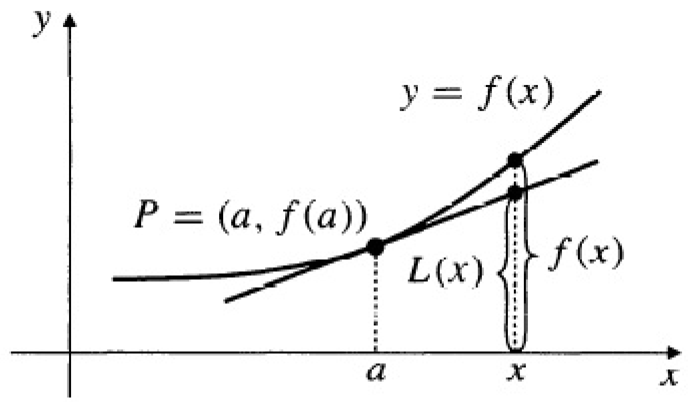
§12.6. Linear Approximations, Differentiability, and Differentials

The tangent line of the graph $y = f(x)$ at $x = a$ provides a convenient approximation for values of $f(x)$ for x near a:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$



The linearization of f at $x = a$

Here, $L(x)$ is the **linearization** of f at a; its graph is the tangent line to $y = f(x)$ there.

Similarly, the tangent plane to the graph of $z = f(x, y)$ at (a, b) is $z = L(x, y)$, where

$$f(x, y) \approx L(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b)$$

is the **linearization** of f at (a, b).

Example 1. Find an approximate value for $f(x, y) = \sqrt{2x^2 + e^{2y}}$ at $(2.2, -0.2)$.

It is convenient to use the linearization at $(2, 0)$, where the values of f and its partials are easily evaluated:

$$f_1(x, y) = \frac{2x}{\sqrt{2x^2 + e^{2y}}},$$

$$f_2(x, y) = \frac{e^{2y}}{\sqrt{2x^2 + e^{2y}}},$$

$$f(2, 0) = 3, f_1(2, 0) = \frac{4}{3}, f_2(2, 0) = \frac{1}{3}.$$

Thus, $L(x, y) = 3 + \frac{4}{3}(x - 2) + \frac{1}{3}(y - 0)$, and

$f(2.2, -0.2) \approx L(2.2, -0.2) = 3 + \frac{4}{3}(2.2 - 2) + \frac{1}{3}(-0.2 - 0) = 3.2$

Definition. We say that the function $f(x, y)$ is **differentiable** at the point (a, b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a + h, b + k) - f(a, b) - hf_1(a, b) - kf_2(a, b)}{\sqrt{h^2 + k^2}} = 0.$$

The function $f(x, y)$ is differentiable at the point (a,b) if and only if the surface $z = f(x, y)$ has a nonvertical tangent plane at (a, b). This implies that $f_1(a, b)$ and $f_2(a, b)$ must exist and that f

must be continuous at (a, b). In particular, the function is continuous whenever it is differentiable.

Theorem 1. If f_1 and f_2 are continuous in neighbourhood of the point (a, b), then f is differentiable at (a, b).

Differentials

If the first partial derivatives of a function $z = f(x_1, x_2, ..., x_n)$ exist at a point, we may construct a differential dz or df of the function at that point in a manner similar to that used for functions of one variable:

$$dz = df = \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + ... + \frac{\partial z}{\partial x_n} dx_n$$

$$= f_1(x_1, x_2, ..., x_n) dx_1 + ... + f_n(x_1, x_2, ..., x_n) dx_n.$$

Here, the differential dz is considered to be a function of the $2n$ independent variables $x_1, x_2, ..., x_n, dx_1, dx_2, ..., dx_n$.

For a differentiable function f, the differential df is an approximation to the change $\triangle f$ in value of the function given by,

$$df \approx \triangle f = f(x_1 + dx_1, ..., x_n + dx_n) - f(x_1, x_2, ..., x_n).$$

The error in this approximation is small compared with the distance between the two points in the domain of f; that is,

$$\frac{\triangle f - df}{\sqrt{(dx_1)^2 + ... + (dx_n)^2}} \to 0$$

if all $dx_i \to 0$, $(1 \le i \le n)$.

In this sense, differentials are just another way of looking at linearization.

Example 2. Estimate the percentage change in the period $T = 2\pi\sqrt{\frac{L}{g}}$ of a simple pendulum if the length, L, of the pendulum increases by 2 percent and acceleration of gravity, g, decreases by 0.6 percent.

We calculate the differential of T:

$$dT = \frac{\partial T}{\partial L}dL + \frac{\partial T}{\partial g}dg$$

$$= \frac{2\pi}{2\sqrt{Lg}}dL - \frac{2\pi\sqrt{L}}{2g^{3/2}}dg.$$

We know that $dL = \frac{2}{100}L$ and $dg = -\frac{6}{1000}g$. Thus,

$$dT = \frac{1}{100}2\pi\sqrt{\frac{L}{g}} - (-\frac{6}{1000})\frac{2\pi}{2}\sqrt{\frac{L}{g}} = \frac{13}{1000}T.$$

Therefore, the period T of the pendulum increases by 1.3 percent.