$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

M, N are both homogeneous of the *same* degree m.

$$u = \frac{y}{x} \qquad\qquad dy = x\,du + u\,dx$$

$$M(x, y) = M(x, xu) = x^m M(1, u)$$

$$N(x, y) = x^m N(1, u).$$

$$x^m \{[M(1, u) + N(1, u) \cdot u]\,dx + xN(1, u)\,du\} = 0.$$

$$\mu = \frac{1}{x^{m+1}[M(1, u) + N(1, u) \cdot u]},$$

$$\mu = \frac{1}{xM(x, y) + yN(x, y)}.$$

$$xM + yN \neq 0$$

$$\frac{M}{N} = -\frac{y}{x}, \quad y\,dx - x\,dy = 0.$$

Example.

$$(x - y)\,dx + (x + y)\,dy = 0,$$

$$\mu = \frac{1}{x(x-y) + y(x+y)} = \frac{1}{x^2 + y^2}.$$

$$\frac{x\,dx + y\,dy}{x^2 + y^2} + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0,$$

$$\frac{1}{2}\,d\ln(x^2 + y^2) + d\arctan\frac{y}{x} = 0,$$

$$\sqrt{x^2 + y^2} = Ce^{-\arctan\frac{y}{x}}.$$

Example. Solve $(x+3y^2)dx+2xydy=0$.

$$M(x,y)=x+3y^2 \qquad \Rightarrow \qquad \frac{\partial M}{\partial y}=6y,$$

$$N(x,y)=2xy \qquad \Rightarrow \qquad \frac{\partial N}{\partial x}=2y.$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2xy}(6y-2y)=\frac{2}{x}$

$$\mu=e^{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx} \qquad \Rightarrow \qquad \mu=e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2.$$

Multiplying the given DE by the above integrating factor gives

$$x^2(x+3y^2)dx+x^2(2xy)dy=0 \quad \Rightarrow \quad (x^3+3x^2y^2)dx+2x^3ydy=0.$$

Check, $M(x,y)=x^3+3x^2y^2 \qquad \Rightarrow \qquad \frac{\partial M}{\partial y}=6x^2y,$

$$N(x,y)=2x^3y \qquad \Rightarrow \qquad \frac{\partial N}{\partial x}=6x^2y.$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the given DE is reduced to exact one.

$$M=\frac{\partial f}{\partial x}=x^3+3x^2y^2 \qquad \Rightarrow \qquad f=\frac{x^4}{4}+x^3y^2+g(y),$$

$$\frac{\partial f}{\partial y}=2x^3y+g'(y), \qquad \text{but} \qquad \frac{\partial f}{\partial y}=N,$$

$$2x^3y + g'(y) = N = 2x^3y \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = C_1,$$

$$f = \frac{x^4}{4} + x^3y^2 + C_1, \qquad \text{but} \qquad f = C_2,$$

$$\frac{x^4}{4} + x^3y^2 + C_1 = C_2 \quad \Rightarrow \quad \frac{x^4}{4} + x^3y^2 = C_3, \qquad [C_3 = C_2 - C_1]$$

or $\qquad x^4 + 4x^3y^2 = C. \qquad [C = 4C_3]$

Example. Solve $(\sin y + x^2 + 2x)dx = \cos y dy$.

$$(\sin y + x^2 + 2x)dx = \cos y dy \quad \Rightarrow \quad (\sin y + x^2 + 2x)dx - \cos y dy = 0.$$

$$M(x,y) = \sin y + x^2 + 2x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \cos y,$$

$$N(x,y) = -\cos y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 0.$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\qquad \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{-\cos y}(\cos y - 0) = -1$

$$\mu = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx} \quad \Rightarrow \quad \mu = e^{\int (-1)\,dx} = e^{-x}.$$

