GTU, Fall 2020, MATH 101

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Preliminaries, Combining Functions

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* The easier way is to use basic operations in the obvious sense.
* Let f and g be two functions. Then for any x in both $\mathcal{D}(f)$ and $\mathcal{D}(g)$ we define the functions $f \pm g$, $f.g$ and f/g by

$$(f \pm g)(x) = f(x) \pm g(x)$$
$$(f.g)(x) = f(x).g(x)$$
$$(f/g)(x) = f(x)/g(x) \quad \text{provided that} \quad g(x) \neq 0$$

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* Then

$$(f\pm g)(x)=\sqrt{x}+\sqrt{1-x},\quad \mathcal{D}(f\pm g)=[0,1]$$
$$(f.g)(x)=\sqrt{x}.\sqrt{1-x}=\sqrt{x-x^2},\quad \mathcal{D}(f.g)=[0,1]$$
$$(f/g)(x)=\frac{\sqrt{x}}{\sqrt{1-x}}=\sqrt{\frac{x}{1-x}},\quad \mathcal{D}(f/g)=[0,1)\quad \text{(why?)}$$
$$(g/f)(x)=\frac{\sqrt{1-x}}{\sqrt{x}}=\sqrt{\frac{1-x}{x}},\quad \mathcal{D}(g/f)=(0,1]\quad \text{(why?)}$$

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* (!) Composition of functions may yield strange 'functions'.
* For example, let $f(x) = -\sqrt{x}$. Then

$$(f \circ f)(x) = f(f(x)) = -\sqrt{-\sqrt{x}}$$

makes sense only if $x = 0$. So $\mathcal{D}(f \circ f) = \{0\}$.

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* We have $\sqrt{x}$, so that $x \geq 0$. But also $1-\sqrt{x} \geq 0$ (since it lies in a square root). So that $0 \leq x \leq 1 \implies \mathcal{D}(g \circ f) = [0,1]$

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- Exercise: Find the domain of the other ones.

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* (!!!) In order to say that two functions are equal to each other, they must have the same 'formula' and they must have the same domain.
* At least we can write

$$(f \circ f)(x) = \frac{2x-1}{5-x}, \quad \text{for} \quad x \neq 2, 5$$

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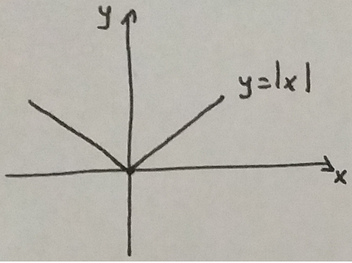
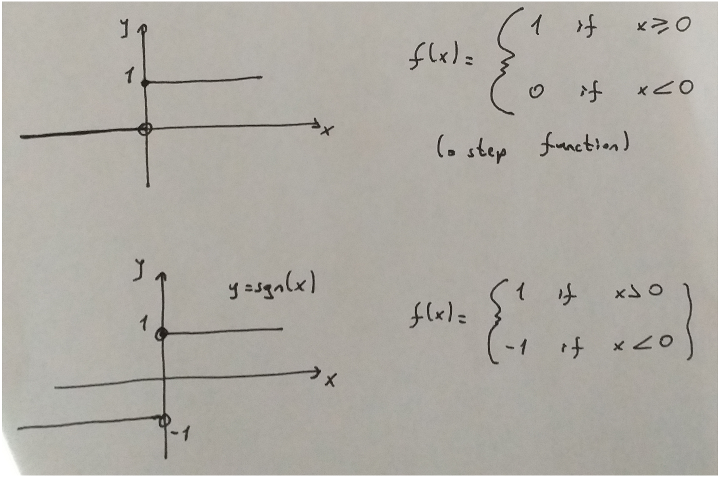
* Such functions are called *piecewise defined functions*.
* The most common example is the absolute value function.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



(sketch $y=x$ for $x \geq 0$,
sketch $y=-x$ for $x<0$)

* Step functions and the sign function ($\mathrm{sgn}(x)$) are also piecewise defined.



Preliminaries, Piecewise defined functions

* We can write lots of piecewise defined functions, for example

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* We can sketch it by sketching each part and then taking the relevant parts of the graphs.