THEOREM 5. Translation Property

Hypothesis. *Suppose f is such that $\mathscr{L}\{f\}$ exists for $s > \alpha$.*

Conclusion. *For any constant a,*

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a) \tag{14}$$

for $s > \alpha + a$, where $F(s)$ denotes $\mathscr{L}\{f(t)\}$.

Proof. $F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$. Replacing s by $s-a$, we have

$$F(s-a) = \int_0^\infty e^{-(s-a)t}f(t)\,dt = \int_0^\infty e^{-st}[e^{at}f(t)]\,dt = \mathscr{L}\{e^{at}f(t)\}.$$

Q.E.D

Example.

Find $\mathscr{L}\{e^{at}t\}$. We apply Theorem 5 with $f(t) = t$.

$$\mathscr{L}\{e^{at}t\} = F(s-a),$$

where $F(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{t\}$. By (3), $\mathscr{L}\{t\} = 1/s^2$ $(s > 0)$. That is, $F(s) = 1/s^2$ and so $F(s-a) = 1/(s-a)^2$. Thus

$$\mathscr{L}\{e^{at}t\} = \frac{1}{(s-a)^2} \qquad (s > a). \tag{15}$$

Example.

Find $\mathscr{L}\{e^{at}\sin bt\}$. We let $f(t) = \sin bt$. Then $\mathscr{L}\{e^{at}\sin bt\} = F(s-a)$, where

$$F(s) = \mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \qquad (s > 0).$$

Thus

$$F(s-a) = \frac{b}{(s-a)^2+b^2}$$

and so

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \qquad (s > a). \tag{16}$$

THEOREM 6.

Hypothesis. *Suppose f is a function satisfying the hypotheses of Theorem 1, with Laplace transform F, where*

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt. \tag{17}$$

Conclusion

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]. \tag{18}$$

Proof. Differentiate both sides of Equation (17) n times with respect to s. This differentiation is justified in the present case and yields

$$\begin{aligned} F'(s) &= (-1)^1 \int_0^{\infty} e^{-st} t f(t)\, dt, \\ F''(s) &= (-1)^2 \int_0^{\infty} e^{-st} t^2 f(t)\, dt, \\ &\vdots \\ F^{(n)}(s) &= (-1)^n \int_0^{\infty} e^{-st} t^n f(t)\, dt, \end{aligned}$$

from which the conclusion (18) is at once apparent. *Q.E.D*

Example.

Find $\mathscr{L}\{t^2 \sin bt\}$. By Theorem 6,

$$\mathscr{L}\{t^2 \sin bt\} = (-1)^2 \frac{d^2}{ds^2}[F(s)],$$

where

$$F(s) = \mathscr{L}\{\sin bt\} = \frac{b}{s^2 + b^2}$$

(using (5)). From this,

$$\frac{d}{ds}[F(s)] = -\frac{2bs}{(s^2 + b^2)^2}$$

and

$$\frac{d^2}{ds^2}[F(s)] = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}.$$

Thus,

$$\mathscr{L}\{t^2 \sin bt\} = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}.$$

In the application of the Laplace transform to certain differential equations problems, we shall need to find the transform of a function having one or more finite discontinuities. In dealing with these functions, we shall find the concept of the so-called unit step function to be very useful.

For each real number $a \geq 0$, the *unit step function* u_a is defined for nonnegative t by

$$u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t > a \end{cases} \tag{19}$$

(see Figure a). In particular, if $a = 0$, this formally becomes

$$u_0(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0; \end{cases}$$

but since we have defined u_a in (19) only for nonnegative t, this reduces to

$$u_0(t) = 1 \quad \text{for } t > 0 \tag{20}$$

(see Figure b).

The function u_a so defined satisfies the hypotheses of Theorem 1, so $\mathscr{L}\{u_a(t)\}$ exists. Using the definition of the Laplace transform, we find

$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st}u_a(t)\,dt = \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$
$$= 0 + \lim_{R\to\infty} \int_a^R e^{-st}\,dt = \lim_{R\to\infty} \left[\frac{-e^{-st}}{s}\right]_a^R$$
$$= \lim_{R\to\infty} \frac{-e^{-sR} + e^{-sa}}{s} = \frac{e^{-as}}{s} \quad \text{for } s > 0.$$

Thus we have

$$\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s} \quad (s > 0). \tag{21}$$

A variety of so-called *step functions* can be expressed as suitable linear combinations of the unit step function u_a. Then, using Theorem 2 (the linear property), and $\mathcal{L}\{u_a(t)\}$, we can readily obtain the Laplace transform of such step functions.

