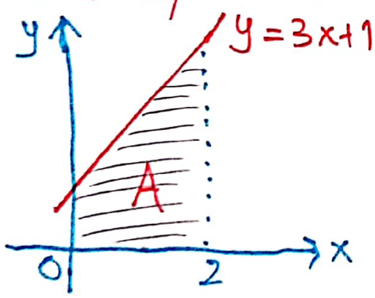
Q8) Evaluate the integrals below by using the properties of the definite integral and interpreting integrals as areas.

a) $\int_0^2 (3x+1)dx$ b) $\int_{-\pi}^{\pi} \sin(x^3)dx$ c) $\int_0^1 \sqrt{4-x^2}\,dx$

Sol: a) Let us sketch the graph of $3x+1=y$.



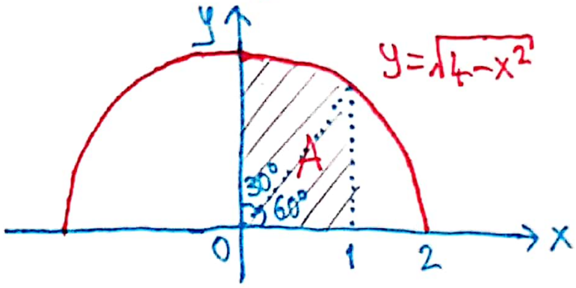
$\int_0^2 (3x+1)dx$ is equal to the area of the region encouraged by $y=3x+1$, $y=0$, $x=0$ and $x=2$.

Also, $A=\int_0^2 (3x+1)dx = \frac{3x^2}{2}+x\Big|_0^2 = 8$ //

b) We know that $\sin(x^3)$ is an odd function and the interval of integration is symmetric about $x=0$.

Therefore, $\int_{-\pi}^{\pi} \sin(x^3)dx = 0$.

c) Let us sketch the graph of $y=\sqrt{4-x^2}$. This is a half circle equation because $y=\sqrt{4-x^2} \Rightarrow x^2+y^2=2^2$, where $y \geq 0$. Thus, we can sketch the graph.



$\int_0^1 \sqrt{4-x^2}\,dx$ is the area of the region in figure.

$$A = \int_0^1 \sqrt{4-x^2}\,dx = \frac{1}{12}(\text{area of circle}) + (\text{area of triangle})$$

$$= \frac{1}{12} \cdot \pi . 2^2 + \frac{1 . \sqrt{3}}{2} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \ //$$

Q9) Given that $\int_0^a x^2\,dx = \frac{a^3}{3}$, evaluate the integral $\int_{-6}^{6} x^2(2+\sin x)\,dx$

Sol: $$\int_{-6}^{6} x^2(2+\sin x)\,dx = \int_{-6}^{6} 2x^2\,dx + \int_{-6}^{6} x^2 . \sin x\,dx$$

$$= 2\int_{-6}^{6} \underbrace{x^2}_{\text{even function}}\,dx + \int_{-6}^{6} \underbrace{x^2 . \sin x}_{\text{odd function}}\,dx$$

$$= 4\int_0^6 x^2\,dx + 0$$

$$= 4 . \frac{6^3}{3} = 288 \ //$$

Q10) Given that $\int_1^x \frac{1}{t}dt = \ln x$. Evaluate $\int_1^2 \frac{1}{x}dx$.

Sol: For $x=2$, $\int_1^2 \frac{1}{t}dt = \int_1^2 \frac{1}{x}.dx = \ln 2$ //

Q11) Find the average value of the function. $f(t)=1+\sin t$ over $[-\pi,\pi]$.

Sol: Average value of f on $[a,b] = \frac{1}{b-a}.\int_a^b f(x)dx$.

