MATH 101 PS-8

Q1) Sketch the graph of a function that has the given properties. Identify any critical points, singular points, local maxima and minima, and inflection points. Assume that f is continuous everywhere unless the contrary is implied or explicitly stated.

$\rightarrow$ $f(-1)=0$, $f(0)=2$, $f(1)=1$, $f(2)=0$, $f(3)=1$, $\lim_{x\to\mp\infty}(f(x)+1-x)=0$, $f'(x)>0$ on $(-\infty,-1)$, $(-1,0)$ and $(2,\infty)$. $f'(x)<0$ on $(0,2)$, $\lim_{x\to -1} f'(x)=\infty$, $f''(x)>0$ on $(-\infty,-1)$ and on $(1,3)$, and $f''(x)<0$ on $(-1,1)$ and on $(3,\infty)$.

Sol: According to given properties:

Oblique asymtote: $y=x+1$

Critical points: $x=0, 2$. Singular points: $x=-1$.

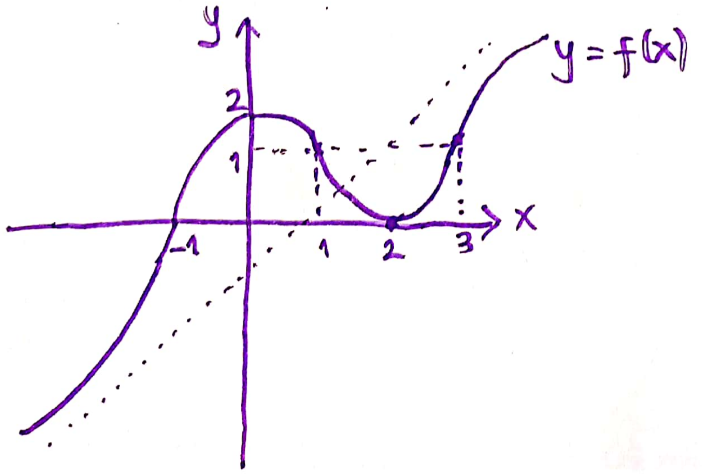
Local max 2 at $x=0$; local min 0 at $x=2$.

		-1		0		2	
f'	$+$		$+$		$-$		$+$
f	$\nearrow$		$\nearrow$	loc max	$\searrow$	loc min	$\nearrow$

Inflection points: $x=-1, 1, 3$.

		-1		1		3	
f''	$+$		$-$		$+$		$-$
f	$\cup$	infp	$\cap$	infp	$\cup$	infp	$\cap$

Since $\lim_{x\to\mp\infty}(f(x)+1-x)=0$, the line $y=x-1$ is an oblique asym.



Q2) Sketch the graphs of given functions, making use of any suitable information you can obtain from the function and its first and second derivatives.

a) $y = \dfrac{x^3}{x^2-1}$

Sol: $y = \dfrac{x^3}{x^2-1} \Rightarrow y' = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$ and $y'' = \dfrac{2x(x^2+3)}{(x^2-1)^3}$.

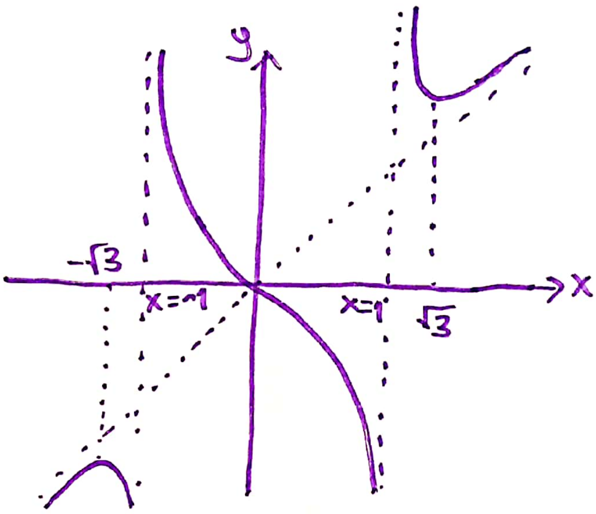
From y, intercept: $(0,0)$. Asymtotes: $x = \mp 1$ (vertical), $y = x$ (oblique). Symmetry: odd func. Other points, $\left(\mp\sqrt{3}, \mp\frac{3\sqrt{3}}{2}\right)$

From y': Critical point: $x=0$, $x=\mp\sqrt{3}$.

