§9.2 Infinite Series

Let $\{a_n\}$ be a sequence. The form $a_1 + a_2 + ... + a_n + ...$ is called series and it is denoted by this notation $\sum_{n=1}^{\infty} a_n$, that is,

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$

We may not obtain the sum of the series $\sum_{n=1}^{\infty} a_n$. But we can calculate the partial sums of the series $\sum_{n=1}^{\infty} a_n$.

The partial sums of the series $\sum_{n=1}^{\infty} a_n$ are of :

$S_1 = a_1$ (only the first term)

$S_2 = a_1 + a_2$ (the sum of the first two terms)

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$S_n = a_1 + a_2 + ... + a_n$ (the sum of the first n terms)

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The sequence S_n is called the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$, constituted by the numbers $S_1, S_2, ..., S_n, ...$

Definition. The series $\sum_{n=1}^{\infty} a_n$ converges to the sum S if $\lim_{n\to\infty} S_n = S$, that is, the sequence of its partial sums converges to S.

We have that $\sum_{n=1}^{\infty} a_n = S$.

We say that a series converges if and only if the sequence of its partial sums converges.

In a similar manner, we get that if the sequence $\{S_n\} = \{\sum_{i=1}^{n} a_i\}$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Geometric Series

Let $a, r \in \mathbb{R}$ and $b_n = ar^{n-1}$. The form $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^{n-1} + ...$ is called geometric series. The number r is called the common ratio of the series: r is obtained by dividing two consecutive terms, that is:

$r = \frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}}$ for each $n = 1, 2, ...$

Now, we will determine whether a geometric series converges or diverges. Consider the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$.

nth partial sum of the series is:

$S_n = a + ar + ar^2 + ... + ar^{n-1}$

$rS_n = ar + ar^2 + ... + ar^n$.

When we calculate $rS_n - S_n$, we get that:

$S_n = \frac{a(1-r^n)}{1-r}$.

If $a = 0$, then $S_n = 0$ and so $\lim_{n\to\infty} S_n = 0$.

Let $a \neq 0$.

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Thus, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$.

If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$. Thus, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \infty$ if $a > 0$; and

$\lim_{n\to\infty} r^n = \infty$. Thus, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = -\infty$ if $a < 0$.

Ler $r = 1$. Then $S_n = a + a + a + ... = na$, therefore;

$\lim_{n\to\infty} S_n = \infty$ if $a > 0$;

$\lim_{n\to\infty} S_n = -\infty$ if $a < 0$.

Let $r = -1$.

$$S_n = a - a + a - a + ... \begin{cases} 0 \; if \; n \; is \; even \\ a \; if \; n \; is \; odd. \end{cases},$$

thus $\lim_{n\to\infty} S_n$ does not exist.

If $r < -1$, then $\lim_{n\to\infty} r^n$ does not exist. Hence, $\lim_{n\to\infty} S_n$ does not exist. We can compile such that :

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} converges \; to \; 0 & if \; a = 0 \\ converges \; to \; \frac{a}{1-r} & if \; |r| < 1 \\ diverges \; to \; \infty & if \; r \geq 1 \; a > 0 \\ diverges \; to \; -\infty & if \; r \geq 1 \; a < 0 \\ diverges & if \; r \leq -1. \end{cases}$$

Example 1. The geometric series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}5}{4^{n-1}} = 5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + ...$ converges to $\frac{5}{1-(-\frac{1}{4})} = 4$ since $|r| = |-\frac{1}{4}| < 1$.

