TRANSCENDENTAL FUNCTIONS

Functions that cannot be constructed are called transcendental functions. e.g. trigonometric fncs, exponential fncs, logarithmic fncs, inverse trigonometric fncs.

Much of the importance of calculus and many of its most useful applications result from its ability to illuminate the behaviour of transcendental fncs that arise naturally when we try to model concrete problems in mathematical terms.

Inverse Functions

Definition: A function f is one to one if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of and $x_1 \neq x_2$ equivalently if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Note: If a fnc. on a single interval is increasing (or decreasing), then it is one-to-one.

Dfn: If f is one-to-one, then it has an <u>inverse function</u> f^{-1}. The value of $f^{-1}(x)$ is the unique number y in the domain of f for which $f(y) = x$. Thus,

$$y = f^{-1}(x) \Leftrightarrow x = f(y).$$

* The domain of f^{-1} is the range of f and vice versa.
* The inverse of one-to-one fnc. is itself one-to-one and so also has an inverse. Not suprizingly, the inverse of f^{-1} is f.

We can substitute either of the eqns. $y = f^{-1}(x)$ or $x = f(y)$ into the other and obtain the cancellation identities:

$$f(f^{-1}(x)) = x \quad , \quad f^{-1}(f(y)) = y$$

If S is any set of real numbers and I_S denotes the identity function on S, defined by

$$I_S(x) = x \quad \forall x \in S.$$

then the cancellation identities say that if $\mathcal{D}(f)$ is the domain of f, then $f \circ f^{-1} = I_{\mathcal{D}(f^{-1})}$ and $f^{-1} \circ f = I_{\mathcal{D}(f)}$

where $f \circ g(x)$ denotes the composition $(f \circ g)(x) = f(g(x))$.

The graph of the functions f^{-1} and f are reflections of each other in $y = x$.

Example: Show that $g(x) = \sqrt{2x+1}$ is invertible, and find its inverse

If $g(x_1) = g(x_2)$ then $\sqrt{2x_1+1} = \sqrt{2x_2+1}$

$$2x_1 + 1 = 2x_2 + 1$$

$$x_1 = x_2$$

$\therefore g$ is one-to-one fnc.

$x = g(y) = \sqrt{2y+1}$

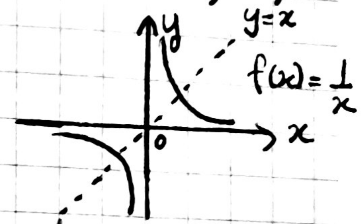
It follows that $x \geq 0$ and $x^2 = 2y+1$

$$y = \frac{x^2 - 1}{2}$$

$$g^{-1}(x) = \frac{x^2-1}{2} \quad \text{for } x \geq 0$$

* The restriction $x \geq 0$ applies since the range of g is $[0, \infty)$.

Definition: A function f is self-inverse if $f^{-1} = f$, that is, if $f(f(x)) = x$ for every x in the domain of f.



e.g. $f(x) = 1/x$ is self-inverse

$$x = \frac{1}{y} = f(y) \Rightarrow y = \frac{1}{x} \text{ so } f^{-1}(x) = \frac{1}{x} = f(x).$$

Derivatives of Inverse Functions

Suppose that the function f is differentiable on an interval (a,b) that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a,b) or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a,b). In either case f is one-to-one on (a,b) and has an inverse f^{-1} defined by

$$y = f^{-1}(x) \iff x = f(y) \quad (a < y < b).$$

$(f'(x) \neq 0)$

Since we are assuming that the graph $y = f(x)$ has a nonhorizontal tangent line at any x in (a,b), its reflection, the graph $y = f^{-1}(x)$, has a nonvertical tangent line at any x in the interval b/w. $f(a)$ and $f(b)$. Therefore, f^{-1} is differentiable at any such x.

Let $y = f^{-1}(x)$. We want to find $\frac{dy}{dx}$.

Solve the eqn. $y = f^{-1}(x)$ for $x = f(y)$ and differentiate implicitly w.r.t. x to obtain, $\frac{d}{dx}(x) = \frac{d}{dx}(f(y))$

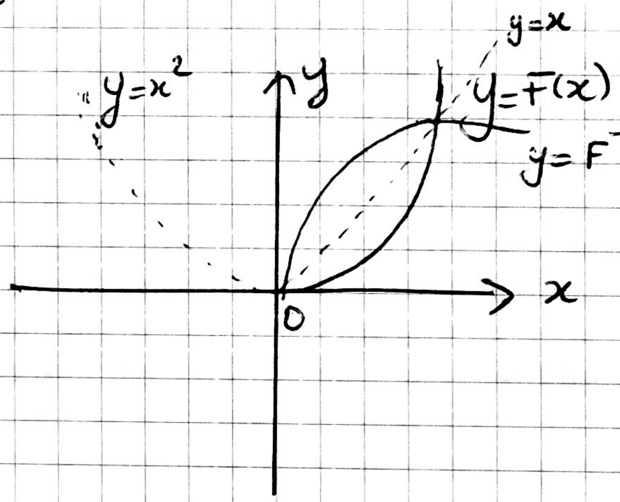
$$1 = f'(y)\frac{dy}{dx} \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

— Inverting Non-One-to-One Functions —

Many important fncs. such as the trigonometric functions are not one-to-one on their whole domains. It is still possible to define an inverse for such a fnc, but we have to restrict the domain of the fnc. artificially so that the restricted ~~function~~ function is one-to-one.

