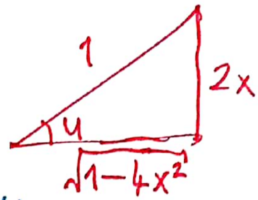
27) $\int \frac{x^2\,dx}{\sqrt{1-4x^2}} = ?$

Sol: Let $2x = \sin u$, then $2dx = \cos u\,du$.



$$\int \frac{x^2\,dx}{\sqrt{1-4x^2}} = \frac{1}{8}\int \frac{\sin^2 u \cos u\,du}{\cos u} = \frac{1}{8}\int \sin^2 u\,du$$

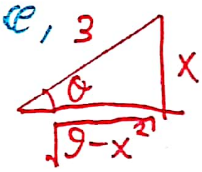
$$= \frac{1}{16}\int (1-\cos 2u)\,du = \frac{u}{16} - \frac{\sin 2u}{32} + C$$

$$= \frac{u}{16} - \frac{2.\sin u.\cos u}{32} + C$$

$$= \frac{1}{16}\arcsin(2x) - \frac{1}{8}x.\sqrt{1-4x^2} + C.$$

28) $\int \frac{dx}{x\sqrt{9-x^2}} = ?$

Sol: Let $x = 3\sin\theta$, then $dx = 3\cos\theta\,d\theta$. Hence,



$$\int \frac{dx}{x\sqrt{9-x^2}} = \int \frac{3\cos\theta\,d\theta}{3\sin\theta\, 3\cos\theta} = \frac{1}{3}\int \operatorname{cosec}\theta\,d\theta$$

$$= \frac{1}{3}\ln|\operatorname{cosec}\theta - \cot\theta| + C = \frac{1}{3}\ln\left|\frac{3}{x} - \frac{\sqrt{9-x^2}}{x}\right| + C$$

29) $\int \frac{x^3\,dx}{\sqrt{9+x^2}} = ?$

Sol: Let $u = 9+x^2$, $du = 2x\,dx$.

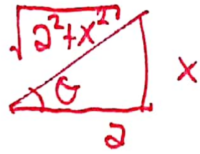
$$\int \frac{x^3\,dx}{\sqrt{9+x^2}} = \frac{1}{2}\int \frac{(u-9)\,du}{\sqrt{u}} = \frac{1}{2}\int (u^{1/2} - 9u^{-1/2})\,du$$

$$= \frac{1}{3}u^{3/2} - 9.\,u^{1/2} + C$$

$$= \frac{1}{3}(9+x^2)^{3/2} - 9\sqrt{9+x^2} + C$$

30) $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}} = ?$

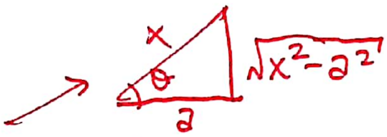
Sol: Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$. Hence,



$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \int \frac{a\sec^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^{3/2}} = \int \frac{a\sec^2\theta}{a^3\sec^3\theta}\, d\theta$$

$$= \frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta + C = \frac{x}{a^2\sqrt{a^2+x^2}} + C$$

31) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = ?$



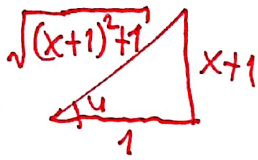
Sol: Let $x = a\sec\theta$ $(a>0)$, $dx = a\sec\theta\tan\theta\, d\theta$

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{a^2\sec^2\theta\; a\tan\theta} = \frac{1}{a^2}\int \cos\theta\, d\theta$$

$$= \frac{1}{a^2}\sin\theta + C = \frac{1}{a^2}\frac{\sqrt{x^2-a^2}}{x} + C.$$

32) $\displaystyle\int \frac{dx}{(x^2+2x+2)^2} = ?$

Sol: $\displaystyle\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{dx}{[(x+1)^2+1]^2}$ $\quad x+1 = \tan u$, $dx = \sec^2 u\, du$



$$= \int \frac{\sec^2 u\, du}{\sec^4 u} = \int \cos^2 u\, du = \frac{1}{2}\int (1+\cos 2u)\, du$$