Multiplying the given DE by the above integrating factor gives

$$e^{-x}(\sin y + x^2 + 2x)dx - e^{-x}\cos y dy = 0.$$

$$N=\frac{\partial f}{\partial y}=-e^{-x}\cos y \qquad \Rightarrow \qquad f=-e^{-x}\sin y+g(x),$$

$$\frac{\partial f}{\partial x}=e^{-x}\sin y+g'(x), \qquad \text{but} \qquad \frac{\partial f}{\partial x}=M,$$

$$e^{-x}\sin y+g'(x)=M=e^{-x}(\sin y+x^2+2x) \quad \Rightarrow \quad g'(x)=x^2e^{-x}+2xe^{-x},$$

$$g(x)=-x^2e^{-x}-2xe^{-x}-2e^{-x}-2xe^{-x}-2e^{-x} \Rightarrow g(x)=-x^2e^{-x}-4xe^{-x}-4e^{-x},$$

$$f=-e^{-x}\sin y-x^2e^{-x}-4xe^{-x}-4e^{-x}, \qquad \text{but} \qquad f=C,$$

$$-e^{-x}\sin y-x^2e^{-x}-4xe^{-x}-4e^{-x}=C_1,$$

or $\quad x^2+4x+4+\sin y=Ce^x. \qquad [C=-C_1]$

Example. Solve $\quad (y+2x)dx+x(y+x+1)dy=0$.

$$M(x,y)=y+2x \qquad \Rightarrow \qquad \frac{\partial M}{\partial y}=1,$$

$$N(x,y)=xy+x^2+x \quad \Rightarrow \quad \frac{\partial N}{\partial x}=y+2x+1.$$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{1-(y+2x+1)}{xy+x^2+x}=\dfrac{-(y+2x)}{x(y+x+1)}$ (is not function of x only)

Check, $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1-(y+2x+1)}{y+2x}=\frac{-(y+2x)}{y+2x}=-1$ (function of y only)

$$\therefore \mu = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dy} \quad \Rightarrow \quad \mu = e^{-\int(-1)dy} = e^{y}.$$

Multiplying the given DE by the above integrating factor gives

$$e^{y}(y+2x)dx + xe^{y}(y+x+1)dy = 0.$$

$$M=\frac{\partial f}{\partial x}=ye^{y}+2xe^{y} \qquad \Rightarrow \qquad f = xye^{y}+x^{2}e^{y}+g(y),$$

$$\frac{\partial f}{\partial y}=xye^{y}+xe^{y}+x^{2}e^{y}+g'(y), \qquad \text{but} \qquad \frac{\partial f}{\partial y}=N,$$

$$xye^{y}+xe^{y}+x^{2}e^{y}+g'(y)=N=xye^{y}+x^{2}e^{y}+xe^{y} \;\Rightarrow\; g'(y)=0 \;\Rightarrow\; g(y)=C_{1},$$

$$f = xye^{y}+x^{2}e^{y}+C_{1}, \qquad \text{but} \qquad f=C_{2},$$

$$xye^{y}+x^{2}e^{y}+C_{1}=C_{2} \quad \Rightarrow \quad xy+x^{2}=Ce^{-y}. \qquad [C=C_{2}-C_{1}]$$

Example. Solve $\cos y\,dx + (2x\sin y - \cos^3 y)\,dy = 0$.

$$M = \cos y \qquad \Rightarrow \qquad \frac{\partial M}{\partial y} = -\sin y,$$

$$N = 2x\sin y - \cos^3 y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2\sin y.$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, then the given DE is not exact.

