§12.7. Gradient and Directional Derivatives

The first partial derivatives of a function into a single vector function is called a gradient.

Definition 1. Let $f(x, y)$ be any function. The gradient vector, denoted by $\nabla f(x, y) = \boldsymbol{grad} f(x, y)$, at any point (x, y) is defined as $\nabla f(x, y) = \boldsymbol{grad} f(x, y) = f_1(x, y)i + f_2(x, y)j$.

The symbol ∇, is called nabla, is a vector differential operator: $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$. When the operator nabla is applied to a function $f(x, y)$, the result is the gradient of the function:
$\nabla f(x, y) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) f(x, y) = f_1(x, y)i + f_2(x, y)j$.

Example 1. Let $f(x, y) = x^2 + y^2$. The gradient of f: $\nabla f(x, y) = 2xi + 2yj$. At the point $(1, 2)$, the gradient vector is: $\nabla f(1, 2) = 2i + 4j$. Note that this vector is perpendicular to the tangent line $x + 2y = 5$ to the circle $x^2 + y^2 = 5$ at $(1, 2)$.

Theorem 1. Let $f(x, y)$ be a differentiable at the point (a, b) and $\nabla f(a, b) \neq 0$. Then $\nabla f(a, b)$ is a normal vector to the level curve of f passing through (a, b).

§Direction Derivatives

The first partial derivatives $f_1(a, b)$ and $f_2(a, b)$ give the rates of changes of $f(x,y)$ at (a,b) measured in the direction of the positive x-axes and y-axes, respectively.

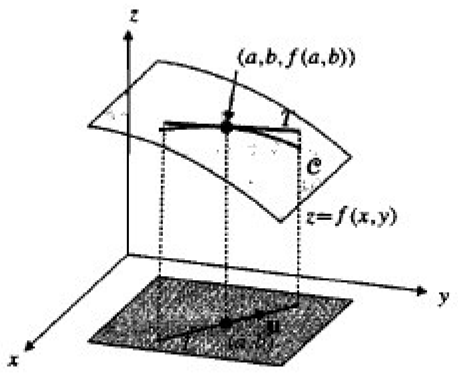
Definition 2. Let $\boldsymbol{u} = ui + vj$ be a unit vector such that $u^2 + v^2 = 1$. The directional derivative of $f(x,y)$ at (a,b) in the direction of $\boldsymbol{u}$ is the rate of change $f(x,y)$ with respect to distance measured at the point along a ray in the direction of $\boldsymbol{u}$ in the xy-plane. This directional derivative is given by:

$$D_{\mathbf{u}} f(a,b) = \lim_{h \to 0+} \frac{f(a+hu, b+hv) - f(a,b)}{h}$$

It is also given by

$$D_{\mathbf{u}} f(a,b) = \frac{d}{dt} f(a+tu, b+tv)\bigg|_{t=0}$$

if the derivative on the right side exists.



Theorem 2. Let Let $f(x,y)$ be a differentiable at the point (a,b)

and $\boldsymbol{u} = ui + vj$ be a unit vector. Then the directional derivative of $f(x, y)$ at the point in the direction of $\boldsymbol{u}$ is given by:
$D_{\boldsymbol{u}} f(a, b) = \boldsymbol{u} \bullet \nabla f(a, b)$.

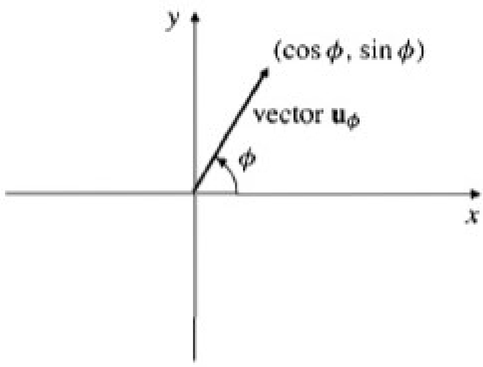
Let $\boldsymbol{v}$ be any nonzero vector. The directional derivative of f at any point (a, b) in the direction of $\boldsymbol{v}$ is:
$D_{\boldsymbol{v}/|\boldsymbol{v}|} f(a, b) = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \bullet \nabla f(a, b)$.

