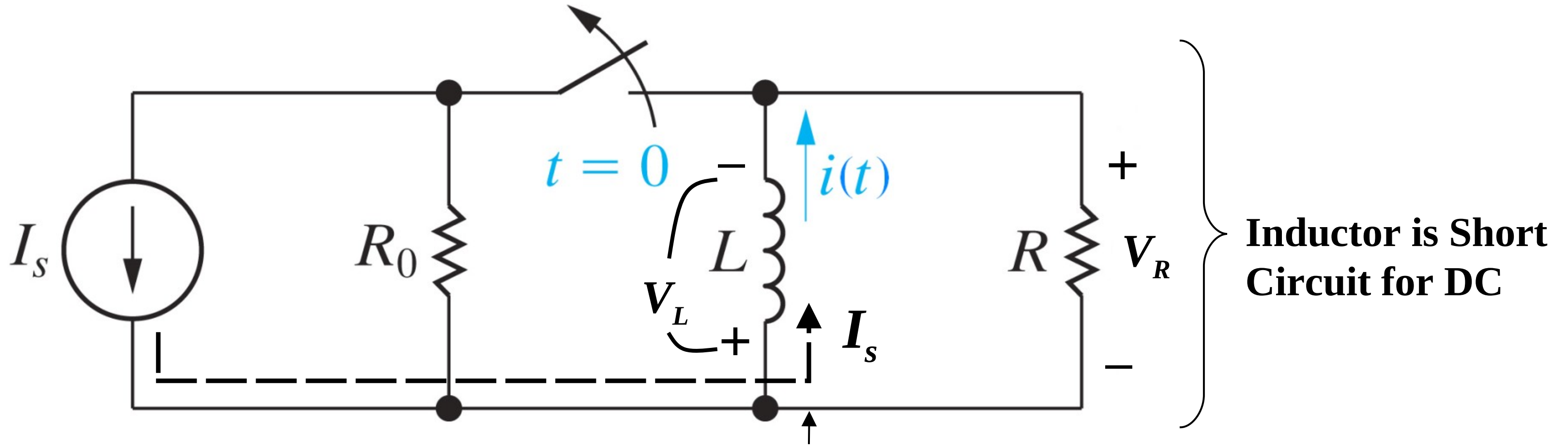
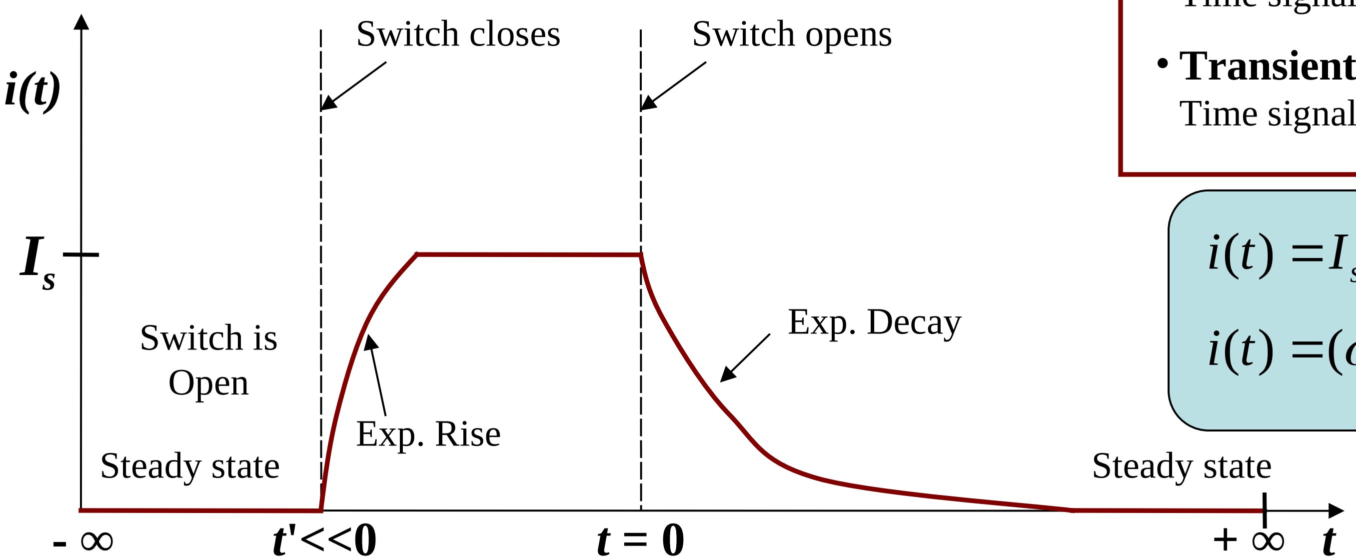
Chapter 5

1st – Order RL and RC Circuits

General Response of RL Circuits



- **Steady State Solution/Response**
  Time signals are constant or periodic
- **Transient Solution/Response**
  Time signals are not constant or periodic



$$i(t) = I_s e^{-(R/L)t} \qquad t \geq 0$$

$$i(t) = (\alpha)\left[1 - e^{-(R/L)[t-t']}\right] \qquad t \leq 0$$

RL ≡ Resistor – Inductor Circuit

RC ≡ Resistor – Capacitor Circuit

Natural Response

Natural Response:

* Response of RL or RC Circuit when dc source is abruptly disconnected.
* Energy is released to resistive circuit
* Recall that Energy is stored in ***L*** and ***C***.

Leaves us with:

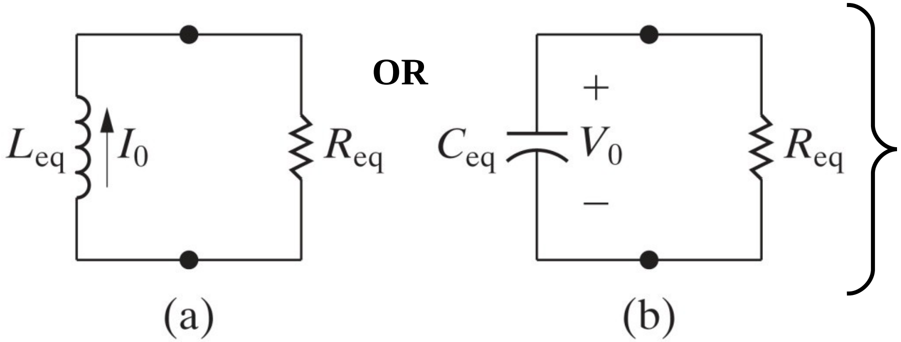


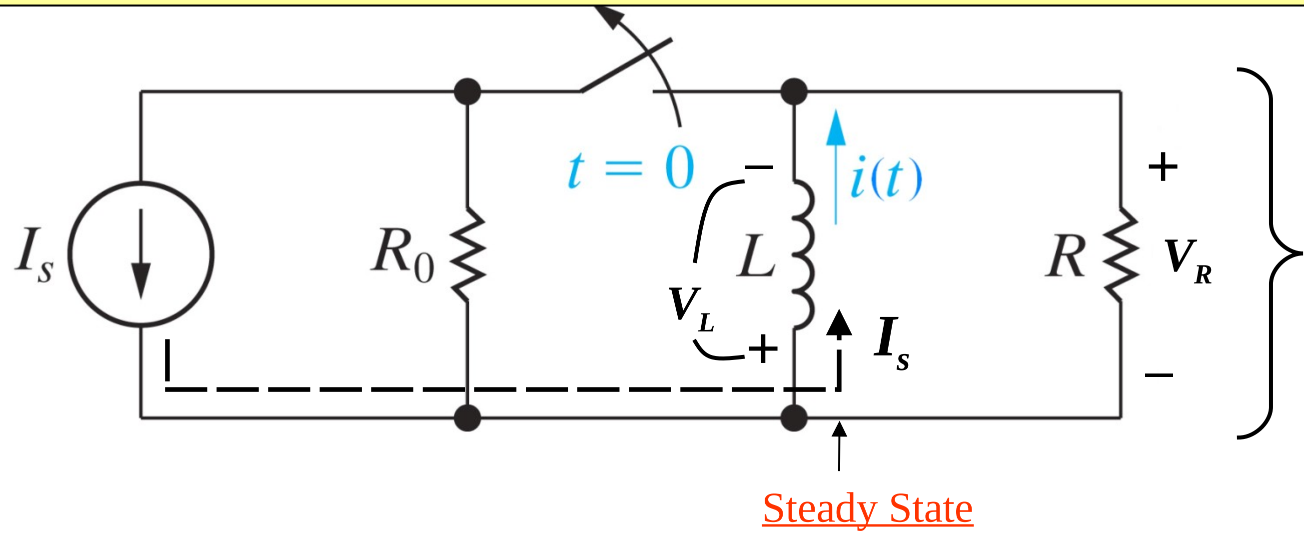
Figure: 07-01

I_0 and V_0 are Initial, Stored Values

1st – Order Circuits

i and *v* are described by **1st – Order Linear Differential Equations**

RL Circuit Natural Response



Note

$V_R = 0$

for $t < 0$

* Assume Steady State long before $t = 0$.

