EXACT DIFFERENTIAL EQUATIONS

DEFINITION

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential *dF of the function F is defined by the formula*

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\,dx + \frac{\partial F(x, y)}{\partial y}\,dy$$

for all $(x, y) \in D$.

Example.

Let F be the function of two real variables defined by

$$F(x, y) = xy^2 + 2x^3y$$

for all real (x, y). Then

$$\frac{\partial F(x, y)}{\partial x} = y^2 + 6x^2y, \qquad \frac{\partial F(x, y)}{\partial y} = 2xy + 2x^3,$$

and the total differential dF is defined by

$$dF(x, y) = (y^2 + 6x^2y)\,dx + (2xy + 2x^3)\,dy$$

for all real (x, y).

DEFINITION

The expression

$$M(x, y)\,dx + N(x, y)\,dy \tag{1}$$

is called an exact differential *in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x, y) \in D$. That is, expression (1) is an exact differential in D if there exists a function F such that*

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad and \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

for all $(x, y) \in D$.

If $M(x, y)\, dx + N(x, y)\, dy$ *is an exact differential, then the differential equation*

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is called an exact differential equation.

Example.

The differential equation

$$y^2\, dx + 2xy\, dy = 0 \tag{2}$$

is an exact differential equation, since the expression $y^2\, dx + 2xy\, dy$ is an exact differential. Indeed, it is the total differential of the function F defined for all (x, y) by $F(x, y) = xy^2$, since the coefficient of dx is $\partial F(x, y)/(\partial x) = y^2$ and that of dy is $\partial F(x, y)/(\partial y) = 2xy$. On the other hand, the more simple appearing equation

$$y\, dx + 2x\, dy = 0,$$

obtained from (2) by dividing through by y, is *not* exact.

It is clear that we need a simple test to determine whether or not a given differential equation is exact. This is given by the following theorem.

THEOREM.

Consider the differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0, \tag{3}$$

where M *and* N *have continuous first partial derivatives at all points* (x, y) *in a rectangular domain* D.

1. If *the differential equation* (3) *is exact in D*, then

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \tag{4}$$

for all $(x, y) \in D$.

2. Conversely, if

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

for all $(x, y) \in D$, then *the differential equation* (3) *is exact in D.*

Proof. If the differential equation (3) is exact in D, then $M\,dx + N\,dy$ is an exact differential in D. By definition of an exact differential, there exists a function F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

for all $(x, y) \in D$. Then

$$\frac{\partial^2 F(x, y)}{\partial y\, \partial x} = \frac{\partial M(x, y)}{\partial y} \quad \text{and} \quad \frac{\partial^2 F(x, y)}{\partial x\, \partial y} = \frac{\partial N(x, y)}{\partial x}$$

for all $(x, y) \in D$. But, using the continuity of the first partial derivatives of M and N, we have

$$\frac{\partial^2 F(x, y)}{\partial y\, \partial x} = \frac{\partial^2 F(x, y)}{\partial x\, \partial y}$$

and therefore

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

for all $(x, y) \in D$.

Conversely, let us suppose that

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

for all $(x, y) \in D$, and set out to show that $M\,dx + N\,dy = 0$ is exact in D. This means that we must prove that there exists a function F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \tag{5}$$

and

$$\frac{\partial F(x, y)}{\partial y} = N(x, y) \tag{6}$$

Let us assume that

$$F(x, y) = \int M(x, y)\,\partial x + \phi(y). \tag{7}$$

Then

$$\frac{\partial F(x, y)}{\partial x} = M(x, y).$$

Differentiating (7) partially with respect to y,

$$\frac{\partial F(x, y)}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\,\partial x + \frac{d\phi(y)}{dy}.$$

Then

$$\frac{d\phi(y)}{dy} = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, \partial x.$$

That is

$$N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, \partial x$$

must be independent of x.

