§9.5 POWER SERIES

Definition. (**Power Series**) Let $\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ...$ This form series is called a power series of $(x-c)$ or a power series about c. $a_0, a_1, a_2, a_3...$ are called the coeeficients of the power series. c is called the centre of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$.

Note that a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges at $x = c$. Also, it is clear that each term of a power series is a function of x and so the series may converge or diverges for each x. Thus, we say that the series is a function of x such that the series converges for those values of x.

For instance, we consider the power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$ Actually, this series presents a geometric series with $r = x$. Then it converges for all x in $(-1, 1)$ ($|r| = |x| < 1$) and it converges to $\frac{1}{1-x}$, that is; the power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$ is defined by the function $\frac{1}{1-x}$ for the interval $-1 < x < 1$. But the series diverges for $|x| \geq 1$ and so the function can not represent the series.

Theorem 1. One of the following cases holds for a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$:

i. The series may converge only at $x = c$.

ii. The series may converge for all $x \in \mathbb{R}$.

iii. There may be a positive real number R such that the series converges for all x with $|x - c| < R \Rightarrow (c - R, c + R)$ and diverges for all x with $|x - c| > R$.

But the series may or not may converges at the endpoints of $(c - R, c + R)$. Consequently, we say that interval of convergence of a power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ may be of the point c, $(-\infty, \infty)$, $[c - R, c + R]$, $[c - R, c + R)$, $(c - R, c + R]$ or $(c - R, c + R)$. We call the number R the radius of convergence of the power series.

Then the radius of convergence of a power series may be of $R = 0$, $R = \infty$.

We can find the radius of convergence of $\sum_{n=0}^{\infty} a_n(x - c)^n$ by ratio test:

$\rho = \lim_{n\to\infty} |\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}| = (\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|)|x - c|$.

If $\rho < 1$, then $|x - c| < \frac{1}{\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|} = R$ such that the series converges for all x.

Briefly, let $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$. The the radius of convergence of the power series is of $R = \frac{1}{L}$.

If $L = 0$, then $R = \infty$, that is, the series converges on $\mathbb{R}$.

If $L = \infty$, then $R = 0$, that is, the series converges at only the point c.

Example 1. Determine the centre, radius and the interval of convergence of the following series.

i. $\sum_{n=0}^{\infty} \frac{(3x+2)^n}{2^n(n+1)}$ ii. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(2n)!}$ iii. $\sum_{n=0}^{\infty} n!(x-2)^n$

i. $\sum_{n=0}^{\infty} \frac{(3x+2)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} (\frac{3}{2})^n \frac{1}{(n+1)}(x+\frac{2}{3})^n$. The centre of convergence is $c = -\frac{2}{3}$. Then $L = \lim_{n\to\infty} \frac{(\frac{3}{2})^{n+1}\frac{1}{(n+2)}}{(\frac{3}{2})^n\frac{1}{(n+1)}} = \lim_{n\to\infty} \frac{3}{2}(\frac{n+1}{n+2}) = \frac{3}{2}$ and so $R = \frac{2}{3}$. Thus, we get that the series converges on $(-\frac{4}{3}, 0)$ and diverges $(-\infty, -\frac{4}{3})$ and $(0, \infty)$. Now, we will check the series at the endpoints $x = -\frac{4}{3}$, $x = 0$. Let $x = -\frac{4}{3}$. Then $\sum_{n=0}^{\infty} (\frac{3}{2})^n \frac{1}{(n+1)}(-\frac{4}{3}+\frac{2}{3})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}$ converges by alternating series test.

Let $x = 0$. Then $\sum_{n=0}^{\infty} (\frac{3}{2})^n \frac{1}{(n+1)}(0+\frac{2}{3})^n = \sum_{n=0}^{\infty} \frac{1}{(n+1)}$ diverges since it is harmonic series.

Thus the series converges on $[-\frac{4}{3}, 0)$.

ii. Consider $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(2n)!}$. The centre of convergence is $c = -2$. Then $L = \lim_{n\to\infty} \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0$ and so $R = \infty$. Thus the series converges for all real numbers.

iii. The centre of convergence is $c = 2$. $L = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$ and so $R = 0$, that is, the series converges only at $c = 2$.

Algebraic operations on power series

Consider the following power series with centre of convergence 0.

$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$

A power series of the form $\sum_{n=0}^{\infty} a_n (y-c)^n$ has any properties satisfied by $\sum_{n=0}^{\infty} a_n x^n$ because the series $\sum_{n=0}^{\infty} a_n (y-c)^n$ is obtained by changing of variable $x = y - c$.

Theorem 2. Assume that $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are two power series with radius of convergence R_a and R_b, respectively and $c \in \mathbb{R}$.

1. $\sum_{n=0}^{\infty} (ca_n) x^n = c \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R_a and converges on its interval of convergence.

2. $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ has radius of convergence R such that $R \geq min\{R_a, R_b\}$.

