UYGULAMA HAFTA 14 **Section 14.5-Üç Katlı İntegraller**

Section 14.6-Üç Katlı İntegrallerde Değişken Dönüşümleri

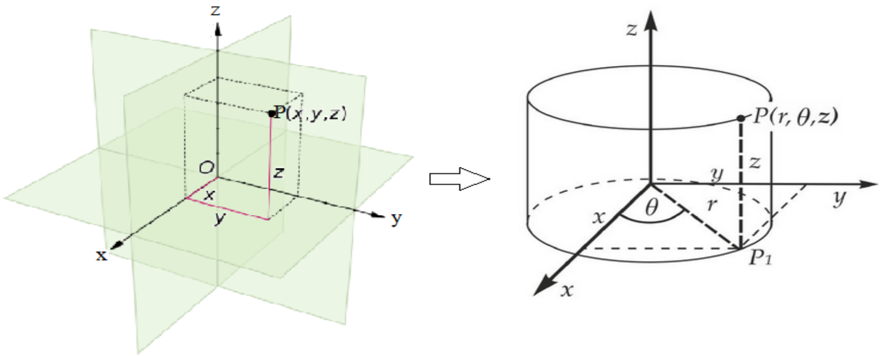
HATIRLATMALAR

- **Teorem: Silindirik Koordinatlar**

  Üç boyutlu uzayda bir (x, y, z) noktasının dikey kartezyen koordinat sisteminden silindirsel koordinatlara dönüşümü,

$$\tau : \mathbb{R}^3 \to \mathbb{R}^3 \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

denklemleri ile verilir.



T cismi üstten $z = v(r, \theta)$ ve alttan $z = u(r, \theta)$ yüzeyleri ile sınırlı olsun.

1. T nin xy - düzlemi üzerine D izdüşümü kutupsal koordinatlarda verilsin.

2. $f(x, \theta, z)$ fonksiyonu S üzerinde sürekli olsun.

3.
$$r = \left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

ise,

$$\iiint_S f(r, \theta, z)dV = \iint_D \int_{u(r,\theta)}^{v(r,\theta)} f(r, \theta, z)rdzdrd\theta$$

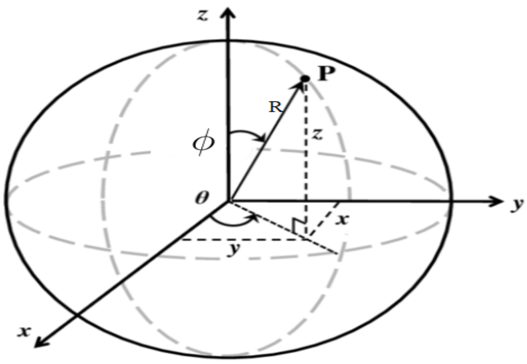
eşitliği sağlanır.

- **Teorem: Küresel Koordinatlar**

  Üç boyutlu uzayda bir $P(x,y,z)$ noktasının dikey kartezyen koordinat sisteminden küresel koordinatlardaki (R,ϕ,θ) noktasına dönüştüren η dönüşümü,

$$\eta : \mathbb{R}^3 \to \mathbb{R}^3 \begin{cases} x = R\cos\theta\sin\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\phi \end{cases}$$

  denklemleri ile verilir. (R $= \sqrt{x^2+y^2+z^2}$ açık.)



1. Küresel koordinatlarda gösterilen S sınırlı cismi üzerinde $\xi = f(R,\phi,\theta)$ fonksiyonu sürekli,

2. η dönüşümü geçerli,

3. η dönüşümünün Jacobiyanı,

$$R^2\sin\phi = \left|\frac{\partial(x,y,z)}{\partial(R,\phi,\theta)}\right| = \begin{vmatrix} x_R & x_\phi & x_0 \\ y_R & y_\phi & y_\theta \\ z_R & z_\phi & z_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & R\cos\phi\cos\theta & -R\sin\phi\sin\theta \\ \sin\phi\sin\theta & R\cos\phi\sin\theta & -R\sin\phi\cos\theta \\ \cos\phi & -R\sin\phi & 0 \end{vmatrix}$$

ise

$$\iiint_S f(x,y,z)dV = \iiint_{S^*} f(R,\phi,\theta)R^2\sin\phi dRd\phi d\theta$$

eşitliği sağlanır.