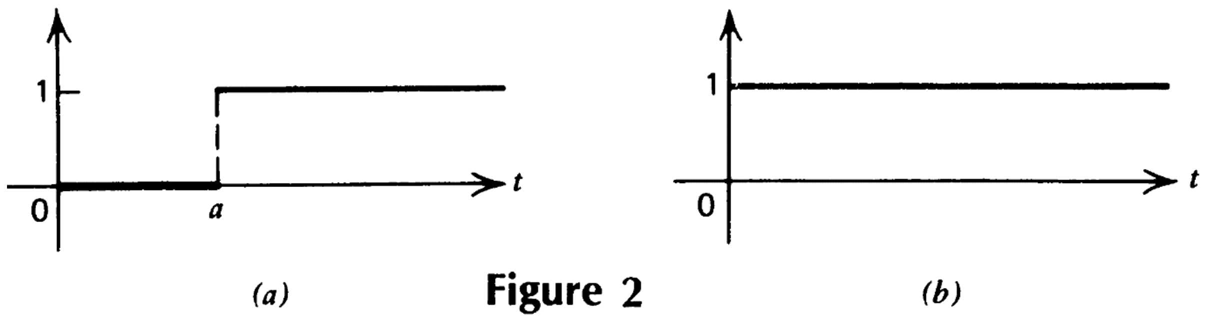

Figure 2

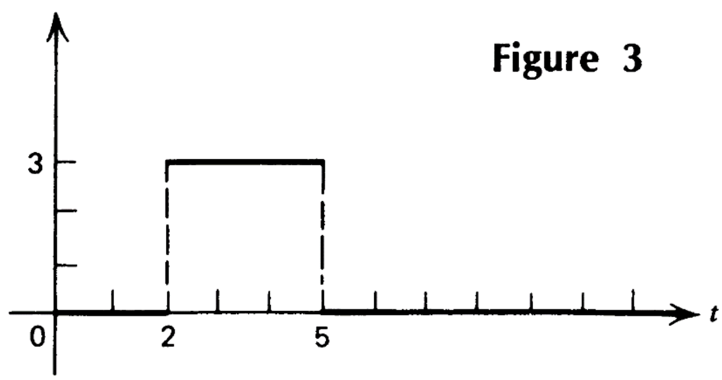

Figure 3

Example.

Consider the step function defined by

$$f(t) = \begin{cases} 0, & 0 < t < 2, \\ 3, & 2 < t < 5, \\ 0, & t > 5. \end{cases}$$

The graph of f is shown in Figure 3. We may express the values of f in the form

$$f(t) = \begin{cases} 0 - 0, & 0 < t < 2, \\ 3 - 0, & 2 < t < 5, \\ 3 - 3, & t > 5. \end{cases}$$

Hence we see that f is the function with values given by

$$\begin{cases} 0, & 0 < t < 2, \\ 3, & t > 2, \end{cases}$$

minus the function with values given by

$$\begin{cases} 0, & 0 < t < 5, \\ 3, & t > 5. \end{cases}$$

Thus $f(t)$ can be expressed as the linear combination

$$3u_2(t) - 3u_5(t)$$

of the unit step functions u_2 and u_5. Then using Theorem 2 and formula (21), we find

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{3u_2(t) - 3u_5(t)\} = \frac{3e^{-2s}}{s} - \frac{3e^{-5s}}{s} = \frac{3}{s}[e^{-2s} - e^{-5s}].$$

Another useful property of the unit step function in connection with Laplace transforms is concerned with the translation of a given function a given distance in the positive direction. Specifically, consider the function f with values $f(t)$ defined for $t > 0$ (see Figure a). Suppose we consider the new function that results from translating the given function f a distance of a units in the positive direction (that is, to the right) and then assigning the value 0 to the new function for $t < a$. Then this new function is defined by

$$\begin{cases} 0, & 0 < t < a, \\ f(t - a), & t > a \end{cases} \tag{22}$$

(see Figure 4b). Then since the unit step function u_a is defined by

$$u_a(t) = \begin{cases} 0, & 0 < t < a, \\ 1, & t > a, \end{cases}$$

we see that the function defined by (22) is $u_a(t)f(t - a)$. That is,

$$u_a(t)f(t - a) = \begin{cases} 0, & 0 < t < a, \\ f(t - a), & t > a \end{cases} \tag{23}$$

(note Figure 4b again).

Concerning the Laplace transform of this function we have the following theorem.

