MATH 101.2 PS-7

Q1) How fast is the surface area of a cube changing when the volume of the cube is $64\ cm^3$ and is increasing at $2\ cm^3/s$?

Sol: The volume V, surface area S and edge length x of a cube are related by

$V = x^3$ and $S = 6x^2$. Hence, (x depends on time $=t$)

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}, \quad \frac{dS}{dt} = 12x.\frac{dx}{dt}.$$

If $V = 64cm^3$ and $\frac{dV}{dt} = 2\ cm^3/s$, then $x^3 = 64 \Rightarrow x = 4$

and $2 = 3.4^2.\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{1}{24}$. Therefore

$\frac{dS}{dt} = 12x.\frac{dx}{dt} = 12.4.\frac{1}{24} = 2$. The surface area is increasing at $2\ cm^2/s$.

Q2) The volume of right circular cylinder is $60\ cm^3$ and is insreasing at $2\ cm^3/min$ at a time when the radius is $5cm$ and increasing at $1cm/min$. How fast is the height of the cylinder changing at that time?

Sol: Let V, r and h denote the volume, radius and height of the cylinder at time t. Thus,

$V = \pi r^2 h$ and $\frac{dV}{dt} = \pi.\frac{d}{dt}(r^2 h) = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$.

If $V = 60$, $\frac{dV}{dt} = 2$, $r = 5$, $\frac{dr}{dt} = 1$, then we write

$$h = \frac{V}{\pi r^2} = \frac{60}{25\pi} = \frac{12}{5\pi} \quad \text{and} \quad \frac{dh}{dt} = \frac{1}{\pi r^2}\left(\frac{dV}{dt} - 2\pi r h \frac{dr}{dt}\right)$$

$$= \frac{1}{25\pi}\left(2 - 10\pi \frac{12}{5\pi}\right) = -\frac{22}{25\pi}$$

The height is decreasing at the rate $\frac{22}{25\pi}$ cm/min.

Q3) The point P moves so that at time t it is at the intersection of the curves $xy=t$ and $y=tx^2$. How fast is the distance of P from the origin changing at time $t=2$?

Sol: We have $xy=t \Rightarrow x\frac{dy}{dt}+y\frac{dx}{dt}=1$ and

$$y=tx^2 \Rightarrow \frac{dy}{dt}=x^2+2xt\frac{dx}{dt}.$$

At $t=2$ we have $xy=2,\ y=2x^2 \Rightarrow 2x^3=2 \Rightarrow x=1,\ y=2.$

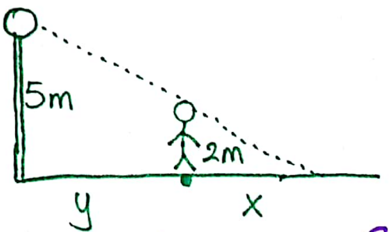
Thus, $\frac{dy}{dt}+2\frac{dx}{dt}=1$, and $1+4\cdot\frac{dx}{dt}=\frac{dy}{dt}$.

So, $1+6\frac{dx}{dt}=1 \Rightarrow \frac{dx}{dt}=0 \Rightarrow \frac{dy}{dt}=1$. Then, Distance D from origin satisfies $D=\sqrt{x^2+y^2}$. So,

$$\frac{dD}{dt}=\frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)=\frac{1}{\sqrt{5}}(1.(0)+2.(1))=\frac{2}{\sqrt{5}}.$$

Q4) A man 2m tall walks toward a lamppost on level ground at a rate of 0.5 m/s. If the lamp is 5m high on the post, how fast is the length of the man's shadow decreasing when he is 3m from the post? How fast is the shadow of his head moving at that time?

