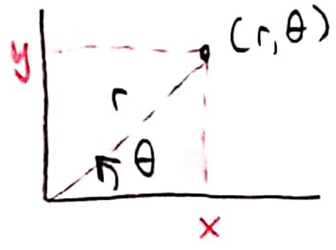
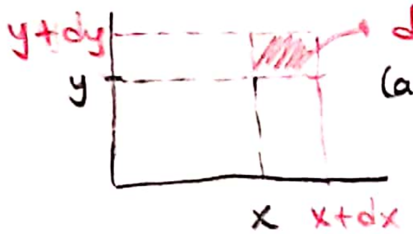
14.4) Double Integrals in Polar Coordinates

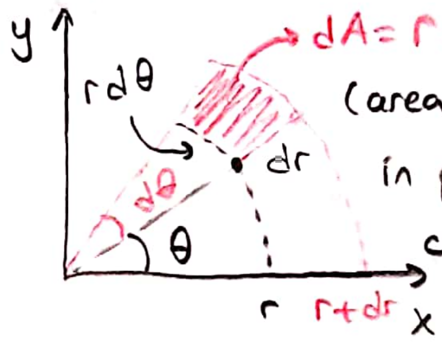


$x = r\cos\theta \qquad r^2 = x^2 + y^2$

$y = r\sin\theta \qquad \tan\theta = \dfrac{y}{x}$

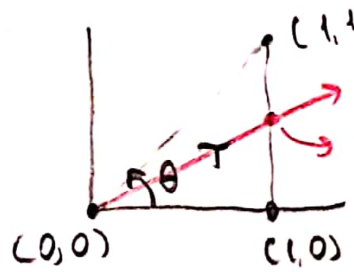


$dA = dx\,dy$ (area element in Cartesian coordinates)



$dA = r\,dr\,d\theta$ (area element in polar coordinates)

Q13) Evaluate $\iint_T (x^2+y^2)\,dA$, where T is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.



$x = 1 \Rightarrow r = \dfrac{1}{\cos\theta} = \sec\theta$

r is from 0 to $\sec\theta$

θ is from 0 to $\dfrac{\pi}{4}$

$$\text{So, } \iint_T (x^2+y^2)\,dA = \int_0^{\pi/4}\int_0^{\sec\theta} (r^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{\sec\theta} r^3\,dr\,d\theta = \int_0^{\pi/4}\left(\frac{r^4}{4}\Big|_0^{\sec\theta}\right)d\theta$$

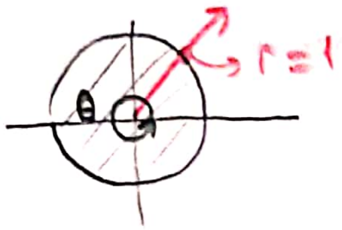
$$= \frac{1}{4}\int_0^{\pi/4} \sec^4\theta\,d\theta = \frac{1}{4}\int_0^{\pi/4}(1+\tan^2\theta)\sec^2\theta\,d\theta$$

$u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$

$u(0) = 0, \quad u(\frac{\pi}{4}) = 1$

$$= \frac{1}{4}\int_0^1 (1+u^2)\,du = \frac{1}{4}\left(u+\frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}$$

Q14) Evaluate $\displaystyle\iint_{x^2+y^2\le 1} \ln(x^2+y^2)\,dA.$



r is from 0 to 1

θ is from 0 to 2π.