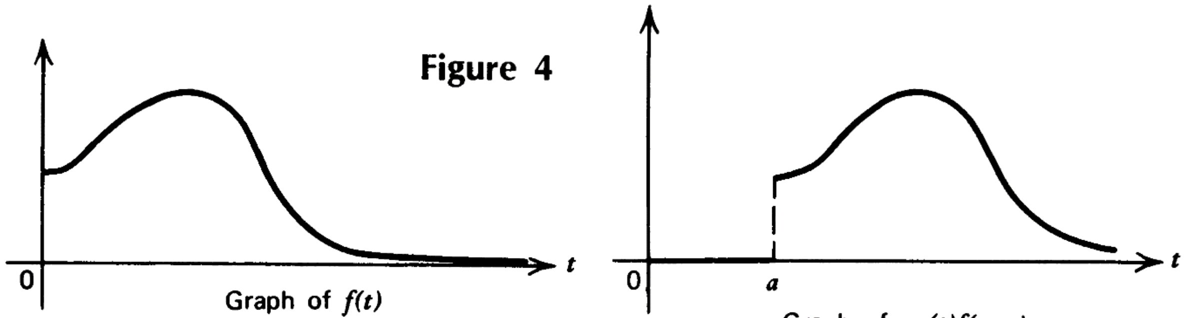

Figure 4

Graph of $f(t)$
(a)

Graph of $u_a(t)f(t-a)$
(b)

THEOREM 7.

Hypothesis. *Suppose f is a function satisfying the hypotheses of Theorem 1 with Laplace transform F so that*

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt;$$

and consider the translated function defined by

$$u_a(t)f(t - a) = \begin{cases} 0, & 0 < t < a, \\ f(t - a), & t > a. \end{cases} \tag{24}$$

Conclusion. *Then,*

$$\mathscr{L}\{u_a(t)f(t - a)\} = e^{-as}\mathscr{L}\{f(t)\}$$

that is, (25)

$$\mathscr{L}\{u_a(t)f(t - a)\} = e^{-as}F(s).$$

Proof

$$\begin{aligned}\mathscr{L}\{u_a(t)f(t-a)\} &= \int_0^\infty e^{-st}u_a(t)f(t-a)\,dt \\ &= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}f(t-a)\,dt \\ &= \int_0^\infty e^{-st}f(t-a)\,dt.\end{aligned}$$

Letting $t - a = \tau$, we obtain

$$\begin{aligned}\int_a^\infty e^{-st}f(t-a)\,dt &= \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau \\ &= e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = e^{-as}\mathscr{L}\{f(t)\}.\end{aligned}$$

Thus

$$\mathscr{L}\{u_a(t)f(t-a)\} = e^{-as}\mathscr{L}\{f(t)\} = e^{-as}F(s). \qquad Q.E.D$$

Example.

Find the Laplace transform of

$$g(t) = \begin{cases} 0 & 0 < t < 5, \\ t-3, & t > 5. \end{cases}$$

Before we can apply Theorem 7 to this translated function, we must express the functional values $t - 3$ for $t > 5$ in terms of $t - 5$, as required by (24). That is, we express $t - 3$ as $(t - 5) + 2$ and write

$$g(t) = \begin{cases} 0, & 0 < t < 5, \\ (t-5)+2, & t > 5. \end{cases}$$

This is now of the form (24), and we recognize it as

$$u_5(t)f(t-5) = \begin{cases} 0, & 0 < t < 5, \\ (t-5)+2, & t > 5, \end{cases}$$

where $f(t) = t + 2$, $t > 0$. Hence we apply Theorem 7 with $f(t) = t + 2$. Using Theorem 2 (the Linear Property) and formulas (2) and (3), we find

$$F(s) = \mathscr{L}\{t+2\} = \mathscr{L}\{t\} + 2\mathscr{L}\{1\} = \frac{1}{s^2} + \frac{2}{s}.$$

Then by Theorem 7, with $a = 5$, we obtain

$$\mathscr{L}\{u_5(t)f(t-5)\} = e^{-5s}F(s) = e^{-5s}\left(\frac{1}{s^2} + \frac{2}{s}\right).$$

This then is the Laplace transform of the given function $g(t)$.

Example.