Hence, $f_{avg} = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi}(1+\sin t)dt = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}1.dt + \underbrace{\int_{-\pi}^{\pi}\sin t\,dt}_{0}\right]$

$= \frac{1}{2\pi}\left[2\pi+0\right] = 1$ //

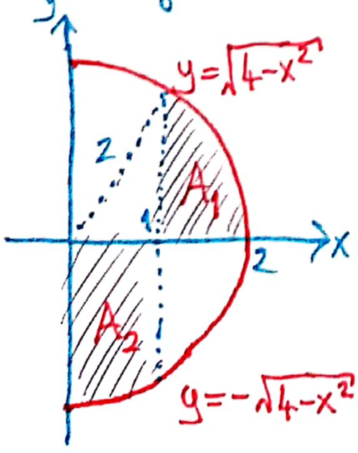
Q12) Find $\int_0^2 g(x)dx$ where $g(x) = \begin{cases} x^2, & \text{if } 0\le x\le 1 \\ x, & \text{if } 1<x\le 2. \end{cases}$

Sol: $\int_0^2 g(x)dx = \int_0^1 g(x)dx + \int_1^2 g(x)dx = \int_0^1 x^2 dx + \int_1^2 x\,dx$

$= \frac{1^3}{3} + \frac{2^2-1^2}{2} = \frac{11}{6}$ //

Q13) Evaluate $\int_0^2 \sqrt{4-x^2}\,\mathrm{sgn}(x-1)dx$

Sol: $\int_0^2 \sqrt{4-x^2}\,\mathrm{sgn}(x-1)dx = \int_0^1(-\sqrt{4-x^2})dx + \int_1^2 \sqrt{4-x^2}\,dx$



$= -\text{Area } A_2 + \text{Area } A_1$

$= -\left[\frac{1}{12}.\pi.2^2 + \frac{1.\sqrt{3}}{2}\right] + \frac{1}{6}\pi.2^2 - \frac{1}{2}.1.\sqrt{3}$

$= \frac{\pi}{3} - \sqrt{3}$ //

Q14) Evaluate the given integrals:

a) $\int_1^2 \left(\frac{2}{x^3} - \frac{x^3}{2}\right) dx$ b) $\int_0^{\pi/3} \sec^2\theta \, d\theta$ c) $\int_0^e a^x \, dx$.

Sol: a) $$\int_1^2 \left(\frac{2}{x^3} - \frac{x^3}{2}\right) dx = \int_1^2 \left(2.x^{-3} - \frac{1}{2}.x^3\right) dx = 2.\frac{x^{-2}}{-2} - \frac{1}{2}.\frac{x^4}{4}\Big]_1^2$$

$$= \left(2.\frac{2^{-2}}{-2} - \frac{1}{2}.\frac{2^4}{4}\right) - \left(-1 - \frac{1}{8}\right) = -\frac{9}{8}.$$

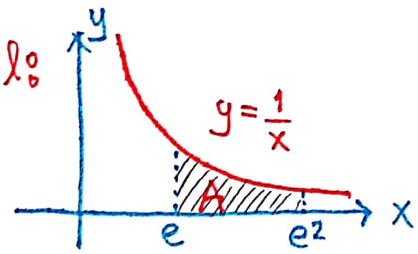
b) $$\int_0^{\pi/3} \sec^2\theta \, d\theta = \tan\theta \Big|_0^{\pi/3} = \tan\frac{\pi}{3} - \tan 0 = \sqrt{3} \, //$$

c) $$\int_0^e a^x dx = \frac{a^x}{\ln a}\Big|_0^e = \frac{a^e - 1}{\ln a} \, //$$

Q15) Find the area of the region R specified in the exercises below. It is helpful to make a sketch of the region.

a) Bounded by $y = \frac{1}{x}$, $y = 0$, $x = e$ and $x = e^2$.

Sol:

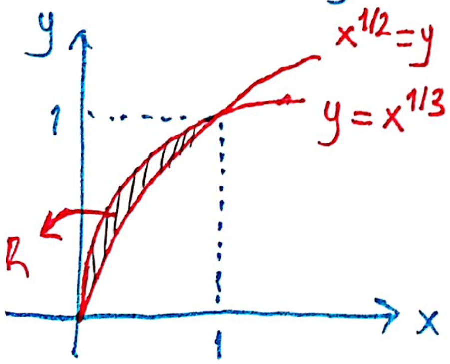


$$A = \text{Area} = \int_e^{e^2} \frac{1}{x} dx = \ln x \Big|_e^{e2}$$

$$= \ln e^2 - \ln e = 2 - 1 = 1 \, //$$

b) Bounded by $y = x^{1/3} - x^{1/2}$, $y = 0$ and $x = 1$.

