10.3) The Cross Product in 3-Space

$u = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$, $v = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

* The cross product:

$$u \times v = \begin{array}{ccc} + & - & + \end{array}$$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

determinant

* $|u \times v| = |u||v| \sin\theta$. θ: angle between u & v.

* The scalar triple product:

$$u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v)$$

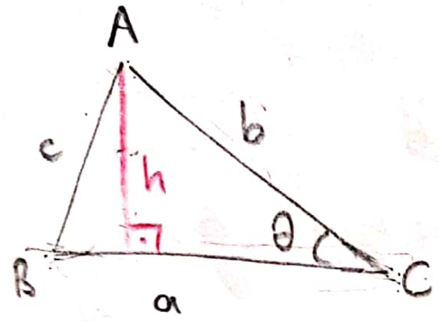
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

NOTE:

$\boxed{3}$ Find the area of the triangle with vertices

$(1,2,0)$, $(1,0,2)$, $(0,3,1)$:

⊕ The area of the triangle:



$$\text{Area} = \frac{1}{2} a h = \frac{1}{2} a b \sin\theta$$

$$= \frac{1}{2} |\vec{BC}| |\vec{AC}| \sin\theta$$

$$= \frac{1}{2} |\vec{BC} \times \vec{AC}|$$

$A = (1,2,0)$, $B = (1,0,2)$, $C = (0,3,1)$

$\Rightarrow \vec{BC} = -\hat{i} + 3\hat{j} - \hat{k}$, $\vec{AC} = -\hat{i} + \hat{j} + \hat{k}$

$$\Rightarrow \vec{BC} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 4\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\Rightarrow \text{Area} = \frac{1}{2} |\vec{BC} \times \vec{AC}|$$

$$= \frac{1}{2} \sqrt{4^2 + 2^2 + 2^2}$$

$$= \frac{1}{2} \sqrt{24}$$

$$= \sqrt{6}$$

$\boxed{6}$ Find a unit vector with positive k component that is perpendicular to both $\underbrace{2\hat{i}-\hat{j}-2\hat{k}}_{u}$ and $\underbrace{2\hat{i}-3\hat{j}+\hat{k}}_{v}$

* $u \times v$ is perpendicular to both u & v.

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -7\hat{i} - 6\hat{j} - 4\hat{k}$$

(a vector with negative k component)

If we divide $u \times v$ by its length and multiply by -1, we get a unit vector with positive k component.

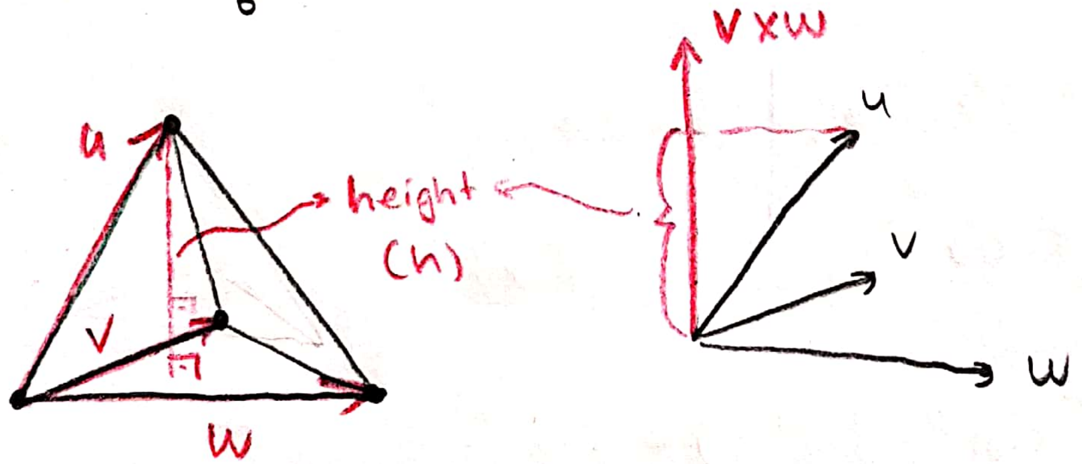
$$|u \times v| = \sqrt{(-7)^2+(-6)^2+(-4)^2} = \sqrt{101}$$

So, $\frac{1}{\sqrt{101}}(7\hat{i}+6\hat{j}+4\hat{k})$ is a unit vector with positive k component that is perpendicular to both $2\hat{i}-\hat{j}-2\hat{k}$ and $2\hat{i}-3\hat{j}+\hat{k}$.