Sol:



Let x and y be the distances shown in the figure. From similar triangles, we can write

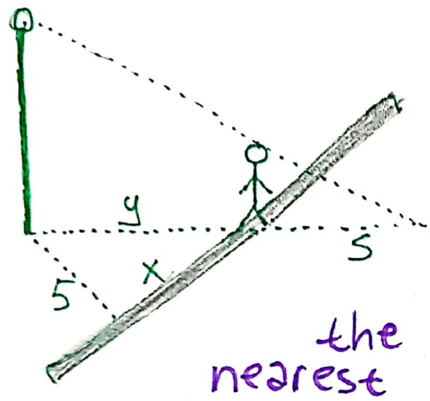
$$\frac{x}{2}=\frac{x+y}{5} \Rightarrow x=\frac{2y}{3} \Rightarrow \frac{dx}{dt}=\frac{2}{3}\frac{dy}{dt}.$$

Since $\frac{dy}{dt}=-\frac{1}{2}$, then $\frac{dx}{dt}=-\frac{1}{3}$ and $\frac{d}{dt}(x+y)=-\frac{1}{2}-\frac{1}{3}=-\frac{5}{6}$.

Hence, the man's shadow is decreasing at $\frac{1}{3}$ m/s and the shadow of his head is moving towards the lamppost at a rate of $\frac{5}{6}$ m/s.

Q5) A woman 6 ft tall is walking at 2 ft/s along a straight path on level ground. There is a lamppost 5 ft to the side of the path. A light 15 ft high on the lamppost casts the woman's shadow on the ground. How fast is the length of her shadow changing when the woman is 12 feet from the point on the path closest to the lamppost?

Sol:



Refer to the figure. s, y and x are, respectively, the length of the woman's shadow, the distanses from the woman to the lamppost, and the distances from the woman to the point on the path nearest the lamppost.

From one of triangles in the figure we have $y^2 = x^2 + 25$. If $x = 12$, then $y = 13$. Moreover, $2y\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$.

We are given that $\frac{dx}{dt} = 2$ ft/s, so $\frac{dy}{dt} = \frac{24}{13}$ ft/s when $x = 12$ ft. Now the similar triangles in the figure show that $\frac{s}{6} = \frac{s+y}{15}$,

so that $s = \frac{2y}{3}$. Hence, $\frac{ds}{dt} = \frac{48}{49}$. The woman's shadow is changing at rate $\frac{48}{49}$ ft/s, when she is 12 ft from the point on the path nearest the lamppost.

Q6) Sawdust is falling onto a pile at a rate of $1/2\ m^3/min$. If the pile maintains the shape of a right circular cone with height equal to half the diameter of its base, how fast is the height of the pile increasing when the pile is 3m high?

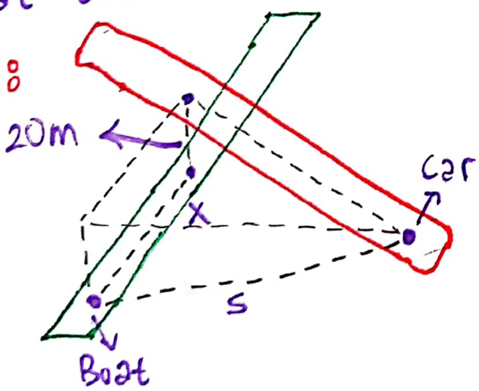
Sol: Let V, r and h be the volume, radius and height of the cone. Since $h=r$, therefore

$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi h^3 \Rightarrow \frac{dV}{dt}=\pi h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{1}{\pi h^2}\frac{dV}{dt}$.

If $\frac{dV}{dt}=\frac{1}{2}$ and $h=3$, then $\frac{dV}{dt}=\frac{1}{18\pi}$. Hence, the height of the pile is increasing at $\frac{1}{18\pi}$ m/min.

Q7) A straight highway and a straight canal intersect at right angles, the highway crossing over the canal on a bridge 20m above the water. A boat travelling at 20 km/h passes under the bridge just as a car travelling at 80 km/h passes over it. How fast are the boat and car separating after one minute?

Sol:



→ Let x and y be the distances travelled from the intersection point by the boat and car respectively in t minutes. Then,

$\frac{dx}{dt}=20\cdot\frac{1000}{60}=\frac{1000}{3}$ m/min.

$\frac{dy}{dt}=80\cdot\frac{1000}{60}=\frac{4000}{3}$ m/min.

