THEOREM 2 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules in Table 5.4.

TABLE 5.4 Rules satisfied by definite integrals

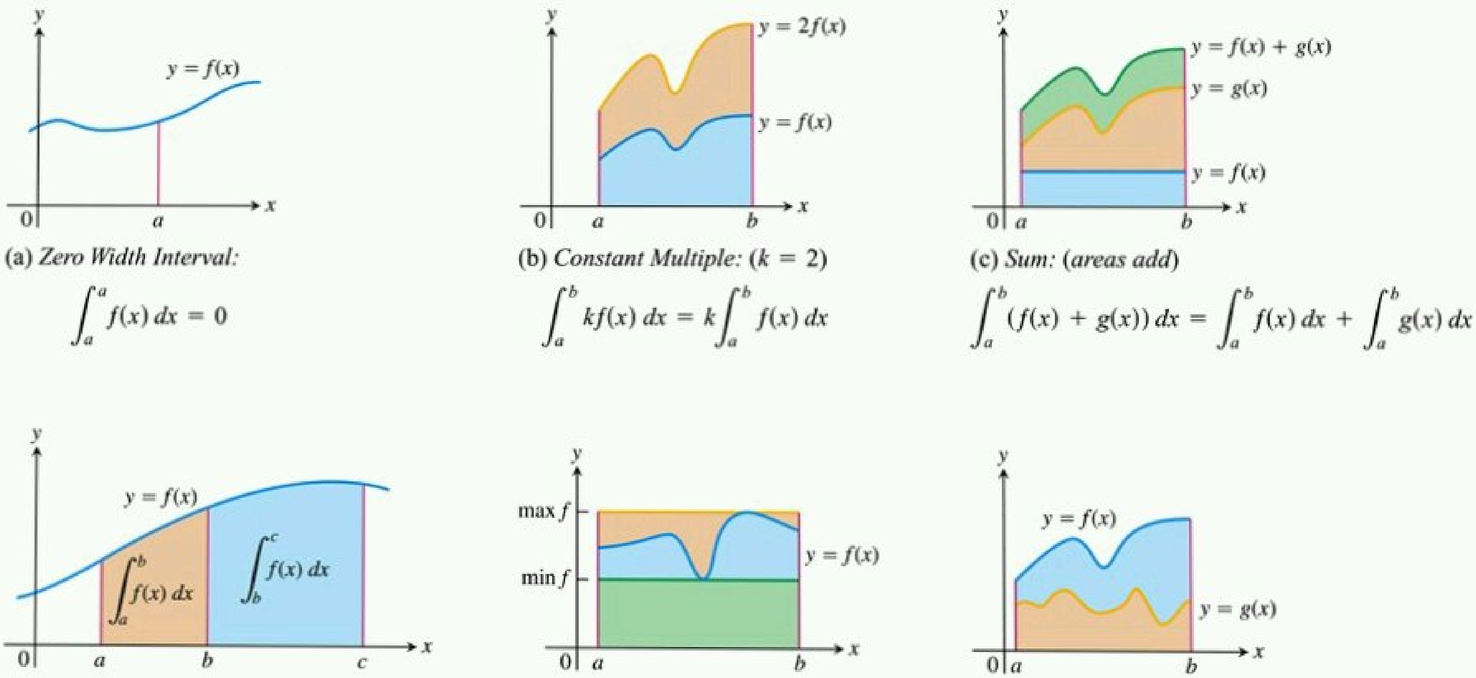
1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A Definition when $f(a)$ exists
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then	

$$\min f \cdot (b - a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b - a).$$

7. *Domination:*

$$f(x) \geq g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$f(x) \geq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0 \quad \text{(Special Case)}$$



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$

(b) *Constant Multiple:* ($k = 2$)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

(c) *Sum:* (*areas add*)

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$

(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

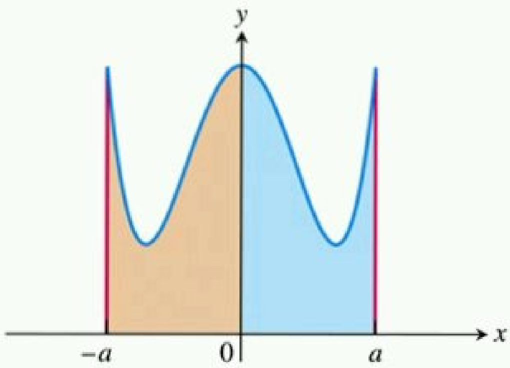
$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.4.

