! An important trigonometric limit:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad (\text{where } \theta \text{ is in radians})$$

Example: Show that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

By using half-angle formula $\cos h = 1 - 2\sin^2(h/2)$

we rewrite $\frac{\cos h - 1}{h} = - \frac{2\sin^2(h/2)}{h}$

$\lim_{h \to 0} - \frac{2\sin^2(h/2)}{h}$ Let $\theta = \frac{h}{2}$ then since $h \to 0$ $\theta \to 0$

$$= -\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \cdot \sin\theta \right) = -\left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \right) \left(\lim_{\theta \to 0} \sin\theta \right)$$

$$= -(1)(0)$$

$$= 0$$

Derivatives of Trigonometric Functions

The trigonometric functions, especially sine and cosine play a very important role in the mathematical modelling of real-world phenomena. In particular, they arise whenever quantities fluctuate in a periodic way. Elastic motions, vibrations, and waves of all kinds naturally involve the trigonometric function, and many physical and mechanical laws are formulated as differential equations having these fncs. as solutions.

The Derivatives of Sine and Cosine

Show that $\frac{d}{dx}\sin x = \cos x$. By using addition formula for sine;

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}$$ Using the rules for combining limits;

$$= \lim_{h\to 0}\sin x\ \lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \underline{\underline{\cos x}}$$

Similarly; we can find $\frac{d}{dx}(\cos x) = \underline{\underline{-\sin x}}$

The Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cdot \cot x$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \frac{d}{dx}\sin x - \sin x \cdot \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Example: Find the tangent and normal lines to the curve $y = \tan(\pi x/4)$ at the point $(1,1)$.

The slope of the tangent to $y = \tan(\pi x/4)$ at $(1,1)$ is;

$$m_T = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{\pi}{4}\sec^2(\pi x/4)\right|_{x=1} = \frac{\pi}{4}\sec^2(\pi/4) = \frac{\pi}{4}(\sqrt{2})^2$$

$$= \frac{\pi}{2}$$

The tangent line is

$$y = 1 + \frac{\pi}{2}(x-1)$$

The normal line is $\left(m_N = -\frac{2}{\pi}\right)$

$$y = 1 - \frac{2}{\pi}(x-1)$$

Example:

Evaluate the derivatives of following functions.

i) $\sin(\pi x) + \cos(3x)$ ii) $x^2 \sin\sqrt{x}$ iii) $\frac{\cos x}{1-\sin x}$

(i) By the sum Rule and Chain Rule:

$$\frac{d}{dx}\sin(\pi x) + \frac{d}{dx}\cos(3x) = \cos(\pi x).\pi - \sin(3x).3$$

(ii) By the Product and Chain Rules:

$$\frac{d}{dx}(x^2 \sin\sqrt{x}) = 2x(\sin\sqrt{x}) + x^2.\cos\sqrt{x}.\frac{1}{2\sqrt{x}}$$

(iii) By the Quotient Rule:

$$\frac{d}{dx}\left(\frac{\cos x}{1-\sin x}\right) = \frac{(-\sin x)(1-\sin x) - (\cos x)(0-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1-\sin x}{(1-\sin x)^2}$$

$$= \frac{1}{1-\sin x}$$

HIGHER - ORDER DERIVATIVES

If the derivative $y' = f'(x)$ of a function $y = f(x)$ is itself differentiable at x, we can calculate its derivative, which we call the second derivative of f and denote by

Also other notations are like;

$$y'' = f''(x) \Rightarrow \left(\frac{d^2y}{dx^2} = y'' = f''(x) = \frac{d}{dx}\frac{d}{dx} f(x) = \frac{d^2 f(x)}{dx^2} \ldots \right)$$

The followings are some applications of derivative;

__Example:__ The velocity of moving object is the (instantaneous) rate of the position of object w.r.t. times, if the object moves along the x-axis and is at position $x = f(t)$ at time t, then its velocity at that time is;

$$v = \frac{dx}{dt} = f'(t)$$

Similarly, the acceleration of the object is the rate of change of the velocity. Thus the acceleration is the second derivative of the position;

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t)$$

In particular;
$f(x) = x^3$
$f'(x) = 3x^2$, $f''(x) = 6x$
$f^{(3)}(x) = 6$
$f^{(4)}(x) = 0$

