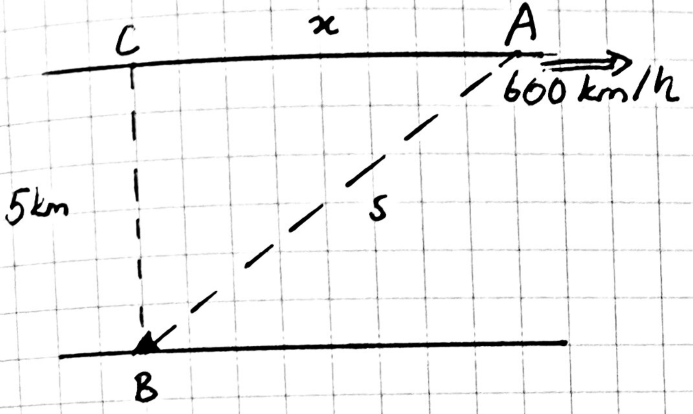
4.1 RELATED RATES

How to solve related rates problems:

1. Try to understand the relationships between the variable quantities. What is given? What is to be found?
2. Make a sketch if appropriate.
3. Define any symbols you want to use that are not defined in the statement of the problem. Express given and required quantities and rates in terms of these symbols.
4. From a careful reading of the problem or consideration of the sketch, identify one or more equations linking the variable quantities
5. Differentiate the equations implicitly with respect to time, regarding all variable quantities as functions of time.
6. Substitute any given values for the quantities and their rates, then solve the resulting equation(s) for the unknown quantities and rates.
7. Make a concluding statement answering the question asked. And check if your answer is reasonable?

Example : An aircraft is flying horizontally at a speed of 600 km/h. How fast is the distance between the aircraft and a radio beacon increasing 1 min after the aircraft passes 5 km directly above the beacon?



$s^2 = x^2 + 5^2$ differentiating implicitey w.r.t. time "t"

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 0$$

It is given that,

$$\frac{dx}{dt} = 600 \text{ km/h} = 10 \text{ km/min}.$$

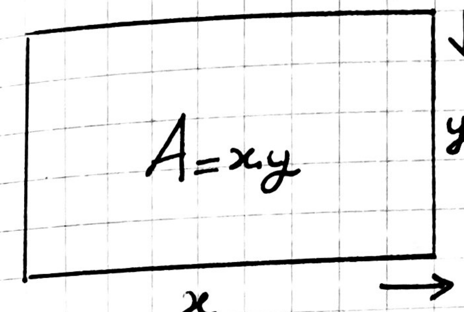
Therefore, $x = 10$ km at time $t = 1$ min. At that time

$s = \sqrt{10^2 + 5^2} = 5\sqrt{5}$ km and increasing at rate of time

$$\frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt} = \frac{10}{5\sqrt{5}} 600 = \frac{1200}{\sqrt{5}} \approx 536{,}7 \text{ km/h}.$$

One minute after the aircraft passes over the beacon, its distance from the beacon is increasing about 537 km/h.

Example : How fast is the area of a rectangle changing if one side is 10 cm long and <u>increasing</u> at a rate of 2 cm/s and the other side is 8 cm long and is <u>decreasing</u> at a rate of 3 cm/s ?



$$A = x \cdot y$$ differentiating implicitly w.r.t. time "t"

$$\frac{dA}{dt}\bigg|_{\substack{x=10cm\\y=8cm}} = \left(\frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt}\right)\bigg|_{\substack{x=10\\y=8}}$$

$$\frac{dx}{dt} = 2 cm/s$$

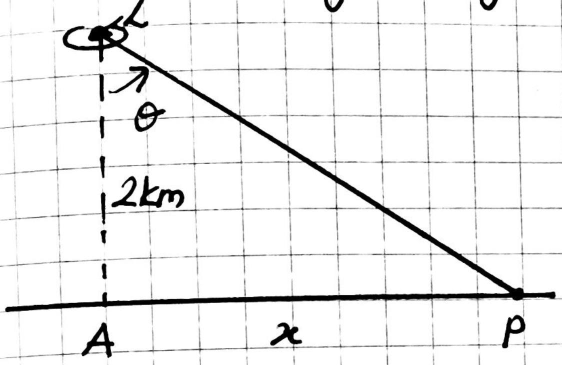
$$\frac{dy}{dt} = -3 cm/s$$

$$\frac{dA}{dt} = 2(8) + 10(-3)$$

$$\frac{dA}{dt} = -14 \ cm^2/s \ .$$

The ~~ar~~ area of rectangle is <u>decreasing</u> at a rate of 14 cm^2/s.

