§14.6 Change of Variables in Triple Integrals: Cylindrical and Spherical Coordinates

Recall that the change of variables $x = x(u,v)$ and $y = y(u,v)$ allow us to evaluate doble integrals over uv-plane;

$$\iint_D g(x,y)\,dx\,dy = \iint_S g(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of the transformation $x = x(u,v)$, $y = y(u,v)$ (Also recall that this transformation maps the region S on the uv-plane to D on the xy-plane).

The change of variables in triple integals can be obtanied in a straightforward way. Consider the change of variables $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$. Suppose that this change of variables transform the domain S in uvw-space to the domain R in the

xyz-space. Then the triple integral in cartesian coordinates equals

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ is the Jacobian (determinant) of the transformation $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ defined as

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

(!) Do not forget to take the absolute value of the Jacobian in the integral.

Example 1: Let $a, b, c > 0$. Find the volume of the ellipsoid $E : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$.

Solution: The volume equals to the triple integral

$$\iiint_E 1\, dV.$$

The transformation $x = au$, $y = bv$ and $z = cw$ transforms the unit ball $S : u^2 + v^2 + w^2 \leq 1$ to E. The Jacobian of this transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

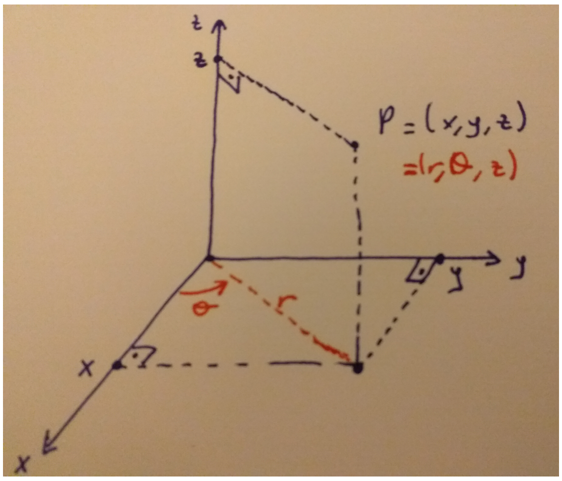
So the volume of E is

$$\iiint_E 1\, dx\, dy\, dz = \iiint_S (abc)\, du\, dv\, dw = abc.(\text{Volume of } S) = \frac{4}{3}\pi abc$$

Cylindrical Coordinates

A common and useful system of coordinates are *cylindrical* coordinates.

A point P in 3-space can be described as $P = (r, \theta, z)$ as shown in the figure. Indeed (r, θ) are the polar coordinates of the projection of P on the xy-plane, and so $r \geq 0$, $\theta \in [0, 2\pi]$. The corresponding transformation is



$$x = r\cos\theta, \;\; y = r\sin\theta, \;\; z = z$$

so in particular z remains the same. The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \geq 0$$

so we have that

$$\iiint\limits_R f(x, y, z)\, dx\, dy\, dz = \iiint\limits_S f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

where the order of integration $dr\,d\theta\,dz$ may be modified when needed.

Example 2: Sketch the region of the iterated integral

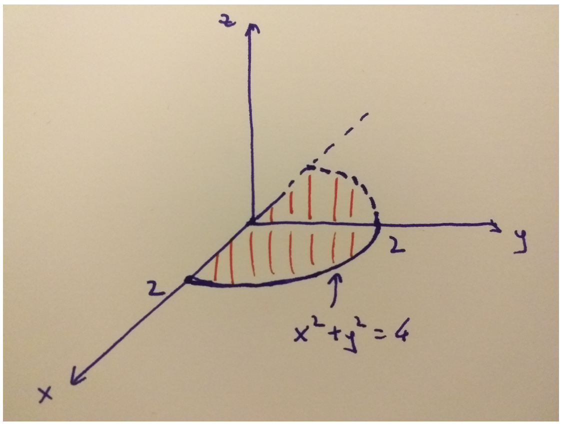
$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} f(x,y,z)\,dz\,dx\,dy$$

