§9.1. Sequences and Convergence

Sequences are fundamental to the study of series(or infinite series) and many applications of mathematics. Now, we will explore general sequence of numbers and the conditions under which they converge.

Definition. A sequence is a function which is defined from positive integer numbers $\mathbb{N}^+$ to real numbers $\mathbb{R}$.

$f : \mathbb{N}^+ \to \mathbb{R}$ such that $f(n) = a_n$.

1. The range of f may be denoted by $\{a_1, a_2, ..., a_n, ...\}$.
2. These elements a_i's are called the terms of the sequence.
3. The general term a_n of a sequence is a general formula which computes all terms of the sequence.
4. The sequence, whose general term is a_n, is denoted by $\{a_n\}$ or (a_n).

That is, a sequence $\{a_n\} = \{a_1, a_2, ..., a_n, ...\}$ is a function from $\mathbb{N}^+$ to $\mathbb{R}$.

Example 1. i. $\{\sqrt{n}\} = \{1, \sqrt{2}, \sqrt{3}, 2, ..., \sqrt{n}, ...\}$ is a sequence whose general term is $a_n = \sqrt{n}$. Actually, $a_n = f(n) = \sqrt{n}$ for $n =$

$1, 2, 3, ...$

ii. $\{(-1)^{n-1}\} = \{cos((n-1)\pi)\} = \{1, -1, 1, -1, ...\}$ is a sequence whose general term is $a_n = (-1)^{n-1}$ or $a_n = cos((n-1)\pi)$.

iii. $\{1/n\} = \{1, 1/2, 1/3, ..., 1/n, ...\}$ is a sequence whose general term is $a_n = 1/n$.

Sometimes, general terms of sequences can not be defined as an explicit function of n. This kind of sequences is called recursively (or inductively), that is, each term is computed from the previous ones rather than as a function of n.

Example 2. i. Consider the sequence $\{a_n\}$ where $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$ for each $n = 1, 2, 3, ...$ Then $\{a_n\} = \{1, \sqrt{7}, \sqrt{6 + \sqrt{7}}, ...\}$.

ii. $\{a_n\} = \{1, 1, 2, 3, 5, ...\}$ is called Fibonacci sequence whose terms, exterior first two terms, are the sum of the previous two terms.

Now, we will describe sequences by terms.

Definition. 1. The sequence $\{a_n\}$ is bounded below by L if $a_n \geq L$ for each $n \in \mathbb{N}^+$. Then L is called a lower bound for $\{a_n\}$. The sequence $\{a_n\}$ is bounded above by M if $a_n \leq M$ for each $n \in \mathbb{N}^+$. Then M is called an upper bound for $\{a_n\}$.

Also, we say that $\{a_n\}$ is bounded if it has both a lower and an upper bounds. In this case, if $\{a_n\}$ is a bounded sequence, then there is a constant U such that $|a_n| \leq U$ for each $n \in \mathbb{N}^+$.

2. The sequence $\{a_n\}$ is positive if $a_n \geq 0$ for each $n \in \mathbb{N}^+$.

   The sequence $\{a_n\}$ is negative if $a_n \leq 0$ for each $n \in \mathbb{N}^+$.

3. The sequence $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ for each $n \in \mathbb{N}^+$.

   The sequence $\{a_n\}$ is decreasing if $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}^+$.

   If a sequence is either increasing or decreasing, then it is called monotonic.

4. The sequence $\{a_n\}$ is alternating if $a_n a_{n+1} < 0$ for each $n \in \mathbb{N}^+$.

   Note that each term of an alternating sequence must be non zero.

