13.2) Extreme Values of Functions Defined on Restricted Domains

Q2) Find the maximum and minimum values of

$f(x,y) = xy - 2x$ on the rectangle $\underbrace{-1 \leq x \leq 1,\ 0 \leq y \leq 1}_{R}$.

I. Critical points $\quad f_x = y - 2 = 0 \Rightarrow y = 2$

$f_y = x = 0$

$\Rightarrow$ the only critical point is $(0,2)$, but $(0,2) \notin R$

So, max / min values of f lies on the boundary of R.

II. Boundary

$x = -1$: $f(-1,y) = -y + 2$ for $0 \leq y \leq 1$

$f_y = 1 \neq 0. \Rightarrow f(-1,y)$ has no critical points.

On endpoints, $\underset{(max)}{f(-1,0) = 2}$, $\underset{(min)}{f(-1,1) = 1}$

$x = 1$: $f(1,y) = y - 2$ for $0 \leq y \leq 1$

$f_y = -1 \neq 0 \Rightarrow$ no critical points.

On endpoints, $\underset{(min)}{f(1,0) = -2}$, $\underset{(max)}{f(1,1) = -1}$

$y=0:\quad f(x,0)=-2x$ for $-1\leq x\leq 1$

$f_x=-2\neq 0 \Rightarrow$ no critical point.

On endpoints, $f(-1,0)=2$, $f(1,0)=-2$
(max) (min)

$y=1:\quad f(x,1)=-x$ for $-1\leq x\leq 1$

$f_x=-1\neq 0 \Rightarrow$ no critical points

On endpoints, $f(-1,1)=1$, $f(1,1)=-1$
(max) (min)

So, the maximum value of f on the rectangle R is 2 and minimum value of f on R is -2.

Q4) Find the max / min values of $f(x,y) = x + 2y$

on the disk $D: x^2 + y^2 \leq 1$.

I) <u>Critical points</u> $f_x = 1 \neq 0, \; f_y = 2 \neq 0$.

$\Rightarrow$ f has no critical points.

II) <u>Boundary</u> $x^2 + y^2 = 1$ (Circle)

Circle can be parametrized as $x = \cos t, \; y = \sin t$

$\Rightarrow f(x,y) = f(\cos t, \sin t) = \cos t + 2\sin t = g(t)$

$g'(t) = -\sin t + 2\cos t = 0 \Rightarrow \tan t = 2$

$\Rightarrow \cos t = \pm \frac{1}{\sqrt{5}} = x, \quad \sin t = \pm \frac{2}{\sqrt{5}} = y$

f has critical points at $\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$ and $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$f\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) = -\sqrt{5}, \quad f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \sqrt{5}$

On D; max value of f is $\sqrt{5}$

min value of f is $-\sqrt{5}$

Q7) Find the max and min values of

$f(x,y) = \sin x \cos y$ on the closed triangle region

bounded by the $\underbrace{\text{coordinate axes and the line } x+y=2\pi}_{T}$

We know that $-1 \leq \sin x, \cos y \leq 1$

$\Rightarrow -1 \leq f(x,y) \leq 1.$

Also, $f\left(\frac{\pi}{2}, 0\right) = 1$, $f\left(\frac{3\pi}{2}, 0\right) = -1$

where $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ are in T

So, f has max / min values on T and

these values are $1, -1$ respectively.

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13.3) Lagrange Multipliers

Q2) Find the shortest distance from the point $(3,0)$ to the parabola $y=x^2$

(a) by reducing to an unconstrained problem in one variable,

(b) by using the method of Lagrange multipliers.

(a) Let D be the <u>square</u> of the distance from the point $(3,0)$ to the point (x,y) on the curve $y=x^2$.

$$\Rightarrow D = (x-3)^2+(y-0)^2 = (x-3)^2+(x^2)^2 = (x-3)^2+x^4$$

$$\Rightarrow \frac{dD}{dx} = 2(x-3)+4x^3 = 0 \Rightarrow 2x^3+x-3=0$$

$x=1$ is a root of this equation.

$$\frac{2x^3+x-3}{x-1} = 2x^2+2x+3 = 0 \text{ has no real roots}$$

since $\Delta = 2^2-4.2.3 = -20 < 0.$

So $x=1$ is the only critical point

$\Rightarrow$ The minimum distance is $\sqrt{D} = \sqrt{(1-3)^2+1^4} = \sqrt{5}$

(b) We want to minimize $D = (x-3)^2 + y^2$

subject to the constraint $y = x^2$.

* Lagrange multipliers method:

To find the points on the curve $g(x,y) = 0$ at which $f(x,y)$ is maximum or minimum, we need to look for critical points of the Lagrangian function

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y) \quad //$$

$$L(x,y,\lambda) = \left[(x-3)^2 + y^2\right] + \lambda\left[x^2 - y\right]$$

Critical points: $$\frac{\partial L}{\partial x} = 2(x-3) + 2\lambda x = 0$$

$$\frac{\partial L}{\partial y} = 2y - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2 - y = 0$$

$\Rightarrow$ $x + \lambda x - 3 = 0$ (1)

$2y - \lambda = 0$ (2)

$x^2 - y = 0$ (3)

From (1) and (2);
$$\begin{cases} x+\lambda x-3=0 \\ 2xy-\lambda x=0 \end{cases}$$

$\Rightarrow x+2xy-3=0 \quad (4)$

Substituting (3) in (4), we have, $2x^3+x-3=0$

$\Rightarrow (x-1)(2x^2+2x+3)=0$

The only real solution is $x=1$. So, the point on $y=x^2$ closest to $(3,0)$ is $(1,1)$.

