Ex: Let $[A] = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $[B] = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $[C] = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

and $[0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ be 2x2 matrices. Find the matrix $[X]$.

a) $2[X] - [A][C] = [B]$

Solution: $$2[X] - \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}}_{= \begin{bmatrix} -1 & 3 \\ -3 & 3 \end{bmatrix}} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$2[X] - \begin{bmatrix} -1 & 3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$2[X] = \begin{bmatrix} -2 & 4 \\ -1 & 4 \end{bmatrix} \qquad \left([X] = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_3 \end{bmatrix} \right.$$

$$2\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 & 2x_2 \\ 2x_3 & 2x_4 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 4 \end{bmatrix}$$

$$\begin{cases} 2x_1 = -2 \\ 2x_2 = 4 \\ 2x_3 = -1 \\ 2x_4 = 4 \end{cases} \Rightarrow [X] = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1/2 & 2 \end{bmatrix} \;//$$

b) $[C][X] - [A][B] = [O]$

Solution: $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}[X] - \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 3 \\ 6 & 3 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x_3 & x_4 \\ -x_1+x_3 & -x_2+x_4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 6 & 3 \end{bmatrix}$$

$$\begin{cases} x_3 = 0 \\ x_4 = 3 \\ -x_1 + x_3 = 6 \\ -x_2 + x_4 = 3 \end{cases}$$

$$\begin{cases} x_1 = -6 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 3 \end{cases}$$

$$[X] = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & 3 \end{bmatrix} //$$

Ex: Let $A=\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}8 & -3 & -5\\ 0 & 1 & 2\\ 4 & -7 & 6\end{bmatrix}$, $C=\begin{bmatrix}0 & -2 & 3\\ 1 & 7 & 4\\ 3 & 5 & 9\end{bmatrix}$

$a=4$. Show that

a) $(A^T)^T = A$

Solution: $A=\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}2 & 0 & -2\\ -1 & 4 & 1\\ 3 & 5 & 4\end{bmatrix} \Rightarrow$

$$(A^T)^T=\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix}=A$$

$\Rightarrow (A^T)^T = A$ //

b) $(A+B)^T = A^T + B^T$

• $(A+B)^T = \left(\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix}+\begin{bmatrix}8 & -3 & -5\\ 0 & 1 & 2\\ 4 & -7 & 6\end{bmatrix}\right)^T = \left(\begin{bmatrix}10 & -4 & -2\\ 0 & 5 & 7\\ 2 & -6 & 10\end{bmatrix}\right)^T = \begin{bmatrix}10 & 0 & 2\\ -4 & 5 & -6\\ -2 & 7 & 10\end{bmatrix}$ //

• $A^T + B^T = \left(\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix}\right)^T + \left(\begin{bmatrix}8 & -3 & -5\\ 0 & 1 & 2\\ 4 & -7 & 6\end{bmatrix}\right)^T = \begin{bmatrix}2 & 0 & -2\\ -1 & 4 & 1\\ 3 & 5 & 4\end{bmatrix}+\begin{bmatrix}8 & 0 & 4\\ -3 & 1 & -7\\ -5 & 2 & 6\end{bmatrix}$

$$= \begin{bmatrix}10 & 0 & 2\\ -4 & 5 & -6\\ -2 & 7 & 10\end{bmatrix} //$$

$\Rightarrow (A+B)^T = A^T + B^T$ //

c) $(aC)^T = aC^T$

- $(aC)^T = \left(4 \cdot \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}\right)^T = \left(\begin{bmatrix} 8 & -4 & 12 \\ 0 & 16 & 20 \\ -8 & 4 & 16 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 & -8 \\ -4 & 16 & 4 \\ 12 & 20 & 16 \end{bmatrix}$ //

//

- $aC^T = 4 \cdot \left(\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}\right)^T = 4\begin{bmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ 3 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 0 & -8 \\ -4 & 16 & 4 \\ 12 & 20 & 16 \end{bmatrix}$ //

$\Rightarrow (aC)^T = aC^T$ //

<u>Ex</u>: $A = \begin{bmatrix} 3 & 0 \\ 2 & -3 \end{bmatrix} \Rightarrow A^{2000} = ?$

$A = \begin{bmatrix} 3 & 0 \\ 2 & -3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 3 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 9 . I_2$

$A^{2000} = (A^2)^{1000} = (9 . I_2)^{1000} = 9^{1000} . \underbrace{I_2^{1000}}_{I_2} = 9^{1000}$ //

since $(I^n = I)$

Ex: Let $A=\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}8 & -3 & -5\\ 0 & 1 & 2\\ 4 & -7 & 6\end{bmatrix}$. Find • $(AB)^T$

