1) a

$$\int_0^{\pi/2} \left(\sin(2x)\sqrt{\sin^2 x + 1}\right) dx$$

$$= \int_0^{\pi/2} \left(2\sin x\cos x\sqrt{\sin^2 x + 1}\right) dx$$

$\sin x = u$

$\cos x\,dx = du$

$\sin 0 = 0$

$\sin \pi/2 = 1$

$$= \int_0^{1} \left(2u\sqrt{u^2+1}\right) du$$

$u^2+1 = y$

$2u\,du = dy$

$$= \int_1^{2} y^{1/2}\,dy$$

$$= \left(\frac{2}{3} y^{3/2}\right)_1^2 = \frac{2}{3}\left(2^{3/2} - 1\right)$$

b $I = \int_1^{e} \sqrt{x}\,\ln x\,dx$?

$\ln x = u$

$\frac{1}{x}dx = du$

$x^{1/2}dx = dv$

$v = \frac{2}{3}x^{3/2}$

$$I = \left(\ln x\,\frac{2}{3}x^{3/2}\right)_1^e - \int_1^e \frac{2}{3}x^{3/2}\,\frac{1}{x}\,dx$$

$$= \frac{2}{3}\left[\ln e \cdot e^{3/2} - 1 \ln(1)\right] - \frac{2}{3}\int_1^e x^{1/2}dx$$

$$= \frac{2}{3}e^{3/2} - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right)_1^e = \frac{2}{3}e^{3/2} - \frac{4}{9}\left(e^{3/2} - 1\right)$$

(c) $\int \frac{x}{x^2+3x+2}\,dx$

$$\frac{x}{x^2+3x+2} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

$$\frac{x}{x^2+3x+2} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$$

$$\left.\begin{aligned} 1 &= A+B \\ 0 &= A+2B \end{aligned}\right\} \quad \boxed{\begin{aligned} B &= -1 \\ A &= 2 \end{aligned}}$$

$$I = 2\int \frac{1}{x+2}\,dx + (-1)\int \frac{1}{(x+1)}\,dx$$

$$I = 2\ln|x+2| - \ln|x+1| + C \quad //$$

(d) $\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$ $\qquad \ln x = u$, $\quad \frac{1}{x}\,dx = du$

$$= \int_0^1 u^{-1/2}\,du = 2\left(\sqrt{u}\right)_0^1 = 2 \;//$$

(e) $\displaystyle\int_0^{\infty} x e^{-x}\,dx$ $\qquad x = u \qquad dv = e^{-x}dx$

$\qquad dx = du \qquad v = -e^{-x}$

$$= \lim_{R\to\infty}\Big[x(-e^{-x})\Big]_0^R - \int_0^{\infty}(-e^{-x})\,dx$$

$$= \lim_{R\to\infty}\left(\frac{-x}{e^{x}}\right)_0^R + \lim_{R\to\infty}\cancel{\int}\left(-e^{-x}\right)_0^R$$

$$= \underbrace{\lim_{R\to\infty}\left(-\frac{R}{e^{R}}-(0)\right)}_{\text{L'Hopital's Rule}} + \lim_{R\to\infty}\left(\cancel{-e^{-R}}^{\,0}-(-e^{0})\right)$$

$$= \lim_{R\to\infty} -\frac{1}{e^{R}} + 1 = 1 \;//$$

(2) (Comparison) Determine if the following integral is convergent or not.

$$\int_1^{\infty}\frac{x-1}{\sqrt{x^5+2}}\,dx$$

$$\frac{x-1}{\sqrt{x^5+2}} < \frac{x}{\sqrt{x^5+2}} < \frac{x}{\sqrt{x^5}} = \frac{1}{x^{3/2}}$$

Since $\displaystyle\int_1^{\infty}\frac{1}{x^{3/2}}\,dx$ $\left(\begin{matrix} p=\frac{3}{2}>1 \\ \text{Convergent} \\ \text{(p-integral)}\end{matrix}\right)$ convergent

by Comparison test $\displaystyle\int_1^{\infty}\frac{x-1}{\sqrt{x^5+2}}\,dx$ is also convergent.

③ (Definition of integral) Evaluate the limit

$$\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \frac{1}{1+(2k/n)^2}$$ by expressing it as

a suitable definite integral.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

$$\Delta x = \frac{b-a}{n} = \frac{2}{n} \Rightarrow b=2 \quad a=0$$

$$f(c_k) = \frac{1}{1+\left(\frac{2}{n}\cdot k\right)^2} \Rightarrow f(x) = \frac{1}{1+x^2}$$

$$\left(c_k = 0 + \frac{2}{n}\cdot k = \frac{2}{n}\cdot k\right)$$

$$\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \frac{1}{1+\left(\frac{2k}{n}\right)^2} = \int_0^2 \frac{1}{1+x^2}dx$$

$$= (\arctan x)\Big|_0^2$$

$$= \tan^{-1}(2)$$ //

④ (Fund. Thm. of Calculus)

Find $F'(x)$ if $F(x) = \int_{x}^{e^x+1} \left(\ln(t-1)\right)^{2018} dt$

$$\boxed{\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x)}$$

$$= \underbrace{\left(\ln\left(e^x+1-1\right)\right)^{2018}}\left(e^x+1\right)' - \left(\ln(x-1)\right)^{2018}\cdot x'$$

$$= x^{2018}\cdot\left(e^x\right) - \left(\ln(x-1)\right)^{2018}$$

⑥ (Max-Min Problem) Find all points (x,y) satisfying $y^2-x^2=1$ and $0\le x\le 3$ that are furthest away from the point $(2,0)$.

(6) $d(x) = \sqrt{(x-2)^2 + \left(\sqrt{1+x^2}\right)^2} = \sqrt{(x-2)^2 + (1+x^2)}$

$(x,y) \begin{cases} \left(x, \sqrt{1+x^2}\right) \\ \left(x, -\sqrt{1+x^2}\right) \end{cases}$ for both cases the distance function

$d(x) = \sqrt{2x^2 - 4x + 5}\ ; \ 0 \le x \le 3.$

$d'(x) = \dfrac{1}{2\sqrt{2x^2-4x+5}} \cdot (4x-4) = 0 \implies$

Critical Point: $\underset{\in [0,3]}{x=1} \implies d(1) = \sqrt{3}$ //

Endpoints: $d(0) = \sqrt{5}$ //

$d(3) = \sqrt{11}$ //

There is no singular point of $d(x)$.

Thus $(3, \sqrt{10})$ and $(3, -\sqrt{10})$ are the points we wanted.