$$\Rightarrow \quad z=\ln 5+\frac{2}{5}(x-1)-\frac{4}{5}(y+2)$$

(c) an equation of the straight line tangent at $(1,-2)$ to the level curve of $f(x,y)$ passing through $(1,-2)$:

The tangent line of the function $f(x,y)=\ln 5$ at $(1,-2)$ can be calculated using the normal vector at $(1,-2)$:

$$f_x(1,-2)(x-1)+f_y(1,-2)(y+2)=0$$

$$\Rightarrow \quad \frac{2}{5}(x-1)-\frac{4}{5}(y+2)=0$$

(If we write $z=\ln 5$ in the equation of the tangent plane above, we get the same result)

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Q6) $f(x,y) = \sqrt{1+xy^2}$. Find

(a) the gradient of $f(x,y)$ at $(2,-2)$:

$$\nabla f(x,y) = \frac{y^2}{2\sqrt{1+xy^2}}\,\hat{\imath} + \frac{2xy}{2\sqrt{1+xy^2}}\,\hat{\jmath}$$

$$\Rightarrow \nabla f(2,-2) = \frac{2}{3}\hat{\imath} - \frac{4}{3}\hat{\jmath}$$

(b) the equation of the plane tangent to the graph of $f(x,y)$ at $(2,-2,f(2,-2))$:

$$z = f(2,-2) + f_x(2,-2)(x-2) + f_y(2,-2)(y+2)$$

$$\Rightarrow z = 3 + \frac{2}{3}(x-2) - \frac{4}{3}(y+2)$$

(c) an equation of straight line tangent at $(2,-2)$ to the level curve of $f(x,y)$ passing through $(2,-2)$:

Using part (b)

$$\frac{2}{3}(x-2) - \frac{4}{3}(y+2) = 0$$

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Q8) Find an equation of the tangent plane to the level surface of $f(x,y,z) = \cos(x+2y+3z)$ at $(\frac{\pi}{2}, \pi, \pi)$:

$$\nabla f(x,y,z) = f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}$$

$$= -\sin(x+2y+3z)(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$$

$$\Rightarrow \nabla f(\tfrac{\pi}{2}, \pi, \pi) = -\sin\frac{11\pi}{2}(\hat{\imath}+2\hat{\jmath}+3\hat{k}) = \hat{\imath}+2\hat{\jmath}+3\hat{k}$$

(normal to the tangent plane)

$$f(\tfrac{\pi}{2}, \pi, \pi) = \cos\frac{11\pi}{2} = 0$$

So, the tangent plane to $f(x,y,z) = 0$ at $(\frac{\pi}{2}, \pi, \pi)$ has equation

$$f_x(\tfrac{\pi}{2},\pi,\pi)(x-\tfrac{\pi}{2}) + f_y(\tfrac{\pi}{2},\pi,\pi)(y-\pi) + f_z(\tfrac{\pi}{2},\pi,\pi)(z-\pi) = 0$$

$$\Rightarrow 1(x-\tfrac{\pi}{2}) + 2(y-\pi) + 3(z-\pi) = 0$$

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Q12) Find the rate of change of

$f(x,y)=\dfrac{x}{1+y}$ at $(0,0)$ in the direction of $\hat{i}-\hat{j}$:

The unit vector in the direction of $\hat{i}-\hat{j}$ is

$$u=\frac{\hat{i}-\hat{j}}{|\hat{i}-\hat{j}|}=\frac{\hat{i}-\hat{j}}{\sqrt{2}}$$

The gradient of $f(x,y)$ is

$$\nabla f(x,y)=\frac{1}{1+y}\hat{i}-\frac{x}{(1+y)^2}\hat{j}$$

$$\Rightarrow \nabla f(0,0)=\hat{i}$$

So, the directional derivative of $f(x,y)$ at $(0,0)$ in the direction of u is

$$D_u f(0,0)=u\cdot\nabla f(0,0)=\frac{\hat{i}-\hat{j}}{\sqrt{2}}\cdot\hat{i}=\frac{1}{\sqrt{2}}$$

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Q17) In what directions at the point $(2,0)$ does the function $f(x,y) = xy$ have rate of change -1?