So, $$\iint_{x^2+y^2\le 1} \ln(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 \ln(r^2)\, r\,dr\,d\theta$$

$w = r^2 \Rightarrow dw = 2r\,dr$

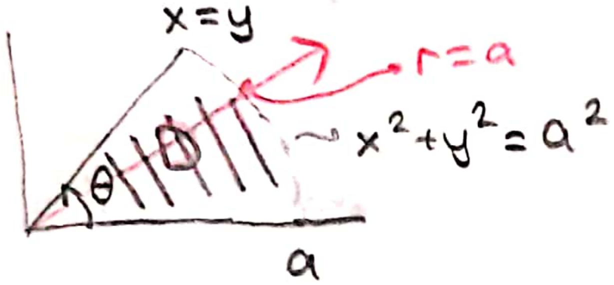
$$= \frac{1}{2}\int_0^{2\pi}\left(\int_0^1 \ln w\,dw\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left(w\ln w\Big|_0^1 - \int_0^1 w\,\frac{dw}{w}\right)d\theta$$

Let $\left[\begin{array}{l} u = \ln w \Rightarrow du = \frac{dw}{w} \\ dv = dw \Rightarrow v = w \end{array}\right]$

$$= \frac{1}{2}\int_0^{2\pi}\left(0 - w\Big|_0^1\right)d\theta = \frac{1}{2}\int_0^{2\pi} - d\theta$$

$$= -\frac{1}{2}\,\theta\Big|_0^{2\pi} = -\pi$$

Q19) Evaluate $\iint_D xy\,dA$, where D is the plane region satisfying $x \geq 0$, $0 \leq y \leq x$, $x^2+y^2 \leq a^2$



θ is from 0 to $\frac{\pi}{4}$

r is from 0 to a.

$$\iint_D xy\,dA = \int_0^{\pi/4}\int_0^a (r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta \sin\theta \left(\int_0^a r^3\,dr\right) d\theta = \int_0^{\pi/4} \frac{\sin 2\theta}{2}\left(\frac{r^4}{4}\Big|_0^a\right) d\theta$$

$$= \int_0^{\pi/4} \frac{\sin 2\theta}{2}\,\frac{a^4}{4}\,d\theta = \frac{a^4}{8}\int_0^{\pi/4} \sin 2\theta\,d\theta$$

$$= \frac{a^4}{16}(-\cos 2\theta)\Big|_0^{\pi/4} = \frac{a^4}{16}$$

Q21) Find the volume lying between the paraboloids

$z = x^2+y^2$ and $3z = 4-x^2-y^2$

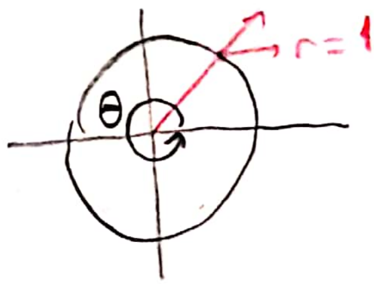
we need to find the region of integration.

$$3x^2+3y^2 = 4-x^2-y^2 \Rightarrow x^2+y^2=1$$

The paraboloids intersect on the cylinder $x^2+y^2=1$.

So we need to calculate the following:

$$\iint\limits_{x^2+y^2\le 1} \left[\frac{4-x^2-y^2}{3} - (x^2+y^2)\right] dA$$



In polar coordinates;

$$= \int_0^{2\pi}\int_0^1 \left[\frac{4-r^2}{3} - r^2\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{4}{3}(1-r^2)\,r\,dr\,d\theta = \frac{4}{3}\int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta$$

$$= \frac{4}{3}\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 d\theta = \frac{4}{3}\int_0^{2\pi}\frac{1}{4}\,d\theta$$

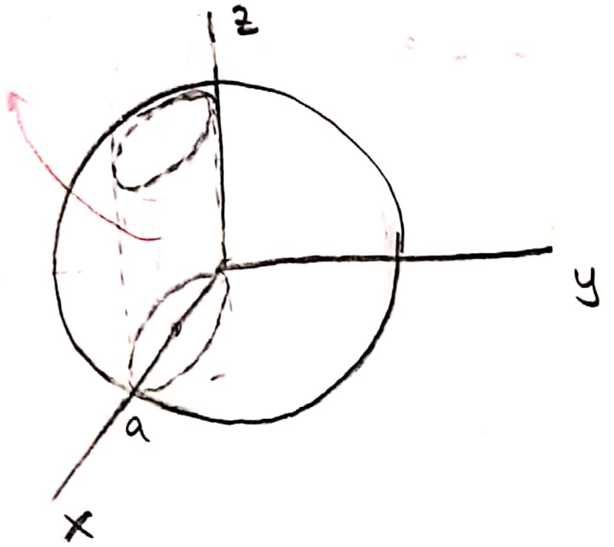
$$= \frac{1}{3}\,\theta\Big|_0^{2\pi} = \frac{2\pi}{3}$$

Q22) Find the volume lying inside both the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$.

