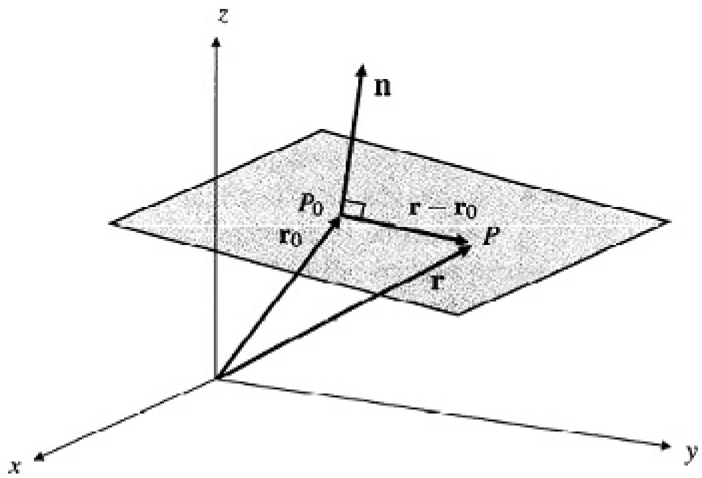
§10.4. Planes and Lines

In this section we will investigate that graphs of **linear equations in three variables**.



Planes in 3-Space

Let $P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$ with position vector,
$\overrightarrow{r}_0 = x_0 \overrightarrow{i} + y_0 \overrightarrow{j} + z_0 \overrightarrow{k}$.
If $\overrightarrow{n} = A\overrightarrow{i} + B\overrightarrow{j} + C\overrightarrow{k}$ is any given nonzero vector, then there exists exactly one **plane** (flat surface) passing through P_0 and perpendicular to $\overrightarrow{n}$. We say that $\overrightarrow{n}$ is a **normal vector** to the plane. The plane is the set of all points P for which $\overrightarrow{P_0P}$ is perpendicular to $\overrightarrow{n}$.
If $P = (x, y, z)$ has position vector $\overrightarrow{r}$, then $\overrightarrow{P_0P} = \overrightarrow{r} - \overrightarrow{r}_0 = \langle x - x_0, y - y_0, z - z_0\rangle$. This vector is perpendicular to $\overrightarrow{n}$ if and only

if $\overrightarrow{n} \bullet (\overrightarrow{r} - \overrightarrow{r}_0) = 0$. This is the equation of the plane in vector form. We can rewrite it in terms of coordinates to obtain the corresponding scalar equation.

The point-normal equation of a plane

The plane having nonzero normal vector $\overrightarrow{n} = A\overrightarrow{i} + B\overrightarrow{j} + C\overrightarrow{k}$, and passing through the point $P_0 = (x_0, y_0, z_0)$ with position vector $\overrightarrow{r}_0$, has equation

$n \bullet (\overrightarrow{r} - \overrightarrow{r}_0) = 0$

in **vector form**, or, equivalently,

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

in **scalar form**.

The scalar form can be written more simply in the **standard form** $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$.

If at least one of the constants A, B, and C is not zero, then the linear equation $Ax + By + Cz = D$ always represents a plane in $\mathbb{R}^3$. For example, if $A \neq 0$, it represents the plane through $(D/A, 0, 0)$ with normal vector $\overrightarrow{n} = A\overrightarrow{i} + B\overrightarrow{j} + C\overrightarrow{k}$.

A vector normal to a plane can always be determined from the coefficients of x, y, and z. If the constant term $D = 0$, then the plane must pass through the origin.

Example 1. (Recognizing and writing the equations of planes)

(a) The equation $x - 2y - 3z = 0$ represents a plane that passes through the origin and is normal (perpendicular) to the vector $\overrightarrow{n} = \overrightarrow{i} - 2\overrightarrow{j} - 3\overrightarrow{k}$.