Example 2. Find the rate of change of $f(x, y) = y^4 + 2xy^3 + x^2y^2$ at $(0, 1)$ measured in each of the the following directions:
a).$i + 2j$ b). $j - 2i$ c). 3i   d). i+j.
$\nabla f(x, y) = (2y^3+2xy^2)i+(4y^3+6xy^2+2x^2y)j$ and $\nabla f(0, 1) = 2i+4j$.
a) The unit vector in the direction of $i + 2j$ is: $\frac{i+2j}{\sqrt{5}}$.
The directional derivative of f at any point $(0, 1)$ in the direction of $i+2j$ is: $\frac{i+2j}{\sqrt{5}} \bullet (2i+4j) = \frac{2+8}{\sqrt{5}} = 2\sqrt{5}$. Note that $i+2j$ is in the same direction as $\nabla f(0, 1)$. Then the directional derivative is positive and equal to the length of $\nabla f(0, 1)$.
b). The unit vector in the direction of $j - 2i$ is: $\frac{-2i+j}{\sqrt{5}}$.
The directional derivative of f at any point $(0, 1)$ in the direction of $-2i+j$ is: $\frac{-2i+j}{\sqrt{5}} \bullet (2i+4j) = \frac{-4+4}{\sqrt{5}} = 0$. Since $j-2i$ is perpendicular to $\nabla f(0, 1)$, it is tangent to the level curve of f through $(0, 1)$ so the directional derivative in that direction is 0.
c). The unit vector in the direction of $3i$ is: i.

The directional derivative of f at any point $(0,1)$ in the direction of $3i$ is: $i \bullet (2i+4j) = 2$. The directional derivative of f in direction of positive x-axis is $f_1(0,1)$.

d). The unit vector in the direction of $i+j$ is: $\frac{i+j}{\sqrt{2}}$.

The directional derivative of f at any point $(0,1)$ in the direction of $i+j$ is: $\frac{i+j}{\sqrt{2}} \bullet (2i+4j) = \frac{2+4}{\sqrt{2}} = 3\sqrt{2}$. If we move along the surface $z = f(x,y)$ through the point $(0,1,1)$ in a direction making horizontal angles of 45^0 with the positive directions of x-and y-axes, we would be rising at a rate of $3\sqrt{2}$ vertical units per horizontal unit moved.



Consider the vector $\boldsymbol{u}$ making angle ϕ with the positive direction of the x-axis corresponds to the unit vector (see the above figure).

Then $\boldsymbol{u}_\phi = \cos\phi i + \sin\phi j$. The directional derivative of f at (x, y) in that direction is :

$D_\phi f(x, y) = D_{\boldsymbol{u}_\phi} f(x, y) = \boldsymbol{u}_\phi \bullet \nabla f(x, y) = f_1(x, y)\cos\phi + f_2(x, y)\sin\phi$.

The symbol $D_\phi f(x, y)$ denotes a derivative of f with respect to distance measured in the direction ϕ.

For any unit vector $\boldsymbol{u}$, $D_{\boldsymbol{u}} f(a, b) = \boldsymbol{u} \bullet \nabla f(a, b) = |\boldsymbol{u}||\nabla f(a, b)|\cos\theta$ where θ is the angle between the vectors $\boldsymbol{u}$ and $\nabla f(a, b)$. Since $-1 \le \cos\theta \le 1$, then $-|\nabla f(a, b)| \le D_{\boldsymbol{u}} f(a, b) \le |\nabla f(a, b)|$.

Consider the following cases:

1. $D_{\boldsymbol{u}} f(a, b) = -|\nabla f(a, b)| \Leftrightarrow \boldsymbol{u}$ points in the opposite direction to $\nabla f(a, b)$ (in this case, $\cos\theta = -1$).

2. $D_{\boldsymbol{u}} f(a, b) = |\nabla f(a, b)| \Leftrightarrow \boldsymbol{u}$ points in the same direction to $\nabla f(a, b)$ (in this case, $\cos\theta = 1$).

