## §10.3. The Cross Product in 3-Space

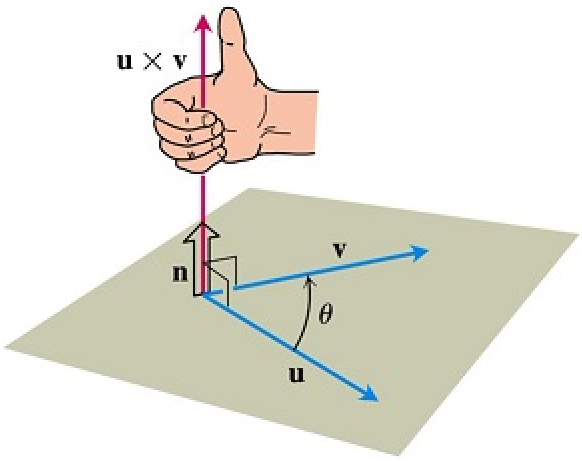
In this section we will see another kind of product of two vectors in 3-space which is called a **cross product** or **vector product**.

**Definition.** For any vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^3$, the cross product $\overrightarrow{u} \times \overrightarrow{u}$ is the unique vector satisfying the following three conditions:

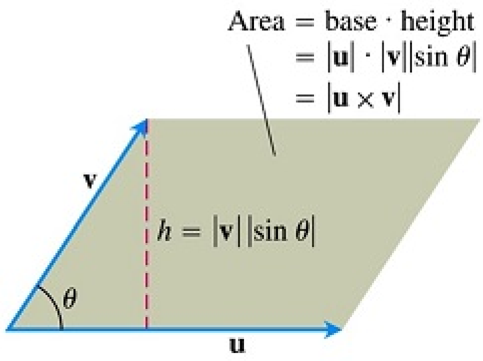
i. $(\overrightarrow{u} \times \overrightarrow{v}) \bullet \overrightarrow{u} = 0$ and $(\overrightarrow{u} \times \overrightarrow{v}) \bullet \overrightarrow{v} = 0$,

ii. $|(\overrightarrow{u} \times \overrightarrow{v})| = |\overrightarrow{u}||\overrightarrow{v}| \sin\theta$, where $\theta$ is the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.

iii. $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{u} \times \overrightarrow{v}$ form a right-handed triad.



From the definition one may say $\overrightarrow{u} \times \overrightarrow{v}$ is perpendicular to both $\overrightarrow{u}$ and $\overrightarrow{v}$ and has length equal to the area of the following shaded parallelogram.




The parallelogram determined by **u** and **v**.

If $\overrightarrow{u}$ and $\overrightarrow{v}$ have their tails at the point $P$, then $\overrightarrow{u} \times \overrightarrow{v}$ is normal (i.e., perpendicular) to the plane through $P$ in which $\overrightarrow{u}$ and $\overrightarrow{v}$ lie. These properties make the cross product very useful for description of tangent planes and normal lines in $\mathbb{R}^3$.

**Theorem 1.** If $\overrightarrow{u} = u_1 \overrightarrow{i} + u_2 \overrightarrow{j} + u_3 \overrightarrow{k}$ and $\overrightarrow{v} = v_1 \overrightarrow{i} + v_2 \overrightarrow{j} + v_3 \overrightarrow{k}$, then

$$\overrightarrow{u} \times \overrightarrow{v} = (u_2 v_3 - u_3 v_2) \overrightarrow{i} + (u_3 v_1 - u_1 v_3) \overrightarrow{j} + (u_1 v_2 - u_2 v_1) \overrightarrow{k}.$$

The formula for the cross product in terms of components may seem awkward and asymmetric however it can be written more easily in terms of a determinant.

**Example 1. (Calculating cross products)**

(a) $\overrightarrow{i} \times \overrightarrow{i} = \overrightarrow{0}$, $\overrightarrow{j} \times \overrightarrow{j} = \overrightarrow{0}$ and $\overrightarrow{k} \times \overrightarrow{k} = \overrightarrow{0}$.

(b) $\overrightarrow{i} \times \overrightarrow{j} = \overrightarrow{k}$, $\overrightarrow{j} \times \overrightarrow{k} = \overrightarrow{i}$ and $\overrightarrow{k} \times \overrightarrow{i} = \overrightarrow{j}$.

(c) $\overrightarrow{j} \times \overrightarrow{i} = \overrightarrow{-k}$, $\overrightarrow{k} \times \overrightarrow{j} = \overrightarrow{-i}$ and $\overrightarrow{i} \times \overrightarrow{k} = \overrightarrow{-j}$.

**Some properties of cross product**

If $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ are any vectors in $\mathbb{R}^3$, and $t$ ia a real number (a scalar), then

(i) $\overrightarrow{u} \times \overrightarrow{u} = \overrightarrow{0}$,

(ii) $\overrightarrow{u} \times \overrightarrow{v} = \overrightarrow{-v} \times \overrightarrow{u}$, (The cross product is **anticommutative**.)

(iii) $(\overrightarrow{u} + \overrightarrow{v}) \times \overrightarrow{w} = \overrightarrow{u} \times \overrightarrow{w} + \overrightarrow{v} \times \overrightarrow{w}$,

(iv) $\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{w}$,

(v) $(t\overrightarrow{u}) \times \overrightarrow{v} = \overrightarrow{u} \times (t\overrightarrow{v}) = t(\overrightarrow{u} \times \overrightarrow{v})$,

(vi) $\overrightarrow{u} \bullet (\overrightarrow{u} \times \overrightarrow{v}) = \overrightarrow{v} \bullet (\overrightarrow{u} \times \overrightarrow{v}) = 0$.

