THEOREM.
Consider the equation

$$(a_1 x + b_1 y + c_1)\,dx + (a_2 x + b_2 y + c_2)\,dy = 0,$$

where a_1, b_1, c_1, a_2, b_2, *and* c_2 *are constants.*

Case 1. *If* $a_2/a_1 \neq b_2/b_1$, *then the transformation*

$$x = X + h,$$

$$y = Y + k,$$

where (h, k) *is the solution of the system*

$$a_1 h + b_1 k + c_1 = 0,$$

$$a_2 h + b_2 k + c_2 = 0,$$

reduces the equation to the homogeneous equation

$$(a_1 X + b_1 Y)\,dX + (a_2 X + b_2 Y)\,dY = 0$$

in the variables X *and* Y.

Case 2. *If* $a_2/a_1 = b_2/b_1 = k$, *then the transformation* $z = a_1 x + b_1 y$ *reduces the equation to a separable equation in the variables* x *and* z.

Example. $(x - 2y + 1)\,dx + (4x - 3y - 6)\,dy = 0.$

Here $a_1 = 1$, $b_1 = -2$, $a_2 = 4$, $b_2 = -3$, and so

$$\frac{a_2}{a_1} = 4 \quad \text{but} \quad \frac{b_2}{b_1} = \frac{3}{2} \neq \frac{a_2}{a_1}.$$

Therefore this is Case 1 of Theorem. We make the transformation

$$x = X + h,$$
$$y = Y + k,$$

where (h, k) is the solution of the system

$$h - 2k + 1 = 0,$$
$$4h - 3k - 6 = 0.$$

The solution of this system is $h = 3$, $k = 2$, and so the transformation is

$$x = X + 3,$$
$$y = Y + 2.$$

This reduces the equation to the homogeneous equation

$$(X - 2Y)\,dX + (4X - 3Y)\,dY = 0.$$

Now we first put this homogeneous equation in the form

$$\frac{dY}{dX} = \frac{1 - 2(Y/X)}{3(Y/X) - 4}$$

and let $Y = vX$ to obtain

$$v + X\frac{dv}{dX} = \frac{1 - 2v}{3v - 4}.$$

This reduces to

$$\frac{(3v-4)\,dv}{3v^2-2v-1} = -\frac{dX}{X}.$$

$$3v^2-2v-1=(3v+1)(v-1)$$

$$\frac{3v-4}{3v^2-2v-1} = \frac{1}{2}\,\frac{6v-2}{3v^2-2v-1} + \frac{-3}{3v^2-2v-1} = \frac{1}{2}\,\frac{6v-2}{3v^2-2v-1} +$$

$$\frac{3}{4}\left(\frac{3}{3v+1} - \frac{1}{v-1}\right) = \frac{1}{2}\,\frac{6v-2}{3v^2-2v-1} + \frac{3}{4}\left(\frac{3}{3v+1} - \frac{3}{3v-3}\right)$$

Integrating , we obtain

$$\frac{1}{2}\ln|3v^2-2v-1| - \frac{3}{4}\ln\left|\frac{3v-3}{3v+1}\right| = -\ln|X| + \ln|c_1|,$$

or

$$\ln(3v^2-2v-1)^2 - \ln\left|\frac{3v-3}{3v+1}\right|^3 = \ln\left(\frac{c_1^4}{X^4}\right),$$

or

$$\ln\left|\frac{(3v+1)^5}{v-1}\right| = \ln\left(\frac{c^4}{X^4}\right),$$

or, finally,

$$X^4|(3v+1)^5| = c|v-1|,$$

where $c = c_1^4$. Now replacing v by Y/X, we obtain the solutions in the form

$$|3Y+X|^5 = c|Y-X|.$$

Finally, replacing X by $x-3$ and Y by $y-2$ from the original transformation, we obtain the solutions in the form

$$|3(y-2)+(x-3)|^5 = c|y-2-x+3|$$

or

$$|x+3y-9|^5 = c|y-x+1|.$$

Example. $(3x - y + 1)\,dx - (6x - 2y - 3)dy = 0$

We can rewrite the equation above with

$$(3x - y + 1)\,dx + (-6x + 2y + 3)dy = 0$$

For the equation above, $a_1 = 3, b_1 = -1, a_2 = -6$ and $b_2 = 2$. So, we get

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = -2$$

Therefore, we can solve the equation by reduced it into separable differential equation using $z = 3x - y$. This is implied that

$$dz = 3dx - dy \text{ or } dy = 3dx - dz.$$

Substitute it into the question, we get

$$(z + 1)\,dx + (-2z + 3)(3dx - dz) = 0$$

$$(z + 1 - 6z + 9)\,dx + (2z - 3)dz = 0$$

$$(-5z + 10)dx + (2z - 3)dz = 0$$

$$dx - \frac{2z - 3}{5z - 10}\,dz = 0$$

By using some manipulations we get solution

$$x - \frac{2}{5}\left(z + \frac{1}{2}\ln|10z - 20|\right) = c$$

$$x - \frac{2}{5}\left(3x - y + \frac{1}{2}\ln|10(3x - y) - 20|\right) = c$$

General Theory of Linear Differential Equations.