__NOTE:__ If $f(x) = x^n$ $(n \in \mathbb{Z}^+)$ then

$$f^{(k)}(x) = n(n-1)\ldots(n-(k-1)) \cdot x^{n-k}$$

$$= \begin{cases} \dfrac{n!}{(n-k)!} x^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

where $n!$ (called n factorial) is defined by;

$0! = 1$

$1! = 0! \times 1 = 1 \times 1 = 1$

$2! = 1! \times 2 = 1 \times 2 = 2$

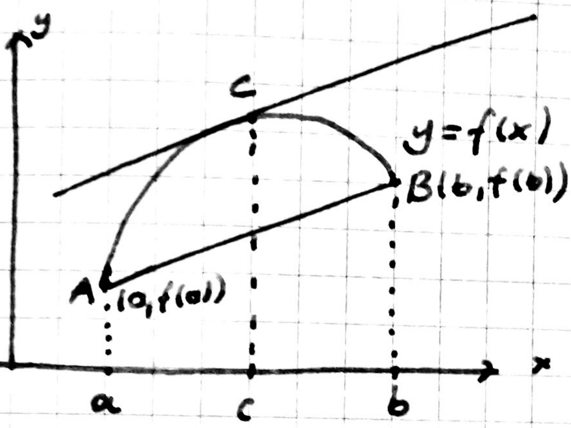
$3! = 2! \times 3 = 2 \times 3 = 6$

$n! = (n-1)! \times n = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$

THE MEAN VALUE THEOREM

Suppose that the function f is continuous on the closed, finite interval $[a,b]$ and that it is differentiable on the interval (a,b). Then there exists a point c in the open interval (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$



This says that the slope of the chord line joining the points $(a,f(a))$ and $(b,f(b))$ equal to the slope of the tangent line to the curve $y=f(x)$ at the point $(c,f(c))$ so the two lines are parallel.

NOTE: The Mean Value Theorem gives us no information on how to find the point c, which it says must exist. For some simple functions it is possible to calculate c, but doing so is usually of no practical value.

As we shall see the Importance of the M.V.T. lies in its use as a theorical tool. It belongs to class of theorems called existence theorems, as do Max-Min Thm and I.V.T.

Example: Verify the conclusion of the M.V.T. for $f(x)=\sqrt{x}$ on the interval $[a,b]$ where $0 \le a < b$.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c}} = \frac{\sqrt{b}-\sqrt{a}}{b-a} = \frac{\sqrt{b}-\sqrt{a}}{(\sqrt{b}-\sqrt{a})(\sqrt{b}+\sqrt{a})} = \frac{1}{\sqrt{b}+\sqrt{a}}$$

$$2\sqrt{c} = \sqrt{a}+\sqrt{b}$$

and $c = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2$ Since $a<b$ we have,

$a = \left(\frac{\sqrt{a}+\sqrt{a}}{2}\right)^2 < c < \left(\frac{\sqrt{b}+\sqrt{b}}{2}\right)^2 = b$ so c lies in the interval (a,b).

Example: Show that $\sin x < x$ $\forall x > 0$.

(Remember that $\forall$ $-1 \le \sin x \le 1$, for all x)

i) If $x > 2\pi$, then $\sin x \le 1 < 2\pi < x$

ii) If $0 < x < 2\pi$, then by the M.V.T. there exists c in $(0, 2\pi)$ such that (We apply M.V.T to $\sin x$ on the interval $[0,x]$)

$$\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x-0} = \frac{d}{dx}\sin x \bigg|_{x=c} = \cos c < 1.$$

Thus $\sin x < x$ in this case too.

Example: Show that $\sqrt{1+x} < 1+\frac{x}{2}$ for $x>0$ and for $-1 \le x < 0$.

If $x>0$, apply the M.V.T. to $f(x)=\sqrt{1+x}$ on $[0,x]$.
There exists $c \in (0,x)$ such that

$$\frac{f(x)-f(0)}{x-0} = \frac{\sqrt{1+x}-1}{x} = f'(c) = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}$$

since $c>0$. Multiplying by the positive number $x>0$ and transposing the -1 gives;

$$\sqrt{1+x} < \frac{x}{2} + 1$$

If $-1 \le x < 0$, we apply M.V.T. to $f(x)=\sqrt{1+x}$ on the interval $[x,0]$. There exists $c \in (x,0)$ such that

$$\frac{1-\sqrt{1+x}}{-x} = \frac{f(0)-f(x)}{0-x} = f'(c) = \frac{1}{2\sqrt{1+c}} > \frac{1}{2}$$

since $c<0$. Then;

$$\sqrt{1+x} - 1 < \frac{x}{2} \implies \sqrt{1+x} < 1+\frac{x}{2}.$$

- Increasing and Decreasing Functions -

Intervals on which the graph of a function f has positive or negative slope provide useful information about the behaviour of f. The M.V.T enables us to determine such intervals by considering the sign of the derivative f'.