Example:



The graph of $F(x)$ is the right half of the parabola $y=x^2$, the graph of f.

Consider $f(x)=x^2$. $D(f)=\mathbb{R}$ and it is not one-to-one since $f(-a)=f(a)$ for any $a\in\mathbb{R}$.

Let us define a new function $F(x)$ equal to $f(x)$ but having a smaller domain, so that is one-to-one.

We can use the interval $[0,\infty)$ as the domain of F:

$$F(x)=x^2 \quad \text{for} \quad 0\leq x<\infty$$

Evidently, F is 1-1, so it has an inverse F^{-1} which is;

Let $y=F^{-1}(x)$ then $x=F(y)=y^2$ and $y\geq 0$.

Thus $y=\sqrt{x}$. Hence $F^{-1}(x)=\sqrt{x}$.

We will use the same method for trigonometric fncs to find their inverse.

ex. Show that $f(x) = x^3 + x$ is one-to-one the whole real line and $f(2) = 10$, find $(f^{-1})'(10)$.

$f'(x) = 3x^2 + 1 > 0 \quad \forall x \in \mathbb{R}$. f is increasing and one to one and so invertible. If $y = f^{-1}(x)$ then,

Differentiating implicitly with respect to x; $1 = 3y^2 y' + y'$

$$x = f(y) = y^3 + y \Rightarrow 1 = (3y^2 + 1) y'$$

$$\left(f^{-1}(x)\right)' = y' = \frac{1}{3y^2 + 1}$$

$x = f(2) = 10$ implies $y = f^{-1}(10) = 2$ Thus,

$$(f^{-1})'(10) = \left.\frac{1}{3y^2+1}\right|_{y=2} = \frac{1}{13} \quad //$$

EXPONENTIAL AND LOGARITMIC FUNCTIONS

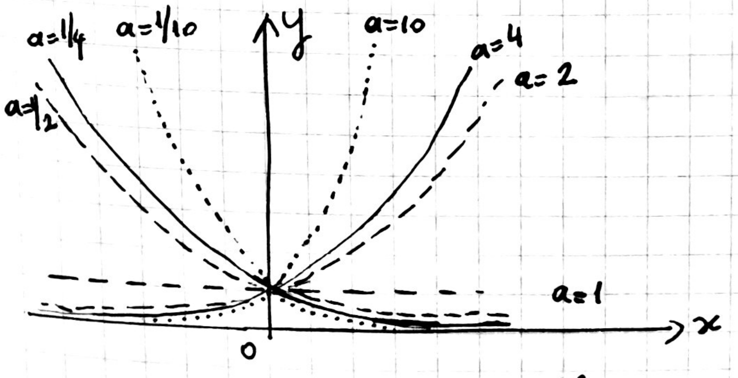
Definition: If $a > 0$, then $a^0 = 1$

$$a^n = \underbrace{a \dots a}_{n \text{ factors}} \quad \text{if } n = 1, 2, 3, \dots$$

$$a^{-n} = \frac{1}{a^n} \quad \text{if } n = 1, 2, 3, \dots$$

$$a^{m/n} = \sqrt[n]{a^m} \quad \text{if } n = 1, 2, 3, \dots \text{ and } m = \pm 1, \pm 2, \pm 3, \dots$$

In this definition $\sqrt[n]{a}$ is the number $b > 0$ that satisfies $b^n = a$.



$- a^x -$

If $a > 1$ then $\lim\limits_{x \to -\infty} a^x = 0$

and

$\lim\limits_{x \to \infty} a^x = \infty$

If $0 < a < 1$ then $\lim\limits_{x \to -\infty} a^x = \infty$ and $\lim\limits_{x \to \infty} a^x = 0$.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The graph of $y = a^x$ has the x-axis as a horizontal asymptote if $a \neq 1$. It is asymptotic on the left (as $x \to -\infty$) if $a > 1$ and on the right (as $x \to +\infty$) if $0 < a < 1$.