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
y'	+		−		−		+		−		+
y	↗	loc max	↘		↘		↘		↘	loc min	↗

From y'', $y''=0$ at $x=0$. $(x=\mp 1)$

		-1		0		1	
y''	−		+		−		+
y	∩		∪		∩		∪



b) $y = \dfrac{x^3-4x}{x^2-1}$

Sol: $y' = \dfrac{x^4+x^2+4}{(x^2-1)^2}$ $\quad y'' = -\dfrac{6x(x^2+3)}{(x^2-1)^3}$

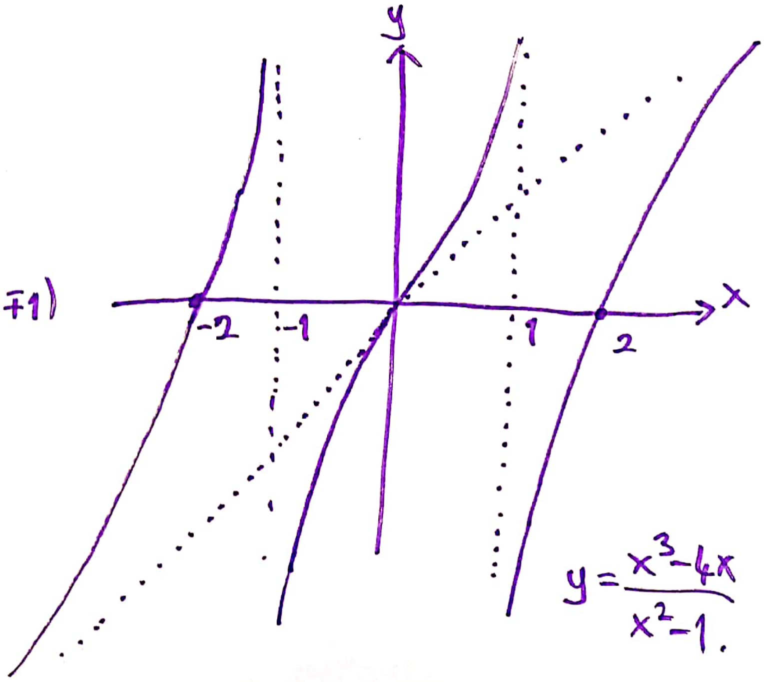
From y: Asymtotes $y=x$ (oblique), $x=\mp1$. Symmetry: odd.
Intercepts $(0,0)$, $(\mp 2, 0)$.

From y', CP: none $(x=\mp 1)$

		-1		1	
y'	+		+		+
y	↗		↗		↗

From y'': $y''=0$ at $x=0$. $(x=\mp1)$

		-1		0		1	
y''	+		−		+		−
y	∪		∩	infp	∪		∩



c) $y = \dfrac{x^3 - 3x^2 + 1}{x^3}$

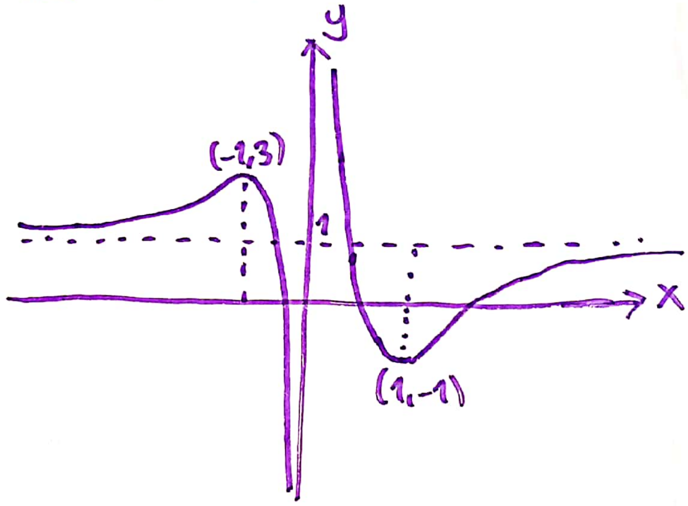
Sol: $y = 1 - \dfrac{3}{x} + \dfrac{1}{x^3}$, $y' = \dfrac{3(x^2-1)}{x^4}$, $y'' = 6\,\dfrac{(2-x^2)}{x^5}$.

