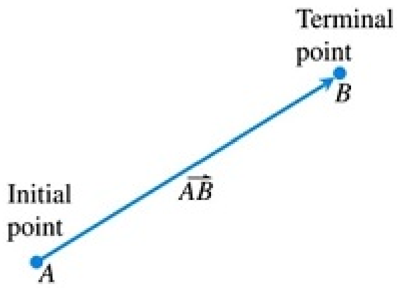
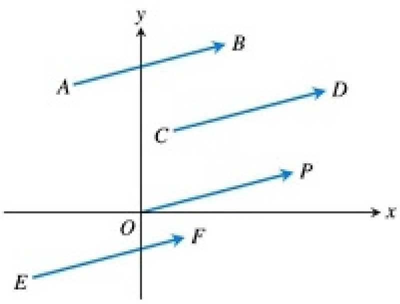
§10.2. Vectors

In this section we show how to represent things that have both magnitude and direction in the plane or space.



Definition. The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and the **length** is denoted by $|\overrightarrow{AB}|$. A quantity such as velocity, force and displacement is called vector.

Two vectors are **equal** if they have the same length and direction regardless of the initial point.

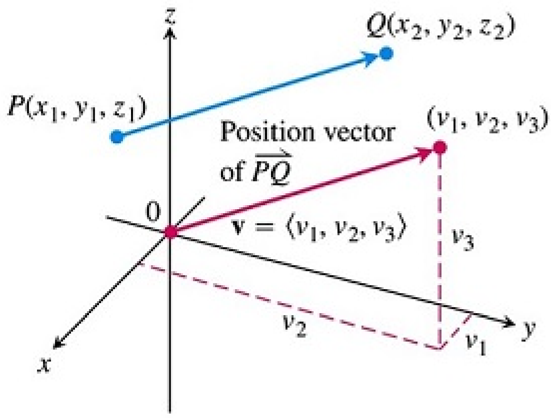


Definition. If $\overrightarrow{v}$ is **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the component form of $\overrightarrow{v}$ is $\overrightarrow{v} = \langle v_1, v_2 \rangle$.

If $\overrightarrow{v}$ is **three-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the component form of $\overrightarrow{v}$ is $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$.

Definition. The **magnitude** or **length** of the vector $\overrightarrow{v} = \overrightarrow{PQ}$ is the nonnegative number;

$$|\overrightarrow{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$



A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

Definition. (Vector Algebra Operations) Let $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar:

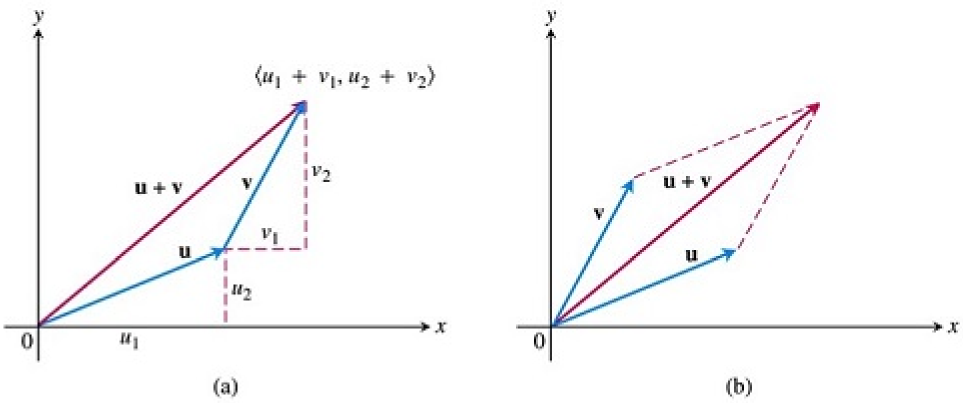
Addition: $\overrightarrow{u} + \overrightarrow{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\overrightarrow{u} = \langle ku_1, ku_2, ku_3 \rangle$

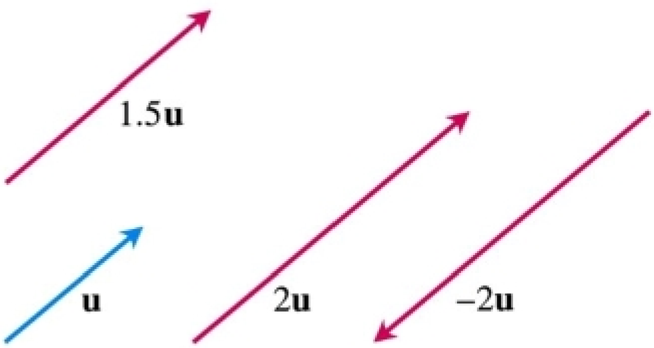
If $k > 0$ then $k\overrightarrow{u}$ has the same direction as $\overrightarrow{u}$,