DEFINITION

A linear ordinary differential equation of order *n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in, the form*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x), \qquad (1)$$

where a_0 is not identically zero. We shall assume that $a_0, a_1, \ldots, a_n$ and F are continuous real functions on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$. The right-hand member $F(x)$ is called the nonhomogeneous term. *If F is identically zero, Equation (1) reduces to*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (2)$$

and is then called homogeneous.

For $n = 2$, Equation (1) reduces to the *second*-order nonhomogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \tag{3}$$

and (2) reduces to the corresponding second-order *homogeneous* equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \tag{4}$$

Here we assume that a_0, a_1, a_2, and F are continuous real functions on a real interval $a \le x \le b$ and that $a_0(x) \neq 0$ for any x on $a \le x \le b$.

Example.

The equation

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3y = e^x$$

is a linear ordinary differential equation of the second order.

Example.

The equation

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} - 5y = \sin x$$

is a linear ordinary differential equation of the third order.

We now state the basic existence theorem for initial-value problems associated with and nth-order linear ordinary differential equation:

THEOREM 1.

Hypothesis

1. Consider the nth-order linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x), \tag{1}$$

where $a_0, a_1, \ldots, a_n$ and F are continuous real functions on a real interval $a \le x \le b$ and $a_0(x) \ne 0$ for any x on $a \le x \le b$.

2. Let x_0 be any point of the interval $a \le x \le b$, and let $c_0, c_1, \ldots, c_{n-1}$ be n arbitrary real constants.

Conclusion. *There exists a unique solution f of (1) such that*

$$f(x_0) = c_0, f'(x_0) = c_1, \ldots, f^{(n-1)}(x_0) = c_{n-1},$$

and this solution is defined over the entire interval $a \le x \le b$.

Example.

Consider the initial-value problem

$$\frac{d^2 y}{dx^2} + 3x\frac{dy}{dx} + x^3 y = e^x,$$

$$y(1) = 2,$$

$$y'(1) = -5.$$

The coefficients $1, 3x$, and x^3, as well as the nonhomogeneous term e^x, in this second-order differential equation are all continuous for all values of x, $-\infty < x < \infty$. The point x_0 here is the point 1, which certainly belongs to this interval; and the real numbers c_0 and c_1 are 2 and -5, respectively. Thus Theorem 1 assures us that a solution of the given problem exists, is unique, and is defined for all x, $-\infty < x < \infty$.

Example.

Consider the initial-value problem

$$2\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} - 5y = \sin x,$$

$$y(4) = 3,$$

$$y'(4) = 5,$$

$$y''(4) = -\tfrac{7}{2}.$$

Here we have a third-order problem. The coefficients $2, x, 3x^2$, and -5, as well as the nonhomogeneous term $\sin x$, are all continuous for all x, $-\infty < x < \infty$. The point $x_0 = 4$ certainly belongs to this interval; the real numbers c_0, c_1, and c_2 in this problem are $3, 5$, and $-\frac{7}{2}$, respectively. Theorem 1 assures us that this problem also has a unique solution which is defined for all x, $-\infty < x < \infty$.

A useful corollary to Theorem 1 is the following:

COROLLARY

Hypothesis. *Let f be a solution of the nth-order* homogeneous *linear differential equation*

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{2}$$

such that

$$f(x_0) = 0, f'(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0,$$

where x_0 is a point of the interval $a \le x \le b$ in which the coefficients $a_0, a_1, \ldots, a_n$ are all continuous and $a_0(x) \neq 0$.

Conclusion. *Then $f(x) = 0$ for all x on $a \le x \le b$.*

Example.

The unique solution f of the third-order homogeneous equation

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + x^2y = 0,$$

which is such that

$$f(2) = f'(2) = f''_{(2)} = 0,$$

is the trivial solution f such that $f(x) = 0$ for *all* x.