From y: Asymtotes: $y=1, x=0$. Symmetry: none. Intercepts: since $\lim_{x\to 0^+} y = \infty$ and $\lim_{x \to 0^-} y = -\infty$, there are no intercepts between -1 and 0, between 0 and 1, and between 2 and 3.

Points: $(-1,3)$, $(1,-1)$, $(2,-3/8)$, $(3, 1/27)$.

From y': $x = \mp 1$ are critical points. $(x=0)$

		-1		0		1	
y'	$+$		$-$		$-$		$+$
y	$\nearrow$	loc max	$\searrow$		$\searrow$	loc min	$\nearrow$



From y'': $y''=0$ at $x = (\mp\sqrt{2})$, $(x=0)$

		$-\sqrt{2}$		0		$\sqrt{2}$	
y''	$+$		$-$		$+$		$-$
y	$\cup$		$\cap$		$\cup$		$\cap$

d) $y = e^{-x} \cdot \sin x$, $(x>0)$.

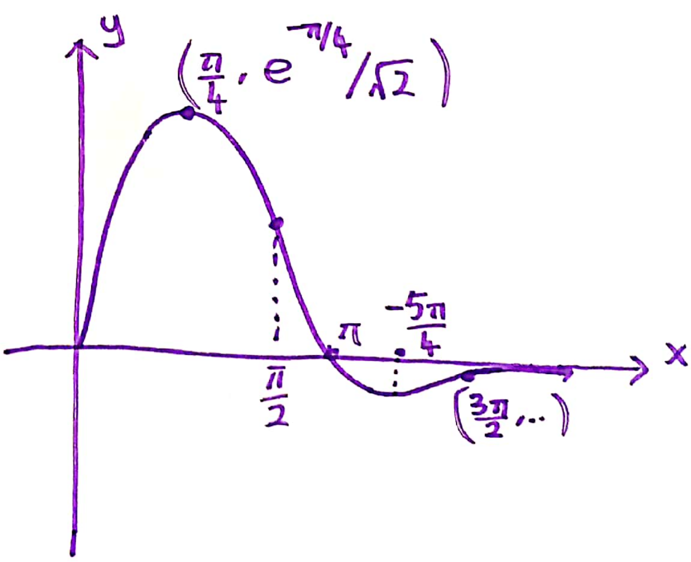
Sol: $y' = e^{-x}(\cos x - \sin x)$, $y'' = -2e^{-x} \cdot \cos x$.

From y: Intercept: $(k\pi, 0)$, where k is an integer.

Asymtotes: $y=0$ as $x \to \infty$.

From y': Critical points $x = \dfrac{\pi}{4} + k\pi$, where k is an integer.

	0		$\frac{\pi}{4}$		$\frac{5\pi}{4}$		$\frac{9\pi}{4}$	
y'		$+$		$-$		$+$		$-$
y		$\nearrow$	abs max	$\searrow$	abs min	$\nearrow$	loc max	$\searrow$



From y'', $y''=0$ at $x = \left(k + \dfrac{1}{2}\right)\pi$

	0		$\frac{\pi}{2}$		$\frac{3\pi}{2}$		$\frac{5\pi}{2}$	
y''		$-$		$+$		$-$		$+$
y		$\cap$	infp	$\cup$	infp	$\cap$	infp	$\cup$

