SUMS AND SIGMA NOTATION

Definition (Sigma notation)

If "m" and "n" are integers with $m \leq n$, and if f is a function defined at the integers. $m, m+1, m+2, .., n$, the symbol $\sum_{i=m}^{n} f(i)$ represents the sum of the values of f at those integers.

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + .. + f(n).$$

The explicit sum appearing on the right side of this equation of the sum represented in sigma notation on the left side.

upper limit

$$\sum_{i=m}^{n} f(i)$$

lower limit

Example: $\sum_{j=1}^{20} j = 1+2+3+ .. + 18+19+20$

$$\sum_{i=0}^{n} x^i = x^0 + x^1 + x^2 + .. + x^{n-1} + x^n$$

$$\sum_{m=1}^{n} 1 = \underbrace{1+1+ .. +1}_{n\text{-terms}}$$

$$\sum_{k=-2}^{3} \frac{1}{k+7} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

1.

Arithmetic Rules for Finite Sums:

$$\sum_{i=m}^{n} (Af(i) + Bg(i)) = A\sum_{i=m}^{n} f(i) + B\sum_{i=m}^{n} g(i).$$

$$\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^{n} f(i+m) = f(m) + f(m+1) + \cdots + f(m+n)$$

"Change of index"

Example: Express $\sum_{j=3}^{17} \sqrt{1+j^2}$ in the form $\sum_{i=1}^{n} f(i)$.

Let $j = i+2$ then $j=3$ corresponds to $i=1$ and $j=17$ corresponds to $i=15$. Thus,

$$\sum_{j=3}^{17} \sqrt{1+j^2} = \sum_{i=1}^{15} f(i) = \sum_{i=1}^{15} \sqrt{1+(i+2)^2}$$

Evaluating Sums:

Theorem (Summation Formulas)

a) $\sum_{i=1}^{n} 1 = \underbrace{(1+1+\cdots+1)}_{n \text{ terms}} = n.$

b) $\sum_{i=1}^{n} i = 1+2+\cdots+(n-1)+n = \dfrac{n(n+1)}{2}.$

c) $\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

d) $\sum_{i=1}^{n} r^{i-1} = r^0+r^1+r^2+\cdots+r^{n-1} = 1+r+r^2+\cdots+r^{n-1} = \dfrac{r^n-1}{r-1}$ if $r \neq 1$.

Example: Evaluate $\sum_{k=m+1}^{n} (6k^2 - 4k + 3)$, where $1 \leq m < n$.

We know that;

$$\sum_{k=1}^{n} (6k^2 - 4k + 3) = 6 \sum_{k=1}^{n} k^2 - 4 \sum_{k=1}^{n} k + 3 \sum_{k=1}^{n} 1$$

$$= 6 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} + 3n$$

$$= 2n^3 + n^2 + 2n$$

Thus;

$$\sum_{k=m+1}^{n} (6k^2 - 4k + 3) = \sum_{k=1}^{n} (6k^2 - 4k + 3) - \sum_{k=1}^{m} (6k^2 - 4k + 3)$$