* $v_L = L\dfrac{di}{dt} = 0$ Short

* Current = 0 in R_0 and R

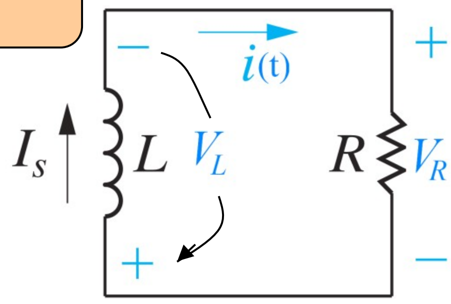
* $i(0^+) = I_s$ in L

Steady State Conditions at t_0

RL Circuit Natural Response (Contd.)

At $t \geq 0$ the circuit becomes:

At $t \geq 0$, Inductor Begins Releasing Energy



Note Polarities

$$V_L + V_R = 0$$ **KVL**

$$\boxed{L\frac{di}{dt} + Ri = 0}$$

$$V_L = L\frac{di}{dt}$$
$$V_R = Ri$$

<u>1st – Order Ordinary Linear Differential Equation With Constant Coefficients</u>

R and ***L*** don't vary with ***i*** or ***t*** **Constant Coefficients**

<u>Two Methods:</u> To solve above differential equation

RL Circuit — Natural Response (Contd.)

(1) Textbook's Method

1 $$\frac{di}{dt} = -\left(\frac{R}{L}\right) i$$ Simplify differential equation

1 $$\frac{di}{dt} dt = -\frac{R}{L} i dt$$ Multiply by dt

1 $$\frac{di}{i} = -\frac{R}{L} dt$$ Separation of Variables

Integrate: * Lets not be "Picky" about Variables
* $\int$ from 0 (initial) to t (some time)

$$\int_{i(0)}^{i(t)} \frac{di}{i} = -\frac{R}{L} \int_0^t dt$$ Integrate 1

$$\ln i \Big|_{i(0)}^{i(t)} = -\frac{R}{L} t \Big|_0^t$$ Anti-derivative

$$\ln i(t) - \ln i(0) = -\frac{R}{L} t$$ Plug in limits

$$\ln \frac{i(t)}{i(0)} = -\frac{R}{L} t$$ ln(.) Identity

[raise e to both sides]

$$\boxed{i(t) = i(0) e^{-\left(\frac{R}{L}\right) t}}$$

RL Circuit Natural Response (Contd.)

Notation

$t = 0^-$ is just Before Switching

$t = 0^+$ is just After Switching

$$i(0^-) \equiv i(0^+) = I_s = I_0$$

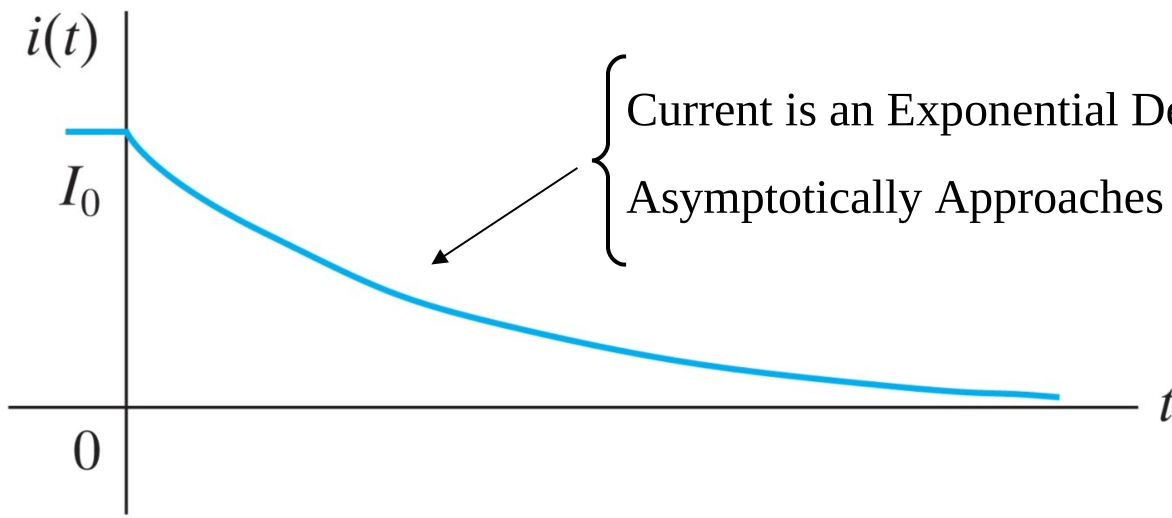
Current can't change instantly

$$\therefore\; i(t) = I_0 e^{-\left(\frac{R}{L}\right)t} \qquad t \geq 0$$

I_0 is initial current

or I_s

Source Current



Current is an Exponential Decay

Asymptotically Approaches Zero as $t \to \infty$

Figure: 07-05

RL Circuit   Natural Response (Contd.)

(2) Alternative Method: Boundary Values

Start with: $\frac{di}{i} = -\frac{R}{L}dt$

$$\int\frac{di}{i} = -\int\frac{R}{L}dt$$ **Integrate**

1 $\ln i(t) = -\frac{R}{L}t + K$ ← **Constant of Integration**

K Determined by "Initial Condition" or Boundary Value

$i(0) = I_0 \equiv I_s$ in our circuit

$$\ln i(0) = \ln I_0 = -\frac{R}{L}(0) + K$$ **Substitute $t = 0$**

2 $\boxed{K = \ln I_0}$

$$\ln i(t) = -\frac{R}{L}t + \ln I_0$$ **Substitute 2 into 1**

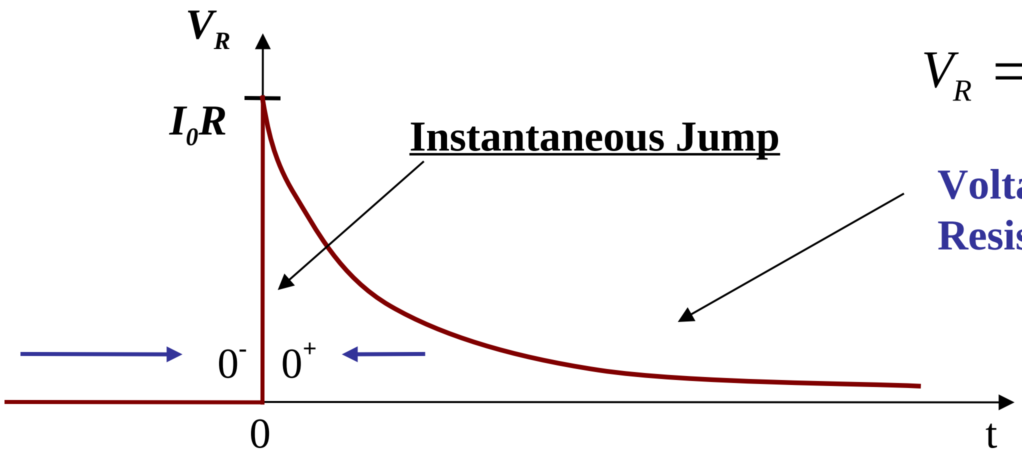
$$\ln i(t) - \ln I_0 = \ln\frac{i(t)}{I_0} = -\frac{R}{L}t$$ **Solve for $i(t)$**

$$\therefore \quad i(t) = I_0 e^{-\left(\frac{R}{L}\right)t} \qquad t \geq 0$$ **Raise to the exponential**

Check Solution:

1. **Plug solution into diff. eqn., and see if it works**
2. **Is $i(t = 0) \equiv I_0$?**

RL Circuit Natural Response (Contd.)



$$V_R = I_0 R e^{-\left(\frac{R}{L}\right)t}$$

Voltage across resistor (Ohm's Law)

$$V_R(0^-) = 0$$

In original circuit resistor is shorted out for *t* < 0

$$p = i^2 R = I_0^2 R e^{-2\left(\frac{R}{L}\right)t}$$

Power dissipated in Resistor

$$w(t) = \int_0^t p\,dt = I_0^2 R \int_0^t e^{-2\left(\frac{R}{L}\right)t}\,dt$$

Integrate Power to obtain Energy

$$w(t) = \frac{1}{2} L I_0^2 \left(1 - e^{-2\left(\frac{R}{L}\right)t}\right)$$

Energy Delivered to the Resistor

$$w(t \to \infty) = \frac{1}{2} L I_0^2 \equiv \text{ Initial Stored Energy in Inductor}$$

Conservation of Energy

All Inductor Energy Transferred to the Resistor

RL Circuit

Time Constant Definition

Solution has the form

$$i(t) = I_0 e^{-(R/L)t}$$

for $t \geq 0$

$$\left. e^{-(R/L)t} \equiv e^{-t/\tau} \right\} \boxed{\tau = \frac{L}{R}}$$

Time Constant

$\tau \equiv$ *Rate at which function approaches zero*

$$\left. \boxed{\therefore \quad i(t) = I_0 e^{-t/\tau}} \right\} \boxed{\tau = \frac{L}{R}}$$

Think of Time Elapsed in Integral Multiples of τ

$\therefore \; t = \tau \quad \equiv 1$ Time Constant $\Rightarrow \; e^{-\tau/\tau} = e^{-1} \equiv 0.37$

$i(t = \tau) = 0.37 I_0$ after 1 Time Constant.

