Q7) State whether the given integral converges or diverges, and justify your claim.

a) $\int_0^{\infty} \frac{dx}{1+\sqrt{x}}$

Sol: We can say $\sqrt{x} \geq 1$ on $[1,\infty)$. Hence,

$2\sqrt{x} \geq 1+\sqrt{x} \Rightarrow \frac{1}{1+\sqrt{x}} \geq \frac{1}{2\sqrt{x}} dx$. Hence,

$$\int_1^{\infty} \frac{1}{1+\sqrt{x}} dx \geq \int_1^{\infty} \frac{1}{2\sqrt{x}} dx = \lim_{R\to\infty} \int_1^{R} \frac{1}{2\sqrt{x}} dx = \lim_{R\to\infty} \sqrt{x}\Big|_1^{R} = \infty.$$

hence, $\int_0^{\infty} \frac{1}{1+\sqrt{x}} dx = \underbrace{\int_0^{1} \frac{1}{1+\sqrt{x}} dx}_{\text{finite}} + \underbrace{\int_1^{\infty} \frac{1}{1+\sqrt{x}} dx}_{\infty} = \infty$ diverges.

b) $\int_0^{\pi} \frac{\sin x}{x} dx$

Sol: Since $\sin x \leq x$ for all $x \geq 0$, thus $\frac{\sin x}{x} \leq 1$. Then

$$I = \int_0^{\pi} \frac{\sin x}{x} dx = \lim_{c\to 0^+} \int_c^{\pi} \frac{\sin x}{x} dx \leq \int_0^{\pi} 1.dx = \pi$$

Also, we can see that $0 \leq I$.

Therefore I converges.

c) $\displaystyle\int_0^{\pi^2} \frac{dx}{1-\cos\sqrt{x}}$

Sol: Since $0 \leq 1-\cos\sqrt{x} = 2\cdot\sin^2\left(\frac{\sqrt{x}}{2}\right) \leq 2.\left(\frac{\sqrt{x}}{2}\right)^2 = \frac{x}{2}$

for all $x \geq 0$, therefore,

$$\int_0^{\pi^2} \frac{dx}{1-\cos\sqrt{x}} \geq 2.\int_0^{\pi^2} \frac{dx}{x} = 2.\lim_{c\to 0^+} \int_c^{\pi^2} \frac{dx}{x} = 2.\lim_{c\to 0^+} \ln x \Big|_c^{\pi^2}$$

$$= 2.\lim_{c\to 0^+} (\ln\pi^2 - \ln c) = \infty.$$

So, given integral diverges.

Q8) Given that $\displaystyle\int_0^{\infty} e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$, evaluate

(a) $\displaystyle\int_0^{\infty} x^2.e^{-x^2} dx$ and (b) $\displaystyle\int_0^{\infty} x^4 e^{-x^2} dx$.

Sol: a) Let us say $u = x$ and $dv = x.e^{-x^2}dx$. Then,
$du = dx$ and $v = -\frac{1}{2}e^{-x^2}$.

$$\int_0^{\infty} x^2.e^{-x^2}dx = \lim_{R\to\infty} \int_0^{R} x^2.e^{-x^2} dx = \lim_{R\to\infty}\left[-\frac{1}{2}x.e^{-x^2}\Big|_0^R + \frac{1}{2}\int_0^R e^{-x^2}dx\right]$$

$$= -\frac{1}{2}\lim_{R\to\infty} R.e^{-R^2} + \frac{1}{2}\int_0^{\infty} e^{-x^2}dx = 0 + \frac{1}{4}\sqrt{\pi} = \frac{\sqrt{\pi}}{4}$$ //

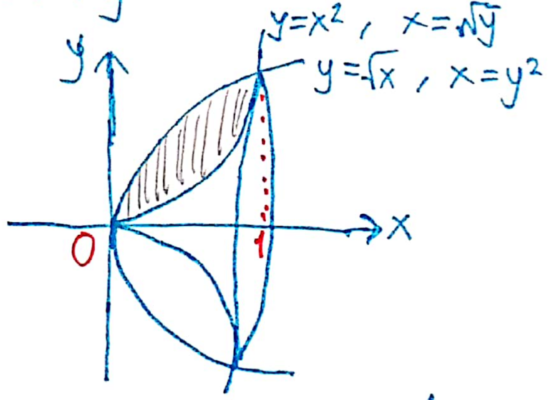
b) Let us say $u = x^3$ and $dv = x.e^{-x^2}dx$. Then,
$du = 3x^2dx$ and $v = -\frac{1}{2}e^{-x^2}$. Hence,