$$= 2n^3 + n^2 + 2n - 2m^3 - m^2 - 2m.$$

AREAS AS LIMITS OF SUMS

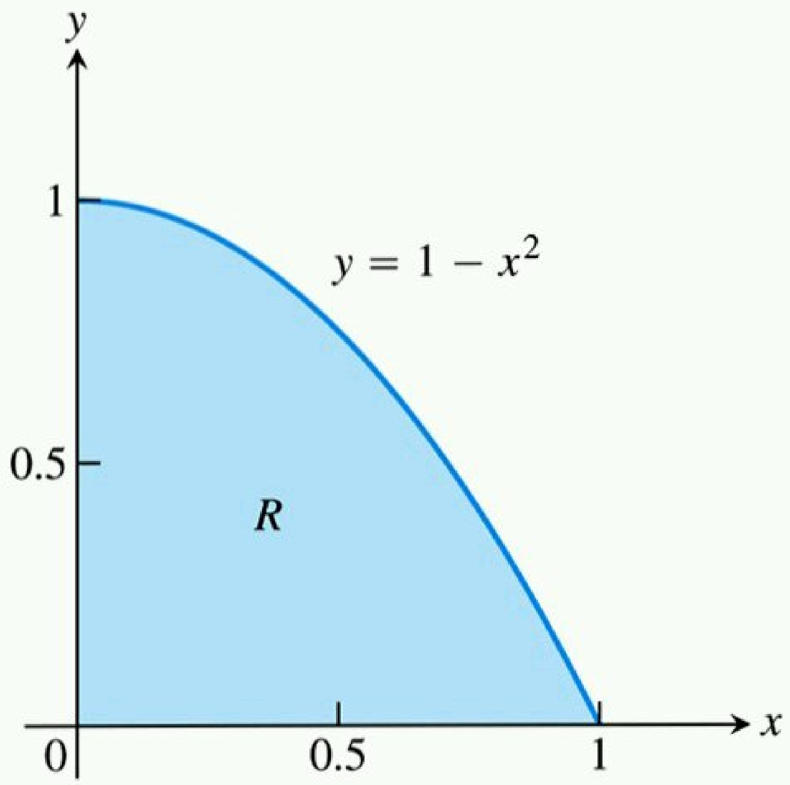


FIGURE 5.1 The area of the region R cannot be found by a simple formula.

