GTU, Spring 2021, MATH 101

Preliminaries, Functions

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* x: independent variable, y: dependent variable

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* We may denote the domain of a function by $\mathcal{D}(f)$.

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* The term inside the square root can not be nagative, so $x \geq 0$ and $1 - x^2 \geq 0$.
* Exercise: Solve these inequalities to find $\mathcal{D}(f)$ (Answer: $\mathcal{D}(f) = [0, 1]$).

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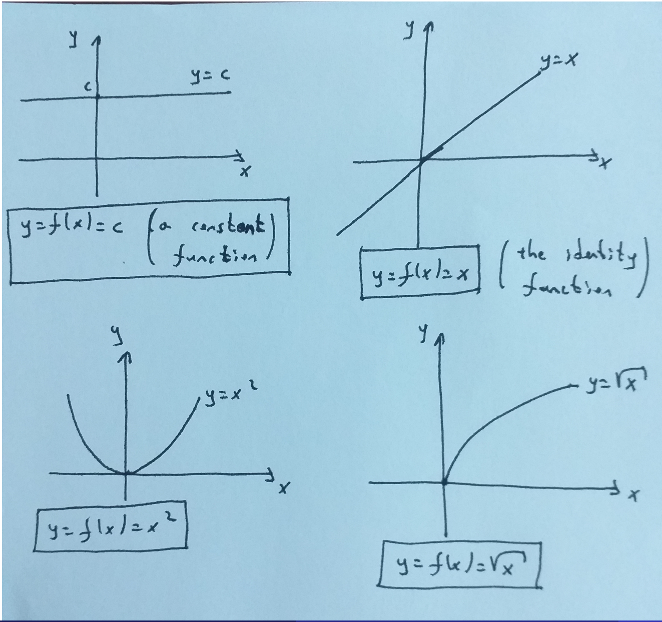
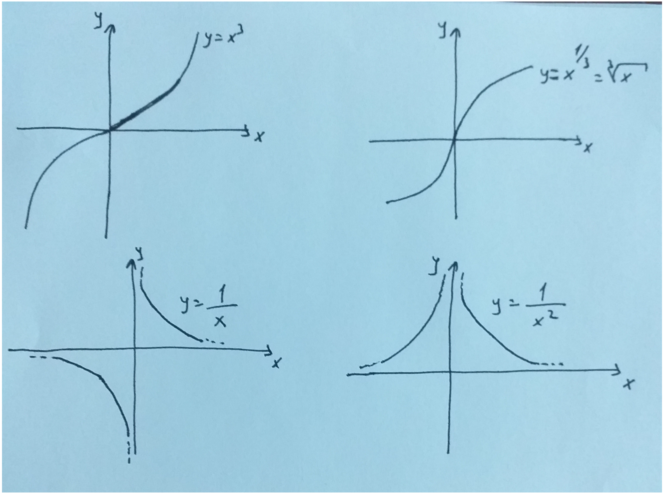
* For now let's see graphs of some basic functions (probably you know these form high school).

Preliminaries, Graphs of Functions




y
x
$y=x^3$
$y=x^{1/3}=\sqrt[3]{x}$
$y=\frac{1}{x}$
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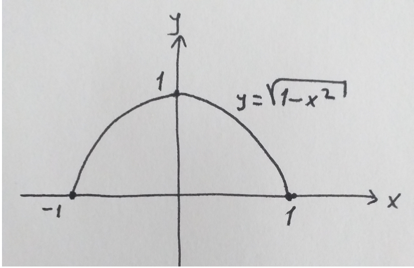

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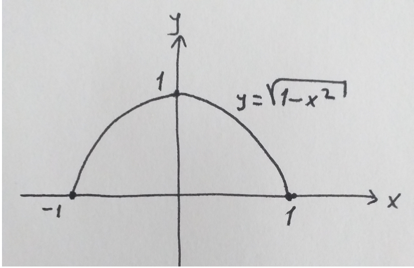

$*$ Exercise: How do the graphs of $y = x^4, x^5, x^6, \ldots$ look like?

