§9.3 CONVERGENCE TEST FOR POSITIVE SERIES

We can not usually find the sum of a convergent series. Also, it is not easy to see whether a series is convergent or divergent. But we can use some techniques for determining whether a series converges or diverges.

In this section, we introduce some tests to determine whether a positive series converges or diverges. Note that these tests hold also for ultimately positive series.

The Integral Test

Let f be a positive, continuous and decreasing on an interval $[N, \infty)$ for some positive integer N and $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(t)dt$ both converge or diverge.

Remark. If we can calculate this integral, we should use integral test.

Example 1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges or diverges.

Consider $f(x) = \frac{1}{1+x^2}$ on $[1, \infty)$. We know that f is positive, continuous and decreasing on the interval since $f'(x) = \frac{-2x}{(1+x^2)^2} < 0$ on the interval. Thus we apply the integral test to this series. Then

$\int_1^{\infty} \frac{1}{1+x^2}dx = \lim_{R\to\infty} \int_1^R \frac{1}{1+x^2}dx = \lim_{R\to\infty} arctanx|_1^R = \lim_{R\to\infty}(arctanR-$

$arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ converges.

By the integral test, $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. But we can not find the sum of this series. (It does not mean that the sum is $\frac{\pi}{4}$)

Example 2. (**p-series**) Show that

$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} converges \quad\quad if\ p > 1 \\ \\ diverges\ to\ \infty\ if\ p \leq 1. \end{cases}$

Let $p > 1$. Note that $f(x) = \frac{1}{x^p}$ is positive, continuous and decreasing on $[1, \infty)$. Then;

$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{R \to \infty} \int_1^R \frac{1}{x^p} dx = \lim_{R \to \infty} (\frac{R^{1-p}}{1-p} - \frac{1}{1-p}) = \frac{1}{p-1}$. Thus the integral converges. By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Let $p = 1$, then $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series and we know that it diverges.

Let $0 < p < 1$. Again, look at the integral above:

$\int_1^{\infty} \frac{1}{x^p} dx = \infty.$

Let $p \leq 0$. Then $\lim_{n \to \infty} \frac{1}{n^p} \neq 0$. By nth-term test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Example 3. Consider $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$. It converges since $p = \frac{3}{2} > 1$.

Example 4. For which p values is the series $\sum_{n=2}^{\infty} \frac{1}{n(lnn)^p}$ convergent?

Note that $f(x) = \frac{1}{x(lnx)^p}$ is positive, continuous and decreasing on $[2, \infty)$. Thus, we can use the integral test. Then,

$\int_2^{\infty} \frac{1}{x(lnx)^p} dx = \int_{ln2}^{\infty} \frac{1}{u^p} du$, in here we use the method of substitution, assume that $u = lnx$, $du = \frac{dx}{x}$. Thus, by p-series, we say that $\sum_{n=1}^{\infty} \frac{1}{n(lnn)^p} \begin{cases} converges & if\ p > 1 \\ diverges\ to\ \infty & if\ 0 < p \leq 1 \end{cases}$.

Homework: Consider the series for $p \leq 0$.

Theorem 1. (**A (direct)comparison test**) Suppose that $0 \leq a_n \leq b_n$ for each $n = 1, 2, ...$ Then;

1. If the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

2. If the series $\sum_{n=1}^{\infty} a_n$ diverges, then the series $\sum_{n=1}^{\infty} b_n$ diverges.

Example 5. We can apply this theorem for the following series:

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3+5n}$. For each n, $n^3 + 5n > n^3 \Rightarrow \frac{1}{n^3+5n} < \frac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is p-series and converges ($p = 3 > 1$), then $\sum_{n=1}^{\infty} \frac{1}{n^3+5n}$ converge by comparison test.

2. Let $\sum_{n=1}^{\infty} \frac{1}{ln(3n)}$. Observe that $ln(3n) < 3n$ for each n, then, $\frac{1}{ln(3n)} > \frac{1}{3n}$. We know that $\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series and so it is divergent. Thus, $\sum_{n=1}^{\infty} \frac{1}{ln(3n)}$ diverges by comprasion test.

3. Let's have a look at $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$. See that $\frac{1}{2^n+1} < \frac{1}{2^n}$ for each

n. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is geometric series and is convergent as $|r| = \frac{1}{2} < 1$, then $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converge by comparison test.

