Question 12 from section 2.4

Find the derivative of $y = \left(2 + |x|^3\right)^{1/3}$.

Question 12 from section 2.4

Find the derivative of $y = \left(2 + |x|^3\right)^{1/3}$.
Solution: The Chain Rule gives $f(g(x))' = f'(g(x))g'(x)$. Also, $\dfrac{d}{dx}|x| = \frac{x}{|x|}$. Therefore,

$$y' = \frac{1}{3}\left(2 + |x|^3\right)^{-2/3}\left(3|x|^2\right)\operatorname{sgn}(x)$$

$$= |x|^2\left(2 + |x|^3\right)^{-2/3}\left(\frac{x}{|x|}\right) = x|x|\left(2 + |x|^3\right)^{-2/3}$$

Question 28 from section 2.4

In Exercises $22 - 29$, express the derivative of the given function in terms of the derivative f' of the differentiable function f.

Question 28 from section 2.4

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Question 28:

$$\frac{d}{dx} f\left(2f\left(3f(x)\right)\right)$$

Question 28 from section 2.4

In Exercises $22 - 29$, express the derivative of the given function in terms of the derivative f' of the differentiable function f.
Question 28:

$$\frac{d}{dx} f\left(2f\left(3f(x)\right)\right)$$

Solution: The Chain Rule gives $f(g(x))' = f'(g(x))g'(x)$. Therefore,

$$\begin{aligned}
\frac{d}{dx} f\left(2f\left(3f(x)\right)\right) &= f'\left(2f\left(3f(x)\right)\right) \cdot 2f'\left(3f(x)\right) \cdot 3f'(x) \\
&= 6f'(x)f'\left(3f(x)\right)f'\left(2f\left(3f(x)\right)\right)
\end{aligned}$$

Question 30 from section 2.4

Find

$$\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\Bigg|_{x=-2}$$

.

Question 30 from section 2.4

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$$\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\Bigg|_{x=-2}$$
.

Solution: We use the Quotient Rule to solve question.
$$(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}.$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\Bigg|_{x=-2} &= \frac{\left(x^2+1\right)\frac{x}{\sqrt{x^2-1}}-\sqrt{x^2-1}(2x)}{\left(x^2+1\right)^2}\Bigg|_{x=-2} \\ &= \frac{(5)\left(-\frac{2}{\sqrt{3}}\right)-\sqrt{3}(-4)}{25} = \frac{2}{25\sqrt{3}}\end{aligned}$$

Question 37 from section 2.4

Find an equation of the tangent line to the curve $y = \left(1 + x^{2/3}\right)^{3/2}$ at $x = -1$

Question 37 from section 2.4

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Solution: Slope of $y = \left(1 + x^{2/3}\right)^{3/2}$ at $x = -1$ is

$$\frac{3}{2}\left(1 + x^{2/3}\right)^{1/2}\left(\frac{2}{3}x^{-1/3}\right)\Big|_{x=-1} = -\sqrt{2}$$

Question 37 from section 2.4

Find an equation of the tangent line to the curve $y = \left(1 + x^{2/3}\right)^{3/2}$ at $x = -1$

Solution: Slope of $y = \left(1 + x^{2/3}\right)^{3/2}$ at $x = -1$ is

$\frac{3}{2}\left(1 + x^{2/3}\right)^{1/2}\left(\frac{2}{3}x^{-1/3}\right)\Big|_{x=-1} = -\sqrt{2}$

The tangent line at $\left(-1, 2^{3/2}\right)$ has equation $y = 2^{3/2} - \sqrt{2}(x + 1)$

Question 16 from section 2.5

Find the derivatives of the functions in Exercises $3-36$. simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.
Question 16:

$$g(\theta) = \tan(\theta \sin \theta)$$

Question 16 from section 2.5

Find the derivatives of the functions in Exercises $3-36$. simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.
Question 16:

$$g(\theta) = \tan(\theta \sin\theta)$$

Solution: From the Chain Rule we know that $(tanu)' = u' \sec^2 u$.
Therefore,

$$g'(\theta) = (\sin\theta + \theta\cos\theta)\sec^2(\theta\sin\theta).$$

Question 36 from section 2.5

Find the derivatives of the functions in Exercises $3 - 36$. simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.

