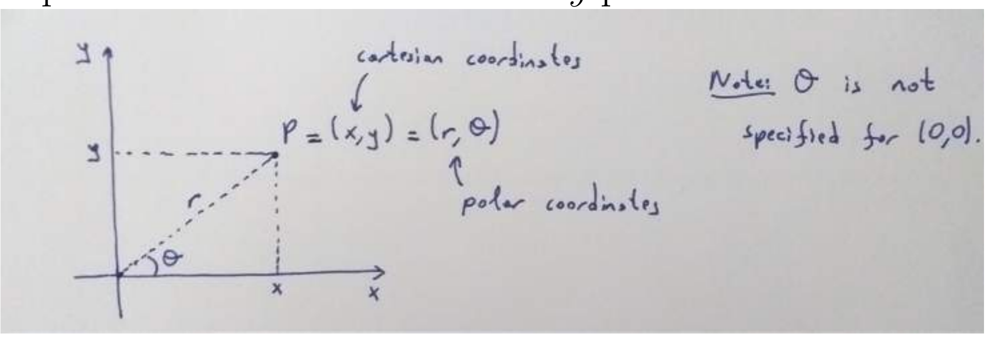
§14.4 Double Integrals in Polar Coordinates

Iteration of some double integrals in cartesian coordinates may be extremely complicated. For example consider the solid below the paraboloid $z = x^2 + y^2$ above D be the unit disc, $D : x^2 + y^2 \leq 1$. The integral that gives the volume of this solid is

$$\iint_D (x^2 + y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$$

which requires considerable effort. But we can ease the computation of the integral by introducing a new *coordinate system* called *polar coordinates.*

A point P can be located on the xy-plane in two different ways;



We may give the x and y coordinates of the point and write $P = (x, y)$, or we may give the distance (denoted by r) of P to the origin and the angle (denoted by θ) made by the positive x-axis of the line passing through the origin and P.

So in the above figure we have that P is represented by (x, y) in Cartesian coordinates and by (r, θ) in polar coordinates. These coordinates are related by

$$r^2 = x^2 + y^2, \; x = r\cos\theta, \; y = r\sin\theta$$

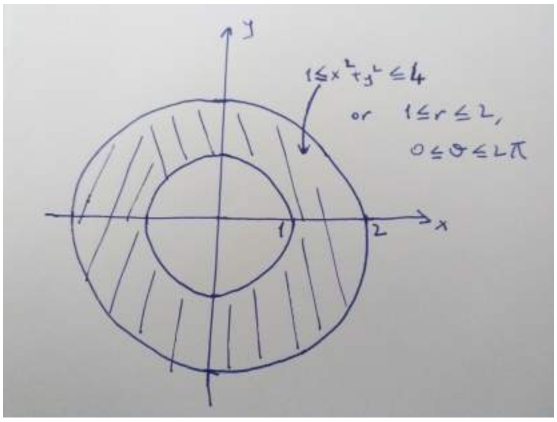
where $\theta \in [0, 2\pi]$ and $r \geq 0$.

Example:

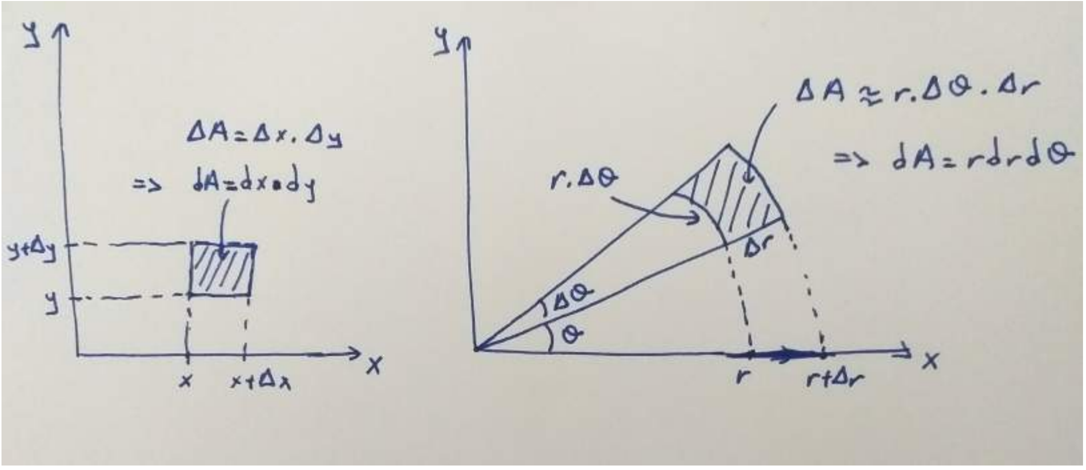
i) $P = (x, y) = (-1, \sqrt{3}) \implies P = (r, \theta) = (2, 2\pi/3)$