The distance s between the boat and car satisfy $s^2=x^2+y^2+20^2$, $s\frac{ds}{dt}=x\frac{dx}{dt}+y\cdot\frac{dy}{dt}$. After one minute,

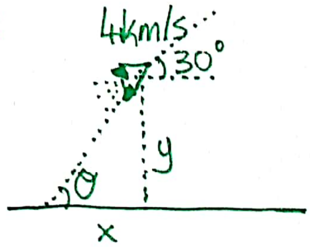
$x=\frac{1000}{3}$, $y=\frac{4000}{3}$ so $s\approx 1374.5$ m. Thus,

$1374.5\frac{ds}{dt}=\frac{1000}{3}\cdot\frac{1000}{3}+\frac{4000}{3}\cdot\frac{4000}{3}\approx 1888889$.

Hence, $\frac{ds}{dt}\approx 1374.2$ m/min ≈ 82.45 km/h after 1 minute.

Q8) (Tracking a rocket) Shortly after launch, a rocket is 100 km high and 50 km downrange. If it is travelling at 4 km/s at an angle of 30° above the horizontal, how fast is its angle of elevation, as measured at the launch site, changing?

Sol:



Let θ be the angle of elevation, and x and y the horizontal and vertical distances from the launch site. We have

$$\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{x\cdot\frac{dy}{dt} - y\cdot\frac{dx}{dt}}{x^2}.$$

At the instant in the question $\frac{dx}{dt} = 4\cdot\cos 30° = 2\sqrt{3}$,

$\frac{dy}{dt} = 4\sin 30° = 2$, $x = 50\text{km}$, $y = 100\text{km}$.

Thus, $\tan\theta = \frac{100}{50} = 2$, $\sec^2\theta = 1+\tan^2\theta = 5$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\,\frac{50.2 - 100.2\sqrt{3}}{50^2} = \frac{1-2\sqrt{3}}{125} \cong -0.0197.$$

Therefore, the angle of elevation is decreasing at about 0.0197 rad/s.

Q9) $\lim\limits_{x\to 0} \frac{1-\cos x}{\ln(1+x^2)} = ?$

Sol: Clearly, $\lim\limits_{x\to 0} \frac{1-\cos x}{\ln(1+x^2)} = \frac{1-\cos 0}{\ln(1+0^2)} = \frac{0}{0}$ indetermined form.

We must use L'Hospital Rule,

$$\lim_{x\to 0}\frac{1-\cos x}{\ln(1+x^2)} \overset{L}{=} \cancel{\lim_{x\to 0}\frac{(1-\cos x)'}{2x}} \overset{L}{=} \lim_{x\to 0}\frac{(1-\cos x)'}{(\ln(1+x^2))'}$$

$$= \lim_{x\to 0}\frac{\sin x}{\frac{2x}{1+x^2}} = \lim_{x\to 0}\frac{x^2+1}{2}\cdot\frac{\sin x}{x}$$

$$= \lim_{x\to 0}[(1+x^2)/2]\cdot\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

Q10) $\lim\limits_{x\to 1} \dfrac{\ln(ex)-1}{\sin(\pi x)} = ?$

Sol: $\lim\limits_{x\to 1} \dfrac{\ln(ex)-1}{\sin(\pi x)} = \dfrac{\ln e - 1}{\sin\pi} = \dfrac{0}{0}$ indetermined form.

$$\lim_{x\to 1} \frac{\ln(ex)-1}{\sin(\pi x)} \overset{L}{=} \lim_{x\to 1} \frac{\frac{e}{ex}}{\pi\cos(\pi x)} = \lim_{x\to 1} \frac{x}{\pi\cos(\pi x)} = -\frac{1}{\pi}$$

Q11) $\lim\limits_{r\to \pi/2} \dfrac{\ln\sin r}{\cos r} = ?$

Sol: $\lim\limits_{r\to \pi/2} \dfrac{\ln\sin r}{\cos r} = \dfrac{\ln\sin(\pi/2)}{\cos(\pi/2)} = \dfrac{\ln 1}{0} = \dfrac{0}{0}$ ↯

$$\lim_{r\to \pi/2} \frac{\ln\sin r}{\cos r} \overset{L}{=} \lim_{r\to \pi/2} \frac{\frac{\cos r}{\sin r}}{-\sin r} = \lim_{r\to \pi/2} \frac{-\cos r}{\sin^2 r} = 0 //$$