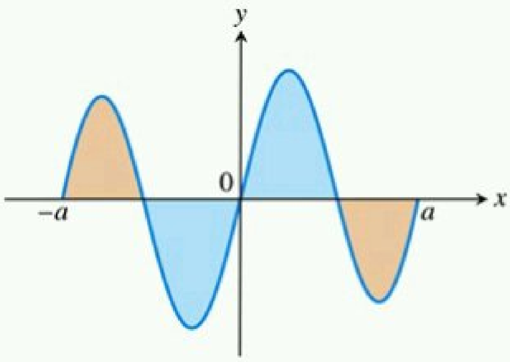
THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\, dx = 0$.



(a)



(b)

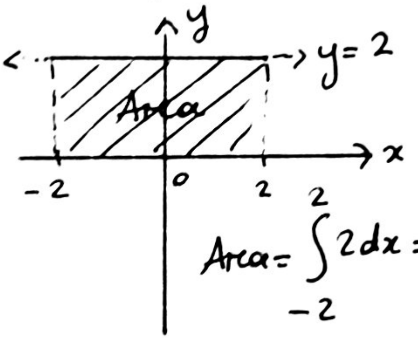
FIGURE 5.24 (a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Example: Evaluate

a) $\int_{-2}^{2} (2+5x)\,dx$, b) $\int_{0}^{3} (2+x)\,dx$ c) $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

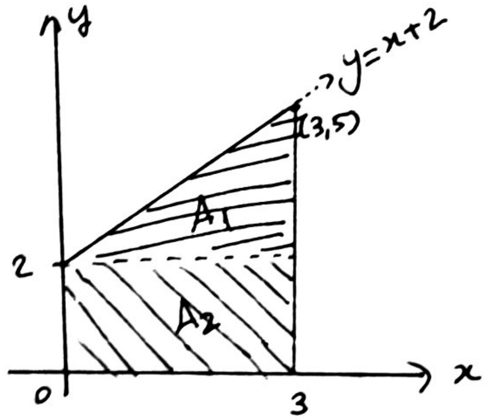
a) $$\int_{-2}^{2} (2+5x)\,dx = \underbrace{\int_{-2}^{2} 2\,dx}_{\Downarrow} + 5\underbrace{\int_{-2}^{2} x\,dx}_{} = 8+0 = 8 \,//$$

$\Longrightarrow$ The integrand "x" is an odd fnc. and interval is symmetric about origin. Thus, the integral $\int_{-2}^{2} x\,dx = 0$.



$$\text{Area} = \int_{-2}^{2} 2\,dx = 4.2 = 8 \,//$$

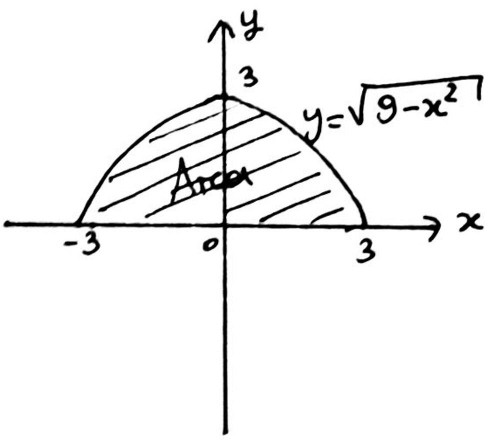
b)



$$\int_{0}^{3} (2+x)\,dx = A_1 + A_2 = \left(\frac{3\times3}{2}\right) + \left(3\times2\right)$$

$$= \frac{21}{2} \,//$$

c)



$$A = \int_{-3}^{3} \sqrt{9-x^2} = \frac{\pi(3^2)}{2} = \frac{9\pi}{2}$$

A Mean-Value Theorem for Integrals

Let f be a function continuous on $[a,b]$. Then f assumes a minimum value "m" and a maximum value "M" on the interval, say at points $x = l$ and $x = u$; respectively:

