MATH 101.2 PS-1

Q1) Express the set of all real numbers x satisfying $x \geq 0$ and $x \leq 5$ as an interval

Sol: Let us show these numbers on the real number line.

$x \geq 0$

$x \leq 5$

$x \geq 0$ and $x \leq 5$

So, $x \geq 0$ and $x \leq 5$ define the interval $[0,5]$.

Q2) Solve the inequality $\dfrac{3}{x-1} < \dfrac{2}{x+1}$.

Sol: $\dfrac{3}{x-1} < \dfrac{2}{x+1} \Rightarrow \dfrac{3}{\underset{(x+1)}{x-1}} - \dfrac{2}{\underset{(x-1)}{x+1}} < 0$

$$\Rightarrow \frac{3(x+1)-2(x-1)}{(x+1)(x-1)} < 0 \Rightarrow \frac{x+5}{(x+1)(x-1)} < 0$$

The root of numarator $x=-5$ and the roots of denominator are $x=-1$ and $x=1$.

Let us make a table by using this information.

	$-\infty$	-5		-1		1	∞
$x+5$	−	ϕ	+		+		+
$x+1$	−		−	ϕ	+		+
$x-1$	−		−		−	ϕ	+
$\dfrac{x+5}{(x+1)(x-1)}$	−	ϕ	+	ϕ	−	ϕ	+

Therefore, the solution set is $(-\infty,-5) \cup (-1,1)$.

Q3) Solve the inequality $|x+1| > |x-3|$.

Sol: * Remember $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ and $|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$

$|x+1| > |x-3| \Rightarrow |x+1| - |x-3| > 0$

$x+1 = 0 \Rightarrow x = -1$ and $x-3 = 0 \Rightarrow x = 3$

$|x+1| = \begin{cases} x+1, & x \geq -1 \\ -x-1, & x < -1 \end{cases}$ and $|x-3| = \begin{cases} x-3, & x \geq 3 \\ -x+3, & x < 3 \end{cases}$

Let us make a table by using this information.

	-1	3	
$\lvert x+1\rvert$	$-x-1$	$x+1$	$x+1$
$-\lvert x-3\rvert$	$x-3$	$x-3$	$-x+3$
$\lvert x+1\rvert-\lvert x-3\rvert$	-4	$2x-2$	4

Therefore, $|x+1| - |x-3| = \begin{cases} -4, & x \leq -1 \\ 2x-2, & -1 < x \leq 3 \\ 4, & x > 3 \end{cases} \rightarrow$ **

** $2x-2 > 0 \Rightarrow 2x > 2$ and $x > 1$.

Hence, the solution set is $(1, \infty)$.

Q4) Write an equation for the line through the points $(-2,1)$ and $(2,-2)$.

Sol: The slope of the line passing through the points (x_1, y_1) and (x_2, y_2) is found by the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Therefore, $m = \dfrac{(-2)-1}{(2)-(-2)} = \dfrac{-3}{4}$.

The equation for the line with slope m and passing through (a, b) is $y - b = m(x-a)$.

If we choose the point $(-2,1)$, the equation of the line is $y - 1 = -\frac{3}{4}(x+2)$. Hence, we can find the line equation as $3x + 4y = -2$.

Q5) For what value of k is the line $2x+ky=3$ perpendicular to the line $4x+y=1$? For what value of k are the lines parallel?

Sol: If the line equation has the form $y = mx+n$ where $m,n \in \mathbb{R}$, then the slope of the line is m.

Given two lines $d_1 : y = m_1 x + n_1$ and $d_2 : y = m_2 x + n_2$

* d_1 and d_2 are perpendicular $\Longleftrightarrow$ $m_1 . m_2 = -1$
* d_1 and d_2 are parallel $\Longleftrightarrow$ $m_1 = m_2$

→ We can write $2x+ky=3$ as $y = \frac{-2}{k}x + \frac{3}{k}$ $\Rightarrow m_1 = -\frac{2}{k}$

→ We can write $4x+y=1$ as $y = -4x+1$ $\Rightarrow m_2 = -4$

If the lines are perp., then $m_1.m_2 = \frac{-2}{k}.(-4) = -1 \Rightarrow k = -8$ //

If the lines are parallel, then $m_1 = m_2 \Rightarrow -\frac{2}{k} = -4 \Rightarrow k = \frac{1}{2}$ //

Q6) Write an equation for the circle with center $C(3,-4)$ and radius $r=5$

Sol: The equation for a circle with center $C(a,b)$ and radius r $(r>0)$ is $(x-a)^2 + (y-b)^2 = r^2$.

