§12.3. Partial Derivatives

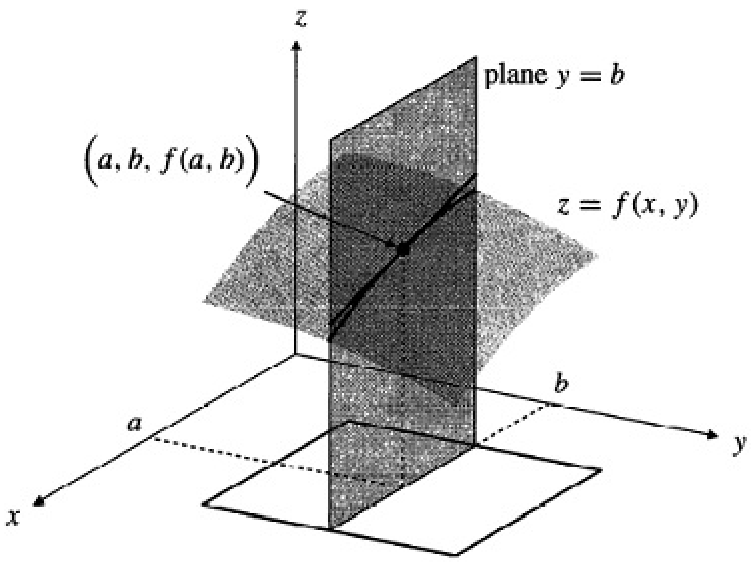
Definition. The first partial derivatives of the function $f(x, y)$ with respect to the variables x and y are the functions $f_1(x, y)$ and $f_2(x, y)$ given by,

$$f_1(x) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

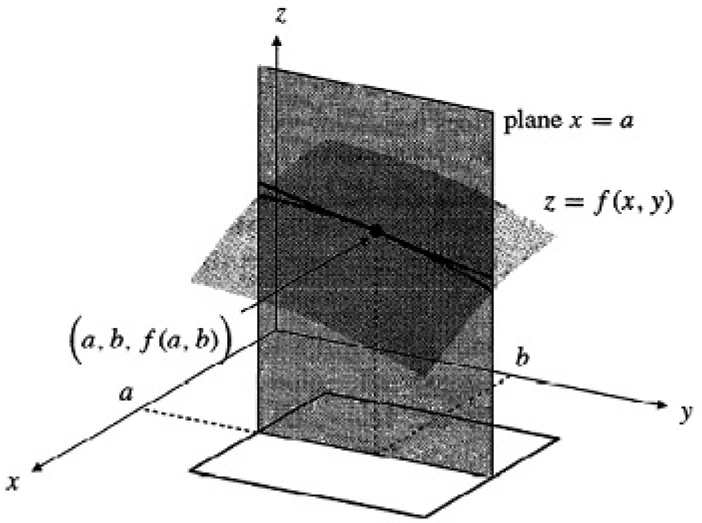
$$f_2(x) = \lim_{h \to 0} \frac{f(x, y+k) - f(x, y)}{k},$$

provided these limits exists.

The partial derivatives $f_1(a, b)$ measures the rate of change of $f(x, y)$ with respect to x at $x = a$ while y is held fixed at b. In graphical terms, the surface $z = f(x, y)$ intersects the vertical plane $y = b$ in a curve $z = f(x, b)$ whose slope at $x = a$ is $f_1(a, b)$.
Similarly, $f_2(a, b)$ represents the rate of change of $f(x, y)$ with respect to y at $y = b$ while x is held fixed at a. The surface $z = f(x, y)$ intersects the vertical plane $x = a$ in a curve $z = f(a, y)$ whose slope at $y = b$ is $f_2(a, b)$.



$f_1(a, b)$ is the slope of the curve of intersection of $z = f(x, y)$ and the vertical plane $y = b$ at $x = a$



$f_2(a, b)$ is the slope of the curve of intersection of $z = f(x, y)$ and the vertical plane $x = a$ at $y = b$

Notations for first derivatives

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_1(x,y) = D_1 f(x,y) = D_x f(x,y)$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_2(x,y) = D_2 f(x,y) = D_y f(x,y)$$

The symbol $\frac{\partial}{\partial x}$ can be read as "partial with respect to x". The reason for distinguishing ∂ from the d of ordinary derivatives of single-variable functions will be clear later.