We shall show that

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, \partial x\right] = 0.$$

We at once have

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, \partial x\right] = \frac{\partial N(x, y)}{\partial x} - \frac{\partial^2}{\partial x\, \partial y} \int M(x, y)\, \partial x$$

$$= \frac{\partial N(x, y)}{\partial x} - \frac{\partial^2}{\partial y\, \partial x} \int M(x, y)\, \partial x = \frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}.$$

But $\quad \dfrac{\partial M(x, y)}{\partial y} = \dfrac{\partial N(x, y)}{\partial x} \quad$ for all $(x, y) \in D$.

Thus

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, \partial x\right] = 0$$

Thus we may write

$$\phi(y) = \int \left[N(x, y) - \int \frac{\partial M(x, y)}{\partial y} \, \partial x \right] dy.$$

Substituting this into Equation (7), we have

$$F(x, y) = \int M(x, y) \, \partial x + \int \left[N(x, y) - \int \frac{\partial M(x, y)}{\partial y} \, \partial x \right] dy.$$

This $F(x, y)$ thus satisfies both (5) and (6) for all $(x, y) \in D$, and so $M \, dx + N \, dy = 0$ is exact in D.

This completes the proof.

Example.

For the equation

$$y^2 \, dx + 2xy \, dy = 0$$

we have

$$M(x, y) = y^2, \qquad N(x, y) = 2xy,$$

$$\frac{\partial M(x, y)}{\partial y} = 2y = \frac{\partial N(x, y)}{\partial x}$$

for all (x, y). Thus equation is exact in every rectangular domain D. On the other hand, for the equation

$$y\,dx + 2x\,dy = 0,$$

we have

$$M(x, y) = y, \qquad N(x, y) = 2x,$$

$$\frac{\partial M(x, y)}{\partial y} = 1 \neq 2 = \frac{\partial N(x, y)}{\partial x}$$

for all (x, y). Thus equation is not exact in any rectangular domain D.

Example.

Consider the differential equation

$$(2x \sin y + y^3 e^x)\,dx + (x^2 \cos y + 3y^2 e^x)\,dy = 0.$$

Here

$$M(x, y) = 2x \sin y + y^3 e^x,$$

$$N(x, y) = x^2 \cos y + 3y^2 e^x,$$

$$\frac{\partial M(x, y)}{\partial y} = 2x \cos y + 3y^2 e^x = \frac{\partial N(x, y)}{\partial x}$$

in every rectangular domain D. Thus this differential equation is exact in every such domain.

Example. Solve $2xy + 6x + \left(x^2 - 4\right) y' = 0$.

$$
\begin{aligned}
\left(x^2 - 4\right) y' &= -2xy - 6x, \\
&= -2xy - 6x, \\
\frac{y'}{y+3} &= -\frac{2x}{x^2 - 4}, \qquad x \neq \pm 2 \\
\ln(|y+3|) &= -\ln\left(\left|x^2 - 4\right|\right) + C, \\
\ln(|y+3|) + \ln\left(\left|x^2 - 4\right|\right) &= C,
\end{aligned}
$$

$$
\begin{aligned}
\left|(y+3)\left(x^2 - 4\right)\right| &= A, \\
(y+3)\left(x^2 - 4\right) &= A, \\
y + 3 &= \frac{A}{x^2 - 4}, \\
y &= -3\,.
\end{aligned}
$$

Example.

Solve the equation

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0.$$

Our first duty is to determine whether or not the equation is exact. Here

$$M(x, y) = 3x^2 + 4xy, \qquad N(x, y) = 2x^2 + 2y,$$

$$\frac{\partial M(x, y)}{\partial y} = 4x, \qquad \frac{\partial N(x, y)}{\partial x} = 4x,$$

for all real (x, y), and so the equation is exact in every rectangular domain D. Thus we must find F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) = 3x^2 + 4xy \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y) = 2x^2 + 2y.$$

From the first of these,

$$F(x, y) = \int M(x, y)\,\partial x + \phi(y) = \int (3x^2 + 4xy)\,\partial x + \phi(y)$$
$$= x^3 + 2x^2y + \phi(y).$$

Then

$$\frac{\partial F(x, y)}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy}.$$