$$\int_0^{\infty} x^4.e^{-x^2}dx = \lim_{R\to\infty}\int_0^R x^4 e^{-x^2}dx = \lim_{R\to\infty}\left[-\frac{1}{2}x^3 e^{-x^2}\Big|_0^R + \frac{3}{2}\int_0^R x^2 e^{-x^2}dx\right]$$

$$= -\frac{1}{2}\lim_{R\to\infty} R^3 e^{-R^2} + \frac{3}{2}\int_0^{\infty} x^2.e^{-x^2}dx$$

$$= 0 + \frac{3}{2}\left(\frac{1}{4}\sqrt{\pi}\right) = \frac{3\sqrt{\pi}}{8}$$ //

Q9) Find the volume of the solid S, generated by rotating about x-axis the region bounded by $y=x^2$ and $y=\sqrt{x}$ between $x=0$ and $x=1$ in two ways, using the method of slicing and the method of cylindirical shells.



Sol: Let us see the graph.

$x^2=\sqrt{x} \Rightarrow x=0$ or $x=1$.

By slicing:

$$V=\pi\int_0^1[(\sqrt{x})^2-(x^2)^2]dx=\pi\int_0^1(x-x^4)dx=\pi\left(\frac{x^2}{2}-\frac{x^5}{5}\right)\Big|_0^1=\frac{3\pi}{10}$$ //

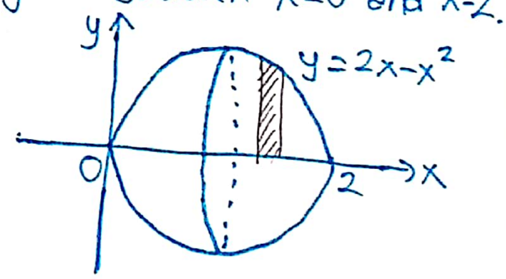
By sheels: $y=x^2 \Rightarrow x=\sqrt{y}$, $y=\sqrt{x}\Rightarrow x=y^2$

$y^2=\sqrt{y} \Rightarrow y=0,\ y=1$. Hence,

$$V=2\pi\int_0^1 y(\sqrt{y}-y^2)dy=2\pi\left(\frac{2\,y^{5/2}}{5}-\frac{y^4}{4}\right)\Big|_0^1=\frac{3\pi}{10}$$ //

Q10) Find the volumes of solids obtained if the plane regions R described belows are rotated about the x-axis and y-axis.

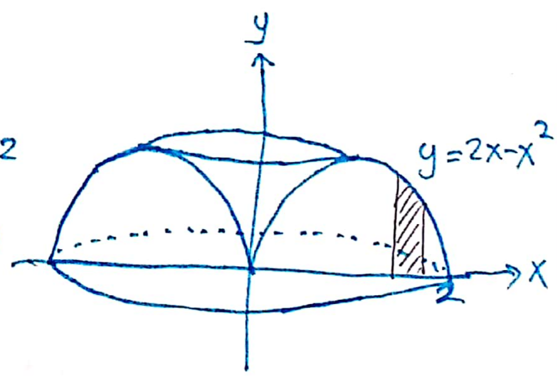
a) R is bounded by $y=x(2-x)$ and $y=0$ between $x=0$ and $x=2$.



Sol: For x-axis, we use slicing.

$$V=\pi\int_0^2 x^2(2-x)^2dx=\pi\int_0^2(4x^2-4x^3+x^4)dx$$

$$=\pi\left(\frac{4}{3}x^3-x^4+\frac{x^5}{5}\right)\Big|_0^2=\frac{16\pi}{5}$$ //

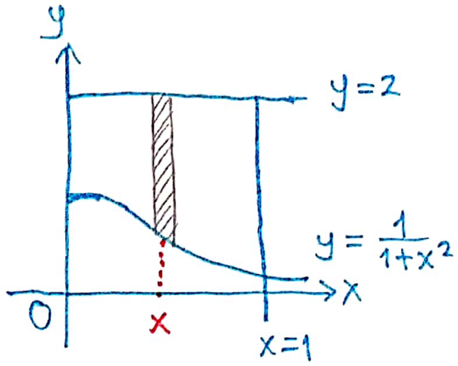


For y-axis, we use shells,

$$V=2\pi\int_0^2 x^2(2-x)dx=2\pi\left(\frac{2x^3}{3}-\frac{x^4}{4}\right)\Big|_0^2$$

$$=2\pi\left(\frac{8.2}{3}-\frac{16}{4}-0\right)=\frac{8\pi}{3}$$ //

b) R is bounded by $y=\frac{1}{1+x^2}$, $y=2$, $x=0$, and $x=1$.



