MATH 101.2 PS-6

Q1) Show that the function $f(x)=\frac{x}{\sqrt{x^2+1}}$ is one-to-one and calculate the inverse function f^{-1}. Specify the domains and ranges f and f^{-1}.

Sol: If $f(x_1)=f(x_2)$, then $\frac{x_1}{\sqrt{x_1^2+1}}=\frac{x_2}{\sqrt{x_2^2+1}}$ (*). Thus,

$x_1^2(x_2^2+1)=x_2^2(x_1^2+1)$. Hence, $x_1^2=x_2^2$. From (*) x_1 and x_2 must have same sign. Then, $x_1=x_2$ and f is 1 to 1. Let $y=f^{-1}(x)$. Then, $x=f(y)=\frac{y}{\sqrt{y^2+1}}$ and $x^2(y^2+1)=y^2$. Hence, $y^2=\frac{x^2}{1-x^2}$. Since $f(y)$ and y have the same sign, we must have $y=\frac{x}{\sqrt{1-x^2}}$, so $f^{-1}(x)=\frac{x}{\sqrt{1-x^2}}$. Also, $D(f)=(-\infty,\infty)$ and $D(f^{-1})=(-1,1)$,
$R(f)=(-1,1)$ and $R(f^{-1})=(-\infty,\infty)$.

Q2) Let f be a one-to-one function with inverse f^{-1}. Calculate the inverse of $p(x)=\frac{1}{1+f(x)}$ in terms of f^{-1}.

Sol: $p(x)=\frac{1}{1+f(x)}$. Let $y=p^{-1}(x)$. Then,

$x=p(y)=\frac{1}{1+f(y)}$ so, $f(y)=\frac{1}{x}-1$, and

$p^{-1}(x)=y=f^{-1}\left(\frac{1}{x}-1\right)$.

Q3) Find $h^{-1}(-3)$ if $h(x) = x.|x|+1$.

Sol: Let us say $h^{-1}(-3) = a$. Then, $h(a) = -3$ and
$h(a) = a|a|+1 = -3 \Rightarrow a|a| = -4 \Rightarrow a = -2$.
So, $h^{-1}(-3) = -2$.

Q4) Find $(f^{-1})'(-2)$ if $f(x) = x\sqrt{3+x^2}$.

Sol: Let $y = f^{-1}(x)$. Then, $x = f(y) = y\sqrt{3+y^2}$, so

$$1 = y'\sqrt{3+y^2} + y\,\frac{2yy'}{2\sqrt{3+y^2}}. \quad \text{Hence,} \quad y' = \frac{\sqrt{3+y^2}}{3+2y^2}.$$

Since $f(-1) = -2$, $f^{-1}(-2) = -1$. we have

$$(f^{-1})'(-2) = \left.\frac{\sqrt{3+y^2}}{3+2y^2}\right|_{y=-1} = \frac{2}{5}.$$

Q5) Simplify the expression $\log_x(x(\log_y y^2))$.

Sol: $\log_x(x(\log_y y^2)) = \log_x(x.[2\underbrace{\log_y y}_{1}]) = \log_x(2x)$

$$= \log_x x + \log_x 2 = 1 + \frac{1}{\log_2 x} \quad //$$

Q6) Solve $2\log_3 x + \log_9 x = 10$ for x.

Sol: $2.\log_3 x + \log_{3^2} x = 10 \Rightarrow 2.\log_3 x + \frac{1}{2}\cdot\log_3 x = 10$.

Hence, $\frac{5}{2}.\log_3 x = 10 \Rightarrow \log_3 x = 4$ and $x = 3^4 = 81$ //

Q7) $\lim_{x\to 0^+} \log_x\left(\frac{1}{2}\right) = ?$

Sol: Note that $\log_x\left(\frac{1}{2}\right) = -\log_x 2 = \frac{-1}{\log_2 x}$. Hence,

$$\lim_{x\to 0^+} \log_x\left(\frac{1}{2}\right) = \lim_{x\to 0^+} \frac{-1}{\log_2 x} = \frac{-1}{\lim_{x\to 0^+} \log_2 x} = \frac{-1}{-\infty} = 0.$$

Q8) $\lim_{x\to 1^-} \log_x 2 = ?$

Sol: Note that $\log_x 2 = \frac{1}{\log_2 x}$. Hence,

$$\lim_{x\to 1^-} \log_x 2 = \lim_{x\to 1^-} \frac{1}{\log_2 x} = \frac{1}{\lim_{x\to 1^-} \log_2 x} = \frac{1}{0^-} = -\infty.$$

Q9) Simplify the expression $2\ln x + 5\ln(x-2)$

Sol: $2\ln x + 5\ln(x-2) = \ln x^2 + \ln(x-2)^5 = \ln\left(x^2(x-2)^5\right)$.

