§12.5. The Chain Rule

The Chain Rule for functions of one variable is a formula that gives the derivative of a composition $f(g(x))$ of two functions f and g:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

The situation for several variables is more complicated.

<u>A version of the Chain Rule</u>

If z is a function of x and y with continuous first partial derivatives, and if x and y are differentiable functions of t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

If we consider a function of two variables, x and y, each of which is in turn a function of two variables, s and t:
$z = f(x, y)$, where $x = u(s, t)$ and $y = v(s, t)$. We can form the composite function

$$z = f(u(s, t), v(s, t)) = g(s, t).$$

Then g has first partial derivatives given by,

$$g_1(s,t) = f_1(u(s,t), v(s,t))u_1(s,t) + f_2(u(s,t), v(s,t))v_1(s,t),$$

$$g_2(s,t) = f_1(u(s,t), v(s,t))u_2(s,t) + f_2(u(s,t), v(s,t))v_2(s,t).$$

Another version of the Chain Rule

If z is a function of x and y with continuous first partial derivatives, and if x and y depend on s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 1. Find $\frac{dw}{dt}$ if $w = xy + z$, $x = cost$, $y = sint$, $z = t$.

Using the Chain Rule for three independent variables, we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$

$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$

$= -\sin^2 t + cos^2 t + 1$

$= 1 + \cos 2t$, so

$(\frac{dw}{dt}_{t=0} = 1 + \cos(0) = 2.$

Example 2. Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if,
$w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.(Here x,y and z are supposed as a functions of r and s)

$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

$=(1)(\frac{1}{s}) + (2)(2r) + (2z)(2)$

$=(\frac{1}{s}) + 4r + (4r)(2)$

$=(\frac{1}{s}) + 12r$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$

$=(1)(-\frac{r}{s^2}) + (2)(\frac{1}{s}) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}$

Homogeneous Functions

A function $f(x_1, x_2, ..., x_n)$ is said to be positively homogenous of degree k if, for every point $(x_1, x_2, ..., x_n)$ in its domain and every real number $t > 0$, we have

$f(tx_1, tx_2, ..., tx_n) = t^k f(x_1, x_2, ..., x_n)$.

For example,

$f(x, y) = x^2 + xy + y^2$ is positively homogenous of degree 2.

$f(x, y) = \frac{2xy}{x^2+y^2}$ is positively homogenous of degree 0.

$f(x, y) = x^2 + y$ is not positively homogenous.

Theorem 1. Euler's Theorem

If $f(x_1, x_2, ..., x_n)$ has continuous first partial derivatives and is positively homogeneous of degree k, then
$\sum_{i=1}^{n} x_i f_i(x_1, x_2, ..., x_n) = kf(x_1, x_2, ..., x_n)$.

Higher-Order Derivatives

Applications of the Chain Rule to higher-order derivatives can become quite complicated. It is important to keep in mind at each stage which variables are independent of one another.

Example 3. Calculate $\frac{\partial^2}{\partial x \partial y} f(x^2 - y^2, xy)$ in terms of partial derivatives of the function f. Assume that the second-order partials of f are continuous.
Let $u = x^2 - y^2$ and $v = xy$.
First differentiate with respect to y:
$\frac{\partial}{\partial y} f(u, v) = -2yf_1(u, v) + xf_2(u, v)$. Now differentiate this result with respect to x.
$\frac{\partial^2}{\partial x \partial y} f(u, v) = -2y(2xf_{11}(u, v) + yf_{12}(u, v)) + f_2(u, v) + x(2xf_{21}(u, v) + yf_{22}(u, v))$
$= f_2(u, v) - 4xyf_{11}(u, v) + 2(x^2 - y^2)f_{12}(u, v) + xyf_{22}(u, v)$.