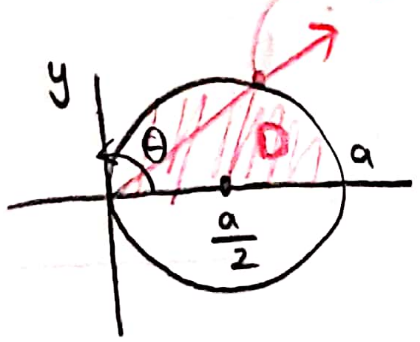
$$x^2+y^2=ax \Rightarrow x^2-ax+\frac{a^2}{4}+y^2=\frac{a^2}{4}$$

So, the equation of the cylinder can be written as

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}.$$ (cylinder along the line $x=\frac{a}{2}, y=0$)



So, we will find the volume between $z=\sqrt{a^2-x^2-y^2}$ and $z=-\sqrt{a^2-x^2-y^2}$ in the region $x^2+y^2=ax$ on xy-plane



$r^2=ar\cos\theta$

$r=a\cos\theta$

r is from 0 to $a\cos\theta$

θ is from 0 to $\frac{\pi}{2}$

So, the volume of the required surface is

$$2\iint_D\left[\sqrt{a^2-x^2-y^2}-\left(-\sqrt{a^2-x^2-y^2}\right)\right]dA$$

$$= 4 \iint_{D} \sqrt{a^2-x^2-y^2}\, dA$$

$$= 4 \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2-r^2}\; r\,dr\,d\theta \qquad u = a^2-r^2 \quad du = -2r\,dr$$

$$= 4 \int_0^{\pi/2} \int_{a^2}^{a^2\sin^2\theta} \sqrt{u}\left(-\frac{1}{2}du\right) d\theta \qquad u(0) = a^2 \quad u(a\cos\theta) = a^2\sin^2\theta$$

$$= -2 \int_0^{\pi/2} \frac{2}{3} u^{3/2} \Big|_{a^2}^{a^2\sin^2\theta} d\theta = -\frac{4}{3} \int_0^{\pi/2} (a^3\sin^3\theta - a^3)\, d\theta$$

$$= \frac{4a^3}{3} \int_0^{\pi/2} (1-\sin^3\theta)\, d\theta$$

$$= \frac{4a^3}{3}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2} (1-\cos^2\theta)\sin\theta\, d\theta\right] \qquad u = \cos\theta \quad du = -\sin\theta\, d\theta$$

$$u(0) = 1 \quad u\left(\frac{\pi}{2}\right) = 0$$

$$= \frac{4a^3}{3}\left[\frac{\pi}{2} + \int_1^0 (1-u^2)\, du\right]$$

$$= \frac{4a^3}{3}\left[\frac{\pi}{2} + \left(u - \frac{u^3}{3}\right)\Big|_1^0\right]$$

$$= \frac{4a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right] = \frac{2}{9} a^3 (3\pi - 4)$$

Change of Variables Formula for Double Integrals

Let $x = x(u,v)$, $y = y(u,v)$ be a 1-1 transformation from a domain S in the uv-plane onto a domain D in the xy-plane. Suppose that x, y and their partial derivatives wrt u & v are continuous in S.

If $f(x,y)$ is integrable on D and if $g(u,v) = f(x(u,v), y(u,v))$, then g is integrable on S and

$$\iint_D f(x,y)\,dx\,dy = \iint_S g(u,v) \underbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv}_{dx\,dy}.$$

Here $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

Note that $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$.