Theorem 2. (**A limit comparison test**) Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.Then;

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges (diverges), then $\sum_{n=1}^{\infty} a_n$ converges (diverges).

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Example 6. Research the convergence or divergence of the following series.

i. $\sum_{n=1}^{\infty} \frac{1}{n^3+5n}$ ii. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$.

i. Remember that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent series (p-series). Since $\lim_{n\to\infty} \frac{\frac{1}{n^3+5n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3+5n} = 1 < \infty$, then $\sum_{n=1}^{\infty} \frac{1}{n^3+5n}$ converges by limit comparison test.

ii. Note that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ diverges (p-series). Since $\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^2+1}}}{\frac{1}{n^{\frac{2}{3}}}} = \lim_{n\to\infty} \frac{n^{\frac{2}{3}}}{n^{\frac{2}{3}}\sqrt{1+\frac{1}{n^2}}} = 1 < \infty$, then $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ diverges by limit comparison test.

In the previous example, we see that both comparison test and limit comparison test are applicable for some series(ex. $\sum_{n=1}^{\infty} \frac{1}{n^3+5n}$). But this may not be valid for all series.

Example 7. Consider $\sum_{n=1}^{\infty} \frac{1}{100n+20000}$. Firstly, we try to apply comparison test. First things that comes to mind is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Then $\frac{1}{100n+20000} < \frac{1}{n}$. We get have any information from comparison test since the harmonic series is divergent. Now, we try limit comparison test. Since $\lim_{n\to\infty} \frac{\frac{1}{100n+20000}}{\frac{1}{n}} = \frac{1}{100} \in (0, \infty)$, then $\sum_{n=1}^{\infty} \frac{1}{100n+20000}$ diverges by limit comparison test.

In this example, you can try comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{20100n}$ and obtain that your series diverges. But we show that for this example, limit comparison test is more useful.

Example 8. Note that $\sum_{n=1}^{\infty} \frac{1+sinn}{n^2}$. Firstly, we try to apply limit comparison test. Take the convergent series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ (p-series). $\lim_{n\to\infty} \frac{\frac{1+sinn}{n^2}}{\frac{2}{n^2}} = \lim_{n\to\infty} \frac{1+sinn}{2}$ does not exist. Thus, this limit comparison test does not work. Now let's try (direct) comparison test. We know that $1 + sinn \leq 2 \Rightarrow \frac{1+sinn}{n^2} \leq \frac{2}{n^2}$ for each n. By comparison test, it converges.

Theorem 3. (**Ratio test**) Let $\sum_{n=1}^{\infty} a_n$ be a (ultimately) positive series and $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then;

1. If $0 \leq \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $1 < \rho \leq \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

3. If $\rho = 1$, then the test is inconclusive.

Example 9. Investigate the convergence of the following series.

i. $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$ ii. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}$ iii. $\sum_{n=1}^{\infty} \frac{4^n(n!)^2}{(2n)!}$.

i. $\rho = \lim_{n\to\infty} \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}} = \lim_{n\to\infty} \frac{1}{3}\frac{2+5/2^n}{1+5/2^n} = \frac{2}{3} < 1$. By ratio test, $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$ converges.

ii. $\rho = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)^{n+1}}}{\frac{(2n)!}{n^n}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{n+1}(\frac{n}{n+1})^n = \lim_{n\to\infty} 2(2n+1)(\frac{n}{n+1})^n = \lim_{n\to\infty} \frac{2(2n+1)}{(1+\frac{1}{n})^n} = \infty$. By ratio test, $\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}$ diverges.

iii. $\rho = \lim_{n\to\infty} \frac{\frac{4^{n+1}((n+1)!)^2}{(2n+2)!}}{\frac{4^n(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{4(n+1)^2}{(2n+2)(2n+1)} = 1$. Thus, we can not decide if the series converges or diverges by ratio test. Note that $\frac{a_{n+1}}{a_n} = \frac{2(n+1)}{2n+1} > 1$ for each n. When $\{a_n\}$ is positive and increasing, $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

Theorem 4. (**Root test**) Let $\sum_{n=1}^{\infty} a_n$ be a (ultimately) positive series and $\sigma = \lim_{n\to\infty}(a_n)^{1/n}$. Then;

1. If $0 \leq \sigma < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $1 < \sigma \leq \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

3. If $\sigma = 1$, then the test is inconclusive.