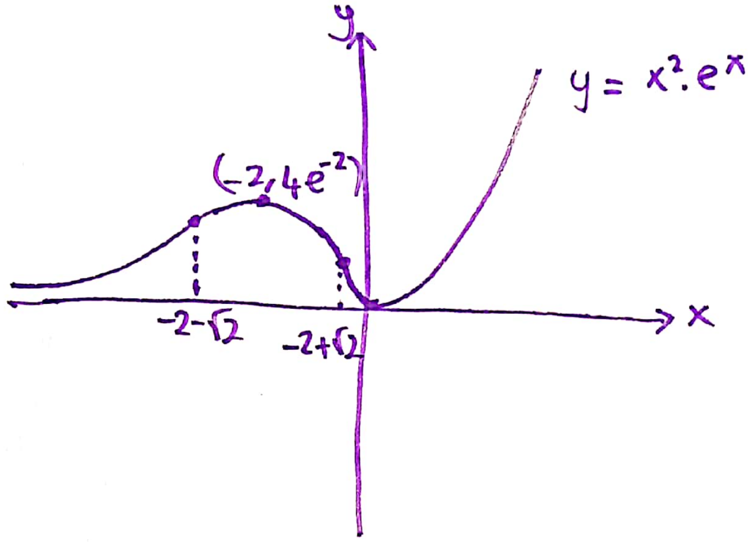
e) $y = x^2 . e^x$

Sol: $y' = (2x+x^2)e^x$, $y'' = (x^2+4x+2)e^x = (x+2-\sqrt{2})(x+2+\sqrt{2})e^x$

From y: Intercept: $(0,0)$. Asymtotes $y=0$ as $x \to -\infty$.

From y': Critical point: $x=0$, $x=-2$.

		-2		0	
y'	$+$		$-$		$+$
y	$\nearrow$	loc max	$\searrow$	loc min	$\nearrow$



From y'': $y''=0$ at $x = -2 \mp \sqrt{2}$.

		$-2-\sqrt{2}$		$-2+\sqrt{2}$	
y''	$+$		$-$		$+$
y	$\cup$	infp	$\cap$	infp	$\cup$

Q3) Two numbers have sum 16. What are the numbers if the product of the cube of one and the fifth power of the other is as large as possible?

Sol: The numbers are x and $16-x$. Let $P(x) = x^3 . (16-x)^5$. Since $P(x) \to -\infty$ as $x \to \mp\infty$, so the maximum must occur at a critical point.

$0 = P'(x) = 3x^2(16-x)^5 - 5x^3(16-x)^4 = x^2(16-x)^4(48-8x)$.

Critical points are $0, 6, 16$. $P(0) = P(16) = 0$ and $P(6) = 216 . 10^5$.

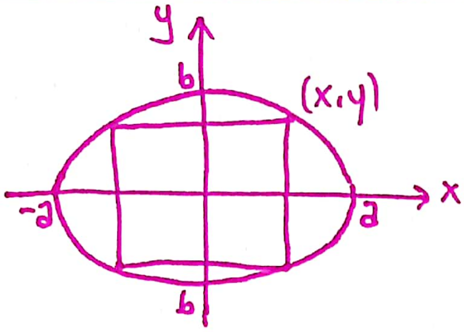
Thus, $P(x)$ is maximum if the numbers are 6 and 10.

Q4) A rectangle with sides parallel to the coordinate axes is inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the largest possible area for this rectangle.

Sol:



Let the upper right corner be (x,y) as shown. Then $x \geq 0$ and $y = b\sqrt{1-\frac{x^2}{a^2}}$ so $x \leq a$.

The area of the rectangle is

$$A(x) = 4xy = 4bx\sqrt{1-\frac{x^2}{a^2}}, \quad (0 \leq x \leq a).$$

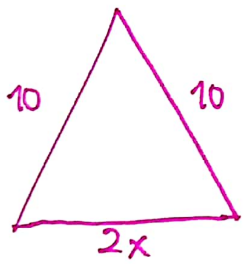
Clearly, $A=0$ if $x=0$ or $x=a$, so maximum A must occur at a critical point.

$$0 = \frac{dA}{dx} = 4b\left(\sqrt{1-\frac{x^2}{a^2}} - \frac{\frac{2x^2}{a^2}}{2\sqrt{1-\frac{x^2}{a^2}}}\right). \text{ Thus, } 1-\frac{x^2}{a^2}-\frac{x^2}{a^2}=0$$

and $x = \frac{a}{\sqrt{2}}$. Thus, $y = \frac{b}{\sqrt{2}}$. The largest area is $4xy = \underline{\underline{2ab}}$

Q5) Find the maximum area of an isosceles triangle whose equal sides are 10 cm in length. Use half the length of the third side of the triangle as the variable in terms of which to express the area of the triangle.

