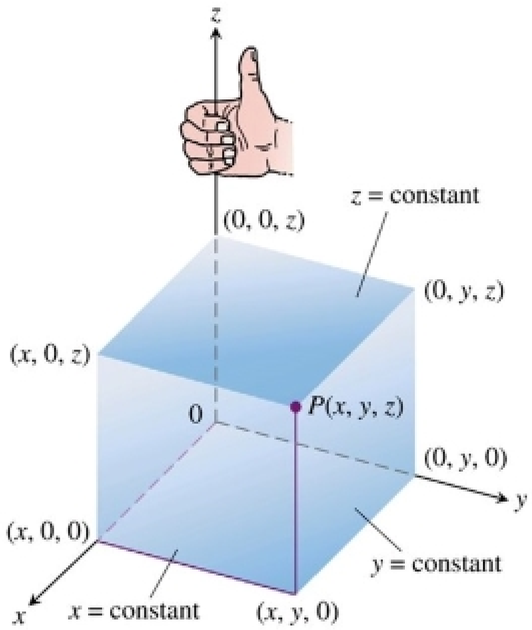
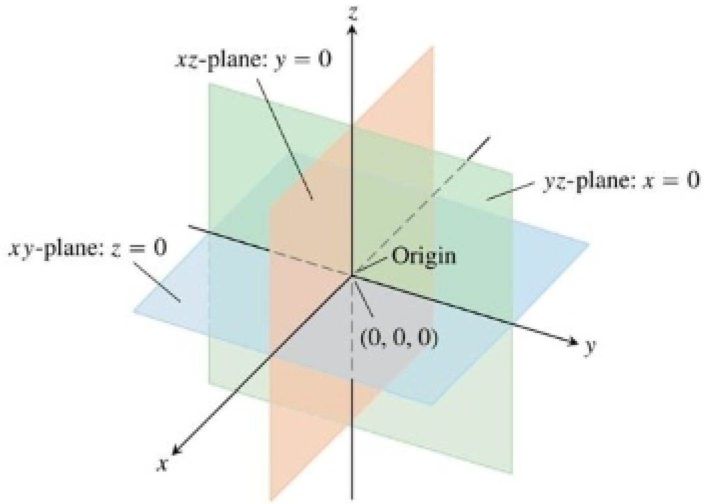
§10.1. Analytic Geometry in Three Dimensions

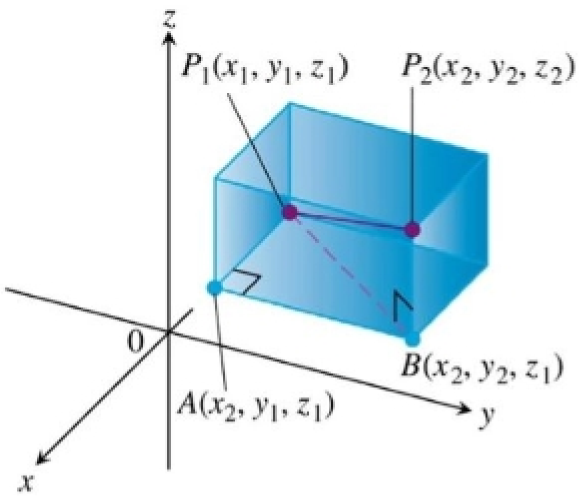
To locate a point $P = (x, y, z)$ in a space, we use three mutually perpendicular coordinate axes, arranged in figure which is the cartesian coordinate system in right-handed.



The planes $x = 0$, $y = 0$ and $z = 0$ divide space into eight octant as in the following figure.



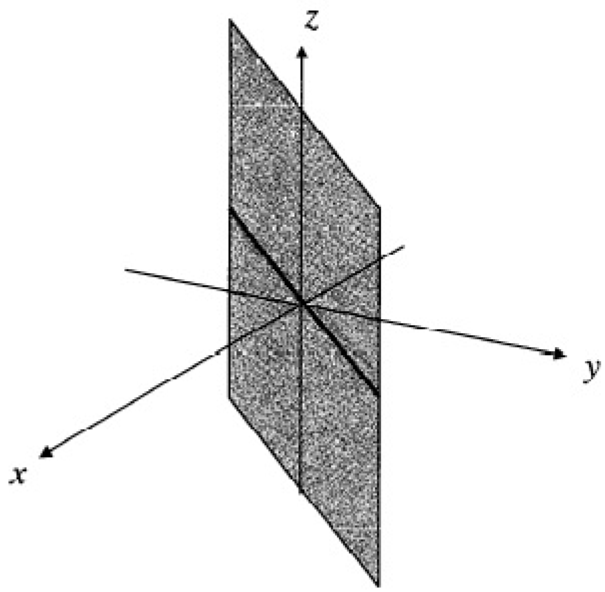
Definition. The distance between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is $r = |P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$. One may find it by applying Phytagorean theorem to the right triangles P_1AB and P_1BP_2 above.



