APPLICATIONS OF INTEGRATION

VOLUMES BY SLICING – SOLIDS OF REVOLUTION

In this section, we show how volumes of certain three-dimensional regions (or solids) can be expressed as definite integrals and thereby determined.

Volumes by Slicing: Knowing the volume of a cyclinder enables us to determine the volumes of some more general solids. We can divide solids into thin "slices" by parallel planes. Each slide is approximately a cyclinder of very small "height"; the height is the thickness of slice. If we know the cross-sectional area of each slice, we can determine its volume and sum these volumes to find the volume of the solid.

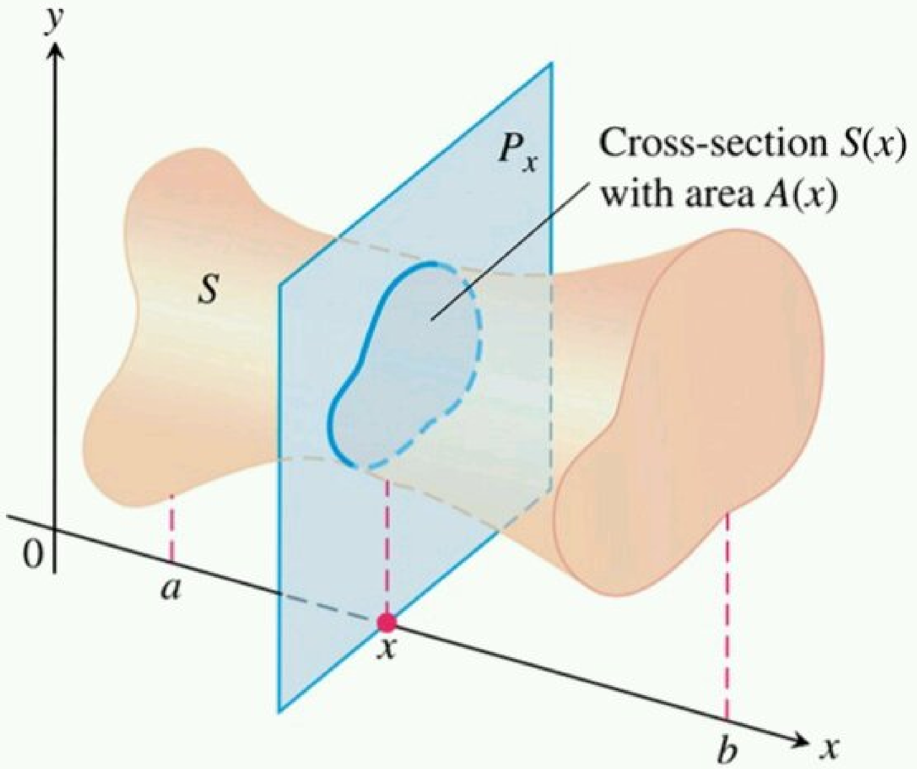

FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

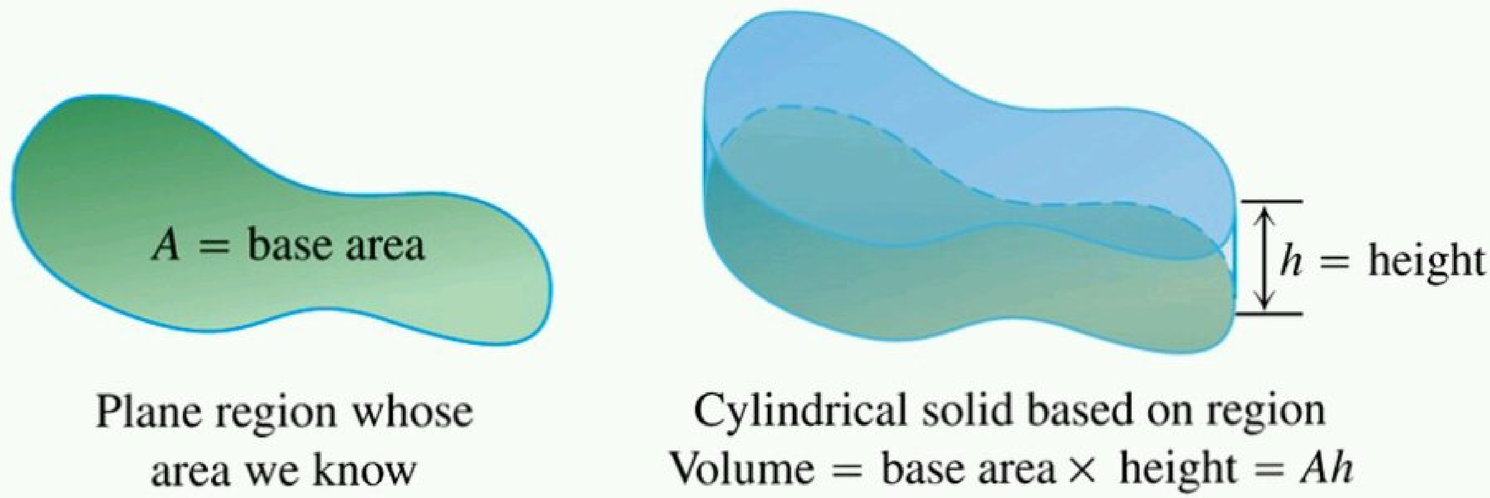

FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

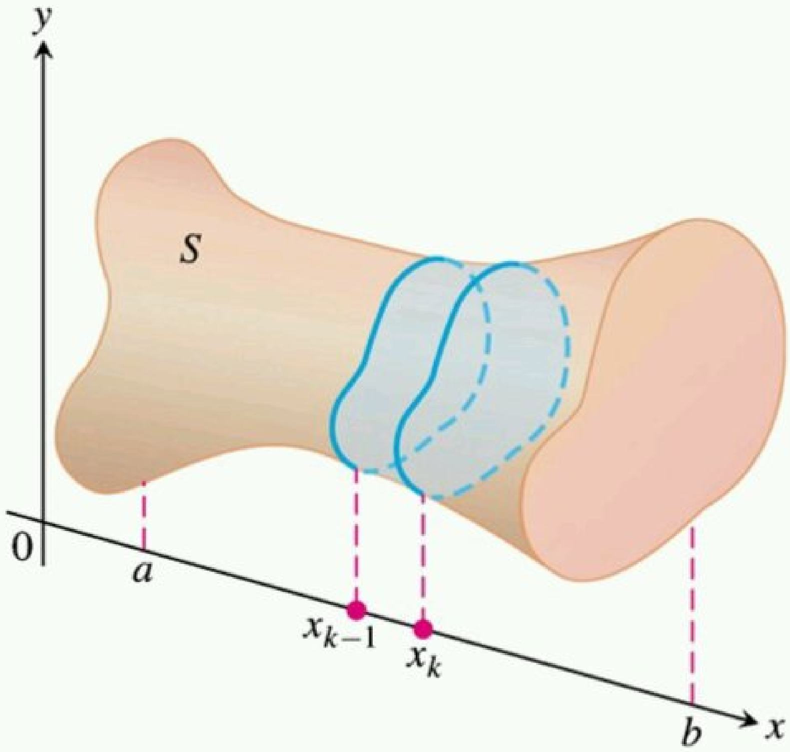

FIGURE 6.3 A typical thin slab in the solid S.

Volume of kth slab $\approx V_k = A(x_k)\Delta x_k = A(x_k)(x_k - x_{k-1})$

The volume of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)(x_k - x_{k-1})$$

This is a Riemann sum for the function $A(x)$ on $[a,b]$.

We expect the approximations from these sums to improve as the norm of the partition of $[a,b]$ goes to zero. Taking a partition of $[a,b]$ into n subintervals with $\|P\| \to 0$ gives

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\Delta x_k = \int_a^b A(x)\,dx$$

Definition: The volume of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx \,.$$

This definition applies whenever $A(x)$ is integrable, and in particular when it is continuous.

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

