1)

$$A-\lambda I=\begin{bmatrix}-4&-2\\3&3\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}-4&-2\\3&3\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}-4-\lambda&-2\\3&3-\lambda\end{bmatrix}.$$

$$\det\begin{bmatrix}-4-\lambda&-2\\3&3-\lambda\end{bmatrix}=(-4-\lambda)(3-\lambda)-3(-2)=0$$

$$\lambda^2+\lambda-6=0.$$

The equation $\lambda^2+\lambda-6=0$ is the characteristic equation of the matrix $A$.

$$\lambda^2+\lambda-6=(\lambda+3)(\lambda-2)=0$$

and so $\lambda=-3$ or $\lambda=2$. The two eigenvalues of $A$ are $\lambda_1=-3$ and $\lambda_2=2$.

We need to solve $(A-\lambda I)X=0$ to get the eigenvectors. Now

$$(A-\lambda I)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-4-\lambda&-2\\3&3-\lambda\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0 \quad \text{where } X=\begin{bmatrix}x\\y\end{bmatrix}.$$

For $\lambda_1=-3$ we have

$$\begin{bmatrix}-4-(-3)&-2\\3&3-(-3)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1&-2\\3&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0.$$

This gives two linear equations

$$x+2y=0$$
$$x=-2y.$$

Let $y=t,\ t\in\mathbb{R}$ then the solution is $x=-2t,\ y=t$ and the eigenvector has the form

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2t\\t\end{bmatrix}=t\begin{bmatrix}-2\\1\end{bmatrix}.$$

So, the eigenvector corresponding to $\lambda_1=-3$ is $X_1=\begin{bmatrix}-2\\1\end{bmatrix}$.

For $\lambda_2=2$, we have

$$\begin{bmatrix}-4-2&-2\\3&3-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-6&-2\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0.$$

This gives two linear equations

$$-6x - 2y = 0$$
$$3x + y = 0.$$

Let $x = t$ then from the second equation, $y = -3t$ and the eigenvector has the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -3t \end{bmatrix} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix},$$

the eigenvalue corresponding to $\lambda_2 = 2$ is $X_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

**2)**

$$\det A = -1\begin{vmatrix} -1 & -2 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ -2 & 5 \end{vmatrix} = 10$$

$$A_{11} = (-1)^2 \cdot \begin{vmatrix} -1 & -2 \\ 5 & 4 \end{vmatrix} = 1 \cdot ((-4) - (-10)) = 6$$

$$A_{12} = (-1)^3 \cdot \begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = -1(8-4) = -4$$

$$A_{13} = (-1)^4 \cdot \begin{vmatrix} 2 & -1 \\ -2 & 5 \end{vmatrix} = 10 - 2 = 8$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} = -1(8-15) = 7$$

$$A_{22} = (-1)^4 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} = (-4) - (-6) = 2$$

$$A_{23} = (-1)^5 \begin{vmatrix} -1 & 2 \\ -2 & 5 \end{vmatrix} = -1((-5) - (-4)) = 1$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = (-4) - (-3) = -1$$

$$A_{32} = (-1)^5 \begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} = -1(2-6) = 4$$

$$A_{33} = (-1)^6 \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - 4 = -3$$

$$\mathrm{Adj}(A) = \begin{bmatrix} 6 & 7 & -1 \\ -4 & 2 & 4 \\ 8 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot \mathrm{Adj}(A) = \frac{1}{10} \cdot \begin{bmatrix} 6 & 7 & -1 \\ -4 & 2 & 4 \\ 8 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3/5 & 7/10 & -1/10 \\ -2/5 & 1/5 & 2/5 \\ 4/5 & 1/10 & -3/10 \end{bmatrix}$$

**3)**

**a)**

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n &= b_s \end{aligned}\right\} \qquad (1)$$

**Kronecker-Capelli theorem.** *A system of linear equations* (1) *is consistent if and only if the rank of the augmented matrix* $\bar{A}$ *is equal to the rank of the matrix* $A$.