Question 36 from section 2.5

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Question 36:

$$f(s) = \cos(s + \cos(s + \cos s))$$

Question 36 from section 2.5

Find the derivatives of the functions in Exercises $3 - 36$. simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.
Question 36:

$$f(s) = \cos(s + \cos(s + \cos s))$$

Solution:

$$f'(s) = -[\sin(s + \cos(s + \cos s))][1 - (\sin(s + \cos s))(1 - \sin s)]$$

Question 45 from section 2.5

Find the points on the curve $y = \tan x, -\pi/2 < x < \pi/2$ where the tangent is parallel to the line $y = 2x$.

Question 45 from section 2.5

Find the points on the curve $y = \tan x, -\pi/2 < x < \pi/2$ where the tangent is parallel to the line $y = 2x$.

Solution: The slope of $y = \tan x$ at $x = a$ is $\sec^2 a$. The tangent there is parallel to $y = 2x$ if $\sec^2 a = 2$, or $\cos a = \pm 1/\sqrt{2}$. The only solutions in $(-\pi/2, \pi/2)$ are $a = \pm\pi/4$. The corresponding points on the graph is $(\pi/4, 1)$ and $(-\pi/4, -1)$.

Question 53 from section 2.5

$$\lim_{x \to 0} \frac{\tan(2x)}{x}$$

Question 53 from section 2.5

$$\lim_{x \to 0} \frac{\tan(2x)}{x}$$

Solution:

$$\lim_{x \to 0} \frac{\tan(2x)}{x} = (\lim_{x \to 0} \frac{\sin 2x}{2x})(\lim_{x \to 0} \frac{2}{\cos 2x}) = 1 \times 2 = 2$$

Question 56 from section 2.5

$$\lim_{x \to 0} \cos\left(\frac{\pi - \pi\cos^2 x}{x^2}\right)$$

Question 56 from section 2.5

$$\lim_{x \to 0} \cos\left(\frac{\pi - \pi\cos^2 x}{x^2}\right)$$

Solution:

$$\lim_{x \to 0} \cos\left(\frac{\pi - \pi\cos^2 x}{x^2}\right) = \lim_{x \to 0} \cos\left(\frac{\pi(1 - cos^2 x)}{x^2}\right)$$

$$= \lim_{x \to 0} \cos\left(\frac{\pi \sin^2 x}{x^2}\right) \qquad \text{use } \cos^2 x + \sin^2 x = 1$$

Question 56 from section 2.5

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Since $\cos x$ is continuous function, using above theorem we get

$$\lim_{x \to 0} \cos\left(\frac{\pi \sin^2 x}{x^2}\right) = \cos\left(\pi \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2\right) = \cos \pi = -1.$$

Question 8 from section 2.6

Find y', y'', and y''' for the function $y = \frac{x-1}{x+1}$

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Find y', y'', and y''' for the function $y = \frac{x-1}{x+1}$

$$y = \frac{x-1}{x+1} = \frac{x+1-2}{x+1} = 1 - \frac{2}{x+1}$$

$$y' = \frac{2}{(x+1)^2}$$

$$y'' = \frac{0-4(x+1)}{(x+1)^4} = -\frac{4}{(x+1)^3}$$

$$y''' = \frac{0-(-4)3(x+1)^2}{(x+1)^6} = \frac{12}{(x+1)^4}$$

Question 11 from section 2.6

Find y', y'', and y''' for the function $y = \cos\left(x^2\right)$

Question 11 from section 2.6

Find y', y'', and y''' for the function $y = \cos\left(x^2\right)$
Solution:

$$\begin{aligned}
&y = \cos\left(x^2\right)\\
&y' = -2x\sin\left(x^2\right)\\
&y'' = -2\sin\left(x^2\right) - 4x^2\cos\left(x^2\right)\\
&y''' = -4x\cos(x^2) - 8x\cos(x^2) + 4x^2 2x\sin(x^2)\\
&= -12x\cos\left(x^2\right) + 8x^3\sin\left(x^2\right)
\end{aligned}$$

Question 17 from section 2.6

In Exercises 13 – 23, calculate enough derivatives of the given function to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.