Sol: Clearly $x^{1/3} \geq x^{1/2}$ on $0 \leq x \leq 1$. Hence,



$$\text{Area } R = \int_0^1 x^{1/3} dx - \int_0^1 x^{1/2} dx$$

$$= \frac{3}{4} x^{4/3}\Big|_0^1 - \frac{2}{3} x^{3/2}\Big|_0^1$$

$$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12} \, //$$

Q16) Find the integral of the piecewise continuous function $\int_0^{3\pi/2} |\cos x|\, dx$.

Sol: We know that $\cos x$ is positive on $[0, \pi/2]$ and negative on $[\pi/2, 3\pi/2]$. Thus,

$|\cos x| = \cos x$ on $[0, \frac{\pi}{2}]$ and $|\cos x| = -\cos x$ on $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

$$\int_0^{3\pi/2} |\cos x| dx = \int_0^{\pi/2} |\cos x| dx + \int_{\pi/2}^{3\pi/2} |\cos x| dx = \int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{3\pi/2} \cos x\, dx$$

$$= \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{3\pi/2} = 1 + 1 + 1 = 3 \;//$$

Q17) Find the indicated derivatives in exercises below.

a) $\dfrac{d}{d\theta} \displaystyle\int_{\sin\theta}^{\cos\theta} \frac{1}{1-x^2}\, dx$ b) $\dfrac{d}{dx} F(\sqrt{x})$, if $F(t) = \displaystyle\int_0^t \cos(x^2)\, dx$

Sol: a) We know that $\left(\displaystyle\int_{u(x)}^{v(x)} f(t)dt \right)' = f(v(x))\, v'(x) - f(u(x))\, u'(x)$.

Hence $\dfrac{d}{d\theta} \displaystyle\int_{\sin\theta}^{\cos\theta} \frac{1}{1-x^2} dx = \frac{1}{1-\cos^2\theta} \cdot (-\sin\theta) - \frac{1}{1-\sin^2\theta} \cos\theta$

$$= \frac{-\sin\theta}{\sin^2\theta} - \frac{\cos\theta}{\cos^2\theta} = -\operatorname{cosec}\theta - \sec\theta.$$

b) $F(t) = \displaystyle\int_0^t \cos(x^2) dx \implies F(\sqrt{x}) = \int_0^{\sqrt{x}} \cos(u^2)\, du.$

$$\frac{d}{dx} F(\sqrt{x}) = \cos x \cdot \frac{1}{2\sqrt{x}} \quad //$$

Q18) Solve the integral equation $f(x) = \pi\left(1 + \int_1^x f(t)\,dt\right)$

Sol: $f(x) = \pi + \pi \int_1^x f(t)\,dt$. Hence,

$f'(x) = (\pi)' + \pi \cdot \left(\int_1^x f(t)\,dt\right)'$. Thus,

$f'(x) = \pi \cdot f(x) \Rightarrow f(x) = c \cdot e^{\pi x}$

$f(1) = \pi = c \cdot e^{\pi} \Rightarrow c = \pi \cdot e^{-\pi}$

$f(x) = \pi e^{\pi(x-1)}$.

