§13.2 Extreme Values of Functions Defined on Restricted Domains

Recall that a continuous function $f(x, y)$ defined on a closed and bounded region D in R^2 takes its absolute maximum/minimum on D. In this section we will see how to find these absolute extreme values.

Theorem 1. Suppose that a function $f(x, y)$ has a local or absolute extreme value at some point (a, b) in its domain. Then (a, b) is one of the following:

i) a critical point, i.e. $\nabla f(a, b) = 0$

ii a singular point, i.e. $\nabla f(a, b)$ does not exists

iii) a boundary point.

So the theorem provides a useful method to find absolute extreme values:

Step 1: Find all critical and singular points of f.

Step 2: Find the extreme values of f on the boundary.

Step 3: Compare the values of f at the boundary points and at critical and singular points.

Example 1. Find the absolute maximum and minimum values of $f(x, y) = x^2 + 2xy - y^2$ on the disc $x^2 + y^2 \leq 1$.

Solution: Since $f(x, y)$ is continuous on the unit disc, f takes its absolute extreme values. First we compute the partial derivatives of f as

$$f_1(x, y) = 2x + 2y, \quad f_2(x, y) = 2x - 2y.$$

So f has no singular point and the only critical point is $(0, 0)$.

Now we consider the boundary points, i.e. the points on the unit circle $x^2 + y^2 = 1$. It is enough to find the absolute extreme values of f on the boundary as we will compare them with $f(0, 0)$.

It is useful to parameterize the boundary whenever it is possible. We can simply take the parametrization

$$x = \cos t, \;\; y = \sin t, \;\; t \in [0, 2\pi].$$

Then we have

$$f(\cos t, \sin t) = \cos^2 t + 2\cos t \sin t - \sin^2 t = \cos(2t) + \sin(2t)$$

So we are reduced to computing the absolute extreme values of

$$g(t) = \cos(2t) + \sin(2t), \;\; t \in [0, 2\pi]$$

The absolute maximum and minimum on the boundary are

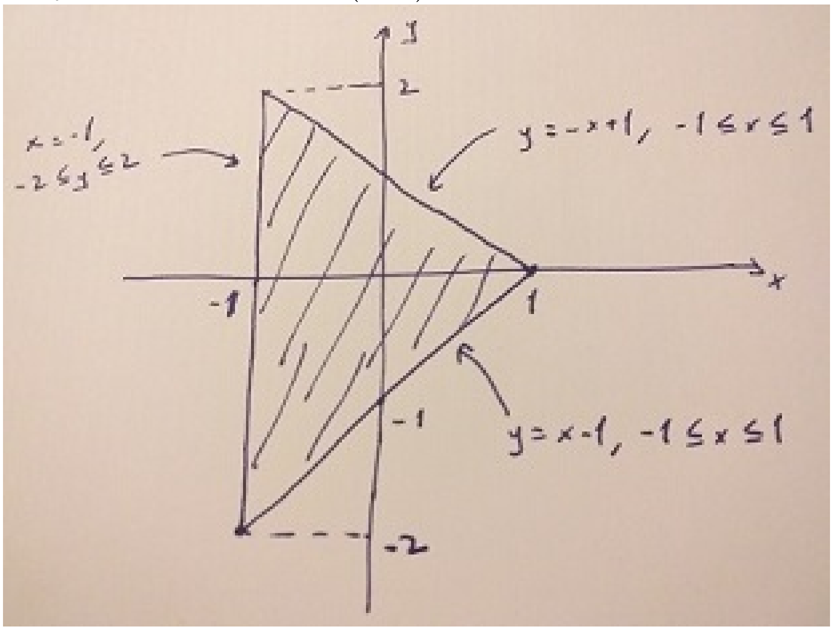
$$g(\pi/8) = \sqrt{2} \text{ and } g(5\pi/8) = -\sqrt{2}$$

respectively (Exercise). Since $f(0,0) = 0$ we conclude that the absolute maximum and minimum of $f(x,y)$ on $x^2 + y^2 \leq 1$ are

$$f(\cos(\pi/8), \sin(\pi/8)) = \sqrt{2}, \quad \text{and} \quad f(\cos(5\pi/8), \sin(5\pi/8)) = -\sqrt{2}$$

Example 2. Find the absolute maximum and minimum values of $f(x, y) = xy + x^2$ on the triangular region D given by $-1 \leq x \leq 1, x - 1 \leq y \leq -x + 1$.

Solution: First by computing the partial derivatives $f_1(x, y) = y + 2x$, $f_2(x, y) = x$ we see that f has no singular points, and also that the only critical point is $(0, 0)$.



The boundary of the domain is the triangle with vertices $(1, 0)$, $(-1, -2)$ and $(-1, 2)$. So the sides of the triangle can be parameterized by the line segments

$$x = -1, -2 \leq y \leq 2$$

$$y = -x + 1, -1 \leq x \leq 1$$

$$y = x - 1, -1 \leq x \leq 1$$

So on each side of the triangle our function f reduces to a function in single variable as

$$g_1(y) = -y + 1, -2 \leq y \leq 2$$

$$g_2(x) = x, -1 \leq x \leq 1$$

$$g_3(x) = 2x^2 - x, -1 \leq x \leq 1$$

If we compute the extreme values of the above functions on the specified domains, we see that the maximum and minimum on the boundary are $f(-1, -2) = 3$ and $f(-1, 2) = -1$ respectively (Exercise). Since $f(0, 0) = 0$, the absolute maximum and minimum of $f(x, y)$ on D are $f(-1, -2) = 3$ and $f(-1, 2) = -1$ respectively.