and express the integral in cylindrical coordinates

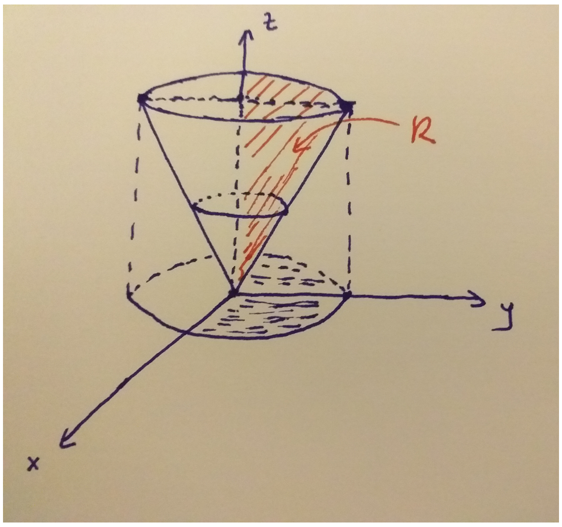
Solution: The iterated integral is the triple integral of f over the region

$$R: 0 \le y \le 2,\ \ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2},\ \ \sqrt{x^2+y^2} \le z \le 2.$$

The limits of the outer and the middle integral (the integrals with respect to $dx\,dy$) determine the projection of R on the xy-plane. So the projection on the xy-plane is given by $0 \le y \le 2$, $-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$ which is a half of the disc with radius 2.



By $\sqrt{x^2+y^2} \leq z \leq 2$ we see that R is bounded by the plane $z = 2$ (from above) and the cone $z = \sqrt{x^2+y^2}$ (from below). Hence R is the right half of the cone $z = \sqrt{x^2+y^2}$ below the plane $z = 2$.



In order to write the given integral in cylindrical coordinates, we need to express R in cylindrical coordinates. We see that

$$R : 0 \leq \theta \leq \pi, \;\; 0 \leq r \leq 2, \;\; r \leq z \leq 2$$

and so

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} f(x,y,z)\, dz\, dx\, dy = \int_0^{\pi} \int_0^2 \int_r^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

(As an exercise evaluate the above integral for $f(x,y,z) = x^2 + y^2$).

Spherical Coordinates

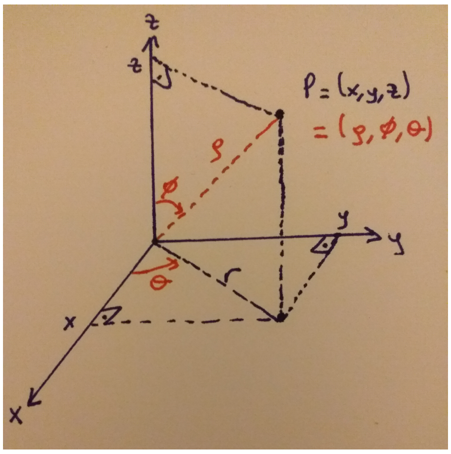
Now we will see another system of coordinates called *spherical* coordinates. In this case we express a point $P = (x, y, z)$ by $P = (\rho, \phi, \theta)$ where ρ is the distance of P to the origin, ϕ is the angle made with the positive z-axis, and θ is the same angle as in cylidrical coordinates. We call (ρ, ϕ, θ) as spherical coordinates.

We relate to cartesian coordinates by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta,$$

$$z = \rho \cos\phi,$$

$$\theta \in [0, 2\pi], \phi \in [0, \pi], \rho \geq 0$$



In particular we have $\rho^2 = x^2 + y^2 + z^2$. The Jacobian of this transformation can be computed as