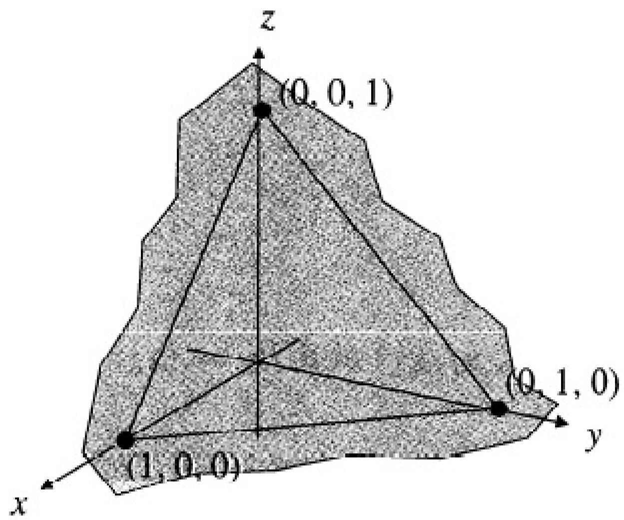
An equation or inequality involving three variables x, y and z defines a subset of points in 3-space whose coordinates satisfy the equation or inequality. A single equation usually represents a surface (a two-dimensional object) in 3-space.

Example 1. a. The equation $z = 0$ represents all points with coordinates $(x, y, 0)$, that is, the xy-plane. The equation $z = -2$ represents all points with coordinates $(x, y, -2)$, that is the horizontal plane passing through the point $(0, 0, -2)$ on the z-axis.

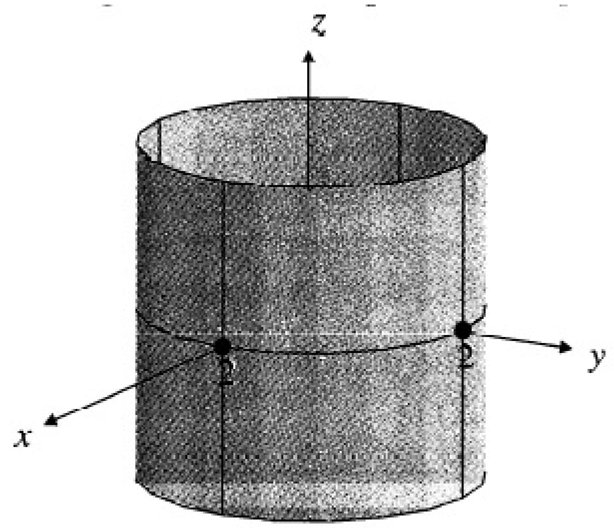
b. The equation $x = y$ represents all points with coordinates (x, x, z). This is a vertical plane containing the straight with equation $x = y$ in the xy-plane. The plane also coordinates the z-axis.



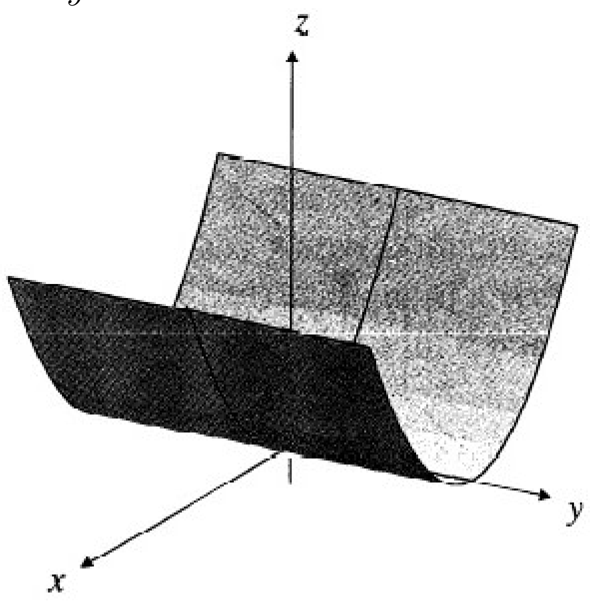
c. The equation $x + y + z = 1$ represents all points the sum of whose coordinates is 1. The set is a plane passes through the three points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. These points are not collinear (they do not lie on a straight line), so there is only one plane passing through all three. The equation $x + y + z = 0$ represents a plane parallel to the one with equation $x + y + z = 1$ but passing through the origin.