Question 17:

$$f(x) = \frac{1}{a+bx} = (a+bx)^{-1}.$$

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In Exercises $13 - 23$, calculate enough derivatives of the given function to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.
Question 17:

$$f(x) = \frac{1}{a+bx} = (a+bx)^{-1}.$$

$$f'(x) = -b(a+bx)^{-2}$$
$$f''(x) = 2b^2(a+bx)^{-3}$$
$$f'''(x) = -3!b^3(a+bx)^{-4}$$

Question 17 from section 2.6

Guess:

$$f^{(n)}(x) = (-1)^n n! b^n (a + bx)^{-(n+1)} \tag{1}$$

Question 17 from section 2.6

Guess:

$$f^{(n)}(x) = (-1)^n n! b^n (a+bx)^{-(n+1)} \tag{1}$$

Proof: (**??**) holds for $n = 1, 2, 3$. Assume (**??**) holds for $n = k$: $f^{(k)}(x) = (-1)^k k! b^k (a+bx)^{-(k+1)}$. Then

$$\begin{aligned} f^{(k+1)}(x) &= (-1)^k k! b^k (-(k+1))(a+bx)^{-(k+1)-1}(b) \\ &= (-1)^{k+1}(k+1)! b^{k+1}(a+bx)^{-(k+2)} \end{aligned}$$

So (**??**) holds for $n = k+1$ if it holds for $n = k$. Therefore, (**??**) holds for $n = 1, 2, 3, 4, \ldots$ by induction.

Question 21 from section 2.6

In Exercises 13 – 23, calculate enough derivatives of the given function to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.
$f(x) = x\sin(ax)$

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$f(x) = x\sin(ax)$

$f'(x) = \sin(ax) + ax\cos(ax)$

$f''(x) = 2a\cos(ax) - a^2x\sin(ax)$ $\quad$ $f^4(x) = -4a^3\cos(ax) + a^4x\sin(ax)$

$f'''(x) = -3a^2\sin(ax) - a^3x\cos(ax)$

Question 21 from section 2.6

In Exercises 13 − 23, calculate enough derivatives of the given function to enable you to guess the general formula for $f^{(n)}(x)$. Then verify your guess using mathematical induction.

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$f'(x) = \sin(ax) + ax\cos(ax)$

$f''(x) = 2a\cos(ax) - a^2x\sin(ax) \qquad f^4(x) = -4a^3\cos(ax) + a^4x\sin(ax)$

$f'''(x) = -3a^2\sin(ax) - a^3x\cos(ax)$

This suggests the formula

$$f^{(n)}(x) = \begin{cases} -na^{n-1}\cos(ax) + a^n x\sin(ax) & \text{if } n = 4k \\ na^{n-1}\sin(ax) + a^n x\cos(ax) & \text{if } n = 4k+1 \\ na^{n-1}\cos(ax) - a^n x\sin(ax) & \text{if } n = 4k+2 \\ -na^{n-1}\sin(ax) - a^n x\cos(ax) & \text{if } n = 4k+3 \end{cases}$$

for $k = 0, 1, 2, \ldots$. Differentiating any of these four formulas produces the one for the next higher value of n, so induction confirms the overall formula.

Question 24 from section 2.6

If $y = \tan kx$, show that $y'' = 2k^2 y \left(1 + y^2\right)$

Question 24 from section 2.6

If $y = \tan kx$, show that $y'' = 2k^2 y \left(1 + y^2\right)$

Solution:
If $y = \tan(kx)$, then $y' = k\sec^2(kx)$ and

$$\begin{aligned} y'' &= 2k^2 \sec^2(kx)\tan(kx) \\ &= 2k^2\left(1+\tan^2(kx)\right)\tan(kx) = 2k^2 y\left(1+y^2\right) \end{aligned}$$