UYGULAMA HAFTA 12

Section 14.1-İki Katlı İntegraller

Section 14.2-Kartezyen Koordinatlarda İki Katlı İntegrallerin İterasyonu

HATIRLATMALAR

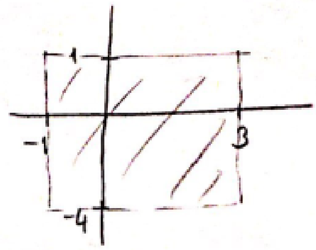
- **Teorem:** $f(x,y)$ fonksiyonu $D = \{(x,y) : a \leq x \leq b; c \leq y \leq$ d) bölgesi üzerinde sürekli ise
$$\iint_D f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$
eşitliği vardır.

- **Teorem:** $D = \{a \leq x \leq b; g_1(x) \leq y \leq g_2(x)\}$ bölgesi üzerinde $f(x,y)$ fonksiyonu sürekli ise
$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$
olur. Değişkenlerin yerleri değiştirilirse, benzer sonuç elde edilebilir:
$D = \{c \leq y \leq d; h_1(y) \leq y \leq h_2(y)\}$ bölgesi üzerinde $f(x,y)$ fonksiyonu sürekli ise
$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$
olur.

Aşağıdaki çift katlı integralleri hesaplayınız.

13) $\iint_R dA$, burada R, $-1 \leq x \leq 3$, $-4 \leq y \leq 1$ dikdörtgenidir.

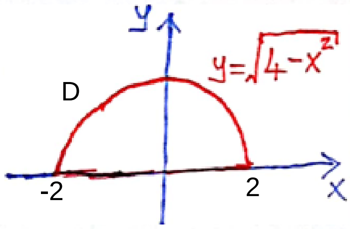
Sol.



$$\iint_R dA = R \text{ bölgesinin alanı}$$
$$= 4 \cdot 5 = 20.$$

14) $\iint_D (x+3)\,dA$; burada D, $0 \leq y \leq \sqrt{4-x^2}$ yarı diskidir.

Sol.



$$\iint_D (x+3)\,dA = \iint_D x\,dA + \iint_D 3\,dA$$
$$= 0 + 3 \cdot (D \text{ bölgesinin alanı})$$
$$= 3 \cdot \frac{\pi \cdot 2^2}{2} = 6\pi .$$

D , $x=0$ etrafında simetrik olduğu için x-in D-üzerindeki integrali sıfırdır.

Aşağıda verilen çift katlı integralleri hesaplayınız.

3) $\int_0^{\pi} \int_{-x}^{x} \cos y \, dy \, dx$

Sol.

$$\int_0^{\pi} \int_{-x}^{x} \cos y \, dy \, dx = \int_0^{\pi} \left(\sin y \Big|_{-x}^{x} \right) dx$$
$$= \int_0^{\pi} (\sin x - \sin(-x))\,dx$$
$$= \int_0^{\pi} 2 \sin x \, dx = -2\cos x \Big|_0^{\pi} = 4.$$

4) $\int_0^2 dy \int_0^y y^2 e^{xy} dx$

Sol.

$$\int_0^2 dy \int_0^y y^2 e^{xy} dx = \int_0^2 y^2 dy \int_0^y e^{xy} dx$$

$$= \int_0^2 y^2 dy \left(\frac{e^{xy}}{y} \Big|_0^y \right)$$

$$= \int_0^2 y \left(e^{y^2} - 1 \right) dy$$

$$= \int_0^4 \left(\frac{e^u - 1}{2} \right) du$$

$$= \left(\frac{e^u}{2} - \frac{u}{2} \right) \Big|_0^4 = \frac{e^4 - 5}{2}$$

D.D.

$y^2 = u \Rightarrow 2y\,dy = du$

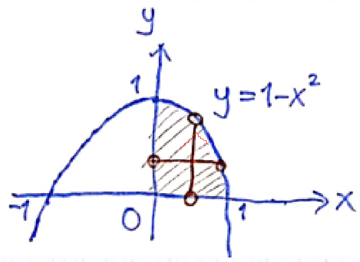
$\Rightarrow \frac{du}{2} = y\,dy$

$y = 0 \Rightarrow u = 0$

$y = 2 \Rightarrow u = 4$

Aşağıda verilen çift katlı integralleri hesaplayınız.

10) $\iint_D x \cos y \, dA$; burada D, birinci çeyrek düzlemde koordinat eksenleri ve $y = 1 - x^2$ eğrisi tarafından sınırlanan sonlu bölgedir.



Sol.

$$\iint_D x \cos y \, dA = \int_0^1 \int_0^{1-x^2} x \cos y \, dy \, dx$$

$$= \int_0^1 x \, dx \int_0^{1-x^2} \cos y \, dy$$

$$= \int_0^1 x \, dx \left(\sin y \Big|_0^{1-x^2} \right)$$

$$= \int_0^1 x \sin(1 - x^2) \, dx$$

$$= \int_1^0 -\sin u \frac{du}{2} = \int_0^1 \frac{\sin u}{2} du$$

$$= -\frac{\cos u}{2} \Big|_0^1 = \frac{1 - \cos(1)}{2}.$$

D.D.

