THE CAUCHY–EULER EQUATION

This is an equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x),$$

where $a_0, a_1, \ldots, a_{n-1}$, a_n are constants. Note the characteristic feature of this equation: each term in the left member is a constant multiple of an expression of the form

$$x^k \frac{d^k y}{dx^k}.$$

THEOREM 1.

The transformation $x = e^t$ reduces the equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$$

to a linear differential equation with constant coefficients.

We shall prove this theorem for the case of the *second*-order Cauchy–Euler differential equation

$$a_0 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = F(x).$$

The proof in the general nth-order case proceeds in a similar fashion. Letting $x = e^t$, assuming $x > 0$, we have $t = \ln x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right) - \frac{1}{x^2}\frac{dy}{dt}$$

$$\frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Thus

$$x\frac{dy}{dx} = \frac{dy}{dt} \quad \text{and} \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

Substituting into equation we obtain

$$a_0\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\frac{dy}{dt} + a_2 y = F(e^t)$$

or

$$A_0\frac{d^2y}{dt^2} + A_1\frac{dy}{dt} + A_2 y = G(t),$$

where

$$A_0 = a_0, \qquad A_1 = a_1 - a_0, \qquad A_2 = a_2, \qquad G(t) = F(e^t).$$

This is a second-order linear differential equation with *constant* coefficients, which was what we wished to show.

Example.

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3.$$

Let $x = e^t$. Then, assuming $x > 0$, we have $t = \ln x$, and

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right) - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Thus

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t}$$

or

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}.$$

The complementary function of this equation is $y_c = c_1e^t + c_2e^{2t}$. We find a particular integral by the method of undetermined coefficients. We assume $y_p = Ae^{3t}$. Then $y_p' = 3Ae^{3t}$, $y_p'' = 9Ae^{3t}$, and substituting into

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

we obtain

$$2Ae^{3t} = e^{3t}.$$

Thus $A = \frac{1}{2}$ and we have $y_p = \frac{1}{2}e^{3t}$. The general solution is

$$y = c_1e^t + c_2e^{2t} + \tfrac{1}{2}e^{3t}.$$

But we are not yet finished! We must return to the original independent variable x. Since $e^t = x$, we find

$$y = c_1x + c_2x^2 + \tfrac{1}{2}x^3.$$

This is the general solution.

Example.

$$x^3 \frac{d^3y}{dx^3} - 4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} - 8y = 4 \ln x.$$

Assuming $x > 0$, we let $x = e^t$. Then $t = \ln x$, and

$$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Now we must consider $\frac{d^3y}{dx^3}$.

$$\frac{d^3y}{dx^3} = \frac{1}{x^2} \frac{d}{dx}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^3y}{dt^3}\frac{dt}{dx} - \frac{d^2y}{dt^2}\frac{dt}{dx}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right).$$

Thus, substituting into equation , we obtain

$$\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) - 4\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 8\left(\frac{dy}{dt}\right) - 8y = 4t$$

or

$$\frac{d^3y}{dt^3} - 7\frac{d^2y}{dt^2} + 14\frac{dy}{dt} - 8y = 4t.$$

The complementary function of the transformed equation is

$$y_c = c_1e^t + c_2e^{2t} + c_3e^{4t}.$$

We procced to obtain a particular integral by the method of undetermined coefficients. We assume $y_p = At + B$. Then $y_p' = A$, $y_p'' = y_p''' = 0$. Substituting into equation, we find

$$14A - 8At - 8B = 4t.$$

Thus

$$-8A = 4, \qquad 14A - 8B = 0,$$

and so $A = -\frac{1}{2}$, $B = -\frac{7}{8}$. Thus

$$y = c_1e^t + c_2e^{2t} + c_3e^{4t} - \tfrac{1}{2}t - \tfrac{7}{8},$$

and so the general solution of equation is

$$y = c_1x + c_2x^2 + c_3x^4 - \tfrac{1}{2}\ln x - \tfrac{7}{8}.$$

The Laplace Transform

DEFINITION

Let f be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

for all values of s for which this integral exists. The function F defined by the integral (1) is called the Laplace transform *of the function f. We shall denote the Laplace transform F of f by $\mathscr{L}\{f\}$ and shall denote $F(s)$ by $\mathscr{L}\{f(t)\}$.*

In order to be certain that the integral (1) does exist for some range of values of s, we must impose suitable restrictions upon the function f under consideration. We shall do this shortly; however, first we shall directly determine the Laplace transforms of a few simple functions.

