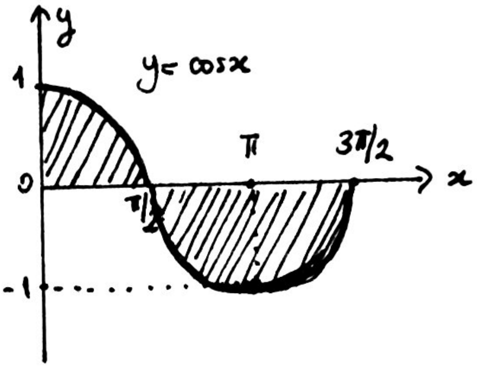
AREAS OF PLANE REGIONS

In this section, we review and extend the use of definite integral to represent plane areas.



The area bounded by $y=\cos x$, $x=0$ (y-axis), $y=0$ (x-axis) and $x=\frac{3\pi}{2}$ is;

$$A=\int_0^{3\pi/2}|\cos x|\,dx$$

$$=\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{3\pi/2}(-\cos x)\,dx$$

$$=\left(\sin x\right)_0^{\pi/2}-\left(\sin x\right)_{\pi/2}^{3\pi/2}$$

$$=(1-0)-(-1-1)$$

$$=3 \text{ square units} \quad //$$

Area Between Two Curves (Integrating w.r.t. "x")

If $f(x)$ and $g(x)$ are continuous with $f(x)\geq g(x)$ throughout $[a,b]$, then the area of the region between the curves $y=f(x)$ and $y=g(x)$ from a to b is the integral of $(f-g)$ from a to b;

$$A=\int_a^b\left[f(x)-g(x)\right]dx.$$

More generally, if the restriction $f(x)\geq g(x)$ is removed, then the vertical rectangle of width dx at position x extending between the graphs of g and f has height $|g(x)-f(x)|$ and hence area $dA=|g(x)-f(x)|\,dx$.

Hence the total area lying between the graphs $y=g(x)$ and $y=f(x)$ and between the lines $x=a$ and $x=b$ is given by;

$$A=\int_a^b |g(x)-f(x)|\,dx$$

In order to evaluate this integral, we have to determine the intervals on which $f(x)>g(x)$ or $g(x)>f(x)$, and break the integral into a sum of integrals over each of these intervals.

<u>Example</u>: Find the area of the bounded, plane region R lying between the curves $y=x^2-2x$ and $y=4-x^2$.

First, we must find the intersections of the curves, so we solve the equations simultaneously:

$$x^2-2x=y=4-x^2$$

$$2x^2-2x-4=0$$

$$2(x-2)(x+1)=0 \Rightarrow x=2 \text{ or } x=-1$$

Since $4-x^2 \geq x^2-2x$ for $-1\leq x\leq 2$, the area A of R is given by

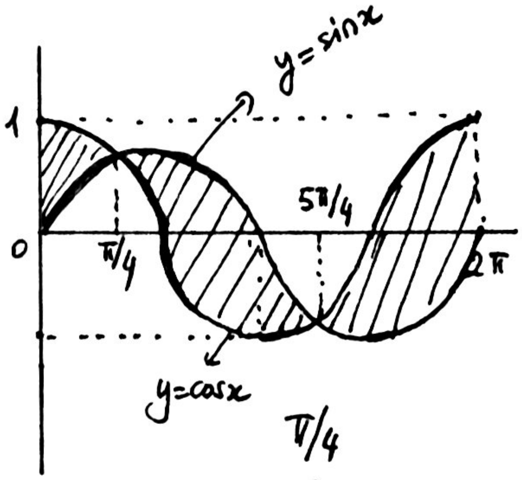
$$A=\int_{-1}^{2}\left[4-x^2-(x^2-2x)\right]dx$$

$$=\int_{-1}^{2}\left(4-2x^2+2x\right)dx$$

$$=\left(4x-\frac{2}{3}x^3+x^2\right)\Big|_{-1}^{2}$$

$$=4(2)-\frac{2}{3}.(8)+4-\left(-4+\frac{2}{3}+1\right)=9 \text{ square units}$$

Example: Find the total area A lying between the curves $y=\sin x$ and $y=\cos x$ from $x=0$ to $x=2\pi$.