Solids of Revolution: Many common solids have circular cross-sections in planes perpendicular to some axis. Such solids are called solids of revolution because they can be generated by rotating a plane region about an axis in that plane so that it sweeps out the solid. For example, a solid ball is generated by rotating a half-disk about the diameter of that half-disk. Similarly, a solid right-circular cone is generated by rotating a right-angled triangle about one of its legs.

If the region R bounded by $y=f(x)$, $y=0$, $x=a$ and $x=b$ is rotated about the x-axis, then the cross section of the solid generated in the plane perpendicular to x-axis at x is a circular disk of radius $|f(x)|$. The area of this cross-section is

$A(x) = \pi (f(x))^2$, so the volume of the solid of revolution is,

$$V = \pi \int_a^b (f(x))^2 \, dx .$$

Example: (The volume of a ball) Find the volume of a solid ball having radius a.

The ball can be generated by rotating the half-disk $0 \le y \le \sqrt{a^2 - x^2}$, $-a \le x \le a$ about the x-axis. Therefore its volume is;

$$V = \pi \int_{-a}^{a} \left(\sqrt{a^2 - x^2}\right)^2 dx = 2\pi \int_{0}^{a} (a^2 - x^2)\, dx$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3}\right)\Big|_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{4}{3}\pi a^3 \text{ cubic units.}$$

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Example: (Volume of a right-circular cone) Find the volume of the right-circular cone of base radius r and height h that is generated by rotating the triangle with vertices $(0,0)$, $(h,0)$, (h,r) about the x-axis.

The line from $(0,0)$ to (h,r) has equation $y = \frac{rx}{h}$. Thus the volume of the cone is;

$$V = \pi \int_{0}^{h} \left(\frac{rx}{h}\right)^2 dx = \pi \left(\frac{r}{h}\right)^2 \left(\frac{x^3}{3}\right)\Big|_0^h = \frac{1}{3}\pi r^2 h \text{ cubic units.}$$

