GTU, Fall 2020, MATH 101

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Preliminaries, Trigonometric Functions

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* There are two different geometrical ways to define trigonometric functions.
* You may either use a right triangle or the unit circle.
* Using a right triangle may be practical in some cases, but it is inadequate for our purposes.
* We will use the unit circle, $x^2 + y^2 = 1$ to define the trigonometric functions.

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- This means that we divide the unit circle into 360 pieces, and we say that the angle of a single piece is 1°.
- But it is more natural (!) to use another measure, namely *radian* for angles instead of degrees.

Preliminaries, Trigonometric Functions

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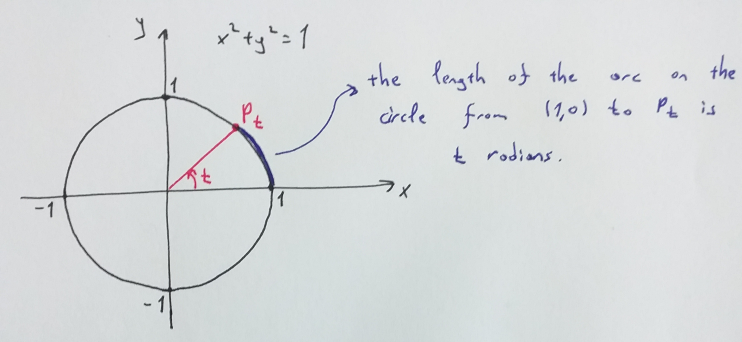

* (!) So indeed radian is a measure of length.

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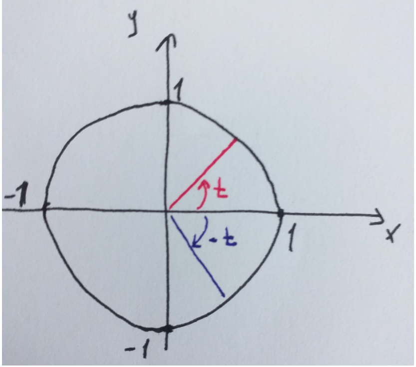

* The counterclockwise direction gives + angles, the clockwise direction gives − angles.

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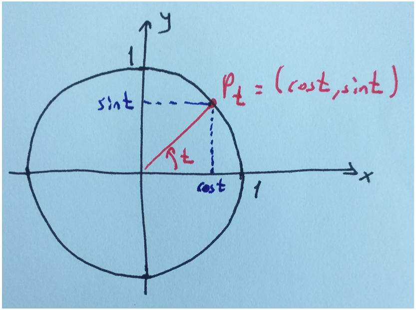
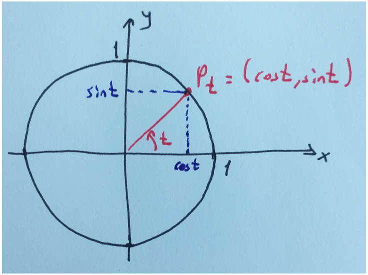


* Move t radian on the unit circle from the point $(1, 0)$ to the point P_t. The functions $\cos t$ and $\sin t$ are defined as the x and y-coordinates of P_t respectively.

Preliminaries, Trigonometric Functions



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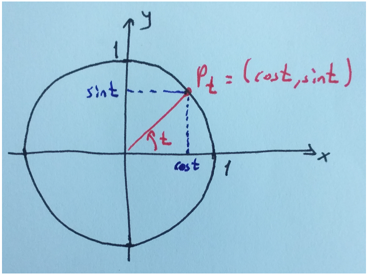
Preliminaries, Trigonometric Functions



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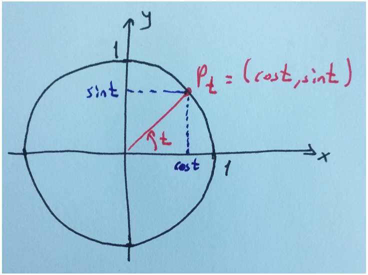

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$$\Longrightarrow \cos(t \pm 2\pi) = \cos t, \ \ \sin(t \pm 2\pi) = \sin t$$

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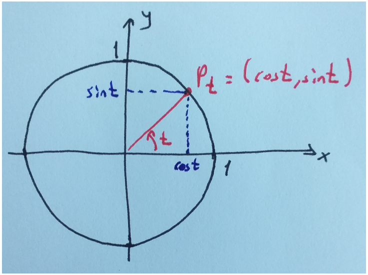