ii) $D : x^2 + y^2 = 1, \; y \geq 0 \implies D : r = 1, 0 \leq \theta \leq \pi$

iii) $D : 1 \leq x^2 + y^2 \leq 4 \implies D : 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi.$



Let's see how we can write double integrals in polar coordinates. Compare the form of area elements in cartesian and polar coordinates.



So the area element dA can be expressed as $dA = dx\,dy$ in rectangular coordinates and $dA = r\,dr\,d\theta$ in polar coordinates. Accordingly we have that

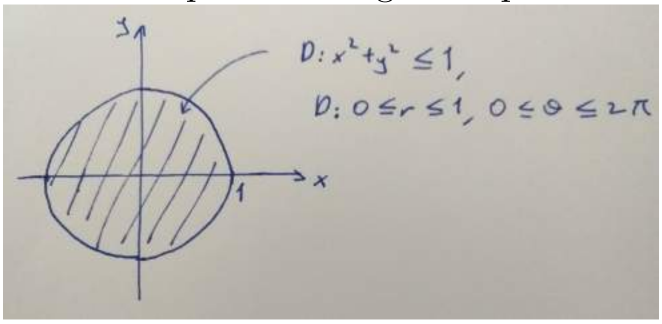
$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Remark: We don't need to sketch regions on *polar ($r\theta$)-plane.* The polar coordinates can be read from the sketch of the region on the xy-plane. Be careful not to forget the additional factor r (!)

Example 1: Find the volume of the solid below the paraboloid $z = x^2 + y^2$ and above $D : x^2 + y^2 \leq 1$.

Solution: We need to evaluate $\iint\limits_D (x^2+y^2)\,dA$. But as we have seen at the beginning of this section evaluating this integral in cartesian coordinates requires considerable effort. Let's try to evaluate in polar coordinates.

First we express the region in polar coordinates.



So we have that $D : 0 \le r \le 1, 0 \le \theta \le 2\pi$.

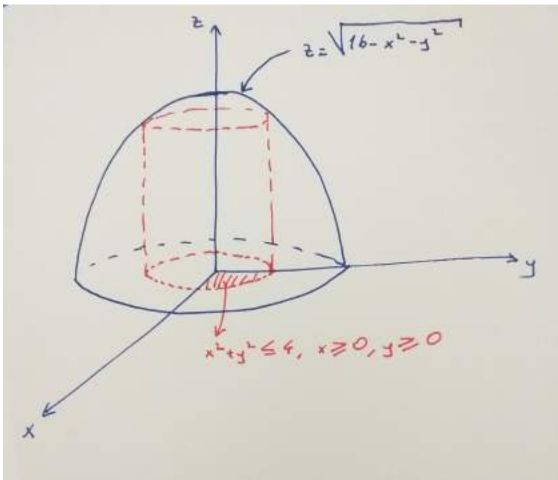
By the transformations $x = r\cos\theta$, $y = r\sin\theta$, $dA = dx\,dy = r\,dr\,d\theta$ we have

$$\iint\limits_D (x^2+y^2)\,dA = \int\limits_0^{2\pi}\int\limits_0^1 r^3\,dr\,d\theta = \int\limits_0^{2\pi}\left(\frac{r^4}{4}\right)\Big|_0^1 d\theta = \pi/2$$

Example 2: Find the volume of the solid lying inside both the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 16$ in the first octant.