$t = 2\tau \quad \equiv 2$ Time Constants $\Rightarrow \; e^{-2\tau/\tau} = e^{-2} \equiv 0.14$

$i(t = 2\tau) = 0.14 I_0$ after 2 Time Constants.

$t = 5\tau \equiv \;$ 5 Time Constants $\Rightarrow \; e^{-5\tau/\tau} = e^{-5} \equiv 0.0067$

$\boxed{i(t = 5\tau) = 0.0067 I_0 \quad \text{after 5 Time Constants.}}$

Current reduced to < 1% of original value after 5 time constants

Numeric Example for Time Constant

$$L = 100\ mH$$

$$R = 1000\ \Omega$$

Example Values

$$\tau = \frac{L}{R} = \frac{0.1}{1000} = 0.1\ (ms)$$

$$i(t) = I_0 e^{-t/\tau} \quad \text{where} \quad 5\tau = 0.5\ (ms)$$

After 0.5 (*ms*)

$$i(t) \cong 0$$

Rule of Thumb for Time Constants

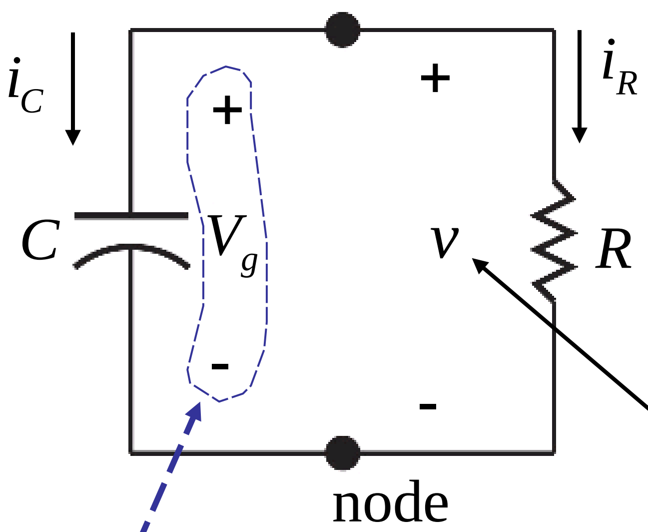
Values of $e^{-t/\tau}$ for t Equal to Integral Multiples of τ			
t	$e^{-t/\tau}$	t	$e^{-t/\tau}$
τ	3.6788×10^{-1}	6τ	2.4788×10^{-3}
2τ	1.3534×10^{-1}	7τ	9.1188×10^{-4}
3τ	4.9787×10^{-2}	8τ	3.3546×10^{-4}
4τ	1.8316×10^{-2}	9τ	1.2341×10^{-4}
5τ	6.7379×10^{-3}	10τ	4.5400×10^{-5}

≈ decayed to zero. } **Rule of Thumb**

RC Circuit

Dual of RL Circuit

- A Capacitor is charged to a Steady-State Voltage V_g at $t = 0$
- Find the time varying expression for $v(t)$ for $t \geq 0$



Initial Voltage

$$i_C = C\frac{dv}{dt}$$ **Capacitor Equation**

V can't Change Instantly

$$\boxed{v = V_g}$$ **KVL**

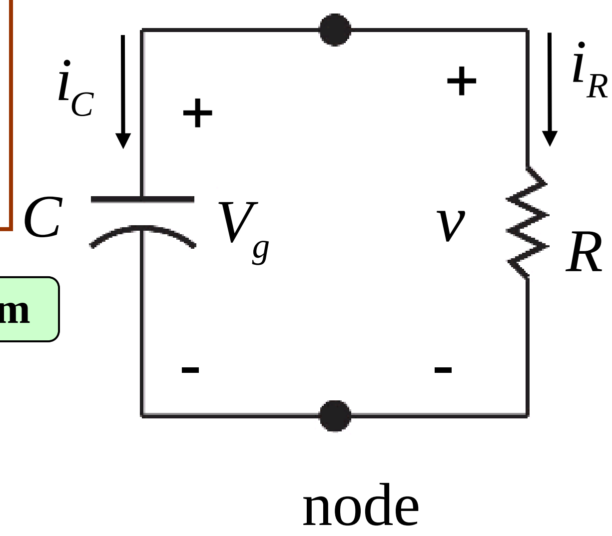
$$\boxed{i_R = \frac{v}{R}}$$ **Ohm's Law**

RC Circuit — Natural Response (Contd.)

Find differential equation for a RC Circuit

$$i_C = C\frac{dv}{dt}$$

$$i_R = \frac{v}{R}$$



Current must be the same in this loop

KCL at Node: $i_C + i_R = 0$ — **From the Circuit Diagram**

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$ — **Substitute for i_C and i_R**

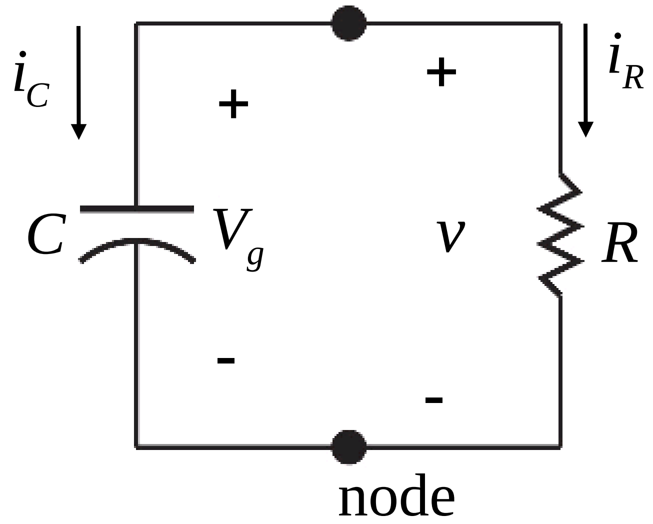
$$\frac{dv}{dt} = \left(-\frac{1}{RC}\right)v$$ — **Differential Equation**

$$\therefore \quad v(t) = v(0)e^{-t/RC}$$ — **Solution of differential equation**

Initial Condition : $v(0^-) = v(0) = v(0^+) = V_g = V_0$

(Contd.) ⟹

RC Circuit Natural Response (Contd.)



$$v(t) = V_0 e^{-t/\tau} \qquad t \geq 0$$ **Solution with τ = *RC***

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \qquad t \geq 0$$ **Current in Resistor**

$$p(t) = v(t) i_R(t) = \frac{V_0^{\,2}}{R} e^{-2t/\tau} \qquad t \geq 0$$ **Power Dissipated in Resistor**

(Contd.)

