GTU, Fall 2020, MATH 101

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Chain rule says that the derivative of the composite is derivative f' of the outside function evaluated at the inside function $g(x)$, multiplied by the derivative $g'(x)$ of the inside function.

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where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

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$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (7u^6)(5) = 35(5x+2)^6.$$

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$$= \frac{7x^4-2x^2-1}{2(x^3-x)^{1/2}}.$$

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An equation for the tangent line is:

$$y - 1 = -6(x - 2) \text{ or } y = -6x + 13.$$