Example.

Consider the function f defined by

$$f(t) = 1, \qquad \text{for } t > 0.$$

Then

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot 1\, dt = \lim_{R\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{1}{s} - \frac{e^{-sR}}{s}\right] = \frac{1}{s}$$

for all $s > 0$. Thus we have

$$\mathscr{L}\{1\} = \frac{1}{s} \qquad (s > 0). \tag{2}$$

Example.

Consider the function f defined by

$$f(t) = t, \qquad \text{for } t > 0.$$

Then

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\,dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot t\,dt = \lim_{R\to\infty} \left[-\frac{e^{-st}}{s^2}(st+1) \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{1}{s^2} - \frac{e^{-sR}}{s^2}(sR+1) \right] = \frac{1}{s^2}$$

for all $s > 0$. Thus

$$\mathscr{L}\{t\} = \frac{1}{s^2} \qquad (s > 0). \tag{3}$$

Example.

Consider the function f defined by

$$f(t) = e^{at}, \qquad \text{for } t > 0.$$

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \lim_{R\to\infty} \int_0^R e^{(a-s)t}\,dt = \lim_{R\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s} \right] = -\frac{1}{a-s} = \frac{1}{s-a} \qquad \text{for all } s > a.$$

Thus

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \qquad (s > a). \tag{4}$$

Example.

Consider the function f defined by

$$f(t) = \sin bt \qquad \text{for } t > 0.$$

$$\mathscr{L}\{\sin bt\} = \int_0^\infty e^{-st} \cdot \sin bt\,dt = \lim_{R\to\infty} \int_0^R e^{-st} \cdot \sin bt\,dt$$

$$= \lim_{R\to\infty} \left[-\frac{e^{-st}}{s^2+b^2}(s \sin bt + b \cos bt) \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{b}{s^2+b^2} - \frac{e^{-sR}}{s^2+b^2}(s \sin bR + b \cos bR) \right]$$

$$= \frac{b}{s^2+b^2} \qquad \text{for all } s > 0.$$

Thus

$$\mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2} \qquad (s > 0). \tag{5}$$

Example.

Consider the function f defined by

$$f(t) = \cos bt \qquad \text{for } t > 0.$$

$$\mathscr{L}\{\cos bt\} = \int_0^\infty e^{-st} \cdot \cos bt \, dt = \lim_{R\to\infty} \int_0^R e^{-st} \cos bt \, dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{-st}}{s^2+b^2}(-s \cos bt + b \sin bt) \right]_0^R$$

$$= \lim_{R\to\infty} \left[\frac{e^{-sR}}{s^2+b^2}(-s \cos bR + b \sin bR) + \frac{s}{s^2+b^2} \right]$$

$$= \frac{s}{s^2+b^2} \qquad \text{for all } s > 0.$$

Thus

$$\mathscr{L}\{\cos bt\} = \frac{s}{s^2+b^2} \qquad (s > 0). \tag{6}$$

In each of the above examples we have seen directly that the integral (1) actually does exist for some range of values of s. We shall now determine a class of functions f for which this is always the case. To do so we first consider certain properties of functions.

DEFINITION

A function f is said to be piecewise continuous (*or* sectionally continuous) *on a finite interval $a \leq t \leq b$ if this interval can be divided into a finite number of subintervals such that (1) f is continuous in the interior of each of these subintervals, and (2) $f(t)$ approaches finite limits as t approaches either endpoint of each of the subintervals from its interior.*

Suppose f is piecewise continuous on $a \leq t \leq b$, and t_0, $a < t_0 < b$, is an endpoint of one of the subintervals of the above definition. Then the finite limit approached by $f(t)$ as t approaches t_0 from the left (that is, through smaller values of t) is called the *left-hand limit* of $f(t)$ as t approaches t_0, denoted by $\lim_{t \to t_0-} f(t)$ or by $f(t_0 -)$. In like manner, the finite limit approached by $f(t)$ as t approaches t_0 from the right (through larger values) is called the *right-hand limit* of $f(t)$ as t approaches t_0, denoted by $\lim_{t \to t_0+} f(t)$ or $f(t_0 +)$. We emphasize that at such a point t_0, both $f(t_0 -)$ and $f(t_0 +)$ are finite but they are not in general equal.