Thus, the minimum distance is $\sqrt{D}=(1-3)^2+1^2=\sqrt{5}$.

Q4) Find the max/min values of the function $f(x,y,z)=x+y-z$ over the sphere $x^2+y^2+z^2=1$.

$$L(x,y,\lambda)=[x+y-z]+\lambda[x^2+y^2+z^2-1]$$

$$\frac{\partial L}{\partial x}=1+2\lambda x=0 \quad (1)$$

$$\frac{\partial L}{\partial y}=1+2\lambda y=0 \quad (2)$$

$$\frac{\partial L}{\partial z}=-1+2\lambda z=0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-1=0 \quad (4)$$

From (1), (2) and (3), $2\lambda x = 2\lambda y = -2\lambda z$

$\Rightarrow \lambda = 0$ or $x = y = -z$

If $\lambda = 0$, $1 + 2\lambda x = 1 + 0 = 0$ not possible.

So, $x = y = -z$ (5)

Substituting (5) in (4) we have

$3x^2 - 1 = 0 \Rightarrow x = \mp \frac{1}{\sqrt{3}}$

$\Rightarrow$ L has critical points at

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \sqrt{3}$ (max)

$f\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\sqrt{3}$ (min)

Q12) Find the max/min values of the function $f(x,y,z) = x^2+y^2+z^2$ on the ellipse formed by the intersection of the cone $z^2=x^2+y^2$ and the plane $x-2z=3$.

$$L(x,y,z,\lambda,\mu) = [x^2+y^2+z^2]+\lambda[x^2+y^2-z^2]+\mu[x-2z-3]$$

$$\frac{\partial L}{\partial x} = 2x+2x\lambda+\mu = 0 \qquad (1)$$

$$\frac{\partial L}{\partial y} = 2y+2y\lambda = 0 \qquad (2)$$

$$\frac{\partial L}{\partial z} = 2z-2z\lambda-2\mu = 0 \qquad (3)$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2-z^2 = 0 \qquad (4)$$

$$\frac{\partial L}{\partial \mu} = x-2z-3 = 0 \qquad (5)$$

From (2), $y=0$ or $\lambda=-1$

If $\lambda=-1$, then by (1), $\mu=0$. By (3), $z=0$. By (4), $x=y=0$. Not possible since $(0,0,0)$ is not a point on the plane $x-2z=3$.

If $y=0$, then by (4), $x=\mp z$

$x=z \Rightarrow$ (5) implies $z=-3$ Critical point: $(-3,0,-3)$

$x=-z \Rightarrow$ (5) implies $z=-1$. Critical point: $(1,0,-1)$

$f(-3,0,-3)=18$, $f(1,0,-1)=2$

(max) (min)

Q22) Find the max / min values of $\overbrace{xy+z^2}^{f(x,y,z)}$ on the ball $\underbrace{x^2+y^2+z^2\le 1}_{B}$. Use Lagrange multipliers to treat the boundary case.

Critical points: $f_x = y = 0$, $f_y = x = 0$, $f_z = 2z = 0$

$\Rightarrow$ $(0,0,0)$ is the only critical point, $(0,0,0) \in B$.

Boundary points: We will use Lagrange multipliers with constraint $x^2+y^2+z^2=1$:

$$L(x,y,z,\lambda) = (xy+z^2) + \lambda(x^2+y^2+z^2-1)$$

Critical points of L:

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$$\frac{\partial L}{\partial x} = y + 2\lambda x = 0 \qquad (1)$$

$$\frac{\partial L}{\partial y} = x + 2\lambda y = 0 \qquad (2)$$

$$\frac{\partial L}{\partial z} = 2z(1+\lambda) = 0 \qquad (3)$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1 = 0 \qquad (4)$$

(3) implies $z=0$ or $\lambda=-1$.

If $\lambda=-1$, (1) and (2) implies that $x=y=0$.

So, (4) implies $z=\mp 1$. Critical points: $(0,0,1)$, $(0,0,-1)$

If $z=0$, then (4) implies that $x^2+y^2=1$ (5)

Also, by (1) & (2), we have

$y+2\lambda x = 0 \rightarrow$ multiply by y
$x+2\lambda y = 0 \rightarrow$ multiply by $-x$
$\Rightarrow x^2 = y^2$. (6)

From (5) and (6), $x^2=y^2=\frac{1}{2}$. So, critical points:

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$f(0,0,0)=0 \qquad f(0,0,\mp 1)=1$

$$f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)=f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)=-\frac{1}{2}$$

$$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)=f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)=\frac{1}{2}$$

So, max of f on B is 1 and min is $-\frac{1}{2}$