• $B^T A^T$

Solution:

• $(AB)^T = \left(\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix}\begin{bmatrix}8 & -3 & -5\\ 0 & 1 & 2\\ 4 & -7 & 6\end{bmatrix}\right)^T = \left(\begin{bmatrix}28 & -28 & 6\\ 20 & -31 & 38\\ 0 & -21 & 36\end{bmatrix}\right)^T$

$$= \begin{bmatrix}28 & 20 & 0\\ -28 & -31 & -21\\ 6 & 38 & 36\end{bmatrix} //$$

• $B^T A^T = \left(\begin{bmatrix}8 & -3 & -5\\ 0 & 1 & 2\\ 4 & -7 & 6\end{bmatrix}\right)^T \cdot \left(\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 5\\ -2 & 1 & 4\end{bmatrix}\right)^T$

$$= \begin{bmatrix}8 & 0 & 4\\ -3 & 1 & -7\\ -5 & 2 & 6\end{bmatrix}\begin{bmatrix}2 & 0 & -2\\ -1 & 4 & 1\\ 3 & 5 & 4\end{bmatrix} = \begin{bmatrix}28 & 20 & 0\\ -28 & -31 & -21\\ 6 & 38 & 36\end{bmatrix} //$$

Observe that $(AB)^T = B^T A^T$ //

Ex: Write each of the following as a linear system in matrix form

a) $x_1\begin{bmatrix}1\\2\end{bmatrix}+x_2\begin{bmatrix}2\\5\end{bmatrix}+x_3\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

Solution: $\begin{bmatrix}x_1\\2x_1\end{bmatrix}+\begin{bmatrix}2x_2\\5x_2\end{bmatrix}+\begin{bmatrix}0\\3x_3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

$$\begin{bmatrix}x_1+2x_2\\2x_1+5x_2+3x_3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{cases}x_1+2x_2=1\\2x_1+5x_2+3x_3=1\end{cases}$$

//

b) $x_1\begin{bmatrix}1\\1\\2\end{bmatrix}+x_2\begin{bmatrix}2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Solution: $\begin{bmatrix}x_1\\x_1\\2x_1\end{bmatrix}+\begin{bmatrix}2x_2\\x_2\\0\end{bmatrix}+\begin{bmatrix}x_3\\2x_3\\2x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{bmatrix}x_1+2x_2+x_3\\x_1+x_2+2x_3\\2x_1+2x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{cases}x_1+2x_2+x_3=0\\x_1+x_2+2x_3=0\\2x_1+2x_3=0\end{cases}$$

Ex: If A is a $n \times n$ matrix, what are the entries on the main diagonal of $A - A^T$?

Solution: Let A be a $n \times n$ matrix, i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix}$$

Hence $A - A^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix}$

$$= \begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} & \cdots & a_{1n}-a_{n1} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} & \cdots & a_{2n}-a_{n2} \\ \vdots & & & & \\ a_{n1}-a_{1n} & - & - & - & 0 \end{bmatrix}$$

The entries on the main diagonal of $A - A^T$ are 0 //

Ex: If $A=[a_{ij}]$ is a $n\times n$ matrix, then the trace of A, $Tr(A)$, is defined as the sum of all elements on the main diagonal of A, $Tr(A)=\sum_{i=1}^{n} a_{ii}$. Show each of the following:

a) $Tr(cA)=c\,Tr(A)$ where c is a real number

Solution: Let $A=[a_{ij}]_{n\times n}$, $B=cA$. We have that

$b_{ij}=c.a_{ij}$ for all $1\leq i\leq n$ and $1\leq j\leq n$.

Hence $Tr(cA)=Tr(B)=\sum_{i=1}^{n} b_{ii}=\sum_{i=1}^{n} c a_{ii}=c\sum_{i=1}^{n} a_{ii}=c\,Tr(A)$

So $Tr(cA)=c\,Tr(A)$ //

b) $Tr(A+B)=Tr(A)+Tr(B)$

Solution: Let A and B be $n\times n$ matrices and let $C=A+B$. We have that $c_{ij}=a_{ij}+b_{ij}$ for all $1\leq i\leq n$, $1\leq j\leq n$. Hence

$$Tr(A+B)=Tr(C)=\sum_{i=1}^{n}(a_{ii}+b_{ii})$$

$$=\sum_{i=1}^{n} a_{ii}+\sum_{i=1}^{n} b_{ii}$$

$$=Tr(A)+Tr(B)$$

Hence $Tr(A+B)=Tr(A)+Tr(B)$ //

c) $Tr(A^T) = Tr(A)$

Solution: Let A be an $n\times n$ matrix and let $B = A^T$.

