MATH 101.2 PS-5

Q1) Find the intervals of increase and decrease of the functions below.

a) $f(x) = (x^2-4)^2$

Sol: $f'(x) = 2.(x^2-4).2x = 4x(x+2)(x-2)$. The roots of $f'(x)$ are $x=-2,0,2$. We can make a table.

	$-\infty$		-2		0		2		∞
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	

$f'(x)>0$ on $(-2,0)\cup(2,\infty)$
$f'(x)<0$ on $(-\infty,-2)\cup(0,2)$

We know that $f(x)$ is increasing if $f'(x)>0$ and $f(x)$ is decreasing if $f'(x)<0$. Therefore, $f(x)$ is increasing on $(-2,0)\cup(2,\infty)$ and decreasing on $(-\infty,-2)\cup(0,2)$

b) $f(x) = \frac{1}{x^2+1}$

Sol: $f'(x) = [(x^2+1)^{-1}]' = -(x^2+1)^{-2}.2x = \frac{-2x}{(x^2+1)^2}$. Clearly,

$f'(x)>0$ if $x<0$ and $f'(x)<0$ if $x>0$. Hence, $f(x)$ is increasing on $(-\infty,0)$ and $f(x)$ is decreasing on $(0,\infty)$.

c) $f(x) = x^3(5-x)^2$

Sol: $f'(x) = 3x^2(5-x)^2 + 2(5-x)(-1).x^3 = x^2(5-x)(15-5x)$
$= 5x^2(5-x)(3-x)$

repeated
The roots of $f'(x)$ are $3,5$ and 0 (double root)
We can make a table.

	$-\infty$		0		3		5		∞
$f'(x)$		$+$	0	$+$	0	$-$	0	$+$	

$f'(x)>0$ on $(-\infty,0)\cup(0,3)\cup(5,\infty)$ and $f'(x)<0$ on $(3,5)$

Therefore, $f(x)$ is increasing on $(-\infty,3)\cup(5,\infty)$ and decreasing on $(3,5)$.

d) $f(x) = x + \sin x$

Sol: $f'(x) = 1 + \cos x$. Since $-1 \leq \cos x \leq 1$, we can say $f'(x) = 1 + \cos x \geqslant 0$. Also, $f'(x) = 0$ only at isolated points $x = \mp \pi, \mp 3\pi, \dots$. Hence, f is increasing everywhere.

Q2) Let $f(x) = x^2 \cdot \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Show that $f'(x)$ exists at every x but f' is not continuous at $x = 0$.

Sol: Let us show $f'(x)$ exists. It must be continuous.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)^2} = \lim_{h \to \mp\infty} \frac{\sin h}{h^2} = 0 = f(0). \quad \left(\begin{array}{l} \frac{1}{x} = h \\ h \to \mp\infty \text{ as } \\ x \to 0 \end{array}\right)$$

Let us find $f'(0)$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{x^2 \cdot \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = \lim_{h \to \mp\infty} \frac{\sin h}{h} = 0.$$

So, $f'(0)$ exists and it is equal to 0.

Let us show that f' is not continuous at $x = 0$.

$$f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) - x^2 \cdot \frac{1}{x^2} \cdot \cos\left(\frac{1}{x}\right) = 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$f'(x)$ is not continuous at $x = 0$ because the limit of $\cos\left(\frac{1}{x}\right)$ does not exist as x goes to 0.

Q3) Find $\frac{dy}{dx}$ in terms of x and y.

a) $x^3 y + x y^5 = 2$

Sol: We must calculate the derivative of both sides of the equation.

$3x^2 \cdot y + x^3 \cdot y' + y^5 + 5xy^4 y' = 0$. Hence,

$$y'(x^3 + 5xy^4) = -3x^2 y - y^5 \implies \frac{dy}{dx} = \frac{-3x^2 y - y^5}{x^3 + 5xy^4}$$

b) $\frac{x-y}{x+y} = \frac{x^2}{y} + 1$

Sol: $\frac{x-y}{x+y} = \frac{x^2}{y} + 1 = \frac{x^2+y}{y} \Rightarrow xy - y^2 = x^3 + x^2y + xy + y^2$ or

$x^3 + x^2y + 2y^2 = 0$. Differentiate with respect to x,

$3x^2 + 2xy + x^2y' + 4yy' = 0 \Rightarrow y'(x^2+4y) = -3x^2 - 2xy$

$\frac{dy}{dx} = -\frac{3x^2+2xy}{x^2+4y}$.

