§14.5 Triple Integrals

We extended the definite integrals of functions in single variable to double integrals in the previous sections. Now we may extend it to triple integrals; i.e. the integrals of functions $f(x, y, z)$ over a domain R in 3-space.

We denote the triple integral of $f(x, y, z)$ on a domain R in 3-space by

$$\iiint\limits_R f(x, y, z)\, dV$$

where dV is called the volume element (three dimensional analogue of dx and of dA).

The definition and properties of double integrals and triple integrals are very similar. The triple integrals are also defined as the limit of Riemann sums. But in this case we need to construct the Riemann sums where a partition of the domain consists of rectangular boxes (recall that for double integrals we divide the domain into small rectangles). We may refer to §14.1 for straightforward extension of properties of double integrals to triple integrals.

For example if R is a domain in 3-space and we take $f(x, y, z) = 1$ then

$$\iiint\limits_R 1\, dV = \text{ Volume of } R$$

(Compare with $\iint\limits_D 1\,dA =$ (Area of D) for a domain D in the xy-plane). But the double integral has a nice geometric interpretation;

$$\iint\limits_D g(x,y)\,dA = \text{ Volume of the solid under } g(x,y) \text{ and above } D$$

whenever $g(x,y)$ is positive. What about the triple integrals? What is the geometric meaning of $\iiint\limits_R f(x,y,z)\,dV$? We can think of it as a *four-dimensional volume* with base R (!) and top the *graph* of $w = f(x,y,z)$ which is impossible to sketch and impractical.

Indeed we have useful applications of triple integrals in practice; for example if $f(x,y,z)$ denotes the density (or charge density) at the point (x,y,z) of a solid represented by the domain R in 3-space, then

$$\iiint\limits_R f(x,y,z)\,dV$$

gives the total mass (or total charge) of this solid. We will not study the applications throughout the course but rather concentrate on the techniques of evaluation of triple integrals. As we did for double integrals, we will evaluate triple integrals as iterated integrals;

$$\iiint\limits_R f(x,y,z)\,dV = \int_{-}^{-}\int_{-}^{-}\int_{-}^{-} f(x,y,z)\,dx\,dy\,dz$$

Example 1: Let R be the rectangular box defined as

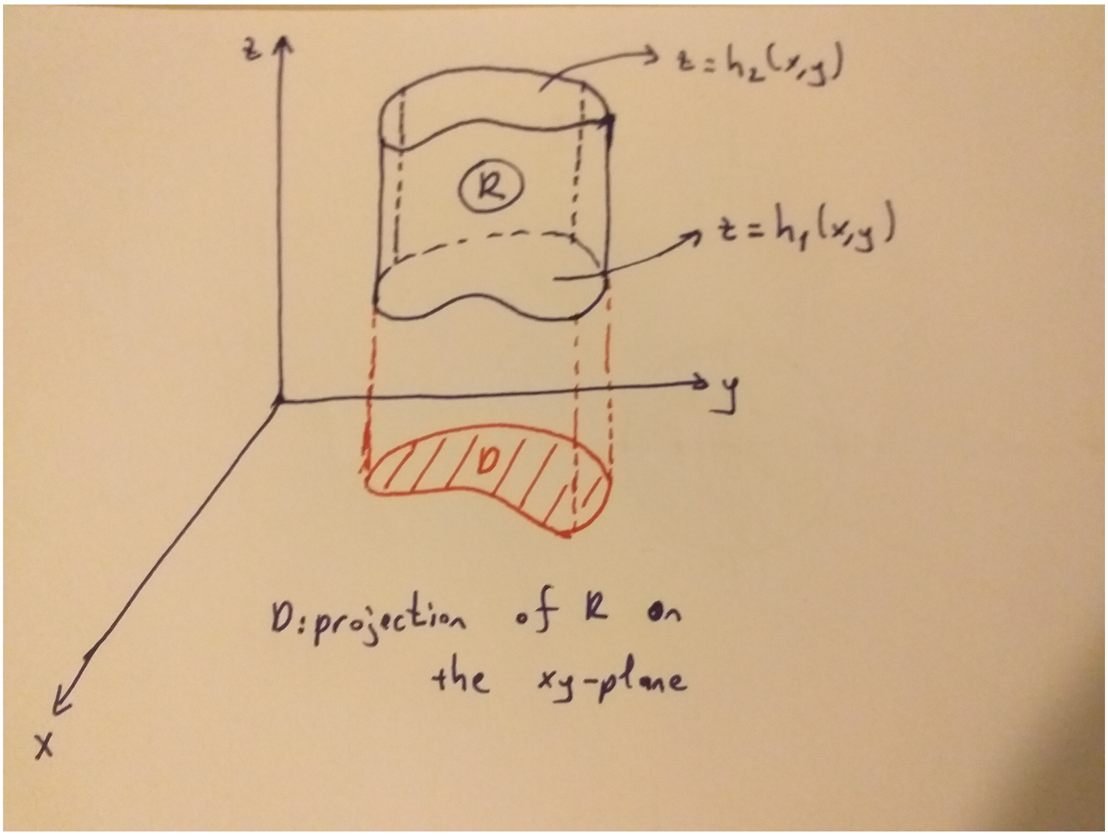
$R : 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$. Then

