§9.4 ABSOLUTE AND CONDITIONAL CONVERGENCE

In the previous section, we study the convergence of (ultimately) positive series. In this section, we investigate the convergence of the other series(ex: negative or alternating). For this case, we first get a positive series from a given series by replacing each term with its absolute value.

Definition. (**Absolute convergence**) A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the corresponding series of absolute values converges, that is; $\sum_{n=1}^{\infty} |a_n|$ converges.

Example 1. Consider $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2^n}$. The corresponding series of absolute values $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{2^n}| = \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with the common ratio $\frac{1}{2}$. Then we know that the geometric series converges since $r < 1$. By the definition, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2^n}$ converges absolutely.

Example 2. We investigate the series $\sum_{n=1}^{\infty} \frac{sinn}{n^3}$. Have a look at $\sum_{n=1}^{\infty} |\frac{sinn}{n^3}| = \sum_{n=1}^{\infty} \frac{|sinn|}{n^3}$. Then $\frac{|sinn|}{n^3} \leq \frac{1}{n^3}$ for each n. By comparison test, $\sum_{n=1}^{\infty} \frac{|sinn|}{n^3}$ converges. Thus, $\sum_{n=1}^{\infty} \frac{sinn}{n^3}$ converges absolutely.

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent, $s = \sum_{n=1}^{\infty} a_n$ and $S = \sum_{n=1}^{\infty} |a_n|$. Since for each n, $-|a_n| \leq a_n \leq |a_n|$, then we get $-\sum_{n=1}^{\infty} |a_n| \leq$

$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|$, that is; $-S \leq s \leq S$. By comparison test, we can say that $\sum_{n=1}^{\infty} a_n$ converges since $\sum_{n=1}^{\infty} |a_n|$ converges (as it is absolutely convergent). According to this explanation, we give the following theorem.

Theorem 1. If a series is absolutely convergent, then it is convergent.

The converse of the previous theorem may not be true, that is; although an absolutely convergent series is convergent, a convergent series may not be an absolutely convergent. For instance, consider $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. We will see in the next topic of alternating series test that it is a convergent series. But it does not converges absolutely since $\sum_{n=1}^{\infty} |(-1)^{n-1}\frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series and so diverges.

Definition. (**Conditional Convergence**) A series $\sum_{n=1}^{\infty} a_n$ is called conditional convergent (or converges conditionally) if it is convergent but is not absolutely convergent.

Remark. To show that a series converges absolutely can be applied convergence test (integral test, comparison test,...) for positive series. Let's apply ratio test to the series $\sum_{n=1}^{\infty} |a_n|$. Then we get that if $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, that is, it converges by the previous theorem. If $\rho > 1$, then $\lim_{n\to\infty}|a_n| = \infty$ and $\sum_{n=1}^{\infty}|a_n|$ diverges by ratio test. Also by n.th test for divergence, $\sum_{n=1}^{\infty} a_n$ diverges since $\lim_{n\to\infty}|a_n| = \infty$ and $\lim_{n\to\infty} a_n \neq 0$. If $\rho = 1$, then we don't have any information about the series $\sum_{n=1}^{\infty}|a_n|$. $\sum_{n=1}^{\infty}|a_n|$ may be divergent or convergent.

Example 3. Investigate the absolutely convergence of the following series.

i. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3n+1}$ ii. $\sum_{n=1}^{\infty}\frac{ncosn\pi}{3^n}$.

i. $\sum_{n=1}^{\infty}|(-1)^{n+1}\frac{1}{3n+1}| = \sum_{n=1}^{\infty}\frac{1}{3n+1}$ if we consider the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent, by limit comparison test $\lim_{n\to\infty}\frac{1}{3n+1}/\frac{1}{n} = \frac{1}{3} < \infty$ $\sum_{n=1}^{\infty}\frac{1}{3n+1}$ also diverges. Thus, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3n+1}$ does not converge absolutely.

ii. Since $\rho = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}|\frac{\frac{(n+1)cos(n+1)\pi}{3^{n+1}}}{\frac{ncosn\pi}{3^n}}| = \lim_{n\to\infty}\frac{n+1}{3n} = \frac{1}{3} < 1$, then the series converges absolutely.

Alternating Series Test

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alternating series. If the followings hold:

1. $a_n \geq 0$ for $n = 1, 2, 3, ...$
2. $a_{n+1} \leq a_n$ for $n = 1, 2, 3, ...$
3. $\lim_{n\to\infty} a_n = 0$,

then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges, otherwise it diverges.

Example 4. Consider $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. Since $a_n = \frac{1}{n}$ holds the conditions of alternating series test, then it is convergent.

Example 5. Note that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n+1}{3n}$. The series does not satisfy the alternating series test (3) since $\lim_{n\to\infty}\frac{3n+1}{3n} = 1$. This it is not convergent.

Example 6. Investigate the following series for absolute and conditional convergence.

i. $\sum_{n=2}^{\infty}\frac{cosn\pi}{lnn}$ ii. $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2}$.

i. The series $\sum_{n=2}^{\infty}\frac{cosn\pi}{lnn}$ holds the test of alternating series. So it converges. Now, we show that it is absolutely convergent. We get that the absolute series $\sum_{n=2}^{\infty}\frac{1}{lnn}$ diverges by comparison test when it compares to the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$. Thus $\sum_{n=2}^{\infty}\frac{cosn\pi}{lnn}$ does not converge absolutely and also it is conditionally convergent.

ii. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is absolutely convergent since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series, $p = 2 > 1$). Thus $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ converges. We could show its convergence using alternating series test.

Example 7. For what values of x does the series $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n^2 2^n}$ converges absolutely, converges conditionally or diverges.

Let's apply ratio test to the series.

$\rho = \lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{(n+1)^2 2^{n+1}}}{\frac{(x-1)^n}{n^2 2^n}}\right| = \lim_{n\to\infty}\frac{n^2}{2(n+1)^2}|x-1| = \frac{|x-1|}{2}$. Since $\rho =$

$\frac{|x-1|}{2} < 1$, then it converges absolutely and so converges on $-1 < x < 3$.

The series diverges for $\rho = \frac{|x-1|}{2} > 1 \Rightarrow x < -1 \ or \ x > 3$.

Let $x = -1$. The series $\sum_{n=1}^{\infty} \frac{(-1-1)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by Example 6 (ii).

Let $x = 3$. Then $\sum_{n=1}^{\infty} \frac{(3-1)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series.

Consequently, the convergence interval of its is $[-1, 3]$.

Example 8. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x)^n}{\sqrt{n}}$ converge absolutely , converge conditionally or diverge. Let's apply ratio test to the series.

$\rho = \lim_{n\to\infty} |\frac{\frac{(x)^{n+1}}{\sqrt{n+1}}}{\frac{(x)^n}{\sqrt{n}}}| = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}}|x| = |x|$. Since $\rho = |x| < 1$, then it converges absolutely and so converges on $-1 < x < 1$.

The series diverges for $\rho = |x| > 1 \Rightarrow x < -1 \ or \ x > 1$.

Let $x = -1$. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by alternating series test.

Let $x = 1$. Then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges by p-series $p = \frac{1}{2} < 1$.

Consequently, the convergence interval of its is $[-1, 1)$.