But we must have

$$\frac{\partial F(x, y)}{\partial y} = N(x, y) = 2x^2 + 2y.$$

Thus

$$2x^2 + 2y = 2x^2 + \frac{d\phi(y)}{dy}$$

or

$$\frac{d\phi(y)}{dy} = 2y.$$

Thus $\phi(y) = y^2 + c_0$, where c_0 is an arbitrary constant, and so

$$F(x, y) = x^3 + 2x^2y + y^2 + c_0.$$

Hence a one-parameter family of solution is $F(x, y) = c_1$, or

$$x^3 + 2x^2y + y^2 + c_0 = c_1.$$

Combining the constants c_0 and c_1 we may write this solution as

$$x^3 + 2x^2y + y^2 = c,$$

where $c = c_1 - c_0$ is an arbitrary constant.

We now consider an alternative procedure.

Method of Grouping. We shall now solve the differential equation of this example by grouping the terms in such a way that its left member appears as the sum of certain exact differentials. We write the differential equation

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$$

in the form

$$3x^2\,dx + (4xy\,dx + 2x^2\,dy) + 2y\,dy = 0.$$

We now recognize this as

$$d(x^3) + d(2x^2y) + d(y^2) = d(c),$$

where c is an arbitrary constant, or

$$d(x^3 + 2x^2y + y^2) = d(c).$$

From this we have at once

$$x^3 + 2x^2y + y^2 = c.$$

Example.

Solve the initial-value problem

$$(2x \cos y + 3x^2 y)\, dx + (x^3 - x^2 \sin y - y)\, dy = 0,$$
$$y(0) = 2.$$

We first observe that the equation is exact in every rectangular domain D, since

$$\frac{\partial M(x, y)}{\partial y} = -2x \sin y + 3x^2 = \frac{\partial N(x, y)}{\partial x}$$

for all real (x, y).

Standard Method. We must find F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) = 2x \cos y + 3x^2 y$$

and

$$\frac{\partial F(x, y)}{\partial y} = N(x, y) = x^3 - x^2 \sin y - y.$$

Then

$$F(x, y) = \int M(x, y)\, \partial x + \phi(y)$$

$$= \int (2x \cos y + 3x^2 y)\, \partial x + \phi(y)$$

$$= x^2 \cos y + x^3 y + \phi(y),$$

$$\frac{\partial F(x, y)}{\partial y} = -x^2 \sin y + x^3 + \frac{d\phi(y)}{dy}.$$

But also

$$\frac{\partial F(x, y)}{\partial y} = N(x, y) = x^3 - x^2 \sin y - y$$

and so

$$\frac{d\phi(y)}{dy} = -y$$

and hence

$$\phi(y) = -\frac{y^2}{2} + c_0.$$

Thus

$$F(x, y) = x^2 \cos y + x^3 y - \frac{y^2}{2} + c_0.$$

Hence a one-parameter family of solutions is $F(x, y) = c_1$, which may be expressed as

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = c.$$

Applying the initial condition $y = 2$ when $x = 0$, we find $c = -2$. Thus the solution of the given initial-value problem is

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = -2.$$

Method of Grouping. We group the terms as follows:

$$(2x \cos y\, dx - x^2 \sin y\, dy) + (3x^2 y\, dx + x^3\, dy) - y\, dy = 0.$$

Thus we have

$$d(x^2 \cos y) + d(x^3 y) - d\left(\frac{y^2}{2}\right) = d(c);$$

and so

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = c$$

is a one-parameter family of solutions of the differential equation. Of course the initial condition $y(0) = 2$ again yields the particular solution already obtained.

Integrating Factors

Given the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

if

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x},$$

then the equation is exact and we can obtain a one-parameter family of solutions by one of the procedures explained above. But if

$$\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x},$$

then the equation is *not* exact and the above procedures do not apply. What shall we do in such a case? Perhaps we can multiply the nonexact equation by some expression that will transform it into an essentially equivalent exact equation.

