14.6) Change of Variables in Triple Integrals.

Let $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ be a 1-1 transformation from a domain S in the uvw-space onto a domain D in the xyz-space. Suppose that x, y, z and their 1st partial derivatives wrt u, v, w are continuous.

If $f(x,y,z)$ is integrable on D and if

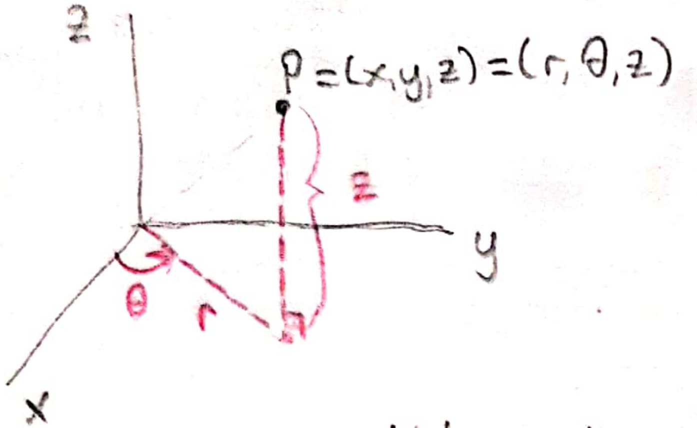
$$g(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w)),$$

then g is integrable on S and

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_S g(u,v,w) \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}_{\text{volume element.}}$$

* We can find volume elements in cylindrical coords. and spherical coords by this formula.

Cylindrical Coordinates (r, θ, z)



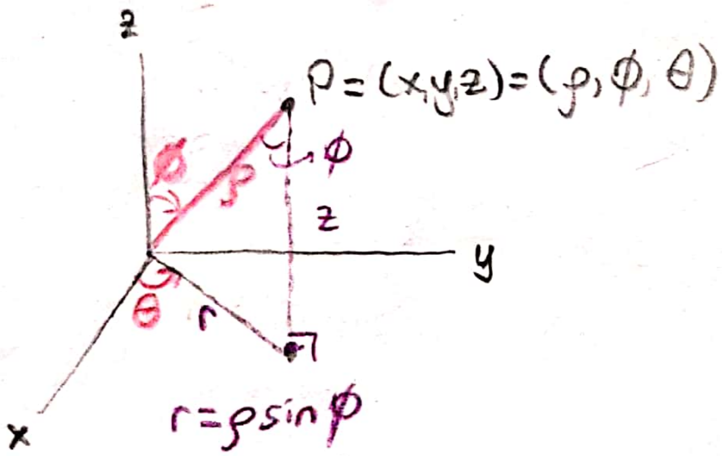
$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$

$x^2+y^2=r^2$, $\tan\theta = \dfrac{y}{x}$

Volume element: $dV = r\,dr\,d\theta\,dz$

$\left|\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$

Spherical Coordinates (ρ, ϕ, θ)



$r = \rho\sin\phi$

$z = \rho\cos\phi$

$$\left.\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}\right\}$$

$\rho^2 = x^2+y^2+z^2 = r^2+z^2$

$\tan\phi = \dfrac{\sqrt{x^2+y^2}}{z}$, $\tan\theta = \dfrac{y}{x}$

Volume element: $dV = \underbrace{\rho^2\sin\phi}\,d\rho\,d\phi\,d\theta$

$\left|\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \rho^2\sin\phi$

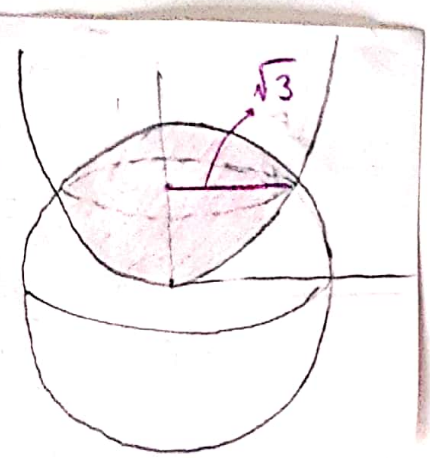
4-6 Find the volumes of the indicated regions.

Q4) Inside the paraboloid $z=x^2+y^2$ and inside the sphere $x^2+y^2+z^2=12$.

