MATH 101.2 PS-9

Q1) Expand the sum $\sum_{j=3}^{n} \frac{(-2)^j}{(j-2)^2}$.

Sol: We know that $\sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$. Thus,

$$\sum_{j=3}^{n} \frac{(-2)^j}{(j-2)^2} = \frac{(-2)^3}{1^2} + \frac{(-2)^4}{2^2} + \dots + \frac{(-2)^n}{(n-2)^2}$$

$$= -2^3 + \frac{2^4}{2^2} + \dots + \frac{(-1)^n \cdot 2^n}{(n-2)^2}.$$

Q2) Write the sum $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots + \frac{(-1)^{n-1}}{n^2}$

using sigma notation.

Sol: Clearly, $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots + \frac{(-1)^{n-1}}{n^2} = \sum_{k=a}^{n} \frac{(-1)^{k-1}}{k^2}$

we must find a. For $k=a$, $1 = \frac{(-1)^{a-1}}{a^2} \Rightarrow a^2 = (-1)^{a-1} \Rightarrow a = 1$.

Hence, $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots + \frac{(-1)^{n-1}}{n^2} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^2}$.

Q3) Find closed form value for the sum $\sum_{i=1}^{n} (2^i - i^2)$.

Sol: $\sum_{i=1}^{n} (2^i - i^2) = \sum_{i=1}^{n} 2^i - \sum_{i=1}^{n} i^2 = (2 + 2^2 + \dots + 2^n) - (1^2 + 2^2 + \dots + n^2)$

$$= (1 + 2^2 + \dots + 2^n - 1) - (1^2 + 2^2 + \dots + n^2)$$

$$= \frac{2^{n+1} - 1}{2 - 1} - 1 - \frac{n(n+1)(2n+1)}{6}$$

$$= 2^{n+1} - 2 - \frac{n(n+1)(2n+1)}{6}$$ //

Q4) $\sum_{i=m}^{2m} \left(\frac{1}{i} - \frac{1}{i+1}\right) = ?$

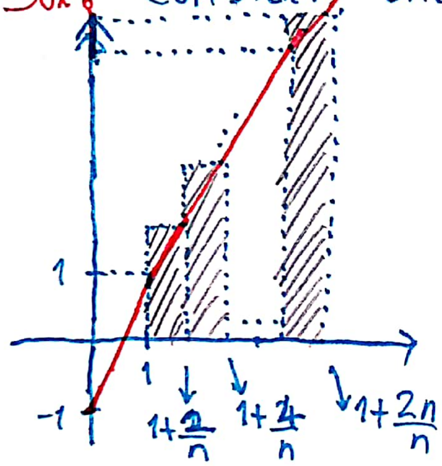
Sol: $\sum_{i=m}^{2m} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+1} - \frac{1}{m+2} + \dots + \frac{1}{2m} - \frac{1}{2m+1}$

$$= \frac{1}{m} - \frac{1}{2m+1} = \frac{m+1}{m(2m+1)}.$$

Q5) Use the techniques with subintervals of equal length to find the areas of the regions specified in the exercises below.

a) Below $y = 2x-1$, above $y=0$, from $x=1$ to $x=3$.

Sol: Consider the figure below.



The area is the limit of the sum of the areas of the rectangles shown in the figure. It is

$$A = \lim_{n\to\infty} \frac{1}{n}\left[2\left(1+\frac{2}{n}\right)-1 + \dots + 2\left(1+\frac{2n}{n}\right)-1\right]$$

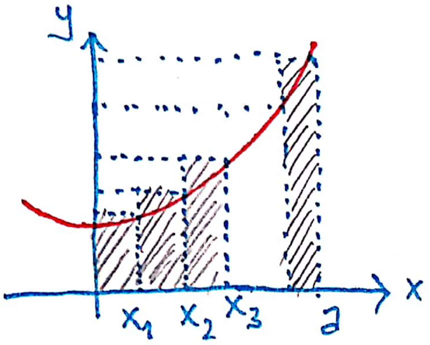
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2+2.\frac{2i}{n}-1\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n} + \frac{8i}{n^2} - \frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{n}.n + \frac{8}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \left(2 + \frac{8}{n^2}\,\frac{n(n+1)}{2}\right) = 6 //$$

b) Below $y=x^2+1$, above $y=0$, from $x=0$ to $x=a>0$.

Sol: Divide $[0,a]$ into n equal subintervals of length $\Delta x = \frac{a}{n}$ by points $x_i = \frac{ia}{n}$, $(0 \le i \le n)$.