Let us consider the equation

$$y\,dx + 2x\,dy = 0,$$

which is *not* exact. However, if we multiply this equation by y, it is transformed into the essentially equivalent equation

$$y^2\,dx + 2xy\,dy = 0,$$

which *is* exact. Since this resulting exact equation is integrable, we call y an *integrating factor*. In general, we have the following definition:

DEFINITION

If the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is not *exact in a domain D but the differential equation*

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0$$

is *exact in D, then $\mu(x, y)$ is called an* integrating factor *of the differential equation* (1).

Example.

Consider the differential equation

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0.$$

$$M(x, y) = 3y + 4xy^2, \qquad N(x, y) = 2x + 3x^2y,$$

$$\frac{\partial M(x, y)}{\partial y} = 3 + 8xy, \qquad \frac{\partial N(x, y)}{\partial x} = 2 + 6xy.$$

$$\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$$

Equation is *not* exact in any rectangular domain D.

Let $\mu(x, y) = x^2y$. Then the corresponding differential equation is

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0.$$

This equation is exact in every rectangular domain D, since

$$\frac{\partial[\mu(x, y)M(x, y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x, y)N(x, y)]}{\partial x}$$

for all real (x, y). Hence $\mu(x, y) = x^2y$ is an integrating factor.

SEPARABLE EQUATIONS

DEFINITION

An equation of the form

$$F(x)G(y)\,dx + f(x)g(y)\,dy = 0 \tag{2}$$

is called an equation with variables separable *or simply a* separable equation.

For example, the equation $(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0$ is a separable equation.

In general the separable equation (2) is not exact, but it possesses an obvious integrating factor, namely $1/f(x)G(y)$.

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0.$$

This equation is exact, since

$$\frac{\partial}{\partial y}\left[\frac{F(x)}{f(x)}\right] = 0 = \frac{\partial}{\partial x}\left[\frac{g(y)}{G(y)}\right].$$

Denoting $F(x)/f(x)$ by $M(x)$ and $g(y)/G(y)$ by $N(y)$, equation takes the form $M(x)\,dx + N(y)\,dy = 0$.

Thus

$$\int M(x)\,dx + \int N(y)\,dy = c,$$

where c is the arbitrary constant.

Since we obtained the separated exact equation from the nonexact equation by the integrating factor $1/f(x)G(y)$, solutions may have been lost or gained in this process. In formally multiplying by the integrating factor $1/f(x)G(y)$, we actually divided by $f(x)G(y)$. We did this under the tacit assumption that neither $f(x)$ nor $G(y)$ is zero. Now, we should investigate the possible loss or gain of solutions that may have occurred in this formal process. In particular, regarding y as the dependent variable as usual, we consider the situation that occurs if $G(y)$ is zero. Writing the original differential equation in the derivative form

$$f(x)g(y)\frac{dy}{dx} + F(x)G(y) = 0,$$

we immediately note the following: If y_0 is any real number such that $G(y_0) = 0$, then $y = y_0$ is a (constant) solution of the original differential equation; and this solution may (or may not) have been lost in the formal separation process.

In finding a one-parameter family of solutions a separable equation, we shall always make the assumption that any factors by which we divide in the formal separation process are not zero. Then we must find the solutions $y = y_0$ of the equation $G(y) = 0$ and determine whether any of these are solutions of the original equation which were lost in the formal separation process.

Example.

Solve the equation

$$(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0.$$

The equation is separable; separating the variables by dividing by x^3y^4, we obtain

$$\frac{(x-4)\,dx}{x^3} - \frac{(y^2-3)\,dy}{y^4} = 0$$

or

$$(x^{-2}-4x^{-3})\,dx - (y^{-2}-3y^{-4})\,dy = 0.$$

Integrating, we have the one-parameter family of solutions

$$-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c,$$

where c is the arbitrary constant.

In dividing by x^3y^4 in the separation process, we assumed that $x^3 \neq 0$ and $y^4 \neq 0$. We now consider the solution $y = 0$ of $y^4 = 0$. It is not a member of the one-parameter family of solutions which we obtained. However, writing the original differential equation of the problem in the derivative form

$$\frac{dy}{dx} = \frac{(x-4)y^4}{x^3(y^2-3)},$$

it is obvious that $y = 0$ *is* a solution of the original equation. We conclude that it is a solution which was lost in the separation process.