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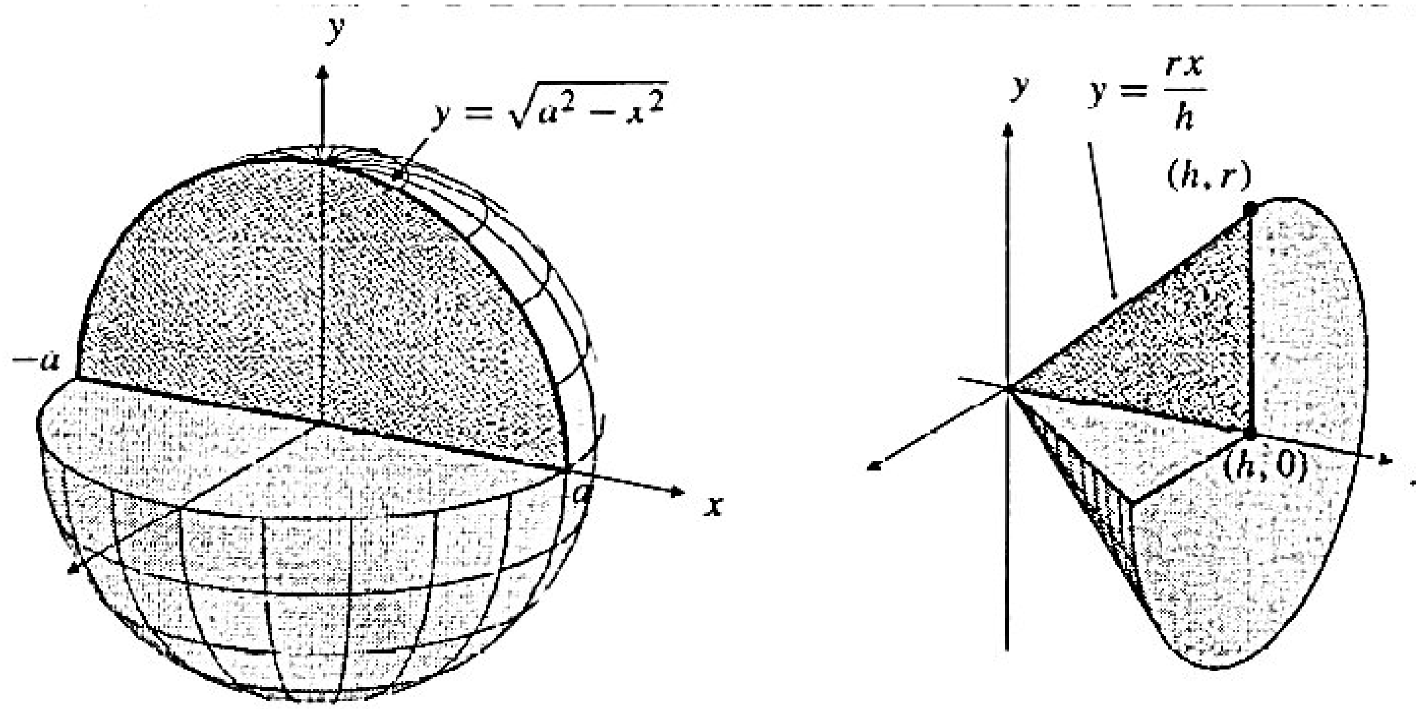


(a)

(b)

Improper integrals can represent volumes of unbounded solids. If the improper integral converges, the unbounded solid has a finite volume.

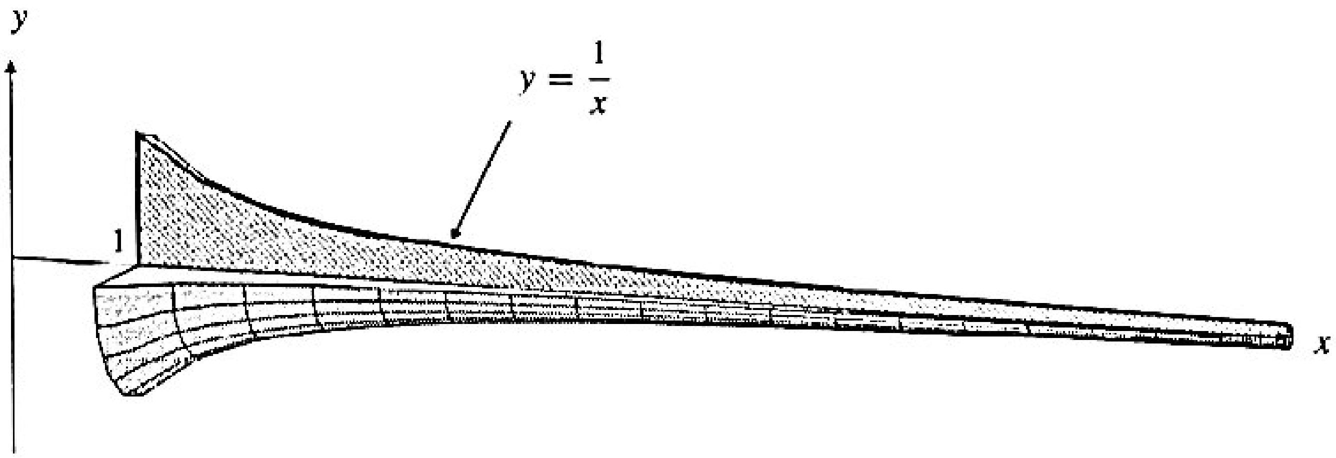
Example: Find the volume of infinitely long horn that is generated by rotating the region bounded by $y=1/x$ and $y=0$ and lying to the right of $x=1$ about the x-axis.

$$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R\to\infty} \int_1^{R} \left(\frac{1}{x}\right)^2 dx$$

$$= -\pi \lim_{R\to\infty} \left(\frac{1}{x}\right)_1^R = -\pi \lim_{R\to\infty} \left(\frac{1}{R} - 1\right)$$

$$= -\pi(-1)$$

$$= \pi \text{ cubic units}$$



y
$y = \frac{1}{x}$
1
x

Example: A ring-shaped solid is generated by rotating the finite plane region R bounded by the curve $y=x^2$ and the line $y=1$ about the line $y=2$. Find its volume.