$$m = f(l) \leq f(x) \leq f(u) = M \quad \text{for all } x \in [a,b].$$

For the 2-point partition P of $[a,b]$ having $x_0 = a$ and $x_1 = b$, we have

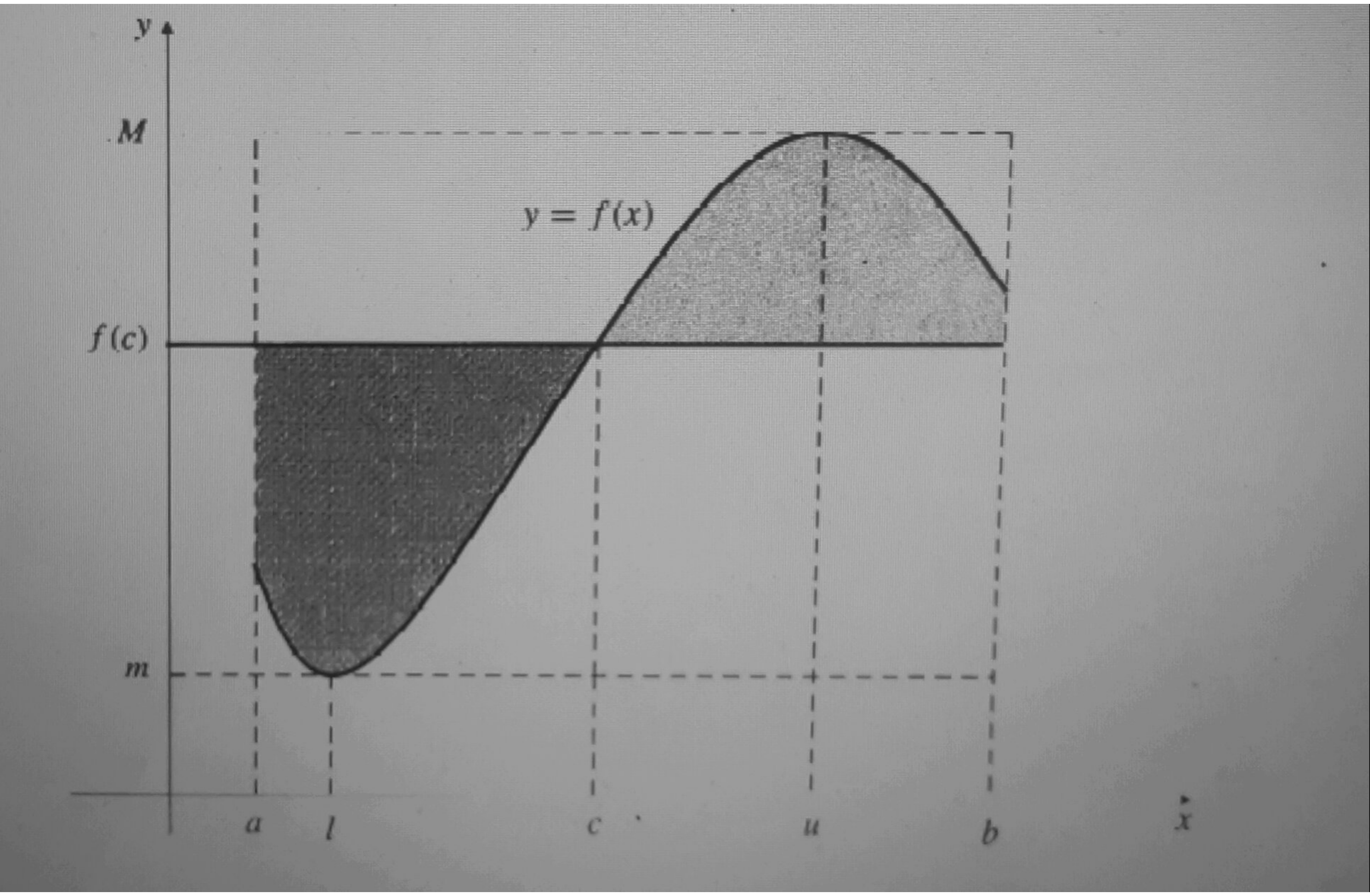
$$m(b-a) = L(f,P) \leq \int_a^b f(x)dx \leq U(f,P) = M.(b-a)$$

