§9.6 TAYLOR AND MACLAURIN SERIES

Theorem 1. Let $\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ...$ converge to $f(x)$ for $|x-c| < R$, $R > 0$. Then $a_k = \frac{f^{(k)}(c)}{k!}$ for all $k = 1, 2, 3, ...$

Definition. (**Taylor and Maclaurin Series**) Let $f(x)$ be differentiable for all orders at $x = c$. Then the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + ...$ is called the Taylor series of f about c (or the Taylor series of f in powers of $(x-c)$). If $c = 0$, then the series is called the Maclaurin series, namely; $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)x + \frac{f''(x)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ...$

Definition. (**Analytic Function**) A function f is called analytic at c if there exists a Taylor series of f about c such that it converges to $f(x)$ in an open interval including c. If f is analytic at each point of an open interval, then it is called analytic on the interval.

Example 1. Find the Taylor series of e^x about c, the interval of its convergence. Where is e^x analytic? Find the Maclaurin series of e^x.
Let $f(x) = e^x$. We know that $f^{(n)}(x) = e^x$ for all $n = 0, 1, 2, 3, ...$
Then the Taylor series for e^x about c is:
$\sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + ...$
The radius of the series is:

$L = \lim_{n\to\infty} |\frac{\frac{e^c}{(n+1)!}}{\frac{e^c}{n!}}| = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow R = \infty$. Thus the series converges for all real numbers. Now, we find the function which to the series converges.

$g(x) = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + ...$

We calculate the derivative of $g(x)$:

$g'(x) = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + ... = g(x)$. Then $g(x) = Ce^x$ and for $x = c$, $g(c) = e^c$. Then $e^c = g(c) = Ce^c \Rightarrow C = 1$, and so $g(x) = e^x$, that is, the Taylor series of e^x about c converges to e^x for all $x \in \mathbb{R}$. Thus e^x is analytic for all real numbers.

The Maclaurin series of e^x is:

$\sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ for all $x \in \mathbb{R}$.

Example 2. Find the Maclaurin series of $sinx$ and $cosx$ and find the intervals of convergence series.

Let $f(x) = sinx$. Then $f'(x) = cosx \Rightarrow f'(0) = 1$, $f''(x) = -sinx \Rightarrow f''(0) = 0$, $f^{(3)}(x) = -cosx \Rightarrow f^{(3)}(0) = -1$, $f^{(4)}(x) = -sinx \Rightarrow f^{(4)}(0) = 0$, $f^{(5)}(x) = cosx \Rightarrow f^{(5)}(0) = 1$,...

The maclaurin series of $sinx$:

$0+x+0-\frac{x^3}{3!}+0+\frac{x^5}{5!}+0-... = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+... = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

$\rho = \lim_{n\to\infty} |\frac{\frac{(-1)^{n+1}}{(2n+3)!}x^{2n+3}}{\frac{(-1)^n}{(2n+1)!}x^{2n+1}}| = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0$, and so the series converges for all x by ratio test.

When the series $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ is differentiated twice, we get that $g(x) = -g''(x)$, that is; $g(x) + g''(x) = 0$. The general solution of such equation is :

$g(x) = Acosx + Bsinx$. We get $g(0) = 0$ and $g'(0) = 1$ and so we obtain $A = 0$ and $B = 1$, namely; $g(x) = sinx$.

Homework Show that $cosx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ for all x.

Some Maclaurin Series

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$ for $-1 < x < 1$

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + ...$ for $-1 < x < 1$

$ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ for $-1 < x \leq 1$

$arctanx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ for $-1 \leq x \leq 1$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$ for all $x \in \mathbb{R}$.

