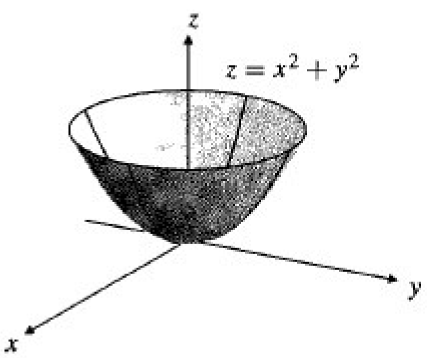
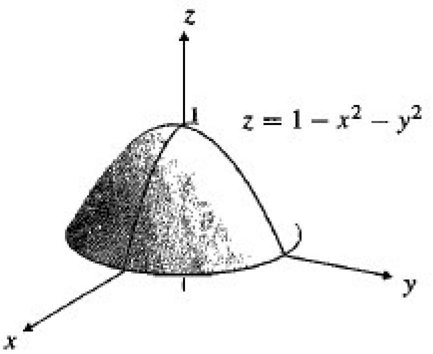
§13.1 Applications of Partial Derivates

Definition 1. Let $z = f(x, y)$ and the point (a, b) in the domain of f. We say that f has a local maximum value at the point (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) sufficiently close to the point (a, b). If the inequality holds for all (x, y) in the domain f, we say that f has absolute maximum value at the point. In a similar way, we say that f has a local minimum value at the point (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) sufficiently close to the point (a, b). If the inequality holds for all (x, y) in the domain f, we say that f has absolute minimum value at the point.





Theorem 1. (***Necessary conditions for extreme values***) A fuction $f(x, y)$ can have local or absolute extreme value at a point (a, b) in its domain only if (a, b) is one of the followings:

1. $\nabla f(a, b) = 0$, that is, (a, b) is a critical point of f.

2. $\nabla f(a,b)$ does not exist, that is, (a,b) is a singular point of f.

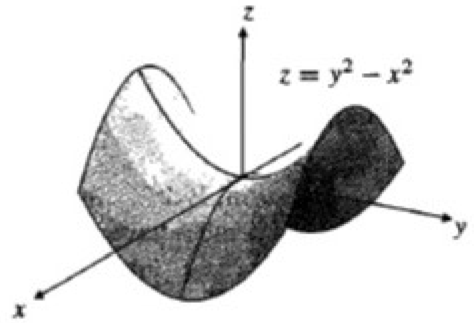
3. a boundary point of the domain of f.

A set $\mathbb{R}^n$ is bounded if it is contained inside some ball $x_1^2 + ... + x_n^2 \leq R^2$ of finite radius R.

Theorem 2. (***Sufficient conditions for extreme values***) If f is a continuous function of n variables whose domain is a closed and bounded set in $\mathbb{R}^n$, then the range of f is a bounded set of real numbers and there are points in its domain where f takes on absolute maximum and minimum values.

Example 1. Consider the function $f(x,y) = x^2 + y^2$. Its gradient vector is $\nabla f = 2xi + 2yj$, and at the point (0,0), it is $\nabla f(0,0) = 0i + 0j$ and so (0,0) is the critical point of f. For all points $(x,y) \neq (0,0)$, $f(x,y) > 0 = f(0,0)$. Then f has absolute minimum value 0 at that point. In a similar way, $g(x,y) = 1 - x^2 - y^2$ has absolute maximum value 1 at its critical point $(0,0)$.

Example 2. The function $h(x,y) = y^2 - x^2$ has a critical point at $(0,0)$ but it has neither a local maximum nor local minimum value at that point. $h(0,0) = 0$ but $h(x,0) < 0$ and $h(0,y) > 0$ for all nonzero points of x and y.

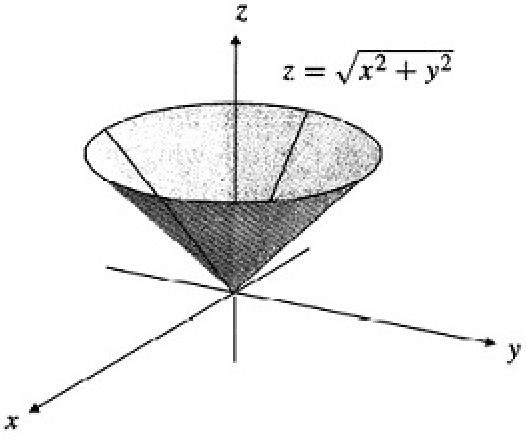


A critical point of the domain of a function f of several variables is called a saddle point if f does not have a local maximum or minimum values there. For the function h, $(0,0)$ is a saddle point.