Sol:



If the sides of the triangle are 10 cm 10 cm and $2x$ cm, then the area of the triangle is $A(x) = x\sqrt{100-x^2}$ cm². where $0 \leq x \leq 10$. Also, $A(0) = A(10) = 0$ and $A(x) > 0$ for $0 < x < 10$.

Thus, A will be maximum at a critical point.

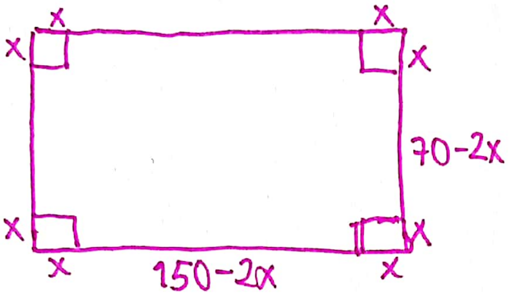
For critical point,

$$0 = A'(x) = \sqrt{100-x^2} - x\left(\frac{-2x}{2\sqrt{100-x^2}}\right) = \frac{100-2x^2}{\sqrt{100-x^2}}$$

Thus, the critical point is given by $2x^2 = 100$, so $x = \sqrt{50}$. The maximum area of the triangle is $A(\sqrt{50}) = 50$ cm².

Q6) A box is to be made from a rectangular sheet of cardboard 70 cm by 150 cm by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. What is the largest possible volume of the box?

Sol:



Let x be the side of the cut-out squares. Then volume of the box is

$V(x) = x(70-2x)(150-2x)$ $(0 \leq x \leq 35)$.

Since $V(0) = V(35) = 0$, the maximum ~~point~~ value will occur at a critical point:

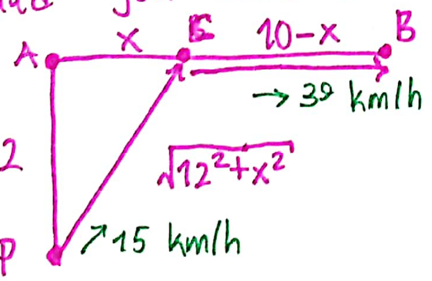
$$0 = V'(x) = 4(2625 - 220x + 3x^2) = 4(3x-175)(x-15)$$

Thus, $x = 15$ or $x = \frac{175}{3}$.

The only critical point in $[0,35]$ is $x=15$. Thus, the largest value for box is $V(15) = 15(70-30)(150-30) = 72000$.

Q7) You are in a dune buggy in the desert 12 km due south of the nearest point A on a straight east-west road. You wish to get to point B on the road 10 km east of A. If your dune buggy can average 15 km/h travelling over the desert and 39 km/h travelling on the road, toward what point on the road should you head in order to minimize your travel time to B?

Sol:



→ Head for C on road x km east of A.

Travel time is $T = \frac{\sqrt{12^2+x^2}}{15} + \frac{10-x}{39}$.

We have $T(0) = \frac{12}{15} + \frac{10}{39} = 1.0564$ hrs.

$T(10) = \frac{\sqrt{244}}{15} = 1.0414$ hrs.

$$0 = T'(x) = \frac{1}{15}\frac{x}{\sqrt{12^2+x^2}} - \frac{1}{39} \Rightarrow x = 5.$$

$$T(5) = \frac{13}{15} + \frac{5}{39} = 0.9949 < \begin{cases} T(0) \\ T(10) \end{cases} \Rightarrow (5, T(5)) \text{ is local min.}$$

To minimaze travel time, head for point 5 km east of A.