The region is shaded in figure. Between 0 and 2π the graph of cosine and sine cross at $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$. The required area is,

$$A=\int_0^{\pi/4}(\cos x-\sin x)\,dx+\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)\,dx+\int_{5\pi/4}^{2\pi}(\cos x-\sin x)\,dx$$

$$=(\sin x+\cos x)\Big|_0^{\pi/4}-(\cos x+\sin x)\Big|_{\pi/4}^{5\pi/4}+(\sin x+\cos x)\Big|_{5\pi/4}^{2\pi}$$

$$=(\sqrt{2}-1)+(\sqrt{2}+\sqrt{2})+(1+\sqrt{2})=4\sqrt{2}\ \text{square units}$$

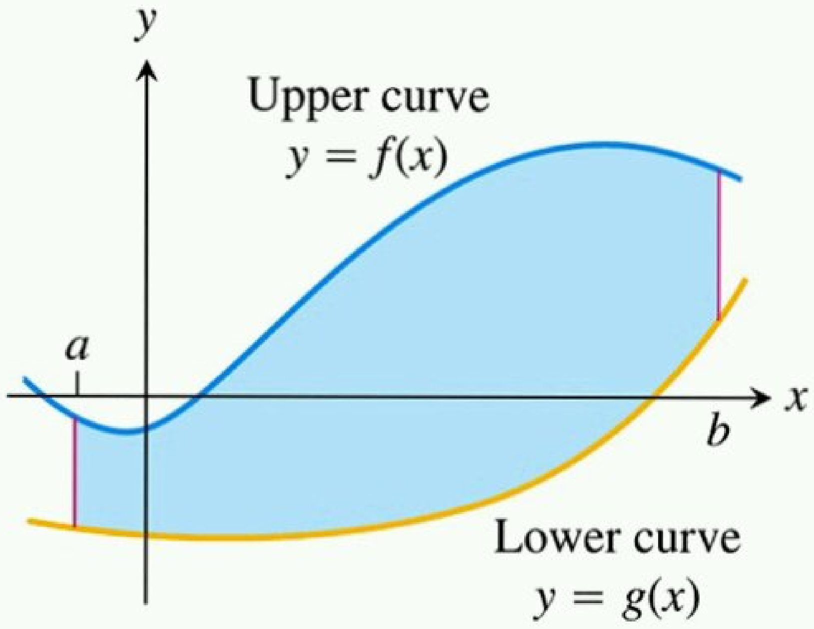


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

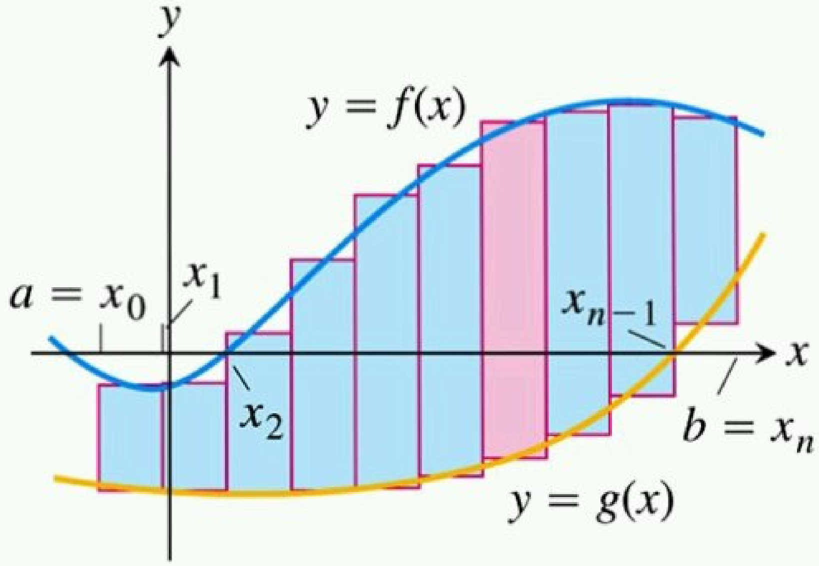


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x-axis.