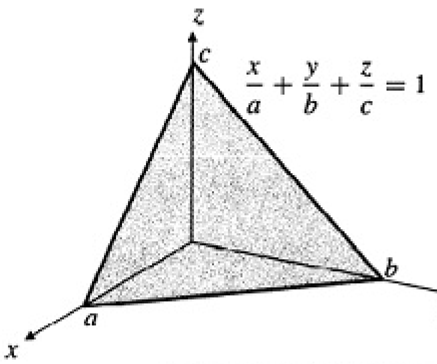
(b) The plane passes through the point $(2, 0, 1)$ and is perpendicular to the straight line passing through the points $(1, 1, 0)$ and $(4, -1, -2)$ has a normal vector $\overrightarrow{n} = (4-1)\overrightarrow{i} + (-1-1)\overrightarrow{j} + (-2-0)\overrightarrow{k} = 3\overrightarrow{i} - 2\overrightarrow{j} - 2\overrightarrow{k}$. Therefore, its equation is $3(x-2) - 2(y-0) - 2(z-1) = 0$, or, more simply, $3x - 2y - 2z = 4$.

(c) The plane with equation $2x - y = 1$ has a normal $2i - j$ that is perpendicular to the z-axis. Thus the plane is parallel to the z-axis. In the xy-plane, the equation $2x - y = 1$ represents a straight line; in 3-space it represents a plane containing that line and parallel to z-axis.

(d) If a, b and c are all nonzero, the plane with intercept a, b and c on the coordinate axes has equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

is called the intercept form on the equation of the plane.



The plane with intercepts a, b, and c on the coordinate axes

Example 2. Find an equation of the plane that passes through the three points $P = (1,1,0)$, $Q = (0,2,1)$ and $R = (3,2,-1)$.

First we will find the normal vector of the plane. Such a vector will be perpendicular to the vector $\overrightarrow{PQ} = -\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$ and $\overrightarrow{PR} = 2\overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$. Therefore, we can use

$$\overrightarrow{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\overrightarrow{i} + \overrightarrow{j} - 3\overrightarrow{k}.$$

Using point P leads to the equation $-2(x-1)+1(y-1)-3(z-0) = 0$, or $2x - y + 3z = 1$.

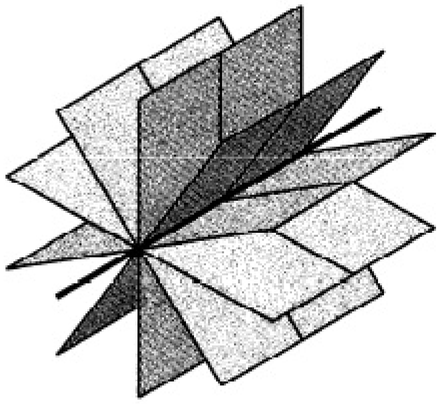
Example 3. Show that the two planes $x - y = 3$ and $x + y + z = 0$

intersect, and find a vector, $\vec{v}$, parallel to their line of intersection.

The two planes have normal vectors

$\vec{n}_1 = \vec{i} - \vec{j}$ and $\vec{n}_2 = \vec{i} + \vec{j} + \vec{k}$, respectively. Since these vectors are not parallel, and they intersect in a straight line perpendicular to both n_1 and n_2. This line must therefore be parallel to $\vec{v} =$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} - \vec{j} + 2\vec{k}.$$



A pencil of planes

A family of planes intersecting in a straight line is called a **pencil of planes**. Such a pencil of planes is determined by any two nonparallel planes in it, since these have a unique line of intersection. If two nonparallel planes have equations

$A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$,

then, for any value of the real number λ, the equation

$A_1x + B_1y + C_1z - D_1 + \lambda(A_2x + B_2y + C_2z - D_2) = 0$

represents a plane in the pencil.