MATH 101.2 PS-10

Q1) $\int e^{2x} \cdot \sin(e^{2x})\, dx = ?$

Sol: Let us say $e^{2x} = u$, then $2 \cdot e^{2x} dx = du$. Hence

$$\cancel{\int e^{2x} \sin 2x} \qquad \int e^{2x} \cdot \sin(e^{2x})\, dx = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + c \ \cancel{\sin}$$

$$= -\frac{1}{2} \cdot \cos(e^{2x}) + C.$$

Q2) $\int x \cdot e^{x^2} dx = ?$

Sol: Let us say $x^2 = u$, then $2x \cdot dx = du$. Hence,

$$\int x \cdot e^{x^2} dx = \frac{1}{2}\int e^u du = \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c.$$

Q3) $\int \frac{x+1}{\sqrt{x^2+2x+3}}\, dx = ?$

Sol: Let $u = x^2+2x+3$, then $du = 2(x+1)$. Hence

$$\int \frac{x+1}{\sqrt{x^2+2x+3}}\, dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + c = \sqrt{x^2+2x+3} + c.$$

Q4) $\int \frac{dx}{e^x+1} = ?$

Sol: $$\int \frac{dx}{e^x+1} = \int \frac{e^x+1-e^x}{e^x+1}\, dx = \int \left(1 - \frac{e^x}{e^x+1}\right) dx$$

$$= \int 1 \cdot dx - \int \frac{e^x}{e^x+1}\, dx \qquad \text{Say } u = e^x+1 \Rightarrow du = e^x dx$$

$$= x - \int \frac{du}{u}$$

$$= x - \ln u + c = x - \ln(e^x+1) + C \; //$$

Q5) $\int \sin(ax)\cdot\cos^2(ax)dx = ?$

Sol: Let us say $\cos(ax) = u$. Then, $-a.\sin(ax) = du$

$$\int \sin(ax)\cos^2(ax)dx = -\frac{1}{a}\int u^2 du = -\frac{u^3}{3a} + c = -\frac{1}{3a}\cos^3(ax) + c$$

Q6) $\int \sec^6 x.\tan^2 x\, dx = ?$

Sol: We know that $\sec^2 x = 1+\tan^2 x$. Hence,

$$\int \sec^6 x.\tan^2 x\, dx = \int \sec^2 x\cdot\tan^2 x.(1+\tan^2 x)^2 dx$$

Let $u=\tan x$, $du = \sec^2 x\, dx$

$$= \int u^2(1+u^2)^2 du = \int (u^2+2u^4+u^6)du$$

$$= \frac{1}{3}u^3 + \frac{2}{5}u^5 + \frac{1}{7}u^7 + c = \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + c$$

Q7) $\int \cos x.\sin^4(\sin x)dx = ?$

Sol: Let us say $u=\sin x$, then $du = \cos x\, dx$. Hence,

$$\int \cos x.\sin^4(\sin x)dx = \int \sin^4 u\, du = \int \left(\frac{1-\cos 2u}{2}\right)^2 du$$

$$= \frac{1}{4}\int\left(1-2\cos 2u + \frac{1+\cos 4u}{2}\right)du$$

$$= \frac{3u}{8} - \frac{\sin 2u}{4} + \frac{\sin 4u}{32} + c$$

$$= \frac{3}{8}\sin x - \frac{1}{4}\sin(2\sin x) + \frac{1}{32}\sin(4\sin x) + c$$

Q8) $\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x} dx = ?$

Sol: Let $\pi.\ln x = u$, then $\frac{\pi}{x}dx = du$. Also, $x=1 \Rightarrow u=0$, $x=\sqrt{e} \Rightarrow u = \frac{\pi}{2}$

$$\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x}dx = \frac{1}{\pi}\int_0^{\pi/2}\sin u\, du = -\frac{1}{\pi}\cos u\Big|_0^{\pi/2}$$

$$= -\frac{1}{\pi}(0-1) = \frac{1}{\pi} //$$

Q9) $\displaystyle\int_{\pi^2/16}^{\pi^2/9} \frac{2^{\sin\sqrt{x}} \cdot \cos\sqrt{x}}{\sqrt{x}}\, dx = ?$

Sol: Let $u=\sin\sqrt{x}$, then, $du = \frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx$. $x=\frac{\pi^2}{16} \Rightarrow u=\frac{1}{\sqrt{2}}$