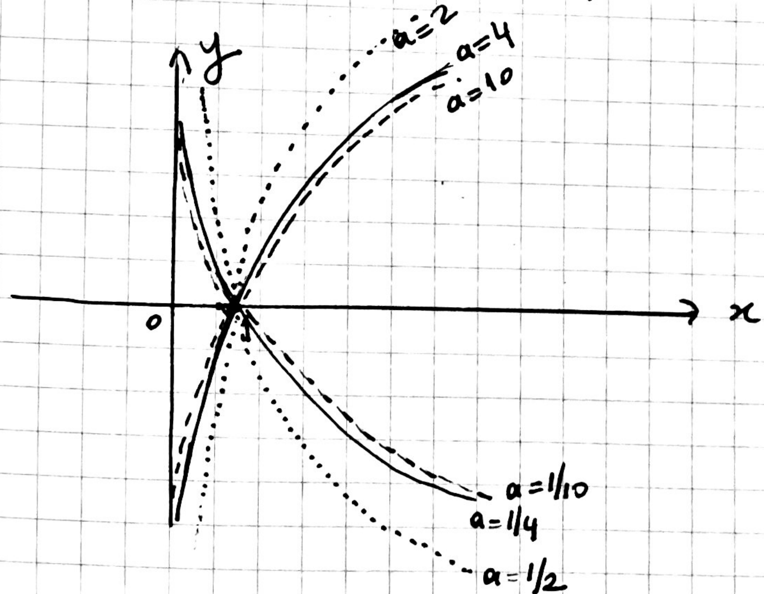
Logarithms: If $a > 0$ and $a \neq 1$, the function $\log_a x$, called the logarithm of x to the base a, is the inverse of the one-to-one function a^x:

$$y = \log_a x \iff x = a^y, \quad (a > 0,\ a \neq 1)$$

Since a^x has domain $(-\infty, \infty)$, $\log_a x$ has range $(-\infty, \infty)$.
Since a^x has range $(0, \infty)$, $\log_a x$ has domain $(0, \infty)$.
Also the following cancellation identities hold:

$\log_a(a^x) = x$ for <u>all real x</u> ($\Rightarrow$ domain of a^x)

$a^{\log_a x} = x$ for <u>all $x > 0$</u> ($\Rightarrow$ domain of $\log_a x$)



The graphs of some typical logarithmic fncs. are shown in figure.

They all pass through the point $(1,0)$.

Each graph is the reflection in the line $y=x$ of the corresponding exponential graph in previous figure.

– $\log_a x$ –

If $a > 1$, then $\lim_{a \to 0^+} \log_a x = -\infty$ and $\lim_{x \to \infty} \log_a x = \infty$

If $0 < a < 1$, then $\lim_{a \to 0^+} \log_a x = \infty$ and $\lim_{x \to \infty} \log_a x = -\infty$

Laws of Logarithms

If $x>0$, $y>0$, $a>0$, $b>0$, $a\neq 1$ and $b\neq 1$ then

i) $\log_a 1 = 0$

ii) $\log_a\left(\frac{1}{x}\right) = -\log_a x$

iii) $\log_a(xy) = \log_a x + \log_a y$

iv) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

v) $\log_a(x^y) = y.\log_a x$

vi) $\log_a x = \dfrac{\log_b x}{\log_b a}$

Example : Simplify

i) $\log_2 10 + \log_2 12 - \log_2 15 = \log_2 \dfrac{10\times 12}{15}$

$= \log_2 8$

$= \log_2 2^3$

$= 3$ (cancellation identity)

ii) $\log_{a^2} a^3 = 3\log_{a^2} a$

$= \dfrac{3}{2}\log_{a^2} a^2$

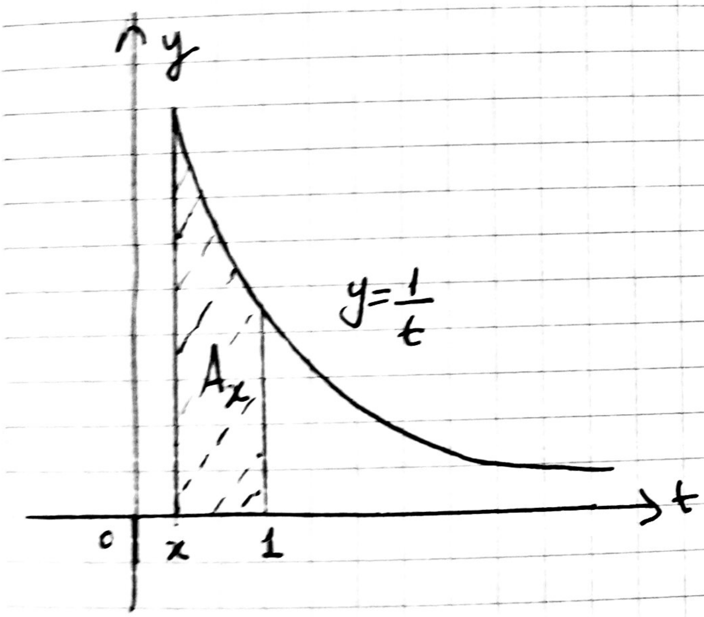
$= \dfrac{3}{2}$ (Cancellation identity)