Example 4. Find an equation of the plane passing through the line of intersection of two planes

$x + y - 2z = 6$ and $2x - y + z = 2$

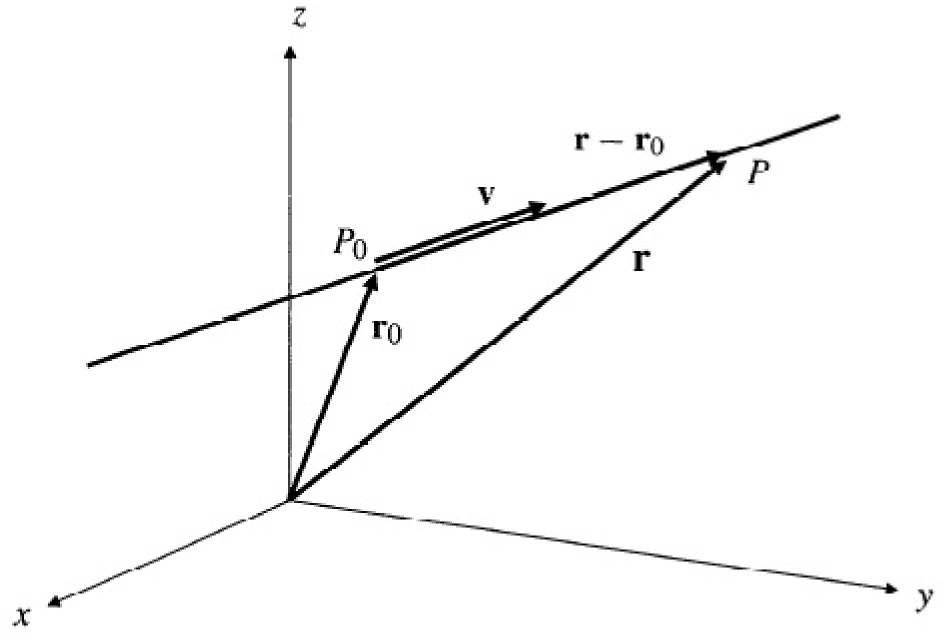
and also passing through the point $(-2, 0, 1)$.

For any constant λ, the equation

$x + y - 2z - 6 + \lambda(2x - y + z - 2) = 0$

represents a plane and is satisfied by the coordinates of all points on the line of intersection of the given planes. This plane passes through the point $(-2, 0, 1)$ if $-2 - 2 - 6 + \lambda(-4 + 1 - 2) = 0$, that is, if $\lambda = -2$. The equation of the required plane therefore simplifies to $3x - 3y + 4z + 2 = 0$.

Lines in 3-Space



As we observed above, any two nonparallel planes in $\mathbb{R}^3$ determine a unique line of intersection, and a vector parallel to this line can be obtained by taking the cross product of normal vectors to the two planes.

Suppose that $\overrightarrow{r}_0 = x_0 \overrightarrow{i} + y_0 \overrightarrow{j} + z_0 \overrightarrow{k}$ is the position vector of the point P_0 and $\overrightarrow{v} = a \overrightarrow{i} + b \overrightarrow{j} + c \overrightarrow{k}$ is a nonzero vector. There is a unique line passing through P_0 parallel to $\overrightarrow{v}$. If $\overrightarrow{r} = x \overrightarrow{i} + y \overrightarrow{j} + z \overrightarrow{k}$ is the position vector of any other point P on the line, then $\overrightarrow{r} - \overrightarrow{r}_0$ lies along the line and so is parallel to $\overrightarrow{v}$. Thus, $\overrightarrow{r} - \overrightarrow{r}_0 = t \overrightarrow{v}$ for some real number t. This equation, usually rewritten in the form

$$\overrightarrow{r} = \overrightarrow{r}_0 + t\overrightarrow{v}$$

is called the **vector parametric equation of the straight line**. All points on the line can be obtained as the parameter t ranges from $-\infty$ to ∞. The vector $\overrightarrow{v}$ is called a **direction vector of the line**. Breaking the vector parametric equation down into its components yields the **scalar parametric equations of the line**:

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

where $t \in \mathbb{R}$.

These appear to be **three** linear equations, but the parameter t can be eliminated to give two linear equations in x, y and z. If $a \neq 0$, $b \neq 0$, and $c \neq 0$, then we can solve each of the scalar equations for t and so obtain,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which is called **the standard form for the equations of the straight line** through (x_0, y_0, z_0) parallel to $\overrightarrow{v}$.