Increasing and Decreasing Functions:

Suppose f is defined on an interval I and that x_1 and x_2 are two points of I.

a- If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$ we say f is increasing on I.

b- If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$ we say f is decreasing on I.

c- If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$ we say f is nondecreasing on I.

d- If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$ we say f is nonincreasing on I.

Note that an increasing (decreasing) fnc. is nondecreasing (nonincreasing) but nondecreasing (nonincreasing) fnc. is not necessarily increasing (decreasing).

THEOREM: Let J be an open interval and let I be an interva[l] consisting of all the points in J and possibly one or both of the endpoints of J. Suppose that f is continuous on I and differentiable on J.

a- If $f'(x) > 0$ for all x in J, then f is increasing on I.

b- If $f'(x) < 0$ for all x in J, then f is decreasing on I.

c- If $f'(x) \geq 0$ for all x in J, then f is nondecreasing on I.

d- If $f'(x) \leq 0$ for all x in J, then f is nonincreasing on I.

PROOF: Let x_1 and x_2 be points in I with $x_2 > x_1$. By the M.V.T. there exists a point c in (x_1, x_2) (an therefore in J) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c); \quad \text{hence} \quad f(x_2) - f(x_1) = (x_2 - x_1) \cdot f'(c)$$

Since $x_2 - x_1 > 0$, the difference has the same sign as $f'(c)$ and may be zero if $f'(c)$ is zero. Thus all four conclusions follow from the corresponding parts of definition of increasing and decreasing fncs.

THEOREM (FERMAT'S THEOREM):

If $f: [a,b] \to \mathbb{R}$ has a local maximum or local minimum at a point $c \in (a,b)$ and f is differentiable at c, then $f'(c)=0$

(If f is continuous on $I([a,b])$, and $f'(x)=0$ at every interior point of I (not a and b) then $f(x) = C$, a constant function, on I)

PROOF: Suppose c is local maximum. then $\exists\, \delta \in \mathbb{R}^+$ such that $|x-c| < \delta$ and $f(x) \leq f(c)$.

If $|h| < \delta$ then $\forall h$, $h+c$ is in this interval again.

$$f(c+h) \leq f(c) \Rightarrow f(c+h) - f(c) \leq 0$$

If $h > 0$ then $\dfrac{f(c+h)-f(c)}{h} \leq 0 \Rightarrow f'(c) \leq 0 \ldots$ (1)

If $h < 0$ then $\dfrac{f(c+h)-f(c)}{h} \geq 0 \Rightarrow f'(c) \geq 0 \ldots$ (2)

Since f is differentiable at c when $h \to 0^-$ and $h \to 0^+$ then

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c).$$

By (1) and (2) $\quad f'(c) = 0$.

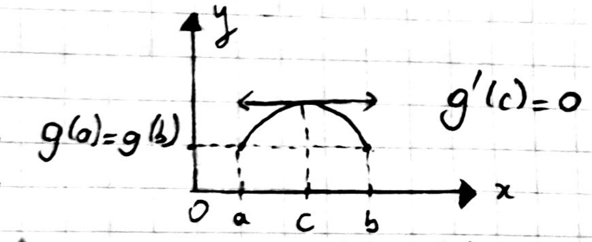
Pick a point x_0 in I and let $C = f(x_0)$. If x is any other point of I then the M.V.T says that $\exists\, c$ btw. x_0 and x such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(c)$$

The point c must belong to I because an interval contains all points btw any two of its points; c cannot be the endpoint of I since $c \neq x_0$ and $c \neq x$. Since $f'(c) = 0$ for all such points c, we have $f(x) - f(x_0) = 0$ $\forall x \in I$ and $f(x) = f(x_0) = C$.

THEOREM (ROLLE'S THEOREM)

Suppose that the function g is continuous on the closed, finite interval $[a,b]$ and that it is differentiable on the open interval (a,b). If $g(a)=g(b)$ then there exists a point c in the open interval (a,b) such that $g'(c)=0$



PROOF: Let suppose $M \to$ maximum
$m \to$ minimum

(i) Let suppose $M=m$ then f is a constant fnc. $\forall x \in [a,b]$
$f'(x)=0$.