$$= \frac{u}{2} + \frac{\sin 2u}{4} + C = \frac{u}{2} + \frac{\sin u\cdot\cos u}{2} + C$$

$$= \frac{1}{2}\arctan(x+1) + \frac{1}{2}\frac{x+1}{x^2+2x+2} + C$$

33) $\displaystyle\int \frac{1+x^{1/2}}{1+x^{1/3}}\,dx = ?$

Sol: Let $x = u^6$, $dx = 6u^5\,du$. Hence,

$$I = \int \frac{1+x^{1/2}}{1+x^{1/3}}\,dx = \int \frac{1+u^3}{1+u^2}\,6u^5\,du = 6.\int \frac{u^8+u^5}{1+u^2}\,du$$

$*$ $u^8 = u^8+u^6-u^6+u^4-u^4+u^2-u^2+1-1 = (u^2+1)(u^6-u^4+u^2-1)+1$

$*$ $u^5 = u^5+u^3-u^3+u-u = (u^2+1)(u^3-u)+u.$

Thus, $\dfrac{u^8+u^5}{u^2+1} = u^6-u^4+u^3+u^2-u-1+\dfrac{u-1}{u^2+1}$. Hence,

$$I = 6.\int\left(u^6-u^4+u^3+u^2-u-1+\frac{u+1}{u^2+1}\right)du$$

$$= 6\left[\frac{u^7}{7}-\frac{u^5}{5}+\frac{u^4}{4}+\frac{u^3}{3}-\frac{u^2}{2}-u+\frac{1}{2}\ln(u^2+1)+\arctan u\right]+C$$

$$= \frac{6}{7}x^{7/6}-\frac{6}{5}x^{5/6}+\frac{3}{2}x^{2/3}+2x^{1/2}-3x^{1/3}-6x^{1/6}+3\ln(1+x^{1/3})+6\arctan x^{1/6}+C$$

34) $\displaystyle\int \frac{dx}{x(3+x^2)\sqrt{1-x^2}} = ?$

Sol: Let $1-x^2 = u^2$. Then, $-2x\,dx = 2u\,du$. Hence,

$$I = \int \frac{dx}{x(3+x^2)\sqrt{1-x^2}} = -\int \frac{du}{(1-u^2)(4-u^2)}$$

$*$ $\dfrac{1}{(1-u^2)(4-u^2)} = \dfrac{A}{1-u}+\dfrac{B}{1+u}+\dfrac{C}{2-u}+\dfrac{D}{2+u} \Longrightarrow A=\dfrac{1}{6},\ B=\dfrac{1}{6},\ C=-\dfrac{1}{12}$

$D=-\dfrac{1}{12}.$

$$I = -\left(\frac{1}{6}\int\frac{du}{1-u}+\frac{1}{6}\int\frac{du}{1+u}\right)-\frac{1}{12}\left(\int\frac{du}{2-u}-\int\frac{du}{2+u}\right)$$

, Therefore,

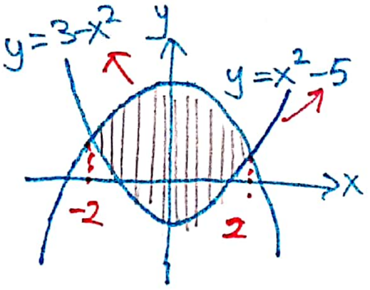
$$I = \frac{1}{6}\ln\left|\frac{1-u}{1+u}\right|+\frac{1}{12}\ln\left|\frac{2+u}{2-u}\right|+C$$

$$= \frac{1}{6}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|+\frac{1}{12}\ln\left|\frac{2+\sqrt{1-x^2}}{2-\sqrt{1-x^2}}\right|+C \quad //$$

MATH 101.2 PS-11

Q1) In the exercises below, find the area of the region bounded by the given curves.

a) $y = x^2 - 5$ and $y = 3 - x^2$



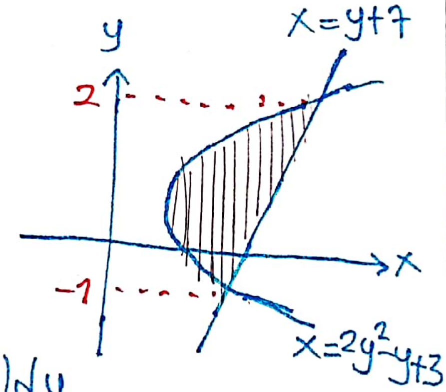
Sol: Let us find the intercection points.