Sol: About x-axis:

$$V=\pi\int_0^1\left[2^2-\left(\frac{1}{1+x^2}\right)^2\right]dx \qquad \text{Let } x=\tan\theta,\ dx=\sec^2\theta\, d\theta$$

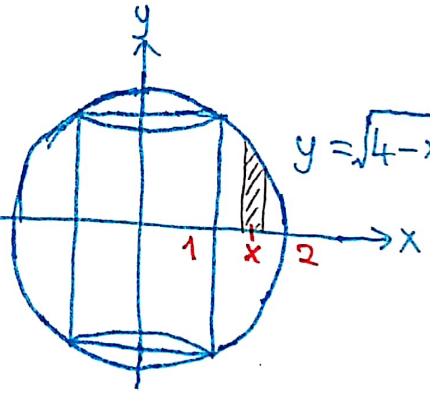
$$=4\pi-\pi\int_0^{\pi/4}\frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$=4\pi-\pi\int_0^{\pi/4}\cos^2\theta\, d\theta=4\pi-\frac{\pi}{2}\left(\theta+\sin\theta\cos\theta\right)\Big|_0^{\pi/4}=\frac{15\pi}{4}-\frac{\pi^2}{8}\ //$$

About y-axis:

$$V=2\pi\int_0^1 x\left(2-\frac{1}{1+x^2}\right)dx=2\pi\left(x^2-\frac{1}{2}\ln(1+x^2)\right)\Big|_0^1=2\pi\left(1-\frac{1}{2}\ln 2\right)\ //$$

Q11) What the percentage of the volume of a ball of radius 2 is removed if a hole of radius 1 drilled through the centre of the ball?



Sol: The volume is

$$V=2\cdot 2\pi\int_1^2 x\cdot\sqrt{4-x^2}\,dx. \qquad \text{Let } u=4-x^2,\ du=-2x\,dx.$$

$$=2\pi\int_0^3\sqrt{u}\,du=\frac{4\pi}{3}u^{3/2}\Big|_0^3=4\pi\sqrt{3}\ //$$

Since the volume of the the ball is $\frac{4}{3}\pi 2^3=\frac{32\pi}{3}$,

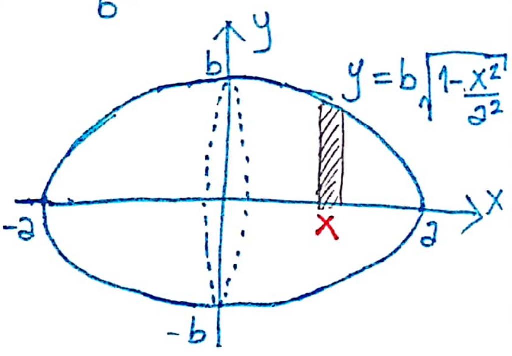
therefore the volume removed is $\frac{32\pi}{3}-4\pi\sqrt{3}$.

The percentage removed is

$$\frac{\frac{32\pi}{3}-4\pi\sqrt{3}}{\frac{32\pi}{3}}\cdot 100=100\left(1-\frac{3\sqrt{3}}{8}\right)\simeq 35.$$

About %35 of the volume removed.

Q12) Find the volume of the ellipsoid of revolution obtained by rotating the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about x-axis.



Sol: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=b\sqrt{1-\frac{x^2}{a^2}}$.

$$V = 2\cdot\pi\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2\pi b^2\left(x-\frac{x^3}{3a^2}\right)\Big|_0^a = \frac{4}{3}\pi a b^2 \;//$$

Q13) The region R bounded by $y=x^{-k}$ and $y=0$ and lying to the right of $x=1$ is rotated about the x-axis. Find all real values of k for which the solid so generated has finite volume.

Sol: The volume is

$$V=\pi\int_1^\infty (x^{-k})^2dx = \pi\int_1^\infty x^{-2k}dx$$

$$=\pi.\lim_{R\to\infty}\frac{x^{1-2k}}{1-2k}\Big|_1^R = \pi.\lim_{R\to\infty}\left(\frac{R^{1-2k}}{1-2k}+\frac{\pi}{2k-1}\right).$$

In order for the solid to have finite volume we need $1-2k<0$, that is $k>\frac{1}{2}$.