Q32) Find $\iint_P (x^2+y^2)\,dA$ where P is the parallelogram bounded by the lines $x+y=1$, $x+y=2$, $3x+4y=5$, $3x+4y=6$

Let $x+y=u$, $3x+4y=v$. That is,
use the transformation $x=4u-v$, $y=v-3u$. Then
P corresponds the square S bounded by $u=1$, $u=2$,
$v=5$, $v=6$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4 & -1 \\ -3 & 1 \end{vmatrix} = 1$$

So, $dA = dx\,dy = 1\,du\,dv$ and we have

$$\iint_P (x^2+y^2)\,dA = \iint_S \left[(4u-v)^2+(v-3u)^2\right] 1.\,du.dv$$

$$= \int_5^6 \int_1^2 (16u^2-8uv+v^2+v^2-6uv+9u^2)\,du\,dv$$

$$= \int_5^6 \int_1^2 (25u^2-14uv+2v^2)\,du\,dv$$

$$= \int_5^6 \left(\frac{25}{3}u^3-7u^2v+2uv^2\right)\Big|_1^2\,dv.$$

$$= \int_5^6 \left(\frac{25}{3} \cdot 7 - 7.3v + 2v^2\right) dv$$

$$= \frac{175}{3} v - \frac{21}{2} v^2 + \frac{2}{3} v^3 \Big|_5^6$$

$$= \frac{175}{3} - \frac{231}{2} + \frac{182}{3} = \frac{357}{3} - \frac{231}{2} = 119 - \frac{231}{2} = \frac{7}{2}$$

Q33) Find the area of the region in the 1^{st} quadrant bounded by the curves $xy = 1$, $xy = 4$, $y = x$, $y = 2x$.

Let $u = xy$, $v = \frac{y}{x}$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2\frac{y}{x} = 2v$$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$$

So the area of the region D is

$$\iint_D dx\,dy = \iint_R \frac{1}{2v}\,du\,dv$$ where R is the rectangle bounded by $u = 1$, $u = 4$, $v = 1$ $v = 2$

$$= \int_1^2 \int_1^4 \frac{1}{2v}\, du\, dv = \int_1^4 \frac{1}{2} \ln v \Big|_1^2 du = \frac{1}{2} \ln 2 \int_1^4 du$$

$$= \left(\frac{1}{2} \ln 2\right) u \Big|_1^4 = \frac{3}{2} \ln 2$$

14.5) Triple Integrals

$\boxed{2-4-9}$ Evaluate the triple integrals over the indicated region.

Q2) $\iiint\limits_B xyz\, dV$ over the box B given by

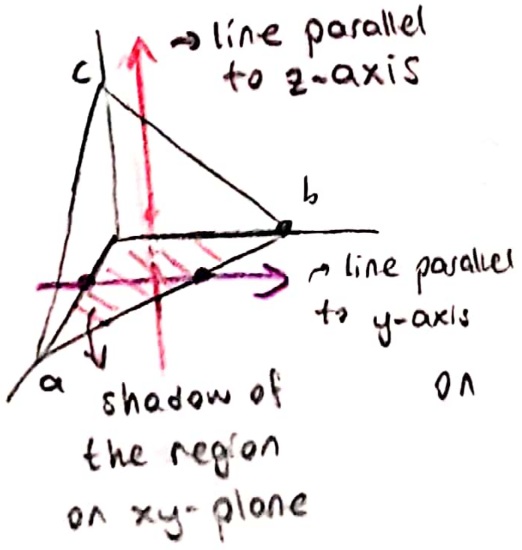
$0 \le x \le 1, \quad -2 \le y \le 0, \quad 1 \le z \le 4.$

$$\int_{z=1}^{z=4}\int_{y=-2}^{y=0}\int_{x=0}^{x=1} xyz\, dx\, dy\, dz = \int_1^4 z\, dz \int_{-2}^{0} y\, dy \int_0^1 x\, dx$$

$$= \int_1^4 z\, dz \int_{-2}^{0} y\, dy \left(\frac{x^2}{2}\Big|_0^1 \right) = \frac{1}{2}\int_1^4 z\, dz \left(\frac{y^2}{2}\Big|_{-2}^{0} \right)$$

$$= \frac{1}{2}(-2)\left(\frac{z^2}{2}\Big|_1^4 \right) = \frac{1}{2}(-2)\left(8 - \frac{1}{2} \right) = -\frac{15}{2}$$

