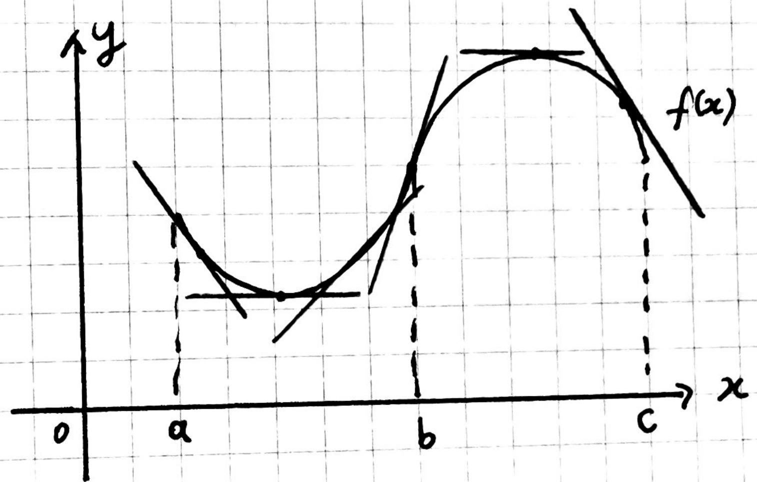
CONCAVITY AND INFLECTIONS

Definition: We say that the function f is concave up on an open interval I if it is differentiable there and the derivative f' is an increasing function on I. Similarly, f is concave down on I if f' exists and is decreasing on I.



f is concave up on (a,b) and concave down on (b,c).

Definition: (Inflection Points) We say that the point $(x_0, f(x_0))$ is an inflection point of the curve $y = f(x)$ (or that the function f has an inflection point at x_0) if the following two conditions are satisfied:

a) the graph of $y = f(x)$ has a tangent line at $x = x_0$, and

b) the concavity of f is opposite on opposite sides of x_0.

Note: A function may or may not have an inflection point at a critical point or singular point.

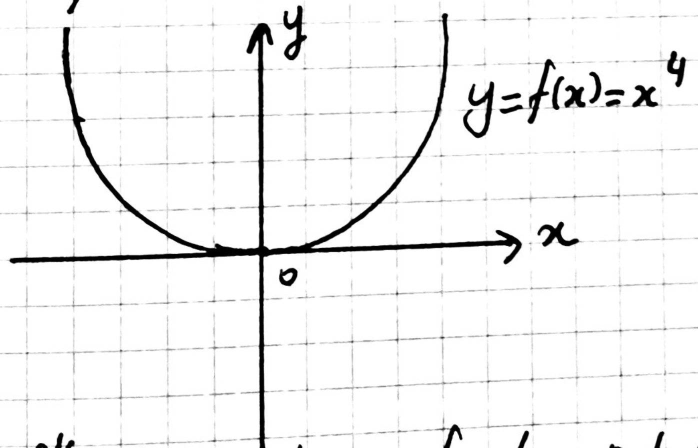
Theorem: (Concavity and the second derivative)

i) If $f''(x) > 0$ on interval I, then f is concave up on I.

ii) If $f''(x) < 0$ on interval I, then f is concave down on I.

iii) If f has an inflection point at x_0 and $f''(x_0)$ exists, then $f''(x_0) = 0$.

Theorem tells us that to find the x-coordinates of inflection points of a twice differentiable function f, we need only look at points where $f''(x) = 0$.
However, not every such point has to be an inflection point. For example $f(x) = x^4$ does not have an inflection point at $x = 0$ even though $f''(0) = 12x^2\big|_{x=0} = 0$.



$f''(0) = 0$ but f does not have an inflection point at 0.

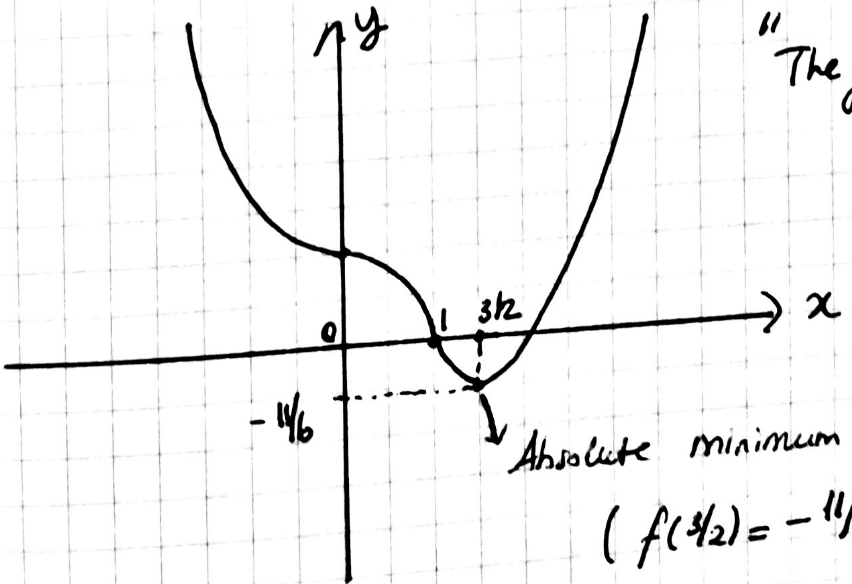
Example: Determine the intervals of increase and decrease, the local extreme values, and the concavity of $f(x) = x^4 - 2x^3 + 1$. Use the information to sketch the graph of f.