Then, the area of the region

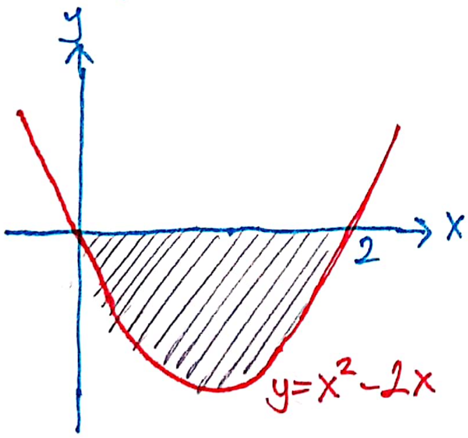
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a}{n}\left[\left(\frac{ia}{n}\right)^2 + 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{a^3}{n^3}\sum_{i=1}^{n} i^2 + \frac{a}{n}\sum_{i=1}^{n} 1\right]$$

Hence, $$A = \lim_{n\to\infty}\left[\frac{a^3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{a}{n}\cdot n\right] = \frac{a^3}{3} + a.$$

c) Above $y=x^2-2x$, below $y=0$.

Sol: The height of the region at position x is $0-(x^2-2x) = 2x-x^2$. The "base" is an interval of length 2, so we approximate using n rectangles of width $2/n$. The shaded area is



$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2\frac{2i}{n} - \frac{4i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{8i}{n^2} - \frac{8i^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left[\frac{8}{n^2}\,\frac{n(n+1)}{2} - \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 4 - \frac{8}{3} = \frac{4}{3} \quad //$$

Q6) In the below exercises, calculate $L(f,P_n)$ and $U(f,P_n)$ for the given function f over the given interval $[a,b]$, where P_n is the partition of the interval into n subintervals of equal length $\Delta x = (b-a)/n$. Show that $\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n)$. Hence f is integrable on $[a,b]$. (Why?) What is $\int_a^b f(x)\,dx$.

a) $f(x) = x$, $[a,b] = [0,1]$ b) $f(x) = e^x$, $[a,b] = [0,3]$.

Sol: a) $f(x) = x$ on $[0,1]$. $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1\}$.

We have $L(f,P_n) = \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n}\right) = \frac{1}{n^2}(1+2+\dots+n-1)$.

Hence, $L(f,P_n) = \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2n}$. Also,

$U(f,P_n) = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \dots + \frac{n}{n}\right) = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n^2}$. Hence,

$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n) = \frac{1}{2}$. If P is any partition of $[0,1]$, then, $L(f,P) \le U(f,P_n) = \frac{n+1}{2n}$ for every n, so $L(f,P) \le \lim_{n\to\infty} U(f,P_n) = 1/2$. Similarly, $U(f,P_n) \geq 1/2$.

If there exists any number I such that $L(f,P) \le I \le U(f,P)$ for all P, then I cannot be lees than $1/2$ and similarly I cannot be greater than $1/2$. Thus $I = \frac{1}{2}$ and $\int_0^1 x\,dx = \frac{1}{2}$.

b) $f(x)=e^x$ on $[0,3]$. $P_n=\{0, \frac{3}{n}, \frac{6}{n}, \dots, \frac{3n-3}{n}, \frac{3n}{n}\}$.

We have (using the result of a))

$$L(f,P_n)=\frac{3}{n}\left(e^{0/n}+e^{3/n}+\dots+e^{3(n-1)/n}\right)=\frac{3}{n}\,\frac{e^{3n/n}-1}{e^{3/n}-1}=\frac{3(e^3-1)}{n(e^{3/n}-1)}.$$

$$U(f,P_n)=\frac{3}{n}\left(e^{3/n}+e^{6/n}+\dots+e^{3n/n}\right)=e^{3/n}\cdot L(f,P_n).$$

By L'Hospital's Rule,

$$\lim_{n\to\infty} n(e^{3/n}-1)=\lim_{n\to\infty}\frac{e^{3/n}-1}{1/n}=\lim_{n\to\infty}\frac{e^{3/n}(-3/n^2)}{-1/n^2}$$

$$=\lim_{n\to\infty}\frac{3\cdot e^{3/n}}{1}=3.$$

Thus, $\lim\limits_{n\to\infty} L(f,P_n)=\lim\limits_{n\to\infty} U(f,P_n)=e^3-1=\int_0^3 e^x\,dx.$

Q7) Express the given sums as a definite integral.

a) $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}}$ b) $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\frac{\pi i}{n}\right)$ c) $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2}$

Sol: a) We have $\frac{1}{n}$, so $b-a=1$. If we choose $a=0$, then $b=0$. $a+i\Delta x=0+i\cdot\frac{1}{n}=\frac{i}{n}$.

Therefore, $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}}=\int_0^1\sqrt{x}\,dx.$

b) We have $\frac{\pi}{n}$, so $b-a=\pi$. If we choose $a=0$, then $b=\pi$. Also, $a+i\cdot\Delta x=0+i\cdot\frac{\pi}{n}=\frac{i\pi}{n}$.

Therefore, $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\frac{\pi i}{n}\right)=\int_0^{\pi}\sin x\,dx.$

c) We must rearrenge given limit as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{1}{1+(i/n)^2}.$$

We have $\frac{1}{n}$, so $b-a=1$. If we choose $a=0$, then $b=1$. Also, $a+i.\Delta x = 0+i.\frac{1}{n} = \frac{1}{n}$.

Hence, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} = \int_0^1 \frac{dx}{1+x^2}$