Solution:

We start by sketching the region (as we must always do). The solid lies below $z = \sqrt{16 - x^2 - y^2}$ and above $D : x^2 + y^2 \leq 4, x \geq 0, y \geq 0$. In polar coordinates D is given by
$0 \leq r \leq 2, 0 \leq \theta \leq \pi/2$.
So the volume of this solid is equal to



$$\iint_D \sqrt{16 - x^2 - y^2}\, dA = \int_0^{\pi/2}\int_0^2 \sqrt{16 - r^2}\ r\, dr\, d\theta = \text{... Exercise ...}$$

$$= \frac{\pi(16^{3/2} - 12^{3/2})}{6}$$

(Hint: Use the substitution $u = 16 - r^2$ to evaluate the inner integral).

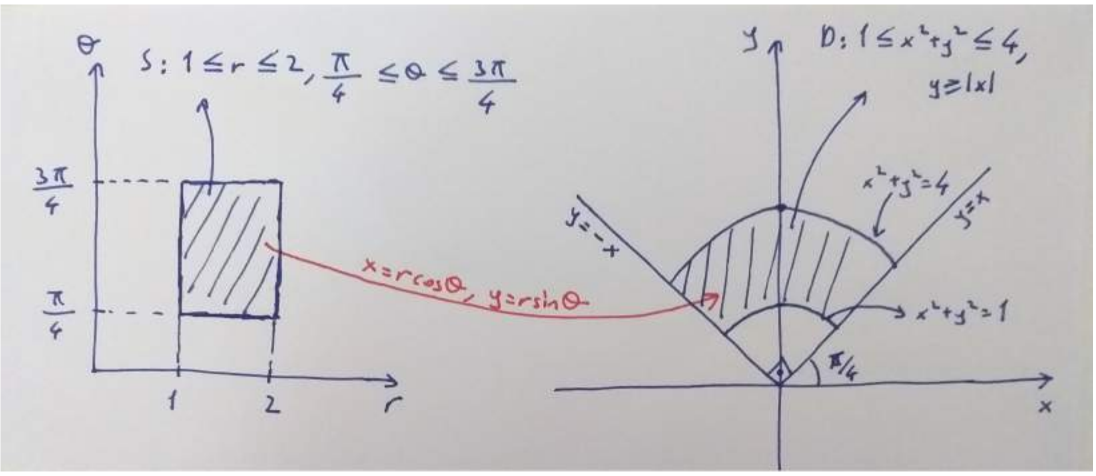
Change of Variables in Double Integrals - The Jacobian Determinant

The transformation of a double integral in cartesian coordinates to polar coordinates is a special case of change of variables in double integrals. Suppose that x and y are substituted by some expressions in u and v;

$$x = x(u, v), \;\; y = y(u, v)$$

(For example to transform to polar coordinates we use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$). We may see the change of variables $x = x(u, v)$, $y = y(u, v)$ as a *transformation* from uv-plane to xy-plane.

For instance consider the domain $S : 1 \leq r \leq 2, \pi/4 \leq \theta \leq 3\pi/4$ on the $r\theta$-plane. S is indeed a (polar) *rectangle* on the $r\theta$-plane.



The transformations $x = r\cos\theta$ and $y = r\sin\theta$ transform S to the domain D lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and the lines $y = x$ and $y = -x$. In particular this transformation is *one-to-one* from S *onto* D, i.e. any point in D is the image of a unique point in S.

In general if we have such a transformation from a domain S on the uv-plane onto a domain D on the xy-plane, then we may apply the change of variables $x = x(u, v)$, $y = y(u, v)$ in the double integral

$$\iint_D f(x, y)\, dA.$$

We must also modify the area element $dA = dx\, dy$ and express in terms of du and dv. We give it and summarize the change of variables rule in the next theorem whose proof is omitted.