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3) = 0 \text{ at } x=0 \text{ and } x = 3/2.$$

$$f''(x) = 12x^2 - 12x = 12x(x-1) = 0 \text{ at } x=0 \text{ and } x=1.$$

Thus, the behaviour of the $f(x)$:

x		C.P 0		1		C.P $3/2$	
f'	$-$		$-$		$-$		$+$
f''	$+$		$-$		$+$		$+$
f	$\searrow$ $\cup$	inf.	$\searrow$ $\cap$	inf.	$\searrow$ $\cup$		$\nearrow$ $\cup$



"The graph of $f(x)$"

Absolute minimum value of $f(x)$ is $-11/6$.

$(f(3/2) = -11/6)$

Theorem : (The Second Derivative Test)

i) If $f'(x_0)=0$ and $f''(x_0)<0$, then f has a local maximum value at x_0

ii) If $f'(x_0)=0$ and $f''(x_0)>0$, then f has a local minimum value at x_0.

iii) If $f'(x_0)=0$ and $f''(x_0)=0$, no conclusion can be drawn; f may have a local maximum at x_0 or a local minimum, or it may have an inflection point instead.

Example: Find an classify the critical points of $f(x)=x^2e^{-x}$.

$$f'(x)= 2x.e^{-x} - x^2e^{-x} = e^{-x}(2x-x^2) = e^{-x}x(2-x)=0$$

at $x=0$ and $x=2$.

$$f''(x)= (2-2x)e^{-x} - e^{-x}(2x-x^2)$$

$$= e^{-x}(2-2x-2x+x^2)$$

$$= e^{-x}(2-4x+x^2)$$

$f''(0)=2>0$, $\quad f''(2)=-2e^{-2}<0$

Thus f has a local minimum value at $x=0$ and a local maximum value at $x=2$.

SKETCHING THE GRAPH OF A FUNCTION

Asymptotes:

Vertical Asymptote: The graph of $y=f(x)$ has a vertical asymptote at $x=a$ if either

$\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$ or both.

Horizontal Asymptote: The graph of $f(x)$ has a horizontal asymptote at $y=L$ if either

$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, or both.

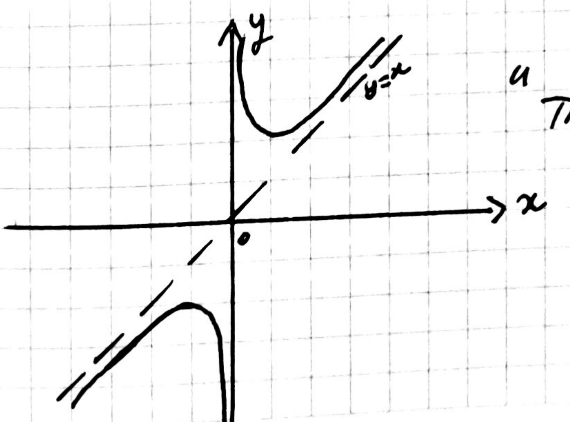
Oblique Asymptote: The straight line $y=ax+b$ (where $a\neq 0$) is an oblique asymptote of the graph of $y=f(x)$ if either

$\lim_{x\to-\infty} \big(f(x)-(ax+b)\big)=0$ or $\lim_{x\to\infty} \big((f(x)-(ax+b)\big)=0$,

or both.

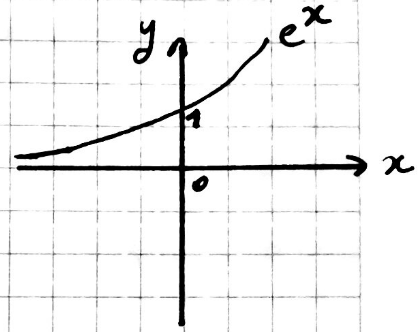
Example: Consider the function $f(x)=\frac{x^2+1}{x} = x+\frac{1}{x}$, the straight line $y=x$ is a two-sided oblique asymptote of the graph of f because

$$\lim_{x\to\mp\infty}\big(f(x)-x\big) = \lim_{x\to\mp\infty}\frac{1}{x} = 0.$$



"The graph of $f(x)=x+\frac{1}{x}$"

Example: The graph of $y=\frac{xe^x}{1+e^x}$ has a horizontal asymptote $y=0$ at the left and an oblique asymptote $y=x$ at the right.