$x=\frac{\pi^2}{9} \Rightarrow u=\frac{\sqrt{3}}{2}$

$$\int_{\pi^2/16}^{\pi^2/9} \frac{2^{\sin\sqrt{x}} \cdot \cos\sqrt{x}}{\sqrt{x}} = 2\cdot\int_{1/\sqrt{2}}^{\sqrt{3}/2} 2^u\, du$$

$$= \frac{2\cdot 2^u}{\ln 2}\Big|_{1/\sqrt{2}}^{\sqrt{3}/2} = \frac{2}{\ln 2}\left(2^{\sqrt{3}/2} - 2^{1/\sqrt{2}}\right).$$

Q10) $\int (x^2-2x)e^{kx}\, dx = ?$

Sol: Let us say $u=x^2-2x$ and $v=$ ~~...~~ e^{kx}. Then, $du = 2x-2$ and $v=\frac{1}{k}e^{kx}$. Hence

$$\int (x^2-2x)e^{kx}\,dx = \frac{1}{k}(x^2-2x)e^{kx} - \frac{1}{k}\int (2x-2)e^{kx}\,dx$$

$u=x-1 \Rightarrow du=dx$
$dv=e^{kx} \Rightarrow v=\frac{1}{k}e^{kx}$ $\rightarrow$ $$= \frac{1}{k}(x^2-2x)e^{kx} - \frac{2}{k}\left[\frac{1}{k}(x-1)^{kx} - \frac{1}{k}\int e^{kx}\,dx\right]$$

$$= \frac{1}{k}(x^2-2x)e^{kx} - \frac{2}{k^2}(x-1)^{kx} + \frac{2}{k^3}e^{kx} + c$$

Q11) $\int x.(\ln x)^3\, dx = ?$

Sol: Let us say $\int x.(\ln x)^3\,dx = I_3$ where $I_n = \int x(\ln x)^n\,dx$.
Let $u=(\ln x)^n$ and $dv = x\,dx$. Hence, $du = \frac{n}{x}x^2(\ln x)^{n-1}\,dx$ and $v=\frac{1}{2}x^2$. Thus,

$$I_n = \int x.(\ln x)^3\,dx = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1}\,dx = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}$$

$$I_3 = \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}I_2 = \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}\left[\frac{1}{2}x^2(\ln x)^2 - \frac{2}{2}I_1\right]$$

$$= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{2}\left[\frac{1}{2}x^2\ln x - \frac{1}{2}I_0\right] \quad \left(I_0 = \int x\,dx = \frac{x^2}{2}\right)$$

$$= \frac{x^2}{2}\left[(\ln x)^3 - \frac{3}{2}(\ln x)^2 + \frac{3}{2}(\ln x) - \frac{3}{4}\right] + C.$$

Q12) $\int \arctan x \, dx = ?$

Sol: Let us say $u = \arctan x$ and $dx = dv$. Then

$du = \frac{1}{1+x^2} dx$ and $x = v$.

$$\int \arctan x \, dx = x.\arctan x - \int \frac{x}{1+x^2} dx = x.\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

Q13) $\int \tan^2 x . \sec x dx = ?$

Sol: Let us say $u = \tan x$ $dv = \sec x . \tan x dx$. Then,

$du = \sec^2 x dx$ and $v = \sec x$. Hence,

$$I = \int \tan^2 x \sec x dx = \tan x . \sec x - \int \sec^3 x \, dx$$

$$= \tan x . \sec x - \int (1 + \tan^2 x) \sec x \, dx$$

$$= \tan x . \sec x - \int \sec x dx - \int \tan^2 x . \sec x dx$$

$$= \tan x \sec x - \ln|\sec x + \tan x| - I$$

Hence, $I = \frac{1}{2}\left[\tan x \sec x - \ln|\sec x + \tan x|\right] + C$

Q14) $\int_1^e \sin(\ln x) dx = ?$

Sol: Let us say $\sin(\ln x) = u$ and $dx = dv$. Then,

$du = \frac{\cos(\ln x)}{x} dx$ and $x = v$. Hence,

$$I = \int_1^e \sin(\ln x) dx = x . \sin(\ln x)\Big|_1^e - \int_1^e \cos(\ln x) dx$$

$u = \cos(\ln x)$, $dx = dv$

$du = -\frac{\sin(\ln x)}{x}$, $x = v$

$$= e . \sin(1) - \left[x . \cos(\ln x)\Big|_1^e + \int_1^e \sin(\ln x) dx \right]$$

Hence, $I = e.\sin(1) - e.\cos(1) + 1 - I$.