Q12) $\lim\limits_{x\to \infty} x(2\tan^{-1}x - \pi) = ?$

Sol: $\lim\limits_{x\to \infty} x(2\tan^{-1}x-\pi) = \infty\cdot(2\tan^{-1}(\infty)-\pi) = \infty\cdot(2\cdot\frac{\pi}{2}-\pi) = \infty\cdot 0$ ↯

$$\lim_{x\to \infty} x(2\tan^{-1}x-\pi) = \lim_{x\to\infty} \frac{2\tan^{-1}x-\pi}{\frac{1}{x}} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{2}{1+x^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} -\frac{2x^2}{1+x^2} = -2.$$

Q13) $\lim\limits_{x\to 0^+} (\operatorname{cosec} x)^{\sin^2 x} = ?$

Sol: Clearly, we have the indetermined form ∞^0.

Let $y = (\operatorname{cosec} x)^{\sin^2 x}$. Then, $\ln y = \sin^2 x \cdot \ln(\operatorname{cosec} x)$.

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(\operatorname{cosec} x)}{\operatorname{cosec}^2 x} \overset{L}{=} \lim_{x\to 0^+} \frac{-\frac{\operatorname{cosec} x\cdot\cot x}{\operatorname{cosec} x}}{-2\cdot\operatorname{cosec}^2 x\cdot\cot x}$$

$$= \lim_{x\to 0^+} \frac{1}{2\operatorname{cosec}^2 x} = 0$$

Thus, $\lim\limits_{x\to 0^+} (\operatorname{cosec} x)^{\sin^2 x} = e^0 = 1.$

Q14) $\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=?$

Sol: We have the indetermined form $\infty-\infty$.

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\frac{x\cdot\ln x-x+1}{(x-1)\cdot\ln x}\overset{L}{=}\lim_{x\to 1^+}\frac{\ln x+x\cdot\frac{1}{x}-\cancel{1}}{\ln x+\frac{1}{x}(x-1)}$$

$$=\lim_{x\to 1^+}\frac{\ln x}{\ln x+1-\frac{1}{x}}\overset{L}{=}\lim_{x\to 1^+}\frac{\frac{1}{x}}{\frac{1}{x}-\frac{1}{x^2}}$$

$$=\lim_{x\to 1^+}\frac{x}{x+1}=\frac{1}{2}\,//$$

Q15) $\lim_{t\to 0}(\cos 2t)^{1/t^2}=?$

Sol: We have the indetermined form 1^∞.

Let $y=(\cos 2t)^{1/t^2}$. Then, $\ln y=\frac{\ln(\cos 2t)}{t^2}$. We have

$$\lim_{t\to 0}\ln y=\lim_{t\to 0}\frac{\ln(\cos 2t)}{t^2}\overset{L}{=}\lim_{t\to 0}\frac{-2\tan 2t}{2t}$$

$$\overset{L}{=}-\lim_{t\to 0}\frac{2\sec^2 2t}{1}=-2.$$

Q16) Evoluate $\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ if f is a twice differentiable function.

Sol: Clearly, we have $\frac{0}{0}$ indetermined form.

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}\overset{L}{=}\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

$$\overset{L}{=}\lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}=\frac{2f''(x)}{2}=f''(x).$$

Q17) Determine whether the given functions has any local or absolute extreme values, and find those values if possible.

a) $f(x)=x^2-1$ on $(2,3)$ b) $f(x)=\frac{1}{x-1}$ on $[2,3]$

Sol: a) $f(x)=x^2-1 \Rightarrow f'(x)=2x$ and $f'(x)>0$ for all $x\in(2,3)$. It means $f(x)$ is increasing on $(2,3)$. Since, $x=2$ and $x=3$ are not in $(2,3)$, $f(x)$ has no max or min values.

b) $f(x)=\frac{1}{x-1} \Rightarrow f'(x)=\frac{-1}{(x-1)^2}<0$ for all $x\in[2,3]$.