Check, $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{-\sin y - 2\sin y}{2x\sin y - \cos^3 y} = \dfrac{-3\sin y}{2x\sin y - \cos^3 y}$ (is not function of x only)

Check, $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{-\sin y - 2\sin y}{\cos y} = \dfrac{-3\sin y}{\cos y}$, (function of y only)

$$\mu = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dy} \quad \Rightarrow \quad \mu = e^{-\int \frac{-3\sin y}{\cos y}dy} = e^{-3\ln\cos y} = \cos^{-3} y = \frac{1}{\cos^3 y}.$$

Multiplying the given DE by the above integrating factor gives

$$\frac{1}{\cos^3 y}(\cos y)\,dx + \frac{1}{\cos^3 y}(2x\sin y - \cos^3 y)\,dy = 0 \Rightarrow \frac{1}{\cos^2 y}\,dx + \left(\frac{2x\sin y}{\cos^3 y} - 1\right)dy = 0$$

$$M = \frac{\partial f}{\partial x} = \frac{1}{\cos^2 y} \qquad \Rightarrow \qquad f = \frac{x}{\cos^2 y} + g(y),$$

$$\frac{\partial f}{\partial y} = -2x\cos^{-3}.(-\sin y) + g'(y) = \frac{2x\sin y}{\cos^3 y} + g'(y), \qquad \text{but} \qquad \frac{\partial f}{\partial y} = N,$$

$$\frac{2x\sin y}{\cos^3 y} + g'(y) = N = \frac{2x\sin y}{\cos^3 y} - 1 \quad \Rightarrow \quad g'(y) = -1 \quad \Rightarrow \quad g(y) = -y,$$

$$f = \frac{x}{\cos^2 y} - y, \qquad \text{but} \qquad f = C,$$

$$\frac{x}{\cos^2 y} - y = C, \qquad \text{or} \qquad x = (y + C)\cos^2 y.$$

Example. $y(x^2 + y^2)dx - x(x^2 + 2y^2)dy = 0$.

In this case $M = y(x^2+y^2)$ and $N = -x(x^2+2y^2)$ are homogeneous functions of degree 3. Using the integrating factor $\mu(x, y) = (xM + yN)^{-1} = -1/(xy^3)$, we can conclude that there exists a function $F(x, y)$ such that

$$dF = -\frac{x^2 + y^2}{xy^2}dx + \frac{x^2 + 2y^2}{y^3}dy.$$

Then, the solution is given by $F(x, y) = C$, where C is an integration constant.

Comparing with the chain rule,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

it follows that

$$\frac{\partial F}{\partial x} = -\frac{x^2 + y^2}{xy^2}, \qquad \frac{\partial F}{\partial y} = \frac{x^2 + 2y^2}{y^3}.$$

Integrating the first equation yields

$$F(x,y) = -\frac{x^2}{2y^2} - \ln|x| + f(y)\,,$$

where $f(y)$ is a function of y alone.

$$\frac{\partial F}{\partial y} = \frac{x^2}{y^3} + \frac{df}{dy}\,.$$

Comparing with

$$\frac{\partial F}{\partial y} = \frac{x^2+2y^2}{y^3}\,,$$

we obtain

$$\frac{df}{dy} = \frac{2}{y}\,, \qquad \Longrightarrow \qquad f(y) = 2\ln|y| + C'\,,$$

where C' is an integration constant. That is, $F(x,y) = \ln|y^2/x| - x^2/(2y^2) + C'$ and we have therefore verified that

$$dF = d\left(\ln\left|\frac{y^2}{x}\right| - \frac{x^2}{2y^2}\right) = -\frac{x^2+y^2}{xy^2}dx + \frac{x^2+2y^2}{y^3}dy\,.$$

Thus, the solution is

$$\ln\left|\frac{y^2}{x}\right| - \frac{x^2}{2y^2} = C\,.$$

Example. $(5xy + 4y^2 + 1)\, dx + (x^2 + 2xy)\, dy = 0$

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial (5xy + 4y^2 + 1)}{\partial y} = 5x + 8y$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial (x^2 + 2xy)}{\partial x} = 2x + 2y$$

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] = \frac{1}{x^2 + 2xy}[5x + 8y - (2x + 2y)]$$

$$= \frac{1}{x^2 + 2xy}[3x + 6y] = \frac{3}{x}$$

Because $\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$ depends on x only, then the integrating factor of equation above is

$$e^{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] dx} = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$$