$$dV = \left(\pi(2-x^2)^2 - \pi(1)^2\right)dx = \pi(3-4x^2+x^4)\,dx$$

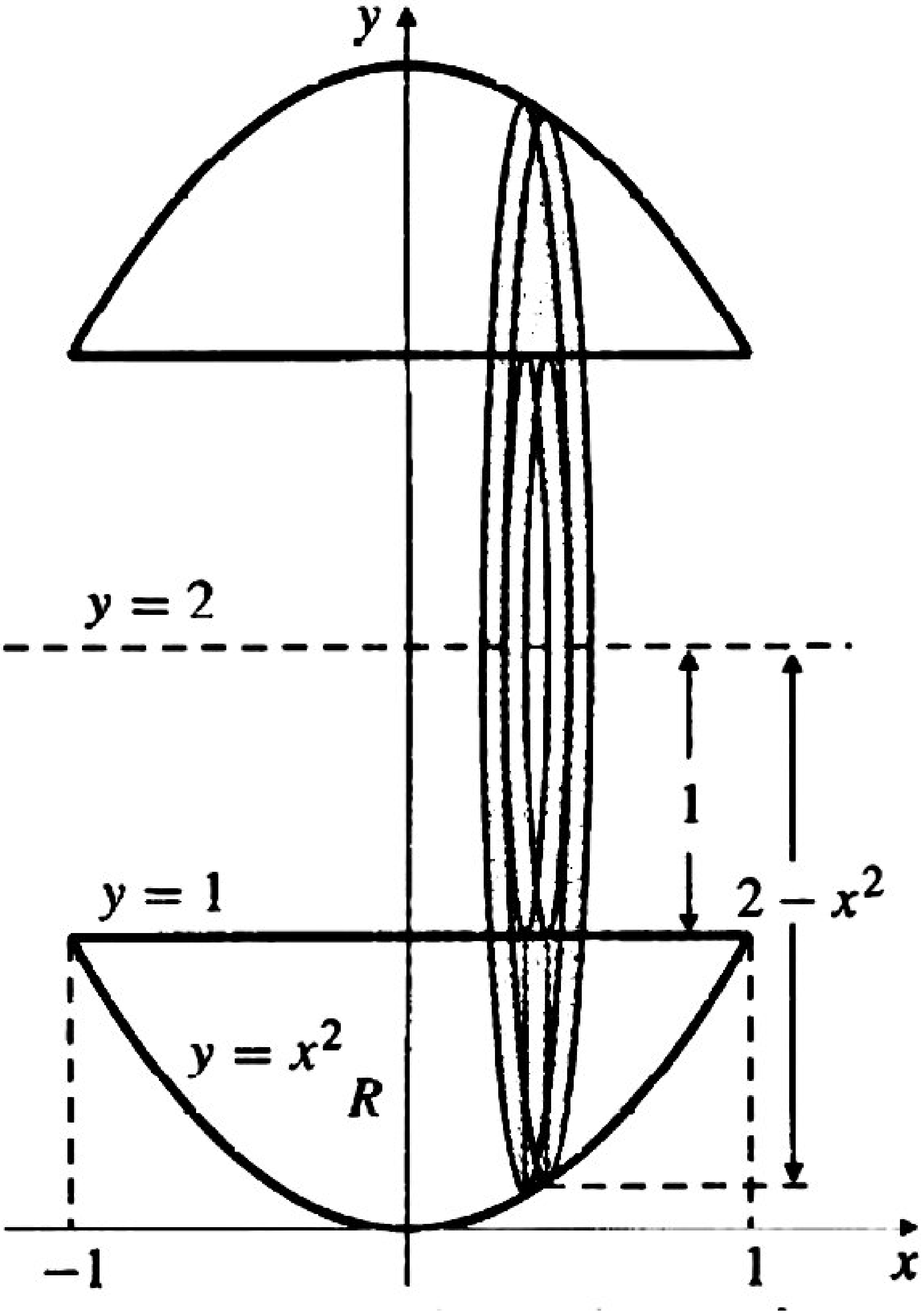
Since the solid extends from $x=-1$ to $x=1$, its volume is

$$V = \pi\int_{-1}^{1}(3-4x^2+x^4)\,dx = 2\pi\int_{0}^{1}(3-4x^2+x^4)\,dx$$

$$= 2\pi\left(3x-\frac{4x^3}{3}+\frac{x^5}{5}\right)\Bigg|_0^1$$

$$= 2\pi\left(3-\frac{4}{3}+\frac{1}{5}\right)$$

$$= \frac{56\pi}{15} \text{ cubic units.} \quad //$$



y
y = 2
1
y = 1
2 − x^2
y = x^2
R
−1
1
x

Sometimes we want to rotate a region bounded by curves with equations of the form $x = g(y)$ about the y-axis. In this case, the roles of x and y are reversed, and we use horizontal slices instead of vertical ones.

__Example:__ Find the volume of the solid generated by rotating the region to the right of the y-axis and to the left of the curve $x = 2y - y^2$ about the y-axis.

For intersections of $x = 2y - y^2$ and $x = 0$, we have

$2y - y^2 = 0 \Rightarrow y = 0$ or $y = 2$.

The solid lies between the horizontal planes at $y = 0$ and $y = 2$.