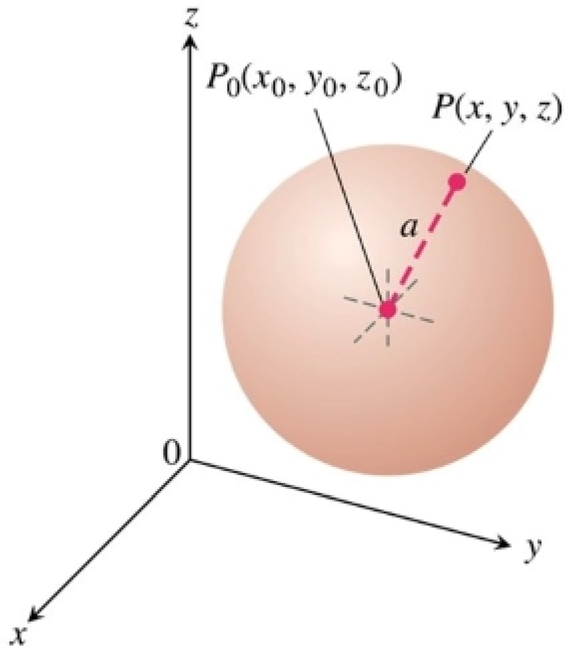
d. The equation $x^2 + y^2 = 4$ represents all points on the vertical circular cylinder containing the circle with the equation $x^2 + y^2 = 4$ in the xy-plane. The circular has radius 2 and along the z-axis.



e. The equation $z = x^2$ represents all the point with coordinates (x, y, x^2). This surface is a parabolic cylinder tangent to xy-plane along the y-axis.

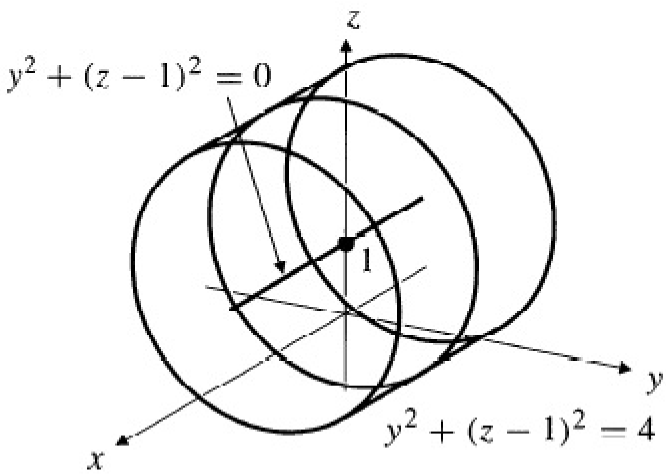


f. The equation $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2$ represents all points (x, y, z) at distance a from the center $P_0 = (x_0, y_0, z_0)$. This set of points is a sphere of radius a centered at $P_0 = (x_0, y_0, z_0)$.



Observe that equations in x, y and z need not involve each variable explicitly. When one of the variables is missing from the equation, the equation represents a surface parallel to the axis of missing variable. Such a surface may be a plane or a cylinder. For example, if z is absent from the equation, the equation represents in 3-space a vertical (i.e., parallel to the z-axis) surface containing the curve with the same equation in the xy-plane.

A single equation may not represent a two-dimensional object (a surface). It can represent a one-dimensional object (a line or a curve), a zero-dimensional object (one or more points), or even nothing at all.



Example 2.

a. Since x is absent, the equation $y^2 + (z-1)^2 = 4$ represents and object parallel to the x-axis. In the yz-plane the equation represents a circle of radius 2 centered at $(y, z) = (0, 1)$. In 3-space it represents a horizontal circular cylinder parallel to x-axis.

b. Since squares can not be negative $y^2 + (z-1)^2 = 0$ implies that represents $y = 0$ and $z = 1$, so it represents points $(x, 0, 1)$. All the points lie on the line parallel to x-axis and one unit above it.

