12.3) Partial Derivatives

$\boxed{6-9}$ Find all the partial derivatives of the given function. Evaluate them at the given point.

Q6) $w = \ln(1+e^{xyz})$; $(2,0,-1)$

* Partial derivative with respect to x : $\frac{\partial w}{\partial x}$

Consider y and z as constants

$\Rightarrow \frac{\partial w}{\partial x} = \frac{yze^{xyz}}{1+e^{xyz}}$; $\frac{\partial w}{\partial x}(2,0,-1) = 0$

* $\frac{\partial w}{\partial y} = \frac{xze^{xyz}}{1+e^{xyz}}$; $\frac{\partial w}{\partial y}(2,0,-1) = \frac{2(-1)e^0}{1+e^0} = -1$

* $\frac{\partial w}{\partial z} = \frac{xye^{xyz}}{1+e^{xyz}}$; $\frac{\partial w}{\partial z}(2,0,-1) = 0$

Q7) $f(x,y) = \sin(x\sqrt{y})$; $(\frac{\pi}{3}, 4)$

* $\frac{\partial f}{\partial x} = \sqrt{y}\cos(x\sqrt{y})$; $\frac{\partial f}{\partial x}(\frac{\pi}{3},4) = 2\cos\frac{2\pi}{3} = -1$

* $\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}}\cos(x\sqrt{y})$; $\frac{\partial f}{\partial y}(\frac{\pi}{3},4) = \frac{\pi}{12}\cos\frac{2\pi}{3} = -\frac{\pi}{24}$

Q9) $w = x^{(y \ln z)}$; $(e, 2, e)$

* $\frac{\partial w}{\partial x} = (y \ln z) x^{(y \ln z - 1)}$; $\frac{\partial w}{\partial x}(e, 2, e) = 2e$

* $\frac{\partial w}{\partial y} = (\ln z)(\ln x) x^{(y \ln z)}$; $\frac{\partial w}{\partial y}(e, 2, e) = e^2$

* $\frac{\partial w}{\partial z} = \frac{y}{z} (\ln x) x^{(y \ln z)}$; $\frac{\partial w}{\partial z}(e, 2, e) = \frac{2}{e} e^2 = 2e$

$\boxed{12}$ Calculate the first partial derivatives of

$$f(x,y) = \begin{cases} \dfrac{x^2 - 2y^2}{x - y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

at $(0,0)$. You will have to use the following definition:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

Using the definition,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2}{h}-0}{h} = 1.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{-2k^2}{-k}-0}{k} = 2$$

$\boxed{16}$ Find equations of the tangent plane and normal line to the graph of $f(x,y)=e^{xy}$ at $(2,0)$.

A normal vector to $z=f(x,y)$ at $(a,b,f(a,b))$

$$\vec{n} = f_x(a,b)\,\hat{\imath} + f_y(a,b)\,\hat{\jmath} - \hat{k}$$

An equation of the tangent plane to $z=f(x,y)$ at $(a,b,f(a,b))$

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)-(z-f(a,b))=0$$

$f(x,y) = e^{xy} \Rightarrow f(2,0) = e^{2.0} = 1$

$f_x = y e^{xy} \Rightarrow f_x(2,0) = 0 . e^{2.0} = 0$

$f_y = x e^{xy} \Rightarrow f_y(2,0) = 2 . e^{2.0} = 2$

Then, $\vec{n} = 0.\hat{\imath} + 2\hat{\jmath} - \hat{k} = 2\hat{\jmath} - \hat{k}$

The equation of the tangent plane:

Passing through $(2,0,1)$, $\vec{n} = 2\hat{\jmath} - k$

$\Rightarrow 0.(x-2) + 2(y-0) - (z-1) = 0$

$\Rightarrow z = 1 + 2y$

The equation of the normal line:

Passing through $(2,0,1)$ and parallel to $\vec{n} = 2\hat{\jmath} - \hat{k}$.