We have that $b_{ij} = a_{ji}$ for $1 \le i \le n$, $1 \le j \le n$.

Hence $Tr(A^T) = Tr(B) = \sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n} a_{ii} = Tr(A)$ //

Ex: Compute the trace of each of the following:

a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ b) $B = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & -2 & -5 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$Tr(A) = 4$ // $Tr(B) = 1$ // $Tr(C) = 3$ //

Recall: • If the transpose of a matrix is equal to itself, that matrix is said to be symmetric.

• A is skew-symmetric $\Leftrightarrow A^T = -A$

Ex: If A is a skew-symmetric matrix, what type of matrix is A^T? Justify your answer.

Solution: Let A be a skew-symmetric matrix. Then we have $A^T = -A$ which gives us

$(A^T)^T = (-A)^T = -A^T \Rightarrow (A^T)^T = -A^T$ $(A^T := B)$

$\Rightarrow B^T = -B$ so B is skew-symmetric.

Hence A^T is skew-symmetric.

Ex: Show that if A is skew-symmetric, then the elements on the main diagonal of A are all zero.

Solution: Let A be a skew-symmetric $n \times n$ matrix. Then we have $A^T = -A$.

Let (a_{ij}) be the entries on A and (b_{ij}) be the entries of A^T. Then we have that $b_{ij} = a_{ji}$ $\forall\, 1 \leq i \leq n,\ 1 \leq j \leq n$

Since $A^T = -A$ we have $b_{ij} = -a_{ij}$. Thus

$$a_{ji} = -a_{ij} \quad \forall\, 1 \leq i \leq n,\ 1 \leq j \leq n$$

In particular, if we take $i = j$ we have $a_{ii} = -a_{ii}$ $\forall\, 1 \leq i \leq n$ which gives us that $a_{ii} = 0$ $\forall\, 1 \leq i \leq n$

Hence the diagonal entries of A are all 0 //

Ex: Show that if A is any $m \times n$ matrix then AA^T and A^TA are symmetric.

Solution: Let A be an $m \times n$ matrix.

Consider the $m \times m$ matrix $B = AA^T$ and
$n \times n$ matrix $C = A^TA$

We have that $B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B$
$\Rightarrow B^T = B \Rightarrow B = AA^T$ is symmetric.

We have that $C^T = (A^TA)^T = A^T (A^T)^T = A^TA = C$
$\Rightarrow C^T = C \Rightarrow C = A^TA$ is symmetric.

Ex: Let $A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ and $C = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$

a) Compute $(AB^T)C$

$$A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 3 & -2 & 4 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 8 \\ -3 & 2 & -4 \\ 9 & -6 & 12 \end{bmatrix}$$

Now $(AB)^T C = \begin{bmatrix} 6 & -4 & 8 \\ -3 & 2 & -4 \\ 9 & -6 & 12 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -34 \\ 17 \\ -51 \end{bmatrix}$ //

b) Compute $B^T C$ and $(B^T C)A$

$$B = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 3 & -2 & 4 \end{bmatrix}$$

$$B^T C = \begin{bmatrix} 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -17 \end{bmatrix} // \quad = -17$$

$$(B^T C) A = -17 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -37 \\ 17 \\ -51 \end{bmatrix} //$$

Ex: Show that $(A-B)^T = A^T - B^T$

Solution: Let A and B be matrices of appropriate sizes.

We have that
$$(A-B)^T = (A+(-B))^T$$
$$= (A+(-1)B)^T$$
$$= A^T + ((-1)B)^T$$
$$= A^T + (-1)B^T$$
$$= A^T - B^T \;//$$

Ex: Find a 2x2 matrix $B \neq 0$ and $B \neq I_2$ such that $AB = BA$ where $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. How many such matrices B are there?

Solution: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $AB = BA$ then $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$$

$$\Rightarrow \begin{cases} a+2c = a \\ c = c \\ b+2d = 2a+b \\ d = 2c+d \end{cases} \Rightarrow \begin{cases} c = 0 \\ a = d \\ c = 0 \end{cases} \Rightarrow B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

There are infinitely many B matrices.

Ex: Which of the following functions are linear transformations?

a) $L: M_{nn} \longrightarrow M_{nn}$ defined by $L(A) = A^T$

b) $L: M_{nn} \longrightarrow M_{nn}$ defined by $L(A) = A^{-1}$. (not now)

a) Consider $L: M_{nn} \longrightarrow M_{nn}$ defined as $L(A) = A^T$.