Cylindrical coordinates: $r^2=x^2+y^2$

$\Rightarrow$ $z=r^2$ and $r^2+z^2=12$ intersect on

$r^2+r^4=12 \Rightarrow r^4+r^2-12=0 \Rightarrow r=\sqrt{3}$



r is from 0 to $\sqrt{3}$

θ is from 0 to 2π

z is from r^2 to $\sqrt{12-r^2}$

So, the volume of the region can be calculated by

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r^2}^{\sqrt{12-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r\,dr \int_{r^2}^{\sqrt{12-r^2}} dz$$

$$= 2\pi \int_0^{\sqrt{3}} r\,dr \left(z\Big|_{r^2}^{\sqrt{12-r^2}} \right) = 2\pi \int_0^{\sqrt{3}} \left(\sqrt{12-r^2}\,r - r^3\right) dr$$

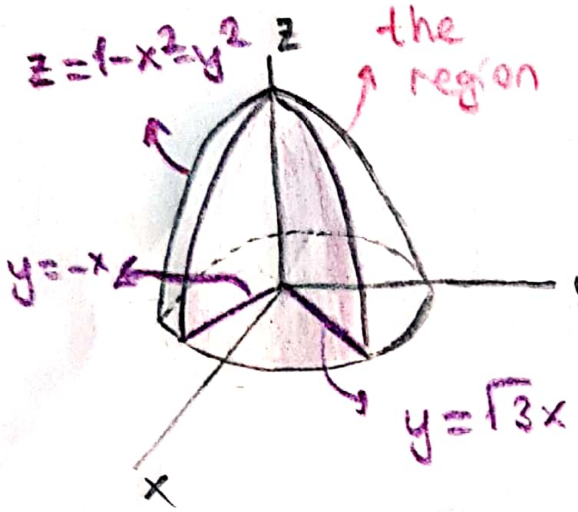
$$\doteq 2\pi\left(\int_{12}^{9} -\frac{1}{2}\sqrt{u}\,du - \int_0^{\sqrt{3}} r^3\,dr\right)$$

$\{ u=12-r^2,\ du=-2r\,dr,\ u(0)=12,\ u(\sqrt{3})=9$

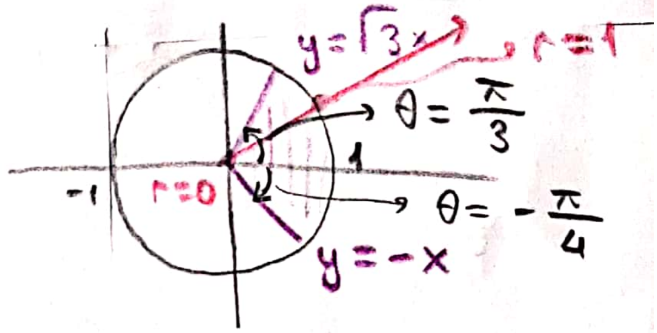
$$= 2\pi\left(-\frac{1}{3}u^{3/2}\Big|_{12}^{9} - \frac{r^4}{4}\Big|_0^{\sqrt{3}}\right) = 2\pi\left(-9 + \frac{12^{3/2}}{3} - \frac{9}{4}\right)$$

$$= 2\pi\left(-9 + 8\sqrt{3} - \frac{9}{4}\right) = -\frac{45\pi}{2} + 16\sqrt{3}\pi$$

Q6) Above the xy-plane, under the paraboloid $z = 1 - x^2 - y^2$ and in the wedge $-x \leq y \leq \sqrt{3}x$



On xy-plane, the integration region becomes



Paraboloid: $z = 1 - x^2 - y^2 = 1 - r^2$

So the volume of the region can be calculated by

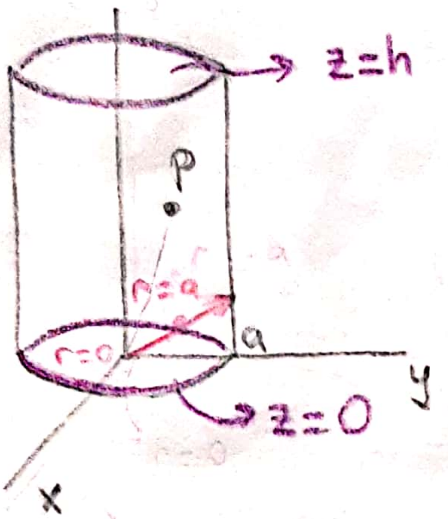
$$\int_{-\pi/4}^{\pi/3}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \int_{-\pi/4}^{\pi/3} d\theta \int_0^1 r\,dr \int_0^{1-r^2} dz$$

$$= \frac{7\pi}{12}\int_0^1 r\,dr\left(z\Big|_0^{1-r^2}\right) = \frac{7\pi}{12}\int_0^1 r(1-r^2)\,dr$$

$$= \frac{7\pi}{12}\int_0^1 (r - r^3)\,dr = \frac{7\pi}{12}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{7\pi}{48}$$

$\boxed{10}$ Evaluate $\iiint_R (x^2+y^2+z^2)\,dV$ where

R is the cylinder $0 \le x^2+y^2 \le a^2$, $0 \le z \le h$.



- Cylindrical coordinates: (r, θ, z)

  $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le h$

- Volume element: $dV = r\,dr\,d\theta\,dz$

- $x^2+y^2+z^2 = r^2+z^2$.