Q4) Find an equation of the tangent to the given curve at the given point.

a) $2x^2 + 3y^2 = 5$ at $(1,1)$

Sol: Let us find $\frac{dy}{dx}$. $4x + 3.2.yy' = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{3y}$

At $(1,1)$, $\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-2.1}{3.1} = -\frac{2}{3}$. Hence,

$y - 1 = \frac{-2}{3}(x-1) \Rightarrow 2x + 3y = 5$.

b) $\frac{x}{y} + \left(\frac{y}{x}\right)^3 = 2$ at $(-1,-1)$

Sol: Let us find $\frac{dy}{dx}$, but we must rearrange the equation. $\frac{x}{y} + \frac{y^3}{x^3} = 2 \Rightarrow x^4 + y^4 = 2x^3y$. Then,

$4x^3 + 4y^3y' = 6x^2y + 2x^3y' \Rightarrow \frac{dy}{dx} = \frac{4x^3 - 6x^2y}{2x^3 - 4y^3}$.

$\left.\frac{dy}{dx}\right|_{(-1,-1)} = \frac{4(-1)^3 - 6(-1)^2(-1)}{2(-1)^3 - 4(-1)^3} = \frac{-4+6}{-2+4} = 1$. Tangent line:

$y + 1 = 1(x+1) \Rightarrow y = x$.

c) $\tan(xy^2) = \frac{2xy}{\pi}$ at $\left(-\pi, \frac{1}{2}\right)$.

Sol: $[1+\tan^2(xy^2)][y^2 + 2xyy'] = \frac{2}{\pi}(y + xy')$. At $\left(-\pi, \frac{1}{2}\right)$,

$$\left[1+\tan^2\left(\frac{-\pi}{4}\right)\right]\left[\frac{1}{4} + 2(-\pi)\cdot\frac{1}{2}\cdot y'\right] = \frac{2}{\pi}\left(\frac{1}{2} + (-\pi)y'\right).$$ Hence,

$$2\cdot\left(\frac{1}{4} - \pi y'\right) = \frac{2}{\pi}\left(\frac{1}{2} - \pi y'\right) \Rightarrow y' = \frac{\pi - 2}{4\pi(\pi - 1)}.$$ The tangent

line has the equation $y - \frac{1}{2} = \frac{\pi - 2}{4\pi(\pi - 1)}(x + \pi)$.

Q5) Let $x^3 - y^2 + y^3 = x$. Find y'' in terms of x and y.

Sol: $3x^2 - 2yy' + 3y^2y' = 1 \Rightarrow y' = \frac{1 - 3x^2}{3y^2 - 2y}$. (*)

$$6x - 2(y')^2 - 2yy'' + 6y(y')^2 + 3y^2y'' = 0.$$ Hence,

$$y'' = \frac{(2 - 6y)(y')^2 - 6x}{3y^2 - 2y}. \quad (**)$$

Let us substitute (*) into (**). Then,

$$y'' = \frac{(2-6y)\left(\frac{1-3x^2}{3y^2-2y}\right)^2 - 6x}{3y^2 - 2y} = \frac{(2-6y)(1-3x^2)^2}{(3y^2-2y)^3} - \frac{6x}{3y^2-2y}.$$

Q6) For $x^2 + y^2 = a^2$ show that $y'' = -\frac{a^2}{y^3}$.

Sol: $x^2 + y^2 = a^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$. Also,

$$2 + 2(y')^2 + 2yy'' = 0.$$ Hence,

$$y'' = \frac{-2-2(y')^2}{2y} = -\frac{2+2\left(-\frac{x}{y}\right)^2}{2y} = -\frac{1+\frac{x^2}{y^2}}{y} = -\frac{x^2+y^2}{y^3} = -\frac{a^2}{y^3} \quad //$$

Q7) Find the indefinite integral $\int \frac{1+\cos^3 x}{\cos^2 x}\,dx$

Sol: $\int \frac{1+\cos^3 x}{\cos^2 x}\,dx = \int\left(\frac{1}{\cos^2 x} + \frac{\cos^3 x}{\cos^2 x}\right)dx = \int(\sec^2 x + \cos x)dx$

$= \tan x + \sin x + C.$

Q8) $\int \frac{6(x-1)}{x^{4/3}}\,dx = ?$

Sol: $\int \frac{6(x-1)}{x^{4/3}}\,dx = \int \frac{6x-6}{x^{4/3}}\,dx = \int\left(\frac{6x}{x^{4/3}} - \frac{6}{x^{4/3}}\right)dx$

$= \int(6.x^{-1/3} - 6.x^{-4/3})dx = 9x^{2/3} + 18x^{-1/3} + C$

Q9) Find the given indefinite integrals. This may require guessing the form of an antiderivative and then checking by differentiation.

a) $\int \frac{dx}{(1+x)^2} = ?$ b) $\int 2x.\sin(x^2)\,dx = ?$

Sol: a) The derivative of $c(1+x)^k$ is $ck(1+x)^{k-1}$.

If $k-1=-2$, then $k=-1$ and $c=-1$. Hence,

$\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x} + C$

b) We know that derivative of $f(x^2)$ is $2x.f'(x^2)$.

Also, $(-\cos x)' = \sin x$. Hmm,

We can conclude

$\int 2x.\sin(x^2)\,dx = -\cos(x^2) + C.$