$1 - x^2 = u \Rightarrow -2x\,dx = du$

$\Rightarrow x\,dx = -\frac{du}{2}$

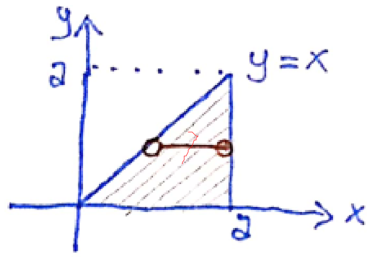
$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 0$

12) $\iint\limits_T \sqrt{a^2-y^2}\, dA$; burada T, $(0,0)$, $(a,0)$ ve (a,a) tepeli üçgendir.

Sol.

$$\iint\limits_T \sqrt{a^2-y^2}\, dA = \int_0^a \int_y^a \sqrt{a^2-y^2}\, dx\, dy$$



$$= \int_0^a \sqrt{a^2-y^2}\, dy \int_y^a dx$$

$$= \int_0^a \sqrt{a^2-y^2}\, dy \left(x \Big|_a^y \right)$$

$$= \int_0^a (a-y)\sqrt{a^2-y^2}\, dy$$

$$= \underbrace{a\int_0^a \sqrt{a^2-y^2}\, dy}_{I_1} - \underbrace{\int_0^a y\sqrt{a^2-y^2}\, dy}_{I_2}$$

$$I_1 = \int_0^a \sqrt{a^2-y^2}\, dy$$

$\longrightarrow$ DD $\boxed{\begin{array}{l} y = a\sin u \Rightarrow dy = a\cos u\, du \\ y=0 \Rightarrow u=0,\ y=a \Rightarrow u=\frac{\pi}{2} \end{array}}$

$$= \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2 u}\; a\cos u\, du$$

$$= \int_0^{\pi/2} a^2\cos^2 u\, du = a^2\int_0^{\pi/2} \left(\frac{1+\cos 2u}{2}\right) du$$

$$= \frac{a^2}{2}\left(u + \frac{\sin 2u}{2}\right)\Big|_0^{\pi/2} = \frac{a^2\pi}{4}$$

$$I_2 = \int_0^a y\sqrt{a^2-y^2}\, dy$$

$\longrightarrow$ DD $a^2-y^2 = u \Rightarrow y\,dy = -\frac{du}{2}$

$y=0 \Rightarrow u=a^2$, $y=a \Rightarrow u=0$

$$= \int_{a^2}^0 -\frac{\sqrt{u}}{2}\, du$$

$$= \int_0^{a^2} \frac{\sqrt{u}}{2}\, du = \frac{1}{3}u^{3/2}\Big|_0^{a^2} = \frac{1}{3}a^3.$$

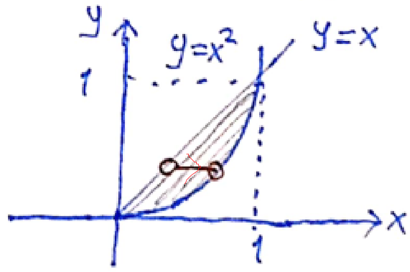
Dolayısıyla;

$$\iint_T \sqrt{a^2-y^2}\,dA = a\,\frac{a^2\pi}{4} - \frac{a^3}{3} = a^3\left(\frac{\pi}{4}-\frac{1}{3}\right)$$

13) $\displaystyle\iint_R \frac{x}{y}e^y\,dA$; burada R, $0\le x\le 1$, $x^2\le y\le x$ bölgesidir.

Sol.

$$x = x^2 \Rightarrow x(x-1)=0 \Rightarrow \begin{matrix} x=0 \\ x=1 \end{matrix}$$



$$\iint_R \frac{x}{y}e^y\,dA = \int_0^1\int_y^{\sqrt{y}} \frac{x}{y}e^y\,dx\,dy$$

$$= \int_0^1 \frac{e^y}{y}\,dy \int_y^{\sqrt{y}} x\,dx$$

$$= \int_0^1 \frac{e^y}{y}\,dy \left(\frac{x^2}{2}\Big|_y^{\sqrt{y}}\right)$$

$$= \int_0^1 \frac{1}{2}(1-y)e^y\,dy$$

Kiy

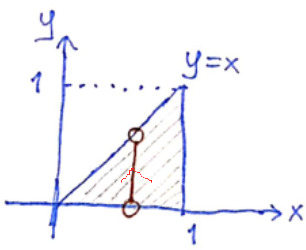
- $1-y=u$, $du=-dy$
- $dv = e^y\,dy$, $v=e^y$

$$= \frac{1}{2}\left((1-y)e^y\Big|_0^1 + \int_0^1 e^y\,dy\right)$$

$$= -\frac{1}{2} + \frac{1}{2}(e-1) = \frac{e}{2} - 1$$

14) $\displaystyle\iint_T \frac{xy}{1+x^4}\,dA$; burada T, $(0,0)$, $(1,0)$ ve $(1,1)$ tepeli üçgendir.

