Math102 PS Week 3

Gebze Technical University

March 10

Outline

9.3-Question 24

Determine whether the given series converges or diverges by using any appropriate test:

$$\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$$

9.3-Question 24

Determine whether the given series converges or diverges by using any appropriate test:

$$\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$$

$$\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{n+1} > \frac{1}{2n}$$

and we know that

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

is a harmonic series, so it diverges.

By comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$ diverges.

9.3-Question 39

Use the Root Test to test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

9.3-Question 39

Use the Root Test to test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

Recall the Root Test: Suppose that $\sigma = \lim_{n\to\infty} {a_n}^{1/n}$ exists or is ∞. Then,

$$\sigma < 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$\sigma > 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

9.3-Question 39

Use the Root Test to test the following series for convergence:

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$$

Recall the Root Test: Suppose that $\sigma = \lim_{n\to\infty} a_n{}^{1/n}$ exists or is ∞. Then,

$$\sigma < 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$\sigma > 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

$$\sigma = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n} = e^{-1} < 1$$

By the root test, the series converges.

- **Absolute convergence:**

$$\sum_{n=1}^{\infty} a_n \quad \text{is absolutely convergent if} \quad \sum_{n=1}^{\infty} |a_n| \quad \text{converges.}$$

Theorem. If a series converges absolutely, then it converges.

$$\left(\text{If} \quad \sum_{n=1}^{\infty} |a_n| \quad \text{converges, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges.}\right)$$

- **Conditional convergence:**

If a series converges but not absolutely, then it is conditionally convergent.

$$\left(\quad \sum_{n=1}^{\infty} |a_n| \quad \text{diverges, but} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges.}\right)$$

The Alternating Series Test.

Suppose that $\{a_n\}$ is a sequence whose terms satisfy

1. $a_n a_{n+1} < 0$ for $n \geq N$ (alternating)
2. $|a_{n+1}| < |a_n|$ for $n \geq N$ (decreasing)
3. $\lim_{n\to\infty} a_n = 0$

for some integer N. Then

$$\sum_{n=1}^{\infty} a_n$$

converges.

Question 1

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}.$$

Question 1

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}.$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{diverges by p-test.}$$

$$\text{So,} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \quad \text{is not absolutely convergent.}$$

But $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}$ is alternating, $|a_n| = \frac{1}{\sqrt{n}}$ is decreasing, and $\lim_{n\to\infty} a_n = 0$.

By alternating series test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}.$$

converges. So, this series converges conditionally.

Question 2

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}.$$

Question 2

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}.$$

$$\left| \frac{(-1)^n}{n^2 + \ln n} \right| = \frac{1}{n^2 + \ln n} \leq \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges by p-test.}$$

So, $$\sum_{n=1}^{\infty} \frac{1}{n^2 + \ln n} \quad \text{converges by comparison test.}$$

So, $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n} \quad \text{converges absolutely.}$$

Question 5

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n^2-1)}{n^2+1}.$$

Question 5

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n^2-1)}{n^2+1}.$$

The general term

$$a_n = \frac{(-1)^n(n^2-1)}{n^2+1}$$

diverges. So by n-th term test, this series diverges.

Question 6

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}.$$

Question 6

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}.$$

$$|a_n| = \left| \frac{(-2)^n}{n!} \right| = \frac{2^n}{n!}.$$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

So, by the Ratio test, $\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

So, $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ converges absolutely.

Question 10

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}.$$

Question 10

Determine whether the series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}.$$

$$|a_n| = \left|\frac{100\cos(n\pi)}{2n+3}\right| = \left|\frac{100(-1)^n}{2n+3}\right| = \frac{100}{2n+3}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{100}{2n+3}$ is divergent, so the given series does not converge absolutely.

But a_n is alternating, $|a_n|$ is decreasing and $\lim_{n\to\infty} a_n = 0$. So by alternating series test

$$\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3} \quad \text{converges.}$$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$ converges conditionally.

Power series about c:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

a_n : coefficients of the power series, c : centre of convergence.

One of the following must hold:

1. the series may converge only at $x = c$
2. the series may converge at every real number x
3. there exists a number $R > 0$ such that the series
   - converges at every x satisfying $|x-c| < R$ (on $(c-R, c+R)$),
   - diverges at every x satisfying $|x-c| > R$,
   - may or may not converge at $x = c \pm R$.

The radius of convergence:

Suppose that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{exists or is } \infty.$$

Then, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is $R = \frac{1}{L}$.

Interval of convergence:

$$\text{Find} \quad L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

1. $L = \infty$

   Radius of convergence $= 0$, interval of convergence is the point c.

2. $L = 0$

   Radius of convergence $= \infty$, interval of convergence: $\mathbb{R}$

3. L is a real number (except 0)

   Radius of convergence $= \frac{1}{L}$, interval of convergence: $(c - R, c + R)$ or $[c - R, c + R)$ or $(c - R, c + R]$ or $[c - R, c + R]$.

Note that the given power series converges absolutely on the open interval $(c - R, c + R)$.

We need to check the endpoints separately.

