EXTREME VALUES

Critical Points, Singular Points, and Endpoints

A function $f(x)$ can have local extreme values only at points x of three special types.

i) critical points of f (points x in $D(f)$) where $f'(x)=0$.

ii) singular points of f (points x in $D(f)$) where $f'(x)$ is not defined, and

iii) endpoints of the domain of f (points in $D(f)$ that do not belong to any open interval contained in $D(f)$).

Theorem: (Locating extreme values) If the function f is defined on an interval I and has a local maximum (or local minimum) value at point $x=x_0$ in I, then x_0 must be either a critical point of f, a singular point of f, or an endpoint of I.

Finding Absolute Extreme Values

Example: Find the maximum and minimum values of the function $g(x)=x^3-3x^2-9x+2$ on the interval $-2\leq x\leq 2$.

Since $g(x)$ is a polynomial it is differentiable for all x. Thus it can have no singular points.

For critical points, we calculate

$$g'(x)=3x^2-6x-9=3(x^2-2x-3)=3(x+1)(x-3)=0$$

if $x=-1$ or $x=3$.

However, $x=3 \notin D(f)$ so we can ignore it.

We need to consider only the values of f at the critical point $x=-1$ and at endpoints $x=-2$ and $x=2$.

$$f(-2)=0 \quad , \quad f(-1)=7 \quad , \quad f(2)=-20$$

The maximum value of $f(x)$ on $-2 \leq x \leq 2$ is 7, at the critical point $x=-1$, and the minimum value is -20, at the endpoint $x=2$.

<u>Example:</u> Find the maximum and minimum values of $h(x)=3x^{2/3}-2x$ on the interval $[-1,1]$.

The derivative of h is;

$$h'(x)=3\left(\frac{2}{3}\right)x^{-1/3}-2=2(x^{-1/3}-1)$$

Note that $x^{-1/3}$ is not defined at the point $x=0$ in $D(f)$, so $x=0$ is a singular point of $h(x)$.

Also, $h(x)$ has a critical point where $x^{-1/3}=1$, that is, at $x=1$, which also happens to be an endpoint of the domain.

Thus,

$$h(-1)=5$$

$$h(0)=0$$

$$h(1)=1$$

The function h has maximum value 5 at the endpoint -1 and minimum value 0 at the singular point $x=0$.

Theorem:
(The First Derivative Test)

PART I: Testing interior critical points and singular points.

Suppose that f is continuous at x_0, and x_0 is not an endpoint of the domain of f.

a) If there exists an open interval (a,b) containing x_0 such that $f'(x)>0$ on (a,x_0) and $f'(x)<0$ on (x_0,b), then f has a local maximum value at x_0.

b) If there exists an open interval (a,b) containing x_0 such that $f'(x)<0$ on (a,x_0) and $f'(x)>0$ on (x_0,b), then f has a local minimum value at x_0.

PART II: Testing endpoints of the domain.

Suppose a is a left endpoint of the domain of f and f is right continuous at a.

c) If $f'(x)>0$ on some interval (a,b) then f has a local minimum value at a.

d) If $f'(x)<0$ on some interval (a,b) then f has a local maximum value at a.

Suppose b is right endpoint of the domain of f and f is left continuous at b.

e) If $f'(x)>0$ on some interval (a,b), then f has a local maximum value at b.

f) If $f'(x)<0$ on some interval (a,b), then f has a local minimum value at b.

<u>Remark:</u> If f' is positive (or negative) on both sides of a critical or singular point, then f has neither a maximum nor a minimum value at that point.

<u>Example:</u> Find the local and absolute extreme values of $f(x) = x^4 - 2x^2 - 3$ on the interval $[-2, 2]$.

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$$

The critical points are $0, -1$, and 1.

The corresponding values are

$$f(0) = -3$$

$$f(-1) = f(1) = -4$$

There are no singular points.

The values of f at the endpoints -2 and 2 are

$f(-2) = f(2) = 5$.

		E.P		C.P		C.P		C.P		E.P	
x		-2		-1		0		1		2	
f'	$-$		$-$		$+$		$-$		$+$		$+$
f	$\searrow$		$\searrow$		$\nearrow$		$\searrow$		$\nearrow$		$\nearrow$
		$f'(x)<0$ Absolute max.		Absolute min.		Local max.		Absolute min.		$f'(x)>0$ Absolute max	

Functions Not Defined on Closed, Finite Intervals

Theorem: (Existence of extreme values on open intervals)
If f is continuous on the open interval (a,b), and if
$\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to b^-} f(x) = M$,
then the following conclusions hold:

i) If $f(u) > L$ and $f(u) > M$ for some u in (a,b), then f has an absolute maximum value on (a,b).

ii) If $f(v) < L$ and $f(v) < M$ for some v in (a,b), then f has an absolute minimum value on (a,b).

In this theorem "a" may be $-\infty$, in which case $\lim_{x\to a^+}$ should be replaced with $\lim_{x\to -\infty}$ and b may be ∞, in which case $\lim_{x\to b^-}$ should be replaced with $\lim_{x\to\infty}$. Also, either or both of L and M may be either ∞ or $-\infty$.

Example: Show that $f(x) = x + \frac{4}{x}$ has an absolute minimum value on the interval $(0,\infty)$, and find the minimum value.

We have
$\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$.
Since $f(1) = 5 < \infty$. Theorem guarantees that f must have an absolute minimum value at some point in $(0,\infty)$.

To find the minimum value we must check the values of f at any critical points or singular points in the interval. We have

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = \frac{(x-2)(x+2)}{x^2}$$

which equals 0 only at $x=2$ and $x=-2$.

Since f has domain $(0,\infty)$, it has no singular points and only one critical point, namely, $x=2$, where f has the value $f(2)=4$. This must be the minimum value of f on $(0,\infty)$.