i) $\iiint\limits_R 1\,dV =$ Volume of $R = abc$

ii)

$$\iiint\limits_R (3x^2yz)\,dV = \int_0^c\int_0^b\int_0^a (3x^2yz)\,dx\,dy\,dz = \int_0^c\int_0^b (x^3yz)\Big|_{x=0}^{a}\,dy\,dz$$
$$= a^3\int_0^c\int_0^b yz\,dy\,dz = a^3b^2c^2/4$$

If the domain R is not a rectangular box then we may reduce triple integrals to double integrals. For example consider the domain $R = \{(x,y,z) : (x,y) \in D,\ h_1(x,y) \le z \le h_2(x,y)\}$ where D is a region on the xy-plane, and h_1 and h_2 are some functions.



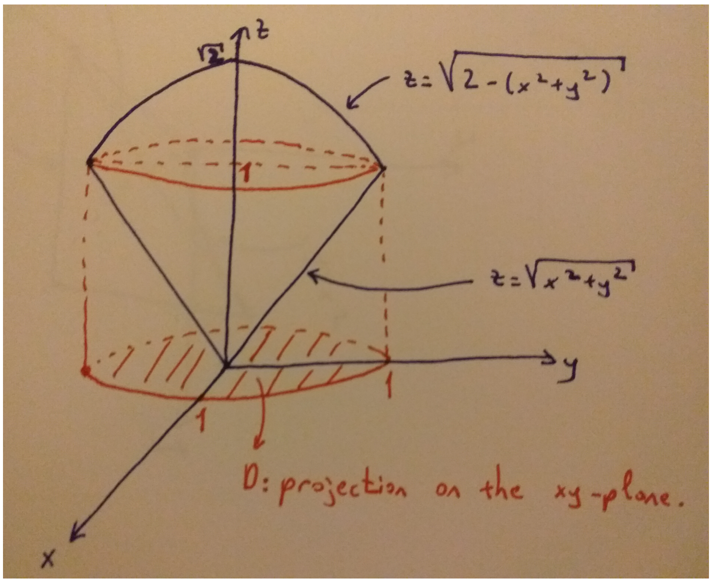
The region D is the *projection* of R on the xy-plane. Then we have

$$\iiint\limits_R f(x,y,z)\,dV = \iint\limits_D \left(\int\limits_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz \right) dA.$$

The inner integral is with respect to dz so we regard x and y as constants. Hence the evaluation of this inner integral reduces the triple integral to a double integral in the form $\iint\limits_D (......)\,dA$ which can be computed in an appropraite order; $dx\,dy$ or $dy\,dx$ (or can be evaluated in polar coordinates if possible).

Example 2: Find the volume of the solid R lying inside both the sphere $x^2+y^2+z^2=2$ and the cone $z=\sqrt{x^2+y^2}$.

Solution: By solving $z=\sqrt{x^2+y^2}=\sqrt{2-(x^2+y^2)}$ we see that the two surfaces intersect at the circle $x^2+y^2=1$, $z=1$ as given in the below figure.