Find the Laplace transform of

$$g(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \sin t, & t > \dfrac{\pi}{2}. \end{cases}$$

Before we can apply Theorem 7, we must express $\sin t$ in terms of $t - \pi/2$, as required by (24). We observe that $\sin t = \cos(t - \pi/2)$ for all t, and hence write

$$g(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \cos\left(t - \dfrac{\pi}{2}\right), & t > \dfrac{\pi}{2}. \end{cases}$$

This is now of the form (24), and we recognize it as

$$u_{\pi/2}(t)f(t - \pi/2) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2}, \\ \cos\left(t - \dfrac{\pi}{2}\right), & t > \dfrac{\pi}{2}, \end{cases}$$

where $f(t) = \cos t$, $t > 0$. Hence we apply Theorem 7 with $f(t) = \cos t$. Using formula (6) with $b = 1$, we obtain

$$F(s) = \mathscr{L}\{\cos t\} = \frac{s}{s^2 + 1}.$$

Then by Theorem 7, with $a = \pi/2$, we obtain

$$\mathscr{L}\{g(t)\} = \mathscr{L}\{u_{\pi/2}(t)f(t - \pi/2)\} = \frac{se^{-(\pi/2)s}}{s^2 + 1}.$$

THEOREM 8.

Hypothesis. *Suppose f is a periodic function of period P which satisfies the hypotheses of Theorem 1.*

Conclusion. *Then*

$$\mathscr{L}\{f(t)\} = \frac{\int_0^P e^{-st} f(t)\,dt}{1 - e^{-Ps}}. \tag{26}$$

Proof. By definition of the Laplace transform,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt. \tag{27}$$

The integral on the right can be broken up into the infinite series of integrals

$$\int_0^P e^{-st} f(t)\,dt + \int_P^{2P} e^{-st} f(t)\,dt + \int_{2P}^{3P} e^{-st} f(t)\,dt + \cdots$$

$$+ \int_{nP}^{(n+1)P} e^{-st} f(t)\,dt \cdots. \tag{28}$$

We now transform each integral in this series. For each $n = 0, 1, 2, \ldots$, let $t = u + nP$ in the corresponding integral

$$\int_{nP}^{(n+1)P} e^{-st} f(t)\,dt.$$

Then for each $n = 0, 1, 2, \ldots$, this becomes

$$\int_0^P e^{-s(u+nP)} f(u + nP)\, du. \tag{29}$$

But by hypothesis, f is periodic of period P. Thus $f(u) = f(u + P) = f(u + 2P) = \cdots = f(u + nP)$ for all u for which f is defined. Also $e^{-s(u+nP)} = e^{-su}e^{-nPs}$, where the factor e^{-nPs} is independent of the variable of integration u in (29). Thus for each $n = 0, 1, 2, \ldots$, the integral in (29) becomes

$$e^{-nPs} \int_0^P e^{-su} f(u)\, du.$$

Hence the infinite series (28) takes the form

$$\begin{aligned}\int_0^P e^{-su} f(u)\, du &+ e^{-Ps} \int_0^P e^{-su} f(u)\, du \\ &+ e^{-2Ps} \int_0^P e^{-su} f(u)\, du + \cdots + e^{-nPs} \int_0^P e^{-su} f(u)\, du + \cdots \\ &= [1 + e^{-Ps} + e^{-2Ps} + \cdots + e^{-nPs} + \cdots] \int_0^P e^{-su} f(u)\, du. \quad (30)\end{aligned}$$

Now observe that the infinite series in brackets is a geometric series of first term 1 and common ratio $r = e^{-Ps} < 1$. Such a series converges to $1/(1 - r)$, and hence the series in brackets converges to $1/(1 - e^{-Ps})$. Therefore the right member of (30), and hence that of (28), reduces to

$$\frac{\int_0^P e^{-su} f(u)\, du}{1 - e^{-Ps}}.$$

Then, since this is the right member of (27), upon replacing the dummy variable u by t, we have

$$\mathscr{L}\{f(t)\} = \frac{\int_0^P e^{-st} f(t)\, dt}{1 - e^{-Ps}} \qquad \textit{Q.E.D.}$$

Example.