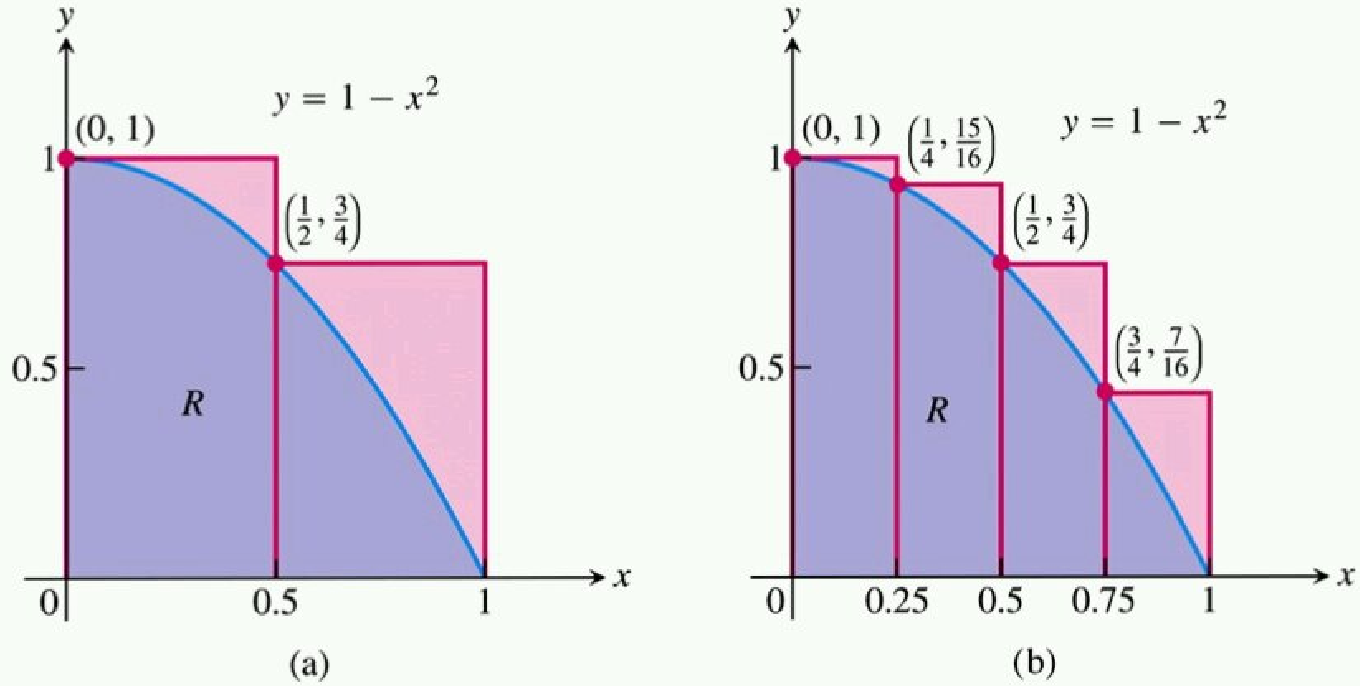


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