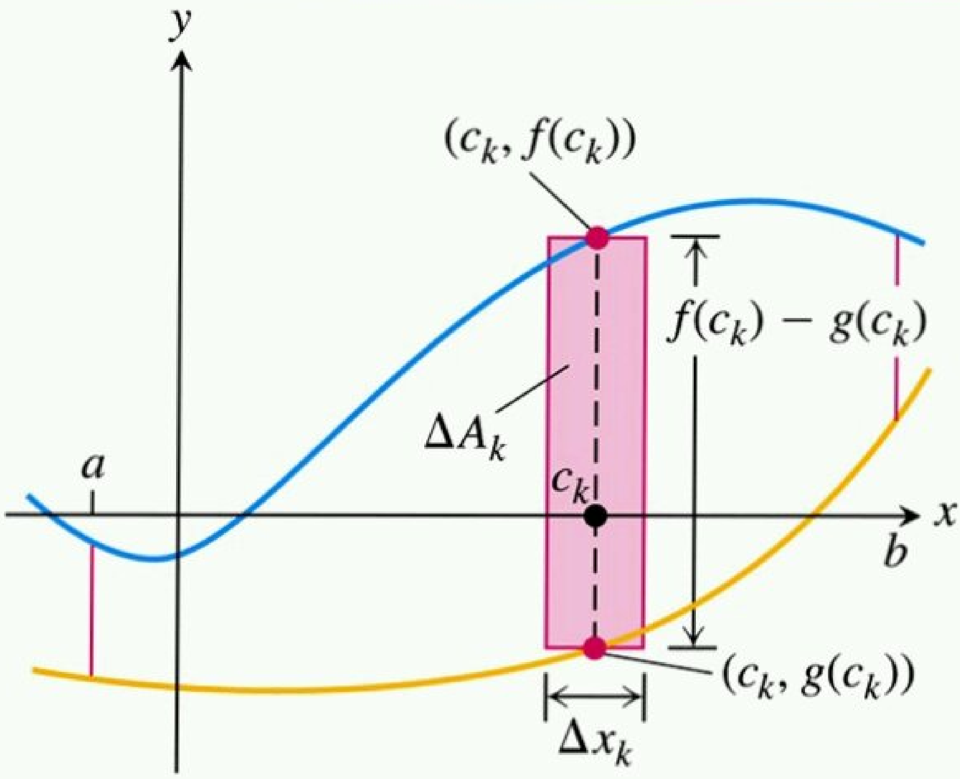
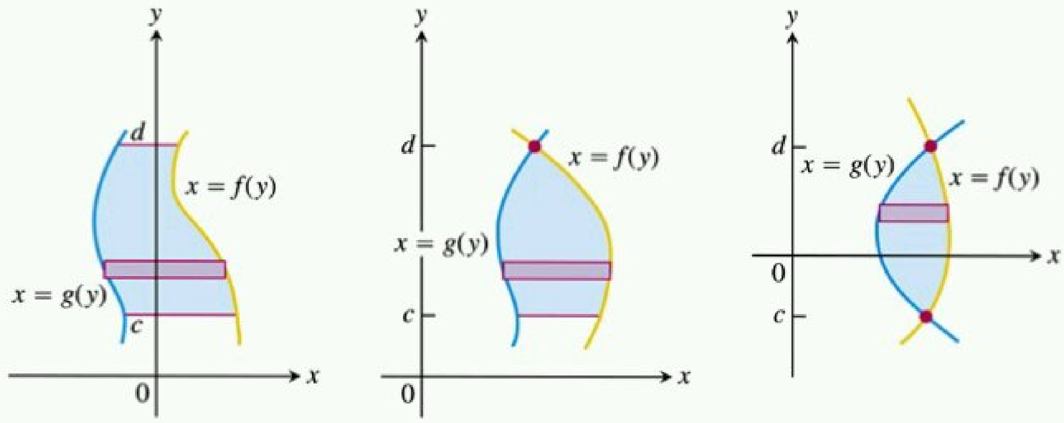