$x^2 - 5 = 3 - x^2 \Rightarrow x^2 = 4 \Rightarrow x = \mp 2$

$$\text{Area} = \int_{-2}^{2} [(-x^2+5)+(3-x^2)]\,dx = \int_{-2}^{2} (8-2x^2)\,dx$$

$$= 8x - 2\frac{x^3}{3}\Big]_{-2}^{2} = 16 - \frac{16}{3} + 16 - \frac{16}{3} = \frac{64}{3}.$$

b) $x - y = 7$ and $x = 2y^2 - y + 3$



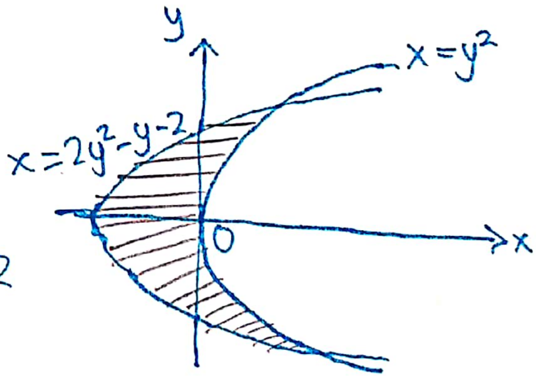
Sol: For intersections, $x = 7 + y$, $x = 2y^2 - y + 3$.

$7 + y = 2y^2 - y + 3 \Rightarrow 2(y-2)(y+1) = 0 \Rightarrow \begin{matrix} y = -1 \\ y = 2 \end{matrix}$

$$\text{Area} = \int_{-1}^{2} [(7+y) - (2y^2 - y + 3)]\,dy = 2\int_{-1}^{2} (2 + y - y^2)\,dy$$

$$= 2\left(2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_{-1}^{3} = 9 \,//$$

c) $x = y^2$, $x = 2y^2 - y - 2$



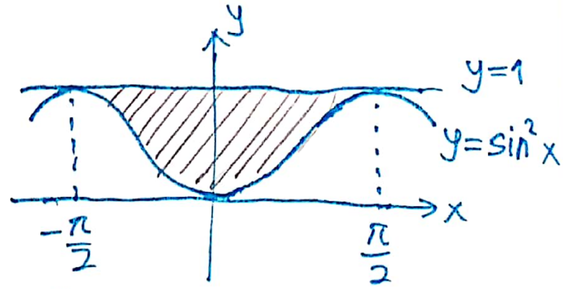
Sol: For intersections, $y^2 = 2y^2 - y - 2$.

$y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0 \Rightarrow y = -1, 2$

$$\text{Area} = \int_{-1}^{2} [y^2 - (2y^2 - y - 2)]\,dy = \int_{-1}^{2} (2 + y - y^2)\,dy$$

$$= \left(2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_{-1}^{2} = \frac{9}{2} \,//$$

Q2) Find the area of the region bounded by $y=\sin^2 x$ and $y=1$ and between two consecutive intersections of these curves.



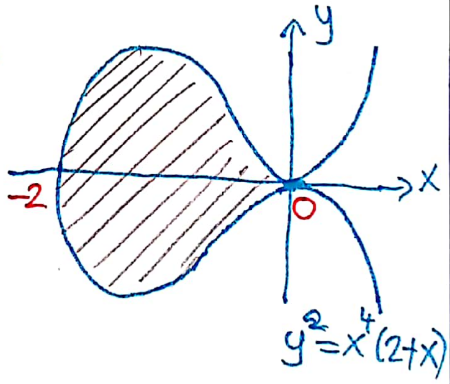
Sol: Let us sketch a graph.