THE NATURAL LOGARITHM AND EXPONENTIAL

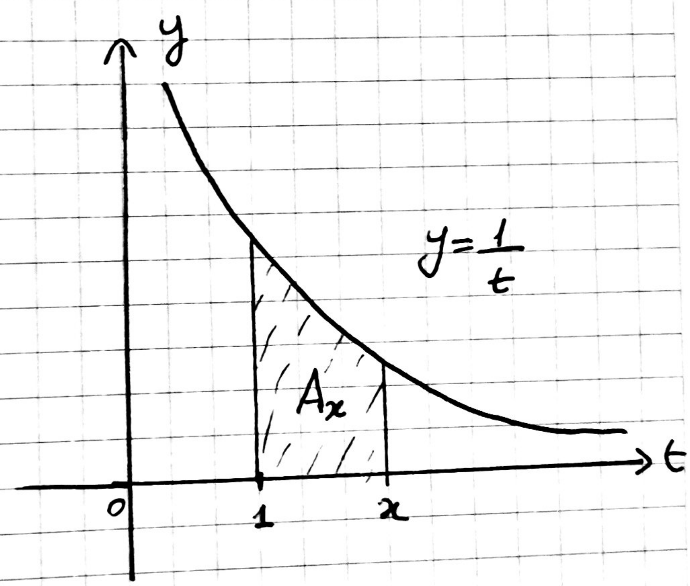
Definition: (The natural logarithm)

For $x>0$, let A_x be the area of the plane region bounded by the curve $y=\frac{1}{t}$, the t-axis, and the vertical lines $t=1$ and $t=x$. The function $\ln x$ is defined by

$$\ln x = \begin{cases} A_x & \text{if } x \geq 1, \\ -A_x & \text{if } 0<x<1. \end{cases}$$



$\ln x = -$ area A_x if $0<x<1$



$\ln x =$ area A_x if $x \geq 1$

The definition implies that $\ln 1 = 0$, that $\ln x > 0$ if $x>1$, that $\ln x < 0$ if $0<x<1$, and that $\ln x$ is one-to-one fnc.

Theorem: If $x>0$, then $\frac{d}{dx} \ln x = \frac{1}{x}$.

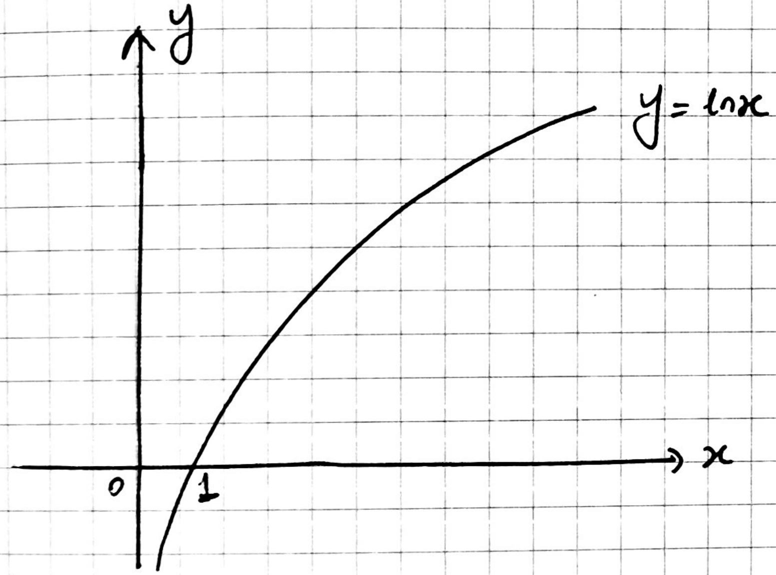
Theorem: (Properties of Natural Logarithm)

i) $\ln(xy) = \ln x + \ln y$

ii) $\ln\left(\frac{1}{x}\right) = -\ln x$

iii) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

vi) $\ln(x^r) = r\ln x$



"The graph of $\ln x$"

As you can see from the graph;

$$\lim_{x\to\infty} \ln x = \infty$$

$$\lim_{x\to 0^+} \ln x = -\infty$$

Example: Show that $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for any $x \neq 0$. Hence find $\int \frac{1}{x}\,dx$.

If $x > 0$ then $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(x) = \frac{1}{x}$

If $x < 0$ then $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$

Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$, and on any interval not containing $x = 0$;

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Example: Find the derivatives of the followings.

a)
$$\frac{d}{dx}\ln|\cos x| = \frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)$$
$$= \frac{1}{\cos x}(-\sin x)$$
$$= -\tan x.$$

b)
$$\frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) = \frac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)$$
$$= \frac{1}{\left(x+\sqrt{x^2+1}\right)}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$
$$= \frac{1}{\left(x+\sqrt{x^2+1}\right)}\,\frac{2\left(\sqrt{x^2+1}+x\right)}{2\sqrt{x^2+1}}$$
$$= \frac{1}{\sqrt{x^2+1}}$$

Definition: (The Exponential Function)

The natural logarithm function "$\ln x$" is one-to-one on its domain, the interval $(0,\infty)$, so it has an inverse there. We call this inverse $\exp x$. Thus,

$$y=\exp x \iff x=\ln y \quad (y>0).$$

Since $\ln 1=0$, we have $\exp 0=1$.

The domain of $\exp x$ is $(-\infty,\infty)$ the range of $\ln x$.

The range of $\exp x$ is $(0,\infty)$ the domain of $\ln x$.

And the cancellation identities are:

* $\ln(\exp x) = x$ for all real x.
* $\exp(\ln x) = x$ for $x > 0$.