We point out that if f is continuous on $a \leq t \leq b$ it is necessarily piecewise continuous on this interval. Also, we note that if f is piecewise continuous on $a \leq t \leq b$, then f is integrable on $a \leq t \leq b$.

Example.

Consider the function f defined by

$$f(t) = \begin{cases} -1, & 0 < t < 2, \\ 1, & t > 2. \end{cases}$$

f is piecewise continuous on every finite interval $0 \leq t \leq b$, for every positive number b. At $t = 2$, we have

$$f(2-) = \lim_{t \to 2-} f(t) = -1,$$

$$f(2+) = \lim_{t \to 2+} f(t) = +1.$$

The graph of f is shown in Figure 1.

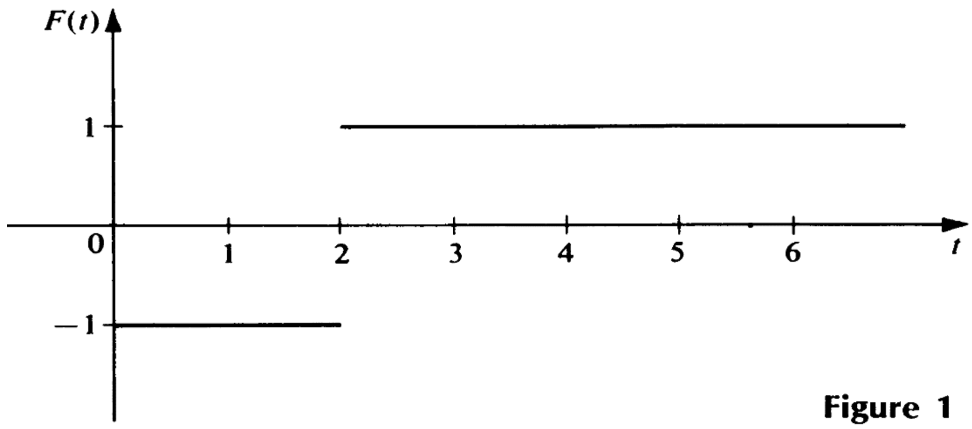

Figure 1

DEFINITION

A function f is said to be of exponential order *if there exists a constant α and positive constants t_0 and M such that*

$$e^{-\alpha t}|f(t)| < M \tag{7}$$

for all $t > t_0$ at which $f(t)$ is defined. More explicitly, if f is of exponential order corresponding to some definite constant α in (7), *then we say that f is of* exponential order $e^{\alpha t}$.

In other words, we say that f is of exponential order if a constant α exists such that the product $e^{-\alpha t}\,|f(t)|$ is bounded for all sufficiently large values of t. From (7) we have

$$|f(t)| < Me^{\alpha t} \tag{8}$$

for all $t > t_0$ at which $f(t)$ is defined. Thus if f is of exponential order and the values $f(t)$ of f become infinite as $t \to \infty$, these values cannot become infinite more rapidly than a multiple M of the corresponding values $e^{\alpha t}$ of some exponential function. We note that if f is of exponential order $e^{\alpha t}$, then f is also of exponential order $e^{\beta t}$ for any $\beta > \alpha$.

Example.

The function f such that $f(t) = e^{at} \sin bt$ is of exponential order, with the constant $\alpha = a$. For we then have

$$e^{-\alpha t}|f(t)| = e^{-at}e^{at}|\sin bt| = |\sin bt|,$$

which is bounded for all t.

Example.

Consider the function f such that $f(t) = t^n$, where $n > 0$. Then $e^{-\alpha t}|f(t)|$ is $e^{-\alpha t}t^n$. For any $\alpha > 0$, $\lim_{t\to\infty} e^{-\alpha t}t^n = 0$. Thus there exists $M > 0$ and $t_0 > 0$ such that

$$e^{-\alpha t}|f(t)| = e^{-\alpha t}t^n < M$$

for $t > t_0$. Hence $f(t) = t^n$ is of exponential order, with the constant α equal to any positive number.

