TECHNIQUES OF INTEGRATION

INTEGRATION BY PARTS

Suppose that $u(x)$ and $v(x)$ are two differentiable functions.
According to Product Rule,

$$\frac{d}{dx}\left(u(x)\,v(x)\right) = u(x)\frac{dv}{dx} + v(x)\cdot\frac{du}{dx}$$

Integrating both sides of the equation and transposing terms,
we obtain

$$\int u(x)\frac{dv}{dx} = u(x)v(x) - \int v(x)\frac{du}{dx}$$

or, more simply,

$$\int u\,dv = u.v - \int v\,du.$$

Example: $\int x e^x\,dx$

Let $u = x$, $dv = e^x dx$
Then $du = dx$, $v = e^x$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$ //

Example: (i) $\int \ln x\,dx$

Let $u = \ln x$, $dv = dx$
Then $du = \frac{1}{x}dx$, $v = x$

$$= x.\ln x - \int x\cdot\frac{1}{x}\,dx$$

$$= x\ln x - x + C$$ //

(ii) $\int x^2 \sin x \, dx \implies$ Let $u = x^2$, $dv = \sin x \, dx$
then, $du = 2x\,dx$, $v = -\cos x$.

$= -x^2 \cos x + 2\int x \cos x \, dx \implies$ Let $u = x$, $dv = \cos x \, dx$
$du = dx$, $v = \sin x$

$= -x^2 \cos x + 2\left(x\sin x - \int \sin x \, dx \right)$

$= -x^2 \cos x + 2x \sin x + 2\cos x + C$ //

(iii) $\int x \tan^{-1} x \, dx \implies$ Let $u = \tan^{-1} x$, $dv = x\,dx$
then, $du = \frac{1}{1+x^2} dx$, $v = \frac{1}{2} x^2$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$ //

(iv) $\int \sin^{-1} x \, dx \implies$ Let $u = \sin^{-1} x$, $dv = dx$
then $du = \frac{1}{\sqrt{1-x^2}} dx$, $V = x$

$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx \implies$ Let $u = 1 - x^2 \Rightarrow du = -2x\,dx$

$= x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} du = x \sin^{-1} x + \frac{1}{2} \cdot 2 \cdot u^{1/2} + C$

$= x \sin^{-1} x + \sqrt{1-x^2} + C$ //

The followings are two useful rules of thumb for choosing u and dv:

(i) If the integrand involves a polynomial multiplied by an exponential, a sine or cosine, or some other readily integrable function, try u equals the polynomial and dv equals the rest.

(ii) If the integrand involves a logarithm, an inverse trigonometric function, or some other function that is not readily integrable but whose derivative is readily calculated, try that function for u and let dv equals the rest.

(Of course, these "rules" come with no guarantee. They may fail to be helpful if "the rest" is not of a suitable form. There remain many functions that cannot be antidifferentiated by any standard techniques)

Example: Evaluate $I = \int \sec^3 x \, dx$.

$$I = \int \sec x \sec^2 x \, dx \implies$$ Let $u = \sec x$, $dv = \sec^2 x \, dx$

then $du = \sec x \tan x \, dx$,

$v = \tan x$

$$= \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - I + \ln|\sec x + \tan x|$$

Thus, $2I = \sec x \tan x + \ln|\sec x \tan x|$ we have;

$$\int \sec^3 x \, dx = I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x \tan x| + C$$

Example: (A definite integral)

$$\int_1^e x^3 (\ln x)^2 \, dx \implies$$ Let $u = (\ln x)^2$, $dV = x^3 dx$
then $du = 2\frac{\ln x}{x} dx$, $V = \frac{x^4}{4}$

$$= \left[\frac{x^4}{4} (\ln x)^2\right]_1^e - \frac{1}{2}\int_1^e x^3 \ln x \, dx \implies$$ Let $u = \ln x$, $dv = x^3 dx$
then $du = \frac{1}{x} dx$, $v = \frac{x^4}{4}$

$$= \left[\frac{e^4}{4} (1)^2 - (0)\right] - \frac{1}{2}\left(\left[\frac{x^4}{4} \ln x\right]_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{1}{x} \, dx\right)$$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\left(\frac{x^4}{4}\right)_1^e = \frac{e^4}{8} + \frac{e^4}{32} - \frac{1}{32}$$

