9.1) Sequences and Convergence

$\boxed{1,4}$ Determine whether the given sequence is

(a) bounded (above or below)

(b) positive or negative (ultimately)

(c) increasing, decreasing or alternating

(d) convergent, divergent, divergent to ∞ or $-\infty$.

Q1) $\left\{ \frac{2n^2}{n^2+1} \right\}$

(a) $0 \leq \frac{2n^2}{n^2+1} \leq \frac{2n^2}{n^2} = 2$

$\Rightarrow$ Bounded below by 0, bounded above by 2

(c) Let $f(n) = \frac{2n^2}{n^2+1}$. Then,

$$f'(n) = \frac{4n(n^2+1) - 2n^2(2n)}{(n^2+1)^2} = \frac{4n}{(n^2+1)^2} > 0$$

$\Rightarrow$ Increasing

(d) $\lim\limits_{n\to\infty} \frac{2n^2}{n^2+1} = \lim\limits_{n\to\infty} \frac{2n^2/n^2}{(n^2+1)/n^2} = 2$

(b) $n^2 \geq 0$ for $n=1,2,\ldots$ So, both numerator and denominator of each term is positive $\Rightarrow$ positive sequence.

Q4) $\left\{ \sin \frac{1}{n} \right\}$

(a) $-1 \leq \sin x \leq 1$ for $x \in \mathbb{R}$ $\Rightarrow$ bounded

(b) $\sin x$ is positive for $x \in [0, \frac{\pi}{2}]$ and $\frac{1}{n} \in [0, \frac{\pi}{2}]$ for $n = 1, 2, \ldots$ $\Rightarrow$ positive

(c) Let $f(n) = \sin \frac{1}{n}$. Then

$f'(n) = -\frac{1}{n^2} \cos \frac{1}{n} < 0 \quad \forall n \Rightarrow$ decreasing

(d) $\lim\limits_{n \to \infty} \sin \frac{1}{n} = 0$ since $\frac{1}{n} \to 0$ as $n \to \infty$

and $\sin x \to 0$ as $x \to 0$.

$\boxed{21-27}$ Evaluate, whenever possible, the limit of the sequence $\{a_n\}$.

Q21) $a_n = \left(\frac{n-3}{n}\right)^n : \quad [1^\infty]$

$$\lim_{n\to\infty} \ln\left[\left(\frac{n-3}{n}\right)^n\right] = \lim_{n\to\infty}\left(n \ln\left(\frac{n-3}{n}\right)\right) \quad [\infty \cdot 0]$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n-3}{n}\right)}{\frac{1}{n}} \underset{\substack{\uparrow \\ \text{L'Hopital}}}{\overset{[\frac{0}{0}]}{=}} \lim_{n\to\infty} \frac{\frac{n}{n-3}\left(\frac{n-(n-3)}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\left(\frac{-3n}{n-3}\right) = -3 \quad \Rightarrow \quad \boxed{\lim_{n\to\infty} a_n = e^{-3}}$$

Q24) $a_n = n - \sqrt{n^2-4n} \quad : \quad [\infty - \infty]$

$$\lim_{n\to\infty}\left(n - \sqrt{n^2-4n}\right) = \lim_{n\to\infty} \frac{\left(n-\sqrt{n^2-4n}\right)\left(n+\sqrt{n^2-4n}\right)}{n+\sqrt{n^2-4n}}$$

$$= \lim_{n\to\infty} \frac{n^2-(n^2-4n)}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty} \frac{4n}{n\left(1+\sqrt{1-\frac{4}{n}}\right)} = 2$$

$$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 2}$$

Q27) $a_n = \dfrac{(n!)^2}{(2n)!}$:

$$0 \le \frac{(n!)^2}{(2n)!} = \frac{n!}{n!} \cdot \frac{1}{n+1} \cdot \underbrace{\frac{2}{n+2}}_{\le \frac{1}{2}} \cdots \underbrace{\frac{n}{2n}}_{\le \frac{1}{2}} \le \frac{1}{2^n} \xrightarrow{n\to\infty} 0$$

$\le \frac{1}{2}$ $\le \frac{1}{2}$ $\le \frac{1}{2}$

[For example,

$n=1 \Rightarrow a_1 = \frac{1}{1+1} \le \frac{1}{2}$, $n=2 \Rightarrow a_2 = \frac{1}{2+1} \frac{1}{2+2} \le \frac{1}{2^2}$]

So, by Squeeze theorem, $\boxed{\lim_{n\to\infty} a_n = 0}$

Q30) Let $a_1 = 1$, $a_{n+1} = \sqrt{1+2a_n}$ $(n=1,2,\dots)$.

Show that $\{a_n\}$ is increasing and bounded above (hence it is convergent) and find its limit.

[Hint 3 is an upper bound]

We will use the mathematical induction in order to show that $\{a_n\}$ is increasing and bounded.

I) $\{a_n\}$ is increasing:

•(Base step) Show that $a_2 > a_1$:

$$a_2 = \sqrt{1+2a_1} = \sqrt{3} > 1 = a_1 \Rightarrow a_2 > a_1$$

•(Inductive step) Suppose that $\underbrace{a_{k+1} > a_k}_{(*)}$. Show that $a_{k+2} > a_{k+1}$:

$$a_{k+2} = \sqrt{1+2a_{k+1}} \overset{\text{by } (*)}{>} \sqrt{1+2a_k} = a_{k+1} \Rightarrow a_{k+2} > a_{k+1}$$

So, $\{a_n\}$ is increasing

II) $\{a_n\}$ is bounded above:

• Show that $a_1 < 3$: $a_1 = 1 \Rightarrow a_1 < 3$.

• Suppose that $a_k < 3$. Show that $\underbrace{a_{k+1} < 3}_{(**)}$:

$$a_{k+1} = \sqrt{1+2a_k} \overset{\text{by } (**)}{<} \sqrt{1+2.3} = \sqrt{7} < 3 \Rightarrow a_{k+1} < 3.$$

So, $\{a_n\}$ is bounded above.

III) Increasing and bounded above $\Rightarrow$ Convergent

Consider some of the terms: $a_1 = 1$, $a_2 = \sqrt{3}$

$a_3 = \sqrt{1+2\sqrt{3}}$, $a_4 = \sqrt{1+2\sqrt{1+2\sqrt{3}}}$, $a_5 = \sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{3}}}}$

-- $a_n = \sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{\cdots}}}}$

Since this sequence is convergent,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$$

$$L = \sqrt{1+2\underbrace{\sqrt{1+2\sqrt{\cdots}}}_{L}} \Rightarrow L = \sqrt{1+2L}$$

$$\Rightarrow L^2 - 2L - 1 = 0 \Rightarrow L = 1 \mp \sqrt{2}$$

L must be positive by definition of $\{a_n\}$.

So, $\boxed{\lim_{n\to\infty} a_n = 1+\sqrt{2}}$