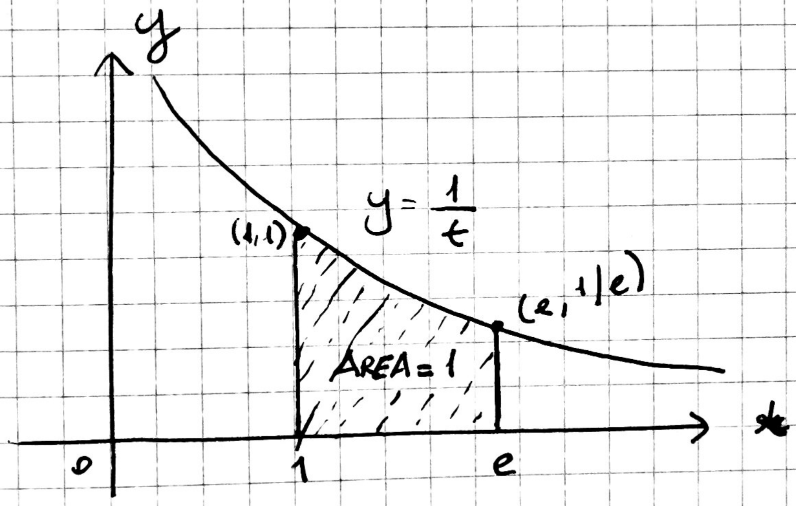
Theorem: Properties of the exponential function

i) $(\exp x)^r = \exp(rx)$

ii) $\exp(x+y) = \exp x \cdot \exp y$

iii) $\exp(-x) = \dfrac{1}{\exp(x)}$

iv) $\exp(x-y) = \dfrac{\exp(x)}{\exp(y)}$



The definition of "e":

Let $e = \exp(1)$. The number "e" satisfies $\ln e = 1$, so the area bounded by the curve $y = 1/t$, the t-axis and the vertical lines $t = 1$ and $t = e$ must be equal to 1 square unit.

The number "e" is irrational like "π" and not a zero of any polynomial with rational coeffcients such numbers are called transcandental.

$$e = 2,718281828459045\ldots$$

$\exp x = e^x$ for all real x.

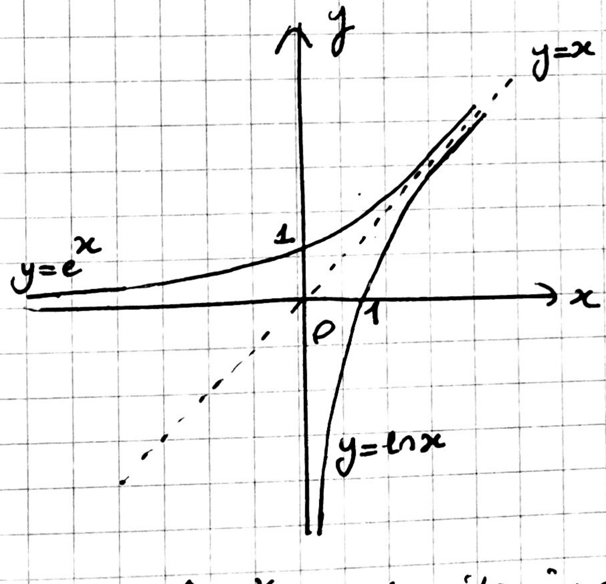
Previous theorem can now be restated in terms of e^x:

i) $(e^x)^y = e^{xy}$

ii) $e^{-x} = \dfrac{1}{e^x}$

iii) $e^{x+y} = e^x e^y$

iv) $e^{x-y} = \dfrac{e^x}{e^y}$



$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to \infty} e^x = \infty$$

- The graph of e^x and its inverse $\ln x$ -

Since $\exp x = e^x$ actually is an exponential fnc., its inverse must actually be a logarithm $\ln x = \log_e x$.

The derivative of $y=e^x$ is calculated by implicit differentiation.

$$y=e^x \Rightarrow x=\ln y$$

$$\Rightarrow 1=\frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=y=e^x$$

$$\therefore, \frac{d}{dx}(e^x)=e^x$$

and $$\int e^x dx = e^x + C.$$

Example: Find the derivatives of the followings.

a) $$\frac{d}{dx}e^{x^2-3x} = e^{x^2-3x}\frac{d}{dx}(x^2-3x) = e^{x^2-3x}(2x-3)$$

b) $$\frac{d}{dx}\sqrt{1+e^{2x}} = \frac{1}{2\sqrt{1+e^{2x}}}\frac{d}{dx}(1+e^{2x}) = \frac{2e^{2x}}{2\sqrt{1+e^{2x}}}$$

$$= \frac{e^{2x}}{\sqrt{1+e^{2x}}}$$

c) $$\frac{d}{dx}\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right) = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{(e^x)^2+2e^xe^{-x}+(e^{-x})^2-(e^x)^2+2e^xe^{-x}-(e^{-x})^2}{(e^x+e^{-x})^2}$$

$$= \frac{4e^xe^{-x}}{(e^x+e^{-x})^2} = \frac{4}{(e^x+e^{-x})^2}$$

Example: Let $f(t) = e^{at}$. Find (a) $f^{(n)}(t)$ and (b) $\int f(t)\,dt$.