A horizontal area element at height y and having thickness dy rotates about the y-axis to generate a thin disk-shaped volume element of radius $2y - y^2$ and thickness dy. Its volume is

$$dV = \pi(2y - y^2)^2\, dy = \pi(4y^2 - 4y^3 + y^4)\, dy$$

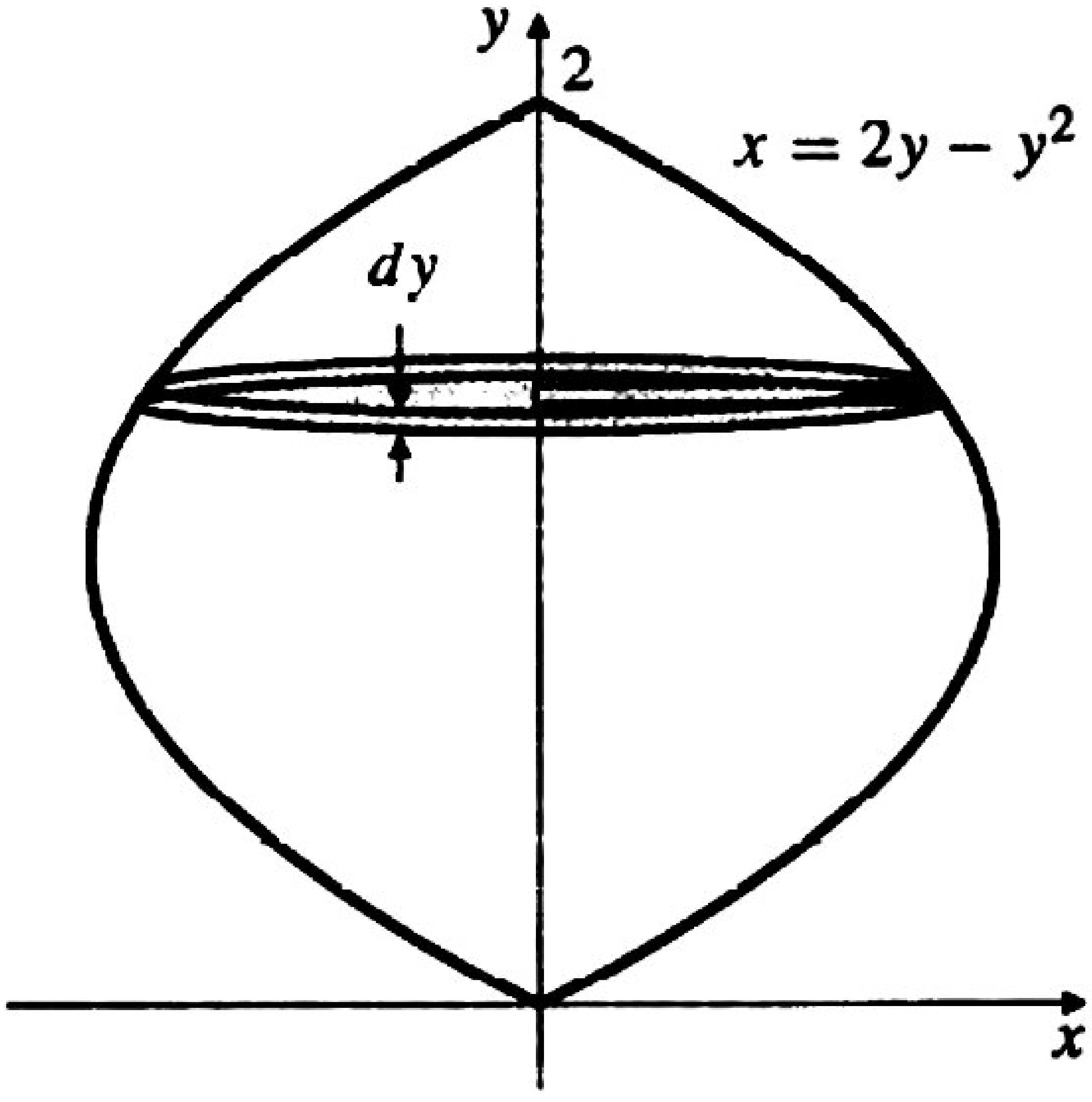
Thus, the volume of the solid is,

$$V = \pi \int_0^2 (4y^2 - 4y^3 + y^4)\, dy$$

$$= \pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right)\Bigg|_0^2$$

$$= \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right)$$

$$= \frac{16}{15}\pi \text{ cubic units.}$$



y
2
$x = 2y - y^2$
dy
x

Cyclindrical Shells:

Suppose that the region R bounded by $y=f(x)\geq 0$, $y=0$, $x=a\geq 0$, and $x=b>a$ is rotated about the y-axis to generate a solid of revolution. In order to find the volume of the solid using (plane) slices, we would need to know the cross-sectional area $A(y)$ in each plane of height y, and this would entail solving the equation $y=f(x)$ for one or more solutions of the form $x=g(y)$. In practice this can be inconvenient or impossible.

The standard area element of R at position x is a vertical strip of width dx, height $f(x)$, and area $dA=f(x)dx$. When R is rotated about the y-axis, this strip sweeps out a volume element in the shape of a circular cyclindrical shell having radius x, height $f(x)$, and thickness dx. Regard this shell as a rolled-up rectangular slab with dimensions $2\pi x$, $f(x)$, and dx; evidently it has volume

$$dV = 2\pi x\, f(x)\, dx.$$

The volume of the solid of revolution is the sum (integral) of the volumes of such shells with radius ranging from a to b;