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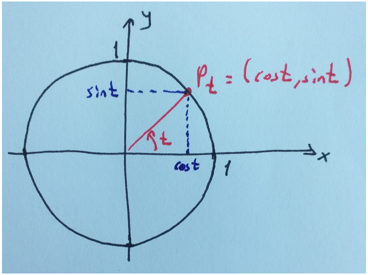

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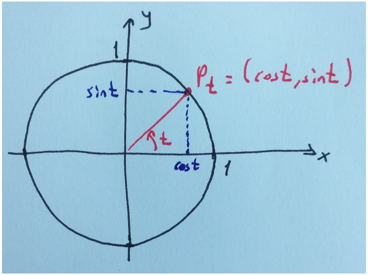

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$*$ $-1 \leq \cos t, \sin t \leq 1$.

Exercise: Observe the following identities by sketching the relevant angles.

i) $\cos(\pi - t) = -\cos t, \quad \sin(\pi - t) = \sin t$

ii) $\cos(\pi + t) = -\cos t, \quad \sin(\pi + t) = -\sin t$

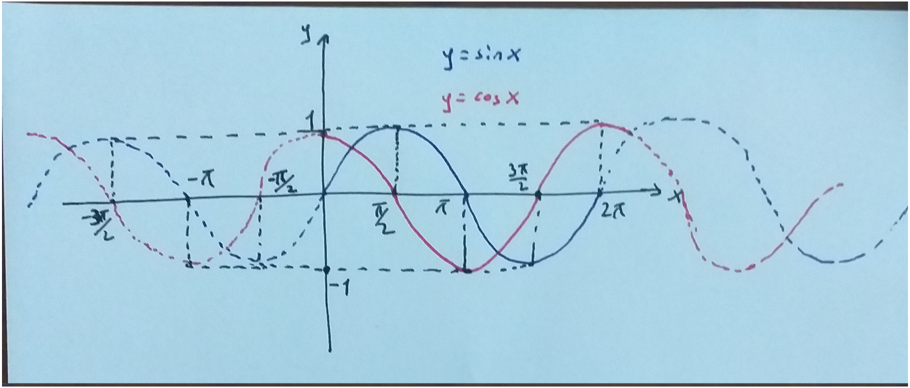
iii) $\cos\left(\frac{\pi}{2} - t\right) = \sin t, \quad \sin\left(\frac{\pi}{2} - t\right) = \cos t$

Exercise: Fill the following table using relevant identites
($360^\circ = 2\pi$ radian).

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\cos t$	1				0				
$\sin t$	0				1			1/2	

(For example; $\sin(5\pi/6) = \sin(\pi - \pi/6) = \sin(\pi/6) = \sin(30^\circ) = 1/2$).

* The graphs of sin and cos:



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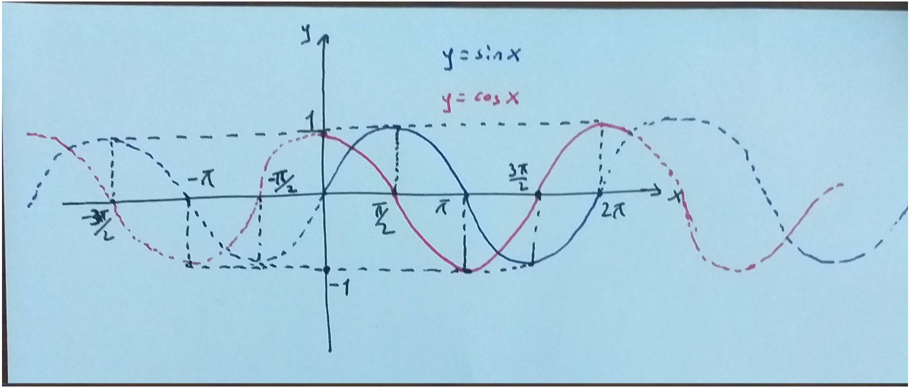

* Note: $\sin t = \cos\left(\frac{\pi}{2} - t\right) = \cos\left(t - \frac{\pi}{2}\right)$. So we just shift the graph of $y = \cos x$ to right by $\pi/2$ units to obtain the graph of $y = \sin x$.