$\Rightarrow x = 2 + 0t, \quad y = 0 + 2t, \quad z = 1 - t$

$\Rightarrow x = 2, \quad y = 2t, \quad z = 1 - t$

12.4) Higher Order Derivatives

$\boxed{4}$ Find all the second partial derivatives of

$$z = \sqrt{3x^2 + y^2} .$$

$$z_x = \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{3x^2+y^2}}(6x) = 3x(3x^2+y^2)^{-1/2}$$

$$z_y = \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{3x^2+y^2}}(2y) = y(3x^2+y^2)^{-1/2}$$

$$z_{xx} = \frac{\partial}{\partial x} z_x = \frac{\partial^2 z}{\partial x^2}$$

$$= 3(3x^2+y^2)^{-1/2} + 3x\left(-\frac{1}{2}\right)(3x^2+y^2)^{-3/2}(6x)$$

$$= (9x^2+3y^2)(3x^2+y^2)^{-3/2} - 9x^2(3x^2+y^2)^{-3/2}$$

$$= 3y^2(3x^2+y^2)^{-3/2}$$

$$z_{xy} = \frac{\partial}{\partial y} z_x = \frac{\partial^2 z}{\partial y \partial x}$$

$$= 3x\left(-\frac{1}{2}\right)(3x^2+y^2)^{-3/2}(2y)$$

$$= -3xy(3x^2+y^2)^{-3/2}$$

$$z_{yx} = \frac{\partial}{\partial x} z_y = \frac{\partial^2 z}{\partial x \partial y}$$

$$= y\left(-\frac{1}{2}\right)(3x^2+y^2)^{-3/2}(6x)$$

$$= -3xy(3x^2+y^2)^{-3/2}$$

$$z_{yy} = \frac{\partial}{\partial y} z_y = \frac{\partial^2 z}{\partial y^2}$$

$$= 1(3x^2+y^2)^{-1/2} + y\left(-\frac{1}{2}\right)(3x^2+y^2)^{-3/2}(2y)$$

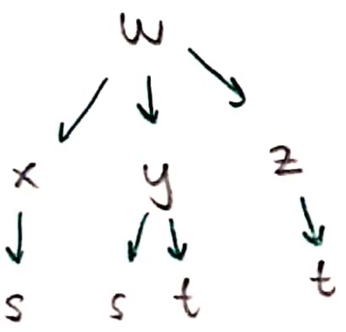
$$= (3x^2+y^2)(3x^2+y^2)^{-3/2} - y^2(3x^2+y^2)^{-3/2}$$

$$= 3x^2(3x^2+y^2)^{-3/2}$$

12.5) The Chain Rule

$\boxed{2,3}$ Write appropriate versions of the Chain Rule for the indicated derivatives.

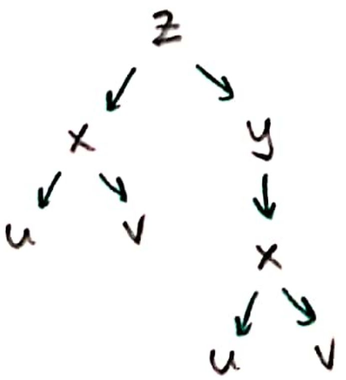
Q2) $\frac{\partial w}{\partial t}$ if $w=f(x,y,z)$, where $x=g(s)$, $y=h(s,t)$ and $z=k(t)$



We need to follow all paths from w to t

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Q3) $\frac{\partial z}{\partial u}$ if $z=g(x,y)$ where $y=f(x)$ and $x=h(u,v)$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{dy}{dx}\frac{\partial x}{\partial u}$$

$\boxed{6}$ Use two different methods to calculate $\dfrac{\partial u}{\partial t}$ if

$u = \sqrt{x^2+y^2}$, $x = e^{st}$ and $y = 1 + s^2 \cos t$.

I) By direct substitution:

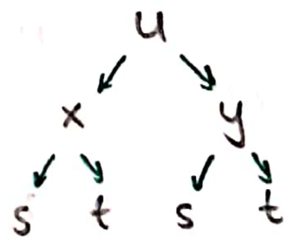
$$u = \sqrt{x^2+y^2} = \sqrt{e^{2st} + (1+s^2\cos t)^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{1}{2\sqrt{e^{2st}+(1+s^2\cos t)^2}} \left(2se^{2st} + 2(1+s^2\cos t)(-s^2 \sin t)\right)$$

$$= \frac{se^{2st} - s^2 \sin t\,(1+s^2\cos t)}{\sqrt{e^{2st}+(1+s^2\cos t)^2}}$$

$$= \frac{x^2 s - ys^2 \sin t}{\sqrt{x^2+y^2}}$$

II) Using the chain rule:



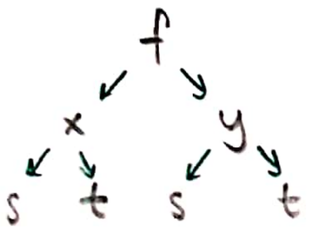
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\right)(se^{st}) + \left(\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\right)(-s^2 \sin t)$$

$$= \frac{xse^{st} - ys^2 \sin t}{\sqrt{x^2+y^2}}$$

$\boxed{17, 18}$ Assume that f has continuous partial derivatives of all orders.