$\sin^2 x = 1 \Rightarrow \sin x = \mp 1 \Rightarrow x = \mp \frac{\pi}{2}$

$$\text{Area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2 x)\,dx = \int_{-\frac{\pi}{2}}^{\pi/2} \left[1 - \frac{1-\cos(2x)}{2}\right] dx$$

$$= \frac{x}{2} + \frac{\sin(2x)}{4} \Big|_{-\frac{\pi}{2}}^{\pi/2} = \frac{\pi}{2} \;//$$

Q3) Find the area of the closed loop of the curve $y^2 = x^4(2+x)$ that lies to the left of the origin.



Sol: For $y=0$, $x^4(2+x)=0 \Rightarrow x=0,\ x=-2$.

Since, $y^2 = x^4(2+x)$ and $(-y)^2 = x^4(2+x)$. The curve is symmetric with respect to x-axis. For positive part, $y = x^2\sqrt{2+x}$

$$\text{Loop area} = 2\int_{-2}^{0} x^2\sqrt{2+x}\,dx \quad \rightarrow \quad \begin{matrix} 2+x=u^2 \\ dx = 2u\,du \end{matrix}$$

$$= 2\int_{0}^{\sqrt{2}} (u^2-2)^2 u \cdot 2u\,du$$

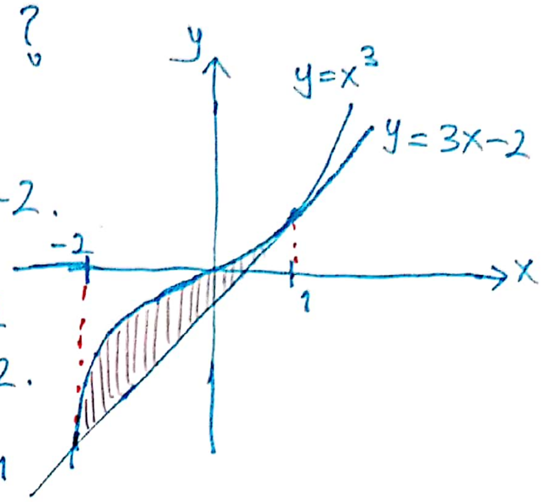
$$= 4\int_{0}^{\sqrt{2}} (u^6 - 4u^4 + 4u^2)\,du$$

$$= 4\left(\frac{1}{7}u^7 - \frac{4}{5}u^5 + \frac{4}{3}u^3\right)\Big|_0^{\sqrt{2}}$$

$$= \frac{256}{105}\sqrt{2} \;//$$

Q4) Find the area of the finite plane region bounded by the curve $y=x^3$ and the tangent line to ~~the~~ that curve at the point $(1,1)$.

Sol: Tangent line to $y=x^3$ at $(1,1)$?



$y' = 3x^2$, $y'(1) = 3 = m$. Hence,

tangent line is $y-1 = 3(x-1) \Rightarrow y = 3x-2$.

The intersections of $y=x^3$ and $y=3x-2$ are $x^3 = 3x-2 \Rightarrow x^3-3x-2=0 \Rightarrow x=1,-2$.

$$\text{Area} = \int_{-2}^{1} (x^3-3x+2)\,dx = \left(\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right)\Big|_{-2}^{1}$$

$$= -\frac{15}{4} - \frac{3}{2} + 6 + 2 + 4 = \frac{27}{4} \; //$$

Q5) Evaluate the given integral or show that it diverges.

a) $\displaystyle\int_3^\infty \frac{1}{(2x-1)^{2/3}}\,dx$

Sol: Let $2x-1 = u$. Then, $du = 2.dx$. Also $x=3 \Rightarrow u=5$, $x=\infty \Rightarrow u=\infty$.

$$\int_3^\infty \frac{1}{(2x-1)^{2/3}}\,dx = \int_5^\infty \frac{du}{u^{2/3}} = \frac{1}{2}\lim_{R\to\infty}\int_5^R u^{-2/3}\,du$$

$$= \frac{1}{2}\cdot\lim_{R\to\infty} 3u^{1/3}\Big|_5^R = \frac{1}{2}\cdot\lim_{R\to\infty}\left(3.R^{1/3} - 3.5^{1/3}\right) = \infty$$

diverges

b) $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2+1}$

Sol: $$\int_{-\infty}^{-1}\frac{dx}{x^2+1} = \lim_{R\to-\infty}\int_R^{-1}\frac{dx}{x^2+1} = \lim_{R\to-\infty} \arctan x\Big|_R^{-1}$$