If $k < 0$ then $k\overrightarrow{u}$ has the opposite direction as $\overrightarrow{u}$

If $k = 0$ then $k\overrightarrow{u}$ has zero length and therefore no particular direction. It is the zero vector denoted $\overrightarrow{0}$.



(a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.



In $\mathbb{R}^2$ we single out two particular vectors for special attention.

They are;

i. the vector $\overrightarrow{i}$ from the origin to the point (1,0), and

ii. the vector $\overrightarrow{j}$ from the origin to the point (0,1).

These vectors are called the **standard basis vectors** in the plane. The vector $\overrightarrow{r}$ from the origin to the point (x, y) has components x and y and can be expressed in the form $\overrightarrow{r} = \langle x, y \rangle = x\overrightarrow{i} + y\overrightarrow{j}$.

In the first form we specify the vector by listing its components between angle brackets; in the second we write $\overrightarrow{r}$ as a **linear combination** of the standard basis vectors i and j.

The vector r is called the **position vector** of the point (x,y). A position vector has its tail at the origin and its head at the point whose position it is specifying. The length of $\overrightarrow{r}$ is $|\overrightarrow{r}| = \sqrt{x^2 + y^2}$.

More generally, the vector $\overrightarrow{AP}$ from $A = (a, b)$ to $P = (p, q)$ can also be written as a list of components or as a linear combination of the standard basis vectors:

$$\overrightarrow{AP} = \langle p - a, q - b \rangle = (p - a)\overrightarrow{i} + (q - b)\overrightarrow{j}.$$

The zero vector is $\overrightarrow{0} = 0\overrightarrow{i} + 0\overrightarrow{j}$. It has length zero and no specific direction.
For any vector $\overrightarrow{u}$ we have $0u = \overrightarrow{0}$. Here 0 is the scalar of the multiplication.
A unit vector is a vector of length 1. Thus, the standard basis vectors $\overrightarrow{i}$ and $\overrightarrow{j}$ are unit vectors.

Given any nonzero vector $\overrightarrow{v}$, we can form a unit vector $\widehat{v}$ in the same direction as $\overrightarrow{v}$ by multiplying $\overrightarrow{v}$ by the reciprocal of its length (which a scalar):
$\widehat{v} = \frac{1}{|\overrightarrow{v}|}\overrightarrow{v}$

Example 1. If $A = (2, -1)$, $B = (-1, 3)$, and $C = (0, 1)$, express $\overrightarrow{AC}$, $\overrightarrow{CB}$ and $2\overrightarrow{AC} - 3\overrightarrow{CB}$.
$\overrightarrow{AC} = (0-2)\overrightarrow{i} + (1-(-1))\overrightarrow{j} = -2\overrightarrow{i} + 2\overrightarrow{j}$,
$\overrightarrow{CB} = (-1-0)\overrightarrow{i} + (3-1)\overrightarrow{j} = -1\overrightarrow{i} + 2\overrightarrow{j}$,
$2\overrightarrow{AC} - 3\overrightarrow{CB} = 2(-2\overrightarrow{i} + 2\overrightarrow{j}) - 3(-\overrightarrow{i} + 2\overrightarrow{j}) = -\overrightarrow{i} - 2\overrightarrow{j}$.

Properties of Vector Operations:

i. $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$

ii. $(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$

iii. $\overrightarrow{u} - \overrightarrow{v} = \overrightarrow{u} + (-1)\overrightarrow{v}$

iv. $t(\overrightarrow{u}+\overrightarrow{v}) = t\overrightarrow{u}+t\overrightarrow{v}$

Vectors in 3-Space

Given a Cartesian coordinate system in 3-space, we define three standard basis vectors, $\overrightarrow{i}$, $\overrightarrow{j}$, and $\overrightarrow{k}$, represented by arrows from the origin to the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, respectively. Any vector in 3-space can be written as a linear combination of these basis vectors; for instance, the position vector of the point (x,y,z) is given by;
$\overrightarrow{r} = \langle x,y,z\rangle = x\overrightarrow{i}+y\overrightarrow{j}+z\overrightarrow{k}$.