(a) We have $f'(t) = a e^{at}$

$$f''(t) = a^2 e^{at}$$

$$f'''(t) = a^3 e^{at}$$

$$\vdots$$

$$f^{(n)}(t) = a^n e^{at}$$

(b) Also, $\int f(t)\,dt = \int e^{at}\,dt = \frac{1}{a} e^{at} + C$ $\left(\text{since } \frac{d}{dt}\frac{1}{a}e^{at} = e^{at}\right)$

Definition: The general exponential a^x

$$a^x = e^{x\ln a} \qquad (a>0,\ x\in\mathbb{R})$$

Derivative of a^x

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x\ln a} = e^{x\ln a}\cdot \ln a = a^x \cdot \ln a$$

Example: Show that the graph of $f(x) = x^{\pi} - \pi^{x}$ has a negative slope at $x=\pi$. (We need to show $f'(\pi) < 0$)

$$f'(x) = \pi x^{\pi-1} - \pi^x \ln\pi$$

$$f'(\pi) = \pi.\pi^{\pi-1} - \pi^{\pi}\ln\pi = \pi^{\pi}(1-\ln\pi)$$

Since $\pi > 3 > e$, we have $\ln\pi > \ln e = 1$ so $1-\ln\pi < 0$.

Since $\pi^{\pi} = e^{\pi\ln\pi} > 0$ we have $f'(\pi) < 0$.

The derivative of $\log_a x$ is given by;

If $y = \log_a x$ then $x = a^y$ and, differentiating implicitly w.r.t. x we get;

$$1 = a^y \ln a \frac{dy}{dx} = x \ln a \frac{dy}{dx}$$

$$\frac{d}{dx} \log_a x = \frac{dy}{dx} = \frac{1}{x \ln a}.$$

Since $\log_a x$ can be expressed in terms of logarithms to any other base, say "e",

$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}.$$

Logarithmic Differentiation

In this part we will differentiate a function of the form $y = (f(x))^{g(x)}$ (for $f(x) > 0$) by taking natural logarithms of both sides of the equation and differentiating implicitely.

Example: Let $y = x^x$ find $\frac{dy}{dx}$.

$$\ln y = x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} \Rightarrow \frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dx} = x^x (\ln x + 1)$$

Example: Find $\frac{dy}{dt}$ if $y = (\sin t)^{\ln t}$ where $0 < t < \pi$.

$\ln y = \ln t \ln(\sin t)$ if we differentiate implicitly w.r.t. "t",

$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{t}\ln(\sin t) + \ln t \, \frac{1}{\sin t} \cdot \frac{d}{dt}(\sin t)$$

$$\frac{dy}{dt} = y \cdot \left(\frac{\ln(\sin t)}{t} + \ln t \cdot \frac{\cos t}{\sin t} \right)$$

$$\frac{dy}{dt} = (\sin t)^{\ln t} \left(\frac{\ln(\sin t)}{t} + \ln t \cdot \cot(t) \right)$$

Example: Differentiate $y = \left[(x+1)(x+2)(x+3)\right] / (x+4)$.

$$\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$

$$\frac{dy}{dx} = \frac{(x+1)(x+2)(x+3)}{(x+4)} \cdot \left[\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} \right]$$

Example: Find $\left.\frac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)} = \left((x+1)(x^2+1)(x^3+1)\right)^{1/2}$

$$\ln u = \frac{1}{2}\left(\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\right)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{2}\left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1} \right)$$ At $x=1$

we have $u = \sqrt{8} = 2\sqrt{2}$ Hence; $\frac{du}{dx} = \sqrt{2}\left(\frac{1}{2} + 1 + \frac{3}{2}\right)$

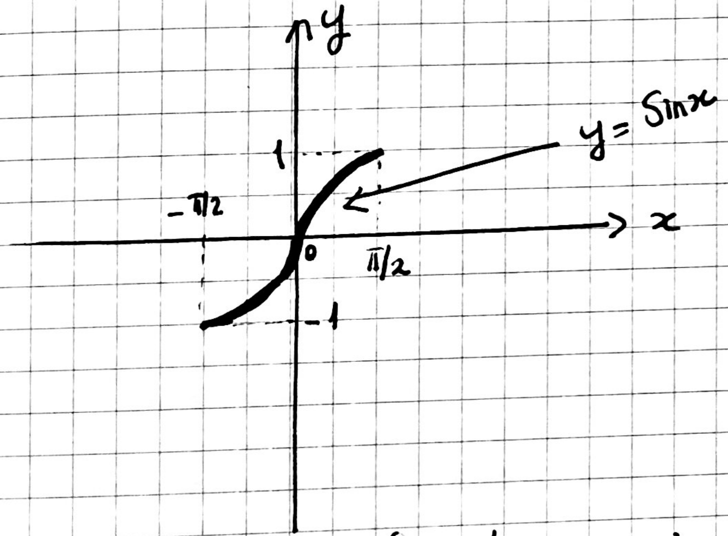
$$= 3\sqrt{2} \quad //$$

THE INVERSE TRIGONOMETRIC FUNCTIONS

The six trigonometric fncs. are periodic and, hence, not one-to-one. However as we did for the parabola x^2 we can restrict their domains in such a way that the restricted functions are one-to-one

