UYGULAMA HAFTA 13

Section 14.4-Kutupsal Koordinatlarda İki Katlı İntegraller

HATIRLATMALAR

- **Teorem:**

$$T : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

dönüşümü ile xy- dikey kartezyen koordinat sisteminden (r, θ) kutupsal koordinat sistemine geçiş yapılır.

$$r^2 = x^2 + y^2, x = r\cos\theta, y = r\sin\theta$$

ve

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

çıkar. Öyleyse,

$D = \{(r, \theta) : a \le r \le b, \alpha \le \theta \le \beta\}$ bölgesinde f fonksiyonu sürekli ise

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta$$

eşitliği sağlanır.

- **Teorem:** 1. $f(x, y)$ fonksiyonu xy-düzlemindeki bir D bölgesinde sürekli olsun.

2. Düzlemde tanımlı bir T dönüşümü

$$T : \begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

parametrik biçiminde verilsin.

3. g ve h fonksiyonları D^* üzerinde sürekli olsunlar.

4. T dönüşümü bire-bir örten olacak biçimde D bölgesini D^* bölgesine dönüştürsün.

5.

$$|J(u,v)| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

olsun. Bu durumda;

$$\iint_D f(x,y)dA = \iint_{D^*} f(g(u,v), h(u,v))|J(u,v)|dudv$$

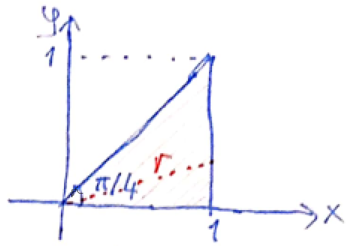
dir.

13) T, (0,0), (1,0) ve (1,1) tepeli üçgen olmak üzere

$\iint_T (x^2+y^2)\,dA$ -yı hesaplayınız.

Sol.

Kartezyen koordinatlardan $((x,y))$, kutupsal koordinatlara $((r,\theta))$ geçiş yapalım.



$$0 \le \theta \le \frac{\pi}{4}, \quad 0 \le r \le \sec\theta$$

$$\iint_T (x^2+y^2)\,dA = \int_0^{\pi/4}\int_0^{\sec\theta} r^2\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \left(\frac{r^4}{4}\right)\Big|_0^{\sec\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} \sec^4\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} (1+\tan^2\theta)\sec^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{1} (1+u^2)\,du$$

$$= \frac{1}{4}\left(u+\frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{3}.$$

D.D.

$\tan\theta = u \Rightarrow \sec^2\theta\,d\theta = du$

$\theta = 0 \rightarrow u = 0$

$\theta = \frac{\pi}{4} \rightarrow u = 1$

14) $\iint_{x^2+y^2\le 1} \ln(x^2+y^2)\,dA$ -yı hesaplayınız.

Sol.

$x^2+y^2 \le 1 \longrightarrow$ Merkezi orijin olan birim disk.

$\Rightarrow 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$\iint_{x^2+y^2\le 1} \ln(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^{1} \ln(r^2)\, r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^{1} r\ln r\,dr\,d\theta$$

K.İ.Y

$u = \ln r \Rightarrow du = \frac{dr}{r}$

$r\,dr = dv$

$v = \frac{r^2}{2}$

$$= 2\int_0^{2\pi} \left(\frac{r^2}{2}\ln r \Big|_0^1 - \int_0^1 \frac{r}{2}\, dr \right) d\theta$$

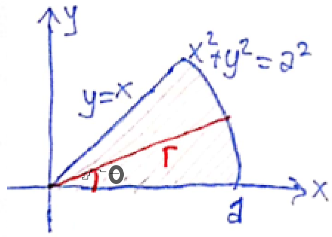
$$= 2\int_0^{2\pi} \left(-\lim_{r\to 0^+} \left(\frac{r^2}{2}\ln r\right) - \frac{r^2}{4}\Big|_0^1 \right) d\theta$$

$$= 2\int_0^{2\pi} -\frac{1}{4}\, d\theta = 2\left(-\frac{1}{4}\right)2\pi = -\pi.$$

19) D, $x \geq 0$, $0 \leq y \leq x$ ve $x^2+y^2 \leq a^2$ -yi sağlayan düzlem olmak üzere $\iint_D xy\, dA$ -yı hesaplayınız.

Sol.