The projection of R on the xy-plane is $D: x^2+y^2\le 1$. Hence the volume of R is

$$\iiint\limits_R 1\,dV = \iint\limits_D \left(\int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-(x^2+y^2)}} 1\,dz \right) dA$$

$$= \iint\limits_D \left(\sqrt{2-(x^2+y^2)} - \sqrt{x^2+y^2}\right) dA.$$

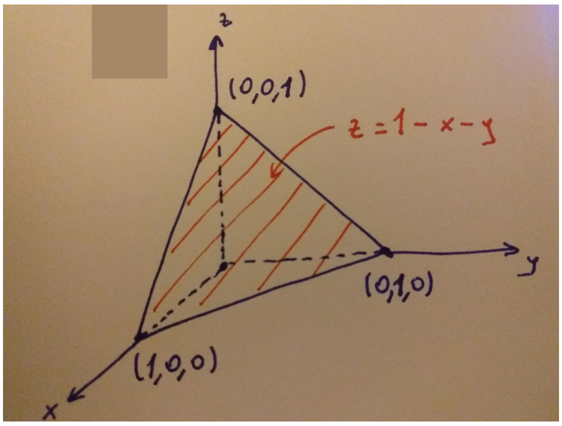
It is clearly easier to evaluate the double integral in polar coordinates, so we set $x = r\cos\theta$, $y = r\sin\theta$ and $dA = r\,dr\,d\theta$. Then we have

$$\iiint\limits_R 1\,dV = \int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2} - r\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 r\sqrt{2-r^2}\,dr\,d\theta - \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \ldots \text{ Exercise } \ldots.$$

Example 3: Let R be the region bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ (R is called a tedrahedron). Evaluate $\iiint\limits_R x\,dV$.

Solution:

The plane $x + y + z = 1$ cuts the x, y and z axes at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. The fourth vertex of R is the origin. The projection of R on the xy-plane is the triangle with vertices $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$. Call this triangle by D. The region D on the xy-plane is given by $D : 0 \le x \le 1$, $0 \le y \le 1 - x$. Also the domain of the



triple integral is

$$R = \{(x, y, z) : (x, y) \in D,\ 0 \le z \le 1 - x - y\}.$$

So we have that

$$\iiint\limits_R x\, dV = \iint\limits_D \left(\int\limits_0^{1-x-y} x\, dz \right) dA = \int\limits_0^1 \int\limits_0^{1-x} \int\limits_0^{1-x-y} x\, dz\, dy\, dx$$

$$= \int\limits_0^1 x \left(\int\limits_0^{1-x} (1 - x - y)\, dy \right) dx = \int\limits_0^1 x \left(y - xy - y^2/2 \right) \Big|_{y=0}^{1-x} dx$$

$$= \int\limits_0^1 \left(\frac{x^3}{2} - x^2 + \frac{x}{2} \right) dx = \frac{1}{24}$$

Note on Example 3: We could also compute the triple integral of Example 2 in a different order. For example we can describe R as $R = \{(x, y, z) : (x, z) \in D',\ 0 \le y \le 1 - x - z\}$ where D' is the triangle on the xz-plane (!) with vertices $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$. So we have

$$\iiint\limits_R x\, dV = \iint\limits_{D'} \left(\int\limits_0^{1-x-z} x\, dy \right) dA = \int\limits_0^1 \int\limits_0^{1-z} \int\limits_0^{1-x-z} x\, dy\, dx\, dz.$$

Example 4: Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in the order of $dx\,dz\,dy$.

Solution: We can write

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx = \iiint_R f(x,y,z)\,dV$$

where $R : 0 \leq x \leq 1$, $0 \leq y \leq x^2$, $0 \leq z \leq y$. We need to write the triple integral in the form

$$\iiint_R f(x,y,z)\,dV = \int_{-}^{-} \int_{-}^{-} \int_{-}^{-} f(x,y,z) dx\,dz\,dy.$$

Note that outers integral can not have limits depending on the variable of integration of the inner integrals. So in particular the limits of the integration with respect to dy must be constant, and the limits of the integration with respect to dz may only depend on y.