Definition: (The restricted function Sinx)

$$\text{Sin}x = \sin x \quad \text{if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$



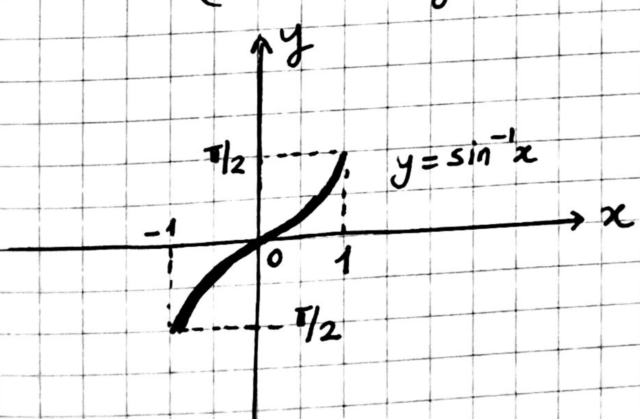
"Restricted sine fnc."

Being one-to-one Sin has an inverse function which is denoted $\sin^{-1}$ or arcsin. which is called inverse sine or arcsine function.

Definition: (The inverse sine function $\sin^{-1}x$ or $\arcsin x$)

$$y = \sin^{-1}x \iff x = \text{Sin}\,y$$

$$\iff x = \sin y \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$



"Arcsine function"

Cancellation Identities :

$\sin^{-1}(\sin x) = \arcsin(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

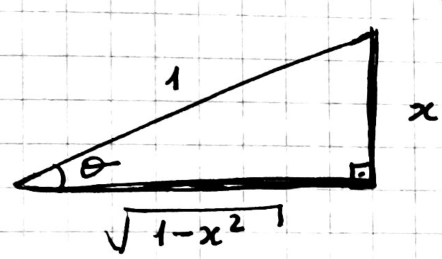
$\sin(\sin^{-1} x) = \sin(\arcsin x) = x$ for $-1 \le x \le 1$.

Example: Simplify the expression $\tan(\sin^{-1} x)$.

We want the tangent of an angle whose sine is x.

Suppose first that $0 \le x < 1$. Let $\sin^{-1} x = \theta \Rightarrow \sin\theta = \frac{x}{1}$

so we draw a right triangle



and we have $\tan(\sin^{-1} x) = \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$.

Because both sides of the above equation are odd fncs. of x, the same result holds for $-1 < x < 0$.

The derivative of arcsine function :

If $y = \sin^{-1} x$ then $x = \sin y$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Differentiating implicitly w.r.t. x ;

$$1 = \cos y \cdot \frac{dy}{dx}$$

(since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ we know $\cos y \ge 0$
therefore $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$)

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad //$$

Example: Find the derivative of $\sin^{-1}\left(\frac{x}{a}\right)$ and hence evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$ where $a>0$.

By the Chain Rule,

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

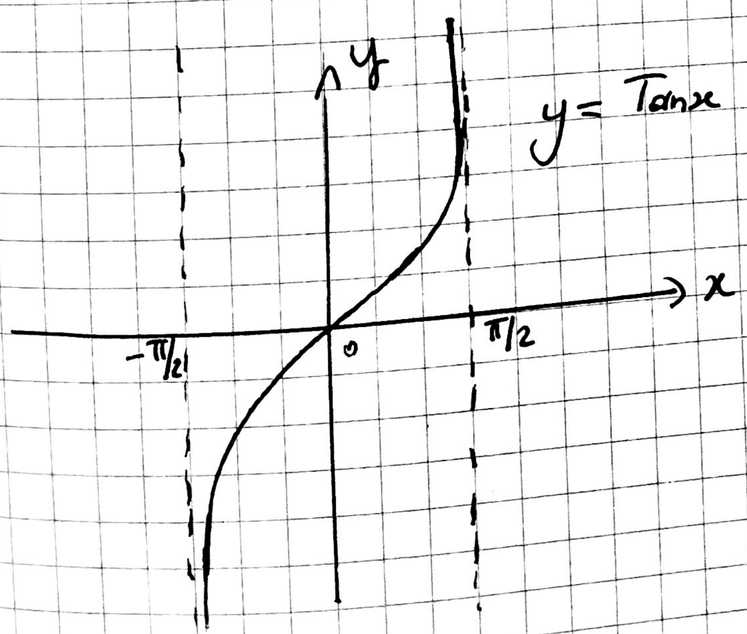
$$= \frac{1}{\sqrt{a^2-x^2}} \quad \text{if } a>0.$$

Hence

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (a>0).$$

Definition: (The restricted function $\operatorname{Tan} x$)

$$\operatorname{Tan} x = \tan x \quad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