$0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq r \leq a$. $x = r\cos\theta$, $y = r\sin\theta$



$$\iint_D xy\, dA = \int_0^{\pi/4}\int_0^{a} r^2\cos\theta\sin\theta\, r\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\int_0^{a} r^3 \sin 2\theta\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sin 2\theta \left(\frac{r^4}{4}\Big|_0^a\right) d\theta$$

$$= \frac{a^4}{8}\int_0^{\pi/4} \sin 2\theta\, d\theta = \frac{a^4}{8}\left(-\frac{\cos 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{a^4}{16}.$$

21) $z = x^2+y^2$ ve $3z = 4-x^2-y^2$ paraboloidleri arasında kalan hacmi bulunuz.

Sol.

$3(x^2+y^2) = 4-x^2-y^2 \Rightarrow x^2+y^2 = 1$
Kesişim

$\Rightarrow 0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$

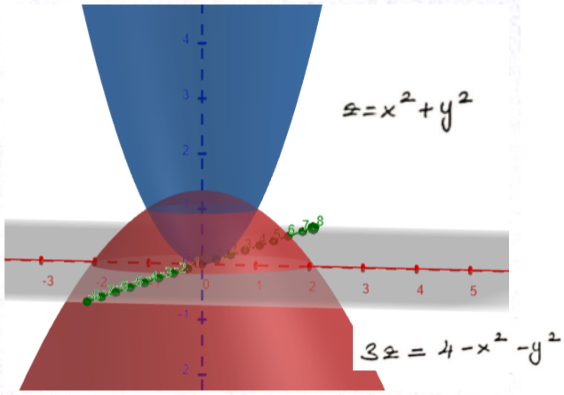
$$V=\iint\limits_{x^2+y^2\le 1}\left[\frac{4-x^2-y^2}{3}-(x^2+y^2)\right]dA$$

$$=\int_0^{2\pi}\int_0^1\left(\frac{4-r^2}{3}-r^2\right)r\,dr\,d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^1(4-r^2-3r^2)r\,dr\,d\theta$$

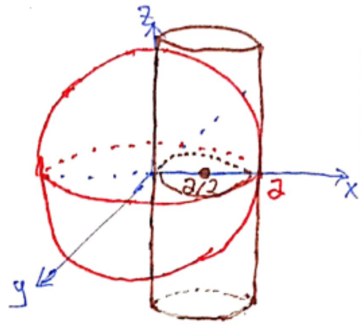
$$=\frac{4}{3}\int_0^{2\pi}\int_0^1(r-r^3)\,dr\,d\theta=\frac{4}{3}\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_0^1 d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}d\theta=\frac{2\pi}{3}$$



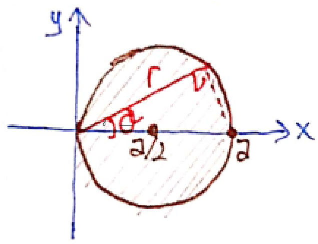
22) Her $x^2+y^2+z^2=a^2$ küresi hem $x^2+y^2=ax$ silindiri içinde kalan hacmi bulunuz.

Sol.



$$x^2+y^2=ax \Rightarrow x^2-ax+\frac{a^2}{4}+y^2=\frac{a^2}{4}$$

$$\Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$



$$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\ ,\quad 0\le r\le a\cos\theta$$

Ayrıca, $z=\sqrt{a^2-x^2-y^2}$, $x^2+y^2=r^2$.

O halde;

$$V=2\cdot\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$$=-\int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\theta}\sqrt{u}\,du\,d\theta$$

$$=-\frac{2}{3}\int_{-\pi/2}^{\pi/2}\left(u^{3/2}\Big|_{a^2}^{a^2\sin^2\theta}\right)d\theta$$

$$=\frac{2}{3}a^3\int_{-\pi/2}^{\pi/2}(1-\sin^3\theta)\,d\theta=\frac{4}{3}a^3\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta$$

D.D.

$a^2-r^2=u$

$\Rightarrow r\,dr=-\frac{du}{2}$

$r=0 \rightarrow u=a^2$

$r=a\cos\theta \Rightarrow u=a^2\sin^2\theta$

$$= \frac{4}{3} a^3 \int_0^{\pi/2} [1 - \sin\theta (1 - \cos^2\theta)] \, d\theta$$

$$= \frac{4}{3} a^3 \int_0^{\pi/2} d\theta - \frac{4}{3} a^3 \int_0^{\pi/2} \sin\theta (1 - \cos^2\theta) \, d\theta$$

D.D.