Find the Laplace transform of f defined on $0 \leq t < 4$ by

$$f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ -1, & 2 \leq t < 4, \end{cases}$$

and for all other positive t by the periodicity condition

$$f(t+4) = f(t).$$

The graph of f is shown in Figure 5. Clearly this function f is periodic of period $P = 4$. Applying formula (26) of Theorem 8, we find

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{\int_0^4 e^{-st} f(t)\, dt}{1 - e^{-4s}} \\
&= \frac{1}{1 - e^{-4s}} \left[\int_0^2 e^{-st}(1)\, dt + \int_2^4 e^{-st}(-1)\, dt \right] \\
&= \frac{1}{1 - e^{-4s}} \left[\left. \frac{-e^{-st}}{s} \right|_0^2 + \left. \frac{e^{-st}}{s} \right|_2^4 \right] \\
&= \frac{1}{1 - e^{-4s}} \left(\frac{1}{s} \right) [-e^{-2s} + 1 + e^{-4s} - e^{-2s}]
\end{aligned}$$

$$= \frac{1 - 2e^{-2s} + e^{-4s}}{s(1 - e^{-4s})} = \frac{(1 - e^{-2s})^2}{s(1 - e^{-2s})(1 + e^{-2s})} = \frac{1 - e^{-2s}}{s(1 + e^{-2s})}.$$

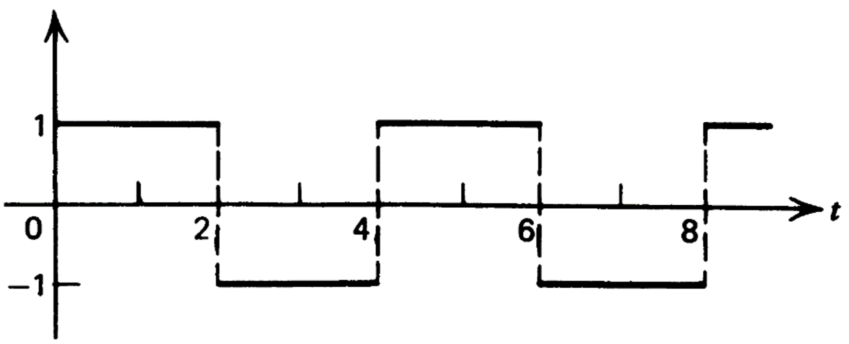

Figure 5

THE INVERSE TRANSFORM AND THE CONVOLUTION

Now consider the inverse problem: Given a function F, to find a function f whose Laplace transform is the given F. We introduce the notation $\mathscr{L}^{-1}\{F\}$ to denote such a function f, denote $\mathscr{L}^{-1}\{F(s)\}$ by $f(t)$, and call such a function an *inverse transform* of F. That is,

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

means that $f(t)$ is such that

$$\mathscr{L}\{f(t)\} = F(s).$$

THEOREM 9.

Hypothesis. *Let f and g be two functions that are continuous for $t \geq 0$ and that have the same Laplace transform F.*

Conclusion. $f(t) = g(t)$ *for all* $t \geq 0$.

Thus if it is known that a given function F has a *continuous* inverse transform f, then f is the *only* continuous inverse transform of F. Let us consider the following example.

Example.

By Equation (2), $\mathscr{L}\{1\} = 1/s$. Thus an inverse transform of the function F defined by $F(s) = 1/s$ is the *continuous* function f defined for all t by $f(t) = 1$. Thus by Theorem 9 there is no other *continuous* inverse transform of the function F such that $F(s) = 1/s$. However, discontinuous inverse transforms of this function F exist. For example, consider the function g defined as follows:

$$g(t) = \begin{cases} 1, & 0 < t < 3, \\ 2, & t = 3, \\ 1, & t > 3. \end{cases}$$

Then

$$\begin{aligned} \mathscr{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t)\, dt = \int_0^3 e^{-st}\, dt + \int_3^\infty e^{-st}\, dt \\ &= \left[-\frac{e^{-st}}{s} \right]_0^3 + \lim_{R \to \infty} \left[-\frac{e^{-st}}{s} \right]_3^R = \frac{1}{s} \qquad \text{if } s > 0. \end{aligned}$$

Thus this discontinuous function g is also an inverse transform of F defined by $F(s) = 1/s$. However, we again emphasize that the only *continuous* inverse transform of F defined by $F(s) = 1/s$ is f defined for all t by $f(t) = 1$. Indeed we write

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$$

with the understanding that f defined for all t by $f(t) = 1$ is the *unique continuous* inverse transform of F defined by $F(s) = 1/s$.

Example.

Using Table 1, find $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + 6s + 13}\right\}$.

Solution. Looking in the $F(s)$ column of Table 1 we would first look for $F(s) = \dfrac{1}{as^2 + bs + c}$. However, we find no such $F(s)$; but we do find $F(s) = \dfrac{b}{(s + a)^2 + b^2}$ (number 11). We can put the given expression $\dfrac{1}{s^2 + 6s + 13}$ in this form as follows:

$$\frac{1}{s^2 + 6s + 13} = \frac{1}{(s + 3)^2 + 4} = \frac{1}{2} \cdot \frac{2}{(s + 3)^2 + 2^2}.$$

Thus, using number 11 of Table 1, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2 + 6s + 13}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s + 3)^2 + 2^2}\right\} = \frac{1}{2}e^{-3t}\sin 2t.$$

Example.