RC Circuit Natural Response (Contd.)

$$w = \int_0^t p(t)dt = \frac{V_0^2}{R}\int_0^t e^{-2t/\tau}dt = \frac{V_0^2}{R}\left[-\frac{\tau}{2}\right]e^{-2t/\tau}\Big|_0^t$$

Integrate $p = \frac{V_0^2}{R}e^{-2t/\tau}$ **to obtain energy**

$$= \frac{V_0^2}{\cancel{R}}\left[-\frac{\cancel{R}C}{2}\right]\left[e^{-2t/\tau} - 1\right]$$

Plug in the limits

$$w = \frac{1}{2}CV_0^2\left[1 - e^{-2t/\tau}\right]$$

Energy Delivered to Resistor

Similar to the Case of the RL Circuit

$$w(t=0) = \frac{1}{2}CV_0^2(1-1) = 0$$

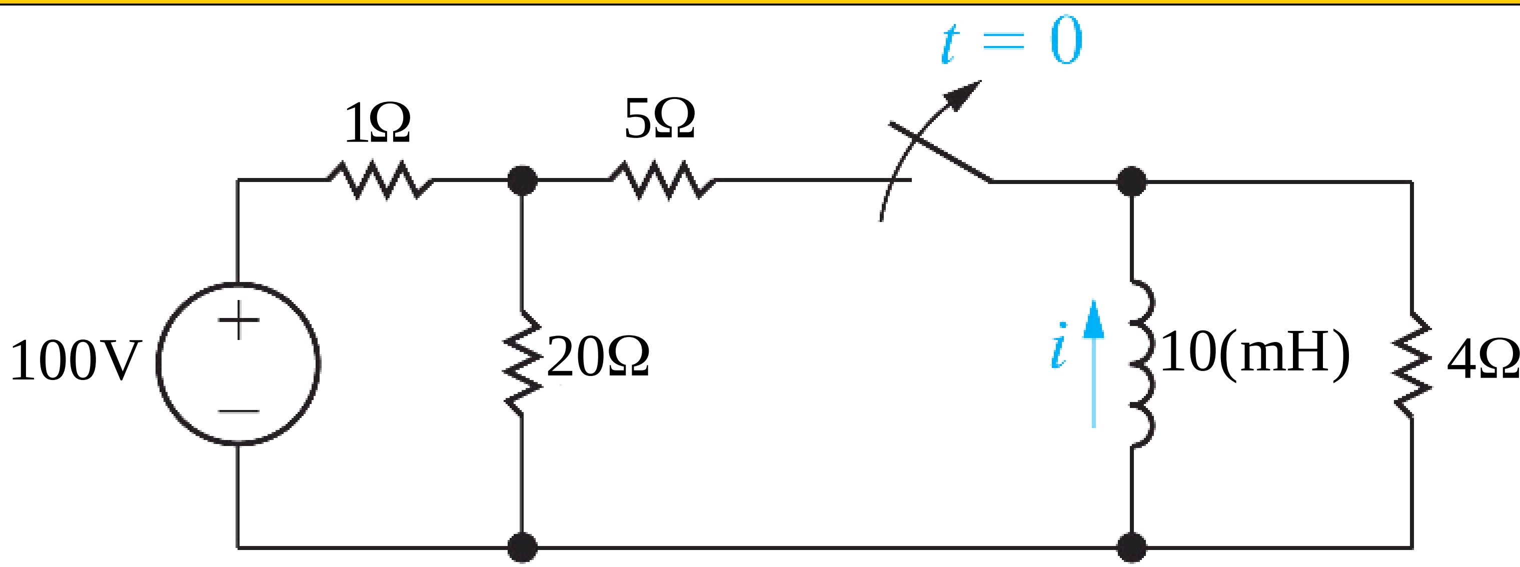
Initial Energy delivered to Resistor

$$w(t=\infty) = \frac{1}{2}CV_0^2(1-0) = \frac{1}{2}CV_0^2$$

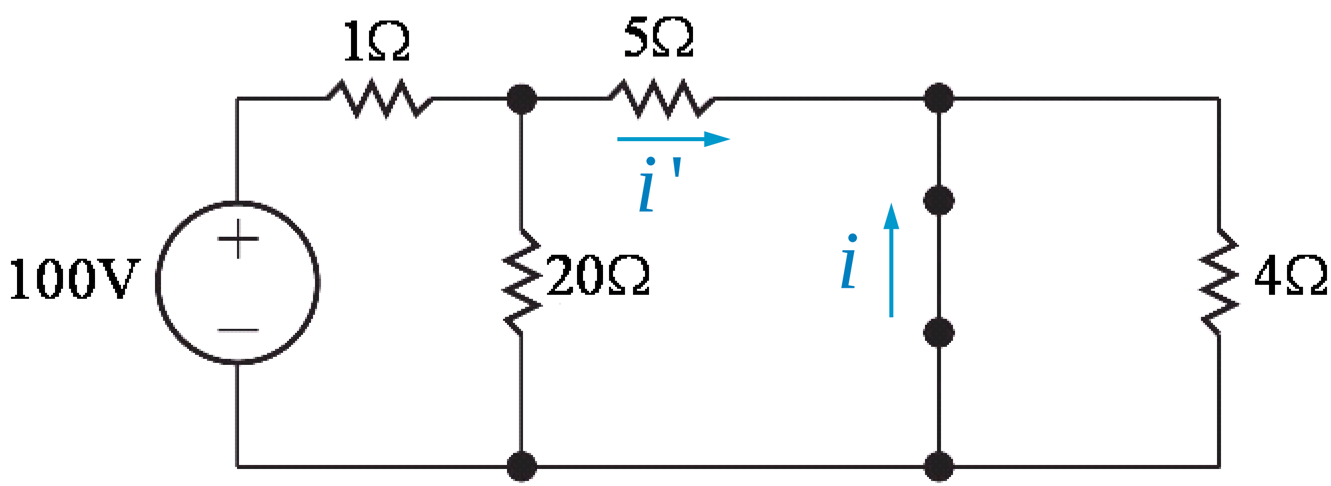
Final Energy delivered to Resistor

$\equiv$ **Initial Energy Stored in Capacitor**

Drill Exercise: Find $i(t)$ for $t \geq 0$



a) Steady State Circuit, $t < 0$: Find $i(0^-)$

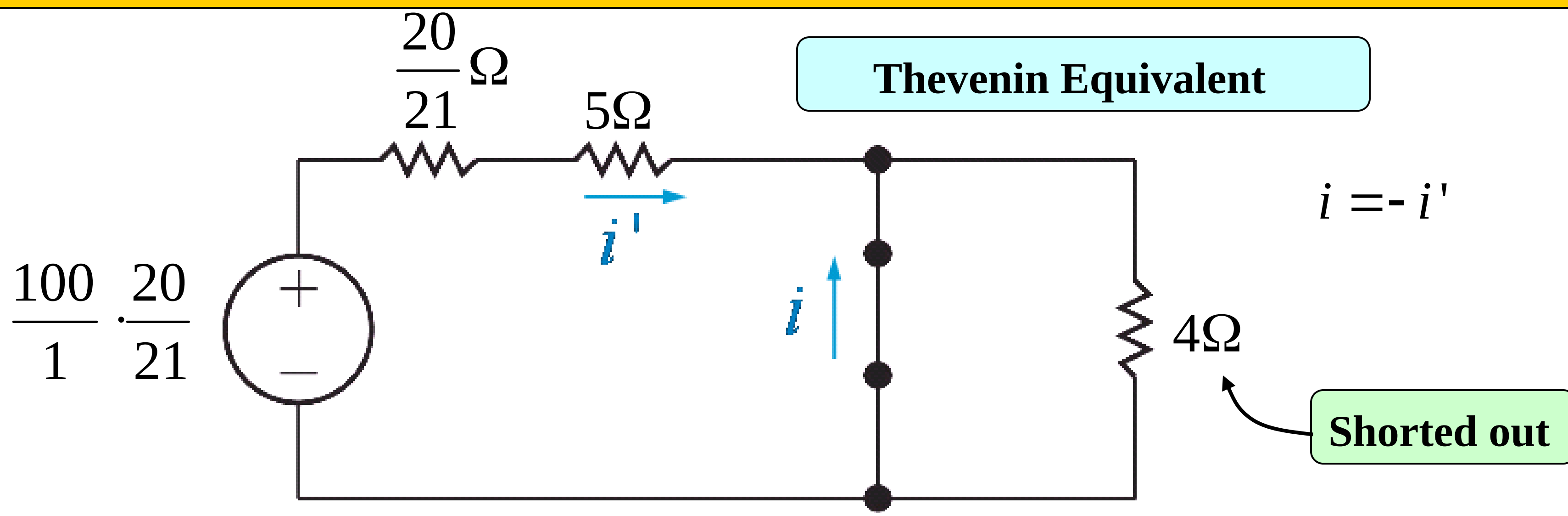


Inductor Shorted

$i' = - i$

No Current in 4Ω Resistor

Drill Exercise: Find $i(0^-)$



$$i = -i'$$

$$i' = \frac{100(20)}{21} \Big/ \left[\frac{20}{21} + 5\right] = 16(A)$$ Ohm's Law

$$\therefore i = -16(A) \rightarrow i(0^-) = -16(A) \equiv I_0$$ Initial Current in Inductor

b)