By the inequalities $0 \leq x \leq 1$, $0 \leq y \leq x^2$ we see that the limits of y are 0 and 1. Also the limits of z are already given by $0 \leq z \leq y$.

So the triple integral is of the form

$$\iiint\limits_R f(x,y,z)\,dV = \int\limits_0^1 \int\limits_0^y \int\limits_{-}^{-} f(x,y,z)dx\,dz\,dy.$$

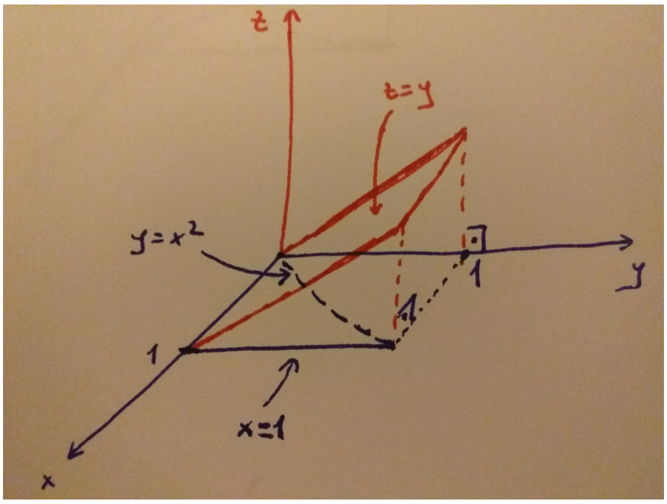
Again by the inequalities $0 \le x \le 1$ and $0 \le y \le x^2$ we see that $\sqrt{y} \le x \le 1$. Hence we have that

$$\iiint\limits_R f(x,y,z)\,dV = \int\limits_0^1 \int\limits_0^y \int\limits_{\sqrt{y}}^1 f(x,y,z)dx\,dz\,dy.$$

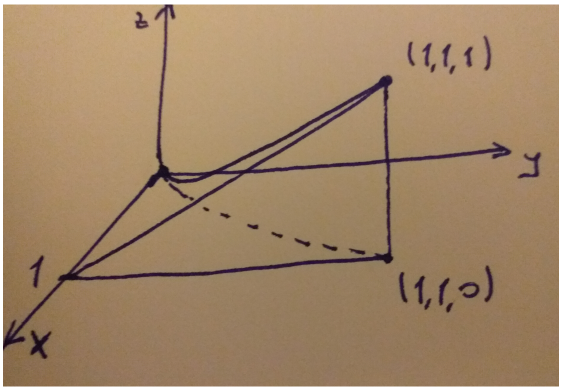
Note on Example 4: We found the limits of the iterated integral *algebraically*, i.e. we didn't use a geometric argument. It is also possible to find the limits of integrations *geometrically* (and this may be necessary if the equations are complicated, and in some cases both approaches may be used together).

We shall start by sketching the region $R: 0 \le x \le 1$, $0 \le y \le x^2$, $0 \le z \le y$ of Example 4. We may start by sketching the projection of R on the xy-plane which is easier than the other projections (Is it related to the order $dz\,dy\,dx$ of the given iterated integral?)

The projection on the xy-plane is the region bounded by the x-axis, $x = 1$ and $y = x^2$. Call this region D. Now we use the inequality $0 \le z \le y$ to see that z lies between the xy-plane and the plane $z = y$. We locate the plane $z = y$.



Finally we strecth the region D by pulling the point $(1,1,0)$ to $(1,1,1)$ (the top point of our region) and obtain the following figure.



So our region is the solid bounded by the xy-plane, the planes $x = 1$ and $z = y$, and the parabolic cylinder $y = x^2$. Now we can obtain the limits of integration in any order. For example if we use the order $dx\,dz\,dy$ then the limits must be $0 \le y \le 1$, $0 \le z \le y$ and $\sqrt{y} \le x \le 1$. As an exercise write the integral of Example 4 in the order $dy\,dx\,dz$ and $dx\,dy\,dz$.