# GTU, Fall 2020, MATH 101

# Differentiation

# Tangent Lines and Their Slopes

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* It is NOT just a line that meets the graph at one point.
* It is the limit of the secant line (a line drawn between two points on the graph) as the distance between the two points goes to zero.

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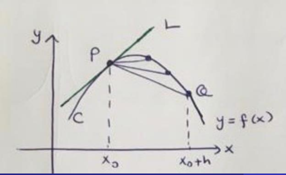

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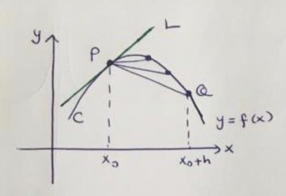
$*$ The slope of the line $PQ$ is

$$\frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(Newton quotient)}$$



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exist. Then

$$y = m(x - x_0) + f(x_0)$$

is called the **tangent line** to the graph of $f$ at $(x_0, f(x_0))$.

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$$y = -(x-1) + 1, \text{ or } y = -x + 2.$$

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- If $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ fails to exist in any other way than by being $\infty$ or $-\infty$, $y = f(x)$ has no tangent line at $(x_0, f(x_0))$.

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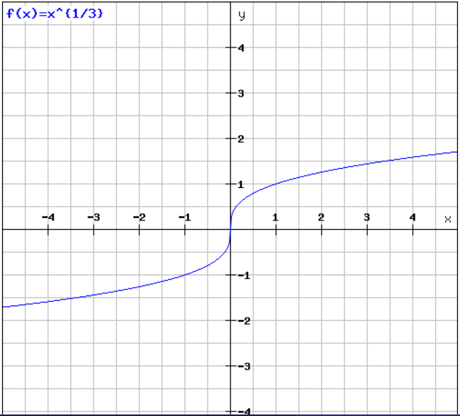
$\implies$ The $y-$axis is tangent to $y = \sqrt[3]{x}$.



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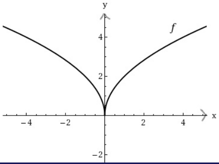
The graph of $f$ has no tangent at $(0, 0)$.



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