The Homogeneous Equation

We now consider the fundamental results concerning the homogeneous equation

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0. \qquad (2)$$

We first state the following basic theorem:

THEOREM 2. BASIC THEOREM ON LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

Hypothesis. *Let $f_1, f_2, \ldots, f_m$ be any m solutions of the homogeneous linear differential equation (2).*

Conclusion. *Then $c_1f_1 + c_2f_2 + \cdots + c_mf_m$ is also a solution of (2), where $c_1, c_2, \ldots, c_m$ are m arbitrary constants.*

DEFINITION

If $f_1, f_2, .., f_m$ are m given functions, and $c_1, c_2, \ldots, c_m$ are m constants, then the expression

$$c_1f_1 + c_2f_2 + \cdots + c_mf_m$$

is called a linear combination *of $f_1, f_2, \ldots, f_m$.*

In terms of this concept, Theorem 2 may be stated as follows:

THEOREM 2. (RESTATED)

Any linear combination of solutions of the homogeneous linear differential equation (2) *is also a solution of* (2).

In particular, any linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$$

of m solutions $f_1, f_2, \ldots, f_m$ of the *second*-order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

is also a solution of this equation.

Example.

$\sin x$ and $\cos x$ are solutions of

$$\frac{d^2y}{dx^2} + y = 0.$$

Theorem 2 states that the linear combination $c_1 \sin x + c_2 \cos x$ is also a solution for any constants c_1 and c_2. For example, the particular linear combination

$$5 \sin x + 6 \cos x$$

is a solution.

Example .

e^x, e^{-x}, and e^{2x} are solutions of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0.$$

Theorem 2 states that the linear combination $c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$ is also a solution for any constants c_1, c_2, and c_3. For example, the particular linear combination

$$2e^x - 3e^{-x} + \tfrac{2}{3}e^{2x}$$

is a solution.

We now consider what constitutes the so-called general solution of (2). To understand this we first introduce the concepts of *linear dependence* and *linear independence.*

DEFINITION

The n functions $f_1, f_2, \ldots, f_n$ are called linearly dependent *on $a \le x \le b$ if there exist constants $c_1, c_2, \ldots, c_n$,* not all zero, *such that*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x such that $a \le x \le b$.

In particular, two functions f_1 and f_2 are linearly dependent *on $a \le x \le b$ if there exist constants c_1, c_2,* not both zero, *such that*

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x such that $a \le x \le b$.

Example.

We observe that x and $2x$ are linearly dependent on the interval $0 \leq x \leq 1$. For there exist constants c_1 and c_2, *not both zero*, such that

$$c_1 x + c_2(2x) = 0$$

for all x on the interval $0 \leq x \leq 1$. For example, let $c_1 = 2, c_2 = -1$.

Example.

We observe that $\sin x$, $3 \sin x$, and $-\sin x$ are linearly dependent on the interval $-1 \leq x \leq 2$. For there exist constants c_1, c_2, c_3, *not all zero*, such that

$$c_1 \sin x + c_2(3 \sin x) + c_3(-\sin x) = 0$$

for all x on the interval $-1 \leq x \leq 2$. For example, let $c_1 = 1, c_2 = 1, c_3 = 4$.

DEFINITION

The n functions $f_1, f_2, \ldots, f_n$ are called linearly independent *on the interval $a \leq x \leq b$ if they are* not *linearly dependent there. That is, the functions $f_1, f_2, \ldots, f_n$ are linearly independent on $a \leq x \leq b$ if the relation*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x such that $a \leq x \leq b$ implies that

$$c_1 = c_2 = \cdots = c_n = 0.$$

In other words, the only linear combination of $f_1, f_2, \ldots, f_n$ that is identically zero on $a \leq x \leq b$ is the trivial linear combination

$$0 \cdot f_1 + 0 \cdot f_2 + \cdots + 0 \cdot f_n.$$

In particular, two functions f_1 and f_2 are linearly independent *on $a \le x \le b$ if the relation*

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x on $a \le x \le b$ implies that

$$c_1 = c_2 = 0.$$

Example.

We assert that x and x^2 are linearly independent on $0 \le x \le 1$, since $c_1 x + c_2 x^2 = 0$ for *all* x on $0 \le x \le 1$ implies that both $c_1 = 0$ and $c_2 = 0$. We may verify this in the following way. We differentiate both sides of $c_1 x + c_2 x^2 = 0$ to obtain $c_1 + 2c_2 x = 0$, which must also hold for all x on $0 \le x \le 1$. Then from this we also have $c_1 x + 2c_2 x^2 = 0$ for all such x. Thus we have both

$$c_1 x + c_2 x^2 = 0 \quad \text{and} \quad c_1 x + 2c_2 x^2 = 0$$

for *all* x on $0 \le x \le 1$. Subtracting the first from the second gives $c_2 x^2 = 0$ for all x on $0 \le x \le 1$, which at once implies $c_2 = 0$. Similarly $c_1 = 0$.