(ii) Let suppose $M \neq m$ then since $f(a)=f(b)$ then the local maximum and local minimum can not be at endpoints, (By Fermat Thm) then $\exists c \in (a,b)$ such that $f'(c)=0$.

Remark: Rolle's Theorem is a special case of the M.V.T in which the chord line has slope 0, so the corresponding parallel tangent line must also have slope 0.

THEOREM (THE GENERALIZED MEAN-VALUE THEOREM)

If functions f and g are both continuous on $[a,b]$ and differentiable on (a,b), and if $g'(x) \neq 0$ for every $x \in (a,b)$, then there exists a number $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

IMPLICIT DIFFERENTIATION

Curves are generally the graphs of equations in two variables. Such equations can be written in the form $F(x,y)=0$ where $F(x,y)$ denotes an expression involving the two variables x and y. Sometimes we can solve an eqn. $F(x,y)=0$ for y and so find explicit formulas for one or more functions $y=f(x)$ defined by the eqn, but we are not able to solve the eqn. However, we can still regard it as defining y as one or more functions of x implicitly, even if we cannot solve for these functions explicitly. Moreover, we still find the derivative dy/dx of these implicit solutions by a technique called implicit differentiation. The idea is to differentiate the given equation w.r.t. x regarding y as a function of x.

Example: Find the slope of circle $x^2+y^2=25$ at the point $(3,-4)$.

The circle is not the graph of a single fnc. of x. Again, it combines the graphs of two fncs, $y_1=\sqrt{25-x^2}$ and $y_2=-\sqrt{25-x^2}$. The point $(3,-4)$ lies on the graph of y_2, so we can find the slope by calculating explicitly;

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25-x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25-9}} = \frac{3}{4}$$

We can also solve the problem more easily by differentiating the given equation of the circle implicitly w.r.t. x;

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(25)$$

$$2x+2y\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

The slope at $(3,-4)$

$$\left.-\frac{x}{y}\right|_{(3,-4)}=\frac{3}{4}$$

Example:

Find $\dfrac{dy}{dx}$ if $y\sin x = x^3 + \cos y$.

By implicit differentiation,

$$\frac{d}{dx}(y\sin x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(\cos y)$$

$$\left(\frac{dy}{dx}\right)\sin x + y.\cos x = 3x^2 - (\sin y).\frac{dy}{dx}$$

$$(\sin x + \sin y)\frac{dy}{dx} = 3x^2 - y\cos x$$

$$y' = \frac{dy}{dx} = \frac{3x^2 - y\cos x}{\sin x + \sin y}$$

NOTE: To find $\frac{dy}{dx}$ by implicit differentiation:

1. Differentiate both sides of the equation w.r.t. x, regarding y as a function of x and using the Chain Rule to differentiate function of y.

2. Collect terms with dy/dx on one side of the equation and solve for dy/dx by dividing by its coefficient.

HIGHER - ORDER DERIVATIVES

Find $y'' = \frac{d^2y}{dx^2}$ if $xy+y^2=2x$.

$1.y + x.y' + 2y.y' = 2$

$y' + y' + xy'' + 2.y'.y' + 2y.y'' = 0$

$$y' = \frac{2-y}{x+2y}$$

$$y'' = -\frac{8}{(x+2y)^3}.$$

ANTIDERIVATIVES AND INITIAL-VALUE PROBLEMS

Throughout this chapter we have been concerned with the problem of finding the derivative f' of a given function f. The reverse problem — given the derivative f', find f — is also interesting and important. It is the problem studied in integral calculus and is generally more difficult to solve than the problem of finding a derivative. We'll take a preliminary look at this problem in this part.

Antiderivatives:

An antiderivative of a function f on the interval I is another function F satisfying $F'(x) = f(x) \ \forall x \in I$.

ex. $F(x) = x$ is an antiderivative of the fnc $f(x) = 1$ on any interval because $F'(x) = f(x) = 1$ everywhere.

The Indefinite Integral:

The indefinite integral of $f(x)$ on interval I is

$$\int f(x)\,dx = F(x) + C \quad \text{on } I,$$

provided $F'(x) = f(x) \quad \forall x \in I$.

A few simple antiderivatives based on the known derivatives of elementary functions;

$$\int dx = \int 1\,dx = x + C \qquad \int x^2\,dx = \frac{x^3}{3} + C$$

$$\int x\,dx = \frac{x^2}{2} + C \qquad \int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

$$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$