Example 3. i. $\{3^{n-1}\} = \{1, 3, 9, ...\}$ is positive, increasing and bounded below with 1 since $a_n \geq 1$ for all $n \in \mathbb{N}^+$.
ii. $\{1 - 2n\} = \{-1, -3, -5, ...\}$ is negative, decreasing and bounded above with -1 since $a_n \leq -1$ for all $n \in \mathbb{N}^+$.
iii. Consider the sequence $\{\frac{3n+4}{n+1}\}$. It is bounded since:
$\frac{3n+4}{n+1} = 3 + \frac{1}{n+1} \geq 3$ and $\frac{3n+4}{n+1} = 3 + \frac{1}{n+1} \leq 3 + \frac{1}{2} = \frac{7}{2}$ (since $\frac{1}{n+1} \leq \frac{1}{2}$) for each $n \in \mathbb{N}^+$, that is, $3 \leq a_n \leq \frac{7}{2}$.
iv. $\{(-\frac{1}{2})^n\} = \{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, ...\}$ is alternating and bounded. It is clear that $a_n a_{n+1} < 0$ for each $n \in \mathbb{N}^+$. Since $a_n \geq -\frac{1}{2}$ and $a_n \leq \frac{1}{4}$ for each

n, then it is bounded.

v. Note that $\{(-1)^n(n+1)\} = \{-2, 3, -4, 5, ...\}$ is alternating but is not bounded.

Remark. If we show that $a_{n+1} - a_n \geq 0$ for each n, then we say that $\{a_n\}$ is increasing. Also, if we show that $\frac{a_{n+1}}{a_n} \geq 1$ for each n and each $a_n \neq 0$, then $\{a_n\}$ is increasing. Furthermore, if we show that $a_{n+1} - a_n \leq 0$ for each n (or $\frac{a_{n+1}}{a_n} \leq 1$ for each n and each $a_n \neq 0$), then we say that $\{a_n\}$ is decreasing.

Remark. Let f be a differentiable function on $(1, \infty)$ and $a_n = f(n)$ for each $n \in \mathbb{N}^+$. We say that $\{a_n\}$ is increasing (decreasing) if f is increasing (decreasing) on $[1, \infty)$. We know that if $f'(x) \geq 0$ ($f'(x) \leq 0$) for each $x \in [1, \infty)$, then f is increasing (decreasing) on $[1, \infty)$.

Example 4. Show that $\{\frac{n}{n^2+1}\}$ is decreasing.

$f(n) = a_n = \frac{n}{n^2+1}$. Consider $f(x) = \frac{x}{x^2+1}$ for $x \in [1, \infty)$. $f'(x) = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \in [1, \infty)$. Thus, f is decreasing on $x \in [1, \infty)$ and so $\{\frac{n}{n^2+1}\}$ is decreasing.

Definition. The sequence $\{a_n\}$ is ultimately increasing if $a_n \leq a_{n+1}$ for each $n \geq K$, for some $K \in \mathbb{N}^+$.

The sequence $\{a_n\}$ is ultimately decreasing if $a_n \geq a_{n+1}$ for each

$n \geq K$, for some $K \in \mathbb{N}^+$.

The sequence $\{a_n\}$ is ultimately positive if $a_n \geq 0$ for each $n \geq K$, for some $K \in \mathbb{N}^+$.

The sequence $\{a_n\}$ is ultimately negative if $a_n \leq 0$ for each $n \geq K$, for some $K \in \mathbb{N}^+$.

The sequence $\{a_n\}$ is ultimately alternating if $a_n a_{n+1} < 0$ for each $n \geq K$, for some $K \in \mathbb{N}^+$.

Example 5. 1. The sequence $\{\frac{n^2}{2^n}\} = \{\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, ...\}$ is positive. Since $a_{n+1} \leq a_n$ for $n \geq 3$, then it is ultimately decreasing.

2. $\{n - 100\} = \{-99, -98, ..., -1, 0, 1, 2, ...\}$ is ultimately positive since $a_n \geq 0$ for $n \geq 99$.

3. $\{(-1)^n + \frac{4}{n}\} = \{3, 3, \frac{1}{3}, 2, -\frac{1}{5}, \frac{5}{3}, -\frac{3}{7}, \frac{3}{2}...\}$ is ultimately alternating since $a_n a_{n+1} < 0$ for each $n \geq 4$.