We need to find a unit vector u such that

$$D_u f(2,0) = u \cdot \nabla f(2,0) = -1$$

(I) $u = u_1 \hat{i} + u_2 \hat{j}$ is a unit vector $\Rightarrow u_1^2 + u_2^2 = 1$

(II) $\nabla f(x,y) = y\hat{i} + x\hat{j} \Rightarrow \nabla f(2,0) = 2\hat{j}$

$$\Rightarrow \quad u \cdot \nabla f(2,0) = u_1 \cdot 0 + u_2 \cdot 2 = 2u_2 = -1$$

So we get $u_2 = -\frac{1}{2} \overset{\text{from (I)}}{\Rightarrow} u_1 = \mp \frac{\sqrt{3}}{2}$

$f(x,y)$ has a rate of change -1 in the directions of $u = \mp \frac{\sqrt{3}}{2} \hat{i} - \frac{1}{2} \hat{j}$ at $(2,0)$

Question. Are there directions in which the rate is -3?

$$u \cdot \nabla f(2,0) = 2u_2 = -3 \Rightarrow u_2 = -\frac{3}{2}$$

$\Rightarrow u_1^2 = -\frac{5}{4}$ not possible.

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So, there is no direction in which f changes at rate -3 at $(2,0)$.

Question. The direction in which the rate is -2?

$u \cdot \nabla f(2,0) = 2u_2 = -2 \Rightarrow u_2 = -1$

$\Rightarrow u_1 = 0 \quad \Rightarrow u = 0\hat{i} - 1\hat{j} = -\hat{j}$

At $(2,0)$, f has rate of change in the direction of $u = -\hat{j}$.

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Q22) Find an equation of the curve in the xy-plane that passes through the point $(1,1)$ and intersects all the level curves of the function $f(x,y)=x^4+y^2$ at right angles.

Let $y=g(x)$ $(y-g(x)=0)$ be the curve. Then

$\nabla(y-g(x)) = -g'(x)\hat{i}+\hat{j}$ is a normal at (x,y)

The level curves of $f(x,y)$ can be written as

$f(x,y)=C$, C is some real number. Then

$\nabla f(x,y) = 4x^3\hat{i}+2y\hat{j}$ is a normal at (x,y).

These curves intersect at right angles if their normals are perpendicular. So

$$\nabla(y-g(x)) \cdot \nabla f(x,y) = 0$$

$\Rightarrow (-g'(x))4x^3 + 1 \cdot 2y = 0$. Since $y=g(x)$,

$$-4x^3 g'(x) + 2g(x) = 0.$$

$$\Rightarrow 4x^3 g'(x) = 2g(x)$$

$$\Rightarrow \frac{g'(x)}{g(x)} = \frac{1}{2x^3} \quad \text{(Integrate both sides)}$$

$$\Rightarrow \int \frac{g'(x)}{g(x)} dx = \int \frac{1}{2x^3} dx$$

$$\left[u = g(x) \Rightarrow du = g'(x)dx \Rightarrow \int \frac{du}{u} = \ln u \right]$$

$$\Rightarrow \ln(g(x)) = -\frac{1}{4x^2} + \underbrace{\ln C}_{\text{a constant}} \qquad (*)$$

Since $y = g(x)$ passes through $(1,1)$, we take $x=1$, $g(1)=1$.

$$\Rightarrow \ln(g(1)) = -\frac{1}{4 \cdot 1^2} + \ln C$$

$$\Rightarrow \ln 1 = -\frac{1}{4} + \ln C \Rightarrow \ln C = \frac{1}{4}$$

So, $(*)$ becomes $\ln(g(x)) = -\frac{1}{4x^2} + \frac{1}{4}$

$$\Rightarrow g(x) = e^{\left(-\frac{1}{4x^2} + \frac{1}{4}\right)}$$

$$\Rightarrow \text{Curve: } y = e^{\left(-\frac{1}{4x^2} + \frac{1}{4}\right)}$$