$$= \frac{5}{32} e^4 - \frac{1}{32}$$

INTEGRATION OF RATIONAL FUNCTIONS

Example: Evaluate $\displaystyle\int \frac{x^3+3x^2}{x^2+1}\,dx$

degree of numerator $= 3 >$ degree of denominator $= 2$

By using long division;

$$
\begin{array}{r|l}
x^3+3x^2 & x^2+1 \\ \cline{2-2}
-\;x^3+x & x+3 \\ \cline{1-1}
3x^2-x & \\
-\;3x^2+3 & \\ \cline{1-1}
-x-3 &
\end{array}
\qquad \Rightarrow \qquad \frac{x^3+3x^2}{x^2+1} = (x+3) - \frac{x+3}{x^2+1}
$$

Thus,

$$\int \frac{x^3+3x^2}{x^2+1}\,dx = \int (x+3)\,dx - \int \frac{x}{x^2+1}\,dx - 3\int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + 3x - \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + C$$

Example: Evaluate $\displaystyle\int \frac{x}{2x-1}\,dx$

(degree of numerator = 1 = degree of denominator)

$$\frac{x}{2x-1} = \frac{1}{2}\,\frac{2x}{2x-1} = \frac{1}{2}\,\frac{2x-1+1}{2x-1} = \frac{1}{2}\left(1+\frac{1}{2x-1}\right)$$

$$\int \frac{x}{2x-1}\,dx = \frac{1}{2}\int\left(1+\frac{1}{2x-1}\right)dx$$

$$= \frac{x}{2} + \frac{1}{4}\ln|2x-1| + C$$

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Examples:

1. $$I = \int \frac{x+4}{x^2-5x+6}\,dx = ?$$

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

$$= \frac{(Ax-3A)+(Bx-2B)}{(x-2)(x-3)}$$

$$\Rightarrow \left.\begin{array}{l} A+B=1 \\ -3A-2B=4 \end{array}\right\} \quad \begin{array}{l} A=-6 \\ B=7 \end{array}$$

$$I = \int \frac{x+4}{x^2-5x+6}\,dx = -6\int \frac{1}{x-2}\,dx + 7\int \frac{1}{x-3}\,dx$$

$$= -6\ln|x-2| + 7\ln|x-3| + C \,//$$

2. $$I = \int \frac{x^3+2}{x^3-x}\,dx = \int \frac{x^3-x+x+2}{x^3-x}\,dx = \int \left(1+\frac{x+2}{x^3-x}\right)dx$$

$$= x + \int \frac{x+2}{x^3-x}\,dx$$

$$\frac{x+2}{x^3-x} = \frac{x+2}{x(x-1)(x+1)}$$

$$= \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

$$= \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x(x-1)(x+1)}$$

$$\left.\begin{array}{l} A+B+C=0 \\ B-C=1 \\ -A=2 \end{array}\right\} \quad \begin{array}{l} A=-2 \\ B=3/2 \\ C=1/2 \end{array}$$

$$I = x - 2\int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= x - 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

③ $$I = \int \frac{2+3x+x^2}{x(x^2+1)}\,dx = ?$$

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx^2+Cx}{x(x^2+1)}$$

$$\left.\begin{aligned} A+B &= 1 \\ C &= 3 \\ A &= 2 \end{aligned}\right\} \quad \begin{aligned} A&=2 \\ B&=-1 \\ C&=3 \end{aligned}$$

$$I = 2\int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx + 3\int \frac{1}{x^2+1}\,dx$$

$$= 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C$$

Completing the Square

Example: $$I = \int \frac{1}{x^3+1}\,dx = ?$$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{A(x^2-x+1) + B(x^2+x) + C(x+1)}{(x+1)(x^2-x+1)}$$

$$\Rightarrow \left.\begin{array}{r} A+B=0 \\ -A+B+C=0 \\ A+C=1 \end{array}\right\} \begin{array}{l} A=1/3 \\ B=-1/3 \\ C=2/3 \end{array}$$

$$I = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

The second integral we complete the square in the denominator $x^2-x+1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}$, and make a similar modification in the numerator.