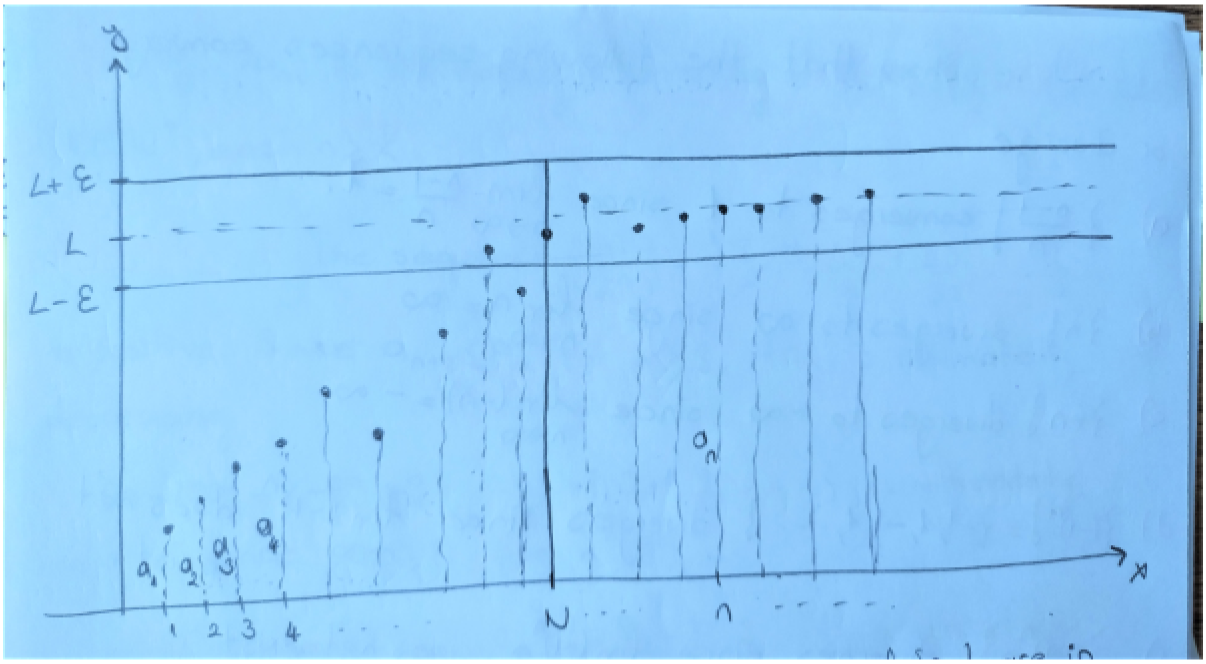
§Convergence of Sequences

Definition. It is said that the sequence $\{a_n\}$ converges to the number L if $\lim_{n\to\infty} a_n = L$.

If $\{a_n\}$ does not converge, then it diverges, that is:

$$\{a_n\} \text{ diverges if } \begin{cases} \lim_{n\to\infty} a_n \text{ does not exist}, \\ \\ \lim_{n\to\infty} a_n = \infty \text{ or } -\infty. \end{cases}$$

Definition. (Formal Definition) We say that the sequence $\{a_n\}$ converges to the number L if for every positive real number ϵ, there is an integer N such that if $n \geq N$, then $|a_n - L| \leq \epsilon$.



According to the above graphic, all terms of $\{a_n\}$ are in $(L-\epsilon, L+\epsilon)$ except for finite number terms.

Example 6. Show that $\{\frac{1}{n}\}$ converges to 0.

Let $\epsilon \geq 0$ be given. It must be shown that there exists an integer N such that $n \geq N$ for each n and $|\frac{1}{n} - 0| = \frac{1}{n} \leq \epsilon$. Thus, $n \geq \frac{1}{\epsilon}$. Thus, $N \geq \frac{1}{\epsilon}$, that is, for given ϵ we have a positive N corresponds ϵ.

Example 7. Show that the following sequences converge or diverge.

1. $\{-\frac{1}{n}\}$ converges to 0 since $\lim_{n\to\infty}(-\frac{1}{n}) = 0$.

2. $\{\frac{n^3}{5-3n^3}\}$ converges to $-\frac{1}{3}$ since $\lim_{n\to\infty}\frac{n^3}{5-3n^3} = -\frac{1}{3}$.