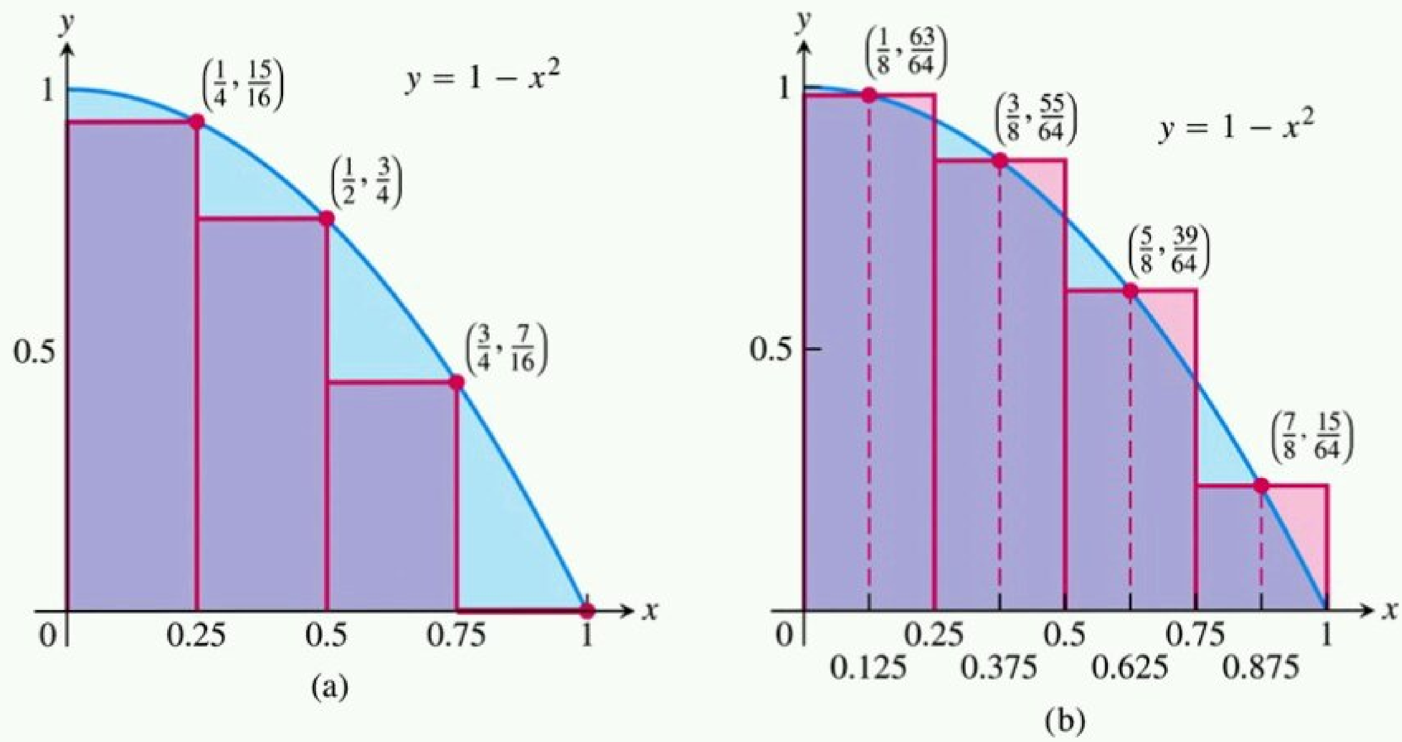
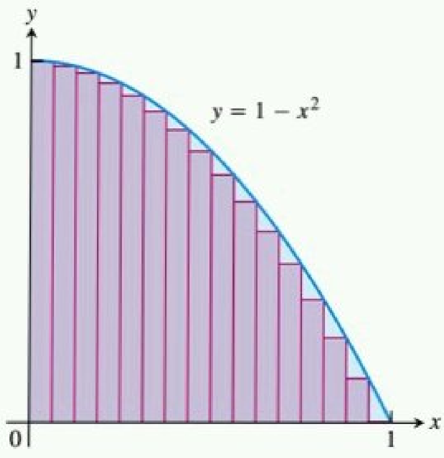
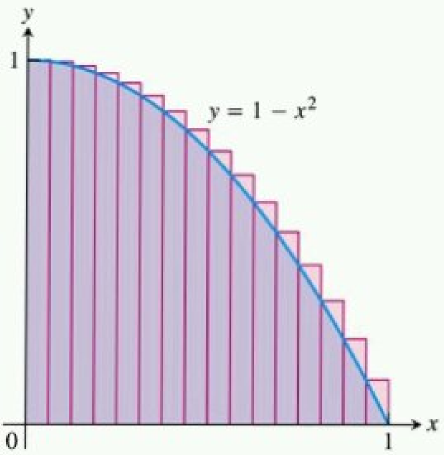