The volume of the solid obtained by rotating the plane region $0\leq y\leq f(x)$, $0\leq a<x<b$ about the y-axis is

$$V = 2\pi\int_a^b x\, f(x)\, dx.$$

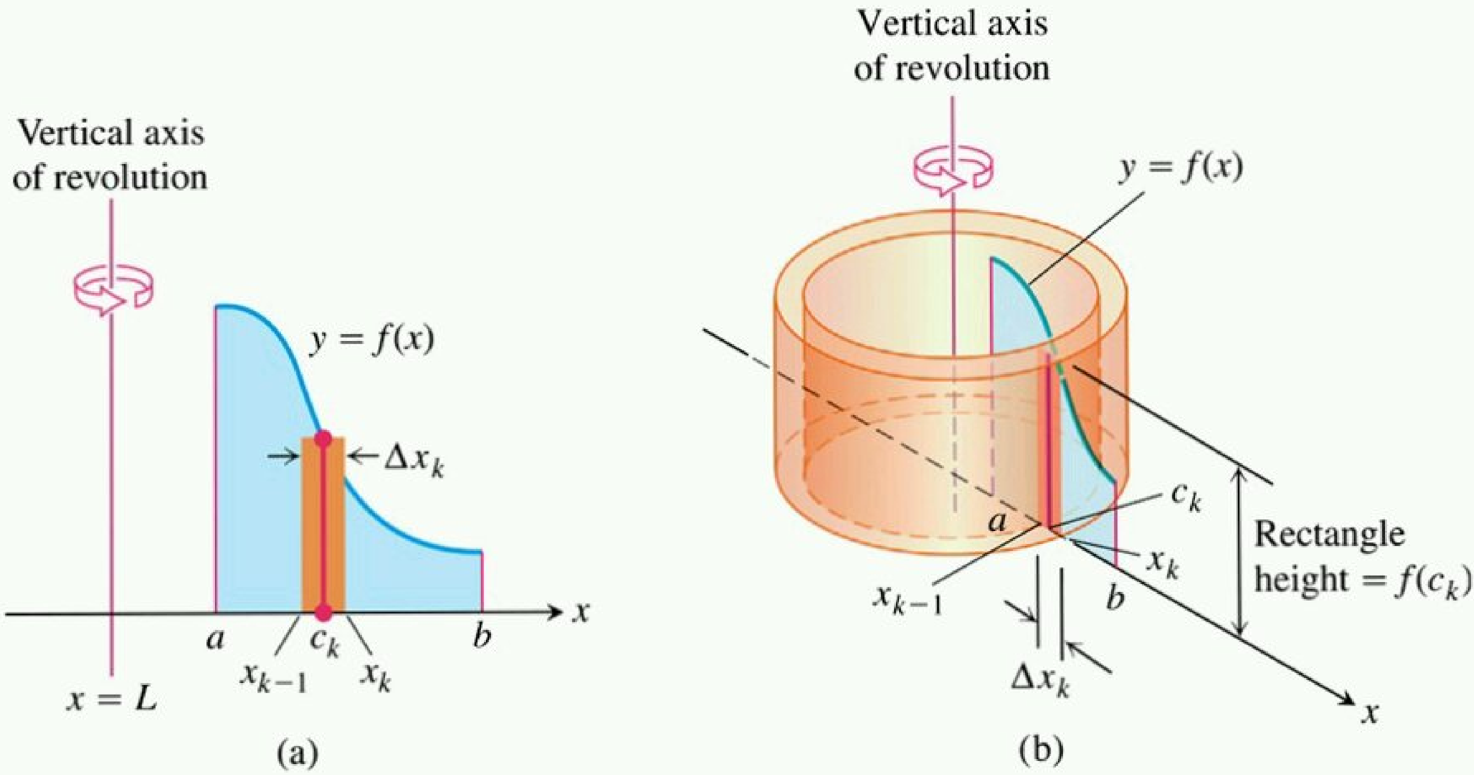


FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

Example: (The volume of a torus)

A disk of radius "a" has centre at the point $(b, 0)$ where $b > a > 0$. The disk is rotated about the y-axis to generate a torus. Find its volume.

The circle with centre at $(b, 0)$ and having radius "a" has equation $(x-b)^2 + y^2 = a^2$, so its upper semicircle is the graph of the function

$$f(x) = \sqrt{a^2 - (x-b)^2}$$

We will double the volume of the upper half of the torus, which is generated by rotating the half-disk $0 \leq y \leq \sqrt{a^2 - (x-b)^2}$, $b - a \leq x \leq b + a$ about the y-axis. The volume of the complete torus is;

