§12.8. Implicit Functions

In $\mathbb{R}^2$, an implicit function is given such an equation $F(x, y(x)) = 0$ where x is independent variable and y depends on x.

For example, $x^2 + y^2 = 1$.

Let the equation $F(x, y) = 0$ be satisfied at the point (a, b) and the first partial derivatives of F be continuous at all near (a, b). Then F is a differentiable at that point.

Can the equation be solved for y as a function of x near (a, b)? That is, is there a function $y(x)$ defined in some interval $I = (a - h, a + h)$ where $h > 0$ satisfying $y(a) = b$ and such that $F(x, y(x)) = 0$ holds for all x in I?

If there is such a function $y(x)$, we can try to find its derivative at $x = a$ by differentiating the equation $F(x, y) = 0$ implicitly with respect to x and evaluating the result at (a, b).

$F_1(x, y) + F_2(x, y)\frac{dy}{dx} = 0$ so that $\frac{dy}{dx}|_{x=a} = -\frac{F_1(a,b)}{F_2(a,b)}$ if $F_2(a, b) \neq 0$.

This condition implies that the level curve $F(x, y) = F(a, b)$ has nonvertical tangent lines near (a, b).

A similar situation holds for equations involving several variables. Let $F(x, y, z) = 0$ define z as a function of x and y (say $z = z(x, y)$) near some point $P_0 = (x_0, y_0, z_0)$ satisfying the equation. If F has continuous first partial near P_0, then the partial derivates of

z can be found at (x_0, y_0) by implicit differantiation of the equation $F(x, y, z) = 0$ with respect to x and y:

$F_1(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial x} = 0$ and $F_2(x, y, z) + F_3(x, y, z)\frac{\partial z}{\partial y} = 0$, so that:

$\frac{\partial z}{\partial x}|_{(x_0,y_0)} = -\frac{F_1(x_0,y_0,z_0)}{F_3(x_0,y_0,z_0)}$ and $\frac{\partial z}{\partial y}|_{(x_0,y_0)} = -\frac{F_2(x_0,y_0,z_0)}{F_3(x_0,y_0,z_0)}$ provided $F_3(x_0, y_0, z_0) \neq 0$.

Since F_3 is the z component of the gradient vector of F, this condition implies that the level surface of F passing through P_0 does not have a horizontal normal vector, so it is not vertical (it is not parallel to the z-axis).

Example 1. Near what points on the sphere $x^2 + y^2 + z^2 = 1$ can the equation of the sphere be solved for z as a function of x and y? Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at such points.

Let $F(x, y, z) = x^2 + y^2 + z^2 - 1$. The sphere is the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

The above equation can be solved for $z = z(x, y)$ near P_0 provided P_0 is not on the equator of the sphere $x^2 + y^2 = 1$ and $z = 0$ because the equator consists of those points that satisfy $F_3(x, y, z) = 0$. If P_0 is not on the equator, then it is on either the upper or the lower hemisphere, represented by $z = z(x, y) = \sqrt{1 - x^2 - y^2}$ and $z = z(x, y) = -\sqrt{1 - x^2 - y^2}$, respectively.

If $z \neq 0$, we can compute the first partial derivates of the solution $z = z(x, y)$ by implicity differentiating the equation of the sphere $x^2 + y^2 + z^2 = 1$:

$2x + 2z\frac{\partial z}{\partial x} = 0 \quad \text{so} \quad \frac{\partial z}{\partial x} = -\frac{x}{z};$

$2y + 2z\frac{\partial z}{\partial y} = 0 \quad \text{so} \quad \frac{\partial z}{\partial y} = -\frac{y}{z}.$

§Systems of Equations

Consider the system of equations: $\begin{cases} F(x, y, z, w) = 0 \\ G(x, y, z, w) = 0 \end{cases}.$

It might possess near some point that satisfies them solutions of one or more of the forms :

$$\begin{cases} x = x(z, w) \\ y = y(z, w) \end{cases}, \begin{cases} x = x(y, w) \\ z = z(y, w) \end{cases}, \begin{cases} x = x(y, z) \\ w = w(y, z) \end{cases}.$$

$$\begin{cases} y = y(x, w) \\ z = z(x, w) \end{cases}, \begin{cases} y = y(x, z) \\ w = w(x, z) \end{cases}, \begin{cases} z = z(x, y) \\ w = w(x, y) \end{cases}.$$

Where such solutions exist, we should be able to differentiate the given system of equations implicity to find partial derivates of the solutions. For example, let x and y depend on z and w, that is; $\begin{cases} x = x(z, w) \\ y = y(z, w) \end{cases}.$

Compute $\frac{\partial x}{\partial z}$.