Example.

$1, \; x, \; x^2, \; \ldots, \; x^n$

$(-\infty, \; +\infty)$.

$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n = 0$

Example.

$$\varphi_1 = \sin^2 x, \quad \varphi_2 = \cos^2 x, \quad \varphi_3 = 1.$$

$$\alpha_1 = 1, \ \alpha_2 = 1, \ \alpha_3 = -1,$$

$$(-\infty < x < \infty):$$

$$\sin^2 x + \cos^2 x - 1 = 0.$$

The next theorem is concerned with the existence of sets of linearly independent solutions of an nth-order homogeneous linear differential equation and with the significance of such linearly independent sets.

THEOREM 3.

The nth-order homogeneous linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{2}$$

always possesses n solutions that are linearly independent. Further, if $f_1, f_2, \ldots, f_n$ are n linearly independent solutions of (2), then every solution f of (2) can be expressed as a linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

of these n linearly independent solutions by proper choice of the constants $c_1, c_2, \ldots, c_n$.

Example.

We have observed that $\sin x$ and $\cos x$ are solutions of

$$\frac{d^2y}{dx^2} + y = 0$$

for all x, $-\infty < x < \infty$. Further, one can show that these two solutions are linearly independent. Now suppose f is *any* solution. Then by Theorem 3 f can be expressed as a certain linear combination $c_1 \sin x + c_2 \cos x$ of the two linearly independent solutions $\sin x$ and $\cos x$ by proper choice of c_1 and c_2. That is, there exist two particular constants c_1 and c_2 such that

$$f(x) = c_1 \sin x + c_2 \cos x$$

for all x, $-\infty < x < \infty$. For example, one can easily verify that $f(x) = \sin(x + \pi/6)$ is a solution. Since

$$\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x,$$

we see that the solution $\sin(x + \pi/6)$ can be expressed as the linear combination

$$\frac{\sqrt{3}}{2}\sin x + \tfrac{1}{2}\cos x$$

of the two linearly independent solutions $\sin x$ and $\cos x$.

Now let $f_1, f_2, \ldots, f_n$ be a set of n linearly independent solutions of (2). Then by Theorem 2 we know that the linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,$$

where $c_1, c_2, \ldots, c_n$ are n *arbitrary* constants, is also a solution of (2). On the other hand, by Theorem 3 we know that if f is *any* solution of (2), then it can be expressed as a linear combination of the n linearly independent solutions $f_1, f_2, \ldots, f_n$, by a suitable choice of the constants $c_1, c_2, \ldots, c_n$. Thus a linear combination of the n linearly independent solutions $f_1, f_2, \ldots, f_n$ in which $c_1, c_2, \ldots, c_n$ are *arbitrary* constants must include *all* solutions of (2). For this reason, we refer to a set of n linearly independent solutions of (2) as a "fundamental set" of (2) and call a "general" linear combination of n linearly independent solutions a "general solution" of (2), in accordance with the following definition:

DEFINITION

If $f_1, f_2, \ldots, f_n$ are n linearly independent solutions of the nth-order homogeneous linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{2}$$

on $a \leq x \leq b$, then the set $f_1, f_2, \ldots, f_n$ is called a fundamental set *of solutions of (2) and the function f defined by*

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \qquad a \leq x \leq b,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants, is called a general solution *of (2) on $a \leq x \leq b$.*

Therefore, if we can find n linearly independent solutions of (2), we can at once write the general solution of (2) as a general linear combination of these n solutions.

Example.

We have observed that sin x and cos x are solutions of

$$\frac{d^2y}{dx^2} + y = 0$$

for all x, $-\infty < x < \infty$. Further, one can show that these two solutions are linearly independent. Thus, they constitute a fundamental set of solutions of the given differential equation, and its general solution may be expressed as the linear combination

$$c_1 \sin x + c_2 \cos x,$$

where c_1 and c_2 are arbitrary constants. We write this as $y = c_1 \sin x + c_2 \cos x$.

Example.

The solutions e^x, e^{-x}, and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

may be shown to be linearly independent for all x, $-\infty < x < \infty$. Thus, e^x, e^{-x}, and e^{2x} constitute a fundamental set of the given differential equation, and its general solution may be expressed as the linear combination

$$c_1 e^x + c_2 e^{-x} + c_3 e^{2x},$$

where c_1, c_2, and c_3 are arbitrary constants. We write this as

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}.$$