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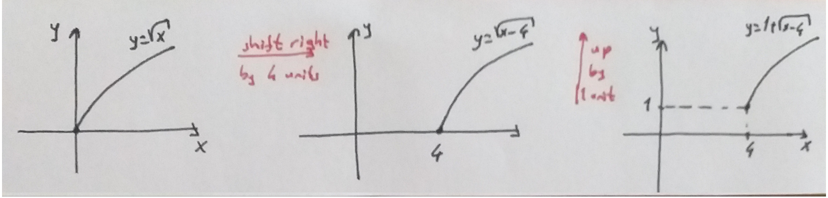
* $f(x)$ is similar to $\sqrt{x}$.



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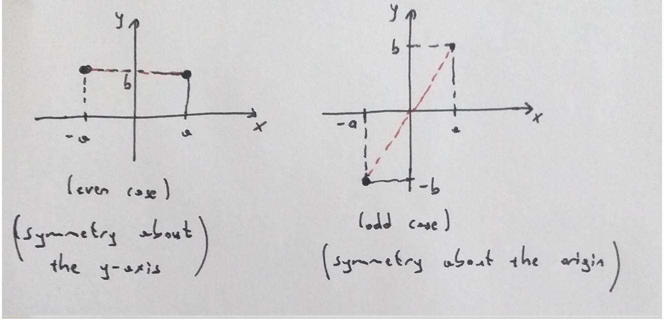

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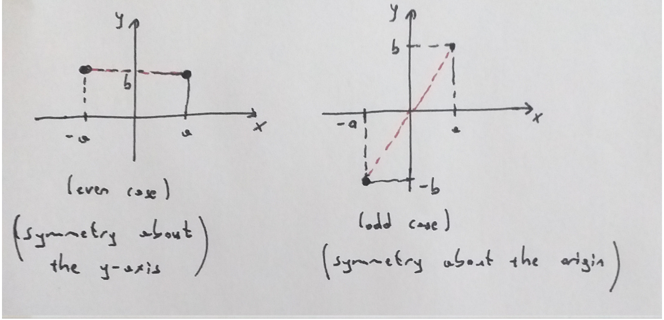

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* For example compare the graphs of $y = x^2$ and $y = x^3$.

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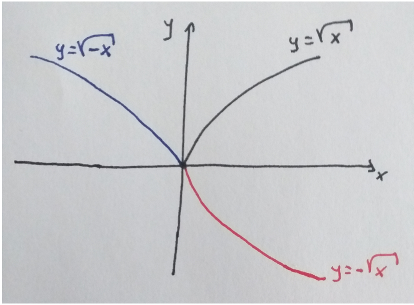

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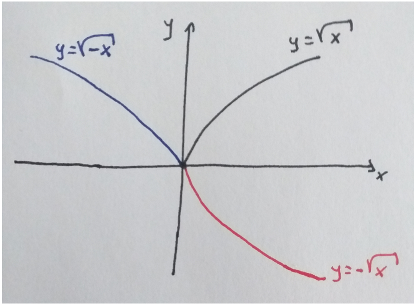

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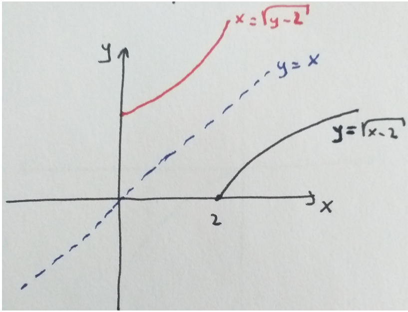

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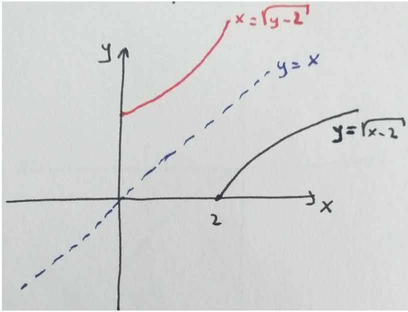

* Exercise: Sketch the graphs of $y = (x-1)^3 - 5$ and $y = \dfrac{x-2}{x+1}$.