So, the equation of the circle is $(x-3)^2 + (y+4)^2 = 5^2$.

Q7) Figure 1 shows the graph $y=x^2$ and four shifted versions are as follows. Write equations for the shifted versions.

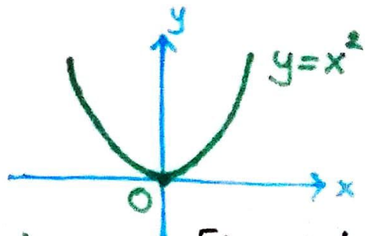
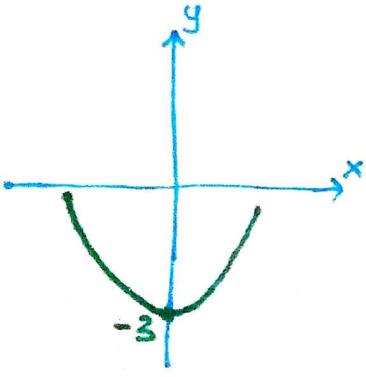


Figure 1

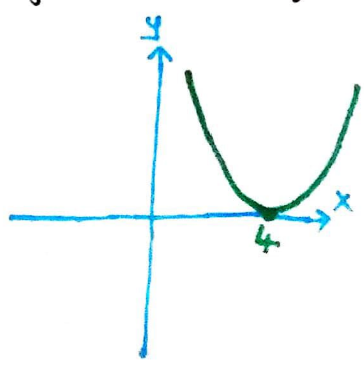


Version (a)

↓ 3 units

$(y+3) = x^2$

$y = x^2 - 3$ //

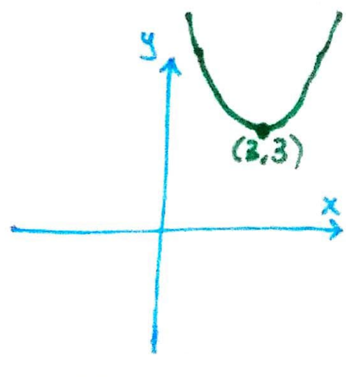


Version (b)

→ 4 units

$y = (x-4)^2$

$y = x^2 - 8x+16$ //

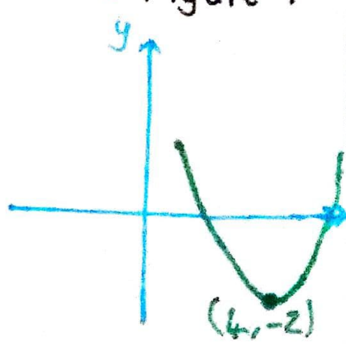


Version (c)

↑ 3 and → 3

$(y-3) = (x-3)^2$

$y = x^2 - 6x + 12$ //



Version (d)

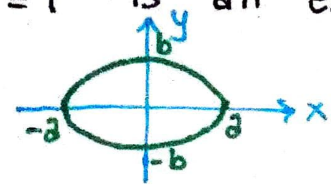
↓ 2 and → 4

$(y+2) = (x-4)^2$

$y = x^2 - 8x + 14$ //

Q8) Sketch the curve represented by $\frac{(x-3)^2}{9} + \frac{(y+2)^2}{4} = 1$.

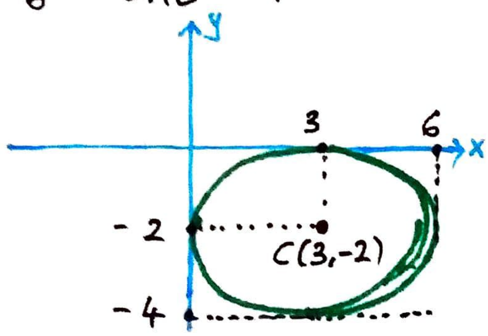
Sol: Remember: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse equation centered at the origin. (a>b).



Here, the length of major and minor axises are $2a$ and $2b$ respectively. Also, if we consider the equation $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$, then this is the same ellipse equation but its center at the point (x_0, y_0).

$\Rightarrow$ We can write the equation $\frac{(x-3)^2}{9} + \frac{(y+2)^2}{4} = 1$ as $\frac{(x-3)^2}{3^2} + \frac{(y+2)^2}{2^2} = 1$. The center of this ellipse is $(3,-2)$.