Using Table 1, find $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2 + 1)}\right\}$.

Solution. No entry of this form appears in the $F(s)$ column of Table 1. We employ the method of partial fractions. We have

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}$$

and hence

$$1 = (A + B)s^2 + Cs + A.$$

Thus

$$A + B = 0, \qquad C = 0, \quad \text{and} \quad A = 1.$$

TABLE 1 LAPLACE TRANSFORMS

	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
1	1	$\frac{1}{s}$
2	e^{at}	$\frac{1}{s-a}$
3	$\sin bt$	$\frac{b}{s^2+b^2}$
4	$\cos bt$	$\frac{s}{s^2+b^2}$
5	$\sinh bt$	$\frac{b}{s^2-b^2}$
6	$\cosh bt$	$\frac{s}{s^2-b^2}$
7	$t^n (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
8	$t^n e^{at} (n = 1, 2, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$
9	$t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
10	$t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
11	$e^{-at} \sin bt$	$\frac{b}{(s+a)^2+b^2}$
12	$e^{-at} \cos bt$	$\frac{s+a}{(s+a)^2+b^2}$
13	$\frac{\sin bt - bt \cos bt}{2b^3}$	$\frac{1}{(s^2+b^2)^2}$
14	$\frac{t \sin bt}{2b}$	$\frac{s}{(s^2+b^2)^2}$
15	$u_a(t)$	$\frac{e^{-as}}{s}$
16	$u_a(t) f(t-a)$ [see Theorem 7]	$e^{-as} F(s)$

From these equations, we have the partial fractions decomposition

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

By number 1 of Table 1, $\mathscr{L}^{-1}\{1/s\} = 1$ and by number 4, $\mathscr{L}^{-1}\{s/(s^2+1)\} = \cos t$. Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

We now give two examples of finding the inverse transform of a function that involves one or more terms of the form $e^{-as}F(s)$.

Example.

Find

$$\mathscr{L}^{-1}\left\{\frac{5}{s} - \frac{3e^{-3s}}{s} - \frac{2e^{-7s}}{s}\right\}.$$

Solution. By number 1 of Table 1, we at once have

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1.$$

By number 15, we see that

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u_a(t). \tag{31}$$

Here u_a is the unit step function [see Equations (19) and following] defined for $a > 0$ by

$$u_a(t) = \begin{cases} 0, & 0 < t < a, \\ 1, & t > a, \end{cases} \tag{32}$$

and for $a = 0$ by

$$u_0(t) = 1 \quad \text{for } t > 0.$$

Applying (31) and (32) with $a = 3$ and $a = 7$, respectively, we have

$$\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} = u_3(t) = \begin{cases} 0, & 0 < t < 3, \\ 1, & t > 3, \end{cases} \tag{33}$$

and

$$\mathscr{L}^{-1}\left\{\frac{e^{-7s}}{s}\right\} = u_7(t) = \begin{cases} 0, & 0 < t < 7, \\ 1, & t > 7. \end{cases} \tag{34}$$

Thus we obtain

$$\mathscr{L}^{-1}\left\{\frac{5}{s} - \frac{3e^{-3s}}{s} - \frac{2e^{-7s}}{s}\right\} = 5 - 3u_3(t) - 2u_7(t).$$

Now using (33) and (34), we see that this equals

$$\begin{cases} 5 - 0 - 0, & 0 < t < 3, \\ 5 - 3 - 0, & 3 < t < 7, \\ 5 - 3 - 2, & t > 7; \end{cases}$$

and hence

$$\mathscr{L}^{-1}\left\{\frac{5}{s} - \frac{3e^{-3s}}{s} - \frac{2e^{-7s}}{s}\right\} = \begin{cases} 5, & 0 < t < 3, \\ 2, & 3 < t < 7, \\ 0, & t > 7. \end{cases}$$

Example.
Find

$$\mathscr{L}^{-1}\left\{e^{-4s}\left(\frac{2}{s^2} + \frac{5}{s}\right)\right\}.$$

Solution. This is of the form $\mathscr{L}^{-1}\{e^{-as}F(s)\}$, where $a = 4$ and $F(s) = 2/s^2 + 5/s$. By number 16 of Table 1, we see that