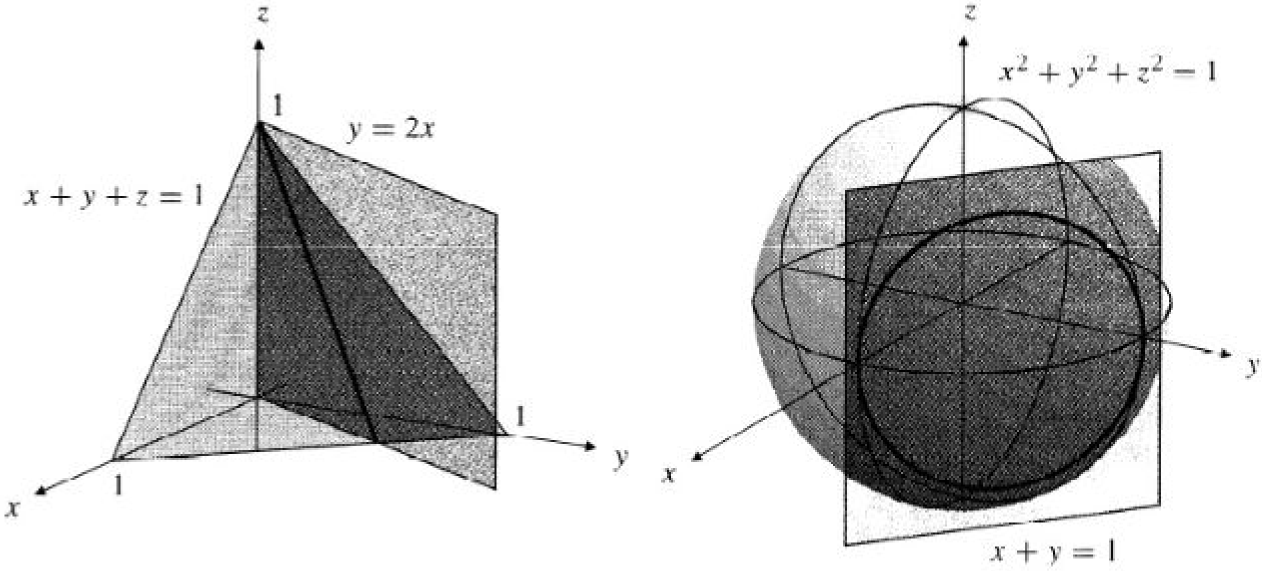
Note that $x^2 + y^2 + z^2 = -3$ is not satisfied by any real number x, y and z so it represents no points at all!

A single inequality x, y, and z typically represents points lying on one side of the surface represented by the corresponding equation (together with points on the surface if the inequality is not strict).

Example 3. a. The inequality $z < 0$ represents all points below the xy-plane.
b. The inequality $x^2 + y^2 \geq 4$ says that the square of the distance from x, y, z to the nearest point $(0, 0, z)$ on the z-axis is at least 4. This inequality represents all points lying on or outside the cylinder of Example 2(d).
c. The inequality $x^2 + y^2 + z^2 \leq a^2$ says that the square of the distance from (x, y, z) to the origin is no longer than a^2. It represents the solid ball of radius a centered at the origin, which contains of all points lying inside or on the sphere of Example 2(f).

Two equations in x, y and z normally represent a one-dimensional object, the line or curve along which the two surfaces represented by two equations intersect. Any point whose coordinates satisfy both equations must lie on both the surfaces, so must lie on their intersection.

Example 4. What sets of points in 3-space are represented by the following pairs of equations?



(a) (b)

a. $\begin{cases} x + y + z = 1 \\ y - 2x = 0 \end{cases}$

The equation $x + y + z = 1$ represents the oblique plane of Example 2(c), and the equation $y - 2x = 0$ represents a vertical plane through the origin and the point $(1, 2, 0)$. Together these two equations represent the line of intersection of the two planes. This lines passes through, for example, the points $(0,0,1)$ and $(\frac{1}{3},\frac{2}{3},0)$.

b. $\begin{cases} x^2+y^2+z^2=1 \\ x+y=1 \end{cases}$

The equation $x^2+y^2+z^2=1$ represents a sphere of radius 1 with centre at the origin, and $x+y=1$ represents a vertical plane through the points $(1,0,0)$ and $(0,1,0)$. The two surfaces intersect in a circle. The line from $(1,0,0)$ to $(0,1,0)$ is a diameter of the circle, so the centre of the circle is $(\frac{1}{2},\frac{1}{2},0)$, and its radius is $\frac{\sqrt{2}}{2}$.

§Euclidean n-Space

The dealing with higher order dimensional spaces is to regard the points in n-space as being ordered n-tuples of real numbers; that is, $(x_1,x_2,...,x_n)$ is a point n-space instead of just being the coordinates of such a point. We stop thinking of points as existing in physical space and start thinking of them as algebraic objects. We denote n-space by the symbol $\mathbb{R}^n$ to show that its points are n-tuples of real numbers.