$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -3 & 4 \\ 1 & -12 & 17 \end{bmatrix}$, $\det A = 2\begin{vmatrix} -3 & 4 \\ -12 & 17 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 4 \\ 11 & 17 \end{vmatrix} +$

$(-1)\begin{vmatrix} 1 & -3 \\ 1 & -12 \end{vmatrix} = -6 + 27 - 21 = 0$

$\begin{vmatrix} -3, & 4 \\ -12 & 17 \end{vmatrix} = -3 \neq 0 \qquad \operatorname{rank} A = 2$

$\bar{A} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & -3 & 4 & 2 \\ 11 & -12 & 17 & 3 \end{bmatrix}$, $\begin{vmatrix} 1 & -1 & 1 \\ -3 & 4 & 2 \\ -12 & 17 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 2 \\ 17 & 3 \end{vmatrix} -$

$(-1)\begin{vmatrix} -3 & 2 \\ -12 & 3 \end{vmatrix} + 1\begin{vmatrix} -3 & 4 \\ -12 & 17 \end{vmatrix} = (12 - 34) + (-9 + 24) +$

$(-51 + 48) = -22 + 15 - 3 \neq 0$

$\Rightarrow \operatorname{rank} \bar{A} = 3$.

$\operatorname{ran} A \neq \operatorname{rank} \bar{A} \Rightarrow$ The system is inconsistent.

**b)**

$$\Delta = \begin{vmatrix} 5 & -1 & -1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 5(4-9) + (2-12) - (3-8) = -25 - 10 + 5 = -30;$$

$$\Delta_1 = \begin{vmatrix} 0 & -1 & -1 \\ 14 & 2 & 3 \\ 16 & 3 & 2 \end{vmatrix} = (28-48) - (42-32) = -20 - 10 = -30.$$

$x_1 = \Delta_1/\Delta = 1;$

$$\Delta_2 = \begin{vmatrix} 5 & 0 & -1 \\ 1 & 14 & 3 \\ 4 & 16 & 2 \end{vmatrix} = 5(28-48) - (16-56) = -100 + 40 = -60.$$

$x_2 = \Delta_2/\Delta = 2;$

$$\Delta_3 = \begin{vmatrix} 5 & -1 & 0 \\ 1 & 2 & 14 \\ 4 & 3 & 16 \end{vmatrix} = 5(32-42) + (16-56) = -50 - 40 = -90.$$

$x_3 = \Delta_3/\Delta = 3.$

**4)**

Theorem. A linear transformation $A: V_1 \to V_2$ is one-to-one if and only if $\mathcal{N}(A) = 0$

a) $A(x_1, x_2) = (0, 0) \Rightarrow$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$\mathcal{N}(A) = \{t \in \mathbb{R} \mid t(1, -1)\}$

A is not one-to-one => not invertible.

b) $A(x_1, x_2) = (0, 0) \Rightarrow$

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 = 0 \Rightarrow \mathcal{N}(A) = \{(0,0)\}$$

=> A is one-to-one => invertible.

$$\begin{cases} x_1 + x_2 = y_1 \\ x_1 - x_2 = y_2 \end{cases} \Rightarrow \begin{matrix} x_1 = \frac{y_1 + y_2}{2} \\ x_2 = \frac{y_1 - y_2}{2} \end{matrix} \Rightarrow$$

$$A^{-1}(x_1, x_2) = \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}\right)$$

**5)**

$$(\cos x - x \sin x + y^2)\,dx + 2xy\,dy = 0$$

The differential equation is exact because

$$\frac{\partial}{\partial y}[\cos x - x \sin x + y^2] = 2y = \frac{\partial}{\partial x}[2xy].$$

Because $N(x, y)$ is simpler than $M(x, y)$, it is better to begin by integrating $N(x, y)$.

$$f(x, y) = \int N(x, y)\,dy = \int 2xy\,dy = xy^2 + g(x)$$

$$f_x(x, y) = \frac{\partial}{\partial x}[xy^2 + g(x)] = y^2 + g'(x) = \cos x - x \sin x + y^2$$

Thus, $g'(x) = \cos x - x \sin x$ and

$$g(x) = \int (\cos x - x \sin x)\,dx$$

$$= x \cos x + C_1$$

which implies that $f(x, y) = xy^2 + x \cos x + C_1$, and the general solution is

$$xy^2 + x \cos x = C.$$