GTU, Fall 2020, MATH 101

The Derivative

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* If $f'(x_0)$ exist then the eq. of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

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* f is differentiable on $[a, b]$ if $f'(x)$ exists for all x in (a, b) and $f'_{+}(a)$ and $f'_{-}(b)$ exist.

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General Power Rule: If $f(x) = x^r$, then

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Exercise: $f(x) = x^n$ where $n = 1, 2, 3, \ldots$
Show that $f'(x) = nx^{n-1}$.

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Then $f'(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases} = \frac{x}{|x|} = sgn(x).$

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  * $\frac{d}{dx}x^2 = 2x$ (the derivative with respect to x of x^2 is $2x$)

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The Newton quotient

$$\frac{f(x+h)-f(x)}{h}$$

can be written in the form $\Delta y/\Delta x$ where $\Delta y = f(x+h) - f(x)$ is the increment in y, and $\Delta x = (x+h) - x = h$. Using symbols:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

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The figure below shows the relationship between the increment Δy and the differential dy.

Δy represent the change in height of the curve $y = f(x)$.

dy represent the change in height of the tangent line when x changes by an amount $dx = \Delta x$.