Example.

The function f such that $f(t) = e^{t^2}$ is *not* of exponential order, for in this case $e^{-\alpha t}|f(t)|$ is $e^{t^2-\alpha t}$ and this becomes unbounded as $t \to \infty$, no matter what is the value of α.

We shall now proceed to obtain a theorem giving conditions on f that are sufficient for the integral (1) to exist. To obtain the desired result we shall need the following two theorems from advanced calculus, which we state without proof.

THEOREM A Comparison Test for Improper Integrals

Hypothesis

1. Let g and G be real functions such that

$$0 \le g(t) \le G(t) \qquad \text{on} \quad a \le t < \infty.$$

2. Suppose $\int_a^\infty G(t)\,dt$ exists.

3. Suppose g is integrable on every finite closed subinterval of $a \le t < \infty$.

Conclusion. *Then $\int_a^\infty g(t)\,dt$ exists.*

THEOREM B

Hypothesis

1. Suppose the real function g is integrable on every finite closed subinterval of $a \le t \le \infty$.

2. Suppose $\int_a^\infty |g(t)|\,dt$ exists.

Conclusion. *Then $\int_a^\infty g(t)\,dt$ exists.*

We now state and prove an existence theorem for Laplace transforms.

THEOREM 1.

Hypothesis. *Let f be a real function that has the following properties:*

1. f is piecewise continuous in every finite closed interval $0 \leq t \leq b$ $(b > 0)$.

2. f is of exponential order; that is, there exists α, $M > 0$, and $t_0 > 0$ such that

$$e^{-\alpha t}|f(t)| < M \qquad \text{for} \quad t > t_0.$$

Conclusion. *The Laplace transform*

$$\int_0^{\infty} e^{-st} f(t)\, dt$$

of f exists for $s > \alpha$.

Proof. We have

$$\int_0^{\infty} e^{-st} f(t)\, dt = \int_0^{t_0} e^{-st} f(t)\, dt + \int_{t_0}^{\infty} e^{-st} f(t)\, dt.$$

By Hypothesis 1, the first integral of the right member exists. By Hypothesis 2,

$$e^{-st}|f(t)| < e^{-st} M e^{\alpha t} = M e^{-(s-\alpha)t}$$

for $t > t_0$. Also

$$\begin{aligned}
\int_{t_0}^{\infty} M e^{-(s-\alpha)t}\, dt &= \lim_{R\to\infty} \int_{t_0}^{R} M e^{-(s-\alpha)t}\, dt = \lim_{R\to\infty} \left[-\frac{M e^{-(s-\alpha)t}}{s-\alpha} \right]_{t_0}^{R} \\
&= \lim_{R\to\infty} \left[\frac{M}{s-\alpha} \right] [e^{-(s-\alpha)t_0} - e^{-(s-\alpha)R}] \\
&= \left[\frac{M}{s-\alpha} \right] e^{-(s-\alpha)t_0} \qquad \text{if} \quad s > \alpha.
\end{aligned}$$

Thus

$$\int_{t_0}^{\infty} M e^{-(s-\alpha)t}\, dt \qquad \text{exists for} \quad s > \alpha.$$

Finally, by Hypothesis 1, $e^{-st}|f(t)|$ is integrable on every finite closed subinterval of $t_0 \leq t < \infty$. Thus, applying Theorem A with $g(t) = e^{-st}|f(t)|$ and $G(t) = M e^{-(s-\alpha)t}$, we see that

$$\int_{t_0}^{\infty} e^{-st}|f(t)|\, dt \qquad \text{exists if} \quad s > \alpha.$$

In other words,

$$\int_{t_0}^{\infty} |e^{-st} f(t)|\, dt \qquad \text{exists if} \quad s > \alpha,$$

and so by Theorem B

$$\int_{t_0}^{\infty} e^{-st} f(t)\, dt$$

also exists if $s > \alpha$. Thus the Laplace transform of f exists for $s > \alpha$. *Q.E.D.*

Let us look back at this proof for a moment. Actually we showed that if f satisfies the hypotheses stated, then

$$\int_{t_0}^{\infty} e^{-st} |f(t)|\, dt \qquad \text{exists if} \quad s > \alpha.$$