$$I = \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x-\frac{1}{2}-\frac{3}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \implies \text{Let } u = x-\frac{1}{2} \text{ then; } du=dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(u^2+\frac{3}{4}\right) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{3}\right) + C$$

Denominators with Repeated Factors

<u>Example</u>: $I = \int \frac{1}{x(x-1)^2}\,dx = ?$

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x^2-2x+1) + B(x^2-x) + Cx}{x(x-1)^2}$$

$$\left.\begin{aligned} A+B&=0 \\ -2A-B+C&=0 \\ A&=1 \end{aligned}\right\} \begin{aligned} A&=1 \\ B&=-1 \\ C&=1 \end{aligned}$$

$$I = \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \ln|x| - \ln|x-1| - \frac{1}{(x-1)} + C$$

$$= \ln\left|\frac{x}{x-1}\right| - \frac{1}{(x-1)} + C \quad //$$

Example: $I = \int \frac{x^2+2}{4x^5+4x^3+x}\,dx = ?$

$$\frac{x^2+2}{x(2x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$$

$$= \frac{A(4x^4+4x^2+1) + B(2x^4+x^2) + C(2x^3+x) + Dx^2 + Ex}{x(2x^2+1)^2}$$

$$\left.\begin{aligned} 4A+2B&=0 \\ 2C&=0 \\ 4A+B+D&=1 \\ C+E&=0 \\ A&=2 \end{aligned}\right\} \quad \begin{aligned} A&=2 \\ B&=-4 \\ C&=0 \\ D&=-3 \\ E&=0 \end{aligned}$$

$$I = 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2} \longrightarrow \begin{aligned} \text{Let } u&=2x^2+1 \\ du&=4x\,dx \end{aligned}$$

$$= 2\ln|x| - \int \frac{du}{u} - \frac{3}{4}\int \frac{du}{u^2}$$

$$= 2\ln|x| - \ln|u| + \frac{3}{4u} + C$$

$$= 2\ln|x| - \ln(2x^2+1) + \frac{3}{4}\cdot\frac{1}{2x^2+1} + C$$

$$= \ln\left(\frac{x^2}{2x^2+1}\right) + \frac{3}{4}\,\frac{1}{2x^2+1} + C \quad //$$

INVERSE SUBSTITUTIONS

The inverse Trigonometric Substitutions

Three very useful inverse substitutions are:

$x = a\sin\theta$, $x = a\tan\theta$ and $x = a\sec\theta$

These correspond to the direct substitutions:

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$, $\theta = \tan^{-1}\left(\frac{x}{a}\right)$, and $\theta = \sec^{-1}\left(\frac{x}{a}\right) = \cos^{-1}\left(\frac{a}{x}\right)$

The inverse sine substitution

Integrals involving $\sqrt{a^2-x^2}$ (where $a>0$) can frequently be reduced to a simpler form by mean of the substitution

$x = a\sin\theta$ or equivalently $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

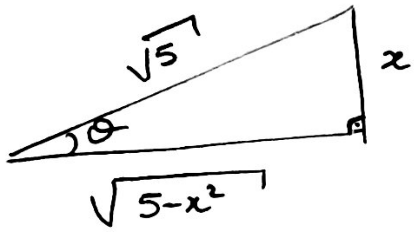
Observe that $\sqrt{a^2-x^2}$ makes sense only if $-a \le x \le a$, which corresponds to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Since $\cos\theta \ge 0$ for such θ, we have

$$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta .$$

$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a} \quad \text{and} \quad \tan\theta = \frac{x}{\sqrt{a^2-x^2}} .$$