§Describing Sets in the Plane, 3-Space, and n-Space

In this section we collect some definitions of terms used to describe sets of points in $\mathbb{R}^n$ for $n \geq 2$. These terms belong to the branch of mathematics called topology, and they generalize the no-tions of open and closed intervals and endpoints used to describe sets on the real line $\mathbb{R}$. We state the definitions for $\mathbb{R}^n$, but we are most interested in the cases where $n = 2$ or $n = 3$.

A **neighbourhood** of a point P in $\mathbb{R}^n$ is a set of the form
$B_r(P) = \{Q \in \mathbb{R}^n$:distance from Q to $P < r\}$ for some $r > 0$.
For $n = 1$, if $p \in \mathbb{R}$, then $B_r(P)$ is the **open interval** $(p - r, p + r)$ centered at p.
For $n = 2$, $B_r(P)$ is the **open disk** of radius r centered at point P.
For $n = 3$, $B_r(P)$ is the **open ball** of radius r centered at point P.

Definition. A set is **open** in $\mathbb{R}^n$ if every point of S has neighbourhood contained in S.
Every neighbourhood is itself an open set. Other examples of open sets in $\mathbb{R}^2$ include the sets of points (x, y) such that $x > 0$, or such that $y > x^2$, or even such that $y \neq x^2$.
Typically, sets defined by strict inequalities (using $>$ and $<$) are open. Examples in $\mathbb{R}^3$ include the sets of points (x, y, z) satisfying $x + y + z > 2$, or $1 < x < 3$.

The whole space $\mathbb{R}^n$ is an open set in itself. For technical reasons, the empty set (containing no points) is also considered to be open.

(No point in the empty set fails to have a neighbourhood contained in the empty set.)

The **complement** of a set S in $\mathbb{R}^n$; S^c; is the set of all points in $\mathbb{R}^n$ that do not belong to S.

For example, the complement of the set of the points (x, y) in $\mathbb{R}^2$ such that $x > 0$ is the set of points for which $x \leq 0$.

A set is said to be **closed** if its complement is open.

Typically, sets defined by nonstrict inequalities (using $\geq$ and $\leq$) are closed. Closed intervals are closed sets in $\mathbb{R}$. Since the whole space and the empty set are both open in $\mathbb{R}^n$ and are complements of each other, they are also both closed. They are the only sets that are both opened and closed.

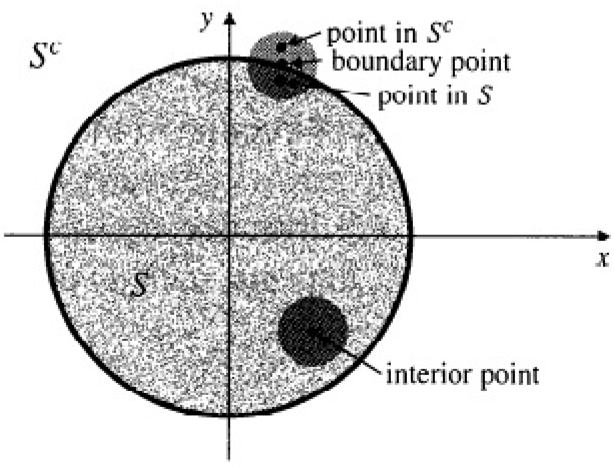
A point P is called a **boundary point** of a set S if every neighbourhood of P contains both points in S and points in S^c.

The boundary; brdy(S); of a set S is the set of all boundary points of S. For example, the boundary of the closed disk $x^2 + y^2 \leq 1$ in $\mathbb{R}^2$ is the circle $x^2 + y^2 = 1$. A closed set contains all its boundary points. An open set contains none of its boundary points.

A point P is an **interior point** of a set S if it belongs to S but not the boundary of S.

P is an **exterior point** of S if it belongs to complement of S but not to the boundary of S.

The **interior**; $int(S)$; and **exterior**; $ext(S)$; of S consist of all the interior points and exterior points S, respectively. Both $int(S)$ and $ext(S)$ are open sets. If S is open, then $int(S) = S$. If S is closed, then $ext(S) = S^c$.



The closed disk S consisting of points $(x, y) \in \mathbb{R}^2$ that satisfy $x^2 + y^2 \leq 1$. Note the shaded neighbourhoods of the boundary point and the interior point.
bdry(S) is the circle $x^2 + y^2 = 1$
int(S) is the open disk $x^2 + y^2 < 1$
ext(S) is the open set $x^2 + y^2 > 1$