Question 1

Determine the centre, radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$$

Question 1

Determine the centre, radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$$

$$a_n = \frac{1}{\sqrt{n+1}} \quad \text{(coefficients)}$$

Center: $c = 0$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = 1$$

Radius: $R = \frac{1}{L} = 1$.

The series converges absolutely when $|x| < 1$, diverges when $|x| > 1$.

Question 1 (cont.)

Check the convergence at endpoints $x = \pm 1$:

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^{2n}}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

diverges by p-test.

Interval: $(-1, 1)$.

Question 3

Determine the centre, radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{x+2}{2} \right)^n$$

Question 3

Determine the centre, radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{x+2}{2} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n n} (x+2)^n \quad \Rightarrow \quad a_n = \frac{1}{2^n n} \quad \text{(coefficients)}$$

Center: -2.

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{1}{2^{n+1}} \frac{1}{n+1}}{\frac{1}{2^n} \frac{1}{n}} = \frac{1}{2}$$

Radius $= \frac{1}{L} = 2$.

Converges absolutely when $|x+2| < 2$ $(-4 < x < 0)$

Question 3 (cont.)

Check convergence at the endpoints:

$$x = -4 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{-4+2}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

converges by the Alternating Series Test.

$$x = 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{0+2}{2} \right)^n = \sum_{n=0}^{\infty} \frac{1}{n}$$

is a harmonic series, it diverges.

Interval: $[-4, 0)$.

Question 8

Determine the centre, radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \left(\frac{4x-1}{n} \right)^n$$

Question 8

Determine the centre, radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \left(\frac{4x-1}{n}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{4}{n}\right)^n \left(x - \frac{1}{4}\right)^n \quad \Rightarrow \quad a_n = \left(\frac{4}{n}\right)^n$$

Center: $\frac{1}{4}$

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{4}{n+1}\right)^{n+1}}{\left(\frac{4}{n}\right)^n} = \lim_{n\to\infty} \left[\frac{4}{n+1}\left(\frac{n}{n+1}\right)^n\right] = 0.e^{-1} = 0$$

Radius $= \frac{1}{L} = \infty$.

Interval of convergence is $\mathbb{R}$.

Question 12

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\frac{1}{2-x} \quad \text{in powers of } x:$$

Question 12

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ... = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\frac{1}{2-x} \quad \text{in powers of } x:$$

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\frac{1}{1-\frac{x}{2}} = \frac{1}{2}(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + ...) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

Question 12 (cont.)

$$\frac{1}{2-x} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$ is a power series about $x = 0$. $\quad a_n = \dfrac{1}{2^n}$.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2} \quad \Rightarrow \quad R = 2$$

.

The power series absolutely converges when $-2 < x < 2$.

Question 12 (cont.)

Check the endpoints:

$$x = -2 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{-2}{2} \right)^n = \sum_{n=0}^{\infty} (-1)^n \quad \text{diverges.}$$

$$x = 2 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{2}{2} \right)^n = \sum_{n=0}^{\infty} 1 \quad \text{diverges.}$$

So, the interval of convergence of the power series is $(-2, 2)$, that is, the representation

$$\frac{1}{2-x} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n$$

is valid on $(-2, 2)$.

Question 13

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ..., \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\frac{1}{(2-x)^2} \quad \text{in powers of } x:$$

Question 13

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ..., \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\frac{1}{(2-x)^2} \quad \text{in powers of } x:$$

$$\begin{aligned}\frac{1}{(2-x)^2} &= \left(\frac{1}{2-x}\right)' = \frac{1}{2}(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + ...)' \\ &= \frac{1}{2}(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + ...)' \\ &= \frac{1}{2}(\frac{1}{2} + \frac{2x}{4} + \frac{3x^2}{8} + ...)\end{aligned}$$

Question 13 (cont.)

$$\frac{1}{2}(\frac{1}{2}+\frac{2x}{4}+\frac{3x^3}{8}+...) = \frac{1}{4}(1+2\frac{x}{2}+3\left(\frac{x}{2}\right)^2+...)$$
$$= \frac{1}{4}\sum_{n=0}^{\infty}(n+1)\left(\frac{x}{2}\right)^n$$

$$a_n = \frac{n+1}{2^n}, \quad L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{n+2}{2^{n+1}}}{\frac{n+1}{2^n}} = \frac{1}{2}$$

Radius of convergence: $R = 2$, absolute convergence: $-2 < x < 2$.

Question 13 (cont.)

Check the endpoints:

$$x = -2 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (n+1)(-1)^n \quad \text{diverges.}$$

$$x = 2 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (n+1) \quad \text{diverges.}$$

So, the representation

$$\frac{1}{(2-x)^2} = \frac{1}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{x}{2}\right)^n$$

is valid on $(-2, 2)$.

Question 15

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ..., \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\ln(2-x) \quad \text{in powers of } x:$$

Question 15

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ..., \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\ln(2-x) \quad \text{in powers of } x:$$

$$\text{We know that} \quad -\int_0^x \frac{dt}{2-t} = \ln(2-x) - \ln 2$$

$$\ln(2-x) = \ln 2 - \int_0^x \frac{dt}{2-t}$$

Question 15 (cont.)