Example: Evaluate $I = \int \frac{1}{(5-x^2)^{3/2}}\,dx$

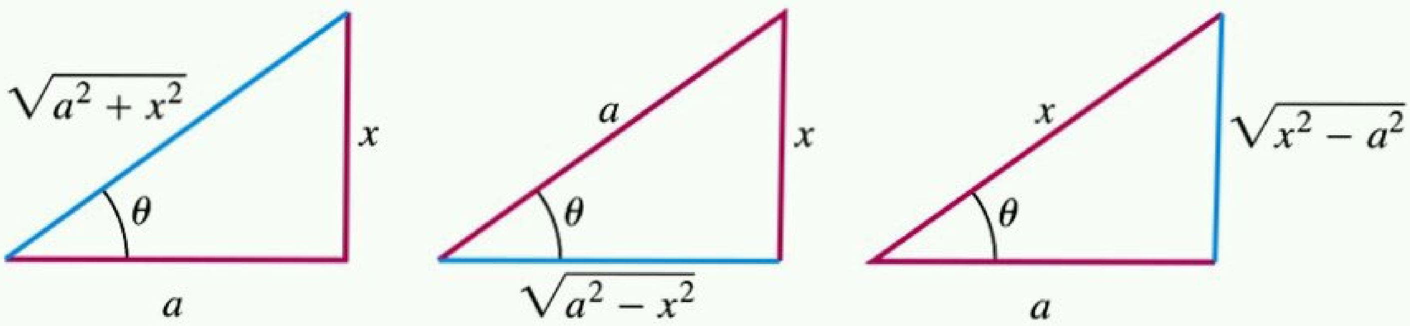
Let $x = \sqrt{5}\sin\theta$
$dx = \sqrt{5}\cos\theta\,d\theta$



$$I = \int \frac{\sqrt{5}\cos\theta}{5^{3/2}\cdot(\cos^2\theta)^{3/2}}\,d\theta = \frac{1}{5}\int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \frac{1}{5}\int \sec^2\theta\,d\theta = \frac{1}{5}\tan\theta + C$$

$$= \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C$$

//



$x = a \tan \theta$ $\qquad$ $x = a \sin \theta$ $\qquad$ $x = a \sec \theta$

$\sqrt{a^2 + x^2} = a|\sec \theta|$ $\qquad$ $\sqrt{a^2 - x^2} = a|\cos \theta|$ $\qquad$ $\sqrt{x^2 - a^2} = a|\tan \theta|$

FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

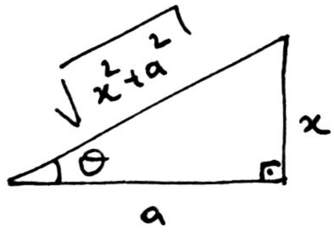
The inverse tangent substitution

Integrals involving $\sqrt{a^2+x^2}$ or $\frac{1}{x^2+a^2}$ (where $a>0$) are often simplified by the substitution;

$$x = a\tan\theta \quad \text{equivalently}, \quad \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

Since x can take any real value, we have $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $\sec\theta > 0$ and

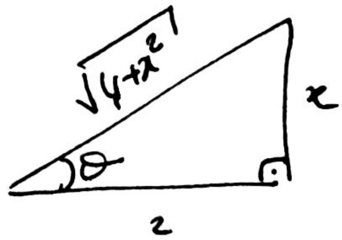
$$\sqrt{a^2+x^2} = a\sqrt{1+\left(\frac{x}{a}\right)^2} = a\sqrt{1+\tan^2\theta} = a\sec\theta$$



$$\sin\theta = \frac{x}{\sqrt{a^2+x^2}} \quad \text{and} \quad \cos\theta = \frac{a}{\sqrt{a^2+x^2}}.$$

Example: Evaluate $J = \int \frac{1}{\sqrt{4+x^2}}\,dx$.

Let $x = 2\tan\theta$ then $dx = 2\sec^2\theta\,d\theta$



$$J = \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int \sec\theta\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

The inverse secant substitution

Integrals involving $\sqrt{x^2-a^2}$ (where $a>0$) can frequently be simplified by using the substitution;

$$x = a\sec\theta \quad \text{or, equivalently,} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

$$\sqrt{x^2-a^2} = a\sqrt{\left(\frac{x}{a}\right)^2 - 1} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a\left|\tan\theta\right|,$$

we cannot always drop the absolute value from the tangent.

Observe that $\sqrt{x^2-a^2}$ makes sense for $x \geq a$ and for $x \leq -a$.

If $x \geq a$, then $0 \leq \theta = \sec^{-1}\left(\frac{x}{a}\right) = \arccos\frac{a}{x} < \frac{\pi}{2}$, and $\tan\theta \geq 0$.