Also, the length of major and minor axises are 6 and 4 ($2a = 2.3 = 6$ and $2b = 2.2 = 4$).



The graph of $\frac{(x-3)^2}{9} + \frac{(y+2)^2}{4} = 1$

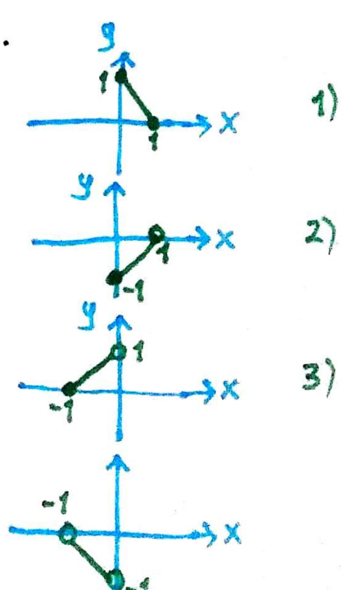
Q9) Sketch the graph of $|x| + |y| = 1$.

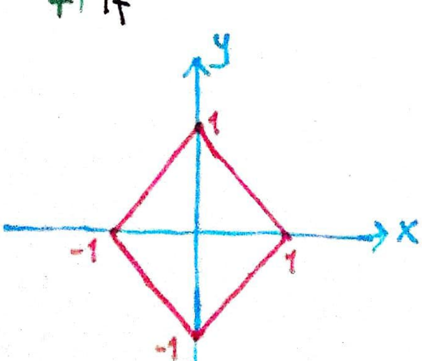
Sol: 1) If $x \geq 0$ and $y \geq 0$, $x + y = 1$

2) If $x \geq 0$ and $y < 0$, $x - y = 1$

3) If $x < 0$ and $y \geq 0$, $-x + y = 1$

4) If $x < 0$ and $y < 0$, $-x - y = 1$





is the graph of $|x| + |y| = 1$.

Q10) What (if any) symmetry does the graph of $f(x) = x^2+1$ possess? In particular, is f even or odd?

Sol: Remember: A function $f(x)$ is even if $f(x) = f(-x)$. f is odd if $-f(x) = f(-x)$. Even functions are symmetric with respect to y-axis. Also, odd functions are symmetric with respect to origin.

$\Rightarrow f(-x) = (-x)^2+1 = x^2+1 = f(x)$. So, f is an even function. Hence, f is symmetric wrt. y-axis.

Q11) Let $f(x) = \frac{2}{x}$ and $g(x) = \frac{x}{1-x}$. Construct the following composite functions and specify the domain of each.

a) $f \circ f(x)$ b) $f \circ g(x)$ c) $g \circ f(x)$ d) $g \circ g(x)$

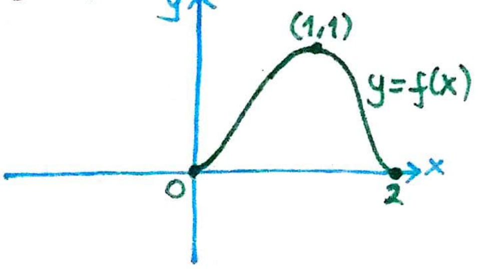
Sol: a) $f \circ f(x) = \frac{2}{f(x)} = \frac{2}{2/x} = x$, $D(f \circ f) = \{x \in \mathbb{R} \mid x \neq 0\}$

b) $f \circ g(x) = \frac{2}{g(x)} = \frac{2}{x/(1-x)} = \frac{2(1-x)}{x}$, $D(f \circ g) = \{x \in \mathbb{R} \mid x \neq 0, 1\}$

c) $g \circ f(x) = \frac{f(x)}{1-f(x)} = \frac{2/x}{1-2/x} = \frac{2}{x-2}$, $D(f \circ g) = \{x \in \mathbb{R} \mid x \neq 0, 2\}$

d) $g \circ g(x) = \frac{g(x)}{1-g(x)} = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{x}{1-2x}$, $D(g \circ g) = \{x \in \mathbb{R} \mid x \neq \frac{1}{2}, 2\}$

Q11) f refers to the function with domain $[0,2]$ and range $[0,1]$, whose graph is shown below. Sketch the graph of $1+f(-x/2)$ and specify its domain and range.