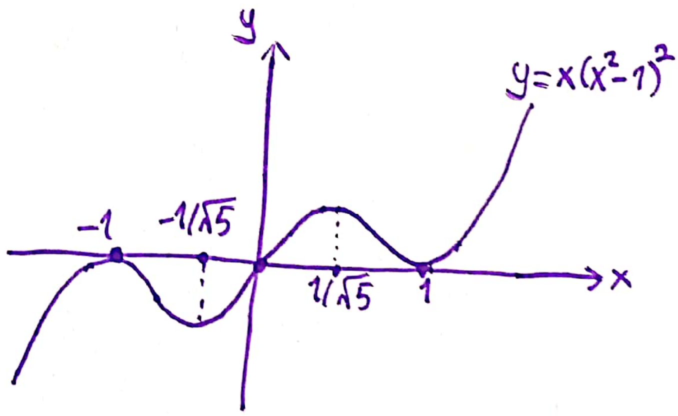
Since $f(x)$ is ~~increasing~~ decreasing on $[2,3]$, we say $f(3)=\frac{1}{2}$ is ~~[illegible]~~ absolute min and $f(2)=1$ is abs. max.

Q18) Locate and classify all local extreme values of the given function. Determine whether any of these extreme values are absolute. Sketch the graph of the function.

a) $f(x)=x\cdot(x^2-1)^2$

Sol: $f'(x)=(x^2-1)^2+2x(x^2-1)\cdot 2x$
$=(x^2-1)(x^2-1+4x^2)$
$=(x-1)(x+1)(\sqrt{5}x-1)(\sqrt{5}x+1)$ $f'(x)=0$ at $x=\mp 1, \mp 1/\sqrt{5}$

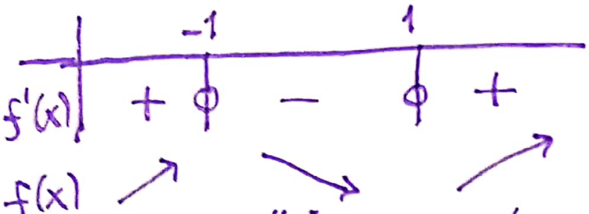
~~f(x)~~		-1		$-1/\sqrt{5}$		$1/\sqrt{5}$		1	
$f'(x)$	+	0	−	0	+	0	−	0	+
$f(x)$	↗	loc max	↘	loc min	↗	loc max	↘	loc min	↗



$f(\pm 1)=0$, $f(\mp 1/\sqrt{5})=\mp 16/25\sqrt{5}$

b) $f(x) = x - 2\tan^{-1}x$

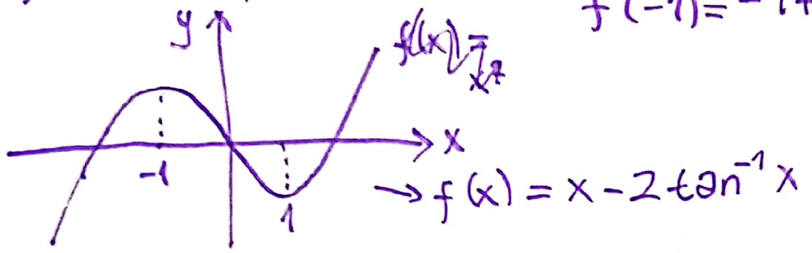
Sol: $f'(x) = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1} = \frac{(x+1)(x-1)}{x^2+1} \Rightarrow x = \mp 1$ are critical points



f(x) has loc max at $x=-1$

" " " min at $x=1$

$f(-1) = -1 + \frac{\pi}{2}$ and $f(1) = 1 - \frac{\pi}{2}$.