Theorem 3. (**Cauchy product of the series**) Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series. Then $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = a_0 b_n + a_1 b_{n-1} + ... + a_n b_0 = \sum_{j=0}^{n} a_j b_{n-j}$. The series $\sum_{n=0}^{\infty} c_n x^n$ is called the Cauchy product of these two series.

Example 2. We know that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$. Now, we will find a power series representation $\frac{1}{(1-x)^2}$ with the Cauchy product of series. Then $(1 + x + x^2 + x^3 + ...)(1 + x + x^2 + x^3 + ...) = 1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=0}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2}$ for $-1 < x < 1$.

Differentiation and Integration of Power Series

Let a power series has a positive radius of convergence. Then it is differentiated or integrated term by term. It is important that the resulting series will converge to the derivative or integral of the sum of the original series everywhere except possibly at the points of the interval of convergence of the original series.

Theorem 4. (**Term by term differentiation and integration of a power series**)

Let the series $\sum_{n=0}^{\infty} a_n x^n$ converge to the function $f(x)$ on an interval $(-R, R)$ with $R > 0$. Then

if $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$ is differentiated on $(-R, R)$, then we get $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ...$ on $(-R, R)$.

Also, if $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$ is integrated on any closed subinterval of $(-R, R)$, then we get $\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + ...$

Theorem 5. (**Abel's Theorem**)

A power series is a continuous function on the interval of convergence of the series. Particularly, if $\sum_{n=0}^{\infty} a_n R^n$ converges for some $R > 0$, then $\lim_{x \to R^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n$ and if $\sum_{n=0}^{\infty} a_n (-R)^n$ converges then $\lim_{x \to -R^+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (-R)^n$.

The following examples show that how the above theorems are applied to get power series representation for functions.

Example 3. Find the power series representations for the following functions.

i. $\frac{1}{(1-x)^2}$ ii. $ln(1+x)$.

Remember that $\frac{1}{1-x} = 1+x+x^2+x^3+... = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$.

i. For $\frac{1}{(1-x)^2}$, we differentiate $f(x) = \frac{1}{1-x}$ for $-1 < x < 1$. Then $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=1}^{\infty} nx^{n-1}$ for $-1 < x < 1$.

ii. Let's replace $-t$ to x in the series $\sum_{n=0}^{\infty} x^n$. Then we get $\frac{1}{1+t} = 1 - t + t^2 - t^3 + ... = \sum_{n=0}^{\infty}(-1)^n t^n$ for $-1 < t < 1$.

Integrate it from 0 to x:

$ln(1+x) = \int_0^x \frac{1}{1+t}dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^n dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ $(-1 < x \leq 1)$. The function $ln(1+x)$ is continuous at $x = 1$ thus the series representation for it must be convergence by Abel's theorem. Therefore, $ln2 = 1 - \frac{1}{2} + \frac{1}{3} - ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

Example 4. Find a power series representation for $arctanx$.

We know that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$.

In this series, when we put $-t^2$ in place of x, we get

$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6... = \sum_{n=0}^{\infty}(-1)^n t^{2n}$ for $-1 < t < 1$ since $0 < t^2 < 1$. Now, we integrate from 0 to x:

$arctanx = \int_0^x \frac{1}{1+t^2}dt = \int_0^x(1-t^2+t^4-t^6...)dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... =$

$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ for $-1 < x < 1$. For $x = -1$ and $x = 1$, we can see that the series converges by alternating series test, that is; the interval of convergence series is $[-1, 1]$.

Example 5. Find the function representation for $\sum_{n=1}^{\infty} n^2x^n$ and calculate the sum of $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$.

Note that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=1}^{\infty} nx^{n-1}$. Multiply it with x:

$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + ... = \sum_{n=1}^{\infty} nx^n$. Then differentiate this new series:

$\frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + ... = \sum_{n=1}^{\infty} n^2x^{n-1}$. Again, multiply with x:

$\frac{x(1+x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + ... = \sum_{n=1}^{\infty} n^2x^n$ for $-1 < x < 1$.

Since $\frac{1}{2} \in (-1, 1)$, then we can put it to the new series:

$\sum_{n=1}^{\infty} n^2(\frac{1}{2})^n = \sum_{n=1}^{\infty}\frac{n^2}{2^n} = \frac{\frac{1}{2}\frac{3}{2}}{\frac{1}{8}} = 6$.

Example 6. Find a series representation of $\frac{1}{2+x}$ in powers of $(x-1)$.

Let $t = x - 1$ If we put $t + 1$ in place of x in the function, we get $\frac{1}{2+x} = \frac{1}{3+t} = \frac{1}{3}\frac{1}{1+\frac{t}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{3^n} = \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{3^{n+1}}$ for $-1 < \frac{t}{3} < 1 \Rightarrow -3 < t < 3$. Thus, $\frac{1}{2+x} = \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{3^{n+1}}$ for $-2 < x < 4$.