Example 2. If $\overrightarrow{u} = -\overrightarrow{i}+3\overrightarrow{j}+\overrightarrow{k}$ and $\overrightarrow{v} = 4\overrightarrow{i}+7\overrightarrow{j}$ are two vectors in 3-space. Find $2\overrightarrow{u}+3\overrightarrow{v}$, $\overrightarrow{u}-\overrightarrow{v}$ and $|\frac{1}{2}\overrightarrow{u}|$.
$2\overrightarrow{u}+3\overrightarrow{v} = 2(-\overrightarrow{i}+3\overrightarrow{j}+\overrightarrow{k})+3(4\overrightarrow{i}+7\overrightarrow{j}) = (-2\overrightarrow{i}+6\overrightarrow{j}+2\overrightarrow{k})+(12\overrightarrow{i}+21\overrightarrow{j}) = 10\overrightarrow{i}+27\overrightarrow{j}+2\overrightarrow{k}$.
$\overrightarrow{u}-\overrightarrow{v} = (-\overrightarrow{i}+3\overrightarrow{j}+\overrightarrow{k})-(4\overrightarrow{i}+7\overrightarrow{j}) = -5\overrightarrow{i}-4\overrightarrow{j}+\overrightarrow{k}$.
$|\frac{1}{2}\overrightarrow{u}| = |\frac{-1}{2}\overrightarrow{i}+\frac{3}{2}\overrightarrow{j}+\frac{1}{2}\overrightarrow{k}| = \sqrt{(\frac{-1}{2})^2+(\frac{3}{2})^2+(\frac{1}{2})^2} = \frac{1}{2}\sqrt{11}$.

The Dot Product and Projections

Definition. Given two vectors, $\overrightarrow{u} = u_1\overrightarrow{i}+u_2\overrightarrow{j}$ and $\overrightarrow{v} = v_1\overrightarrow{i}+v_2\overrightarrow{j}$ in $\mathbb{R}^2$, we define their **dot product** $\overrightarrow{u}\bullet\overrightarrow{v}$ to be the sum of the products of their corresponding components.

$\overrightarrow{u} \bullet \overrightarrow{v} = u_1 v_1 + u_2 v_2.$

The terms **scalar product** and **inner product** are also used in place of dot product. Similarly, for vectors $\overrightarrow{u} = u_1 \overrightarrow{i} + u_2 \overrightarrow{j} + u_3 \overrightarrow{k}$ and $\overrightarrow{v} = v_1 \overrightarrow{i} + v_2 \overrightarrow{j} + v_3 \overrightarrow{k}$ in $\mathbb{R}^3$,

$\overrightarrow{u} \bullet \overrightarrow{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$

The dot product has the following algebraic properties;

$\overrightarrow{u} \bullet \overrightarrow{v} = \overrightarrow{v} \bullet \overrightarrow{u}$

$\overrightarrow{u} \bullet (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \bullet \overrightarrow{v} + u \bullet \overrightarrow{w}$

$(t\overrightarrow{u}) \bullet \overrightarrow{v} = \overrightarrow{u} \bullet (t\overrightarrow{v}) = t(\overrightarrow{u} \bullet \overrightarrow{v})$

$\overrightarrow{u} \bullet \overrightarrow{u} = |u|^2$

Theorem 1. If θ is the angle between the directions of $\overrightarrow{u}$ and $\overrightarrow{v}$ $(0 \leq \theta \leq \pi)$, then

$\overrightarrow{u} \bullet \overrightarrow{v} = |\overrightarrow{u}||\overrightarrow{v}| \cos\theta.$

In particular, $\overrightarrow{u} \bullet \overrightarrow{v} = 0$ if and only if $\overrightarrow{u}$ and $\overrightarrow{v}$ are perpendicular. (Of course, the zero vector is perpendicular to every vector.)

Example 3. Find the angle between $\overrightarrow{u} = \langle 1, -2, -2 \rangle$ and $\overrightarrow{v} =$

$\langle 6, 3, 2\rangle$;

$\overrightarrow{u} \bullet \overrightarrow{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$

$|\overrightarrow{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$

$|\overrightarrow{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$

$\theta = \cos^{-1}(\frac{\overrightarrow{u} \bullet \overrightarrow{v}}{|\overrightarrow{u}||\overrightarrow{v}|}) = \cos^{-1}(\frac{-4}{(3)(7)}) \approx 1.76$ radians.