Q4) $\iiint\limits_R x\, dV$, over the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$



If we choose $dV = dz\, dy\, dx$,

I) Draw a line parallel to z-axis.

II) Find the shadow of the region on xy-plane.

III) On the shadow, x & y limits are as in double integral.

Pink line enters the region at $z=0$ and leaves the region at $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$. (z- limits of integration)

Purple line enters the region at $y=0$ and leaves the region at $y=b\left(1-\frac{x}{a}\right)$. (y-limits of integration)

x is from 0 to a.

$$\Rightarrow \iiint_R x\, dV = \int_{x=0}^{x=a} \int_{y=0}^{y=b\left(1-\frac{x}{a}\right)} \int_{z=0}^{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x \, dz\, dy\, dx$$

$$= \int_0^a x\, dx \int_0^{b\left(1-\frac{x}{a}\right)} dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz$$

$$= \int_0^a x\, dx \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy$$

$$= c\int_0^a x\left(y-\frac{xy}{a}-\frac{y^2}{2b}\right)\Bigg|_{y=0}^{y=b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a x\left[b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right] dx$$

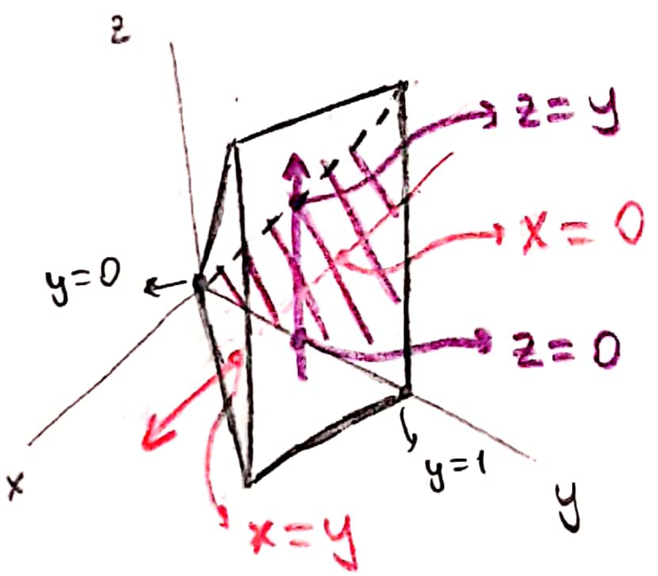
$$= c\int_0^a x\left[b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 x\,dx = \frac{bc}{2}\int_0^a \left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)x\,dx$$

$$= \frac{bc}{2}\left(\frac{x^2}{2}-\frac{2x^3}{3a}+\frac{x^4}{4a^2}\right)\Big|_0^a$$

$$= \frac{bc}{2}\left(\frac{a^2}{2}-\frac{2a^3}{3a}+\frac{a^4}{4a^2}\right) = \frac{a^2bc}{24}$$

Q9) $\iiint\limits_R \sin(\pi y^3)\,dV$ over the pyramid with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,1,1)$, $(1,1,1)$.



Choose $dV = dx\,dz\,dy$

I) Draw a line (pink) parallel to x axis.