3. If $D_{\boldsymbol{u}} f(a, b) = 0$, then $\theta = \pi/2$, thus it is the direction of the tangent line of the level curve of f passing through (a, b).

§Geometric properties of the gradient vector

1. At (a, b), $f(x, y)$ increases most rapidly in the direction of the gradient vector $\nabla f(a, b)$. The maximum rate of increase is $|\nabla f(a, b)|$.

2. At (a,b), $f(x,y)$ decreases most rapidly in the direction of the gradient vector $-\nabla f(a,b)$. The maximum rate of decrease is $|\nabla f(a,b)|$.

3. The rate of change of $f(x,y)$ at (a,b) is 0 in direction tangent to the level curve of f passing through (a,b).

Example 3. The temperature at position (x,y) in a region of the xy-plane is T^0C where $T(x,y) = x + 2e^{-y}$. In what direction at the point $(2,1)$ does the temperature increase most rapidly? What is the rate of increase of f in that direction.

$\nabla T(x,y) = 2xe^{-y}i - x^2e^{-y}j$.

$\nabla T(2,1) = \frac{4}{e}i - \frac{4}{e}j = \frac{4}{e}(i-j)$.

At (2,1), $T(x,y)$ increases most rapidly in the direction of the vector $i-j$. The rate of increase in this direction is $|\nabla T(2,1)| = \frac{4\sqrt{2}}{e}^0C/$unit distance.

Example 4. Find the second directional derivative of $f(x,y)$ in the direction making angle ϕ with the positive x-axis.

The first directional derivative is $D_\phi f(x,y) = (\cos\phi i + \sin\phi j) \bullet \nabla f(x,y) = f_1(x,y)\cos\phi + f_2(x,y)\sin\phi$.

The second directional derivative is :

$D^2_\phi f(x,y) = D_\phi(D_\phi f(x,y)) = (\cos\phi i + \sin\phi j) \bullet \nabla(f_1(x,y)\cos\phi +$

$f_2(x,y)\sin\phi) = (f_{11}(x,y)\cos\phi + f_{21}(x,y)\sin\phi)\cos\phi + (f_{12}(x,y)\cos\phi + f_{22}(x,y)\sin\phi)\sin\phi = f_{11}(x,y)\cos^2\phi + 2f_{12}(x,y)\sin\phi\cos\phi + f_{22}(x,y)\sin^2\phi$.
If $\phi = 0$ or $\phi = \pi$, then the directional derivative is in a direction parallel to the x-axis, namely, $D^2_\phi f(x,y) = f_{11}(x,y)$. Similarly, if $\phi = \pi/2$ or $\phi = 3\pi/2$, then $D^2_\phi f(x,y) = f_{22}(x,y)$.

§The Gradient in Three and More Dimensions

Let $f(x_1, x_2, ..., x_n)$ be a function with n-independent variables. The gradient vector of it is :
$\nabla f(x_1, x_2, ..., x_n) = \frac{\partial f}{\partial x_1}e_1 + \frac{\partial f}{\partial x_2}e_2 + ... + \frac{\partial f}{\partial x_n}e_n$, where e_j is a unit vector from orijin to the unit point on the jth coordinate axis. In particular, for a function of three variables:
$\nabla f(x,y,z) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k.$
The level surface of $f(x,y,z)$ passing through (a,b,c) has a tangent plane there if f is differentiable at (a,b,c) and $\nabla f(a,b,c) \neq 0$. Namely, the vector $\nabla f(P_0)$ is normal to the level surface of f passing through the point P_0 and if f is differentiable at the point, the rate of change of f at the point in the direction of the unit vector $\boldsymbol{u}$ is given by $\boldsymbol{u} \bullet \nabla f(P_0)$.

Example 5. Let $f(x,y,z) = x^2 + y^2 + z^2$. Then;

1. Find $\nabla f(x,y,z)$ and $\nabla f(1,-1,2)$.

2. Find the equation of the tangent plane at the sphere $x^2 + y^2 + z^2 = 6$ at the point $(1, -1, 2)$.

3. What is the maximum rate of increase of f at $(1, -1, 2)$?

4. What is the rate of change with respect to distance of f at $(1, -1, 2)$ measured in the direction from that point toward the point $(3, 1, 1)$?