Therefore, $I = \frac{1}{2}\left[e.\sin 1 - e.\cos(1) + 1\right]$.

15) $\int \frac{\ln(\ln x)}{x}\,dx = ?$

Sol: Let $u=\ln x$, then $du=\frac{1}{x}dx$. Hence,

$\int \frac{\ln(\ln x)}{x}\,dx = \int \ln u . du$. Let us say $U=\ln u$ and $du=dV$

Hence, $dU=\frac{1}{u}du$ and $V=u$. Therefore,

$$\int \frac{\ln(\ln x)}{x}\,dx = \int \ln u\, du = u.\ln u - \int du = u.\ln u - u + c$$
$$= (\ln x)(\ln(\ln x)) - \ln x + c.$$

Q16) $\int_1^2 \operatorname{arcsec} x\,dx = ?$

Sol: We know that $\operatorname{arcsec} x = \arccos\frac{1}{x}$. Hence,

$\int_1^2 \operatorname{arcsec} x\,dx = \int_1^2 \arccos\left(\frac{1}{x}\right)dx$. * Let $U=\arccos\frac{1}{x}$ $dx=dV$

$U = -\frac{1}{\sqrt{1-\frac{1}{x^2}}}\cdot\left(\frac{-1}{x^2}\right)dx$, $x=V$

$$*= x.\arccos\frac{1}{x}\Big|_1^2 - \int_1^2 \frac{dx}{\sqrt{x^2-1}}.$$

$\to$ Let $x=\sec\theta$, $dx=\sec\theta.\tan\theta\, d\theta$

$$= \frac{2\pi}{3} - 0 - \int_0^{\pi/3} \sec\theta\, d\theta = \frac{2\pi}{3} - \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/3}$$
$$= \frac{2\pi}{3} - \ln(2+\sqrt{3}).$$

Q17) $\int x.e^x.\cos x\,dx = ?$

Sol: $\int e^x.\cos dx = \frac{1}{2}e^x(\sin x+\cos x)+c$, $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x-\cos x)+c$

We will use these two facts. (You can show these).

For $\int x.e^x.\cos x\,dx$. Let $u=x$ and $dv=e^x.\cos x\,dx$.

Then, $du=dx$, $v=\frac{1}{2}e^x(\sin x+\cos x)$. Therefore,

$$\int x.e^x.\cos x\,dx = \frac{1}{2}x.e^x(\sin x+\cos x) - \frac{1}{2}\int e^x(\sin x+\cos x)dx$$
$$= \frac{1}{2}x.e^x(\sin x+\cos x) - \frac{1}{4}e^x(\sin x-\cos x+\sin x+\cos x)+C$$
$$= \frac{1}{2}xe^x(\sin x+\cos x) - \frac{1}{2}e^x\sin x + C.$$

Q18) Obtain a reduction formula for $I_n = \int \sin^n x\, dx$ (where $n \geq 2$) and use it to find I_6 and I_7.

Sol: $I_n = \int \sin^n x\, dx$, $(n \geq 2)$.

Let $u = \sin^{n-1} x$ and $dv = \sin x dx$. Hence,
$du = (n-1)\sin^{n-2} x . \cos x\, dx$ and $V = -\cos x$. Thus,

$$I_n = -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x . \cos^2 x\, dx$$

$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x . (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x . \cos x + (n-1) . (I_{n-2} - I_n)$$. Hence,

$$n . I_n = -\sin^{n-1} x . \cos x + (n-1)(I_{n-2})$$, Thus

$$I_n = -\frac{1}{n} . \sin^{n-1} x . \cos x + \frac{n-1}{n} . I_{n-2} .$$

Note that $I_0 = x + C$ and $I_1 = -\cos x + C$.