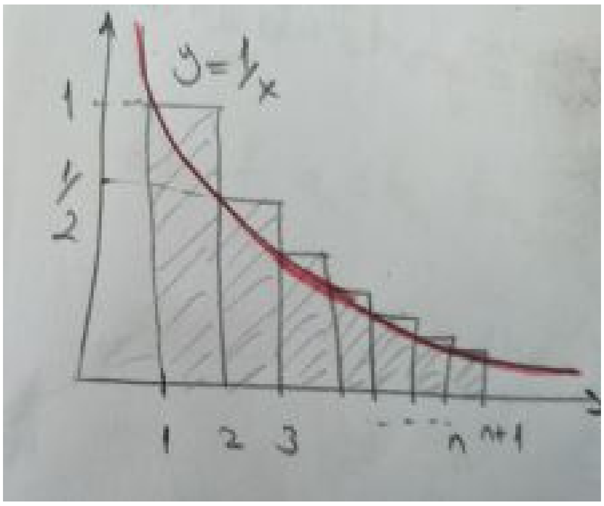
Example 2. The sequence $\sum_{n=1}^{\infty} \sqrt{3}^{n-1}$ diverges to ∞ since $|r| =$

$\sqrt{3} > 1$ and $a = 1 > 0$.

Example 3. (**Telescoping Series**) Let's show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ...$ converges and find its sum.
General term: $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.
nth partial sum : $S_n = a_1 + a_2 + ... + a_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + ... + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$.
Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty}(1 - \frac{1}{n+1}) = 1$. Thus, the series converges to 1; $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

This type series is called telescoping since the partial sums are converted to a simple form when the terms are expanded in partial fractions.

Example 4. (**Harmonic Series**) Show that $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ...$ diverges to ∞.
Note that the sequence of its partial sums $\{S_n\}$. The general term is:
$S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$.



In this image, we obtain that:

The sum of areas of rectangles is greater than the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = n + 1$. Also, note that:

S_n=The sum of areas of rectangles in the image, then $S_n > \int_1^{n+1} \frac{dx}{x} = ln(n + 1)$.

$\lim_{n\to\infty} S_n > \lim_{n\to\infty} ln(n + 1) = \infty$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞.

Theorem 1. (**nth-term test for divergence**) If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Then, we say that if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

But the converse of the previous theorem may not be true, that is, when $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may not be convergent. For this case, consider $\sum_{n=1}^{\infty} \frac{1}{n}$. We know that it is divergent but

$\lim_{n\to\infty} \frac{1}{n} = 0$.

As a result, if $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may or may not be convergent.

Example 5. The series $\sum_{n=1}^{\infty} \frac{3+2^n}{2^{n+2}}$ diverges since $\lim_{n\to\infty} \frac{3+2^n}{2^{n+2}} = \frac{1}{4} \neq 0$.

Theorem 2. A series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for any integer $N \geq 1$.

By the theorem, we understand that to drop from a series $\sum_{n=1}^{\infty} a_n$ any finite number of terms does not affect to convergence. But it affects the actual sum of the series since the sum of the series corresponds to all the terms.

In a similar way, we can obtain that a series $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\sum_{n=N}^{\infty} a_n$ diverges for any integer $N \geq 1$.

Theorem 3. Let $\{a_n\}$ be a (ultimately) positive sequence.

If $\{S_n\}$ is bounded above, then the series $\sum_{n=1}^{\infty} a_n$ converges.

If $\{S_n\}$ is not bounded above, then the series $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

Theorem 4. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to A and B, respectively. Then;

1. The series $\sum_{n=1}^{\infty} ca_n$ converges to cA (c is any constant).

2. The series $\sum_{n=1}^{\infty}(a_n + b_n)$, $\sum_{n=1}^{\infty}(a_n - b_n)$ converge to $A + B$, $A - B$, respectively.

3. If $a_n \leq b_n$ for each $n \in \mathbb{N}^+$, then $A \leq B$.

Example 6. Find the sum of the series $\sum_{n=1}^{\infty} \frac{3+2^n}{5^{n+1}}$.

$\sum_{n=1}^{\infty} \frac{3+2^n}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{3}{5^{n+1}} + \sum_{n=1}^{\infty} \frac{2^n}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{3}{25}(\frac{1}{5})^{n-1} + \sum_{n=1}^{\infty} \frac{2}{25}(\frac{2}{5})^{n-1} = \frac{\frac{3}{25}}{1-\frac{1}{5}} + \frac{\frac{2}{25}}{1-\frac{2}{5}} = \frac{17}{300}$