1. Solutions: 1)a) Since $\sin x$ is a continuous function we have

$$\lim_{n\to\infty} \sin\big(\frac{\pi n}{2n+1}\big) = \sin\big(\lim_{n\to\infty}\frac{\pi n}{2n+1}\big) = sin(\frac{\pi}{2}) = 1 \neq 0.$$

By using n^{th} term test, the series diverges.
b) Let $a_n = \frac{n}{n^3-4n+1}$ and $b_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3}{n^3-4n+1} = 1 \neq 0, \infty$. Since $\sum_{n=2}^{\infty}\frac{1}{n^2}$ convergent p-series $\sum_{n=2}^{\infty}\frac{n}{n^3-4n+1}$ is also convergent by the limit comparison test.

2) Let $u_n = (-1)^n\frac{(x-4)^n}{n^{1/2}}$. By the ratio test

$$\lim_{n\to\infty} \mid \frac{u_{n+1}}{u_n} \mid = \mid x-4 \mid \lim_{n\to\infty}(\frac{n}{n+1})^{\frac{1}{2}} < 1.$$

So $\mid x-4 \mid < 1$, i.e., $3 < x < 5$. We have to check end points. For $x = 3$, we have $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ divergent p-series. For $x = 5$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\frac{1}{2}}}$ does not converge absolutely but $0 < \frac{1}{(n+1)^{1/2}} < \frac{1}{(n)^{1/2}}$ and $\lim_{n\to\infty}\frac{1}{(n)^{1/2}} = 0$. Therefore, using Alternating series test we get series converge conditionally at 5. Then, the interval of convergence is $3 < x \leq 5$. Also the radius of convergence is $R = 1$.
b) Along $x = 0$ $\lim_{y\to 1}\frac{0}{(y-1)^2} = 0$ and along $x = y-1$ $\lim_{x\to 0}\frac{x^2}{2x^2} = \frac{1}{2}$. As $0 \neq \frac{1}{2}$, limit does not exist.
3)

a) Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $\mid x \mid < 1$. Using this series we obtain $\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ for $\mid x \mid < 1$.

b) We can integrate above series in this interval and we get

$$\int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n\int_0^x t^{2n}dt \implies \arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \text{ for } -1 < x < 1.$$

We have to check the end points: For $x = 1$, we get $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ alternating series and

$$\sum_{n=0}^{\infty} \mid \frac{(-1)^n}{2n+1} \mid = \sum_{n=0}^{\infty}\frac{1}{2n+1}$$

does not converge absolutely but it converges conditionally from the Alternating series test because $0 < a_{n+1} = \frac{1}{2n+3} < a_n = \frac{1}{2n+1}$ for all n and $\lim_{n\to\infty}\frac{1}{2n+1} = 0$. For $x = -1$, similarly above case, we get $\sum_{n=0}^{\infty}\frac{(-1)^n+1}{2n+1}$ is also convergent conditionally. Therefore, this representation valid for $-1 \leq x \leq 1$.

c) Using b)we obtain

$$\arctan x^2 = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1} \text{ for } -1 \leq x \leq 1.$$

d) From b) we know that $\arctan x \approx P_3 = x - \frac{x^3}{3}$. Therefore,
$\arctan\frac{1}{2} = \frac{1}{2} - \frac{1}{3}(\frac{1}{2})^3 = \frac{13}{24}$.

4)a)

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (3v^2u)f_x(x,y) + uf_y(x,y).$$

Then,

$$\frac{\partial z}{\partial v}|(1,1) = 3f_x(2,1) + f_y(2,1) = 3.(1) - 2 = 1.$$

b)Let $F(x,y,z)=x^2-y^2-z=0$. Then the normal of the tangent plane at $(-2,1)$ is $\nabla F(-2,1,3)=(-4,-2,-1)$. So the tangent plane at this point is $-4x-2y-z=d$ and $d=-4(-2)+-2(1)-3=3$. Then the tangent plane is $-4x-2y-z=3$.

5)a)These two planes perpendicular to each other if their normals are perpendicular to each other. Therefore the dot product of $n_1=(-1,3,-5)$ and $n_2=(2,-1,-1)$ must be zero. If we product these normals, we get $n_1.n_2=-2-3+5=0$.

b) The plane which is the perpendicular to D_1 and D_2 has the normal n such that its normal is perpendicular to normals of the D_1 and D_2 ,that is, $n=n_1\times n_2=-8i-11j-5k$. As a result of this we have $-8x-11y-5z=d$. The point $(0,1,1)$ holds the equation of the plane therefore we have $d=-16$ and so the equation of the plane is $8x+11y+5z=16$.

c)$d=\frac{|2(1)-1-1-4|}{\sqrt{2^2+(-1)^2+(-1)^2}}=\frac{4}{\sqrt{6}}$.