§12.4. Higher-Order Derivatives

If $z = f(x, y)$, we can calculate four partial derivatives of second order, namely, two pure second partial derivatives with respect to x or y.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial z}{\partial x} = f_{11}(x, y) = f_{xx}(x, y),$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial z}{\partial y} = f_{22}(x, y) = f_{yy}(x, y),$$

and two mixed second partial derivatives with respect to x and y,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = f_{21}(x, y) = f_{yx}(x, y),$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial z}{\partial x} = f_{12}(x, y) = f_{xy}(x, y).$$

The subscript closest to f indicates which differentiation occurs first.

Similarly, if $w = f(x, y, z)$, then

$$\frac{\partial^5 w}{\partial y \partial x \partial y^2 \partial z} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial w}{\partial z} = f_{32212}(x, y, z) = f_{zyyxy}(x, y, z).$$

Example 1. Find the four second partial derivatives of $f(x, y) = x^3 y^4$.

$f_1(x, y) = 3x^2 y^4$, $f_2(x, y) = 4x^3 y^3$,

$f_{11}(x,y) = \frac{\partial}{\partial x}(3x^2y^4) = 6xy^4$, $f_{21}(x,y) = \frac{\partial}{\partial x}(4x^3y^3) = 12x^2y^3$,
$f_{12}(x,y) = \frac{\partial}{\partial y}(3x^2y^4) = 12x^2y^3$, $f_{22}(x,y) = \frac{\partial}{\partial y}(4x^3y^3) = 12x^3y^2$,

Theorem 1. Equality of mixed partials

Suppose that two mixed nth-order partial derivatives of a function f involve the same differentiations but in different orders. If those partials are continuous at a point P, and if f and partial of f of order less than n are continuous in a neighbourhood of P, then the two mixed partials are equal at the point P.

The Laplace and Wave Equations

We will encounter two particular partial differential equations that arise frequently in mathematics and the physical sciences.

Example 2. (Laplace Equation) Show that for any real number k the function $z = e^{kx}cos(ky)$ satisfy the partial differential equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ at every point in the xy-plane.

$$\frac{\partial z}{\partial x} = ke^{kx}\cos(ky),$$
$$\frac{\partial z}{\partial y} = -ke^{kx}\sin(ky),$$

$$\frac{\partial^2 z}{\partial x^2} = k^2 e^{kx} \cos(ky),$$
$$\frac{\partial^2 z}{\partial y^2} = -k^2 e^{kx} \cos(ky).$$

Thus, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = k^2 e^{kx} \cos(ky) - k^2 e^{kx} \cos(ky) = 0$.

Remark. The partial differential equation in the above example is called the (two-dimensional) **Laplace equation**. A function of two variables having continuous second partial derivatives in a region of the plane is said to be **harmonic** there if it satisfies Laplace's equation.

Harmonic functions have many interesting properties. They have derivatives of all orders, and they are analytic; that is, they are the sums of their (multivariable) Taylor series. Laplace's equation, and therefore harmonic functions, can be considered in any number of dimensions.

Example 3. (Wave Equation) If f and g are any twice-differentiable functions of one variable, show that

$$w = f(x - ct) + g(x + ct)$$

satisfies the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

Using the Chain Rule for functions of one variable we obtain, $\frac{\partial w}{\partial t} = -cf'(x-ct) + cg'(x+ct)$, $\frac{\partial w}{\partial x} = f'(x-ct) + g'(x+ct)$, $\frac{\partial^2 w}{\partial t^2} = c^2 f''(x-ct) + c^2 g''(x+ct)$, $\frac{\partial^2 w}{\partial x^2} = f''(x-ct) + g''(x+ct)$. Thus w satisfies the given differential equation.

Remark. The partial differential equation in the above example is called the (one dimensional) **wave equation**. If t measures time, then $f(x-ct)$ represents a waveform travelling to the right along the x-axis with speed c. Similarly, $g(x+ct)$ represents a waveform travelling to the left with speed c.