$u = \cos\theta \Rightarrow du = -\sin\theta \, d\theta$

$\theta = 0 \to u = 1$

$\theta = \pi/2 \to u = 0$

$$= \frac{4}{3} a^3 \cdot \frac{\pi}{2} - \frac{4}{3} a^3 \int_0^1 (1 - u^2) \, d\theta$$

$$= \frac{2\pi}{3} a^3 - \frac{4}{3} a^3 \left(u - \frac{u^3}{3} \right) \Big|_0^1$$

$$= \frac{2\pi}{3} a^3 - \frac{8}{9} a^3 = \frac{2}{9} a^3 (3\pi - 4).$$

32) P, $x+y=1$, $x+y=2$, $3x+4y=5$ ve $3x+4y=6$ tarafından sınırlanan paralel kenar olmak üzere

$$\iint_P (x^2+y^2) \, dA$$ -yı hesaplayınız.

Sol.

$u = x+y$, $v = 3x+4y$ olsun. $\Rightarrow$ $u=1$, $u=2$

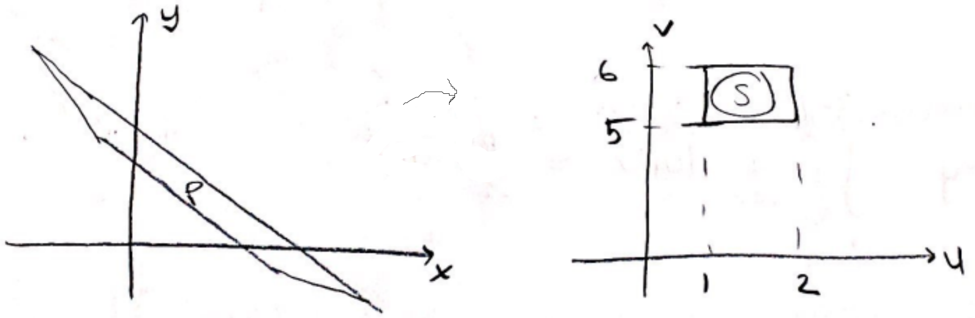
$v=5$, $v=6$.

Ayrıca; $-4/$ $u = x+y$

$v = 3x+4y$

$x = 4u - v$

$y = v - 3u$



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ -3 & 1 \end{vmatrix} = 1$$

$$x^2+y^2 = (4u-v)^2+(v-3u)^2 = 25u^2-14uv+2v^2.$$

O halde;

$$\iint_P (x^2+y^2)\,dA = \int_1^2\int_5^6 (25u^2-14uv+2v^2)\cdot 1\,dv\,du$$

$$= \int_1^2 \left(25u^2v - 7uv^2 + \frac{2v^3}{3}\right)\Big|_5^6 du$$

$$= \int_1^2 \left(25u^2 - 77u + \frac{182}{3}\right) du$$

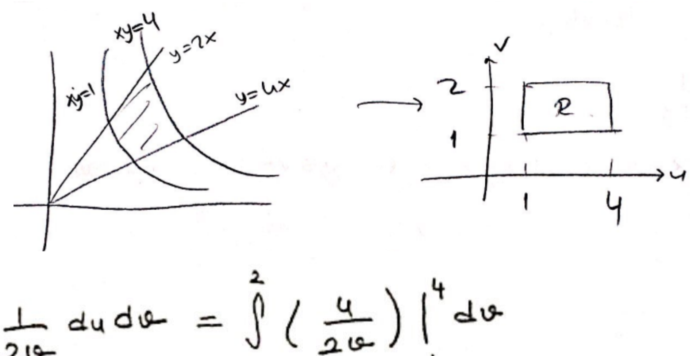
$$= \left(25\frac{u^3}{3} - 77\frac{u^2}{2} + \frac{182}{3}u\right)\Big|_1^2 = \frac{7}{2}.$$

33) $xy=1$, $xy=4$, $y=x$ ve $y=2x$ eğrileri tarafından sınırlanan birinci çeyrek düzlemdeki bölgenin alanını bulunuz.

Sol.

$$u=xy, \quad v=\frac{y}{x} \quad \Rightarrow \quad \begin{matrix} u=1, & u=4 \\ v=1, & v=2 \end{matrix}$$

$$\Rightarrow \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2\frac{y}{x} = 2v \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$$



$$\Rightarrow \iint_D dA = \int_1^2\int_1^4 \frac{1}{2v}\,du\,dv = \int_1^2 \left(\frac{u}{2v}\right)\Big|_1^4 dv$$

$$= \frac{3}{2}\int_1^2 \frac{1}{v}\,dv = \frac{3}{2}\ln v\Big|_1^2 = \frac{3}{2}\ln 2.$$