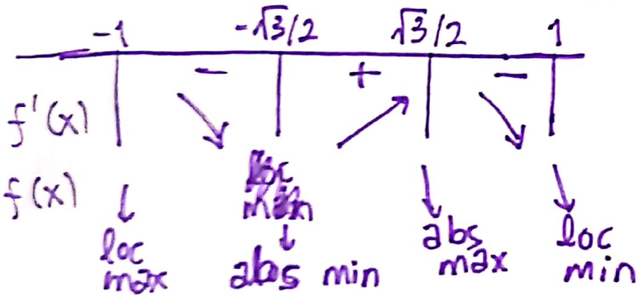
c) $f(x) = 2x - \sin^{-1}x$

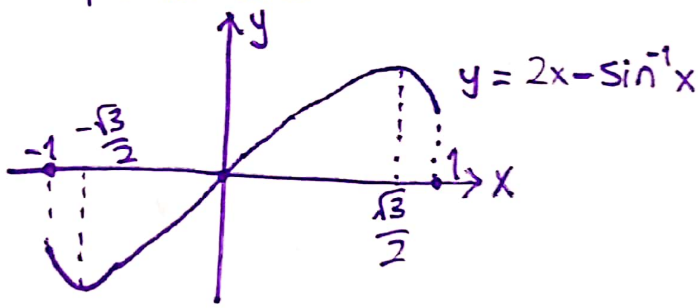
Sol: Clearly, $f(x)$ is defined on $[-1,1]$.

$$f'(x) = 2 - \frac{1}{\sqrt{1-x^2}} = \frac{2\sqrt{1-x^2}-1}{\sqrt{1-x^2}} = \frac{3-4x^2}{\sqrt{1-x^2}\,(2\sqrt{1-x^2}+1)}$$

$(2\sqrt{1-x^2}+1)$

So, $f'(x) = 0$ at $x = \mp\frac{\sqrt{3}}{2}$. End points are $\mp 1 = x$.

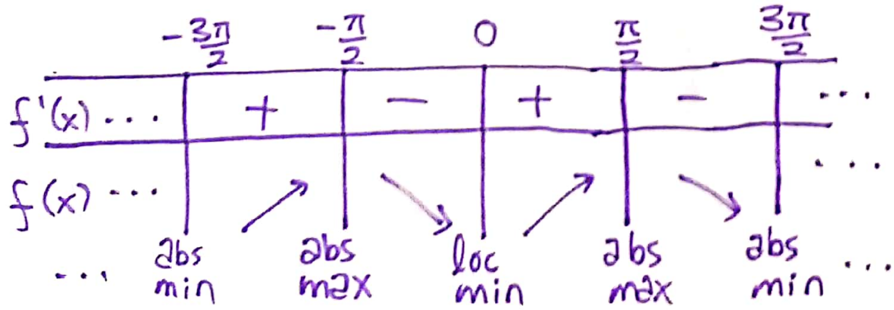


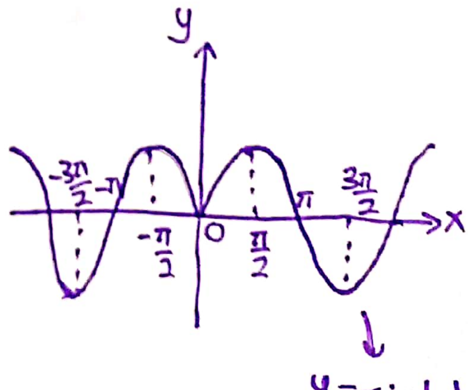


d) $f(x) = \sin|x|$

Sol: $f'(x) = \mathrm{sgn}(x).\cos|x| \rightarrow$ critical points are $x = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots$

$x=0$ is a singular point. Since $f(x)$ is even function, its graph symmetric about the origin.





$y = \sin|x|$

Hence, f has abs max 1 at $x = \mp(4k+1)\frac{\pi}{2}$ and abs min -1 at $x = \mp(4k+3)\frac{\pi}{2}$ and loc min 0 at $x=0$.

Q19) Determine the intervals of constant concavity of the functions given below, and locate any inflection points.

a) $f(x) = 10x^3 - 3x^5$ b) $f(x) = \ln(1+x^2)$ c) $f(x) = (x-1)^{\frac{1}{3}} + (x+1)^{\frac{1}{3}}$

Sol: a) * An inflection point is where it goes from concave up to concave down (or vice versa).