Aşağıdaki alıştırmalarda belirtilen bölge üzerindeki üç katlı integralleri hesaplayınız.

2) $0 \le x \le 1$, $-2 \le y \le 0$, $1 \le z \le 4$ ile verilen B kutusu üzerinde

$$\iiint_B xyz\, dV$$

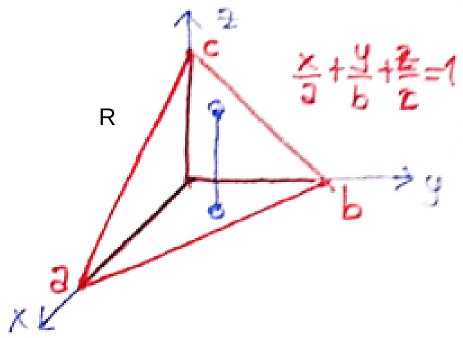
Sol.

$$\iiint_B xyz\, dV = \int_0^1 \int_{-2}^0 \int_1^4 xyz\, dz\, dy\, dx = \int_0^1 x\, dx \int_{-2}^0 y\, dy \int_1^4 z\, dz$$

$$= \frac{x^2}{2}\Big|_0^1 + \frac{y^2}{2}\Big|_{-2}^0 + \frac{z^2}{2}\Big|_0^4 = -\frac{15}{2}.$$

4) Koordinat düzlemleri ve $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ düzlemi tarafından sınırlanan tetrahedron üzerinde $\iiint_R x\, dV$

Sol.



$$\iiint_R x\, dV = \int_0^a x\, dx \int_0^{b(1-\frac{x}{a})} dy \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz$$

$$= c \int_0^a x\, dx \int_0^{b(1-\frac{x}{a})} \left(1 - \frac{x}{a} - \frac{y}{b}\right) dy$$

$$= c \int_0^a x\, dx \left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)\Big|_{y=0}^{y=b(1-\frac{x}{a})}$$

$$= c \int_0^a x \left[b\left(1-\frac{x}{a}\right) - \frac{bx}{a}\left(1-\frac{x}{a}\right) - \frac{b^2\left(1-\frac{x}{a}\right)^2}{2b} \right] dx$$

$$= c \int_0^a x \left[b\left(1-\frac{x}{a}\right)^2 - \frac{b}{2}\left(1-\frac{x}{a}\right)^2 \right] dx$$

$$= \frac{bc}{2} \int_0^a x \left(1 - \frac{x}{a}\right)^2 dx$$

$u = 1 - \frac{x}{a} \rightarrow du = -\frac{dx}{a}$

$x = 0 \rightarrow u = 1$

$x = a \rightarrow u = 0$

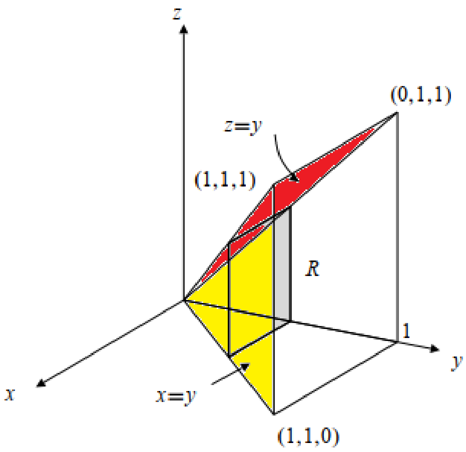
$$= \frac{bc}{2}\int_0^1 a(1-u)\,u^2 a\,du = \frac{a^2bc}{2}\int_0^1 u^2(1-u)\,du$$

$$= \frac{a^2bc}{2}\left(\frac{u^3}{3}-\frac{u^4}{4}\right)\Big|_0^1 = \frac{a^2bc}{24}.$$