FIGURE 5.27 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these

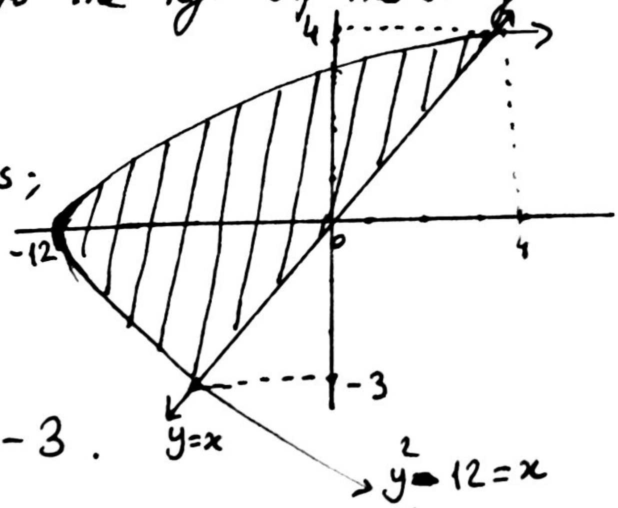


use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Example: Find the area of the plane region lying to the right of the parabola $x=y^2-12$ and to the left of the straight line $y=x$.



For the intersection of the curves;

$$y^2-12=x=y$$

$$y^2-y-12=0$$

$$(y-4)(y+3)=0 \Rightarrow y=4 \text{ or } y=-3.$$

Observe that $y^2-12 \leq y$ for $-3 \leq y \leq 4$. Thus, the area is;

$$A=\int_{-3}^{4}\left(y-(y^2-12)\right)dy=\int_{-3}^{4}(y-y^2+12)\,dy$$

$$=\left(\frac{y^2}{2}-\frac{y^3}{3}+12y\right)\Bigg|_{3}^{4}$$

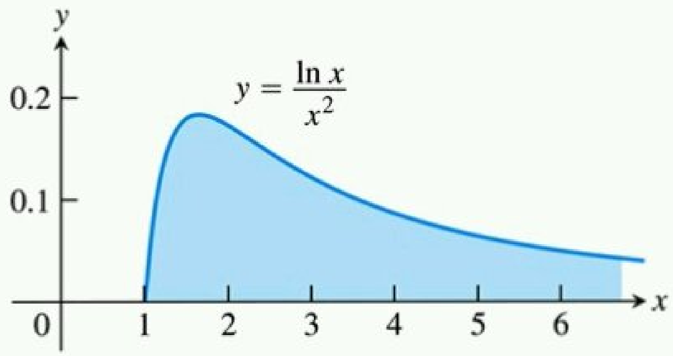
$$=\frac{343}{6} \text{ square units.}$$

Of course, the same result could have been obtained by integrating in the x direction, but the integral would have been more complicated:

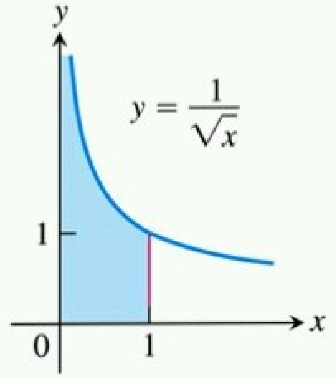
$$A=\int_{-12}^{-3}\left(\sqrt{12+x}-\left(-\sqrt{12+x}\right)\right)dx+\int_{-3}^{4}\left(\sqrt{12+x}-x\right)dx\,;$$

different integrals are required over the integrals where the region is bounded below by the parabola and by the straight line.

Improper Integrals



(a)



(b)

FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

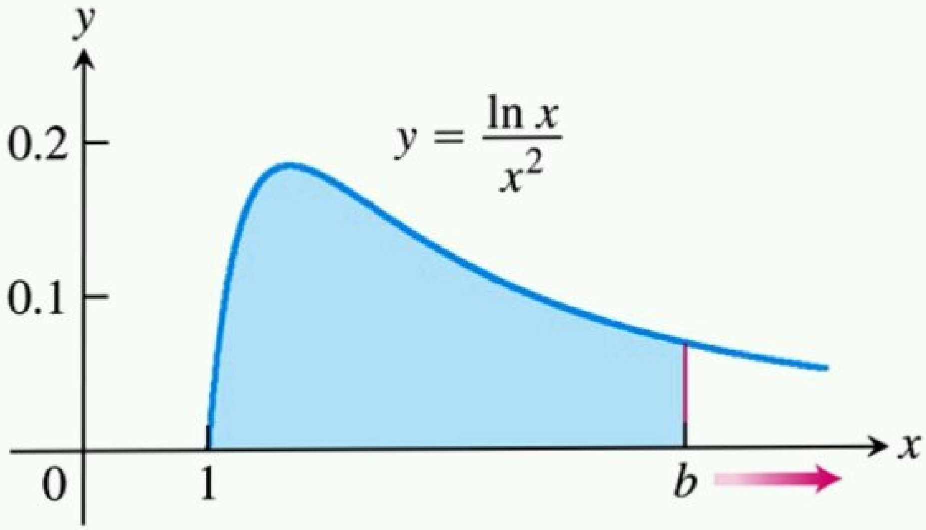


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges.**

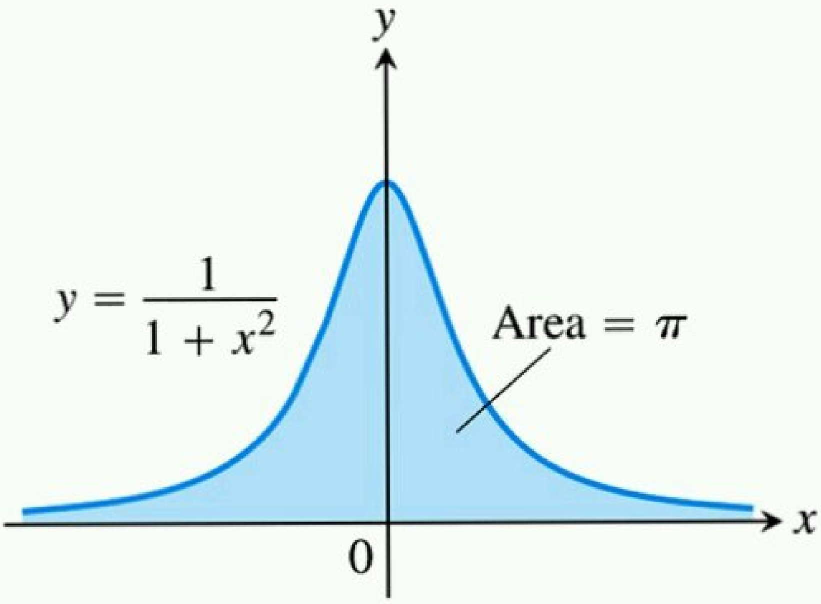


FIGURE 8.15 The area under this curve is finite (Example 2).

IMPROPER INTEGRALS

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$ (Type I)

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \underbrace{2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx}_{\text{Since the integrand is an even fnc.}}$$

$$= 2 \lim_{R\to\infty} \int_{0}^{R} \frac{1}{1+x^2}\,dx$$

$$= 2 \lim_{R\to\infty} \left(\arctan x\right)_{0}^{R}$$

$$= 2 \lim_{R\to\infty} \left(\arctan R - 0\right)$$

$$= 2\left(\frac{\pi}{2}\right) = \pi \text{ square units}$$

DEFINITION **Type II Improper Integrals**

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II.**

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

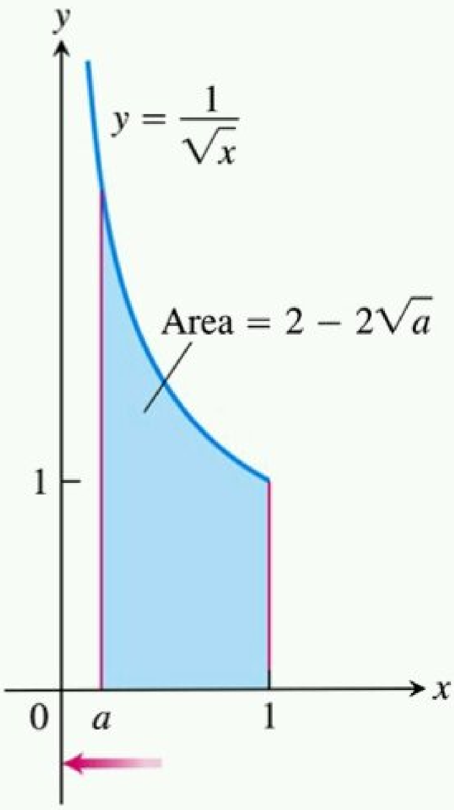


FIGURE 8.16 The area under this curve is an example of an improper integral of the second kind.