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.



(a)



(b)

FIGURE 5.4 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

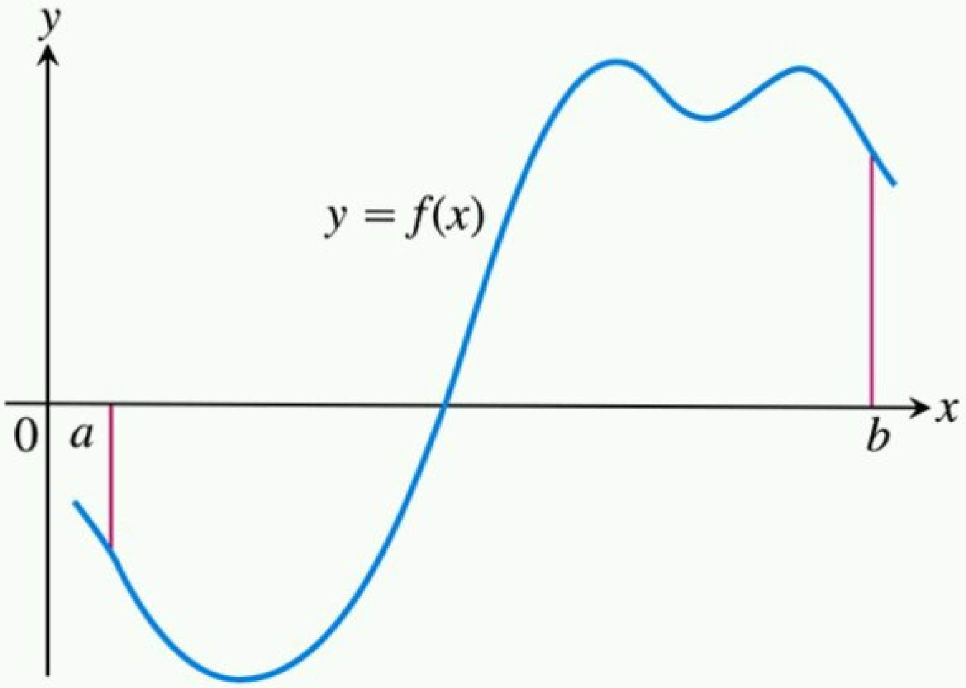
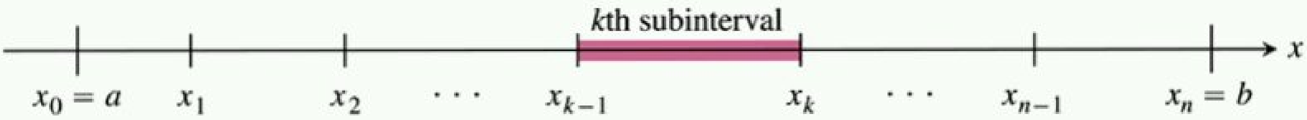
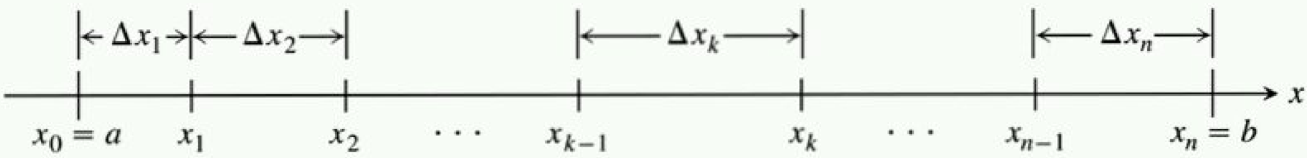