Sol.



$$\iint_T \frac{xy}{1+x^4}\,dA = \int_0^1\int_0^x \frac{xy}{1+x^4}\,dy\,dx$$

$$= \int_0^1 \frac{x}{1+x^4}\,dx \int_0^x y\,dy$$

$$= \int_0^1 \frac{x}{1+x^4}\,dx \left(\frac{y^2}{2}\Big|_0^x\right)$$

$$= \int_0^1 \frac{1}{2}\left(\frac{x^3}{1+x^4}\right)dx$$

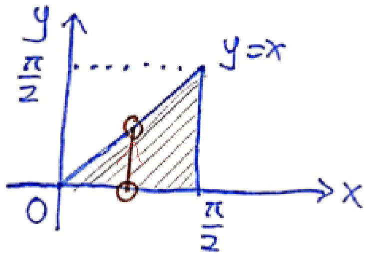
DD. $1+x^4=u \Rightarrow 4x^3\,dx = du$

$x=0 \Rightarrow u=1$, $x=1 \Rightarrow u=2$.

$$= \frac{1}{8}\int_1^2 \frac{du}{u} = \frac{1}{8}\ln u \Big|_1^2 = \frac{\ln 2}{8}.$$

16) İntegrasyon bölgesini çizerek $\displaystyle\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x}\,dx$ tekrarlı integralini hesaplayınız.

Sol.



$$\int_0^{\pi/2}\int_y^{\pi/2} \frac{\sin x}{x}\,dx\,dy = \int_0^{\pi/2}\int_0^{x} \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^{\pi/2} \left(\frac{\sin x}{x}\, y \Big|_0^x \right) dx$$

$$= \int_0^{\pi/2} \sin x\,dx$$

$$= -\cos x \Big|_0^{\pi/2} = 1.$$

Aşağıdaki alıştırmalarda belirtilen katı cisimlerin hacimlerini bulunuz.

20) $z = 1 - x^2$-nin altı ve $0 \le y \le 1$, $0 \le x \le y$ bölgesinin üstü

Sol.

$$V = \int_0^1 dy \int_0^y (1-x^2)\,dx = \int_0^1 \left(x - \frac{x^3}{3} \right)\Big|_0^y dy$$

$$= \int_0^1 \left(y - \frac{y^3}{3} \right) dy$$

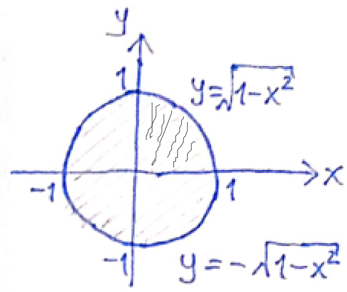
$$= \left(\frac{y^2}{2} - \frac{y^4}{12} \right)\Big|_0^1 = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}.$$

22) $z = 1-y^2$-nin altı ve $z=x^2$-nin üstü

Sol.

$z=1-y^2$ ve $z=x^2$, ($z=1-y^2=x^2 \Rightarrow x^2+y^2=1$), $x^2+y^2=1$ silindiri üzerinde kesişirler.

$$\Rightarrow V = \iint\limits_{x^2+y^2\leq 1} (1-y^2-x^2)\, dA$$



$$= 4\int_0^1 \int_0^{\sqrt{1-x^2}} (1-y^2-x^2)\, dy\, dx$$

$$= 4\int_0^1 \left(y - \frac{y^3}{3} - x^2 y\right)\Big|_0^{\sqrt{1-x^2}} dx$$

$$= 4\int_0^1 \left(\sqrt{1-x^2} - \frac{(1-x^2)^{3/2}}{3} - x^2\sqrt{1-x^2}\right) dx$$

$$= 4\int_0^1 \left(\overbrace{\sqrt{1-x^2}\,(1-x^2)}^{(1-x^2)^{3/2}} - \frac{(1-x^2)^{3/2}}{3}\right) dx$$

$$= \frac{8}{3}\int_0^1 (1-x^2)^{3/2} dx$$

D.D.

$$\boxed{\begin{array}{l} x=\sin u \Rightarrow dx = \cos u\, du \\ x=0 \rightarrow u=0 \\ x=1 \rightarrow u=\frac{\pi}{2} \end{array}}$$

$$= \frac{8}{3}\int_0^{\pi/2} (\overbrace{1-\sin^2 u}^{\cos^2 u})^{3/2} \cos u\, du$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^4 u\, du \quad \rightarrow \quad \cos^2 u = \frac{1+\cos 2u}{2}$$

$$= \frac{8}{3}\int_0^{\pi/2} \frac{(1+\cos 2u)^2}{4}$$

$$= \frac{2}{3}\int_0^{\pi/2} \left(1 + 2\cos 2u + \overbrace{\frac{1+\cos 4u}{2}}^{\cos^2 2u}\right) du$$

$$= \frac{2}{3}\left(u + \sin 2u + \frac{u}{2} + \frac{\sin 4u}{8}\right)\Big|_0^{\pi/2}$$

$$= \pi/2 .$$