Note that the cross product is **not associative**. In general, $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) \neq (\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w}$.

**Determinants**:

In this part we will introduce $2 \times 2$ and $3 \times 3$ determinants. A determinant is an expression that involves the elements of a square array (matrix) of numbers. The determinant of the $2 \times 2$ array of numbers is (about the first row),

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1)^{1+1}ad + (-1)^{1+2}bc = ad - bc.$$

Similarly, the determinant of a $3 \times 3$ array of numbers is (about the first row) defined by,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-1)^{1+1}a(ei-hf)+(-1)^{1+2}b(di-gf)+(-1)^{1+3}c(dh-eg)$$

$$= aei + bfg + cdh - gec - hfa - idb.$$

**Properties of Determinants:**

(i) If two rows of a determinant are interchanged, then the determinant changes sign:

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

(ii) If two rows of the determinant are equal, the determinant has value 0:

$$\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix} = 0.$$

(iii) If the multiple of one row of the determinant is added to another row, the values of the determinant remains unchanged:

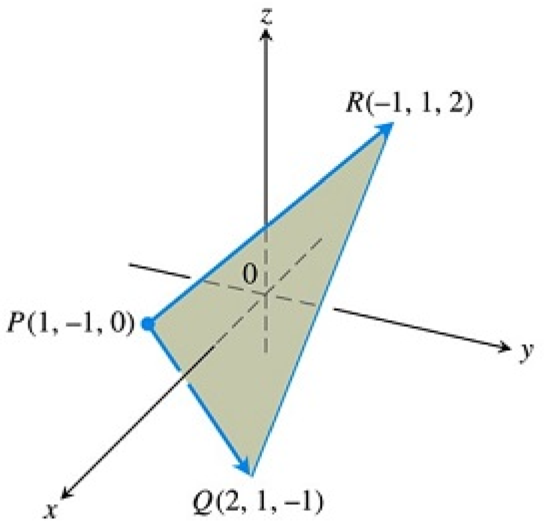
$$\begin{vmatrix} a & b & c \\ d+ta & e+tb & f+tc \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

**The Cross Product as a Determinant:**

The formula for the cross product of $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ presented in the above theorem can be expressed symbolically as a determinant (about the first row) with basis vectors as the elements of the first row :

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}.$$

**Example 2.** Find the area of the triangle with vertices at three points $P = (1, -1, 0)$, $R = (-1, 1, 2)$ and $Q = (2, 1, -1)$.




The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane of triangle $PQR$ (Example 2). The area of triangle $PQR$ is half of $|\overrightarrow{PQ} \times \overrightarrow{PR}|$

Two sides if the triangle as in the figure are given by the vectors:

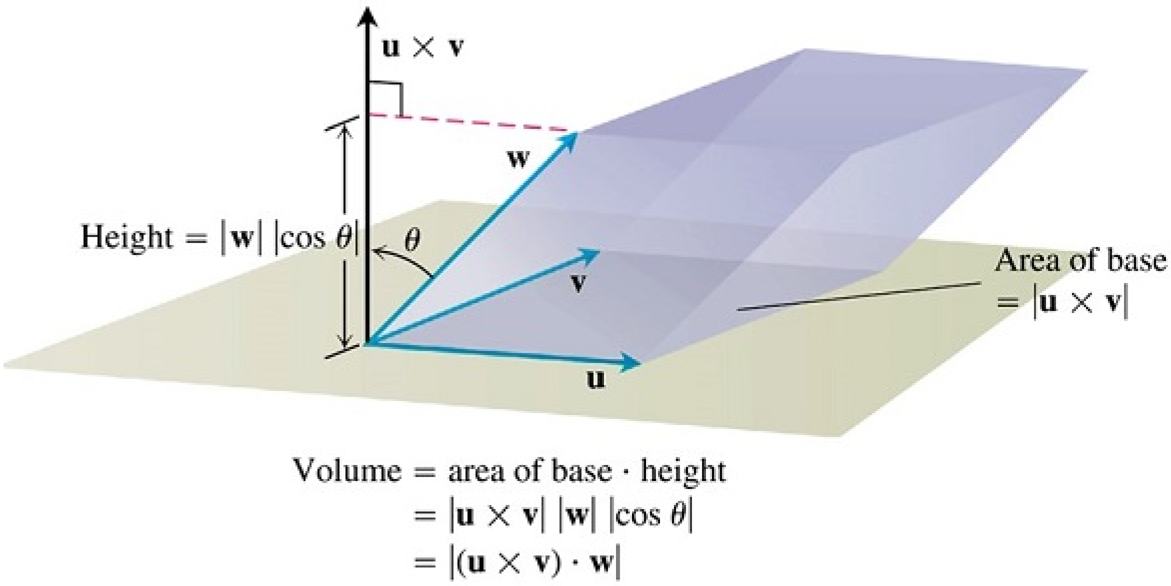
$\overrightarrow{PQ} =< 1, 2, -1 >$ and $\overrightarrow{PR} =< -2, 2, 2 >$.

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \frac{1}{2}| < 6, 0, 6 > | = \frac{1}{2}\sqrt{36 + 36} =$$

$3\sqrt{2}$ square units.

**Definition.** The quantity $\overrightarrow{u} \bullet (\overrightarrow{v} \times \overrightarrow{w})$ is called **the scalar triple product** of the vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$.

The volume of the parallelepiped spanned by the vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ can be find as in the following figure.




The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.