$$\lim_{x\to-\infty}\frac{xe^x}{1+e^x}=\frac{0}{1}=0 \quad \text{and}$$

$$\lim_{x\to\infty}\left(\frac{xe^x}{1+e^x}-x\right)$$

$$=\lim_{x\to\infty}\frac{x(e^x-1-e^x)}{1+e^x}$$

$$=\lim_{x\to\infty}\frac{-x}{1+e^x}=0$$

Asymptotes of a rational function

Suppose that $f(x)=\frac{P_m(x)}{Q_n(x)}$, where P_m and Q_n are polynomials of degree m and n, respectively. Suppose also that P_m and Q_n have no common linear factors. Then,

i) The graph of f has a vertical asymptote at every position x such that $Q_n(x)=0$

ii) The graph of f has a two-sided horizontal asymptote $y=0$ if $m<n$.

iii) The graph of f has a two-sided horizontal asymptote $y=L$, $(L \neq 0)$ if $m=n$. L is the quotient of the coefficients of the highest degree terms in P_m and Q_n.

iv) The graph of f has a two-sided oblique asymptote if $m=n+1$. This asymptote can be found by dividing Q_n into P_m to obtain a linear quotient, $ax+b$, and remainder, R, a polynomial of degree at most $n-1$. That is,

$$f(x) = (ax+b) + \frac{R(x)}{Q_n(x)}.$$

The oblique asymptote is $ax+b$.

v) The graph of f has no horizontal or oblique asymptotes if $m > n+1$.

Example: Sketch the graph of $y = \dfrac{x^2+2x+4}{2x}$.

We can write $y = \frac{x}{2} + 1 + \frac{2}{x}$

$y = \frac{x}{2} + 1$ is an oblique asymptote.

$y' = \dfrac{x^2-4}{2x^2}$ and $y'' = \dfrac{4}{x^3}$.

From y: $D(f) = \mathbb{R} - \{0\}$.

Asymptotes
- Vertical asymptote : $x = 0$
- Oblique asymptote : $y = \frac{x}{2} + 1$.

Symmetry : none (y is neither odd nor even fnc.)

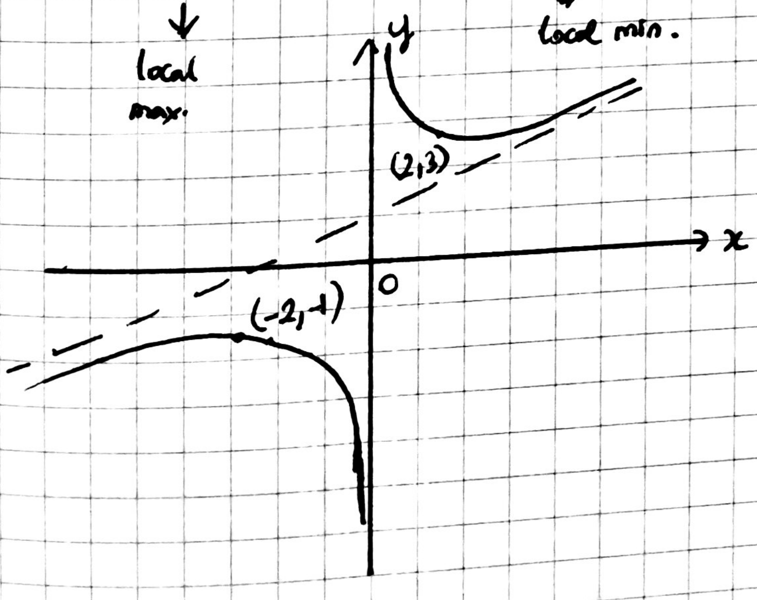
Intercepts: none . $x^2 + 2x + 4 = (x+1)^2 + 3 \geq 3$ for all x, and y is not defined at $x = 0$.

From y': Critical Points : $x = \pm 2$; points $(-2,-1)$ and $(2,3)$.

y' not defined at $x = 0$ (vertical asymptote)

From y'': $y'' = 0$ nowhere , y'' undefined at $x = 0$.

x		C.P. -2		ASY. 0		C.P. 2	
y'	$+$		$-$		$-$		$+$
y''	$-$		$-$		$+$		$+$
y	↗		↘		↘		↗
		local max.				local min.	



Example: Sketch the graph of $f(x)=\dfrac{x^2-1}{x^2-4}$.

$$f'(x)=\frac{-6x}{(x^2-4)^2} \quad \text{and} \quad f''(x)=\frac{6(3x^2+4)}{(x^2-4)^3}$$

From f: $D(f)=\mathbb{R}-\{-2,2\}$

Vertical asymptotes: $x=-2$ and $x=2$.

Horizontal asymptote: $y=1$ (as $x\to\pm\infty$)

Symmetry: about the y-axis (even function)

Intercepts: $(0,1/4)$, $(-1,0)$ and $(1,0)$

From f': Critical points: $x=0$;

f' is not defined at $x=2$ or $x=-2$.

From f'': $f''(x)=0$ nowhere, f'' not defined at $x=2$ or $x=-2$.

x		-2		0		2	
f'	+		+		$-$		$-$
f''	+		$-$		$-$		+
f	↗ ∪		↗ ∩	local max	↘ ∩		↘ ∪