Example 5. (Equations of straight lines)

(a) The equations

$$x = 2 + t$$

$$y = 3$$

$$z = -4t$$

represents the straight line through $(2, 3, 0)$ parallel to the vector $\overrightarrow{i} - 4\overrightarrow{k}$.

(b) The straight line through $(1, -2, 3)$ perpendicular to the plane $x - 2y + 4z = 5$ is parallel to the normal vector $\overrightarrow{i} - 2\overrightarrow{j} + 4\overrightarrow{k}$ of the plane. Therefore, the line has vector parametric equation

$$\overrightarrow{r} = \overrightarrow{i} - 2\overrightarrow{j} + 3\overrightarrow{k} + t(\overrightarrow{i} - 2\overrightarrow{j} + 4\overrightarrow{k}),$$

or scalar parametric equations

$$x = 1 + t$$

$$y = -2 - 2t$$

$$z = 3 + 4t.$$

Its standard form equations are

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{4}.$$

Example 6. Find the direction vector of the line of intersection of the two planes $x + y - z = 0$ and $y + 2z = 6$, and find the set of equations for the line in standard form.

The two planes have respective normals $\overrightarrow{n}_1 = \overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$ and $\overrightarrow{n}_2 = \overrightarrow{j} + 2\overrightarrow{k}$. Thus, a direction vector of their line of intersection is

$$\overrightarrow{v} = \overrightarrow{n}_1 \times \overrightarrow{n}_2 = 3\overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}.$$

We need to know one point on the line in order to write equations in standard form. Taking $z = 0$ in the two equations we are led to $y = 6$ and $x = -6$ so $(-6, 6, 0)$ is one point on the line. Thus, the line has standard form equations,

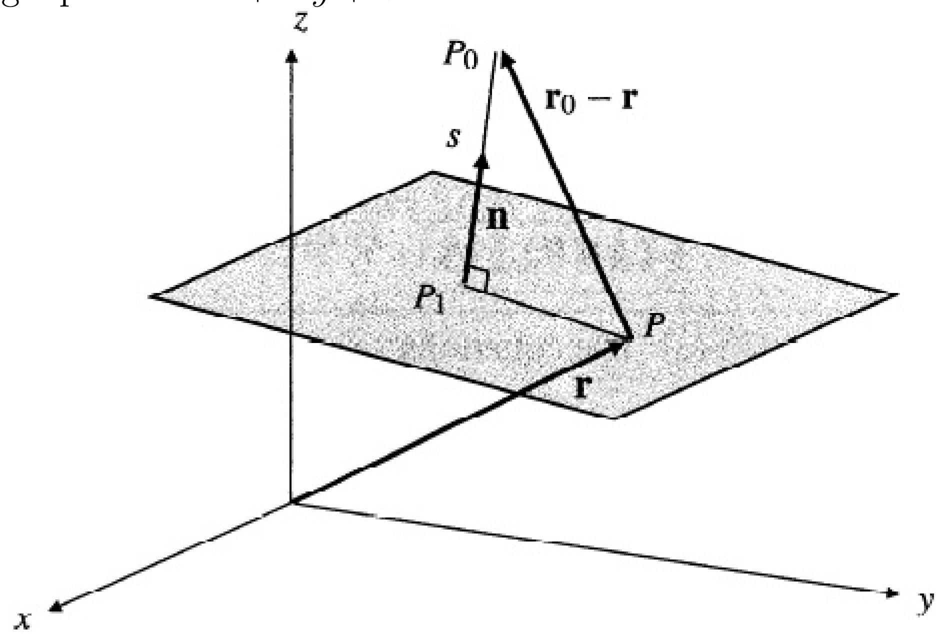
$$\frac{x+6}{3} = \frac{y-6}{-2} = \frac{z-0}{1}.$$

This answer is not unique; the coordinate of any other point on the line could be used in place of $(-6, 6, 0)$.

Distances

Distance from a point to a plane:

Find the distance from the point $P_0 = (x_0, y_0, z_0)$ to the plane $\mathcal{P}$ having equation $Ax + By + Cz = D$.