Multiply both equation with x^3, we get

$$x^3(5xy + 4y^2 + 1)\, dx + x^3(x^2 + 2xy)\, dy = 0$$

$$(5x^4y + 4x^3y^2 + 1)\, dx + (x^5 + 2x^4y)\, dy = 0$$

By using method of grouping as follow

$$[5x^4y\, dx + x^5 dy] + [4x^3y^2\, dx + 2x^4y\, dy] + 1\, dx = 0$$

and then define the total derivative as

$$d(x^5y) + d(x^4y^2) + dx = 0$$

Integrating it, we get the solution as follows

$$x^5y + x^4y^2 + x = c.$$

Exercise . *Find the solution of*

$$3x + \frac{6}{y} + \left(\frac{x^2}{y} + \frac{3y}{x}\right) y' = 0$$

Solution. We only need to consider where $xy \neq 0$. Denote

$$M = 3x + \frac{6}{y}, \qquad N = \frac{x^2}{y} + \frac{3y}{x}.$$

By calculations,

$$M_y = -\frac{6}{y^2} \neq \frac{2x}{y} - \frac{3y}{x^2} = N_x.$$

Hence this equation is not an exact equation. Let $\mu = \mu(x, y)$ such that

$$\mu M + \mu N y' = 0$$

is an exact equation. Then

$$\partial_y \mu M + \mu M_y = \partial_y(\mu M) = \partial_x(\mu N) = \partial_x \mu N + \mu N_x,$$

which is

$$\partial_y\mu\left(3x+\frac{6}{y}\right)-\frac{6}{y^2}\mu=\partial_x\mu\left(\frac{x^2}{y}+\frac{3y}{x}\right)+\mu\left(\frac{2x}{y}-\frac{3y}{x^2}\right).$$

Since $\mu=\mu(x)$ or $\mu(y)$ are not solutions, we consider $\mu=\mu(z)=\mu(xy)$. Then

$$\partial_x\mu=y\mu', \qquad \partial_y\mu=x\mu'.$$

$$(\ln|\mu|)'=\frac{\mu'}{\mu}=\frac{1}{xy}=\frac{1}{z}.$$

Therefore, we can take

$$\mu=z=xy.$$

Now we get the new equation

$$3x^2y+6x+(x^3+3y^2)y'=0,$$

which is exact. You can solve it by finding the potential function Ψ.

More and more specialized results concerning particular types of integrating factors corresponding to particular types of equations are known. However, instead of going into such special cases we shall now proceed to investigate certain useful transformations.

We have already made use of transformations in reducing both homogeneous and Bernoulli equations to more tractable types. Another type of equation that can be reduced to a more basic type by means of a suitable transformation is an equation of the form

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0.$$

We state the following theorem concerning this equation.

THEOREM.

Consider the equation

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0,$$

where a_1, b_1, c_1, a_2, b_2, and c_2 are constants.

Case 1. *If $a_2/a_1 \neq b_2/b_1$, then the transformation*

$$x = X + h,$$

$$y = Y + k,$$

where (h, k) is the solution of the system

$$a_1h + b_1k + c_1 = 0,$$

$$a_2h + b_2k + c_2 = 0,$$

reduces the equation to the homogeneous equation

$$(a_1X + b_1Y)\,dX + (a_2X + b_2Y)\,dY = 0$$

in the variables X and Y.

Case 2. *If $a_2/a_1 = b_2/b_1 = k$, then the transformation $z = a_1x + b_1y$ reduces the equation to a separable equation in the variables x and z.*

Example. $(x - 2y + 1)\,dx + (4x - 3y - 6)\,dy = 0.$

Here $a_1 = 1$, $b_1 = -2$, $a_2 = 4$, $b_2 = -3$, and so

$$\frac{a_2}{a_1} = 4 \quad \text{but} \quad \frac{b_2}{b_1} = \frac{3}{2} \neq \frac{a_2}{a_1}.$$

