§12.2. Limits and Continuity

The concept of the limit of a function of several variables is similar to that for functions of one variable.
We might say that $f(x, y)$ approaches the limit L as the point (x, y) approaches the point (a, b), and write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

You may review the concepts of neighbourhood, open and closed sets, and boundary and interior points for the following formal definition of limit for functions of two variables(Section 10.1).

Definition. We say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$, provided that
(i) every neighbourhood of (a, b) contains points of the domain of f different from (a, b), and
(ii) for every positive number ϵ there exists a positive number $\delta = \delta(\epsilon)$ such that $|f(x, y)-L| < \epsilon$ holds whenever (x, y) is in the domain of f and satisfies $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

All the usual laws of limits extend to several variables in the obvious way. For instance, if $\lim_{(x,y)\to(a,b)} f(x, y) = L$, $\lim_{(x,y)\to(a,b)} g(x, y) = M$, and every neighbourhood of (a, b) contains points in $\mathcal{D}(f) \bigcap \mathcal{D}(g)$

other than (a, b), then

$\lim_{(x,y)\to(a,b)}(f(x, y) \pm g(x, y)) = L \pm M$,

$\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = LM$,

$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ provided $M \neq 0$.

Also, if $F(t)$ is continuous at $t = L$, then

$$\lim_{(x,y)\to(a,b)} F(f(x, y)) = F(L).$$

Example 1. a) $\lim_{(x,y)\to(2,1)}(2x - y^2) = 4 - 1 = 3$,

b) $\lim_{(x,y)\to(a,b)} x^3y = a^3b$,

c) $\lim_{(x,y)\to(\pi/3,2)} x \sin(\frac{x}{y}) = \frac{\pi}{3}\frac{1}{2} = \frac{\pi}{6}$.

Example 2. The function $f(x, y) = \sqrt{4 - x^2 - y^2}$ has limit $f(a, b)$ at all points (a, b) of its domain, the closed disk $x^2 + y^2 \leq 2$, and is therefore considered to be continuous on its domain. Of course (x, y) can approach points of the bounding circle $x^2 + y^2 = 2$ only from within disk.

Now we will give an example to see that the requirement that $f(x, y)$ approach the same limit no matter how (x, y) approaches (a, b) can be very restrictive, and makes limits in two or more variables much more subtle than in the single-variable case.

Example 3. (a) Investigate the limiting behaviour of $f(x, y) = \frac{2xy}{x^2+y^2}$ as (x, y) approaches $(0, 0)$.

Note that $f(x, y)$ is defined at all points of the xy-plane except the origin. We can still ask whether the limit exists as (x, y) approaches $(0, 0)$. If we let (x, y) approaches $(0, 0)$ along x-axis $(y = 0)$, then $f(x, y) = f(x, 0) \to 0$ (because $f(x, 0) = 0$ identically). Thus, $\lim_{(x,y)\to(0,0)} f(x, y)$ must be 0 if it exists at all. Similarly, at all points the y-axis we have $f(x, y) = f(0, y) \to 0$. However, at points of the line $x = y$, f has a different constant value; $f(x, x) = f(y, y) = 1$. Since the limit of $f(x, y)$ is 1 as (x, y) approaches $(0, 0)$ along this line, it follows that $f(x, y)$ **cannot have a unique limit at the origin**. That is,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$$

does not exist.

(b) Investigate the limiting behaviour of $f(x, y) = \frac{2x^2y}{x^4+y^2}$ as (x, y) approaches $(0, 0)$.

$f(x, y)$ vanishes identically on the coordinate axes, so $\lim_{(x,y)\to(0,0)} f(x, y)$ must be 0 if it exists at all. If we examine $f(x, y)$ at points of the ray $y = kx$ (k is any scalar), we obtain

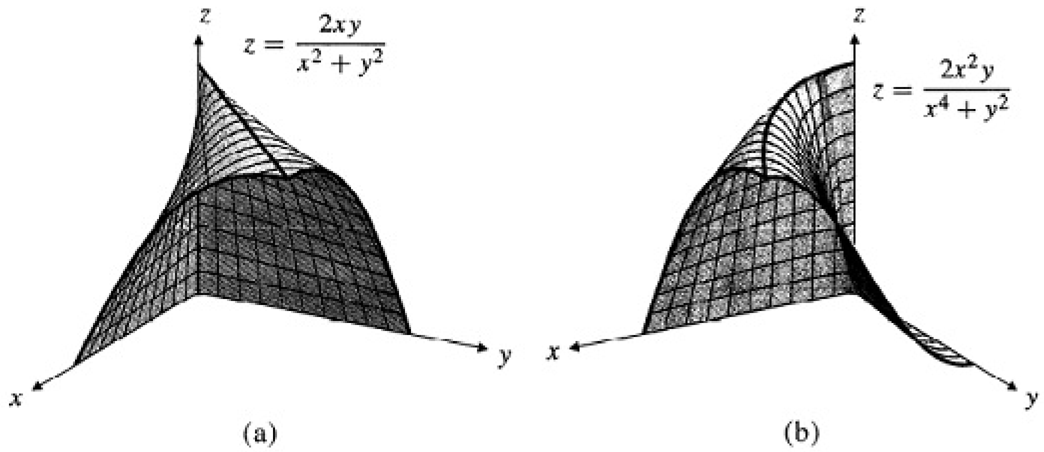
$$f(x, kx) = \frac{2kx^3}{x^4 + k^2x^2} = \frac{2kx}{x^2 + k^2} \to 0,$$

as $x \to 0$ $(k \neq 0)$.

Thus, $f(x, y) \to 0$ as (x, y) approaches $(0, 0)$ along any straight line through the origin. We might be think that limit exists and equals 0 but **this is not correct!**. Observe the behaviour of f along the curve $y = x^2$:

$$f(x, x^2) = \frac{2x^4}{x^4 + x^4} = 1.$$

Thus, $f(x, y)$ does not approach 0 as (x, y) approaches $(0, 0)$ along this curve, so the **limit does not exist!**



Definition. The function $f(x, y)$ is **continuous at the point** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

It remains true that sums, differences, products, quotients, and compositions of continuous functions are continuous.

As for functions of one variable, the existence of a limit of a function at a point does not imply that the function is continuous at that point. For example the function

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

satisfies $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, which is not equal to $f(0, 0) = 1$, so f is not continuous at origin. But we can make f continuous at origin by redefining (continuous extension) its value at that point to be 0.