Therefore,

$$f(l) = m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M = f(u)$$

By the Intermediate Value Theorem, $f(x)$ must take on every value between the two values $f(l)$ and $f(u)$ at some point between l and u. Hence, there is a number c between l and u such that,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$



y
M
y = f(x)
f(c)
m
a
l
c
u
b
x

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

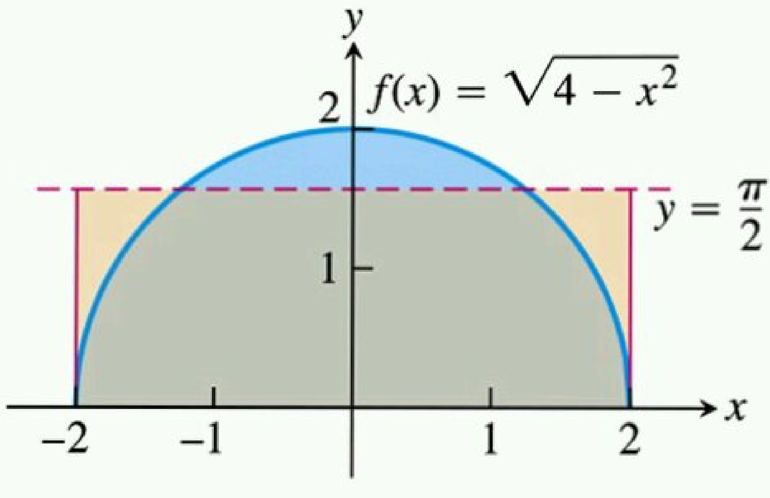


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5).

Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$.

$$\text{avg}(f) = \bar{f} = \frac{1}{2-(-2)} \underbrace{\int_{-2}^{2} \sqrt{4-x^2}\, dx}_{\text{Area of semicircle with radius } r=2.}$$

$$\text{avg}(f) = \frac{1}{4} \cdot \frac{\pi(2^2)}{2}$$

$$\text{avg}(f) = \frac{\pi}{2} \quad //$$

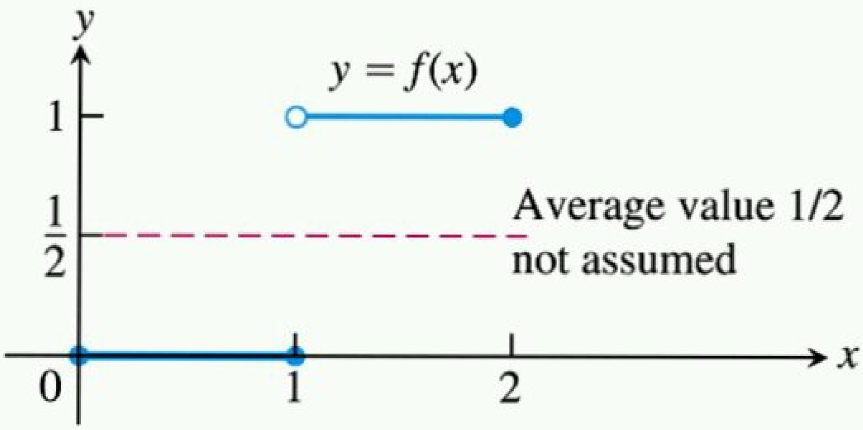


FIGURE 5.17 A discontinuous function need not assume its average value.