Example 1. Find $f_1(0,\pi)$ if $f(x,y) = e^{xy}\cos(x+y)$.
Let's first find $f_1(x,y) = ye^{xy}\cos(x+y) - e^{xy}\sin(x+y)$, now we can compute $f_1(0,\pi) = \pi e^0\cos(\pi) - e^0\sin(\pi) = -\pi$

The single-variable version of the Chain Rule also continues to apply to,say, $f(g(x,y))$, where f is a function of only one variable having derivative f':

$$\tfrac{\partial}{\partial x} f(g(x,y)) = f'(g(x,y))g_1(x,y),\ \tfrac{\partial}{\partial y} f(g(x,y)) = f'(g(x,y))g_2(x,y).$$

Example 2. If f is an everywhere differentiable function of one variable, show that $z = f(x/y)$ satisfies the partial differential equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

By the (single variable) Chain Rule,

$$\frac{\partial z}{\partial x} = f'(\frac{x}{y})(\frac{1}{y})$$

and

$$\frac{\partial z}{\partial y} = f'(\frac{x}{y})(\frac{-x}{y^2}).$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = f'(\frac{x}{y})(x \times \frac{1}{y} + y \times \frac{-x}{y^2}) = 0$$

.

The definition of partial derivatives for functions of two variables can be extend for functions of more than two variables.

Example 3.

$$\frac{\partial}{\partial z}(\frac{2xy}{1+xz+yz}) = -\frac{2xy}{(1+xz+yz)^2}(x+y).$$

Note that, all the standard differentiation rules are applied to calculate partial derivatives.

Remark. If a single-variable function $f(x)$ has a derivative $f'(a)$ at $x = a$, then f is necessarily continuous at $x = a$. This property does not extend to partial derivatives. Even if all the first partial derivatives of a function of several variables exist at a point, the function may still fail to be continuous at that point.

Tangent Planes and Normal Lines If the graph of $z = f(x, y)$ is a "smooth" surface near the point P with coordinates $(a, b, f(a, b))$, then that graph will have a **tangent plane** and **normal line** at P. The normal line is the line through P that is perpendicular to the surface. Any nonzero vector that is parallel to the normal line at P is called a normal vector to the surface at P. The tangent plane to the surface $z = f(x, y)$ at P is the plane through P that is perpendicular to the normal line at P.

Let us assume that the surface $z = f(x, y)$ has a nonvertical tangent plane (and therefore a nonhorizontal normal line) at point P. The tangent plane intersects the vertical plane $y = b$ in a straight line that is tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $y = b$. This line has slope $f_1(a, b)$, so it is parallel to the vector $\overrightarrow{T}_1 = \overrightarrow{i} + f_1(a, b)\overrightarrow{k}$. Similarly, the tangent plane intersects the vertical plane $x = a$ in a straight line having slope $f_2(a, b)$. This line is therefore parallel to the vector $\overrightarrow{T}_2 = \overrightarrow{j} + f_2(a, b)\overrightarrow{k}$. It follows that the tangent plane, and therefore the surface $z = f(x, y)$ itself, has normal vector

$$\overrightarrow{n} = \overrightarrow{T}_2 \times \overrightarrow{T}_1 = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 1 & f_2(a,b) \\ 1 & 0 & f_1(a,b) \end{vmatrix} = f_1(a,b)\overrightarrow{i} + f_2(a,b)\overrightarrow{j} - \overrightarrow{k}.$$

A normal vector to $z = f(x, y)$ at $(a, b, f(a, b))$ is

$$\overrightarrow{n} = f_1(a,b)\overrightarrow{i} + f_2(a,b)\overrightarrow{j} - \overrightarrow{k}.$$

Since the tangent plane passes through $P = (a, b, f(a, b))$ it has equation

$$f_1(a,b)(x-a)+f_2(a,b)(y-b)-(z-f(a,b))=0,$$

or, equivalently,

$$z=f(a,b)+f_1(a,b)(x-a)+f_2(a,b)(y-b).$$

The normal line to $z=f(x,y)$ at $(a,b,f(a,b))$ has a direction vector $\overrightarrow{n}$ and so has equations

$$\frac{x-a}{f_1(a,b)}=\frac{y-b}{f_2(a,b)}=\frac{z-f(a,b)}{-1}.$$

Example 4. Find the tangent plane and normal line to the graph $z=9-x^2-y^2$ at the point $P=(1,2,4)$.

$\frac{\partial z}{\partial x}=-2x$ and $\frac{\partial z}{\partial y}=-2y$ at point $(1,2)$ we have $\frac{\partial z}{\partial x}=-2$ and $\frac{\partial z}{\partial y}=-4$. Therefore, the surface has normal vector $\overrightarrow{n}=-2\overrightarrow{i}-4\overrightarrow{j}-\overrightarrow{k}$ and the tangent plane

$$z=4-2(x-1)-4(y-2),$$

or more simply, $2x + 4y + z = 14$. The normal line has equation

$$\frac{x-1}{-2} = \frac{y-2}{-4} = \frac{z-4}{-1}.$$