Example 3. Find the Maclaurin series for the following functions.

i. e^{3x+1} ii. $cos(2x - \pi)$ iii. $ln(\frac{1-x}{1+x})$

i. We know that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Then $e^{3x+1} = e.e^{3x} = e \sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(3x)^n = e + 3ex + \frac{e(3x)^2}{2!} + \frac{e(3x)^3}{3!} + ...$ for all x.

ii. Consider $cos(2x - \pi) = -cos2x$. Since $cosx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ for all x, we get $-cos2x = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^{2n}}{(2n)!}(x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}4^n}{(2n)!}(x)^{2n}$ for all x.

iii. Note that $ln(\frac{1-x}{1+x}) = ln(1 - x) - ln(1 + x)$. We know that

$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ for $-1 < x \leq 1$.

Then we get $ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ...$ for $-1 < x \leq 1$, and so $ln(\frac{1-x}{1+x}) = ln(1-x) - ln(1+x) = -2x - 2\frac{x^3}{3} - 2\frac{x^5}{5} - ... = -2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$ for $-1 < x \leq 1$.

Example 4. Find the Taylor series representation for the following functions.

i. e^{-2x} about $x = -1$ ii. lnx about $x = 3$.

i. Let $t = x + 1 \Rightarrow x = t - 1$.

$e^{-2x} = e^{-2(t-1)} = e^{-2t}e^2 = e^2 \sum_{n=0}^{\infty} \frac{1}{n!}(-2t)^n = \sum_{n=0}^{\infty} \frac{(-1)^n e^2 2^n}{n!} t^n$ and so $e^{-2x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^2 2^n}{n!}(x+1)^n$ for all real numbers.

ii. Let $t = x - 3 \Rightarrow x = t + 3$.

$lnx = ln(t+3) = ln(3(1+\frac{t}{3})) = ln3 + ln(1+\frac{t}{3})$. Since $ln(1+t) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n}$ for all $-1 < t \leq 1$, then $lnx = ln3 + ln(1+\frac{t}{3}) = ln3 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n3^n} = ln3 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-3)^n}{n3^n}$ for $-1 < \frac{t}{3} \leq 1 \Rightarrow -3 < t \leq 3 \Rightarrow -3 < x - 3 \leq 3 \Rightarrow 0 < x \leq 6$.

Sometimes, it is difficult (if not impossible) to find the general formula of a Maclaurin or a Taylor series. In such cases we usually find first few terms.

Example 5. Find the first three nonzero terms of the Maclaurin series of $lncosx$.

we now that $lncosx = ln(1+-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...) = ln(1+(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...)) = (-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...)-\frac{1}{2}(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...)^2+\frac{1}{3}(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...)^3-... = -\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+....$

The polynomial $P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$ is called the Taylor polynomial of degree n for about f the point $x = c$.

Theorem 2. (**Taylor's Theorem**) Let the function f has the $(n+1)$st derivative on an interval containing c and x and $P_n(x)$ is the Taylor polynomial of degree n for f about the point $x = c$. Then the formula $f(x) = P_n(x)+E_n(x)$ is called Taylor's formula where $E_n(x)$ is the error term given by:
(**Lagrange Remainder**) $E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-c)^{n+1}$ for some s between c and x.

Example 6. Find the Maclaurin series for e^x by applying Taylor's Theorem.
We know $P_n(x) = \sum_{k=0}^{n}\frac{x^k}{k!}$. Now, we calculate the Lagrange remainder for e^x. Note that e^x is positive and increasing. Then $e^s \leq e^{|x|}$ for $s \leq |x|$.
For some s between 0 and x , $|E_n(x)| = |\frac{f^{(n+1)}(s)}{(n+1)!}x^{n+1}| = \frac{e^s}{(n+1)!}|x|^{n+1} \leq \frac{e^{|x|}}{(n+1)!}|x|^{n+1}$. Then $\lim_{n\to\infty}\frac{e^{|x|}}{(n+1)!}|x|^{n+1} = 0$ and so $\lim_{n\to\infty}E_n(x) = 0$ by Squeeze theorem. Thus, $e^x = \lim_{n\to\infty}(\sum_{k=0}^{n}\frac{x^k}{k!}+E_n(x)) =$

$\sum_{k=0}^{\infty} \frac{x^k}{k!}$.