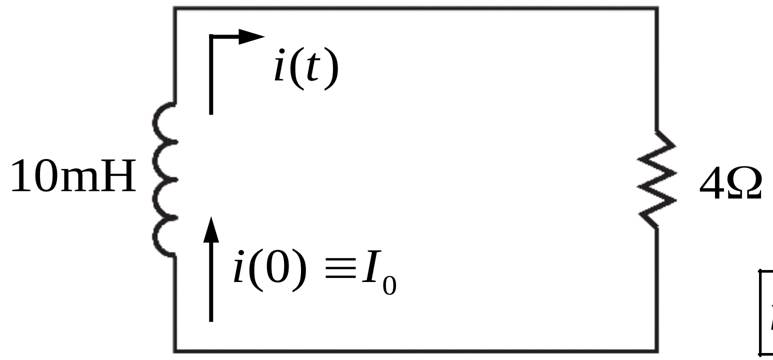
$$w(0^-) = \frac{1}{2} Li^2(0^-) = \frac{1}{2}(0.01)(-16)^2$$

$$w(0^-) = 1.28(J)$$

Initial Energy stored in Inductor

Drill Exercise: Find *i(t)* for *t* ≥ 0

c) Circuit for $t \geq 0$:



$$\tau = \frac{L}{R} = \frac{0.01}{4}$$

$$\boxed{\tau = 2.5(ms)}$$

$$\boxed{i(0^-) = i(0) = I_0 = -16(A)}$$

Inductor current can't change instantly

d) $$i(t) = I_0 e^{-t/\tau}$$

Solution in General

$$i(t) = -16e^{-t/2.5\times10^{-3}} = -16e^{-400t}\,(A)$$

Plug in value for τ

e) $$i(t=5(ms)) = -16e^{-5/2.5} = -16e^{-2} = -2.17(A)$$

$$w(t=5(ms)) = \frac{1}{2}Li^2(5(ms)) = \frac{1}{2}(0.01)(-2.17)^2$$

$\tau = 2.5(ms)$

After 2 Time Constants

$2\times2.5(ms) = 5(ms)$

Drill Exercise (Contd.)

$$w(t=5(ms)) = 23.44(mJ)$$

Energy in the inductor after 2 Time Constants

$$w_{diss} = w(0^-) - w(t=5(ms))$$

Energy dissipated in the inductor after 5 *ms*

$$= 1.28 - 23.44 \times 10^{-3}$$

Plug in numbers

$$w_{diss} = 1.2566\,(J)$$

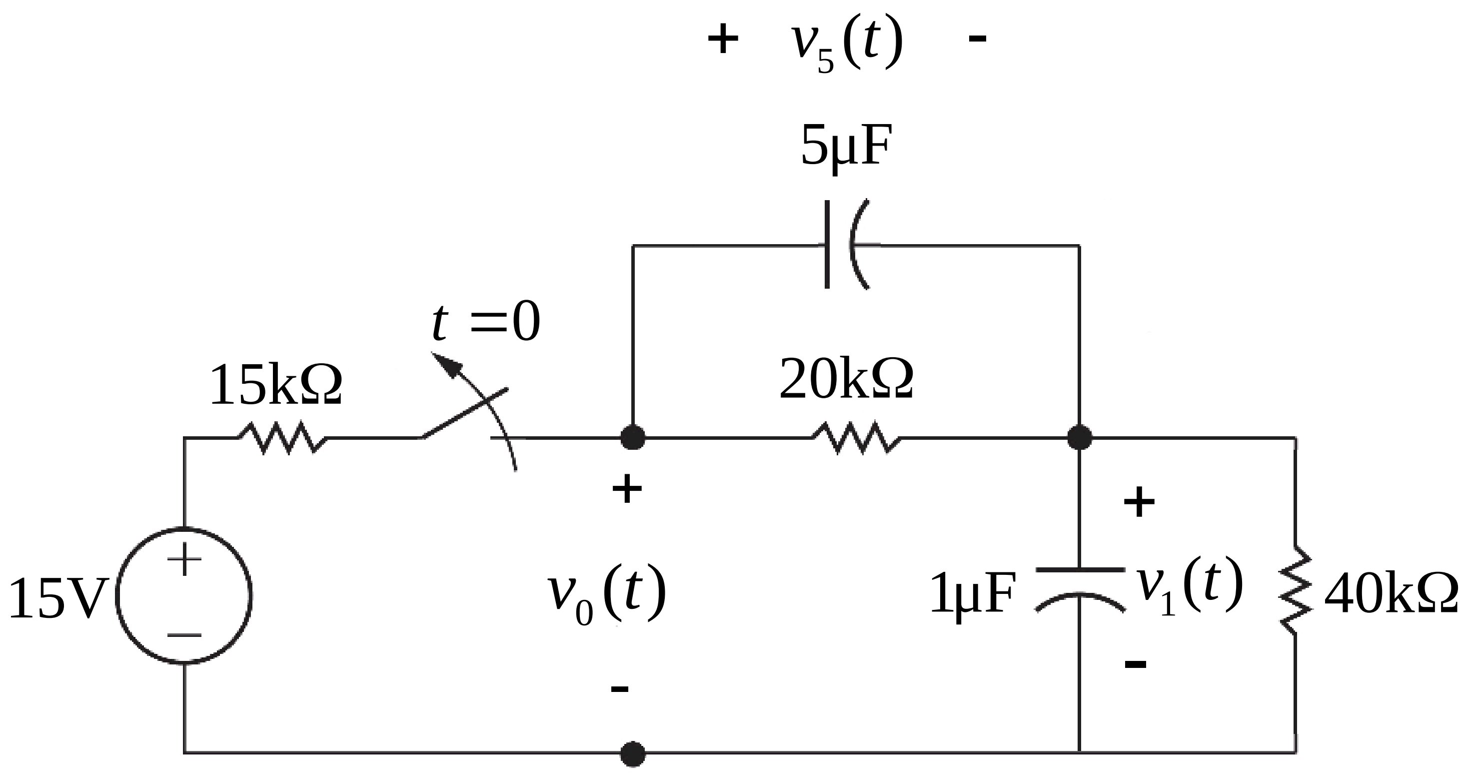
Energy dissipated in the inductor after 5 *ms*

$$\%\,\text{dissipated} = \left(\frac{1.2566}{1.28}\right) \times 100 = 98.17\%$$

After 2 Time Constants

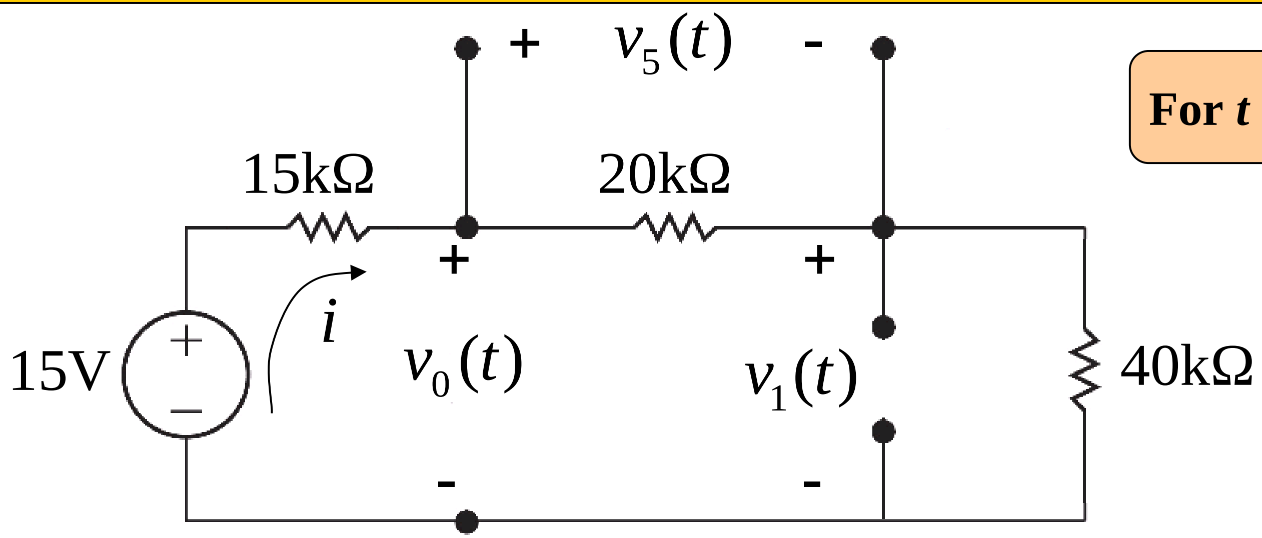
The effects of $\frac{1}{2}Li^2$ are clearly shown

Drill Exercise: Find $v_0(t)$ for $t \geq 0$



$$v_0(t) = v_5(t) + v_1(t)$$ **Solution has two parts**

Drill Exercise: Find $v_1(t)$ and $v_5(t)$ for $t \geq 0$ (Contd.)