Q8) At the altitude of airliners, winds can typicall blow at a speed of about 100 knots (nautical miles per hour) from the west toward the east. A westward flying passenger jet from London, England, on its way to Toronto, flies directly against this wind for 3000 nautical miles. The energy per unit time expended by the airliner is proportional to v^3, where v is the speed of the airliner relative to the air. This reflects the power required to push aside the air exerting ram presure proportional to v^2. What speed uses the least energy on this trip? Estimate the time it would take to fly this route at the resulting optimal speed. Is this a typical speed at which airliners travel? Explain.

Sol: The time for the trip is $\frac{3000}{v-100}$, so the total energy needed for the trip is $E = 3000k\frac{v^3}{v-100}$,

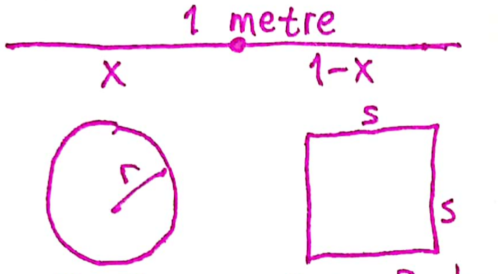
where k is a constant. Evidently, we must have $v > 100$ or the trip would be impossible. The minimum value of E will occur at a critical point where

$$0 = \frac{dE}{dv} = \frac{2v^3 - 300v^2}{(v-100)^2}.$$

The minimum ~~th~~ thus occurs at $v = 150$ knots, and the time for the fight at this speed would be $\frac{3000}{50} = 60$ hours, or 2.5 days. This is much slower than commercial airliners can travel; the time for the flight at that speed is not much shorter than a fast ship could cross the Atlantic ocean.

Q9) A one-metre length of stiff wire is cut into two pieces. One piece is bent into a circle, the other piece into a square. Find the length of the part used for the square if the sum of the areas of the circle and the square is a) maxmum and b) minimum.

Sol:



$x = C = 2\pi r$ $\quad 1-x = P = 4s$

Use x m for the circle and $1-x$ m for the square.

Then, sum of the areas.

$$A = \pi r^2 + s^2 = \frac{\pi x^2}{4\pi^2} + \frac{(1-x)^2}{4} \Rightarrow A(x) = \frac{x^2}{4\pi} + \frac{(1-x)^2}{4^2} \quad (0 \leq x \leq 1).$$

Now, $A(0) = \frac{1}{16}$, $A(1) = \frac{1}{4\pi} > A(0)$. For critical points,

$$0 = \frac{dA}{dx} = \frac{x}{2\pi} - \frac{1-x}{8} \Rightarrow x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = \frac{1}{8} \Rightarrow x = \frac{\pi}{4+\pi}.$$

Since, $\frac{d^2A}{dx^2} = \frac{1}{2\pi} + \frac{1}{8} > 0$, the CP gives local min for A.

a) For max total area use none of wire for the square, i.e., $x = 1$.

b) For min total area use $1 - \frac{\pi}{4+\pi} = \frac{4}{4+\pi}$ m for square.

Q10) Find the shortest distance from the origin to the curve $x^2y^4 = 1$.

Sol: Let (x,y) be a point on $x^2y^4 = 1$. Then $x^2y^4 = 1$ and the square of distance from (x,y) to $(0,0)$ is

$S = x^2 + y^2 = \frac{1}{y^4} + y^2$ $(y \neq 0)$. Clearly, $S \to \infty$ as $y \to 0$

or $y \to \mp\infty$, so minimum S must occur at a CP. For CP:

$0 = \frac{dS}{dy} = \frac{-4}{y^5} + 2y \Rightarrow y^6 = 2 \Rightarrow y = \mp 2^{1/6} \Rightarrow x = \mp \frac{1}{2^{1/3}}$. Thus, the shortest distance from origin to curve is $S = \sqrt{\frac{1}{2^{2/3}} + 2^{1/3}} = \frac{3^{1/2}}{2^{1/3}}$.