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the **kth subinterval of** P is $[x_{k-1}, x_k]$, for k an integer between 1 and n.



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.



In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$ (Figure 5.9).

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k .$$

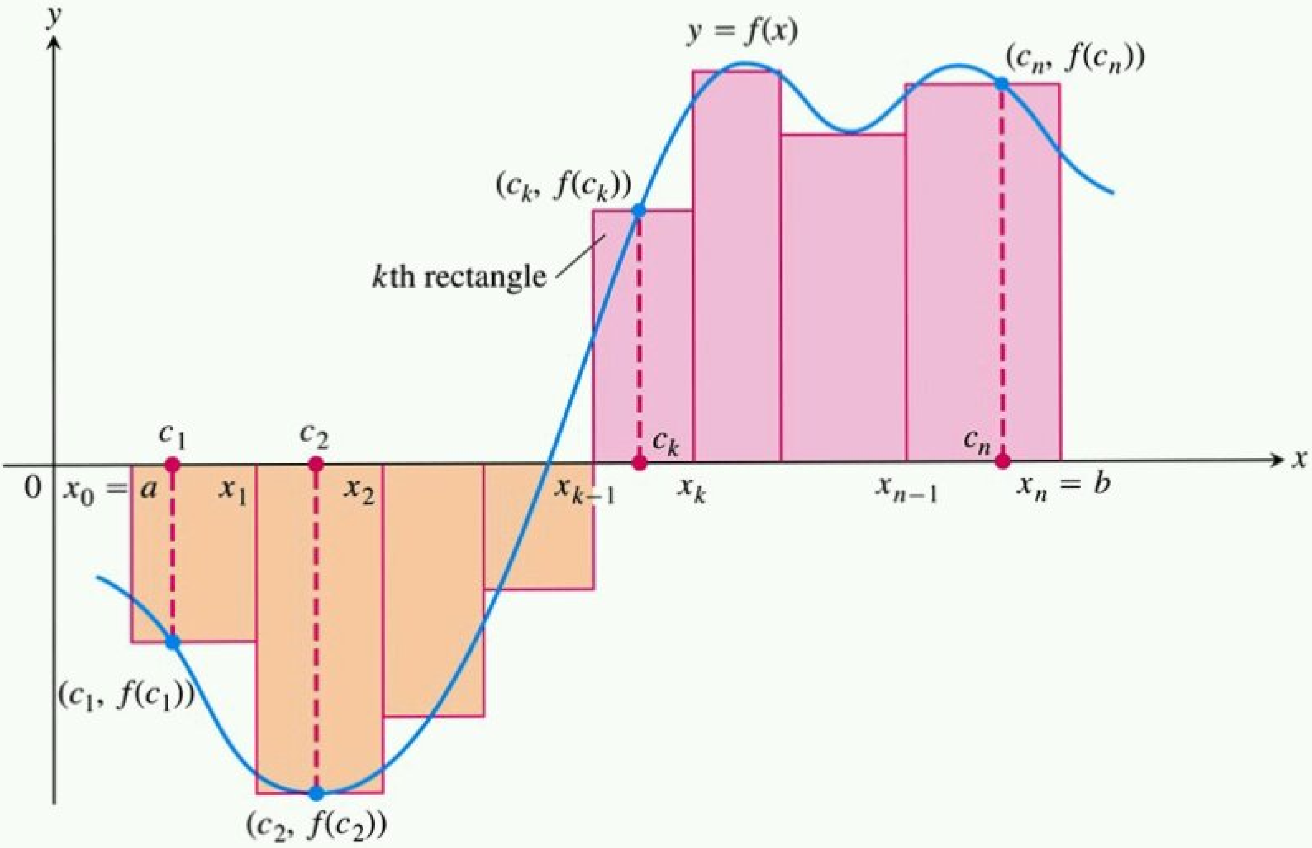


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis. Figure 5.8 has been enlarged to enhance the partition of $[a, b]$ and selection of points c_k that produce the rectangles.

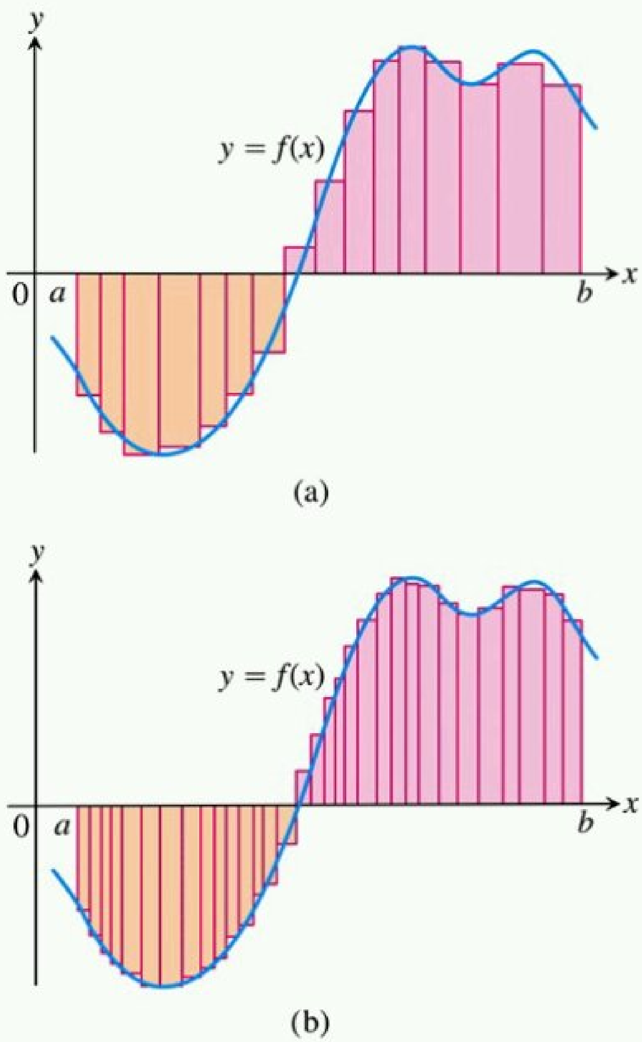
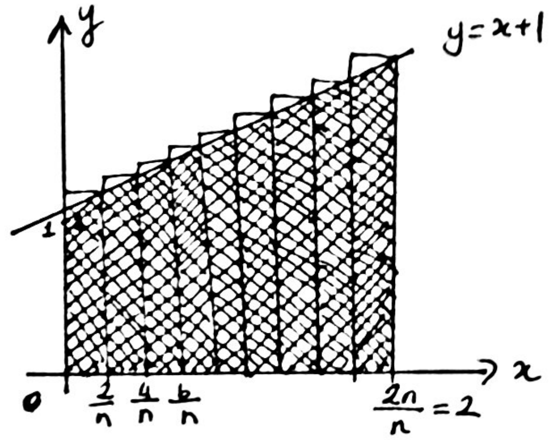


