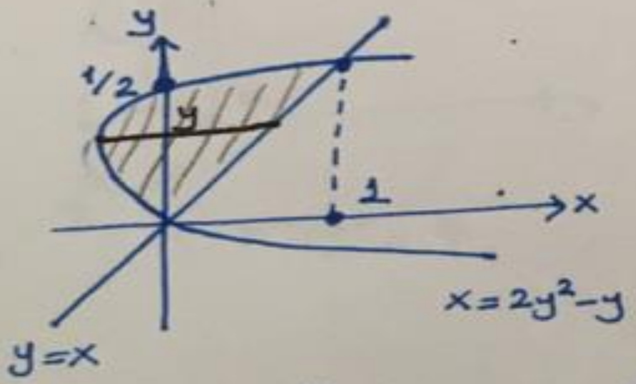
ex: Evaluate $\iint_D 2xy\,dA$ where D is the finite region in the xy-plane bounded by the parabola $x=2y^2-y$ and the line $y=x$.



$y = 2y^2 - y \Rightarrow 2y^2 - 2y = 0$

$2y(y-1) = 0$

$y=0, \quad y=1$

$$\int_0^1 dy \int_{2y^2-y}^{y} 2xy\,dx = \int_0^1 x^2y \Big|_{x=2y^2-y}^{x=y} dy = \int_0^1 \left[y^2-(2y^2-y)^2\right] y\,dy$$

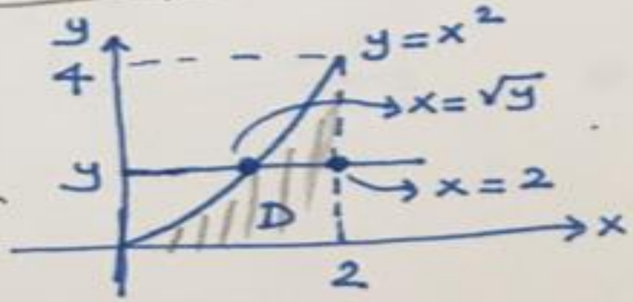
$$= \int_0^1 \left(-4y^4+4y^3\right) y\,dy = \int_0^1 (-4y^5+4y^4)\,dy$$

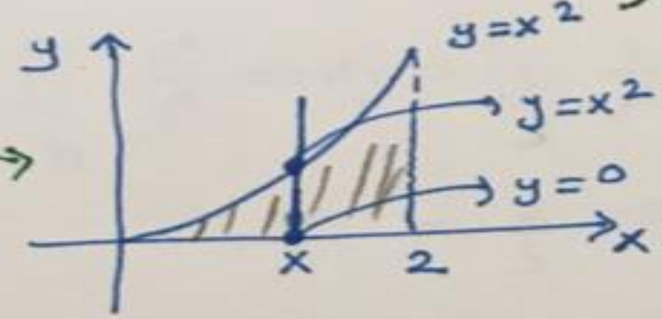
$$= -4\frac{y^6}{6} + 4\frac{y^5}{5}\Big|_0^1 = -\frac{2}{3}+\frac{4}{5} = \frac{2}{15}$$

ex: Evaluate the following integral:

$$\int_0^4 \left(\int_{\sqrt{y}}^2 \frac{y}{\sqrt{4+x^5}}\, dx \right) dy .$$

$$\underbrace{\int_0^4 \left(\int_{\sqrt{y}}^2 \frac{y}{\sqrt{4+x^5}}\, dx \right) dy} = \iint_D \frac{y}{\sqrt{4+x^5}}\, dA \quad \underline{\text{where}} \quad D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le x^2\}$$





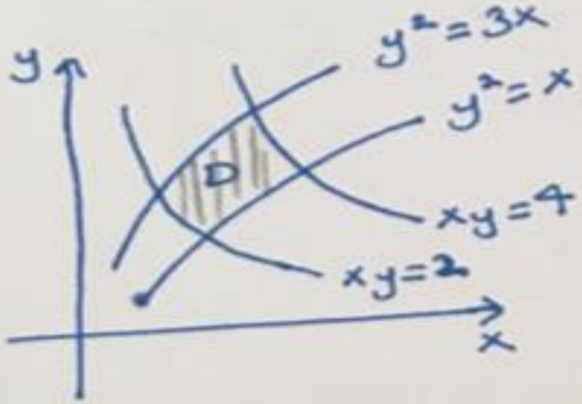
$$= \int_0^2 dx \int_0^{x^2} \frac{y}{\sqrt{4+x^5}}\, dy$$

$$= \int_0^2 \frac{1}{\sqrt{4+x^5}} \left. \frac{y^2}{2} \right|_{y=0}^{y=x^2} dx$$

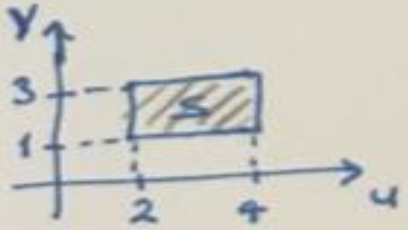
$$= \frac{1}{2} \int_0^2 \frac{x^4}{\sqrt{4+x^5}}\, dx = \frac{1}{2} \frac{2}{5} \left. \sqrt{4+x^5} \right|_0^2$$

$$= \frac{1}{5} \left(\sqrt{36} - \sqrt{4} \right) = \frac{4}{5}$$

ex: Evaluate the double integral $\iint_D \frac{x^2}{y^4}\,dx\,dy$ where D is the region bounded by the hyperbolas $xy=2$, $xy=4$ and the parabolas $y^2=x$, $y^2=3x$.