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

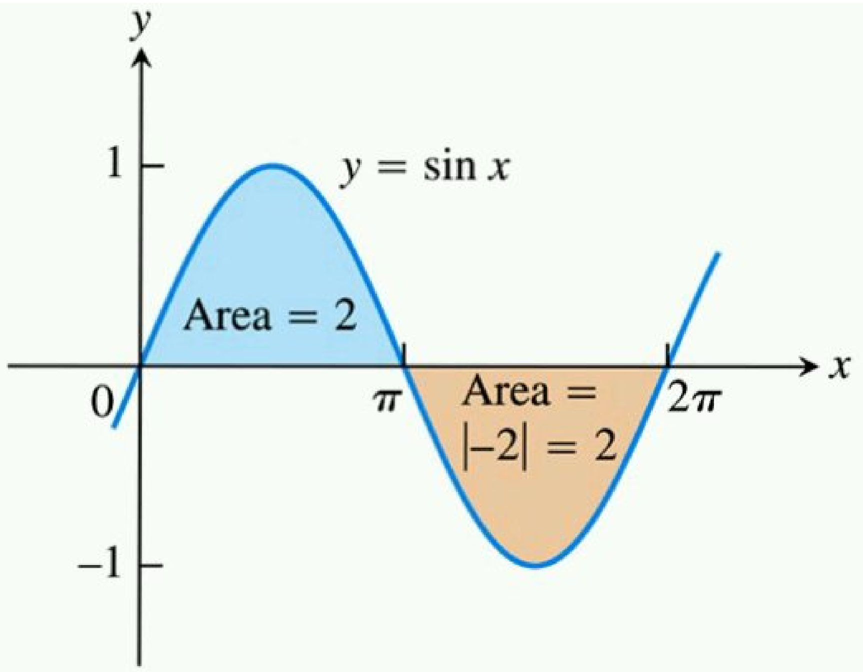


FIGURE 5.21 The total area between $y = \sin x$ and the x-axis for $0 \leq x \leq 2\pi$ is the sum of the absolute values of two integrals (Example 7).

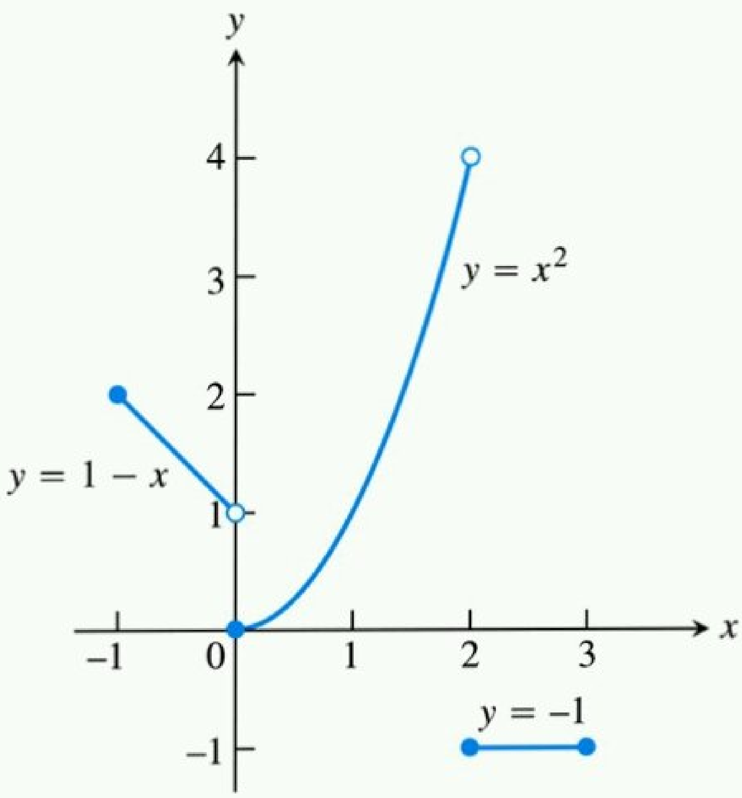


FIGURE 5.31 Piecewise continuous functions like this are integrated piece by piece.

Example: Find the area between piece wise defined function $f(x)$ and x-axis; (Find $\int_{-1}^{3} f(x)dx$) where

$$f(x)= \begin{cases} 1-x & ; \quad -1 \le x < 0 \\ x^2 & ; \quad 0 \le x < 2 \\ -1 & ; \quad 2 \le x \le 3 \end{cases}$$

$$\text{Area} = \left| \int_{-1}^{0} (1-x)\, dx \right| + \left| \int_{0}^{2} x^2 dx \right| + \left| \int_{2}^{3} (-1)\, dx \right|$$

We will compute it after we give integration method.

The Fundamental Theorem of Calculus

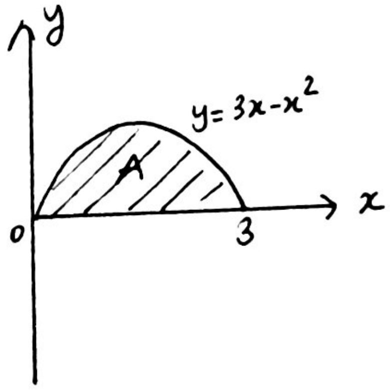
THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2 If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Example: Find the area A of the plane region lying above the x-axis and under the curve $y = 3x - x^2$.



$$3x - x^2 = x(3 - x) = 0 \Rightarrow \begin{matrix} x = 0 \\ x = 3 \end{matrix}$$