Find the area of the region S lying under $y = \frac{1}{\sqrt{x}}$, above the x-axis, between $x=0$ and $x=1$.

$$A = \int_0^1 \frac{1}{\sqrt{x}}\, dx \qquad (\text{Type II})$$

$$A = \lim_{c \to 0^+} \int_c^1 x^{-1/2}\, dx = \lim_{c \to 0^+} \left(2x^{1/2}\right)_c^1$$

$$= \lim_{c \to 0^+} (2 - 2\sqrt{c})$$

$= 2$ square units.

EXAMPLE 3 For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x}$$

$$= \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \ln x\bigg]_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

THEOREM 2—Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Example: Show that $\int_0^\infty e^{-x^2}\,dx$ converges, and find an upper bound for its value.

We can't integrate e^{-x^2} but we can integrate e^{-x}. We would like to use the inequality $e^{-x^2} \le e^{-x}$, but this is only valid for $x \geq 1$. Therefore, we break the integral into two parts.

On $[0,1]$ we have $0 < e^{-x^2} \le 1$, so

$$0 < \int_0^1 e^{-x^2}\,dx \le \int_0^1 dx = 1.$$

On $[1,\infty)$ we have $x^2 \geq x$ so $-x^2 \le -x$ and $0 < e^{-x^2} \le e^{-x}$.

Thus,

$$0 < \int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = \lim_{R\to\infty}\left(\frac{e^{-x}}{-1}\right)_{-1}^{R}$$

$$= \lim_{R\to\infty}\left(\frac{1}{e} - \frac{1}{e^R}\right) = \frac{1}{e} \;//$$

Hence, $\int_0^\infty e^{-x^2}\,dx$ converges and its value is not greater than $1 + \left(\frac{1}{e}\right)$ square units.

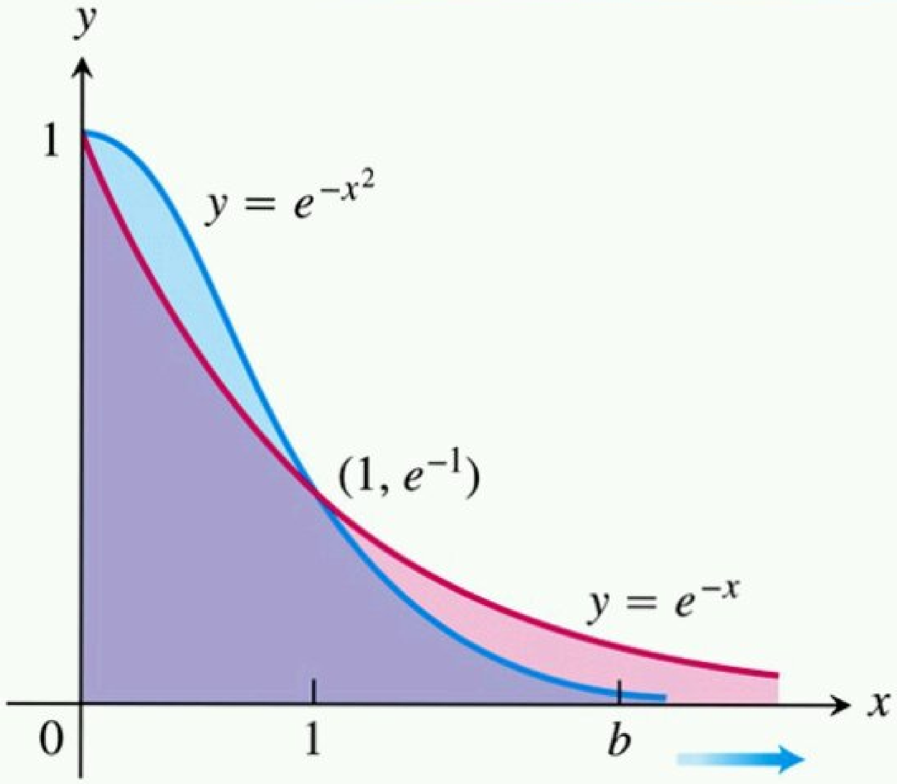


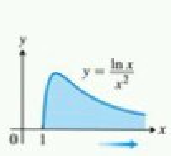
FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$