- Let $u=xy$, $v=\frac{y^2}{x}$



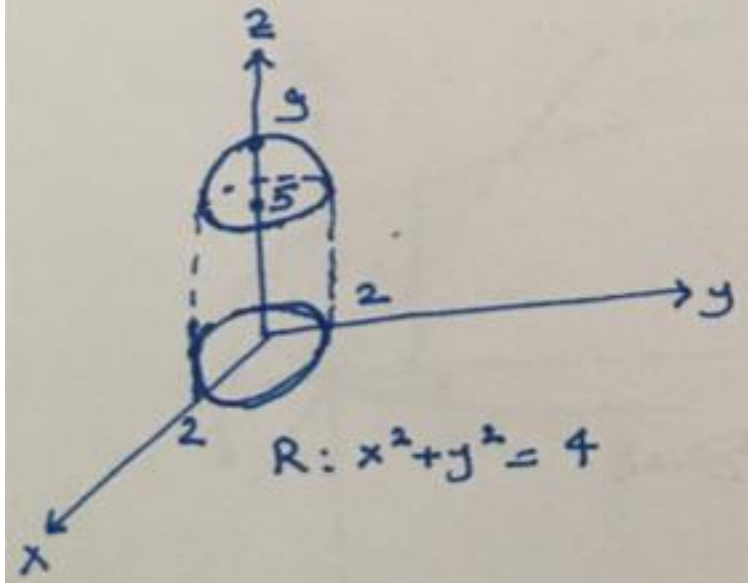
- $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{3y^2}{x}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{x}{3y^2} = \frac{1}{3v}$$

- $$\iint_D \frac{x^2}{y^4}\,dx\,dy = \iint_S \frac{1}{v^2}\,\frac{1}{3v}\,du\,dv = \frac{1}{3}\int_2^4\left(\int_1^3 \frac{1}{v^3}\,dv\right)du$$

$$= \frac{2}{3}\left.\frac{-1}{2v^2}\right|_1^3 = \frac{8}{27}$$

ex: Find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the plane $z = 5$.



$R: x^2+y^2 = 4$

$z = 9 - x^2 - y^2 = 5$

$\Rightarrow x^2 + y^2 = 4$

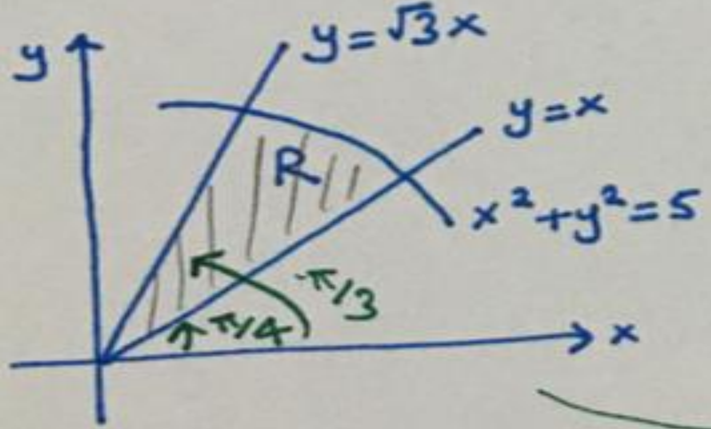
- $V = \iint_R (9 - x^2 - y^2 - 5)\, dA$

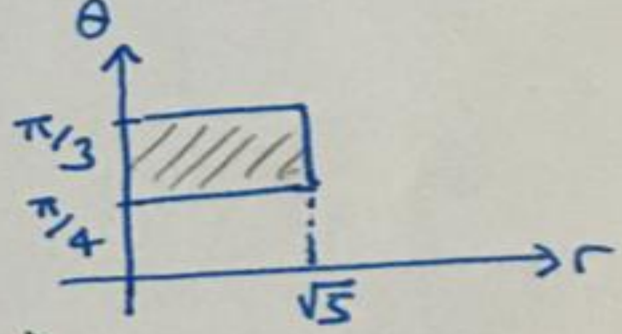
$= \iint_R (4 - r^2)\, r\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta$

$= 2\pi \left[4\frac{r^2}{2} - \frac{r^4}{4} \right]\Big|_0^2 = 8\pi$

ex: Evaluate $\iint_R e^{x^2+y^2}\,dA$ where R is the region in the first quadrant bounded by $y=x$, $y=\sqrt{3}x$ and $x^2+y^2=5$.





in polar coordinates

$$\iint_R e^{x^2+y^2}\,dA = \int_{\pi/4}^{\pi/3}\int_0^{\sqrt{5}} e^{r^2}\, r\,dr\,d\theta = \left(\frac{\pi}{3}-\frac{\pi}{4}\right)\frac{1}{2}e^{r^2}\Big|_0^{\sqrt{5}}$$

$$= \frac{\pi}{24}\left(e^5-1\right)$$