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

$$\text{Area of } R = \lim_{\substack{n\to\infty \\ \max \Delta x_k \to 0}} S_p = \lim_{\substack{n\to\infty \\ \max \Delta x_k \to 0}} \sum_{k=1}^{n} f(c_k)\,\Delta x_k .$$

<u>Example</u>: Find the area A of the region lying under the straight line $y=x+1$, above the x-axis and between the lines $x=0$ and $x=2$.



$x_0=0,\ x_1=\frac{2}{n},\ x_2=\frac{4}{n},\ \dots,\ x_n=\frac{2n}{2}=2.$

The value of $y=x+1$ at $x=x_k$ is $x_k+1=\frac{2k}{n}+1$ and the kth subinterval, $\left[\frac{2(k-1)}{n}, \frac{2k}{n}\right]$ has length $\Delta x_k = \frac{2}{n}$.

Observe that, $\Delta x_k \to 0$ as $n\to\infty$. The sum of the areas of the approximating rectangles shown in figure.

$$S_p = \sum_{k=1}^{n}\left(\frac{2k}{n}+1\right)\frac{2}{n}$$

$$= \left(\frac{2}{n}\right)\left[\frac{2}{n}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right]$$

$$= \left(\frac{2}{n}\right)\left[\frac{2}{n}\,\frac{n(n+1)}{2} + n\right]$$

$$= \frac{2(n+1)}{n} + 2$$

$$A = \lim_{n\to\infty} S_p = \lim_{n\to\infty}\left(\frac{2(n+1)}{n}+2\right) = 2+2=4 \text{ square units.}$$

THE DEFINITE INTEGRAL

Partitions and Riemann Sums:

Let P be a finite set of points arranged in order between "a" and "b" on the real line, say

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\},$$

where $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. Such a set P is called a partition of $[a,b]$; it divides $[a,b]$ into n subintervals of which the kth is $[x_{k-1}, x_k]$. We call these the subintervals of the partition P. The number n depends on the particular partition, so we write $n = n(P)$. The length of the kth subinterval of P is

$$\Delta x_k = x_k - x_{k-1} \quad (\text{for } 1 \le k \le n)$$

and we call the greatest of these numbers Δx_k, the norm of the partition P and denote it $\|P\|$;

$$\|P\| = \max_{1 \le k \le n} \Delta x_k ;$$

Since f is continuous on each subinterval $[x_{k-1}, x_k]$ of P, it takes on maximum and minimum values at points of that interval. Thus there are numbers l_k and u_k in $[x_{k-1}, x_k]$ such that

$$f(l_k) \le f(x) \le f(u_k) \quad \text{whenever} \quad x_{k-1} \le x \le x_k .$$

And $\quad f(l_k)\Delta x_k \le A_k \le f(u_k)\Delta x_k .$

Definition: (Upper and Riemann sums)

The lower (Riemann) sum $\mathcal{L}(f,P)$, and the upper (Riemann) sum, $\mathcal{U}(f,P)$, for the function f and the partition P are defined by ;

$$\mathcal{L}(f,P) = f(l_1)\Delta x_1 + f(l_2)\Delta x_2 + \cdots + f(l_n)\Delta x_n$$

$$= \sum_{k=1}^{n} f(l_k)\Delta x_k ,$$

$$\mathcal{U}(f,P) = f(u_1)\Delta x_1 + f(u_2)\Delta x_2 + \cdots + f(u_n)\Delta x_n$$

$$= \sum_{k=1}^{n} f(u_k)\Delta x_k$$

Example: Calculate lower and upper Riemann sums for the function $f(x) = \frac{1}{x}$ on the interval $[1,2]$, corresponding to the partition P of $[1,2]$ into four subintervals of equal length.