3. $\{n\}$ diverges to ∞ since $\lim_{n\to\infty} n = \infty$.

4. $\{(-1)^n n^2\}$ diverges since $\lim_{n\to\infty}(-1)^n n^2$ does not exist.

Theorem 1. Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences. Then:

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$, $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.

2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.

3. $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ where $\lim_{n\to\infty} b_n \neq 0$.

4. If $a_n \leq b_n$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

5. (**Squeeze Theorem**) If $a_n \leq c_n \leq b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$.

Example 8. Determine whether the following sequences converge or diverge.

1. $\{\frac{cosn}{n}\}$:

   for each n, $-1 \leq cosn \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{cosn}{n} \leq \frac{1}{n}$. Since $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} \frac{cosn}{n} = 0$ by Squeeze Theorem.

2. $\{\frac{lnn}{n}\}$ converges to 0 since $\lim_{x\to\infty} \frac{lnx}{x} = \lim_{x\to\infty} \frac{lnx}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$ by taking account of $f(x) = \frac{lnx}{x}$ and by I'Hospital's Rule.

3. $\{ntan^{-1}(\frac{1}{n})\}$ converges to 1 in a similar way Example 8(2).

4. $\{(1+\frac{1}{n})^n\}$ converges to e since $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

Theorem 2. Every convergent sequence is bounded.

But the converse of the theorem is not true. For example, let's look at the sequence $\{(-1)^n\} = \{-1, 1, -1, 1, ...\}$. The sequence is bounded but it diverges.

Theorem 3. Bounded monotonic sequences converge.

Example 9. Consider the sequence $\{a_n\}$ which $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$ for each $n \in \mathbb{N}^+$. Show that it converges.
By the previous theorem, if $\{a_n\}$ is bounded monotonic, then it is convergent. So we must show that it is bounded and increasing.

We use induction to show that it is bounded and increasing.

Mathematical induction: If a statement $P(n)$ holds the following for every $n \in \mathbb{N}$, then it is proved.

1. Show that P holds for $n = 0$ (or $n = 1$),
2. Assume that P satisfies for $n = k$,
3. Show that P holds for $n = k + 1$ by using the first and the second cases.

Firstly, let's show that it is increasing. Let $P(n) = a_{n+1} \geq a_n$.

1. For $n = 1$, $P(1) = a_2 = \sqrt{7} \geq a_1 = 1$.

2. Assume that $a_{k+1} \geq a_k$, that is, $P(k)$ holds.

3. $P(k+1) = a_{k+2} = \sqrt{6 + a_{k+1}} \geq \sqrt{6 + a_k} = a_{k+1}$.

Thus, $\{a_n\}$ is increasing.

Now, we must show that it is bounded. Since $a_n \geq 1$ for each n and $\{a_n\}$ is increasing, then it is bounded below. Let $P(n) = a_n \leq 3$. By induction:

1. $P(1) = a_1 = 1 \leq 3$.

2. Assume that $P(k) = a_k \leq 3$ for $n = k$.

3. $P(k+1) = a_{k+1} = \sqrt{6 + a_k} \leq \sqrt{6+3} = 3$.

Thus, it is bounded above. Consequently, it converges. Now, we will find the number which to the sequence converges. Let $\lim_{n\to\infty} a_n = a$. It is clear that $\lim_{n\to\infty} a_{n+1} = a$. Thus $a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6+a_n} = \sqrt{6+\lim_{n\to\infty} a_n} = \sqrt{6+a}$ and so $a = 3$ or $a = -2$. a must be 3 since $\{a_n\}$ is increasing where all terms is greater than or equals to 1, that is, $\lim_{n\to\infty} a_n = 3$.

We need the following theorem in the studies on series.

Theorem 4. 1. If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

2. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every $x \in \mathbb{R}$.

Example 10. $\lim_{n\to\infty} \frac{3^n+4^n+5^n}{5^n} = \lim_{n\to\infty}[(\frac{3}{5})^n + (\frac{4}{5})^n + 1] = 1$.