9.6) Taylor and Maclaurin Series.

⊛ Suppose that f has derivatives of all orders.

Then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

is called the Taylor series of f about c.

If $c=0$, we say "Maclaurin" instead of "Taylor"

Maclaurin Series for Some Elementary Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6} + \cdots$$

for all these series radius of convergence is ∞.

So, these representations are valid on $\mathbb{R}$.

$\boxed{5-12}$ Find Maclaurin series representations for the given functions. For what values of x is each representation valid?

Q5) $x^2 \sin\left(\frac{x}{2}\right)$:

We can write

$$\sin\left(\frac{x}{3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{x}{3}\right)^{2n+1}$$

$$\Rightarrow \quad x^2 \sin\left(\frac{x}{3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{3^{2n+1}(2n+1)!} \qquad \forall x \in \mathbb{R}$$

Interval of convergence of this power series is R. Also, given function is defined on R. So, this representation is valid on R.

Q6) $\cos^2\left(\frac{x}{2}\right)$:

We know that

$$\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 + \cos x)$$

$$= \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right)$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \forall x \in \mathbb{R}$$

Q12) $(e^{2x^2}-1)/x^2$:

We can write

$$e^{2x^2} = 1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \cdots \quad \forall x.$$

$$\Rightarrow \frac{e^{2x^2}-1}{x^2} = 2 + 2^2 \cdot \frac{x^2}{2!} + 2^3 \frac{x^4}{3!} + \cdots$$

$$= \sum_{n=0} \frac{2^{n+1} x^{2n}}{(n+1)!}, \quad x \neq 0.$$

Interval of convergence of this power series is R, but given function is defined on R\{0}. So, this representation is valid on R\{0}.

$\boxed{16-26}$ Find the required Taylor series representations of the functions. Where is each series representation valid?

Q16) $f(x) = \sin x$ about $\frac{\pi}{2}$:

$f(\frac{\pi}{2}) = 1$; $f'(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin x$, $f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x$, $f'''(\frac{\pi}{2}) = 0$

$f^{IV}(x) = \sin x$, $f^{IV}(\frac{\pi}{2}) = 1$

$$\Rightarrow f^{(k)}\left(\frac{\pi}{2}\right) = \begin{cases} 1 & k=4i \\ 0 & k=4i+1 \\ -1 & k=4i+2 \\ 0 & k=4i+3 \end{cases} \qquad i=0,1,\dots$$

Taylor series of $\sin x$ about $\frac{\pi}{2}$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!}\left(x-\frac{\pi}{2}\right) = 1 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{4}\right)^2 \dots$$

$$= \sum_{=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k}$$ and interval of conv. is $\mathbb{R}$

$$\Rightarrow \sin x = \sum^{\infty} \frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n} \quad \forall x \in \mathbb{R}.$$

Q24) $f(x) = \frac{x}{1+x}$ in powers of $x-1$:

$$f(1) = \frac{1}{2}; \quad f'(x) = \frac{(1+x)-x}{(1+x)^2} = \frac{1}{(1+x)^2}, \quad f'(1) = \frac{1}{4}$$

$$f''(x) = -\frac{2}{(1+x)^3}, \quad f''(1) = -\frac{2}{2^3}$$

$$f'''(x) = +\frac{2.3}{(1+x)^4}, \quad f'''(1) = +\frac{2.3}{2^4}$$

$\vdots$

$$f^{(k)}(1) = (-1)^{k+1}\frac{k!}{2^{k+1}} \qquad \text{for } k=1,2,\dots$$

Taylor series of $f(x)$ about $x=1$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k!}(-1)^{k+1}\frac{k!}{2^{k+1}}(x-1)^k$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^{k+1}(x-1)^k$$

We can find that the interval of conv is $(-1,3)$

$$\Rightarrow \frac{x}{1+x} = \frac{1}{2} + \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n+1}(x-1)^n \quad \text{for } x \in (-1,3)$$

Q26) $f(x) = xe^x$ in powers of $x+2$:

$f(-2) = -2e^{-2}$; $f'(x) = e^x + xe^x$, $f'(-2) = -e^{-2}$

$f''(x) = e^x + (e^x + xe^x) = 2e^x + xe^x$, $f''(-2) = 0$

$f'''(x) = 2e^x + (e^x + xe^x) = 3e^x + xe^x = e^x$ $f'''(-2) = e^{-2}$

$f^k(-2) = (k-2)e^{-2}$ for $k = 0, 1, 2, \ldots$

Taylor series of $f(x)$ about $x=-2$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(-2)}{k!}(x+2)^k = \sum_{k=0}^{\infty} \frac{(k-2)e^{-2}}{k!}(x+2)^k$$

Interval of conv. is ∞.

So, $f(x) = \sum_{n=0}^{\infty} \frac{(n-2)}{n!} e^{-2}(x+2)^n$ for $x \in \mathbb{R}$

9.7) Applications of Taylor and Maclaurin Series

Theorem Taylor's Theorem

If the $(n+1)$st derivative of f exists on an interval containing c and x, and $P_n(x)$ is the Taylor polynomial of degree n for f about the point $x=c$, then

$$f(x) = P_n(x) + E_n(x) \quad \text{where}$$

$$E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-c)^{n+1} \quad (s \text{ is between } x \text{ and } c)$$

We can write $|f(x) - P(x)| = |E(x)|$

$\boxed{1}$ Estimate the error if the Maclaurin polynomial of degree 5 for $\sin x$ is used to approximate $\sin(0.2)$.

$$\sin(0.2) = \frac{2}{10} - \frac{1}{3!}\left(\frac{2}{10}\right)^3 + \frac{1}{5!}\left(\frac{2}{10}\right)^5 - \frac{1}{7!}\left(\frac{2}{10}\right)^7 + \cdots$$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad \text{for some } s \text{ between } 0 \text{ and } \frac{2}{10}$$

$$|\sin(0.2) - P_5(0.2)| \leq \frac{s^7}{7!} \leq \frac{(0.2)^7}{7!} < (2.6)\,10^{-9}$$

15 Find Maclaurin series for the given function.

$$I(x) = \int_0^x \frac{\sin t}{t}\, dt .$$

$$I(x) = \int_0^x \frac{\sin t}{t}\, dt$$

$$= \int_0^x \frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \right] dt$$

$$= \int_0^x \left[1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots \right] dt$$

$$= \left[t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \cdots \right] \Bigg|_0^x$$

$$= x - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} - \frac{x^7}{7.7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1).(2n+1)!} \qquad \forall x \in \mathbb{R}.$$

[interval of convergence: $(-\infty, \infty)$]

$\boxed{23}$ Evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos(x^2)}{(1-\cos x)^2}$ $\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{1-\left(1-\frac{(x^2)^2}{2!}+\frac{(x^2)^4}{4!}-\frac{(x^2)^6}{6!}+\cdots\right)}{\left[1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)\right]^2}$$

$$= \lim_{x\to 0} \frac{\frac{x^4}{2!}-\frac{x^8}{4!}+\frac{x^{12}}{6!}-\frac{x^{16}}{8!}+\cdots}{\left[\frac{x^2}{2!}-\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots\right]^2}$$

$$= \lim_{x\to 0} \frac{\frac{x^4}{2!}\left(1-[\underbrace{\text{terms containing the powers of } x \text{ in numerator}}_{0 \text{ as } x\to 0}]\right)}{\frac{x^4}{4}\left(1-[\underbrace{\text{terms containing the powers of } x \text{ in numerator}}_{0 \text{ as } x\to 0}]\right)}$$

$$= 2$$

CamScanner ile tarandı