By given figure,

$s = |\overrightarrow{P_1P_0}|$. If $P = (x, y, z)$, having position vector $\overrightarrow{r}$, is any point on $\mathcal{P}$, then s is the length of the projection of $\overrightarrow{PP_0} = \overrightarrow{r}_0 - \overrightarrow{r}$ in the direction of $\overrightarrow{n}$. Thus,

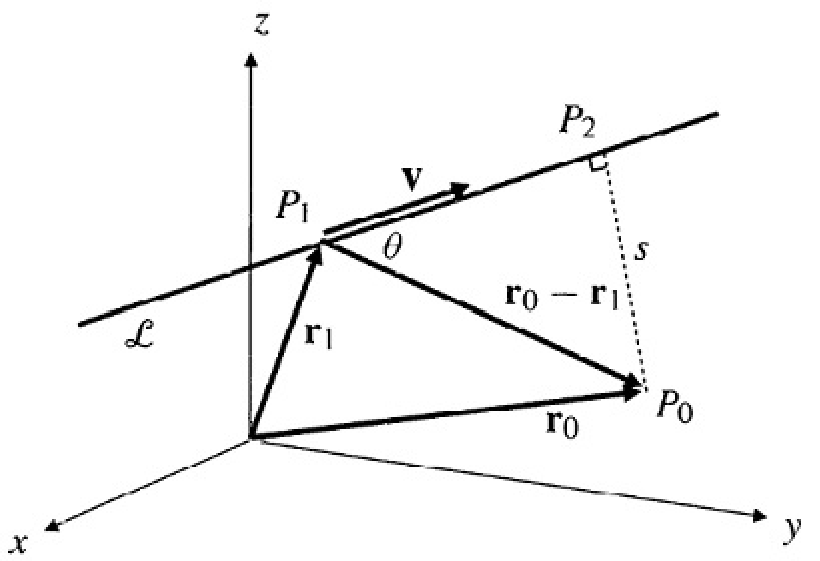
$$s = \left|\frac{\overrightarrow{PP_0} \bullet \overrightarrow{n}}{|\overrightarrow{n}|}\right| = \frac{|(\overrightarrow{r}_0 - \overrightarrow{r}) \bullet \overrightarrow{n}|}{|\overrightarrow{n}|} = \frac{|\overrightarrow{r}_0 \bullet \overrightarrow{n} - \overrightarrow{r} \bullet \overrightarrow{n}|}{|\overrightarrow{n}|}.$$

Since $P(x, y, z)$ lies on $\mathcal{P}$, we have $\overrightarrow{r} \bullet \overrightarrow{n} = Ax + By + Cz = D$. In terms of the coordinates of P_0, we can therefore represent the distance s as;

$$s = \frac{|Ax_0 + BY_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Distance from a point to a line:

Find the distance from the point P_0 to the straight line $\mathcal{L}$ through P_1 parallel to the nonzero vector $\overrightarrow{v}$.



Let $\overrightarrow{r}_0$ and $\overrightarrow{r}_1$ be the position vectors of P_0 and P_1, respectively. The point P_2 on $\mathcal{L}$ that is closest to P_0 is such that P_2P_0 is perpendicular to $\mathcal{L}$. The distance from P_0 to $\mathcal{L}$ is,

$$s = |P_2P_0| = |P_1P_0| \sin\theta = |\overrightarrow{r}_0 - \overrightarrow{r}_1| \sin\theta,$$

where θ is the angle between $\overrightarrow{r}_0 - \overrightarrow{r}_1$ and $\overrightarrow{v}$. Since

$$|(\overrightarrow{r}_0 - \overrightarrow{r}_1) \times \overrightarrow{v}| = |\overrightarrow{r}_0 - \overrightarrow{r}_1||\overrightarrow{v}| \sin\theta,$$

we have

$$s = \frac{|(\overrightarrow{r}_0 - \overrightarrow{r}_1) \times \overrightarrow{v}|}{|\overrightarrow{v}|}.$$