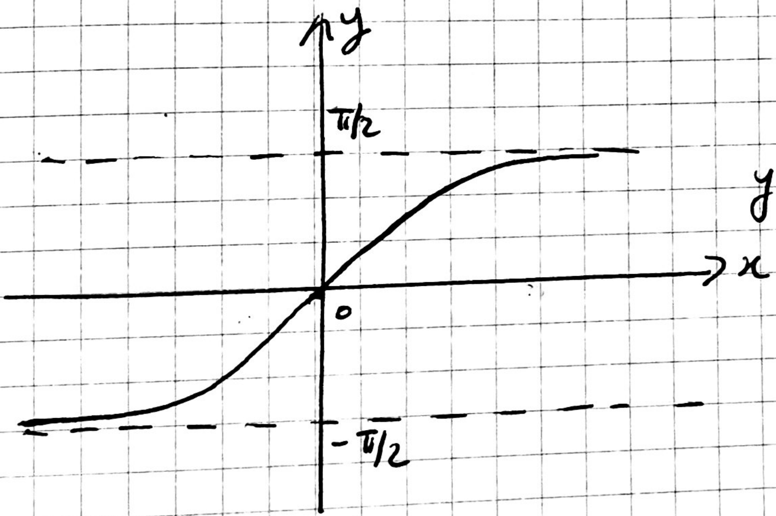


"Restricted tangent fnc."

The inverse of the function Tan is called the inverse tangent function and is denoted $\tan^{-1}$ or arctan.
The domain of $\tan^{-1}$ is the whole real line (the range of Tan). Its range is the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (the domain of Tan).

Definition: (The inverse tangent function $\tan^{-1}x$ or $\arctan x$)

$$y = \tan^{-1}x \iff x = \mathrm{Tan}\, y$$
$$\iff x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$



$y = \tan^{-1}x$

"Arctangent function"

Cancellation Identities:

$\tan^{-1}(\mathrm{Tan}\,x) = \arctan(\mathrm{Tan}\,x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$\mathrm{Tan}(\tan^{-1}x) = \mathrm{Tan}(\arctan x) = x$ for $-\infty < x < \infty$.

The derivative of arctangent function:

If $y = \tan^{-1}x$ then $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Differentiating implicitey w.r.t. x;

$$1 = \sec^2 y \frac{dy}{dx} = (1+\tan^2 y)\frac{dy}{dx} = (1+x^2)\frac{dy}{dx}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{dy}{dx} = \frac{1}{1+x^2} \quad //$$

Example: Find $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)$ and hence evaluate

$$\int \frac{1}{x^2+a^2}\,dx.$$

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{a}{a^2+x^2}.$$

Hence;

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example: Prove that $\tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1}x - \frac{\pi}{4}$ for $x>-1$.

Let $f(x) = \tan^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}x$. On the interval $(-1,\infty)$ we have,

$$f'(x) = \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2}\cdot\frac{d}{dx}\left(\frac{x-1}{x+1}\right) - \frac{1}{1+x^2}$$

$$= \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2}\cdot\frac{(x+1)-(x-1)}{(x+1)^2} - \frac{1}{1+x^2}$$

$$= \frac{(x+1)^2}{(x^2+2x+1)+(x^2-2x+1)}\cdot\frac{2}{(x+1)^2} - \frac{1}{1+x^2}$$

$$= \frac{2}{2x^2+2} - \frac{1}{1+x^2} = 0$$

$\therefore,\ f'(x) = 0$

Hence $f(x) = C$ (constant) on the interval. We can find C by finding $f(0)$:

$$C = f(0) = \tan^{-1}(-1) - \tan^{-1}(0) = -\frac{\pi}{4}.$$

Hence,

$$\tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1}x - \frac{\pi}{4} \quad \text{for } x > -1.$$

Definition: (The inverse casine function $\cos^{-1}x$ or $\arccos x$)

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x \quad \text{for } -1 \leq x \leq 1.$$

Cancellation Identities:

$\cos^{-1}(\cos x) = \arccos(\cos x) = x$ for $0 \leq x \leq \pi$.

$\cos(\cos^{-1}x) = \cos(\arccos x) = x$ for $-1 \leq x \leq 1$.

The derivative of $\cos^{-1}x$:

If $y = \cos^{-1}x$ then $x = \cos y$ and $0 \leq y \leq \pi$.

Differentiating implicitly w.r.t. x;

$$1 = -\sin y \frac{dy}{dx}$$ (Since $0 \leq y \leq \pi$ we know $\sin y \geq 0$ therefore $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$)

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \quad //$$

Definition: (The inverse secant function $\sec^{-1}x$ or arcsex)

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \quad \text{for} \quad |x| \geq 1.$$

We canc calculate the derivative of $\sec^{-1}$ from that of $\cos^{-1}$.

$$\frac{d}{dx}\sec^{-1}x = \frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\sqrt{\frac{x^2}{x^2-1}} = \frac{1}{x^2}\frac{|x|}{\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}.$$

The corresponding integration formula takes different forms on intervals where $x \geq 1$ or $x \leq 1$:

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \begin{cases} \sec^{-1}x + C & \text{on intervals where } x \geq 1. \\ -\sec^{-1}x + C & \text{on intervals where } x \leq -1. \end{cases}$$

Definition: (The inverse cosecant and inverse cotangent functions)

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) \quad (|x| \geq 1);$$

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right), \quad (x \neq 0).$$