Therefore this is Case 1 of Theorem. We make the transformation

$$x = X + h,$$
$$y = Y + k,$$

where (h, k) is the solution of the system

$$h - 2k + 1 = 0,$$
$$4h - 3k - 6 = 0.$$

The solution of this system is $h = 3$, $k = 2$, and so the transformation is

$$x = X + 3,$$
$$y = Y + 2.$$

This reduces the equation to the homogeneous equation

$$(X - 2Y)\,dX + (4X - 3Y)\,dY = 0.$$

Now we first put this homogeneous equation in the form

$$\frac{dY}{dX} = \frac{1 - 2(Y/X)}{3(Y/X) - 4}$$

and let $Y = vX$ to obtain

$$v + X\frac{dv}{dX} = \frac{1 - 2v}{3v - 4}.$$

This reduces to

$$\frac{(3v-4)\,dv}{3v^2-2v-1} = -\frac{dX}{X}.$$

Integrating (we recommend the use of tables here), we obtain

$$\tfrac{1}{2}\ln|3v^2-2v-1| - \tfrac{3}{4}\ln\left|\frac{3v-3}{3v+1}\right| = -\ln|X| + \ln|c_1|,$$

or

$$\ln(3v^2-2v-1)^2 - \ln\left|\frac{3v-3}{3v+1}\right|^3 = \ln\left(\frac{c_1^4}{X^4}\right),$$

or

$$\ln\left|\frac{(3v+1)^5}{v-1}\right| = \ln\left(\frac{c_1^4}{X^4}\right),$$

or, finally,

$$X^4|(3v+1)^5| = c|v-1|,$$

where $c = c_1^4$. Now replacing v by Y/X, we obtain the solutions in the form

$$|3Y+X|^5 = c|Y-X|.$$

Finally, replacing X by $x-3$ and Y by $y-2$ from the original transformation, we obtain the solutions in the form

$$|3(y-2)+(x-3)|^5 = c|y-2-x+3|$$

or

$$|x+3y-9|^5 = c|y-x+1|.$$

Example. $(x + 2y + 3)\,dx + (2x + 4y - 1)\,dy = 0.$

Here $a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 4$, and $a_2/a_1 = b_2/b_1 = 2$. Therefore, this is Case 2 of Theorem. We therefore let

$$z = x + 2y,$$

and the equation transforms into

$$(z + 3)\,dx + (2z - 1)\left(\frac{dz - dx}{2}\right) = 0$$

or

$$7\,dx + (2z - 1)\,dz = 0,$$

which is separable. Integrating, we have

$$7x + z^2 - z = c.$$

Replacing z by $x + 2y$ we obtain the solution in the form

$$7x + (x + 2y)^2 - (x + 2y) = c$$

or

$$x^2 + 4xy + 4y^2 + 6x - 2y = c.$$

Example. $(3x - y + 1)\,dx - (6x - 2y - 3)dy = 0$

We can rewrite the equation above with

$$(3x - y + 1)\,dx + (-6x + 2y + 3)dy = 0$$

For the equation above, $a_1 = 3, b_1 = -1, a_2 = -6$ and $b_2 = 2$. So, we get

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = -2$$

Therefore, we can solve the equation by reduced it into separable differential equation using $z = 3x - y$. This is implied that

$$dz = 3dx - dy \text{ or } dy = 3dx - dz.$$

Substitute it into the question, we get

$$(z + 1)\,dx + (-2z + 3)(3dx - dz) = 0$$

$$(z + 1 - 6z + 9)\,dx + (2z - 3)dz = 0$$

$$(-5z + 10)dx + (2z - 3)dz = 0$$

$$dx - \frac{2z - 3}{5z - 10}\,dz = 0$$

By using some manipulations we get solution

$$x - \frac{2}{5}\left(z + \frac{1}{2}\ln|10z - 20|\right) = c$$

$$x - \frac{2}{5}\left(3x - y + \frac{1}{2}\ln|10(3x - y) - 20|\right) = c$$