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2 \sin\phi$$

and so $dV = dx\, dy\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Example 3: Let R be the upper half of the unit ball, i.e. $R : x^2 + y^2 + z^2 \leq 1,\ z \geq 0$. Evaluate

$$\iiint\limits_R e^{(x^2+y^2+z^2)^{3/2}}\, dV.$$

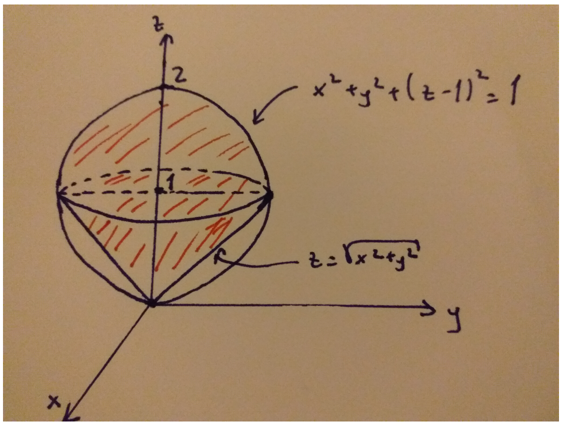
Solution: We can express R in spherical coordinates as $R : 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/2,\ 0 \leq \rho \leq 1$. So we have

$$\iiint\limits_R e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 e^{\rho^3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} e^{\rho^3} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho \quad \text{(Evaluate the trivial integrals first)}$$

$$= 2\pi \int_0^1 e^{\rho^3} \rho^2 \, d\rho = \frac{2\pi}{3}(e-1) \quad \text{(Use the substitution } u = \rho^3\text{)}$$

Example 4 (A challenging example): Let R be the region lying above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+(z-1)^2 = 1$. Express the triple integral of $f(x,y,z) = z$ over R as an iterated integral in **a)** cartesian coordinates, **b)** cylindrical coordinates, **c)** spherical coordinates



Solution: (!) The center of the sphere is $(0,0,1)$. The sphere and the cone intersect at the circle $x^2+y^2=1$, $z=1$ (To see this, plug $x^2+y^2=z^2$ in the equation of the sphere to obtain $z^2+(z-1)^2=1$ and find z).

a) Express R in cartesian coordinates;

$$R : -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},$$
$$\sqrt{x^2+y^2} \le z \le 1+\sqrt{1-(x^2+y^2)}$$

$$\Longrightarrow \iiint_R z dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-(x^2+y^2)}} z\, dz\, dy\, dx$$

b) Express R in cylindrical coordinates;

$$R : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 1 + \sqrt{1 - r^2}$$

$$\implies \iiint\limits_R z dV = \int_0^{2\pi} \int_0^1 \int_r^{1+\sqrt{1-r^2}} zr\, dz\, dr\, d\theta$$

c) To express R in spherical coordinates, first observe that $0 \leq \theta \leq 2\pi$. Also the lower limit of ϕ is 0. The upper limit of ϕ is determined by the cone. Since the cone makes a constant angle of $\pi/4$ with the positive z-axis we have that $0 \leq \phi \leq \pi/4$ (In other words the cone $z = \sqrt{x^2 + y^2}$ is described by $\phi = \pi/4$ in spherical coordinates).

The smallest value of ρ on R is clearly 0. The upper limit of ρ is determined by the sphere. By the change of variables

$$x = \rho \sin\phi \cos\theta,\ \ y = \rho \sin\phi \sin\theta,\ \ z = \rho\cos\phi,\ \ \rho^2 = x^2 + y^2 + z^2$$

we obtain that

$$x^2 + y^2 + (z-1)^2 = 1 \implies x^2 + y^2 + z^2 = 2z$$
$$\implies \rho^2 = 2\rho\cos\phi \implies \rho = 2\cos\phi.$$

(!) The sphere is given by $\rho = 2\cos\phi$ in spherical coordinates, and so $0 \leq \rho \leq 2\cos\phi$.

$$\iiint\limits_R z dV = \int\limits_0^{2\pi} \int\limits_0^{\pi/4} \int\limits_0^{2\cos\phi} \rho^3 \cos\phi \, \sin\phi \, d\rho \, d\phi \, d\theta$$

Note: It is not difficult to compute the integral in **c)** opposed to the integrals in **a)** and **b)**. As an exercise compute it (The answer is $7\pi/6$)