$$V = 2 \times 2\pi \int_{b-a}^{b+a} x\sqrt{a^2 - (x-b)^2}\, dx$$

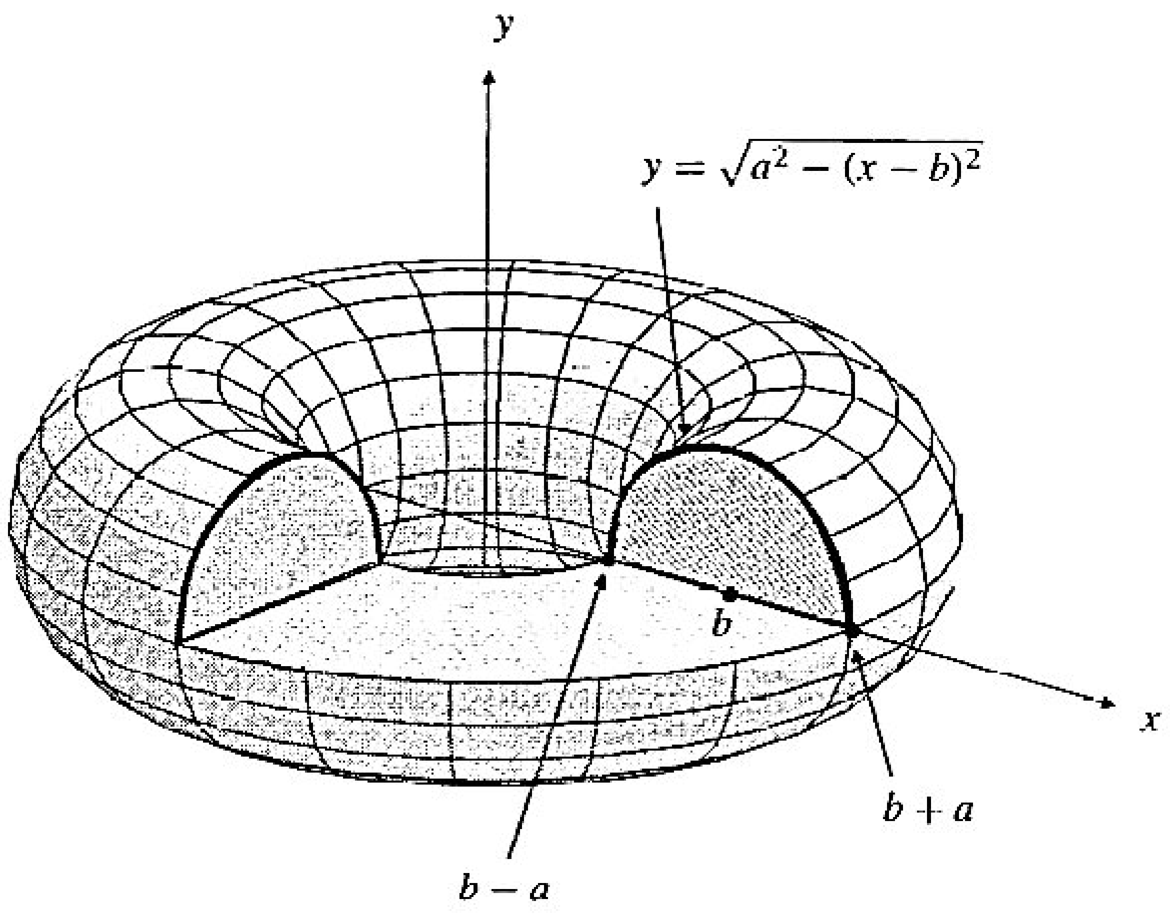
Let $u = x - b$
then $du = dx$

$$= 4\pi \int_{-a}^{a} (u+b)\sqrt{a^2 - u^2}\, du$$

$$= 4\pi \int_{-a}^{a} u\sqrt{a^2 - u^2}\, du + 4\pi b \underbrace{\int_{-a}^{a} \sqrt{a^2 - u^2}\, du}_{\text{Area of a semicircle of radius "a".}}$$

$$= 0 + 4\pi b\left(\frac{\pi a^2}{2}\right)$$

$$= 2\pi^2 a^2 b \quad \text{cubic units.}$$



y
$y = \sqrt{a^2 - (x - b)^2}$
b
x
b + a
b − a

Example: Find the volume of a bowl obtained by revolving the parabolic arc $y = x^2$, $0 \le x \le 1$ about the y-axis.

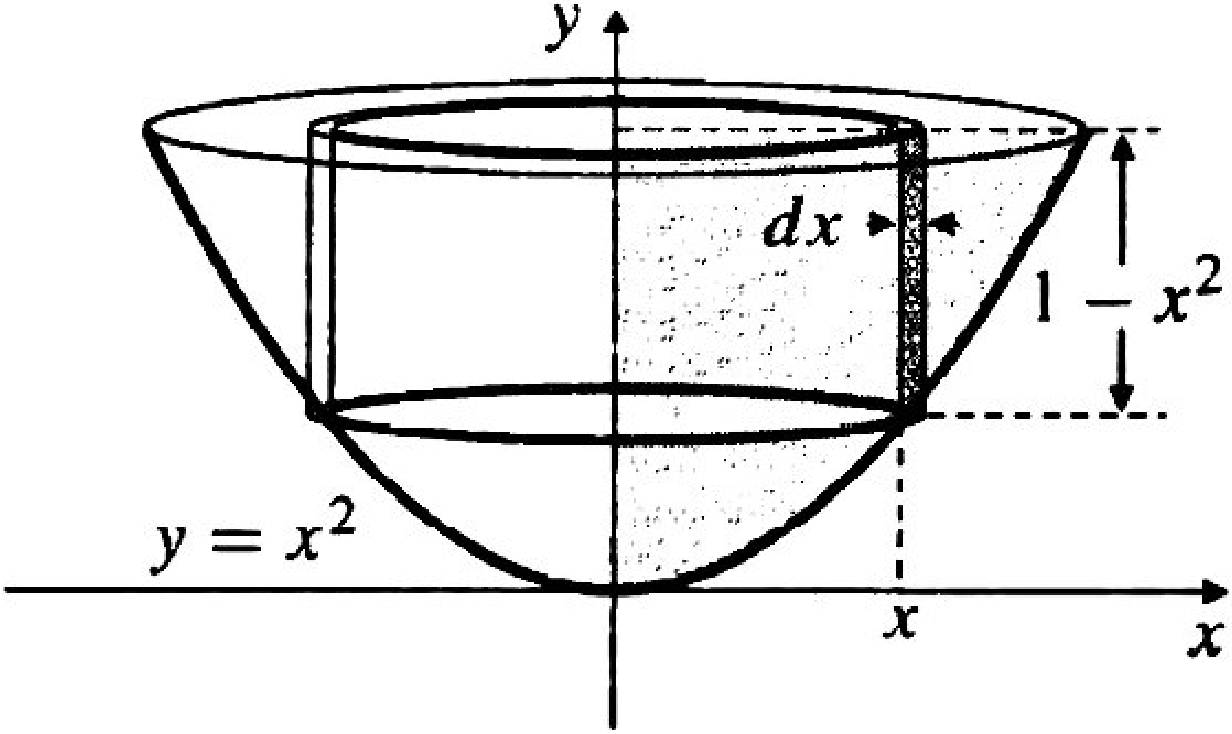
The interior of the bowl corresponds to revolving the region given by $x^2 \le y < 1$, $0 \le x \le 1$ about the y-axis. The area element at position x has height $1 - x^2$ and generates a cylindrical shell of volume