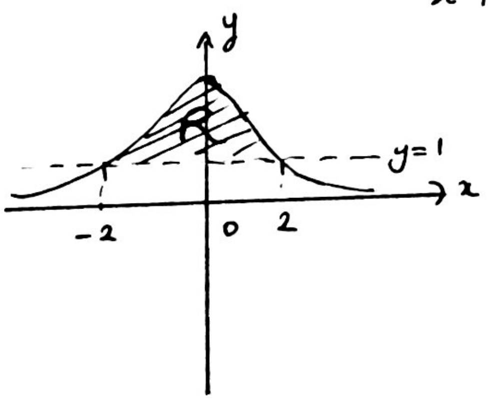
$$A = \int_0^3 (3x - x^2)\, dx$$

$$= \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^3$$

$$= \left(\frac{27}{2} - \frac{27}{3}\right) - (0 - 0)$$

$$= \frac{27}{6} = \frac{9}{2} \text{ square units.}$$

Example: Find the area of the region R lying above the line $y = 1$ and below the curve $y = \dfrac{5}{x^2+1}$.



Intersections of $y = 1$ and $y = \dfrac{5}{x^2+1}$;

$$1 = \frac{5}{x^2+1} \Rightarrow \underline{\underline{x = \pm 2}}$$

$$A = \int_{-2}^{2} \frac{5}{x^2+1}\, dx - \underbrace{4}_{\text{Area of the rectangle under region } R.} = 2\int_0^2 \frac{5}{x^2+1}\, dx - 4$$

$$= 10\left(\tan^{-1} x\right)\Big|_0^2 - 4 \text{ square units.}$$

Substitution Method

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

the **method of substitution**, the integral version of the Chain Rule. If we rewrite the Chain Rule, $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$, in integral form, we obtain

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C.$$

Observe that the following formalism would produce this latter formula even if we did not already know it was true:

Let $u = g(x)$. Then $du/dx = g'(x)$, or in differential form, $du = g'(x)\, dx$. Thus,

$$\int f'(g(x))\, g'(x)\, dx = \int f'(u)\, du = f(u) + C = f(g(x)) + C.$$

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)	12. $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	13. $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
3. $\int \frac{dx}{x} = \ln\lvert x\rvert + C$	14. $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
4. $\int e^x\,dx = e^x + C$	15. $\int \csc x\,dx = -\ln\lvert\csc x + \cot x\rvert + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$	16. $\int \sinh x\,dx = \cosh x + C$
6. $\int \sin x\,dx = -\cos x + C$	17. $\int \cosh x\,dx = \sinh x + C$
7. $\int \cos x\,dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
8. $\int \sec^2 x\,dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
9. $\int \csc^2 x\,dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$
10. $\int \sec x \tan x\,dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
11. $\int \csc x \cot x\,dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

EXAMPLE 3 **(Examples of substitution)** Find the indefinite integrals:

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$, (b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx$, and (c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$.

Solution

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$ Let $u = x^2+1$.
Then $du = 2x\,dx$ and
$x\,dx = \frac{1}{2}\,du$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C.$$

(Both versions of the final answer are equally acceptable.)

(b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx$ Let $u = 3\ln x$.
Then $du = \dfrac{3}{x}\,dx$

$$= \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3\ln x) + C.$$

(c) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$ Let $v = 1+e^x$.
Then $dv = e^x\,dx$

$$= \int v^{1/2}\,dv = \frac{2}{3}v^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$$

EXAMPLE 4 Evaluate (a) $\int \frac{1}{x^2 + 4x + 5}\,dx$ and (b) $\int \frac{dx}{\sqrt{e^{2x} - 1}}$.

Solution

(a) $\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1}$ Let $t = x + 2$. Then $dt = dx$.

$$= \int \frac{dt}{t^2 + 1}$$

$$= \tan^{-1} t + C = \tan^{-1}(x + 2) + C.$$

(b) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$

$$= \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}}$$

Let $u = e^{-x}$.
Then $du = -e^{-x}\,dx$.

$$= -\int \frac{du}{\sqrt{1-u^2}}$$

$$= -\sin^{-1} u + C = -\sin^{-1}(e^{-x}) + C.$$

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If m **is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m **is even and** n **is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both** m **and** n **are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.