$F_1\frac{\partial x}{\partial z} + F_2\frac{\partial y}{\partial z} + F_3 = 0$

$G_1\frac{\partial x}{\partial z} + G_2\frac{\partial y}{\partial z} + G_3 = 0.$

Note that $F_4\frac{\partial w}{\partial z}$ and $G_4\frac{\partial w}{\partial z}$ are not present because w and z are independent variables. Then $\left(\frac{\partial x}{\partial z}\right)_w = -\frac{F_3G_2-F_2G_3}{F_1G_2-F_2G_1}$.

Example 2. Let x, y, u and v be related by the equations $\begin{cases} u = x^2 + xy - y^2 \\ v = 2xy + y^2 \end{cases}$.

Find a). $\left(\frac{\partial x}{\partial u}\right)_v$ and b). $\left(\frac{\partial x}{\partial u}\right)_y$ at the point $(x, y) = (2, -1)$.

a). We assume x and y as functions of u and v and differentiate the given equations with respect to u holding v.

$1 = \frac{\partial u}{\partial u} = (2x + y)\frac{\partial x}{\partial u} + (x - 2y)\frac{\partial y}{\partial u}$

$0 = \frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u} + (2x + 2y)\frac{\partial y}{\partial u}.$

At $(x, y) = (2, -1)$, we get;

$1 = 3\frac{\partial x}{\partial u} + 4\frac{\partial y}{\partial u}$

$0 = -2\frac{\partial x}{\partial u} + 2\frac{\partial y}{\partial u}.$

Eliminating $\frac{\partial y}{\partial u}$, we have $\frac{\partial x}{\partial u} = \frac{1}{7}$.

b). We assume x and v as functions of y and u and differentiate the given equations with respect to u holding y.

$1 = \frac{\partial u}{\partial u} = (2x + y)\frac{\partial x}{\partial u}$

$\frac{\partial v}{\partial u} = 2y\frac{\partial x}{\partial u}.$

At $(x, y) = (2, -1)$, we get $\left(\frac{\partial x}{\partial u}\right)_y = \frac{1}{3}$.

§Jacobian Determinants

The **Jacobian determinant** (or simply **the Jacobian**) of the two functions, $u = u(x, y)$ and $v = v(x, y)$, with respect to two variables, x and y, is the determinant

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Similarly, the Jacobian of two functions, $F(x, y, \ldots)$ and $G(x, y, \ldots)$, with respect to the variables, x and y, is the determinant

$$\frac{\partial(F, G)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix} = \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix}.$$

The definition above can be extended in the obvious way to give the jacobian of n functions with respect to n variables. For example, the jacobian of three functions F, G and H with respect to x, y and z, is the determinant

$$\frac{\partial(F,G,H)}{\partial(x,y,z)} = \begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix}.$$

Example 3. In terms of jacobians, the value of $\left(\frac{\partial x}{\partial z}\right)_w$, obtained from the system of equations $F(x, y, z, w) = 0$ and $G(x, y, z, w) = 0$.

Then $\left(\frac{\partial x}{\partial z}\right)_w = -\frac{\frac{\partial(F,G)}{\partial(z,y)}}{\frac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_3 & F_2 \\ G_3 & G_2 \end{vmatrix}}{\begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix}} = -\frac{F_3G_2-F_2G_3}{F_1G_2-F_2G_1}$. Note that x and y are dependent variables and z, w are independent variables.

The denominator is the jacobian of F and G with respect to the dependent variables x, y and the numerator is the same jacobian except that the dependent variable x is replaced by the independent variable z.