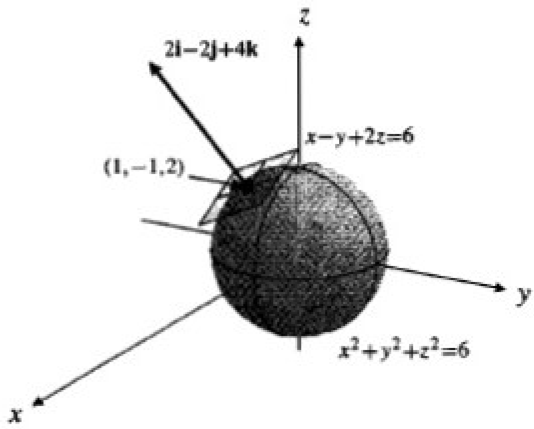
1. $\nabla f(x, y, z) = 2xi + 2yj + 2zk$ and $\nabla f(1, -1, 2) = 2i - 2j + 4k$.

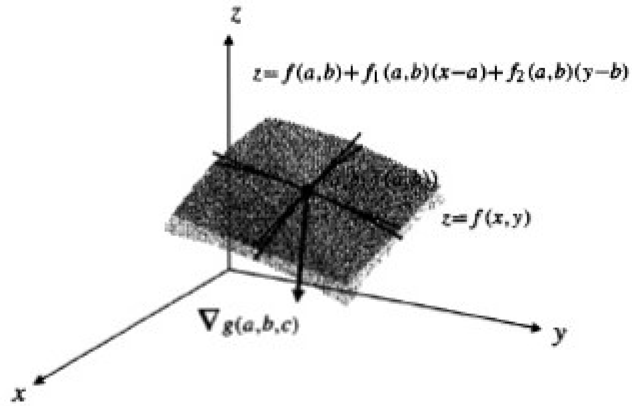
2. $\nabla f(1, -1, 2)$ is normal vector of the required tangent plane. Then its equation is : $2(x - 1) - 2(y + 1) + 4(z - 2) = 0$.

3. The maximum rate of increase of f at $(1, -1, 2)$ is $|\nabla f(1, -1, 2)| = 2\sqrt{6}$ and it occurs in the direction of the vector $i - j + 2k$.

4. The direction from $(1, -1, 2)$ toward $(3, 1, 1)$ is specified by $2i + 2j - k$. The rate of change of f with respect to distance in this direction is :

   $\frac{2i+2j-k}{\sqrt{4+4+1}} \bullet (2i - 2j + 4k) = \frac{4-4-4}{3} = \frac{-4}{3}$.





Example 6. The graph of a function $f(x, y)$ of two variables is the graph of the equation $z = f(x, y)$ in 3-space. This surface is the level surface of $g(x, y, z) = 0$ of the 3-variable function $g(x, y, z) = f(x, y) - z$.

If f is differentiable at (a, b) and $c = f(a, b)$, then g is differentiable at (a, b, c) and $\nabla g(a, b, c) = f_1(a, b)i + f_2(a, b)j - k$ is a normal to $g(x, y, z) = 0$ at (a, b, c). The graph of f has nonvertical tangent plane at (a, b) given by

$f_1(a,b)(x-a)+f_2(a,b)(y-b)-(z-c)=0$ or $z=f_1(a,b)(x-a)+f_2(a,b)(y-b)+c$.

Example 7. Find a vector tangent to the curve of the intersection of the two surfaces $z=x^2-y^2$ and $xyz+30=0$ at the point $(-3,2,5)$.

$n_1=\nabla(x^2-y^2-z)|_{(-3,2,5)}=2xi-2yj-k|_{(-3,2,5)}=-6i-4j-k$.

$n_2=\nabla(xyz+30)|_{(-3,2,5)}=yzi+xzj+xyk|_{(-3,2,5)}=10i-15j-6k$.

For the tangent vector T, we can use the cross product of these normals:

$$T=n_1\times n_2=\begin{vmatrix} i & j & k \\ -6 & -4 & -1 \\ 10 & -15 & -6 \end{vmatrix}=9i-46j+130k.$$