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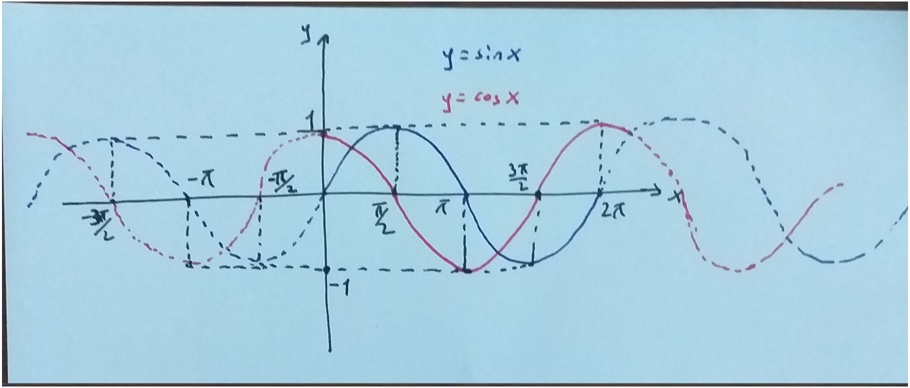

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* (!) Remember these graphs, we will use them many times.

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- If $a = b$ we have the following particular identities:
  $\sin(2a) = 2\sin(a)\cos(a)$
  $\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$

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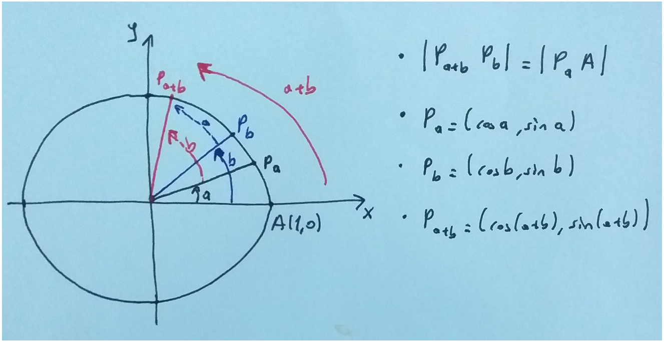

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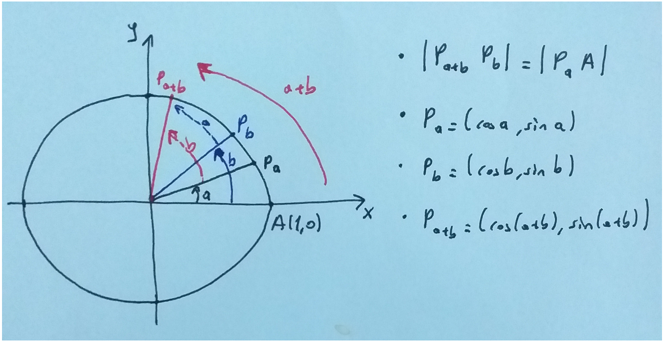

$|P_{a+b}P_b|^2 = (\cos(a+b) - \cos b)^2 + (\sin(a+b) - \sin b)^2 = \ldots =$
$2 - 2\cos(b)\cos(a+b) - 2\sin(b)\sin(a+b)$
$= |P_a A|^2 = (\cos a - 1)^2 + \sin^2 a = 2 - 2\cos a$

Preliminaries, Trigonometric Functions

$$* \implies \cos a = \cos(b)\cos(a+b) + \sin(b)\sin(a+b)$$

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* Exercise: Prove the other addition formulas (Hint: You can adopt the above proof for the other addition formulas, or you can use basic identities to derive them from the identity of $\cos(s-t)$.

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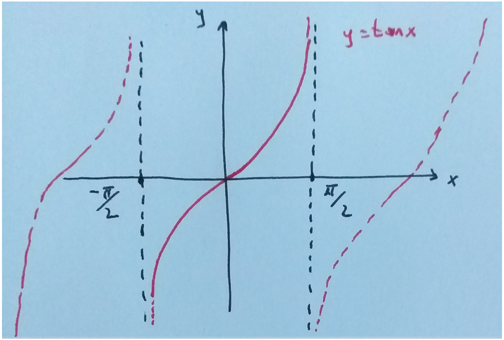
- So it is enough to know the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$.



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* Exercise: Find $\cos x$ and $\tan x$ if $\sin x = -1/2$ and $x \in (-\pi/2, 0)$.

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* Exercise: Find the domains of these functions.
* Exercise: Study the graphs of these functions from your text book.