$$I_6 = -\frac{1}{6} . \sin^5 x . \cos x + \frac{5}{6} I_4 = -\frac{1}{6} \sin^5 \cos x + \frac{5}{6}\left(-\frac{1}{4} \sin^3 x . \cos x + \frac{3}{4} I_2\right)$$

$$= -\frac{1}{6} \sin^5 x . \cos x - \frac{5}{24} \sin^3 x . \cos x + \frac{5}{8}\left(-\frac{1}{2} \sin x . \cos x + \frac{1}{2} I_0\right)$$

$$= -\frac{1}{6} \sin^5 x . \cos x - \frac{5}{24} \sin^3 x . \cos x - \frac{5}{16} \sin x . \cos x + \frac{5}{16} x + C.$$

$$I_7 = -\frac{1}{7} . \sin^6 x . \cos x + \frac{6}{7} I_5 = -\frac{1}{7} \sin^6 x . \cos x + \frac{6}{7}\left(-\frac{1}{5} \sin^4 x . \cos x + \frac{4}{5} I_3\right)$$

$$= -\frac{1}{7} \sin^6 x . \cos x - \frac{6}{35} . \sin^4 x . \cos x + \frac{34}{35}\left(-\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} I_1\right)$$

$$= -\frac{1}{7} \sin^6 x . \cos x - \frac{6}{35} \sin^4 x . \cos x - \frac{8}{35} \sin^2 x . \cos x - \frac{16}{35} \cos x + C$$

Q19) $\int \frac{x^2}{x-4}dx = ?$

Sol: $\int \frac{x^2}{x-4}dx = \int \left(\frac{x^2-16+16}{x-4}\right)dx = \int \left(\frac{x^2-16}{x-4} + \frac{16}{x-4}\right)dx$

$= \int \left(x+4+\frac{16}{x-4}\right)dx = \frac{x^2}{2}+4x+16\ln|x-4|+c$

Q20) $\int \frac{1}{x^2-9}dx = ?$

Sol: $\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3} = \frac{Ax+3A+Bx-3B}{x^2-9}$

$A+B=0\ , \quad 3(A-B)=1 \Rightarrow A=-B\ , \quad -6B=1 \Rightarrow B=\frac{-1}{6}\ , \ A=\frac{1}{6}$

$\int \frac{dx}{x^2-9} = \frac{1}{6}\int\frac{1}{x-3}dx - \frac{1}{6}\int\frac{dx}{x+3} = \frac{1}{6}(\ln|x-3| - \ln|x+3|)+C.$

Q21) $\int \frac{x^2\,dx}{x^2+x-2} = ?$

Sol: $\int \frac{x^2\,dx}{x^2+x-2} = \int\left(1-\frac{x-2}{x^2+x-2}\right)dx = x - \int\frac{x-2}{x^2+x-2}dx$

If $\frac{x-2}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{Ax-A+Bx+2B}{x^2+x-2}$, then

$A+B=1$ and $-A+2B=-2$, so that $A=\frac{4}{3}$ and $B=-\frac{1}{3}$

Thus, $\int\frac{x^2\,dx}{x^2+x-2} = x - \frac{4}{3}\int\frac{dx}{x+2} + \frac{1}{3}\int\frac{dx}{x-1}$

$= x - \frac{4}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$ //

22) $\int \frac{x-2}{x^2+x}dx = ?$

Sol: $\frac{x-2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax+A+Bx}{x^2+x} \Rightarrow \begin{matrix} A+B=1 \\ A=-2 \end{matrix} \Rightarrow B=3.$

$$\int \frac{x-2}{x^2+x}dx = 3.\int \frac{dx}{x+1} - 2\int \frac{dx}{x} = 3\ln|x+1| - 2\ln|x| + C.$$