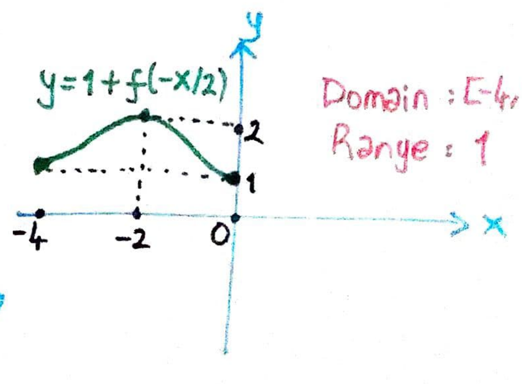
Sol: $1+f(-x/2)$. For domain:

* $0 \leq -\frac{x}{2} \leq 2 \Rightarrow -4 \leq x \leq 0$.

** $+1$ means we must shift the function up to 1 unit.

*** $f(\frac{-x}{2})$ means, we must consider the symmetry of the function wrt. y-axis.

→ If we consider *, ** and ***, we draw this.



Domain : $[-4, 0]$
Range : 1

Q12) Express the rational function as the sum of a polynomial and another rational function whose numerator is either zero or has smaller degree than the denominator.

$$\frac{x^3}{x^2+2x+3}$$

Sol: We must use division algorithm.

$$\begin{array}{r|l} x^3 & x^2+2x+3 \\ \cline{2-2} -\;(x^3+2x^2+3x) & x-2 \\ \hline -2x^2-3x & \\ -\;(-2x^2-4x-6) & \\ \hline x+6 & \end{array}$$

$$\Rightarrow \frac{x^3}{x^2+2x+3} = x-2+\frac{x+6}{x^2+2x+3}.$$

Q13) Express $\sin\left(\frac{3\pi}{2}-x\right)$ in terms of $\sin x$ and $\cos x$.

Sol: $\sin(k\pi \mp x)$, $\cos(k\pi \mp x)$, $\tan(k\pi \mp x)$, $\cot(k\pi \mp x)$

* First of all we consider x as an acute angle.

** We determine the sign of the function (+ or −)

*** If $k=\pi, 2\pi$, the function does not change.

If $k=\frac{\pi}{2}, \frac{3\pi}{2}$, the function changes. ($\sin \rightleftarrows \cos$, $\tan \rightleftarrows \cot$)

→ For $\sin\left(\frac{3\pi}{2}-x\right)$, $\pi < \frac{3\pi}{2}-x < \frac{3\pi}{2}$. So, it is negative.

Also, the function changes. ($\sin \to \cos$). Hence,

$\sin\left(\frac{3\pi}{2}-x\right) = -\cos x$.

Q14) Express $\cos 3x$ in terms of $\sin x$ and $\cos x$.

Sol:

* $\sin(a \mp b) = \sin a.\cos b \mp \cos a.\sin b$

* $\cos(a \mp b) = \cos a.\cos b \pm \sin a.\sin b$

* $\sin 2x = 2.\sin x.\cos x$

* $\cos 2x = 2\cos^2 x - 1$

$$\begin{aligned} \cos 3x = \cos(2x+x) &= \cos 2x.\cos x - \sin 2x.\sin x \\ &= (2\cos^2 x - 1)\cos x - (2.\sin x.\cos x).\sin x \\ &= 2\cos^3 x - \cos x - 2\sin^2 x.\cos x \\ &= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x \\ &= 2\cos^3 x - \cos x + 2\cos^3 x - 2\cos x \\ &= 4\cos^3 x - 3\cos x. \end{aligned}$$

Q15) Find $\sin\theta$ and $\cos\theta$ if $\tan\theta = \frac{1}{2}$, θ in $\left[\pi, \frac{3\pi}{2}\right]$.

Sol: $\tan\theta = \frac{1}{2} \Rightarrow \tan^2\theta = \frac{1}{4}$. Hence,

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{4} \Rightarrow \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{1}{4} \Rightarrow 4-4\cos^2\theta = \cos^2\theta.$$

Thus, $5\cos^2\theta = 4$ and $\cos\theta = \frac{-2}{\sqrt{5}}$ //

! $\cos\theta$ is negative on the interval $\left[\pi, \frac{3\pi}{2}\right]$. It can not be positive.

Also, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{2} \Rightarrow \frac{\sin\theta}{-2/\sqrt{5}} = \frac{1}{2} \Rightarrow \sin\theta = \frac{-1}{\sqrt{5}}$ //