GTU, Fall 2020, MATH 101

Continuity

Continuity at a Point

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  Interior points : $(-1, 1)$ and end points: -1,1
* For $S = (-\infty, 0) \cup (0, \infty)$, all points in S is an interior point.

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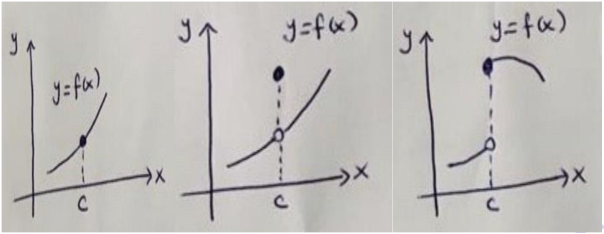
Consider figure below and investigate the continuity of f at c.



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* f is continuous at c iff it is both right continuous and left continuous at c.
* Consider the Heaviside function $H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$

  $H(x)$ is continuous for $x \neq 0$. It is right continuous at 0 since $\lim\limits_{x \to 0+} H(x) = H(0)$.

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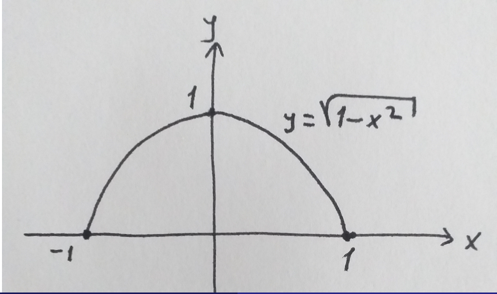
  f is continuous at a right endpoint 1 since $\lim\limits_{x \to 1-} f(x) = 0 = f(1)$.
  f is continuous at a left endpoint -1 since $\lim\limits_{x \to -1+} f(x) = 0 = f(-1)$.



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* $\lim\limits_{x \to 0+} \sqrt{x} = 0$ and $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$ $(\forall a \in (0,\infty)) \implies \sqrt{x}$ is continuous on $[0,\infty)$.

Note that

* all polynomials;
* all rational functions;
* all rational powers $x^{m/n}$;
* the trigonometric functions;
* the absolute value functions $|x|$;

are continuous whenever they are defined.

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- If $f(g(x))$ is defined on an interval containing c, and f is continuous at L and $\lim_{x\to c} g(x) = L$, then

  $$\lim_{x\to c} f(g(x)) = f(L) = f\Big(\lim_{x\to c} g(x)\Big).$$

For example,

* $2x^3 + 5$
* $\frac{x+3}{x^4-6}$
* $\sqrt{x^3 + 2x + 1}$
* $\frac{|x|}{|x+1|}$

are continuous everywhere on their respective domains.

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  For example,
* $f(x) = \begin{cases} x & \text{if} \quad x \neq 1 \\ 2 & \text{if} \quad x = 1 \end{cases}$
  has a removable discontinuity at $x = 1$.

Continuous Functions on Closed, Finite Intervals

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The Max.-Min. Theorem

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* The theorem implies that a function that is continuous on a closed, finite interval is **bounded**.This means there must exist a number K such that

$$|f(x)| \leq K \text{ that is } -K \leq f(x) \leq K.$$

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* Consider $f(x) = \begin{cases} x & \text{if} \quad x \neq 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$
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f is continuous on $[1/2, 1]$. It is bounded. It has both a maximum and a minimum value.

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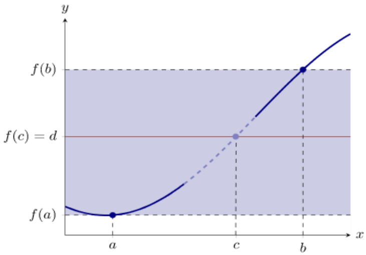

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