14 Volume of a tetrahedron is $\frac{1}{3}Ah$, where A is the area of the base and h is the height measured perpendicular to the base.

If u, v and w are vectors coinciding with the three edges of a tetrahedron that meet at one vertex, show that the volume of the tetrahedron is

$$V = \frac{1}{6}\left| u \cdot (v \times w) \right|$$



So, height can be found by the scalar projection (positive-valued) of u in the direction of v x w

$$\Rightarrow \underset{(h)}{\text{Height}} = \left| \frac{u \cdot (v \times w)}{|v \times w|} \right|$$ ← absolute value

Area of the base = $A = \frac{1}{2}|v \times w|$ (by Q3)

The volume of the tetrahedron:

$$V = \frac{1}{3} Ah = \frac{1}{3}\left(\frac{1}{2}|v \times w|\right)\left(\frac{|u \cdot (v \times w)|}{|v \times w|}\right)$$

$$= \frac{1}{6}|u \cdot (v \times w)|$$

$\boxed{15}$ Find the volume of the tetrahedron with vertices

$\underbrace{(1,0,0)}_{A}$, $\underbrace{(1,2,0)}_{B}$, $\underbrace{(2,2,2)}_{C}$, $\underbrace{(0,3,2)}_{D}$:

$u = \overrightarrow{AB} = 2\hat{j}$, $v = \overrightarrow{AC} = \hat{i} + 2\hat{j} + 2\hat{k}$, $w = \overrightarrow{AD} = -\hat{i} + 3\hat{j} + 2\hat{k}$

$$\text{Volume} = \frac{1}{6}|u \cdot (v \times w)|$$

$$= \frac{1}{6}\left|\begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 2 \\ -1 & 3 & 2 \end{vmatrix}\right|$$

absolute value

determinant

$$= \frac{1}{6}|(-1)\,2\,(2+2)|$$

$$= \frac{1}{6}|-8|$$

$$= \frac{4}{3}$$

$\boxed{17}$ For what value of k do the four points

$\underbrace{(1,1,-1)}_{A}, \underbrace{(0,3,-2)}_{B}, \underbrace{(-2,1,0)}_{C}, \underbrace{(k,0,2)}_{D}$ all lie in a plane?

If these points lie in a plane, then the volume of the tetrahedron with these vertices is 0.

$u = \overrightarrow{AB} = -\hat{i} + 2\hat{j} - \hat{k}$,

$v = \overrightarrow{AC} = -3\hat{i} + \hat{k}$

$w = \overrightarrow{AD} = (k-1)\hat{i} - \hat{j} + 3\hat{k}$

$$\text{Volume} = \frac{1}{6} \left| u \cdot (v \times w) \right| = 0$$

$$\Rightarrow \left| u \cdot (v \times w) \right| = 0$$

$$\Rightarrow \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} -1 & 2 & -1 \\ -3 & 0 & 1 \\ (k-1) & 1 & 3 \end{vmatrix} \end{array} = -1 - 2(-9-k+1) - 1.3 = 0$$

$$\Rightarrow 2k + 12 = 0 \quad \Rightarrow k = -6$$

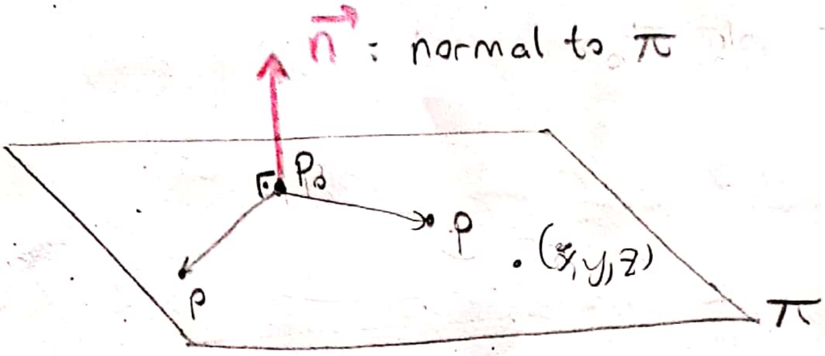
So, A, B, C, D are coplanar if $k = -6$

↑

lie in a plane

10.4) Planes and Lines

* Equation of a plane:



$\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$

$P_0 = (x_0, y_0, z_0)$

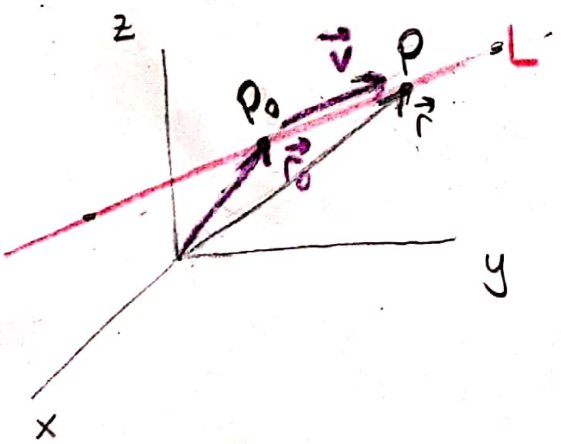
$P = (x, y, z)$

The plane passing through the point P_0 and perpendicular to $\vec{n}$:

$$\vec{n} \cdot (\overrightarrow{P_0P}) = 0 \Rightarrow A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

(All points P satisfying this equation form the plane π)

* Equation of a line:



$P_0 = (x_0, y_0, z_0)$

$\vec{r_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$: position vector of P_0

$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$: direction vector

$P = (x, y, z)$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$: position vector of P

$\vec{r} = \vec{r_0} + t\vec{v}$ gives the line passing through P_0 and parallel to $\vec{v}$.

(All points P with this position vector form the line L)

1) $\vec{r} = \vec{r}_0 + t\vec{v}$ is called the vector parametric equation of the straight line L.

2) The components of the vector parametric equation give the scalar parametric equations of the line:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \quad \text{for } -\infty < t < \infty$$

3) If $a \neq 0$, $b \neq 0$, $c \neq 0$, solve the equations in 2) for t and obtain the standard form for the equations of the straight line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

* If $a \neq 0$, $b \neq 0$ but $c = 0$, then the standard form:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0$$

4-8 Find equations of the planes satisfying the given conditions.

Q4) passing through $(1,2,3)$ and parallel to the plane $3x+y-2z=15$

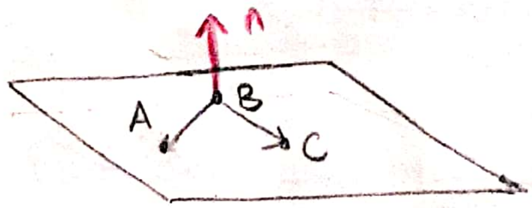
Parallel planes have the same normals and $\vec{n}=3\hat{i}+\hat{j}-2\hat{k}$ is the normal to the plane $3x+y-2z=15$.

The plane passing through $(1,2,3)$ and perpendicular to $\vec{n}=3\hat{i}+\hat{j}-2\hat{k}$ is

$$3(x-1)+1\cdot(y-2)-2(z-3)=0$$

$$\Rightarrow \quad 3x+y-2z=-1$$

Q5) passing through the points $\underbrace{(1,1,0)}_{A}, \underbrace{(2,0,2)}_{B}, \underbrace{(0,3,3)}_{C}$



$$\vec{BA}=-\hat{i}+\hat{j}-2\hat{k}$$
$$\vec{BC}=-2\hat{i}+3\hat{j}+\hat{k}$$

The cross product of two vectors on the plane gives a vector perpendicular to the plane. We can take that vector as the normal to the plane.

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ -2 & 3 & 1 \end{vmatrix} = 7\hat{\imath} + 5\hat{\jmath} - \hat{k} = \vec{n}$$

The plane passing through the point $B = (2,0,2)$ and perpendicular to $\vec{n} = 7\hat{\imath} + 5\hat{\jmath} - \hat{k}$ is

We can choose A or C since those points are also on the plane

$$7(x-2) + 5(y-0) - 1(z-2) = 0$$

$$\Rightarrow \quad 7x + 5y - z = 12$$

Q8) passing through the line of intersection of the planes $2x+3y-z=0$ and $x-4y+2z=-5$, and passing through the point $(-2,0,-1)$.