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = u_a(t)f(t - a). \tag{35}$$

Here u_a is the unit step function defined for $a > 0$ by (32) and $f(t) = \mathscr{L}^{-1}\{F(s)\}$ [see Theorem 7]. By number 1 of Table 1, we again find $\mathscr{L}^{-1}\{1/s\} = 1$; and by number 7 with $n = 1$, we obtain $\mathscr{L}^{-1}\{1/s^2\} = t$. Thus

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{2}{s^2} + \frac{5}{s}\right\} = 2t + 5,$$

and so $f(t-4) = 2(t-4) + 5 = 2t - 3$. Then by (35), with $a = 4$,

$$\mathscr{L}^{-1}\{e^{-4s}F(s)\} = u_4(t)f(t-4);$$

that is,

$$\mathscr{L}^{-1}\left\{e^{-4s}\left(\frac{2}{s^2} + \frac{5}{s}\right)\right\} = u_4(t)[2t-3] = \begin{cases} 0, & 0 < t < 4, \\ 2t - 3, & t > 4. \end{cases}$$

The Convolution

DEFINITION

Let f and g be two functions that are piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order. The function denoted by $f * g$ and defined by

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \tag{36}$$

is called the convolution *of the functions f and g.*

Let us change the variable of integration in (36) by means of the substitution $u = t - \tau$. We have

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(\tau)g(t-\tau)\,d\tau = -\int_t^0 f(t-u)g(u)\,du \\ &= \int_0^t g(u)f(t-u)\,du = g(t) * f(t). \end{aligned}$$

Thus we have shown that

$$f * g = g * f \tag{37}$$

Suppose that both f and g are piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order e^{at}. Then it can be shown that $f * g$ is also piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order $e^{(a+\epsilon)t}$, where ϵ is any positive number. Thus $\mathscr{L}\{f * g\}$ exists for s sufficiently large. More explicitly, it can be shown that $\mathscr{L}\{f * g\}$ exists for $s > a$.

THEOREM 10.

Hypothesis. *Let the functions f and g be piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order e^{at}.*

Conclusion

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f\}\mathscr{L}\{g\} \tag{38}$$

for $s > a$.

Denoting $\mathscr{L}\{f\}$ by F and $\mathscr{L}\{g\}$ by G, we may write the conclusion (38) in the form

$$\mathscr{L}\{f(t) * g(t)\} = F(s)G(s).$$

Hence, we have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau, \tag{39}$$

and using (37), we also have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = g(t) * f(t) = \int_0^t g(\tau)f(t - \tau)\, d\tau. \tag{40}$$

Suppose we are given a function H and are required to determine $\mathscr{L}^{-1}\{H(s)\}$. If we can express $H(s)$ as a product $F(s)G(s)$, where $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$ are known, then we can apply either (39) or (40) to determine $\mathscr{L}^{-1}\{H(s)\}$.

Example.

Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2 + 1)}\right\}$ using the convolution and Table 1.

Solution. We write $1/s(s^2 + 1)$ as the product $F(s)G(s)$, where $F(s) = 1/s$ and $G(s) = 1/(s^2 + 1)$. By Table 1, number 1, $f(t) = \mathscr{L}^{-1}\{1/s\} = 1$, and by number 3, $g(t) = \mathscr{L}^{-1}\{1/(s^2 + 1)\} = \sin t$. Thus by (39),

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2 + 1)}\right\} = f(t) * g(t) = \int_0^t 1 \cdot \sin(t - \tau)\, d\tau,$$

and by (40),

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2 + 1)}\right\} = g(t) * f(t) = \int_0^t \sin\tau \cdot 1\, d\tau.$$

The second of these two integrals is slightly more simple. Evaluating it, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2 + 1)}\right\} = 1 - \cos t.$$

We now consider how the Laplace transform may be applied to solve the initial-value problem consisting of the nth-order linear differential equation with constant coefficients

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b(t), \tag{41}$$

plus the initial conditions

$$y(0) = c_0,\ y'(0) = c_1, \ldots,\ y^{(n-1)}(0) = c_{n-1}. \tag{42}$$

We now take the Laplace transform of both members of Equation (41). By Theorem 2, we have

$$a_0 \mathscr{L}\left\{\frac{d^n y}{dt^n}\right\} + a_1 \mathscr{L}\left\{\frac{d^{n-1} y}{dt^{n-1}}\right\} + \cdots + a_{n-1} \mathscr{L}\left\{\frac{dy}{dt}\right\} + a_n \mathscr{L}\{y(t)\} = \mathscr{L}\{b(t)\}. \tag{43}$$

