GTU, Fall 2020, MATH 101

Limits of Functions

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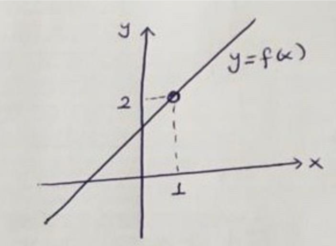
We say $f(x)$ approaches the limit 2 as x approaches 1.



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$*$ Limits are unique;
if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.

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We will see that if f is an elementary function defined at a then

$$\lim_{x\to a} f(x) = f(a).$$

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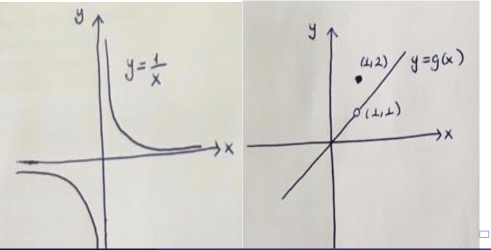

One-Sided Limits

Right-limit: $\lim_{x \to a+} f(x) = L$ is known as the right-limit and means that you should use values of x that are greater than a (to the right of a on the real line) to compute the limit.

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Relationship between one-sided and two-sided limits:

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Relationship between one-sided and two-sided limits:

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = L.$$

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* If m is an integer and n is a positive integer , then

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provided $L > 0$ if n is even, and $L \neq 0$ if $m < 0$.

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Remark: These rules are also valid for right limits and left limits.

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2. If $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$, then

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$$

Example

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b) $lim_{x\to 3}\sqrt{2x+3} = \sqrt{2.3+3} = 3$

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Example

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$\lim_{x\to 1}\frac{1}{(x+1)(\sqrt{x}+1)} = \frac{1}{4}.$

Example

If $f(x) = \frac{|x-3|}{x^2-x-6}$ find $\lim_{x\to 3+} f(x)$, $\lim_{x\to 3-} f(x)$ and $\lim_{x\to 3} f(x)$.

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Observe that

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Limits of Functions

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The limit does not exist since $\lim_{x\to 3+} f(x) \neq \lim_{x\to 3-} f(x)$.

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Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. If

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$\implies$ By Squeeze Theorem, $\lim_{x \to 2} f(x) = -1$.

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