Types of Improper Integrals Discussed in This Section

INFINITE LIMITS OF INTEGRATION: TYPE I

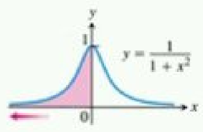
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$



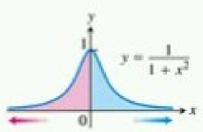
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}$$



3. Both limits

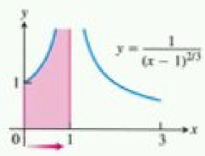
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^c \frac{dx}{1+x^2}$$



INTEGRAND BECOMES INFINITE: TYPE II

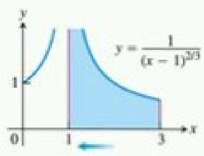
4. Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



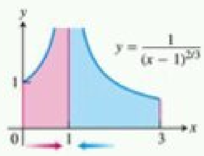
5. Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



EXAMPLE 6 Evaluate each of the following integrals or show that it diverges:

(a) $\displaystyle\int_0^1 \frac{1}{x}\,dx$, (b) $\displaystyle\int_0^2 \frac{1}{\sqrt{2x - x^2}}\,dx$, and (c) $\displaystyle\int_0^1 \ln x\,dx$.

Solution

(a) $$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0+} (\ln 1 - \ln c) = \infty.$$

This integral diverges to infinity.

(b) $$\int_0^2 \frac{1}{\sqrt{2x - x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx \qquad \text{Let } u = x - 1,\ du = dx$$

$$= \int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du$$

$$= 2\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du \qquad \text{(by symmetry)}$$

$$= 2 \lim_{c\to 1-} \int_0^c \frac{1}{\sqrt{1-u^2}}\,du$$

$$= 2 \lim_{c\to 1-} \sin^{-1} u \Big|_0^c = 2 \lim_{c\to 1-} \sin^{-1} c = \pi.$$

This integral converges to π. Observe how a change of variable can be made even before an improper integral is expressed as a limit of proper integrals.

(c) $\displaystyle\int_0^1 \ln x\,dx = \lim_{c\to 0+}\int_c^1 \ln x\,dx$

$$
\begin{aligned}
&= \lim_{c\to 0+} (x\ln x - x)\Big|_c^1 \\
&= \lim_{c\to 0+} (0 - 1 - c\ln c + c) \\
&= -1 + 0 - \lim_{c\to 0+} \frac{\ln c}{1/c} \qquad \left[\frac{-\infty}{\infty}\right] \\
&= -1 - \lim_{c\to 0+} \frac{1/c}{-(1/c^2)} \qquad \text{(by l'Hôpital's Rule)} \\
&= -1 - \lim_{c\to 0+} (-c) = -1 + 0 = -1.
\end{aligned}
$$

The integral converges to -1.

EXAMPLE 9 Determine whether $\int_0^\infty \frac{dx}{\sqrt{x+x^3}}$ converges.

Solution The integral is improper of both types, so we write

$$\int_0^\infty \frac{dx}{\sqrt{x+x^3}} = \int_0^1 \frac{dx}{\sqrt{x+x^3}} + \int_1^\infty \frac{dx}{\sqrt{x+x^3}} = I_1 + I_2.$$

On $(0, 1]$ we have $\sqrt{x+x^3} > \sqrt{x}$, so

$$I_1 < \int_0^1 \frac{dx}{\sqrt{x}} = 2 \qquad \text{(by Theorem 2)}.$$

On $[1, \infty)$ we have $\sqrt{x+x^3} > \sqrt{x^3}$, so

$$I_2 < \int_1^\infty x^{-3/2}\, dx = 2 \qquad \text{(by Theorem 2)}.$$

Hence, the given integral converges, and its value is less than 4.