We now apply Theorem 4 to

$$\mathscr{L}\left\{\frac{d^n y}{dt^n}\right\}, \mathscr{L}\left\{\frac{d^{n-1} y}{dt^{n-1}}\right\}, \ldots, \mathscr{L}\left\{\frac{dy}{dt}\right\}$$

in the left member of Equation (43). Using the initial conditions (42), we have

$$\begin{aligned}
\mathscr{L}\left\{\frac{d^n y}{dt^n}\right\} &= s^n \mathscr{L}\{y(t)\} - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - y^{(n-1)}(0) \\
&= s^n \mathscr{L}\{y(t)\} - c_0 s^{n-1} - c_1 s^{n-2} - \cdots - c_{n-1}, \\
\mathscr{L}\left\{\frac{d^{n-1}}{dt^{n-1}}\right\} &= s^{n-1} \mathscr{L}\{y(t)\} - s^{n-2} y(0) - s^{n-3} y'(0) - \cdots - y^{(n-2)}(0) \\
&= s^{n-1} \mathscr{L}\{y(t)\} - c_0 s^{n-2} - c_1 s^{n-3} - \cdots - c_{n-2}, \\
&\ \vdots \\
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= s \mathscr{L}\{y(t)\} - y(0) = s \mathscr{L}\{y(t)\} - c_0.
\end{aligned}$$

Thus, letting $Y(s)$ denote $\mathscr{L}\{y(t)\}$ and $B(s)$ denote $\mathscr{L}\{b(t)\}$, Equation (43) becomes

$$\begin{aligned}[a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n]Y(s) \\ - c_0[a_0 s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}] \\ - c_1[a_0 s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-2}] \\ - \cdots - c_{n-2}[a_0 s + a_1] - c_{n-1}a_0 = B(s). \quad (44)\end{aligned}$$

Since b is a known function of t, then B, assuming it exists and can be determined, is a known function of s. Thus Equation (44) is an algebraic equation in the "unknown" $Y(s)$. We now solve the algebraic equation (44) to determine $Y(s)$. Once $Y(s)$ has been found, we then find the unique solution

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

of the given initial-value problem using the table of transforms.

Example.

Solve the initial-value problem

$$\frac{dy}{dt} - 2y = e^{5t}, \tag{45}$$

$$y(0) = 3 \tag{46}$$

Step 1. Taking the Laplace transform of both sides of the differential equation (45), we have

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} - 2\mathscr{L}\{y(t)\} = \mathscr{L}\{e^{5t}\}. \tag{47}$$

Using Theorem 4 with $n = 1$ (or Theorem 3) and denoting $\mathscr{L}\{y(t)\}$ by $Y(s)$, we may express $\mathscr{L}\{dy/dt\}$ in terms of $Y(s)$ and $y(0)$ as follows:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0).$$

Applying the initial condition (46), this becomes

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - 3.$$

Using this, the left member of Equation (47) becomes $sY(s) - 3 - 2Y(s)$. From Table 1, number 2, $\mathscr{L}\{e^{5t}\} = 1/(s - 5)$. Thus Equation (47) reduces to the algebraic equation

$$[s - 2]Y(s) - 3 = \frac{1}{s - 5} \tag{48}$$

in the unknown $Y(s)$.

Step 2. We now solve Equation (48) for $Y(s)$. We have

$$[s - 2]Y(s) = \frac{3s - 14}{s - 5}$$

and so

$$Y(s) = \frac{3s - 14}{(s - 2)(s - 5)}.$$

Step 3. We must now determine

$$\mathscr{L}^{-1}\left\{\frac{3s - 14}{(s - 2)(s - 5)}\right\}.$$

We employ partial fractions. We have

$$\frac{3s - 14}{(s - 2)(s - 5)} = \frac{A}{s - 2} + \frac{B}{s - 5},$$

and so $3s - 14 = A(s - 5) + B(s - 2)$. From this we find that

$$A = \tfrac{8}{3} \quad \text{and} \quad B = \tfrac{1}{3},$$

and so

$$\mathscr{L}^{-1}\left\{\frac{3s-14}{(s-2)(s-5)}\right\} = \frac{8}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\}.$$

Using number 2 of Table 1,

$$\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} \quad \text{and} \quad \mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\} = e^{5t}.$$

Thus the solution of the given initial-value problem is

$$y = \tfrac{8}{3}e^{2t} + \tfrac{1}{3}e^{5t}.$$