Given $A, B \in M_{nn}$ we have that

- $L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$

$$\Rightarrow L(A+B) = L(A) + L(B) //$$

Given $A \in M_{nn}$ and a scalar c we have that

- $L(cA) = (cA)^T = cA^T = c\,L(A)$

$$\Rightarrow L(cA) = cL(A) //$$

Hence L is a linear transformation

b) not now

Ex: Find the standard matrix representing each given linear transformation.

a) $L: \mathbb{R}^2 \to \mathbb{R}^2$ defined as $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} -u_2 \\ -u_1 \end{bmatrix}$

b) $L: \mathbb{R}^2 \to \mathbb{R}^2$ defined as $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + ku_2 \\ u_2 \end{bmatrix}$

a) the standard matrix representing L is $A = [L(e_1) \; L(e_2)]$ //

We have $L: \mathbb{R}^2 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} -u_2 \\ -u_1 \end{bmatrix}$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence $A = [L(e_1) \; L(e_2)] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ // : the standard matrix representing L

b) The standard matrix representing L is $A = [L(e_1) \; L(e_2)]$

We have $L: \mathbb{R}^2 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + ku_2 \\ u_2 \end{bmatrix}$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 + k.0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 + k.1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

Hence the standard matrix representing L is

$$A = [L(e_1) \; L(e_2)] = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} //$$

Ex: Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation for which we know that

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\-4\end{bmatrix} \quad L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}3\\-5\end{bmatrix} \quad L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix}$$

What is $L\left(\begin{bmatrix}1\\-2\\3\end{bmatrix}\right)$?

$$L\left(\begin{bmatrix}1\\-2\\3\end{bmatrix}\right)=L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)+L\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right)+L\left(\begin{bmatrix}0\\0\\3\end{bmatrix}\right)$$

$$=L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)+(-2)L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)+(3)L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$=\begin{bmatrix}2\\-4\end{bmatrix}+(-2)\begin{bmatrix}3\\-5\end{bmatrix}+3\begin{bmatrix}2\\3\end{bmatrix}$$

$$=\begin{bmatrix}2-6+6\\-4+10+9\end{bmatrix}=\begin{bmatrix}2\\15\end{bmatrix} //$$

Ex: Let $L: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$L([u_1 \ u_2]) = [u_1 \quad u_1+u_2 \quad u_2])$

Recall: Definition: Let $L: V \to W$ be a linear transformation of a vector space V into a vector space W. The kernel of L, $\operatorname{Ker} L$, is the subset of V consisting of all elements v of V such that $L(v) = 0_W$

Consider $L([u_1 \ u_2]) = [u_1 \quad u_1+u_2 \quad u_2])$. If $u = [u_1 \ u_2] \in \operatorname{Ker} L$ we have that $L(u) = L([u_1 \ u_2]) = [0 \quad 0 \quad 0]$

$L(u) = L([u_1 \quad u_2]) = [u_1 \quad u_1+u_2 \quad u_2] = [0 \quad 0 \quad 0]$

$u_1 = 0$

$u_1+u_2 = 0$ Hence $\ker L = \{[0 \quad 0]\} //$

$u_2 = 0$

Ex: Let $L: \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$L([u_1\ u_2\ u_3\ u_4]) = [u_1+u_2\quad u_3+u_4\quad u_1+u_3]$

Find a basis for ker L.

If $u = [u_1\ u_2\ u_3\ u_4] \in \ker L$ we have that

$L(u) = L([u_1\ u_2\ u_3\ u_4]) = [0\quad 0\quad 0]$

And also $L(u) = L([u_1\ u_2\ u_3\ u_4]) = [u_1+u_2\quad u_3+u_4\quad u_1+u_3]$

So
$$\begin{cases} u_1+u_2=0 \\ u_3+u_4=0 \\ u_1+u_3=0 \end{cases} \Rightarrow \begin{array}{l} u_1=-u_2 \\ u_3=-u_4 \\ u_1=-u_3 \end{array}$$

Hence $\ker L = \{[a\ b\ c\ d] \in \mathbb{R}^4 \mid b=-a,\ c=-a,\ d=a\}$

that is $\ker L = \{[a\quad -a\quad -a\quad a]\}$.

We know $[a\quad -a\quad -a\quad a] = a[1\ -1\ -1\ 1]$

Hence $\ker L = \text{span}\{[1\ -1\ -1\ 1]\}$ //