So, the given integral becomes

$$\int_0^h \int_0^{2\pi} \int_0^a (r^2+z^2)\,r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^h \int_0^a (r^3+z^2r)\,dr\,dz$$

$$= 2\pi \int_0^h \left(\frac{r^4}{4}+\frac{z^2r^2}{2}\right)\Bigg|_{r=0}^{r=a} dz$$

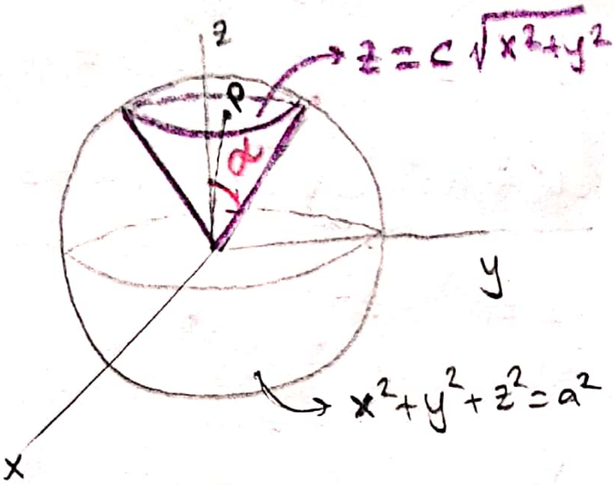
$$= 2\pi \int_0^h \left(\frac{a^4}{4}+\frac{a^2z^2}{2}\right) dz$$

$$= 2\pi \left(\frac{a^4z}{4}+\frac{a^2z^3}{6}\right)\Bigg|_{z=0}^{z=h}$$

$$= \pi\left(\frac{a^4h}{2}+\frac{a^2h^3}{3}\right)$$

$\boxed{13}$ Find $\iiint_R (x^2+y^2+z^2)\,dV$ where

R is the region that lies above the cone $z = c\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = a^2$

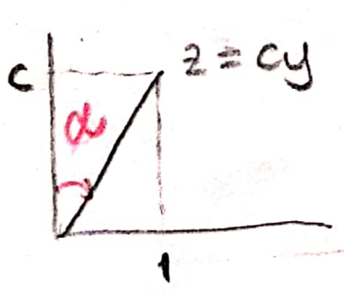


- Spherical coordinates: (ρ, ϕ, θ)
- Volume element: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$
- $x^2+y^2+z^2 = \rho^2$

For any point $P = (\rho, \phi, \theta)$ inside the region,

$0 \le \rho \le a, \quad 0 \le \phi \le \alpha, \quad 0 \le \theta \le 2\pi$

Consider the projection of the cone on yz-plane $(x=0)$



$z = cy \Rightarrow \dfrac{y}{z} = \dfrac{1}{c}$ $\qquad \tan\alpha = \dfrac{y}{z} = \dfrac{1}{c}$

$\Rightarrow \alpha = \arctan\left(\frac{1}{c}\right)$

So we will calculate

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\arctan\left(\frac{1}{c}\right)} \int_{\rho=0}^{\rho=a} (\rho^2) \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{volume element}}$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\arctan\left(\frac{1}{c}\right)} \sin\phi \int_{\rho=0}^{\rho=a} \rho^4 \, d\rho \, d\phi \, d\theta$$

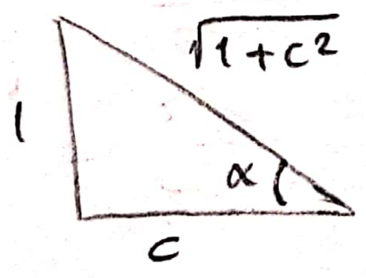
$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\arctan\left(\frac{1}{c}\right)} \sin\phi \left(\frac{\rho^5}{5} \Big|_0^a \right) d\phi \, d\theta$$

$$= \frac{a^5}{5} \int_{\theta=0}^{\theta=2\pi} (-\cos\phi) \Big|_0^{\arctan\left(\frac{1}{c}\right)} d\theta$$

$$= -\frac{a^5}{5} \left[\cos\left(\arctan\left(\frac{1}{c}\right)\right) - 1 \right] \int_0^{2\pi} d\theta$$

$$= -\frac{2\pi a^5}{5} \left[\cos\left(\arctan\left(\frac{1}{c}\right)\right) - 1 \right]$$

$$\left[\quad \right.$$



$$\alpha = \arctan\left(\frac{1}{c}\right) \Rightarrow \tan\alpha = \frac{1}{c}$$

$$\Rightarrow \cos\alpha = \frac{c}{\sqrt{1+c^2}}$$

$$\left. \quad \right]$$

$$= -\frac{2\pi a^5}{5} \left(\frac{c}{\sqrt{1+c^2}} - 1 \right)$$

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