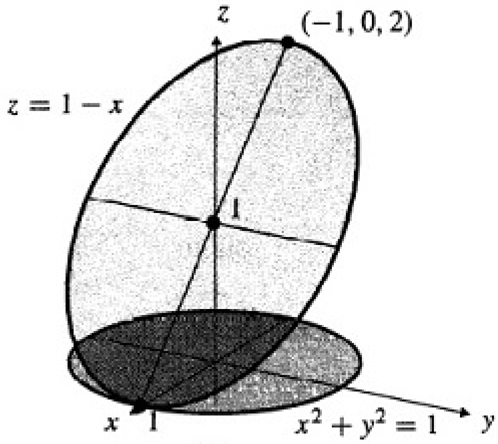
Example 3. The function $f(x,y) = \sqrt{x^2+y^2}$ has no critical points but it has a singular point at $(0,0)$ where it has a local and absolute minimum value 0 at that point.

Example 4. The function $f(x,y) = 1 - x$ is defined everywhere in the xy-plane and it has no critical and singular points. $\nabla f(x,y) = -i$ for each point (x,y).

But if we restrict the domain of f to the points in the disk $x^2+y^2 \leq 1$, then f have absolute maximum and minimum values at boundary points of the disk on the circle $x^2+y^2=1$. The maximum value is 2 at the boundary point (-1,0) and minimum value is 0 at (1,0).



(a)



(b)

§Classifying critical points Let (a, b) be a critical point of $f(x, y)$ and h, k small values. Consider $\triangle f = f(a+h, b+k) - f(a, b)$.

1. if $\triangle f > 0$, then f has local minimum value at that point.
2. if $\triangle f < 0$, then f has local maximam value at that point.
3. if $\triangle f < 0$ for some points (h, k) arbitrarily near (0,0) and $\triangle f > 0$ for other points, then f has a saddle point at that point.

Example 5. Classify the critical points of $f(x, y) = 2x^3 - 6xy + 3y^2$. $\nabla f(x, y) = (6x^2 - 6y)i + (-6x + 6y^2)j = (0, 0)$. Then the critical points of f are $(0, 0)$ and $(1, 1)$. Consider $(0, 0)$. $\triangle f = f(0 + h, 0 + k) - f(0, 0) = 2h^3 - 6hk + 3k^2$. Since $f(h, 0) - f(0, 0) = 2h^3 > 0$ for small positive h and $f(h, 0) - f(0, 0) = 2h^3 < 0$ for small negative h, then f does not have a maximum or minimum value at (0,0). Therefore, (0,0) is a saddle point.

Consider (1,1). $\triangle f = f(1+h, 1+k) - f(1,1) = 3(h-k)^2 + h^2(3+2h)$. If $|h| < 3/2$ and (h,k) is nonzero, then $\triangle f > 0$ for small h and k. Hence f has a local minimum value -1 at (1,1).

§A second derivative test Let (a,b) be a critical point of $f(x,y)$ interior to the domain of f. Assume that the second partial derivatives of f are continuous in a neighbourhood of (a,b) and have the following values at that point.

$A = f_{11}(a,b)$, $B = f_{12}(a,b) = f_{21}(a,b)$ and $C = f_{22}(a,b)$.

1. if $B^2 - AC < 0$ and $A > 0$, then f has a local minimum value at (a,b).

2. if $B^2 - AC < 0$ and $A < 0$, then f has a local maximum value at (a,b).

3. if $B^2 - AC > 0$, then f has a saddle point at (a,b).

4. if $B^2 - AC = 0$, this test does not work.

Example 6. Classify the critical points of $f(x,y) = xye^{-(x^2+y^2)/2}$.

$f_1(x,y) = y(1-x^2)e^{-(x^2+y^2)/2}$

$f_2(x,y) = x(1-y^2)e^{-(x^2+y^2)/2}$

$f_{11}(x,y) = xy(x^2-3)e^{-(x^2+y^2)/2}$

$f_{12}(x,y) = (1-x^2)(1-y^2)e^{-(x^2+y^2)/2}$

$f_{22}(x,y) = xy(y^2-3)e^{-(x^2+y^2)/2}$

Let $f_1 = 0$ and $f_2 = 0$. Then the critical points are $(0,0), (1,1), (1,-1), (-1,1)$ and $(-1,-1)$.

At $(0,0)$, $A = C = 0$ and $B = 1$ so that $B^2 - AC = 1 > 0$. Then f has a saddle point at that point.

At $(1,1)$ and $(-1,-1)$, $A = C = -2/e < 0$, $B = 0$. $B^2 - AC = -4/e^2$. Thus f has local maximum values at these points. The value of f at each point is 1/e.

At $(1,-1)$ and $(-1,1)$, $A = C = 2/e > 0$ and $B = 0$. $B^2 - AC = -4/e^2 < 0$ thus f has local minimum values at these points. The value of f at each point is -1/e.