Theorem 1. Let $x = x(u, v)$, $y = y(u, v)$ be a one-to-one transformation from a domain S on the uv-plane onto a domain D on the xy-plane. Suppose that $x(u, v)$ and $y(u, v)$ have continuous first order partial derivatives on S. Then

$$dA = dx\, dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

where $\dfrac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian (determinant) of the transformation $x = x(u,v)$, $y = y(u,v)$, and so

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Recall that the Jacobian of the transformation $x = x(u,v)$, $y = y(u,v)$ is defined as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Example: The Jacobian of the transformation

$x = r\cos\theta$ and $y = r\sin\theta$ is

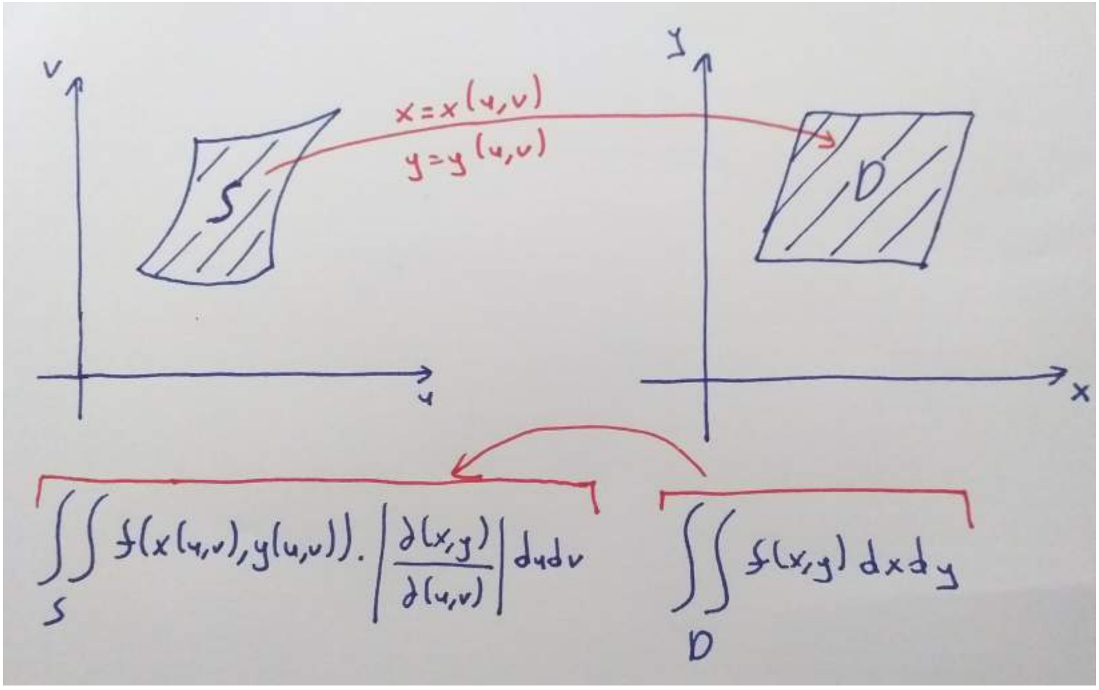
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

So $dA = dx\,dy = r\,dr\,d\theta$.

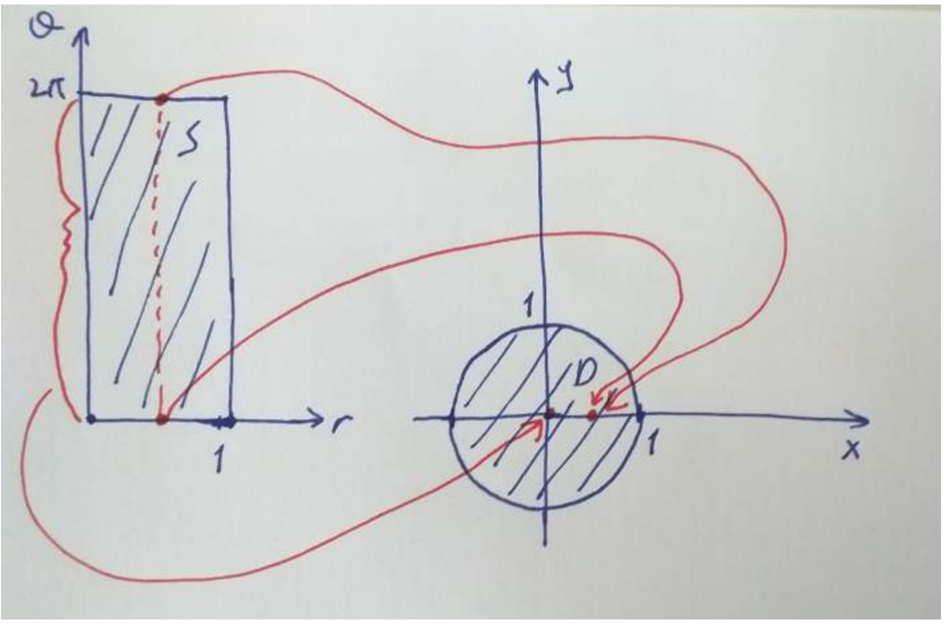
Remarks:

i) The term $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$ is the absolute value of the Jacobian of the transformation $x = x(u, v)$, $y = y(u, v)$.

ii) The transformation $x = x(u, v)$, $y = y(u, v)$ transforms the region S on the uv-plane to the region D on the xy-plane. But in terms of the change of variables, we transform an integral with respect to $dx\,dy$ to an integral with respect to $du\,dv$.



iii) The theorem still holds if the transformation $x = x(u, v)$, $y = y(u, v)$ is not one-to-one on the boundary of S.



For example the domain $S : 0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ on the $r\theta$-plane is mapped onto $D : x^2 + y^2 \leq 1$ on the xy-plane under the transformation $x = r\cos\theta$, $y = r\sin\theta$. The boundary of S is the *rectangle* with sides $r = 0$, $r = 1, \theta = 0, \theta = 2\pi$. The side $r = 0$ is mapped to the origin, and also any two points $(r, 0)$ and $(r, 2\pi)$ are mapped to the same point. But this is not a problem for the double integrals (What is the value of the integral on the boundary?).

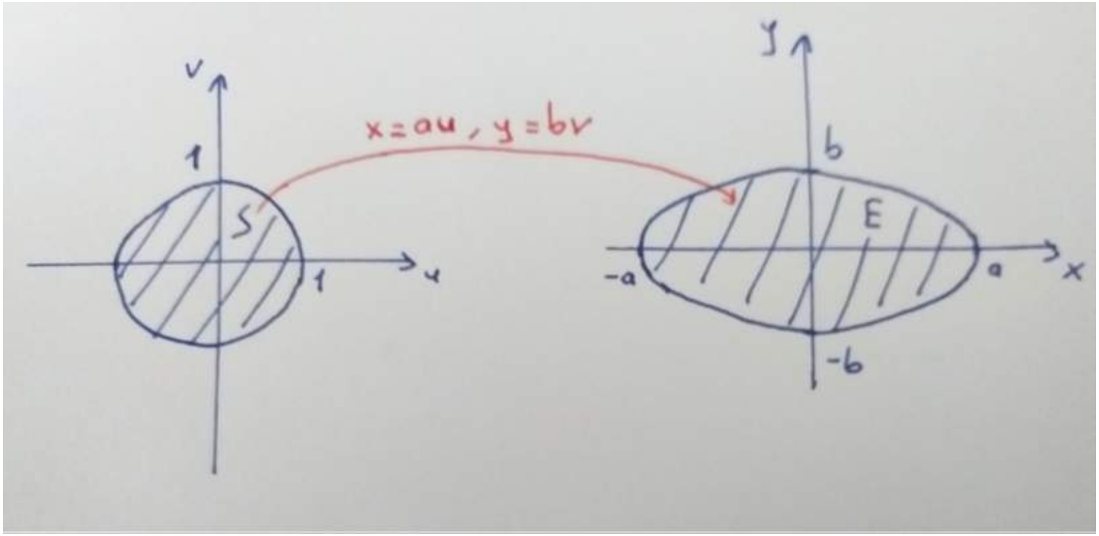
iv) The main aim to use transformations for double integrals is to simplify the domain. In general it will be helpful to look for a transformation which transforms the domain to a simpler domain such as rectangles, discs, etc.

Example 3: Find the area of the elliptic disc $E : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$ for some $a, b > 0$.

Solution: The area in cartesian coordinates is given by

$$\iint_E 1 dA = \int_{-b}^{b} \int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}} 1\, dx\, dy.$$

But consider the transformation $x = au$, $y = bv$ which transforms the unit disc $S : u^2 + v^2 \leq 1$ on the uv-plane onto the elliptic disc E.



The Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

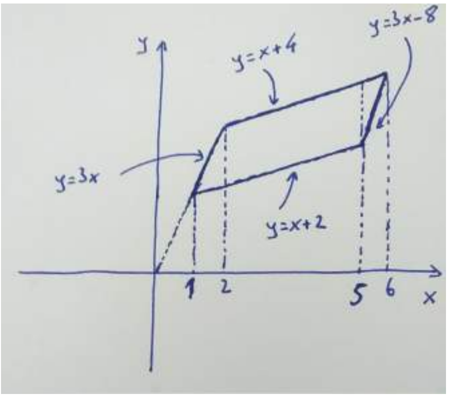
So we have that

$$\text{Area of } E = \iint_E 1\, dx\, dy = \iint_S |ab|\, du\, dv$$
$$= |ab|.(\text{Area of } S) = \pi ab$$

Example 4: Evaluate the double integral of $f(x,y) = 9x^2 - y^2$ over the parallelogram determined by the lines $y = 3x$, $y = x + 2$, $y = 3x - 8$ and $y = x + 4$.

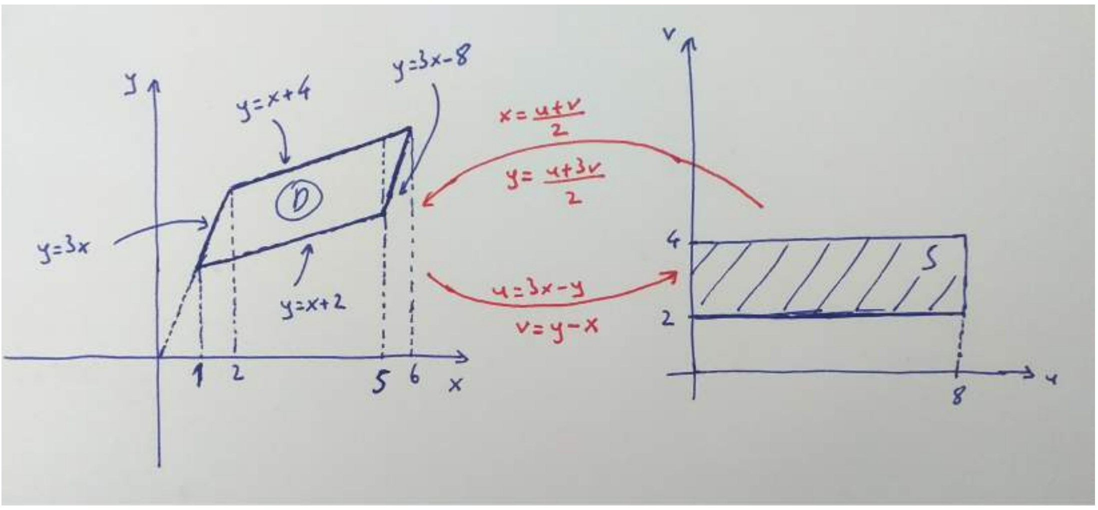
Solution:

First we sketch the domain. It is possible to evaluate this integral in cartesian coordinates. But in this case we need to seperate our region into three parts as



shown.

However it will be easier if we can transform the domain of integration to a rectangle. We can express our domain as $0 \le 3x - y \le 8$, $2 \le y - x \le 4$, so the substitution $u = 3x - y$, $v = y - x$ simplifies the domain to a rectangle. In other words we take the transformation $x = (u + v)/2$, $y = (u + 3v)/2$ (which is equiavalent to $u = 3x - y$, $v = y - x$) which transforms the rectangle $S : 0 \le u \le 8$, $2 \le v \le 4$ on the uv-plane to D.



The Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{vmatrix} = 1/2$$

Then we have

$$\iint\limits_D f(x,y)\,dA = \int\limits_2^4 \int\limits_0^8 f\left(\frac{u+v}{2}, \frac{u+3v}{2}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv =$$

$$= \frac{1}{2}\int\limits_2^4 \int\limits_0^8 (2u^2 + 3uv)\,du\,dv = \text{..... Exercise ......}$$