§The Implicit Function Theorem

Consider a system of n equations in $n + m$ variables,

$$\begin{cases} F_{(1)}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0 \\ F_{(2)}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0 \\ \quad\vdots \\ F_{(n)}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0, \end{cases}$$

and a point $P_0 = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ that satisfies the system. Suppose each of the functions $F_{(i)}$ has continuous first partial derivatives with respect to each of the variables x_j and y_k, $(i = 1, \ldots, n,\ j = 1, \ldots, m,\ k = 1, \ldots, n)$, near P_0. Finally, suppose that

$$\left.\frac{\partial(F_{(1)}, F_{(2)}, \ldots, F_{(n)})}{\partial(y_1, y_2, \ldots, y_n)}\right|_{P_0} \neq 0.$$

Then the system can be solved for $y_1, y_2, \ldots, y_n$ as functions of $x_1, x_2, \ldots, x_m$ near P_0. That is, there exist functions

$$\phi_1(x_1, \ldots, x_m), \ldots, \phi_n(x_1, \ldots, x_m)$$

such that

$$\phi_j(a_1, \ldots, a_m) = b_j, \qquad (j = 1, \ldots, n),$$

and such that the equations

$$F_{(1)}\Big(x_1, \ldots, x_m, \phi_1(x_1, \ldots, x_m), \ldots, \phi_n(x_1, \ldots, x_m)\Big) = 0,$$
$$F_{(2)}\Big(x_1, \ldots, x_m, \phi_1(x_1, \ldots, x_m), \ldots, \phi_n(x_1, \ldots, x_m)\Big) = 0,$$
$$\vdots$$
$$F_{(n)}\Big(x_1, \ldots, x_m, \phi_1(x_1, \ldots, x_m), \ldots, \phi_n(x_1, \ldots, x_m)\Big) = 0,$$

hold for all $(x_1, \ldots, x_m)$ sufficiently near $(a_1, \ldots, a_m)$.

Moreover,

$$\frac{\partial \phi_i}{\partial x_j} = \left(\frac{\partial y_i}{\partial x_j}\right)_{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m} = -\frac{\dfrac{\partial(F_{(1)}, F_{(2)}, \ldots, F_{(n)})}{\partial(y_1, \ldots, x_j, \ldots, y_n)}}{\dfrac{\partial(F_{(1)}, F_{(2)}, \ldots, F_{(n)})}{\partial(y_1, \ldots, y_i, \ldots, y_n)}}.$$

Example 4. Show that the system $\begin{cases} xy^2 + xzu + yv^2 = 3 \\ x^3yz + 2xv - u^2v^2 = 2 \end{cases}$ can be solved for (u, v) as a function of (x, y, z) near the point $P_0 = (1, 1, 1, 1, 1)$ and find the value of $\frac{\partial v}{\partial y}$ for the solution at $(x, y, z) = (1, 1, 1)$.

Let $\begin{cases} F(x, y, z, u, v) = xy^2 + xzu + yv^2 - 3 \\ G(x, y, z, u, v) = x^3yz + 2xv - u^2v^2 - 2 \end{cases}$.

$\frac{\partial(F,G)}{\partial(u,v)}|_{P_0} = \begin{vmatrix} xz & 2yv \\ -2uv^2 & 2x - 2u^2v \end{vmatrix}_{P_0} = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4$. Since the jacobian is nonzero, the theorem assures us that the given equations can be solved for u and v as functions of x,y and z that is, $(u, v) = f(x, y, z)$.

Since $\frac{\partial(F,G)}{\partial(u,y)}|_{P_0} = \begin{vmatrix} xz & 2xy + v^2 \\ -2uv^2 & x^3z \end{vmatrix}_{P_0} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7$, then $\left(\frac{\partial v}{\partial y}\right)_{x,z} = -\frac{7}{4}$.

Example 5. If the equations $x = u^2 + v^2$ and $y = uv$ are solved for u and v in terms of x and y, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$. Show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$.

Let $F(u, v, x, y) = u^2 + v^2 - x = 0$, $G(u, v, x, y) = uv - y = 0$ and $J = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} 2u & 2v \\ v & u \end{vmatrix} = 2(u^2 - v^2) = \frac{\partial(x,y)}{\partial(u,v)}$.

If $u^2 \neq v^2$, then $J = \frac{\partial(F,G)}{\partial(u,v)} \neq 0$

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Thus, $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{J^2}\begin{vmatrix} u & -2v \\ -v & 2u \end{vmatrix} = \frac{J}{J^2} = \frac{1}{J} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$.