$$P = \left\{ x_0 = 1,\ x_1 = \frac{5}{4},\ x_2 = \frac{3}{2},\ x_3 = \frac{7}{4},\ x_4 = 2 \right\}$$

Since $\frac{1}{x}$ is decreasing on $[1,2]$, its minimum and maximum values on the kth subinterval $[x_{k-1}, x_k]$ are $\frac{1}{x_k}$ and $\frac{1}{x_{k-1}}$, respectively. Thus, the lower and upper Riemann sums are,

$$\mathcal{L}(f,P) = \frac{1}{4}\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right) = \frac{533}{840} \approx 0,6345.$$

$$\mathcal{U}(f,P) = \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) = \frac{319}{420} \approx 0,7595.$$

Definition: (The Definite Integral)

Suppose there is exactly one number I such that for every partition P of $[a,b]$ we have

$$\mathcal{L}(f,P) \leq I \leq \mathcal{U}(f,P)$$

Then we say that the function f is integrable on $[a,b]$, and we call I the definite integral of f on $[a,b]$. The definite integral is denoted by the symbol

$$I = \int_a^b f(x)\,dx.$$

Limits of integration.

differential of x.

integral sign.

integrand.

General Riemann Sums

Let $P = \{x_0, x_1, x_2, \ldots, x_k\}$ where $a = x_0 < x_1 < x_2 \cdots < x_k = b$, be a partition of $[a,b]$ having norm $\|P\| = \max_{1 \le i \le k} \Delta x_i$. In each subinterval $[x_{i-1}, x_i]$ of P pick a point c_i (called a tag). Let $c = \{c_1, c_2, \ldots, c_k\}$ denote the set of these tags. The sum

$$R(f, P, c) = \sum_{i=1}^{k} f(c_i)\,\Delta x_i$$

$$= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_k)\Delta x_k$$

is called the Riemann Sum of f on $[a,b]$ corresponding to partition P and tags c. The Riemann sum satisfies

$$L(f, P) \le R(f, P, c) \le U(f, P)$$

Therefore, if f is integrable on $[a,b]$, then its integral is the limit of such Riemann sums, where the limit is taken as the number $n(P)$ of subintervals of P increases to infinity in such a way that the lengths of all subintervals approach zero. That is;

$$\lim_{\substack{n(P) \to \infty \\ \|P\| \to 0}} R(f, P, c) = \int_a^b f(x)\,dx.$$

Theorem: If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Example: Express the limit $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n}\left(1 + \frac{2i-1}{n}\right)^{1/3}$ as a definite integral.

We want to interpret the sum as a Riemann sum for $f(x) = (1+x)^{1/3}$. The factor $\frac{2}{n}$ suggests that the interval of integration has length 2 and is partitioned into n equal subintervals, each of length $\frac{2}{n}$. Thus, the interval is $[0,2]$, and the points of the partition are $x_i = \frac{2i}{n}$, (If we let $c_i = \frac{2i-1}{n}$ for $i = 1, 2, \ldots, n$, as $n \to \infty$, $c_1 = \frac{1}{n} \to 0$ and $c_n = \frac{2n-1}{n} \to 2$.) Observe that $x_{i-1} = \frac{2i-2}{n} < c_i < \frac{2i}{n} = x_i$ for each i, so that the sum indeed a Riemann sum for $f(x)$ over $[0,2]$. Since f is continuous on $[0,2]$

, it is integrable there, and

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1+\frac{2i-1}{n}\right)^{1/3} = \int_0^2 (1+x)^{1/3}\,dx .$$