Let $\pi_1 = 2x+3y-z$, $\pi_2 = x-4y+2z+5$

Then, the required plane will have an equation of the form

$$\pi_1 + t\pi_2 = 0 \Rightarrow (2x+3y-z) + t(x-4y+2z+5) = 0 \qquad (*)$$

The point $(-2,0,-1)$ is on this plane. So, it must satisfy equation $(*)$:

$2(-2)+3.0-(-1)+t(-2-4.0+2(-1)+5)=0$

$\Rightarrow t=3.$

So the required plane is

$(2x+3y-z)+3(x-4y+2z+5)=0$

$\Rightarrow 5x-9y+5z=-15$

$\boxed{18}$ Find the equation of the line through the point $(2,-1,-1)$ and parallel to each of the two planes

$\underbrace{x+y=0}_{\pi_1}$ and $\underbrace{x-y+2z=0}_{\pi_2}$.

$\vec{n}_1=\hat{i}+\hat{j}$ is the normal of π_1.

$\vec{n}_2=\hat{i}-\hat{j}+2\hat{k}$ is the normal of π_2.

If the line is parallel to π_1 and π_2, then it is perpendicular to n_1 and n_2

$\Rightarrow$ the line is parallel to $n_1 \times n_2$.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 2\hat{i}-2\hat{j}-2\hat{k}$$

The line through $(2,-1,-1)$ and parallel to the vector

$\vec{n}_1 \times \vec{n}_2 = 2\hat{i} - 2\hat{j} - 2\hat{k}$ is

$(2\hat{i} - \hat{j} - \hat{k}) + t(2\hat{i} - 2\hat{j} - 2\hat{k})$

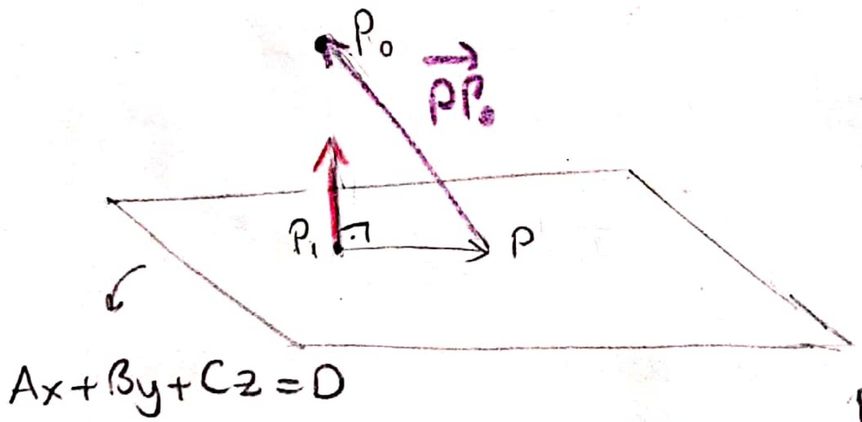
$\Rightarrow (2+2t)\hat{i} + (-1-2t)\hat{j} + (-1-2t)\hat{k}$ (vector parametric form)

$\Rightarrow x = 2+2t, \quad y = -1-2t, \quad z = -1-2t$ (scalar parametric form)

$\Rightarrow x-2 = -y-1 = -z-1$ (Standard form)

Distances

* Distance from a point to a plane:



$P = (x,y,z)$ is any point on the plane

P_1 is the point on the plane closest to $P_0 = (x_0, y_0, z_0)$

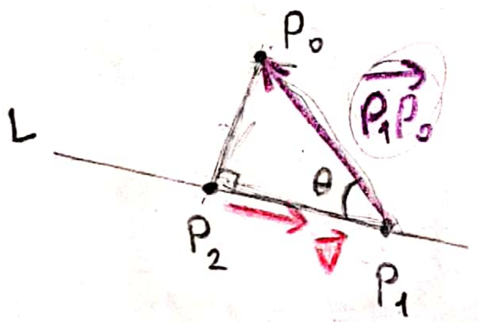
The distance between P_0 and P_1 gives the distance from P_0 to the plane. So we need to take the length of the projection of $\vec{PP_0}$ in the direction of $\vec{n}$.

$$\text{Distance} = \left| \frac{\overrightarrow{PP_0} \cdot \vec{n}}{|\vec{n}|} \right| = \frac{|A(x-x_0)+B(y-y_0)+C(z-z_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|Ax+By+Cz-(Ax_0+By_0+Cz_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$\Rightarrow \text{Distance} = \frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}$$

⊛ Distance from a point to a line:



L is a line through P_1 parallel to $\vec{v}$.