Q10) Find the domain of the function $f(x) = \ln \frac{x}{x-2}$.

Sol: $f(x)$ is defined if $\frac{x}{x-2} > 0$. The root of numerator is 0 and the root of denominator is 2.

We can make a table.

		0		2	
x	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$
$\frac{x}{x-2}$	>0		<0		>0

$\frac{x}{x-2} > 0$ on $(-\infty,0)\cup(2,\infty)$

Therefore $D(f) = (-\infty,0)\cup(2,\infty)$.

Q11) Differentiate the given functions. If possible, simplify your answers.

a) $y = x^2 e^{\frac{x}{2}}$

$\Rightarrow y' = (x^2)' e^{\frac{x}{2}} + x^2 (e^{\frac{x}{2}})' = 2x . e^{\frac{x}{2}} + x^2 . \frac{1}{2} . e^{\frac{x}{2}}$

$\Rightarrow y' = e^{\frac{x}{2}} (2x + \frac{x^2}{2})$.

b) $y = \ln(\ln x)$

$\Rightarrow y' = \frac{(\ln x)'}{\ln x} = \frac{\frac{1}{x}}{\ln x} \Rightarrow y' = \frac{1}{x \ln x}$.

c) $y = 2^{x^2-3x+8}$

$\Rightarrow y' = (x^2-3x+8)' . 2^{x^2-3x+8} . \ln 2 \Rightarrow y' = \ln 2 . (2x-3) . 2^{x^2-3x+8}$

d) $y = \ln|\sec x + \tan x|$

$\Rightarrow y' = \frac{(\sec x + \tan x)'}{\sec x + \tan x} = \frac{\sec x . \tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} \Rightarrow y' = \sec x$

Q12) Find an equation of the straight line tangent to the curve $x . e^y + y - 2x = \ln 2$ at the point $(1, \ln 2)$.

Sol: We must calculate the derivative of both sides.

$\Rightarrow e^y + x . e^y y' + y' - 2 = 0 \Rightarrow y' = \frac{dy}{dx} = \frac{2 - e^y}{x e^y + 1}$.

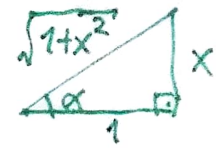
At $(1, \ln 2)$, $\left.\frac{dy}{dx}\right|_{(1,\ln 2)} = \frac{2 - e^{\ln 2}}{1 . e^{\ln 2} + 1} = \frac{2-2}{2+1} = 0$.

Therefore, $y - \ln 2 = 0 (x-1) \Rightarrow y = \ln 2$.

Q13) ~~Simpl~~ Simplify the given expressions.

a) $\cos(\tan^{-1}x)$ b) $\tan(\sec^{-1}x)$

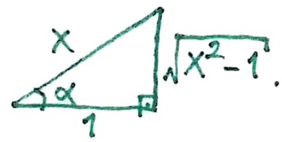
Sol: a) Let $\tan^{-1}x = \alpha$. Then, $\tan\alpha = x$.



Hence, $\cos(\tan^{-1}x) = \cos\alpha = \dfrac{1}{\sqrt{1+x^2}}$

b) Let $\sec^{-1}x = \alpha$. Then, $\sec\alpha = x \Rightarrow \dfrac{1}{\cos\alpha} = x$.

Hence, $\cos\alpha = \dfrac{1}{x}$.



Thus,

$\tan(\sec^{-1}x) = \tan\alpha = \sqrt{x^2-1}$.

Q14) Differentiate the function $f(x) = x.\sin^{-1}x$ and simplify the answer whenever possible.

Sol: $f'(x) = (x)'.\sin^{-1}x + x.(\sin^{-1}x)'$

$= \sin^{-1}x + \dfrac{x}{\sqrt{1-x^2}}$.

Q15) Find equations of two straight lines tangent to the graph of $y=\sin^{-1}x$ and having slope 2.

Sol: If $y = \sin^{-1}x$, then $y' = \dfrac{1}{\sqrt{1-x^2}}$. If the slope is 2 then $\dfrac{1}{\sqrt{1-x^2}} = 2$ so that $\dfrac{1}{1-x^2} = 4 \Rightarrow 1 = 4-4x^2$.

Hence, $4x^2 = 3 \Rightarrow x = \mp\dfrac{\sqrt{3}}{2}$. ~~Thus~~

At $x=\dfrac{\sqrt{3}}{2}$, $y = \dfrac{\pi}{3}$ and at $x = -\dfrac{\sqrt{3}}{2}$, $y = -\dfrac{\pi}{3}$

The tangent linen are

$y-\dfrac{\pi}{3} = 2\left(x-\dfrac{\sqrt{3}}{2}\right)$ and $y+\dfrac{\pi}{3} = 2\left(x+\dfrac{\sqrt{3}}{2}\right)$.