Example : A lighthouse L is located on a small island 2km from the nearest point A on a long, straight shoreline. If the lighthouse lamp rotates at 3 revolutions per minute, how fast is the illuminated spot P on the shoreline moving along the shoreline when it is 4km from A?



$$\tan\theta = \frac{x}{2}$$

$$x = 2\tan\theta \qquad \text{"differentiating implicitly w.r.t time"}$$

$$\frac{dx}{dt} = 2\sec^2\theta \, \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = (3\,rev/min)\left(2\pi\,\frac{radians}{rev}\right) = 6\pi\,\frac{radians}{min}$$

When $x = 4$ we have $\tan\theta = \frac{4}{2} = 2$ and $\sec^2\theta = 1 + \tan^2\theta$

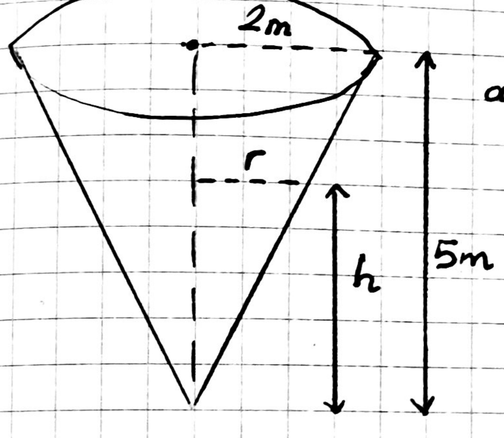
$$\sec^2\theta = 1 + 2^2 = 5$$

Thus,

$$\frac{dx}{dt} = (2).(5)(6\pi) = 60\pi \approx 188,5 \text{ km/min}$$

The spot of light is moving along the shoreline at a rate of about 189 km/min when it is 4km from A.

Example: A leaky water tank is the shape of an inverted right circular cone with depth 5m and top radius 2m. When the water in the tank is 4 m deep, it is leaking out at a rate of $1/12\ m^3/min$. How fast is the water level in the tank dropping at that time?



The volume of water V (in m^3) in the tank at time "t";

$$V = \frac{1}{3}\pi r^2 h$$

Using similar triangles,

$$\frac{r}{h} = \frac{2}{5} \quad \text{so} \quad r = \frac{2h}{5} \quad \text{and}$$

$$V(h) = V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 \cdot h = \frac{4\pi}{75}h^3$$

Differentiating implicitly w.r.t time:

$$\frac{dV}{dt} = \frac{4\pi}{75} \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{4\pi}{25}h^2\frac{dh}{dt}$$

Since $\frac{dV}{dt} = -\frac{1}{12}\ m^3/min$ when $h=4$, we have

$$-\frac{1}{12} = \frac{4\pi}{25} \cdot 4^2\frac{dh}{dt} \quad \text{So} \quad \frac{dh}{dt} = -\frac{25}{768\pi}\ m/min.$$

When the water in the tank is 4m deep, its level is dropping at a rate of $\frac{25}{768\pi}\frac{m}{min}$.

INDETERMINATE FORMS

Types of indeterminate forms

Type	Example
$[0/0]$	$\lim_{x\to 0} \frac{\sin x}{x}$
$[\infty/\infty]$	$\lim_{x\to 0} \frac{\ln(1/x^2)}{\cot(x^2)}$
$[0.\infty]$	$\lim_{x\to 0^+} x\ln\left(\frac{1}{x}\right)$
$[\infty-\infty]$	$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \left(\tan x - \frac{1}{\pi-2x}\right)$
$[0^0]$	$\lim_{x\to 0^+} x^x$
$[\infty^0]$	$\lim_{x\to\left(\frac{\pi}{2}\right)^-} (\tan x)^{\cos x}$
$[1^\infty]$	$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

You can evaluate many indeterminate forms of type [0/0] with simple algebra, typically by cancelling common factors. We will now develop another method called L'Hôpital's Rule for evaluating limits of indeterminate forms of the types [0/0] and [∞/∞].