P_2 is the point on the line closest to P_0.

$$|\overrightarrow{P_2P_0}| = |\overrightarrow{P_1P_0}| \sin\theta \Rightarrow \sin\theta = \frac{|\overrightarrow{P_2P_0}|}{|\overrightarrow{P_1P_0}|} \quad (1)$$

$$|\overrightarrow{P_0P_1} \times \vec{v}| = |\overrightarrow{P_0P_1}|\,|\vec{v}| \sin\theta \quad (2)$$

$$\Rightarrow \sin\theta = \frac{|\overrightarrow{P_0P_1} \times \vec{v}|}{|\overrightarrow{P_0P_1}|\,|\vec{v}|} \quad (2)$$

By (1) and (2), $|P_0P_2| = \dfrac{|\overrightarrow{P_0P_1} \times \vec{v}|}{|\vec{v}|}$

The distance from P_0 to the line is given by $|\vec{P_0 P}|$...

$$\Rightarrow \text{Distance} = \frac{|\vec{P_0P_1} \times v|}{|\vec{v}|}$$

$\boxed{27, 28}$ Find the required distances.

Q27) from $(1,2,0)$ to the plane $3x-4y-5z=2$.

$P_0 = (1,2,0)$, $A=3$, $B=-4$, $C=-5$, $D=2$

$$\Rightarrow \text{Distance} = \frac{|3.1-4.2-5.0-2|}{\sqrt{3^2+(-4)^2+(-5)^2}} = \frac{7}{5\sqrt{2}}$$

Q28) from the origin to the line $\begin{cases} x+y+z=0 \\ 2x-y-5z=1 \end{cases}$

We need to find a point on the line and a vector parallel to the line.

Choose $z=0$. Then

$$\left.\begin{aligned} x+y&=0 \\ 2x-y&=1 \end{aligned}\right\} \Rightarrow x=\frac{1}{3}, \; y=-\frac{1}{3}$$

So, $\underbrace{\left(\frac{1}{3}, -\frac{1}{3}, 0\right)}_{P_0}$ is a point on the line.

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$\vec{n}_1 = \hat{i} + \hat{j} + \hat{k}$ is the normal to the plane $x+y+z=0$

$\vec{n}_2 = 2\hat{i} - \hat{j} - 5\hat{k}$ is the normal to the plane $2x-y-5z=1$

The line is parallel to these plane So it must be perpendicular to $\vec{n}_1$ and $\vec{n}_2$.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & -5 \end{vmatrix} = -4\hat{i} + 7\hat{j} - 3\hat{k}$$

the line is parallel to $\vec{n}_1 \times \vec{n}_2$

$$|\vec{n}_1 \times \vec{n}_2| = \sqrt{74}$$

If $r_0 = \frac{1}{3}\hat{i} - \frac{1}{3}\hat{j}$ is the vector from the origin to P_0,

$$r_0 \times (n_1 \times n_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ -4 & 7 & -3 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k}$$

$$|r_0 \times (n_1 \times n_2)| = \sqrt{3}$$

$$\Rightarrow \text{Distance} = \frac{|r_0 \times (n_1 \times n_2)|}{|n_1 \times n_2|} = \sqrt{\frac{3}{74}}$$

$\boxed{30}$ Show that the line $x-2 = \frac{y+3}{2} = \frac{z-1}{4}$

is parallel to the plane $2y - z = 1$.

What is the distance between the line and the plane?

$x = 2+t$, $y = -3+2t$, $z = 1+4t$.

So the line is passing through $(2,-3,1)$ and parallel

to $\vec{v} = \hat{i} + 2\hat{j} + 4\hat{k}$.

This line is parallel to the plane if it is perpendicular

to the normal of the plane.

$\vec{n} = 2\hat{j} - \hat{k}$, $\vec{v} \cdot \vec{n} = 0+4-4=0$

$\Rightarrow$ $\vec{v}$ and $\vec{n}$ are perpendicular. So, the line is

parallel to the plane.

Since they are parallel, distance from any point on the

line to the plane is always the same.

$P_0 = (2,-3,1)$, $A=0$, $B=2$, $C=-1$, $D=1$

$$\Rightarrow \text{Distance} = \frac{|0.2 + 2(-3) - 1.1 - 1|}{\sqrt{0^2 + (-3)^2 + 1^2}} = \frac{8}{\sqrt{5}}$$