For $t < 0$

- Steady State
- Caps Open

To Use RC Solution, We Have 2 Parts

$$i = \frac{15}{(15+20+40)(k\Omega)} = \frac{15}{75(k\Omega)} = 0.2(mA)$$ **Ohm's Law**

Ohm's Law

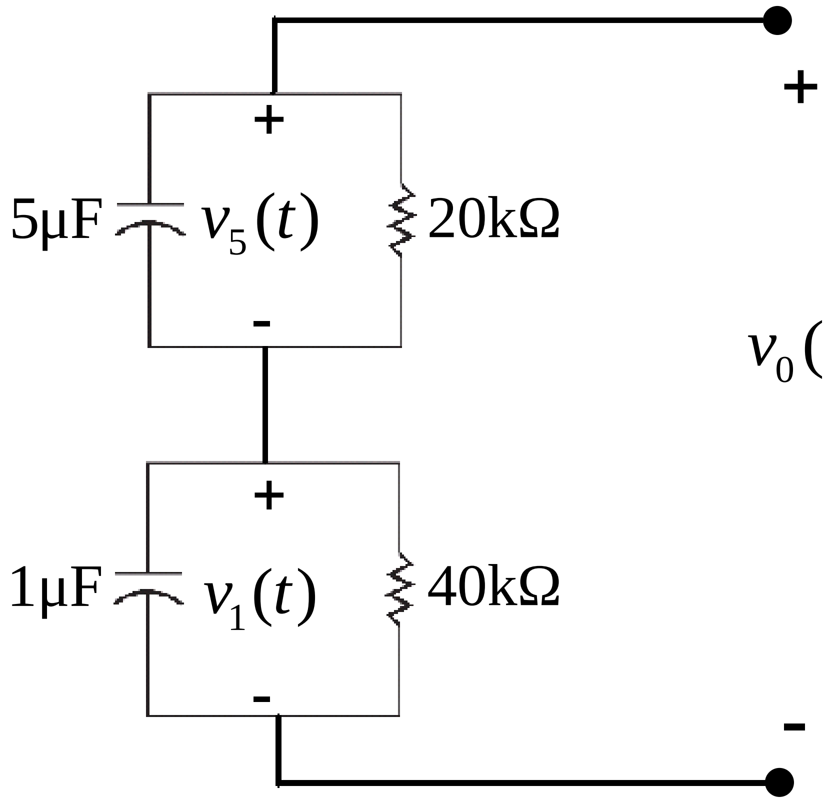
$$v_1(0^-) = 40(k\Omega)\times 0.2(mA) \longrightarrow v_1(0^-) = 8(V)$$

$$v_5(0^-) = 20(k\Omega)\times 0.2(mA) \longrightarrow v_5(0^-) = 4(V)$$

KVL $v_0(0) = 8+4 = 12(V)$ → Use as a Check after we Obtain $v_0(t)$.

Drill Exercise (Contd.)

For $t \geq 0$ the circuit becomes



$$v_0(t) = v_1(t) + v_5(t)$$

2 RC Circuits In Series

$$\tau_1 = RC = 40(k\Omega) \cdot 1(\mu F) \longrightarrow \tau_1 = 40(ms)$$

$$\tau_5 = RC = 20(k\Omega) \cdot 5(\mu F) \longrightarrow \tau_5 = 100(ms)$$

Find the Time Constants

Drill Exercise (Contd.)

$$v(t) = V_0 e^{-t/\tau}$$ **General Formula RC Parallel Circuit**

$$v_1(t) = v_1(0^-)e^{-t/\tau_1} = 8e^{-t/40(ms)}$$ **Plug in numbers**

1 $$v_1(t) = 8e^{-25t}\,(V)$$ **Simplify**

$$v_5(t) = v_5(0^-)e^{-t/\tau_5} = 4e^{-t/100(ms)}$$ **Plug in numbers**

2 $$v_5(t) = 4e^{-10t}\,(V)$$ **Simplify**

3 $$v_0(t) = v_1(t) + v_5(t)$$ **KVL**

$$v_0(t) = [8e^{-25t} + 4e^{-10t}](V) \qquad t \geq 0$$ **Substitute 1 and 2 into 3**

$$v_0(t=0) = 8 + 4 = 12(V)$$ **<u>Checks</u>**

Drill Exercise (Contd.)

b) After t = 60 (ms), What % of Energy is Dissipated ? ⇨ **Use** $w = \frac{1}{2}Cv^2$

Initial Energy Stored in Circuit :

$$w_1(0^-) = \frac{1}{2}Cv_1^{\ 2}(0^-) = \frac{1}{2}\left(1(\mu F)\right)(8)^2 = 32(\mu J)$$ **Initial Energy in 1μF Capacitor**

$$w_5(0^-) = \frac{1}{2}Cv_5^{\ 2}(0^-) = \frac{1}{2}\left(5(\mu F)\right)(4)^2 = 40(\mu J)$$ **Initial Energy in 5μF Capacitor**

$$w_{\text{Total}}(0^-) = w_1(0^-) + w_5(0^-) = \boxed{72(\mu J)}$$ **Total Initial Energy**

$$v_1(t) = 8e^{-25t} \qquad v_5(t) = 4e^{-10t}$$ **Analytical Expression**

$$v_1\left(60(ms)\right) = 1.79(V) \qquad v_5\left(60(ms)\right) = 2.20(V)$$ **Plug in 60ms in Analytical Expression**

$$w_1\left(60(ms)\right) = \frac{1}{2}\left(1(\mu F)\right)(1.79)^2 = 1.59(\mu J)$$ **Energy in 1μF Capacitor at 60ms**

$$w_5\left(60(ms)\right) = \frac{1}{2}\left(5(\mu F)\right)(2.20)^2 = 12.05(\mu J)$$ **Energy in 5μF Capacitor at 60ms**

$$w_{\text{Total}}\left(60(ms)\right) = w_1\left(60(ms)\right) + w_5\left(60(ms)\right) = \boxed{13.64(\mu J)}$$ **Total Energy at 60ms**

Drill Exercise (Contd.)

$$w_{diss} = w_{\text{Total}}(0^-) - w_{\text{Total}}\left(60(ms)\right)$$ Energy Dissipated from $t = 0$ to $t = 60$ *ms*

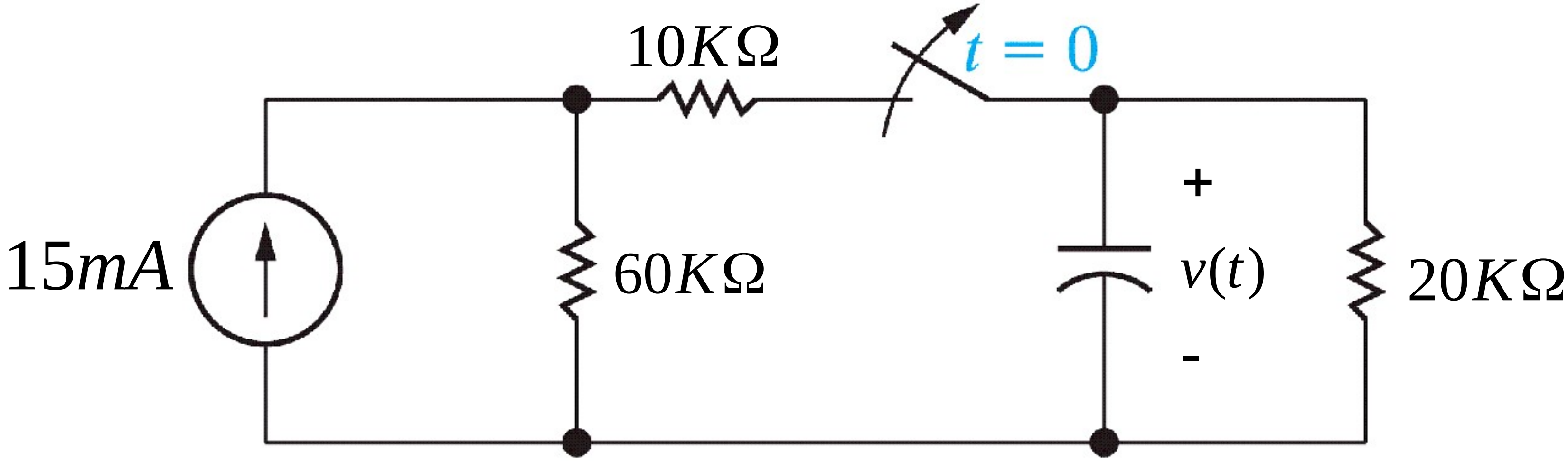
$$= 72(\mu J) - 13.64(\mu J)$$ Plug in the numbers

$$w_{diss} = 58.36(\mu J)$$ **Energy dissipated by the capacitors after 60(*ms*)**

$$\%\,\text{dissipate} = \frac{w_{diss}}{w_{Total}(0^-)} \times 100\% = \frac{58.36}{72} \times 100\%$$ Plug in the numbers