If $x \leq -a$, then $\frac{\pi}{2} < \theta = \sec^{-1}\left(\frac{x}{a}\right) = \arccos\frac{a}{x} \leq \pi$, and $\tan\theta \leq 0$.

In the first case $\sqrt{x^2-a^2} = a\tan\theta$; in the second case $\sqrt{x^2-a^2} = -a\tan\theta$.

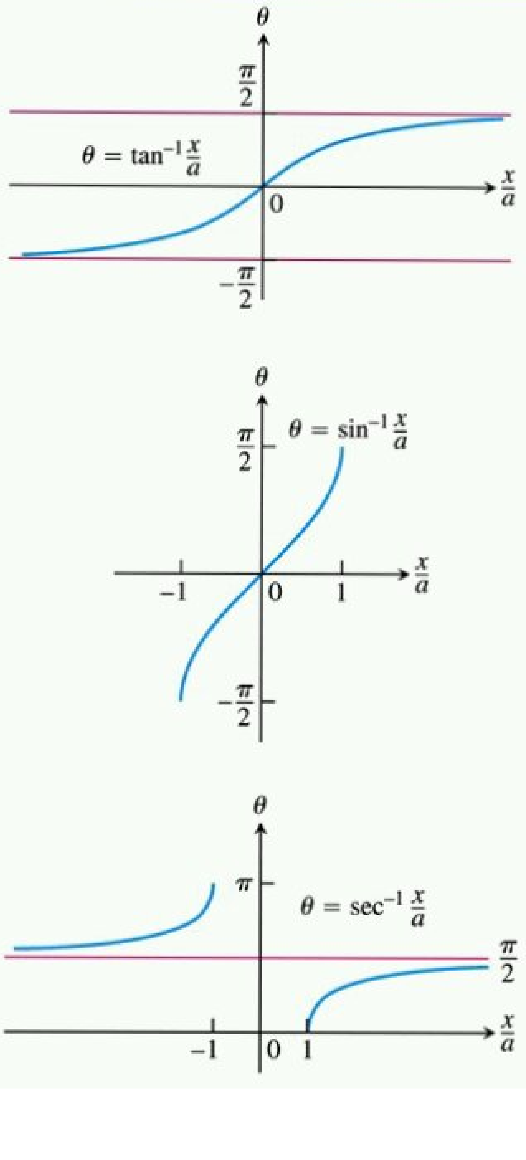


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

Procedure For a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

EXAMPLE 8

$$\int \frac{1}{1+\sqrt{2x}}\,dx \qquad \text{Let } 2x = u^2, \quad 2\,dx = 2u\,du$$

$$= \int \frac{u}{1+u}\,du$$

$$= \int \frac{1+u-1}{1+u}\,du$$

$$= \int \left(1 - \frac{1}{1+u}\right) du \qquad \text{Let } v = 1+u, \quad dv = du$$

$$= u - \int \frac{dv}{v} = u - \ln|v| + C$$

$$= \sqrt{2x} - \ln\left(1+\sqrt{2x}\right) + C$$

EXAMPLE 9

$$\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx \qquad \text{Let } 3x+2=u^3,$$
$$3\,dx = 3u^2\,du$$

$$= \int_1^2 \frac{u^3-2}{3u}\,u^2\,du$$

$$= \frac{1}{3}\int_1^2 (u^4-2u)\,du = \frac{1}{3}\left(\frac{u^5}{5}-u^2\right)\Bigg|_1^2 = \frac{16}{15}.$$

EXAMPLE 10 Evaluate $\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx$.

Solution We can eliminate both the square root and the cube root by using the inverse substitution $x = u^6$. (The power 6 is chosen because 6 is the least common multiple of 2 and 3.)

$$\int \frac{dx}{x^{1/2}(1+x^{1/3})} \qquad \text{Let } x = u^6, \quad dx = 6u^5\,du$$

$$= 6\int \frac{u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2}{1+u^2}\,du = 6\int \left(1 - \frac{1}{1+u^2}\right) du$$

$$= 6\,(u - \tan^{-1} u) + C = 6\,(x^{1/6} - \tan^{-1} x^{1/6}) + C.$$