9) $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(1,1,1)$ ve $(0,1,1)$ tepeli piramit üzerinde $\iiint_R \sin(\pi y^3)\,dV$

Sol.

$$\iiint_R \sin(\pi y^3)\,dV = \int_0^1 \sin(\pi y^3)\,dy \int_0^y dz \int_0^y dx$$



$$= \int_0^1 y^2 \sin(\pi y^3)\,dy \qquad u = y^3 \rightarrow du = 3y^2dy$$

$$= \int_0^1 \frac{\sin(u\pi)}{3}\,du$$

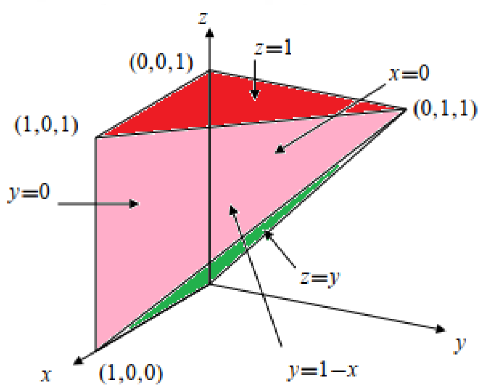
$$= -\frac{\cos(u\pi)}{3\pi}\Big|_0^1 = -\frac{1}{3\pi}(-1-1) = \frac{2}{3\pi}.$$

28) $\displaystyle\int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\,dz$?

Sol.

$$\int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\,dz$$

$$= \iiint_R \frac{\sin(\pi z)}{z(2-z)}\,dV$$



$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\,dz \int_0^z dy \int_0^{1-y} dx$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \int_0^z (1-y)\,dy$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \underbrace{\left(z - \frac{z^2}{2}\right)}_{\frac{z(2-z)}{2}} dz = \frac{1}{2}\int_0^1 \sin(\pi z)\,dz$$

$$= \frac{1}{2}\left(-\frac{\cos(\pi z)}{\pi}\right)\Big|_0^1 = \frac{1}{\pi}.$$

Aşağıda belirtilen bölgelerin hacimlerini hesaplayınız.

4) $z = x^2+y^2$ paraboloidinin ve $x^2+y^2+z^2=12$ küresinin içinde

Sol.

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow z = r^2$$

$0 \le \theta \le 2\pi$

$0 \le r \le \sqrt{3}$

$r^2 + z^2 = 12$

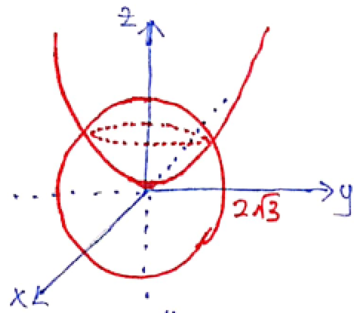
$r^2 + r^4 = 12$

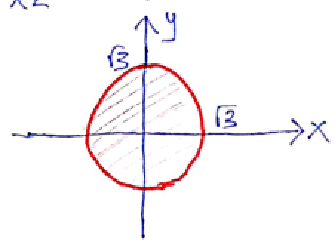
$r^4 + r^2 - 12 = 0$

$(r^2 = t) \quad t^2 + t - 12 = 0$

$\Rightarrow t = 3 \Rightarrow r^2 = 3$

$\Rightarrow r = \sqrt{3}$





Ayrıca; $z^2 = 12 - x^2 - y^2 = 12 - r^2$; $z = x^2+y^2$

$\Rightarrow z = \sqrt{12-r^2}$ $\qquad \Rightarrow z = r^2$

Dolayısıyla;

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left(\sqrt{12-r^2} - r^2\right) r\,dr$$