$$\%\,\text{dissipated} = 81.056\%$$

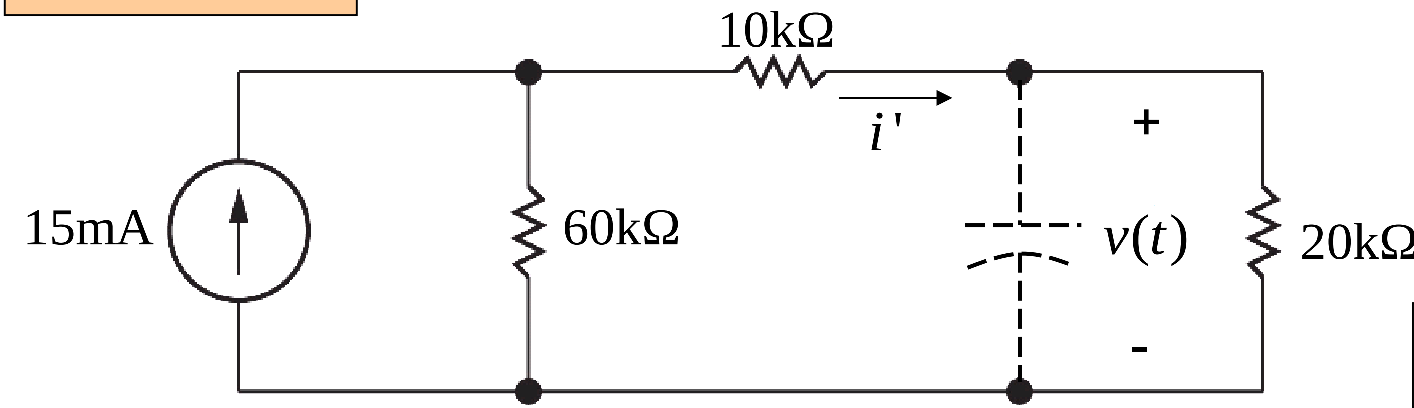
Drill Exercise: Find *v(t)* for $t \geq 0$



At Steady State ⇨ Capacitor is Open

Drill Exercise: Find $v(0)$ (Contd.)

a) For $t < 0$



Steady State. Capacitor Open.

① $v(0^-) = V_0 = i' \cdot 20(k\Omega)$ **Ohm's Law**

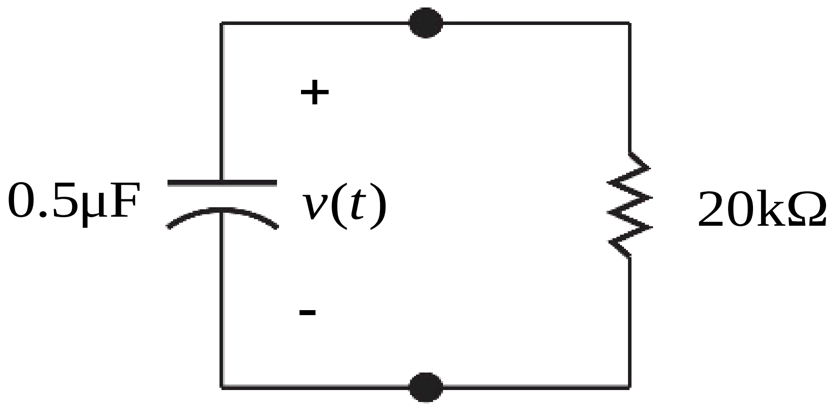
② $i' = 15(mA)\left[\dfrac{60(k\Omega)}{60(k\Omega) + (10(k\Omega) + 20(k\Omega))}\right] = 10(mA)$ **Current Division**

$V_0 = 10(mA) \cdot 20(k\Omega)$ **Substitute ② into ①**

$V_0 = 200(V)$ **Voltage can't change instantly**

Drill Exercise (Contd.)

b) For $t \geq 0$ the circuit becomes



$$\tau = RC = 20(k\Omega)\cdot 0.5(\mu F)$$ **Time Constant Formula**

$$\tau = 10(ms)$$

$$\boxed{V_0 = 200(V)}$$

c) Analytical Expression

$$v(t) = V_0 e^{-t/\tau} = 200e^{-t/10(ms)}$$ **Plug in the numbers**

$$v(t) = 200e^{-100t} \ (V)$$

d) $$w(0^-) = \frac{1}{2}CV_0^2 = \frac{1}{2}(0.5(\mu F))(200(V))^2$$ **Calculate the Initial Energy**

$$w(0^-) = 10(mJ)$$

Drill Exercise (Contd.)

How Long Does it Take for 75% of the Energy to be Dissipated?

e) Analytical Expression for the Energy

$$w(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2}\left(0.5\left(\mu F\right)\right)(200e^{-100t})^2 = 10e^{-200t}\left(mJ\right)$$

If 75% Dissipated, 25% is Left.

Find t' where $w(t')$ = 25% $w(0^-)$

$$\Rightarrow 10e^{-200t'} = \underbrace{0.25 \cdot w(0)}_{\text{25\% of Initial Energy}} = 0.25(10) = 2.5\left(mJ\right) \quad \text{Solve for } t'$$

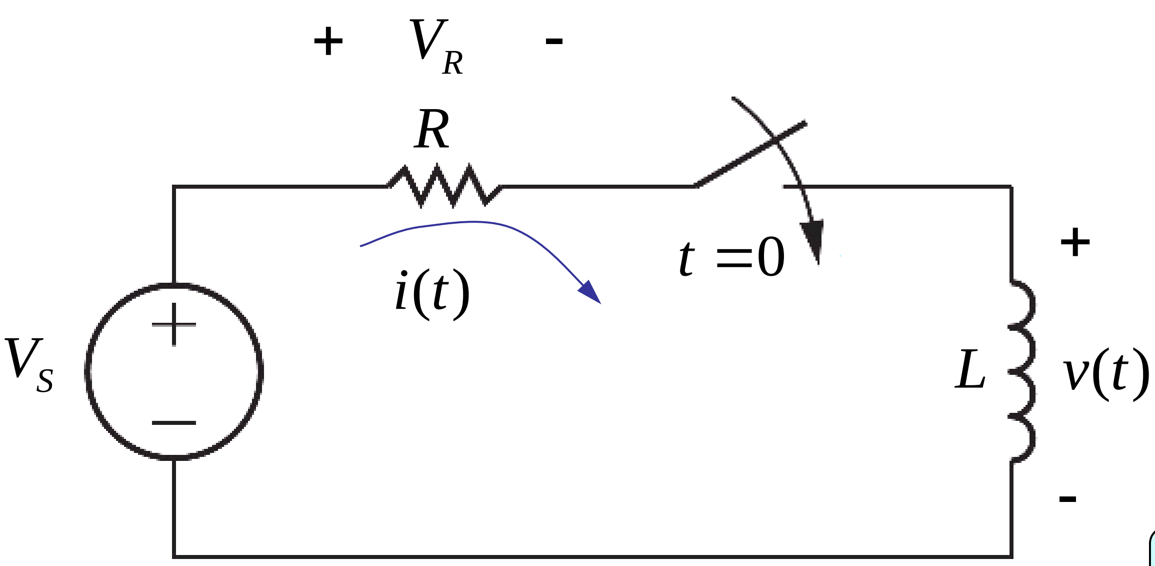
$w(0^-) = 10$

$$\therefore\ t' = 6.93\left(ms\right)$$

It Takes 6.93(*ms*) for 75% of the Energy to be Dissipated.

RL Step Response: Charging Up

Response to Sudden Application of dc ***I*** or ***V*** .



Switch Closed at $t = 0$

L "Charges Up"

i Can't Change Instantly

For $t \geq 0_-$

KVL ⟶ $V_S = V_R + v$ **Substitute** $V_R = iR$ **and** $v = L\dfrac{di}{dt}$

$$V_S = Ri + L\frac{di}{dt}$$ **Differential Equation**

RL Step Response (Contd.)