Further, Hypothesis 1 shows that

$$\int_{0}^{t_0} e^{-st} |f(t)|\, dt \qquad \text{exists.}$$

Thus

$$\int_{0}^{\infty} e^{-st} |f(t)|\, dt \qquad \text{exists if} \quad s > \alpha.$$

In other words, if f satisfies the hypotheses of Theorem 1, then not only does $\mathscr{L}\{f\}$ exist for $s > \alpha$, but also $\mathscr{L}\{|f|\}$ exists for $s > \alpha$. That is,

$$\int_{0}^{\infty} e^{-st} f(t)\, dt \qquad \text{is absolutely convergent for} \quad s > \alpha.$$

We point out that the conditions on f described in the hypothesis of Theorem 1 are not necessary for the existence of $\mathscr{L}\{f\}$. In other words, there exist functions f that do *not* satisfy the hypotheses of Theorem 1, but for which $\mathscr{L}\{f\}$ exists. For instance, suppose we replace Hypothesis 1 by the following less restrictive condition. Let us suppose that f is piecewise continuous in every finite closed interval $a \le t \le b$, where $a > 0$, and is such that $|t^n f(t)|$ remains bounded as $t \to 0^+$ for some number n, where $0 < n < 1$. Then, provided Hypothesis 2 remains satisfied, it can be shown that $\mathscr{L}\{f\}$ still exists. Thus for example, if $f(t) = t^{-1/3}$, $t > 0$, $\mathscr{L}\{f\}$ exists. For although f does *not* satisfy the hypothesis of Theorem 1 [$f(t) \to \infty$ as $t \to 0^+$], it *does* satisfy the less restrictive requirement stated above (take $n = \frac{2}{3}$), and f is of exponential order.

THEOREM 2. The Linear Property

Let f_1 and f_2 be functions whose Laplace transforms exist, and let c_1 and c_2 be constants. Then

$$\mathscr{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathscr{L}\{f_1(t)\} + c_2 \mathscr{L}\{f_2(t)\}. \tag{9}$$

This follows directly from the definition.

Example.

Use Theorem 2 to find $\mathscr{L}\{\sin^2 at\}$. Since $\sin^2 at = (1 - \cos 2at)/2$, we have

$$\mathscr{L}\{\sin^2 at\} = \mathscr{L}\{\tfrac{1}{2} - \tfrac{1}{2}\cos 2at\}.$$

By Theorem 2,

$$\mathscr{L}\{\tfrac{1}{2} - \tfrac{1}{2}\cos 2at\} = \tfrac{1}{2}\mathscr{L}\{1\} - \tfrac{1}{2}\mathscr{L}\{\cos 2at\}.$$

By (2), $\mathscr{L}\{1\} = 1/s$, and by (6), $\mathscr{L}\{\cos 2at\} = s/(s^2 + 4a^2)$. Thus

$$\mathscr{L}\{\sin^2 at\} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2 + 4a^2} = \frac{2a^2}{s(s^2 + 4a^2)}. \tag{10}$$

THEOREM 3.

Hypothesis

1. Let f be a real function that is continuous for $t \ge 0$ and of exponential order $e^{\alpha t}$.

2. Let f' (the derivative of f) be piecewise continuous in every finite closed interval $0 \le t \le b$.

Conclusion. *Then $\mathscr{L}\{f'\}$ exists for $s > \alpha$; and*

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0). \tag{11}$$

Proof. By definition of the Laplace transform,

$$\mathscr{L}\{f'(t)\} = \lim_{R\to\infty} \int_0^R e^{-st} f'(t)\,dt,$$

provided this limit exists. In any closed interval $0 \le t \le R$, $f'(t)$ has at most a finite number of discontinuities; denote these by $t_1, t_2, \ldots, t_n$, where

$$0 \le t_1 < t_2 < \cdots < t_n \le R.$$

Then we may write

$$\int_0^R e^{-st} f'(t)\,dt = \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \cdots + \int_{t_n}^{R} e^{-st} f'(t)\,dt.$$