$$= \lim_{R\to-\infty}\left(\arctan(-1) - \arctan(R)\right)$$

$$= -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4} \; //$$

c) $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{(1-\sin x)^{2/3}}$

$x=0 \Rightarrow u=1$
$x=\frac{\pi}{2} \Rightarrow u=0$

Sol: Let $u=1-\sin x$. Then, $du=-\cos x\,dx$. Hence

$$\int_0^{\pi/2} \frac{\cos x\,dx}{(1-\sin x)^{2/3}} = \int_1^0 \frac{-du}{u^{2/3}} = \int_0^1 u^{-2/3}\,du = \lim_{c\to 0^+} 3u^{1/3}\Big|_c^1$$

$$= \lim_{c\to 0^+}\left(3.1^{1/3} - 3.c^{1/3}\right) = 3 \quad \text{converges.}$$

d) $\displaystyle\int_0^{\infty} x.e^{-x}\,dx$

Sol: Let $u=x$ and $dv=e^{-x}dx$. Then, $du=dx$ and $v=-e^{-x}$.

$$\int_0^{\infty} x.e^{-x}dx = \lim_{R\to\infty}\int_0^R x.e^{-x}dx = \lim_{R\to\infty}\left(-x.e^{-x}\Big|_0^R + \int_0^R e^{-x}dx\right)$$

$$= \lim_{R\to\infty}\left(-\frac{R}{e^R} - \frac{1}{e^R} + 1\right) = 1 // \quad \text{converges.}$$

e) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx$

Sol: $$\int_{-\infty}^{\infty} \frac{x}{1+x^4}dx = \int_{-\infty}^{0} \frac{x.dx}{1+x^4} + \int_0^{\infty} \frac{x}{1+x^4}dx = I_1 + I_2 \quad \text{respectively.}$$

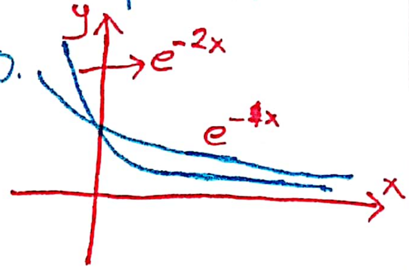
$$I_1 = \lim_{R\to\infty}\int_R^0 \frac{x\,dx}{1+x^4} \underset{\substack{u=x^2\\ du=2xdx}}{=} \lim_{R\to+\infty}\int_R^0 \frac{1}{2}\frac{du}{1+u^2} = \frac{1}{2}\lim_{R\to\infty} \arctan u\Big|_R^0 = -\frac{\pi}{4}$$

$$I_2 = \lim_{R\to\infty}\int_0^R \frac{x\,dx}{1+x^4} \underset{\substack{u=x^2\\ du=2xdx}}{=} \lim_{R\to\infty}\int_0^R \frac{1}{2}\frac{du}{1+u^2} = \frac{1}{2}\lim_{R\to\infty} \arctan u\Big|_0^R = \frac{\pi}{4}$$

Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+x^4}dx = I_1 + I_2 = 0 //$

Q6) Find the area below $y = e^{-x}$, above $y = e^{-2x}$ and to the right of $x = 0$.

Sol: Let us sketch th graph. For intersection points,



$$e^{-x} = e^{-2x} \Rightarrow \ln e^{-x} = \ln e^{-2x} \Rightarrow -x = -2x \Rightarrow x = 0.$$

Also, $e^{-x} \geq e^{-2x}$ for all $x \geq 0$. Hence,

$$\text{Area} = \int_0^{\infty} (e^{-x} - e^{-2x})\,dx = \lim_{R \to \infty} \int_0^{R} (e^{-x} - e^{-2x})\,dx$$

$$= \lim_{R \to \infty} \left(-e^{-x} + \frac{1}{2} e^{-2x}\right)\Big|_0^R = \lim_{R \to \infty} \left(-e^{-R} + \frac{1}{2} e^{-2R} + 1 - \frac{1}{2}\right) = \frac{1}{2} \,//$$