23) $\int \frac{x^2+1}{6x-9x^2}dx = ?$

Sol: $\int \frac{x^2+1}{6x-9x^2} = \frac{1}{9}\int \frac{9x^2-6x+6x+9}{6x-9x^2}dx = \frac{1}{9}\int\left(-1+\frac{6x+9}{6x-9x^2}\right)dx$

$$= -\frac{x}{9} + \int \frac{6x+9}{6x-9x^2}dx$$

Now, $\frac{2x+3}{x(2-3x)} = \frac{A}{x} + \frac{B}{2-3x} = \frac{2A-3Ax+Bx}{x(2-3x)}$

$B-3A=2,\quad 2A=3, \Rightarrow A=\frac{3}{2}$ ve $B=\frac{13}{2}.$

$$\int \frac{x^2+1}{6x-9x^2}dx = -\frac{x}{9} + \frac{1}{6}\int\frac{dx}{x} + \frac{13}{18}\int\frac{dx}{2-3x}$$

$$= -\frac{x}{9} + \frac{1}{6}\ln|x| - \frac{13}{54}\ln|2-3x| + C //$$

Q24) $\int \frac{dx}{x^4-a^4} = ?$

Sol: $\frac{1}{x^4-a^4} = \frac{A}{x-a} + \frac{B}{x+a} + \frac{Cx+D}{x^2+a^2}$

$$= \frac{A(x^3+ax^2+a^2x+a^3)+B(x^3-ax^2+a^2x-a^3)+C(x^3-a^2x)+D(x^2-a^2)}{x^4-a^4}.$$

$\Rightarrow A+B+C=0,\quad aA-aB+D=0,\quad a^2A+a^2B-a^2C=0$

$a^3A-a^3B-a^2D=1. \quad \Rightarrow A=\frac{1}{4a^3},\ B=-\frac{1}{4a^3},\ C=0,\ D=\frac{-1}{2a^2}.$

$$\int \frac{dx}{x^4-a^4} = \frac{1}{4a^3}\int\left(\frac{1}{x-a} - \frac{1}{x+a} - \frac{2a}{x^2+a^2}\right)dx$$

$$= \frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + C //$$

25) $\int \frac{dx}{x^4-3x^3} = ?$

Sol: $\frac{1}{x^4-3x^3} = \frac{1}{x^3(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$

$$= \frac{A(x^3-3x^2)+B(x^2-3x)+C(x-3)+Dx^3}{x^3(x-3)}$$

$\Rightarrow A+D=0, \quad -3A+B=0, \quad -3B+C=0, \quad -3C=1$

$A = \frac{-1}{27}, \quad B = \frac{-1}{9}, \quad C = \frac{-1}{3}, \quad D = \frac{1}{27}$

$$\int \frac{dx}{x^4-3x^3} = -\frac{1}{27}\int \frac{dx}{x} - \frac{1}{9}\int \frac{dx}{x^2} - \frac{1}{3}\int \frac{dx}{x^3} + \frac{1}{27}\int \frac{dx}{x-3}$$

$$= \frac{1}{27}\ln\left|\frac{x-3}{x}\right| + \frac{1}{9x} + \frac{1}{6x^2} + C.$$

26) $\int \frac{dx}{e^{2x}-4e^x+4} = ?$

Sol: Let $e^x = u$, then $e^x dx = du \Rightarrow dx = \frac{du}{u}$.

$$\int \frac{dx}{e^{2x}-4e^x+4} = \int \frac{du}{u(u-2)^2}$$

$$\frac{1}{u(u-2)^2} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{(u-2)^2} = \frac{A(u^2-4u+4)+B(u^2-2u)+Cu}{u(u-2)^2}.$$

$A+B=0, \quad -4A-2B+C=0, \quad 4A=1.$

Hence, $A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = \frac{1}{2}.$

$$\int \frac{du}{u(u-2)^2} = \frac{1}{4}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u-2} + \frac{1}{2}\int \frac{du}{(u-2)^2}.$$

$$= \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u-2| - \frac{1}{2}\cdot\frac{1}{u-2} + c$$

$$= \frac{x}{4} - \frac{1}{4}\ln|e^x-2| - \frac{1}{2(e^x-2)} + C \,//$$