Definition. Scalar and vector projections

The scalar projection s of any vector $\overrightarrow{u}$ in the direction of nonzero vector $\overrightarrow{v}$ is the dot product of $\overrightarrow{u}$ with a unit vector in the direction of $\overrightarrow{v}$. Thus, is the number

$s = \frac{\overrightarrow{u} \bullet \overrightarrow{v}}{|\overrightarrow{v}|} = |\overrightarrow{u}| \cos\theta$,

where θ is the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.

The vector projection, $\overrightarrow{u}_{\overrightarrow{v}}$ of $\overrightarrow{u}$ in the direction of $\overrightarrow{v}$ is the scalar multiple of a unit vector $\widehat{v}$ in the direction of $\overrightarrow{v}$, by the scalar projection of $\overrightarrow{u}$ in the direction of $\overrightarrow{v}$; that is,

$$\overrightarrow{u}_v = \frac{\overrightarrow{u} \bullet \overrightarrow{v}}{|\overrightarrow{v}|} \widehat{v} = \frac{\overrightarrow{u} \bullet \overrightarrow{v}}{|\overrightarrow{v}|^2} \overrightarrow{v}.$$

Note that $|s|$ is the length of the line segment along the line of $\overrightarrow{v}$ obtained by dropping perpendiculars to that line from the tail and head of $\overrightarrow{u}$. Also, s is negative if $\theta > 90^o$.

It is often necessary to express a vector as a sum of two other vectors parallel and perpendicular to a given direction.

Example 4. Express the vector $3\overrightarrow{i}+\overrightarrow{j}$ as a sum of vectors $\overrightarrow{u}+\overrightarrow{v}$, where $\overrightarrow{u}$ is parallel to the vector $\overrightarrow{i}+\overrightarrow{j}$ and $\overrightarrow{v}$ is perpendicular to $\overrightarrow{u}$.

Method-I(Using vector projection) Note that $\overrightarrow{u}$ must be the vector projection of $3\overrightarrow{i}+\overrightarrow{j}$ in the direction of $\overrightarrow{i}+\overrightarrow{j}$. Thus,

$$\overrightarrow{u} = \frac{(3\overrightarrow{i}+\overrightarrow{j})\bullet(\overrightarrow{i}+\overrightarrow{j})}{|\overrightarrow{i}+\overrightarrow{j}|^2}(\overrightarrow{i}+\overrightarrow{j}) = \frac{4}{2}(\overrightarrow{i}+\overrightarrow{j}) = 2\overrightarrow{i}+2\overrightarrow{j}$$

$$\overrightarrow{v} = 3\overrightarrow{i}+\overrightarrow{j}-\overrightarrow{u} = \overrightarrow{i}-\overrightarrow{j}$$

Method-II(From basic principles) Since $\overrightarrow{u}$ is parallel to $\overrightarrow{i}+\overrightarrow{j}$ and $\overrightarrow{v}$ is perpendicular to $\overrightarrow{u}$, we have

$\overrightarrow{u} = t(\overrightarrow{i}+\overrightarrow{j})$ and $\overrightarrow{v}\bullet(\overrightarrow{i}+\overrightarrow{j}) = 0$,

for some scalar t. We want $\overrightarrow{u}+\overrightarrow{v} = 3\overrightarrow{i}+\overrightarrow{j}$. Take the dot product of this equation with $\overrightarrow{i}+\overrightarrow{j}$:

$\overrightarrow{u}\bullet(\overrightarrow{i}+\overrightarrow{j})+\overrightarrow{v}\bullet(\overrightarrow{i}+\overrightarrow{j}) = (3\overrightarrow{i}+\overrightarrow{j})\bullet(\overrightarrow{i}+\overrightarrow{j})$

$t(\overrightarrow{i}+\overrightarrow{j})\bullet(\overrightarrow{i}+\overrightarrow{j})+0 = 4.$

Thus $2t = 4$, so $t = 2$. Therefore,

$\overrightarrow{u} = 2\overrightarrow{i}+2\overrightarrow{j}$ and $\overrightarrow{v} = 3\overrightarrow{i}+\overrightarrow{j}-\overrightarrow{u} = \overrightarrow{i}-\overrightarrow{j}$.