Q17) If $x = t\sin s$ and $y = t\cos s$, find $\dfrac{\partial^2}{\partial s \partial t} f(x,y)$



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= f_x \sin s + f_y \cos s$$

$$\frac{\partial^2 f}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial s}\left(f_x \sin s + f_y \cos s\right)$$

$$= \frac{\partial f_x}{\partial s}\sin s + f_x\left(\frac{\partial}{\partial s}\sin s\right) + \frac{\partial f_y}{\partial s}\cos s + f_y\left(\frac{\partial}{\partial s}\cos s\right)$$

$$= \left(f_{xx}\frac{\partial x}{\partial s} + \underline{f_{xy}}\frac{\partial y}{\partial s}\right)\sin s + f_x \cos s$$

$$+ \left(\underline{f_{yx}}\frac{\partial x}{\partial s} + f_{yy}\frac{\partial y}{\partial s}\right)\cos s - f_y \sin s$$

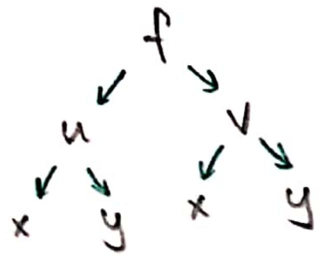
$$= \left(f_{xx}\, t\cos s - \underline{f_{xy}}\, t\sin s\right)\sin s + f_x \cos s$$

$$+ \left(\underline{f_{yx}}\, t\cos s - f_{yy}\, t\sin s\right)\cos s - f_y \sin s$$

$$= f_x \cos s - f_y \sin s + (f_{xx} - f_{yy})\, t\cos s \sin s$$

$$+ f_{xy}\, t\left(\cos^2 s - \sin^2 s\right)$$

Q18) Find $\dfrac{\partial^3}{\partial x \partial y^2} f(\underbrace{2x+3y}_{u}, \underbrace{xy}_{v})$ in terms of partial derivatives of f.



$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$= 3f_u + x f_v$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(3f_u + x f_v) = 3\frac{\partial f_u}{\partial y} + x\frac{\partial f_v}{\partial y}$$

$$= 3\left(\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial y}\right) + x\left(\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$= 3(3f_{uu} + x\underline{f_{uv}}) + x(3\underline{f_{vu}} + x f_{vv})$$

$$= 9f_{uu} + 6x f_{uv} + x^2 f_{vv}$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}(9f_{uu} + 6x f_{uv} + x^2 f_{vv})$$

$$= 9\frac{\partial f_{uu}}{\partial x} + 6f_{uv} + 6x\frac{\partial f_{uv}}{\partial x} + 2x f_{vv} + x^2\frac{\partial f_{vv}}{\partial x}$$

$$= 9\left(f_{uuu}\frac{\partial u}{\partial x} + f_{uuv}\frac{\partial v}{\partial x}\right) + 6f_{uv}$$

$$+ 6x\left(f_{uvu}\frac{\partial u}{\partial x} + \underline{f_{uvv}}\frac{\partial v}{\partial x}\right) + 2x f_{vv}$$

$$+ x^2\left(\underline{f_{vvu}}\frac{\partial u}{\partial x} + f_{vvv}\frac{\partial v}{\partial x}\right)$$

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$$= 9(2f_{uuu} + y f_{uuv}) + 6 f_{uv}$$
$$+ 6x(2 f_{uuv} + y f_{uvv}) + 2x f_{vv}$$
$$+ x^2(2 f_{uvv} + y f_{vvv})$$

$$= 6 f_{uv} + 2x f_{vv} + 18 f_{uuu} + (9y + 12x) f_{uuv}$$
$$+ (6xy + 2x^2) f_{uvv} + x^2 y f_{vvv}$$