By Question 12,

$$\begin{aligned}
\ln(2-x) &= \ln 2 - \frac{1}{2}\int_0^x (1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \ldots)dt \\
&= \ln 2 - \frac{1}{2}(t + \frac{t^2}{2.2} + \frac{t^3}{3.4} + \frac{t^4}{4.8} + \ldots)\Big|_0^x \\
&= \ln 2 - (\frac{t}{2} + \frac{t^2}{2.2^2} + \frac{t^3}{3.2^3} + \frac{t^4}{4.2^4} + \ldots)\Big|_0^x \\
&= \ln 2 - \sum_{n=1}^{\infty} \frac{x^n}{2^n n}
\end{aligned}$$

$$a_n = \frac{1}{2^n n}, \quad L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{2^{n+1} n+1}}{\frac{1}{2^n n}} = \frac{1}{2}$$

Radius of convergence: $R = 2$, absolute convergence: $-2 < x < 2$.

Question 15 (cont.)

Check the endpoints:

$$x = -2 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by the Alternating series test.

$$x = 2 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

So, the representation

$$\ln(2-x) = \ln 2 - \sum_{n=1}^{\infty} \frac{x^n}{2^n n}$$

is valid on $[-2, 2)$.

Question 20

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ..., \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\ln x \quad \text{in powers of } x - 4:$$

Question 20

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ..., \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\ln x \quad \text{in powers of } x - 4:$$

First find $\frac{1}{x}$ in powers of $x-4$.

Question 20

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + ..., \quad -1 < x < 1.$$

Determine power series representations for the given function. On what interval is each representation valid?

$$\ln x \quad \text{in powers of } x - 4 :$$

First find $\frac{1}{x}$ in powers of $x - 4$.

$$\frac{1}{x} = \frac{1}{4 + (x-4)} = \frac{1}{4}\frac{1}{1 - \frac{4-x}{4}} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{4-x}{4}\right)^n.$$

We know that $\displaystyle\int_4^x \frac{dt}{t} = \ln x - \ln 4$

Question 20 (cont.)

$$
\begin{aligned}
\ln x &= \ln 4 + \int_4^x \frac{dt}{t} \\
&= \ln 4 + \frac{1}{4}\int_4^x (1 + \frac{4-t}{4} + \left(\frac{4-t}{4}\right)^2 + \left(\frac{4-t}{4}\right)^3 + \ldots)dt \\
&= \ln 4 + \frac{1}{4}(t - \frac{(4-t)^2}{2.4} - \frac{(4-t)^3}{3.4^2} - \ldots)\Big|_4^x \\
&= \ln 4 + \frac{1}{4}(x - 4 - \frac{(4-x)^2}{2.4} - \frac{(4-x)^3}{3.4^2} - \ldots) \\
&= \ln 4 - (\frac{4-x}{4} + \frac{(4-x)^2}{2.4^2} + \frac{(4-x)^3}{3.4^3} + \ldots) \\
&= \ln 4 - \sum_{n=1}^{\infty} \frac{(4-x)^n}{4^n n}
\end{aligned}
$$

Question 20 (cont.)

$$a_n = \frac{1}{4^n n} \quad \Rightarrow \quad L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{4^{n+1} n+1}}{\frac{1}{4^n n}} = \frac{1}{4}$$

Radius $=$ 4 , interval of absolute convergence:
$-4 < 4 - x < 4 \quad \Rightarrow \quad 0 < x < 8.$

Endpoints:

$$x = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$$x = 8 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges (by alternating series test).}$$

So, the representation

$$\ln x = \ln 4 - \sum_{n=1}^{\infty} \frac{(4-x)^n}{4^n n}$$

is valid on the interval $(0, 8]$.

Question 26

Determine the interval of convergence and the sum of

$$1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

Question 26

Determine the interval of convergence and the sum of

$$1-\frac{x^2}{2}+\frac{x^4}{3}-\frac{x^6}{4}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n+1}$$

$$a_n=\frac{(-1)^n}{n+1} \quad \Rightarrow \quad \text{Radius}=1, \quad \text{absolute convergence: } -1<x<1$$

Endpoints:

$$x=\pm 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{(-1)^n}{n} \quad \text{converges by alternating series test}$$

So the interval of the convergence is $[-1,1]$.

Question 26 (cont.)

Let $x \in [-1, 1]$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1} = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{n+1}}{n+1} = \frac{1}{x^2} \sum_{n=0}^{\infty} \int_0^{x^2} (-1)^n t^n$$

$$= \frac{1}{x^2} \int_0^{x^2} \sum_{n=0}^{\infty} (-t)^n = \frac{1}{x^2} \int_0^{x^2} \frac{1}{1+t}$$

$$= \frac{1}{x^2} \ln(1+x^2) \qquad \text{for } x \neq 0$$

So,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1} = \begin{cases} \ln(1+x^2) & x \neq 0, \\ 1 & x = 0. \end{cases}$$