Theorem: (The First L'Hôpital Rule)

Suppose the functions f and g are differentiable on the interval (a,b) and $g'(x) \neq 0$ there. Suppose also that

i) $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$ and

ii) $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L$ (where L is finite or ∞ or $-\infty$)

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

Similar results hold if every occurence of $\lim\limits_{x\to a^+}$ is replaced by $\lim\limits_{x\to b^-}$ or even $\lim\limits_{x\to c}$ where $a<c<b$.

The cases $a=-\infty$ and $b=\infty$ are also allowed.

<u>Example</u>: Evaluate $\lim\limits_{x\to 0} \dfrac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2}$ $\left[=\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{2\cos x - 2\cos(2x)}{2e^x - 2 - 2x} = \lim_{x\to 0} \frac{\cos x - \cos(2x)}{e^x - 1 - x} \quad \left[=\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x + 2\sin(2x)}{e^x - 1} \quad \left[=\frac{0}{0}\right] = \lim_{x\to 0} \frac{-\cos x + 4\cos(2x)}{e^x}$$

$$= \frac{-1+4}{1} = 3$$

Example: Evaluate the following limits.

a) $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} \dfrac{2x - \pi}{\cos^2 x} \quad \left[= \dfrac{0}{0}\right]$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{2}{2\cos x(-\sin x)} = \frac{2}{-0} = -\infty$$

b) $\lim\limits_{x \to 1^+} \dfrac{x}{\ln x}$ not an indeterminate form.

$$\lim_{x \to 1^+} \frac{x}{\ln x} = \frac{1}{0} = \infty$$

Example: Evaluate $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right) \quad [= \infty - \infty]$

Thus L' Hôpital's Rule cannot be applied.

However, it becomes [0/0] after we combine the fractions into one fraction.

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) [= \infty - \infty]$$

$$= \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \quad \left[= \frac{0}{0}\right]$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} \quad \left[= \frac{0}{0}\right]$$

$$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = -\frac{0}{2} = 0 //$$

Theorem : (The second L'Hôpital Rule)

Suppose that f and g are differentiable on the interval (a,b) and that $g'(x) \neq 0$ there.

i) $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = \pm\infty$ and

ii) $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$ (where L is finite, or ∞ or $-\infty$)

Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

Again, similar results hold for $\lim_{x \to b^-}$ and for $\lim_{x \to c}$ where $a<c<b$. The cases $a=-\infty$ and $b=\infty$ are also allowed.

Example:

a) $\lim_{x \to \infty} \frac{x^2}{e^x}$ $\left[= \frac{\infty}{\infty}\right]$

$= \lim_{x \to \infty} \frac{2x}{e^x}$ $\left[= \frac{\infty}{\infty}\right]$

$= \lim_{x \to \infty} \frac{2}{e^x} = 0$ //

b) $\lim_{x \to 0^+} x^a \ln x$ $(a>0)$ (Indeterminate form is: $[0 \cdot (-\infty)]$)

$= \lim_{x \to 0^+} \frac{\ln x}{x^{-a}}$ $\left(= \frac{-\infty}{\infty}\right)$

$= \lim_{x \to 0^+} \frac{1/x}{-a x^{-a-1}} = \lim_{x \to 0^+} \frac{x^a}{-a} = \frac{0}{-a} = 0$ //

Example: Evaluate $\lim_{x\to 0^+} x^x$. $[=0^0]$

Let $y = x^x$ then

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln x = (0).(1) = 0$$

Hence $\lim_{x\to 0} x^x = e^{\lim_{x\to 0^+} \ln y} = e^0 = 1$

Example: Evaluate $\lim_{x\to\infty} \left(1+\sin\frac{3}{x}\right)^x$. $[=1^\infty]$

Let $y = \left(1+\sin\frac{3}{x}\right)^x$ then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1+\sin\frac{3}{x}\right) \quad [=\infty.0]$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\sin\frac{3}{x}\right)}{\frac{1}{x}} \quad \left[=\frac{0}{0}\right]$$

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{1+\sin\frac{3}{x}}\right)\left(\cos\frac{3}{x}\right)\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3\cos\frac{3}{x}}{1+\sin\frac{3}{x}} = \frac{3}{1} = 3$$

Hence $\lim_{x\to\infty} \left(1+\sin\frac{3}{x}\right)^x = e^3.$