II) Find the shadow of the region on yz-plane.

$$\Rightarrow \iiint\limits_R \sin(\pi y^3)\,dV = \int_{y=0}^{y=1}\int_{z=0}^{z=y}\int_{x=0}^{x=y} \sin(\pi y^3)\,dx\,dz\,dy$$

$$= \int_{y=0}^{y=1}\int_{z=0}^{z=y} \sin(\pi y^3)\left(x\Big|_0^y\right)dz\,dy$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=y} y\sin(\pi y^3)\, dz\, dy = \int_0^1 y\sin(\pi y^3)\left(z\Big|_0^y\right) dy$$

$$= \int_0^1 y^2 \sin(\pi y^3)\, dy$$

Let $u = \pi y^3 \Rightarrow du = 3\pi y^2 dy$

$u(0) = 0$, $u(1) = \pi$.

$$= \frac{1}{3\pi} \int_0^{\pi} \sin u\, du = -\frac{1}{3\pi} \cos u \Big|_0^{\pi} = \frac{2}{3\pi}$$

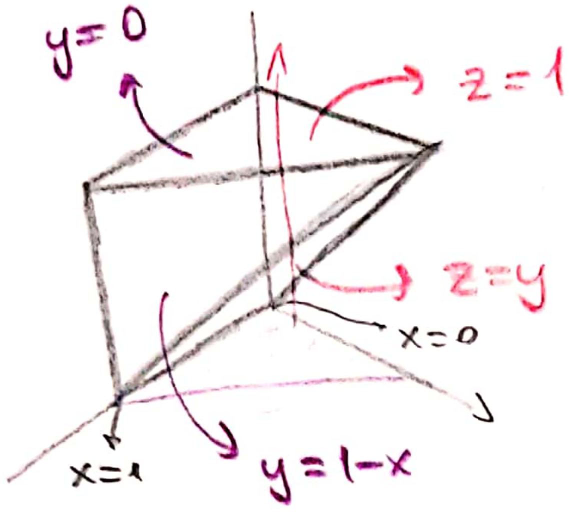
Q28) Evaluate the given iterated integral by reiterating it in a different order:

$$\int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\, dz$$

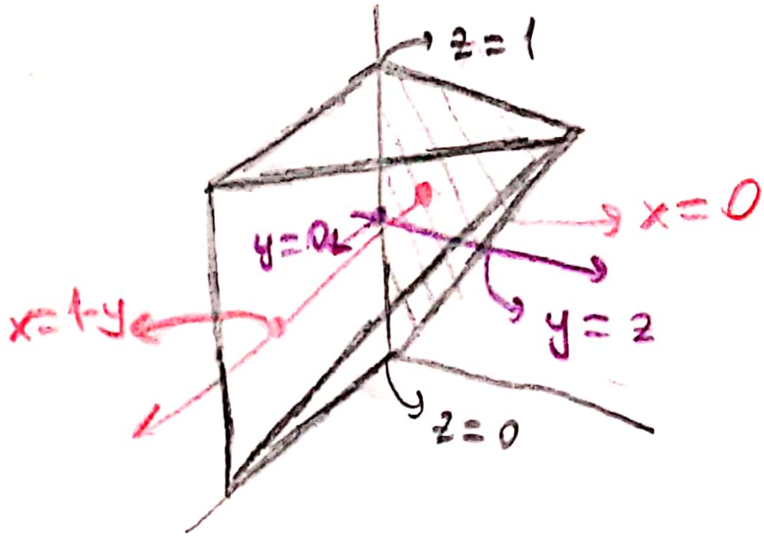
Order: $dz\,dy\,dx$. It is hard to calculate this integral.

To change the order, we need to sketch the region of integration.

Choose $dV = dx\,dy\,dz$.



$dV = dz\,dy\,dx$



$$\Rightarrow \int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\,dz$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\,dz \int_0^z dy \int_0^{1-y} dx$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\,dz \int_0^z (1-y)\,dy$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \left(y - \frac{y^2}{2} \right)\Bigg|_{y=0}^{y=z} dz$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \left(z - \frac{z^2}{2} \right) dz = \frac{1}{2} \int_0^1 \sin(\pi z)\,dz$$

$$= \frac{1}{2\pi} \left(-\cos(\pi z) \Big|_0^1 \right) = \frac{1}{\pi}$$