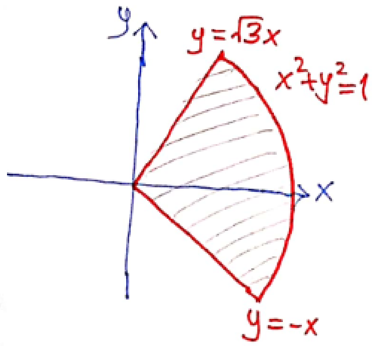
$$= \int_0^{2\pi} d\theta \left(-\frac{1}{3}(12-r^2)^{3/2} - \frac{r^4}{4}\right)\Big|_0^{\sqrt{3}}$$

$$= 2\pi\left(8\sqrt{3} - \frac{45}{4}\right).$$

6) xy - düzlemi üzerinde, $z=1-x^2-y^2$ paraboloidinin altında ve $-x \le y \le \sqrt{3}x$ olacak şekilde

Sol.

$$0=1-x^2-y^2 \Rightarrow x^2+y^2=1$$



$$V=\iint_D (1-x^2-y^2)\,dA$$

$$=\int_{-\pi/4}^{\pi/3} d\theta \int_0^1 (1-r^2)\,r\,dr$$

$$=\frac{7\pi}{12}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1=\frac{7\pi}{48}.$$

10) $\iiint_R (x^2+y^2+z^2)\,dV$ integralini hesaplayınız; burada R

$0 \le x^2+y^2 \le a^2$, $0 \le z \le h$ silindiridir.

Sol.

$$\iiint_R (x^2+y^2+z^2)\,dV=\int_0^{2\pi} d\theta \int_0^a r\,dr \int_0^h (r^2+z^2)\,dz$$

Silindirik koor. (r,θ,z).
(x,y,z)

$$=2\pi\int_0^a r\,dr\left(r^2 z+\frac{z^3}{3}\right)\Big|_{z=0}^{z=h}$$

$$=2\pi\left(\frac{a^4h}{4}+\frac{a^2h^3}{6}\right)=\frac{\pi a^4 h}{2}+\frac{\pi a^2 h^3}{3}$$

13) $\iiint_R (x^2+y^2+z^2)\,dV$ integralini hesaplayınız; burada R

$z=c\sqrt{x^2+y^2}$ konisi üzerinde olan ve $x^2+y^2+z^2=a^2$ küresi içinde uzanan bölgedir.

Sol.

Küresel koordinatlar (R, ϕ, θ)

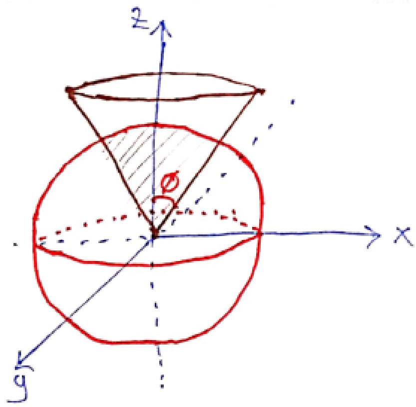
(x, y, z)

$$R = \sqrt{x^2+y^2+z^2} \Rightarrow R^2 = x^2+y^2+z^2.$$

Ayrıca; $r = R\sin\phi$, $z = R\cos\phi$

$$\Rightarrow \tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{c\sqrt{x^2+y^2}} = \frac{1}{c}$$

$$\Rightarrow \phi = \arctan\left(\frac{1}{c}\right).$$



$$\iiint_D (x^2+y^2+z^2)\,dV$$

$$= \int_0^{2\pi}\int_0^{\arctan(\frac{1}{c})}\int_0^{a} R^2 \cdot R^2 \sin\phi \, dR\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\arctan(\frac{1}{c})} \sin\phi \left(\frac{R^5}{5}\right)\Big|_0^{a} d\phi\, d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi}\int_0^{\arctan(\frac{1}{c})} \sin\phi\, d\phi\, d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi} (-\cos\phi)\Big|_0^{\arctan(\frac{1}{c})} d\theta$$

$$= \frac{a^5}{5}\, 2\pi \left(1 - \cos\left(\arctan\left(\frac{1}{c}\right)\right)\right) = \frac{2\pi a^5}{5}\left(1 - \frac{c}{\sqrt{c^2+1}}\right).$$