$\rightarrow f(x) = 10x^3 - 3x^5 \Rightarrow f'(x) = 30x^2 - 15x^4 \Rightarrow f''(x) = 60x - 60x^3.$

Hence, $f''(x) = 60x(1-x)(1+x).$

	$-\infty$		-1		0		1		∞
f''		$+$	0	$-$	0	$+$	0	$-$	
f		$\cup$		$\cap$		$\cup$		$\cap$	
		concave up		concave down		concave up		concave down	

$\Rightarrow f$ has inflection points at $x = -1, 0, 1$.

b) $f(x) = \ln(1+x^2) \Rightarrow f'(x) = \dfrac{2x}{1+x^2} \Rightarrow f''(x) = \dfrac{2(1+x^2) - 2x.2x}{(1+x^2)^2}.$

Hence, $f''(x) = \dfrac{2(1+x)(1-x)}{(1+x^2)^2}.$

		-1		1	
f''	$-$	0	$+$	0	$-$
f	$\cap$		$\cup$		$\cap$
	concave down		concave up		concave down

$\Rightarrow f$ has inflection points at $x=-1$ and $x=1$.

c) $f(x) = (x-1)^{\frac{1}{3}} + (x+1)^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}\left[(x-1)^{-\frac{2}{3}} + (x+1)^{\frac{2}{3}}\right],$

$f''(x) = -\frac{2}{9}\left[(x-1)^{-\frac{5}{3}} + (x+1)^{-\frac{5}{3}}\right].$

$f''(x) = 0 \Leftrightarrow (x-1)^{-\frac{5}{3}} + (x+1)^{-\frac{5}{3}} = 0 \Leftrightarrow x-1 = -x-1 \Leftrightarrow x = 0$

Thus, f has inflection point at $x=0$. Also, $f''(x)$ is undefined at $x = \mp 1$. f is defined at $x = \mp 1$ and $x = \mp 1$ are also inflection points. f is concave up on $(-\infty, -1)$ and $(0,1)$; and concave down on $(-1,0)$ and $(1,\infty)$.

Q20) Classify the critical points of the functions given below. using the Second Derivative Test whenever possible.

a) $f(x)=x+\frac{4}{x}$ b) $f(x)=x.e^x$ c) $f(x)=x^2.e^{-2x^2}$

Sol: a) * Second derivative test

- If $f'(x_0)=0$ and $f''(x_0)<0$, then f has a local maximum value at x_0.
- If $f'(x_0)=0$ and $f''(x_0)>0$, then f has a local minimum value at x_0.

→ $f(x)=x+\frac{4}{x} \Rightarrow f'(x)=1-\frac{4}{x^2} \Rightarrow f''(x)=8.x^{-3}$.

For critical points,

$f'(x)=0 \Rightarrow 1-\frac{4}{x^2}=0 \Rightarrow x=\mp 2$ are critical points

$x=2$, $f''(2)=1>0 \Rightarrow f$ has local min. at $x=2$.

$x=-2$, $f''(-2)=-1<0 \Rightarrow f$ has local max. at $x=-2$.

b) $f(x)=x.e^x \Rightarrow f'(x)=e^x(1+x) \Rightarrow f''(x)=e^x(2+x)$

For critical points, $f'(x)=0 \Rightarrow e^x(1+x)=0 \Rightarrow x=-1$.

$x=-1$ is critical point. $f''(-1)=e^{-1}>0$. Hence, f has a local min. at $x=-1$.

c) $f(x)=x^2e^{-2x^2} \Rightarrow f'(x)=e^{-2x^2}(2x-4x^3)=2e^{-2x^2}(1+\sqrt{2}x)(1-\sqrt{2}x).x$

$\Rightarrow f''(x)=e^{-2x^2}(2-20x^2+16x^4)$.

For critical points, $f'(x)=0 \Rightarrow x=0,\ x=\frac{1}{\sqrt{2}},\ x=\frac{-1}{\sqrt{2}}$.

$f''(0)=2>0 \Rightarrow f$ has local min. at $x=0$.

$f''(\mp\frac{1}{\sqrt{2}})=-\frac{4}{e}<0 \Rightarrow$ f has local max. at $x=\mp\frac{1}{\sqrt{2}}$.