$dV = 2\pi x (1 - x^2)\, dx$

Thus the volume of the bowl is

$$V = 2\pi \int_0^1 x(1 - x^2)\, dx$$

$$= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} \right)_0^1 = \frac{\pi}{2} \text{ cubic units.}$$



y
dx
$1 - x^2$
$y = x^2$
x
x

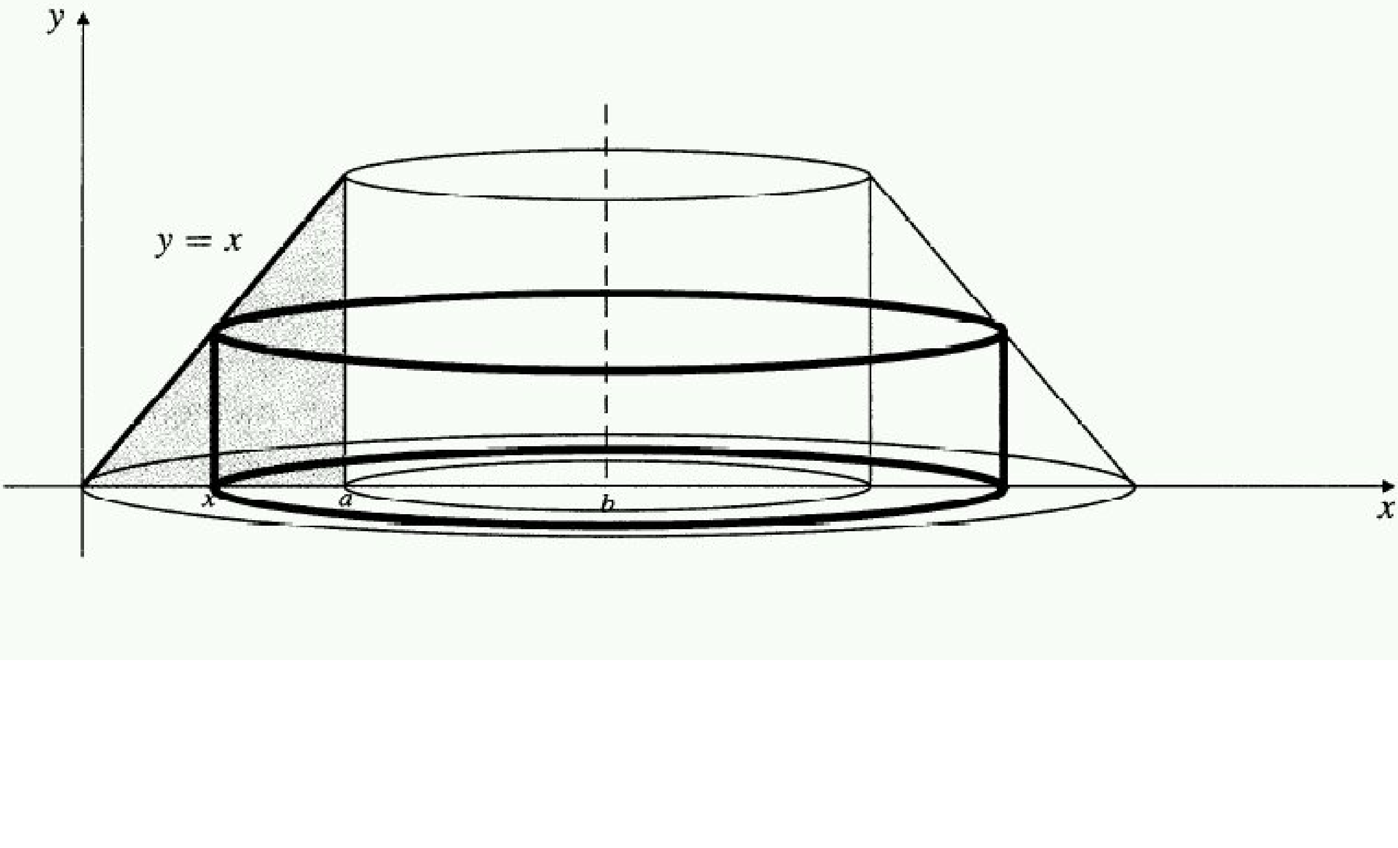
Example: The triangular region bounded by $y=x$, $y=0$ and $x=a>0$ is rotated about the line $x=b>a$. Find the volume of the solid so generated.

Here the vertical area element at x generates a cylindrical shell of radius $b-x$, height x, and thickness dx.

Its volume is $dV = 2\pi(b-x)x\,dx$, and the volume of the solid is

$$V = 2\pi \int_0^a (b-x)\,x\,dx = 2\pi\left(\frac{bx^2}{2} - \frac{x^3}{3}\right)_0^a$$

$$= \pi\left(a^2 b - \frac{2a^3}{3}\right) \text{ cubic units.}$$



y
y = x
x
a
b
x