Now the integrand of each of the integrals on the right is continuous. We may therefore integrate each by parts. Doing so, we obtain

$$\begin{aligned}\int_0^R e^{-st} f'(t)\,dt = \Big[e^{-st} f(t)\Big]_0^{t_1 -} + s\int_0^{t_1} e^{-st} f(t)\,dt + \Big[e^{-st} f(t)\Big]_{t_1 +}^{t_2 -}\\ + s\int_{t_1}^{t_2} e^{-st} f(t)\,dt + \cdots + \Big[e^{-st} f(t)\Big]_{t_n +}^{R -} + s\int_{t_n}^{R} e^{-st} f(t)\,dt.\end{aligned}$$

By Hypothesis 1, f is continuous for $t \ge 0$. Thus

$$f(t_1 -) = f(t_1 +),\ f(t_2 -) = f(t_2 +), \ldots, f(t_n -) = f(t_n +).$$

Thus all of the integrated "pieces" add out, except for $e^{-st} f(t)|_{t=0}$ and $e^{-st} f(t)|_{t=R-}$, and there remains only

$$\int_0^R e^{-st} f'(t)\,dt = -f(0) + e^{-sR} f(R) + s\int_0^R e^{-st} f(t)\,dt.$$

But by Hypothesis 1 f is of exponential order $e^{\alpha t}$. Thus there exists $M > 0$ and $t_0 > 0$ such that $e^{-\alpha t}|f(t)| < M$ for $t > t_0$. Thus $|e^{-sR} f(R)| < Me^{-(s-\alpha)R}$ for $R > t_0$. Thus if $s > \alpha$,

$$\lim_{R\to\infty} e^{-sR} f(R) = 0.$$

Further,

$$\lim_{R\to\infty} s\int_0^R e^{-st} f(t)\,dt = s\mathscr{L}\{f(t)\}.$$

Thus, we have

$$\lim_{R\to\infty}\int_0^R e^{-st}f'(t)\,dt = -f(0) + s\mathscr{L}\{f(t)\},$$

and so $\mathscr{L}\{f'(t)\}$ exists for $s > \alpha$ and is given by (11). *Q.E.D*

Example.

Consider the function defined by $f(t) = \sin^2 at$. This function satisfies the hypotheses of Theorem 3. Since $f'(t) = 2a \sin at \cos at$ and $f(0) = 0$, Equation (11) gives

$$\mathscr{L}\{2a \sin at \cos at\} = s\mathscr{L}\{\sin^2 at\}.$$

By (10),

$$\mathscr{L}\{\sin^2 at\} = \frac{2a^2}{s(s^2 + 4a^2)}.$$

Thus,

$$\mathscr{L}\{2a \sin at \cos at\} = \frac{2a^2}{s^2 + 4a^2}.$$

Since $2a \sin at \cos at = a \sin 2at$, we also have

$$\mathscr{L}\{\sin 2at\} = \frac{2a}{s^2 + 4a^2}.$$

THEOREM 4.

Hypothesis

1. Let f be a real function having a continuous $(n-1)$st derivative $f^{(n-1)}$ (and hence f, $f', \ldots, f^{(n-2)}$ are also continuous) for $t \geq 0$; and assume that $f, f', \ldots, f^{(n-1)}$ are all of exponential order $e^{\alpha t}$.

2. Suppose $f^{(n)}$ is piecewise continuous in every finite closed interval $0 \leq t \leq b$.

Conclusion. *$\mathscr{L}\{f^{(n)}\}$ exists for $s > \alpha$ and*

$$\mathscr{L}\{f^{(n)}(t)\} = s^n\mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0). \tag{12}$$

Example.

We apply Theorem 4, with $n = 2$, to find $\mathscr{L}\{\sin bt\}$, which we have already found directly and given by (5). Clearly the function f defined by $f(t) = \sin bt$ satisfies the hypotheses of the theorem with $\alpha = 0$. For $n = 2$, Equation (12) becomes

$$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0). \tag{13}$$

We have $f'(t) = b \cos bt$, $f''(t) = -b^2 \sin bt$, $f(0) = 0$, $f'(0) = b$. Substituting into Equation (13) we find

$$\mathscr{L}\{-b^2 \sin bt\} = s^2\mathscr{L}\{\sin bt\} - b,$$

and so

$$(s^2 + b^2)\mathscr{L}\{\sin bt\} = b.$$

Thus,

$$\mathscr{L}\{\sin bt\} = \frac{b}{s^2 + b^2} \qquad (s > 0),$$

which is the result (5), already found directly.