Solve for *i(t)* by Separation of Variables

$$\frac{di}{dt} = \frac{V_S - Ri}{L} = -\frac{R}{L}\left[i - \frac{V_S}{R} \right]$$

Solve for $\frac{di}{dt}$

$$\frac{di}{i - \frac{V_S}{R}} = \frac{-R}{L} dt$$

Separate Differentials

Integrate Both Sides

$$\int \frac{di}{i - \frac{V_S}{R}} = \int \frac{-R}{L} dt + K$$

Integration Constant

RL Step Response (Contd.)

$$\int \frac{di}{i - \frac{V_s}{R}} = \int -\frac{R}{L}\,dt + K$$

Need to integrate the LHS

Integral Tables

$$\int \frac{dx}{x - c} = \ln(x - c)$$

where c is a constant

$$\text{1} \quad \ln\left[i(t) - \frac{V_s}{R}\right] = \frac{-R}{L}t + K$$

Use above equation to integrate

$$i(t = 0) \equiv i(0) = I_0$$

Use *i(0)* to find the value of *K*

$$\ln\left[i(0) - \frac{V_s}{R}\right] = \frac{-R}{L}(0) + K$$

Substitute *t* = 0 into 1

$$\boxed{\therefore K = \ln\left[I_0 - \frac{V_s}{R}\right]}$$

After solving for *K*

RL Step Response (Contd.)

$$\ln\left[\frac{i(t)-\frac{V_s}{R}}{I_0-\frac{V_s}{R}}\right]=\frac{-R}{L}t$$

Substitute expression for *K* into 1

$$\therefore i(t)=\frac{V_s}{R}+\left(I_0-\frac{V_s}{R}\right)e^{-t/\tau} \qquad t\geq 0;\ \tau=\frac{L}{R}$$

After solving for *i(t)*

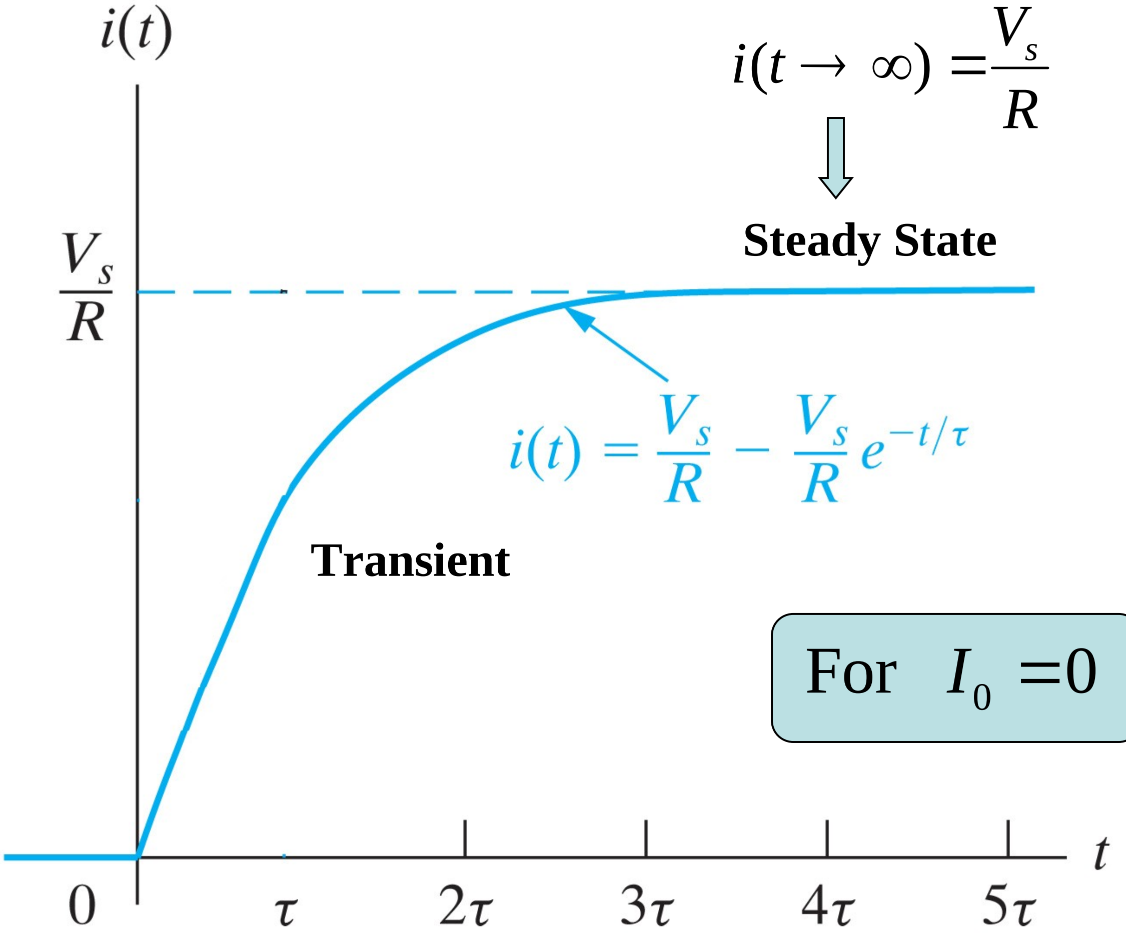
If inductor is not initially charged then $I_0=0$

$$\therefore i(t)=\frac{V_s}{R}\left[1-e^{-\frac{t}{\tau}}\right] \qquad t\geq 0;\ \tau=\frac{L}{R}$$

Expression for *i(t)* with I_0 = 0

Graphical Illustration of Step Response

$$i(t \to \infty) = \frac{V_s}{R}$$



$$\tau = \frac{L}{R}$$

For $I_0 = 0$

RL Step Response: Current in the Inductor

$$v(t) = L\frac{di(t)}{dt}$$ Substitute $i(t) = \frac{V_s}{R} + \left(I_o - \frac{V_s}{R}\right)e^{-\left(\frac{R}{L}\right)t}$

$$\boxed{v(t) = (V_s - I_0 R)e^{-\frac{t}{\tau}}}$$

$v(0^-) = 0$ From original circuit

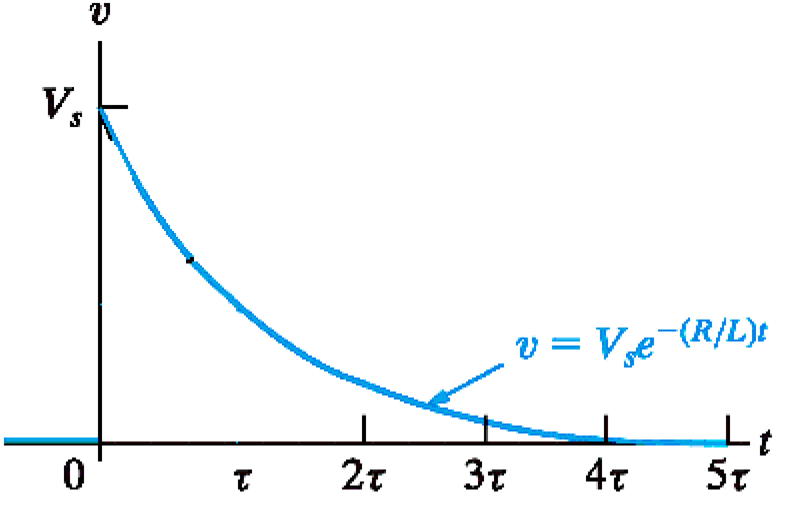
$v(0) = V_s - I_0 R$ Step change in voltage

For $I_0 = 0$

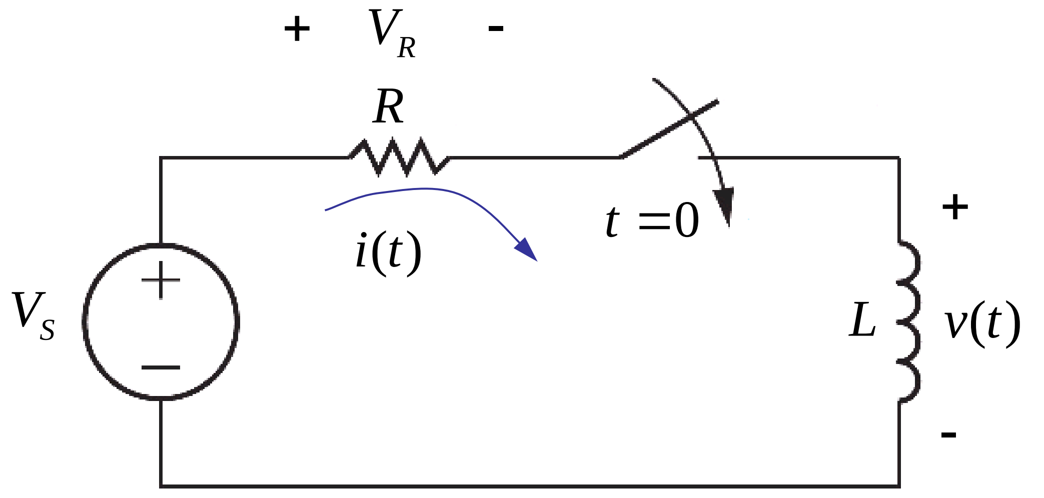
$$\boxed{v(t) = V_s e^{-\frac{t}{\tau}}}$$

$v(0^-) = 0$

$v(0) = V_s$



RL Step Response: Voltage Across Inductor
(Contd.)



Summarize Ideas …After Switch Closes

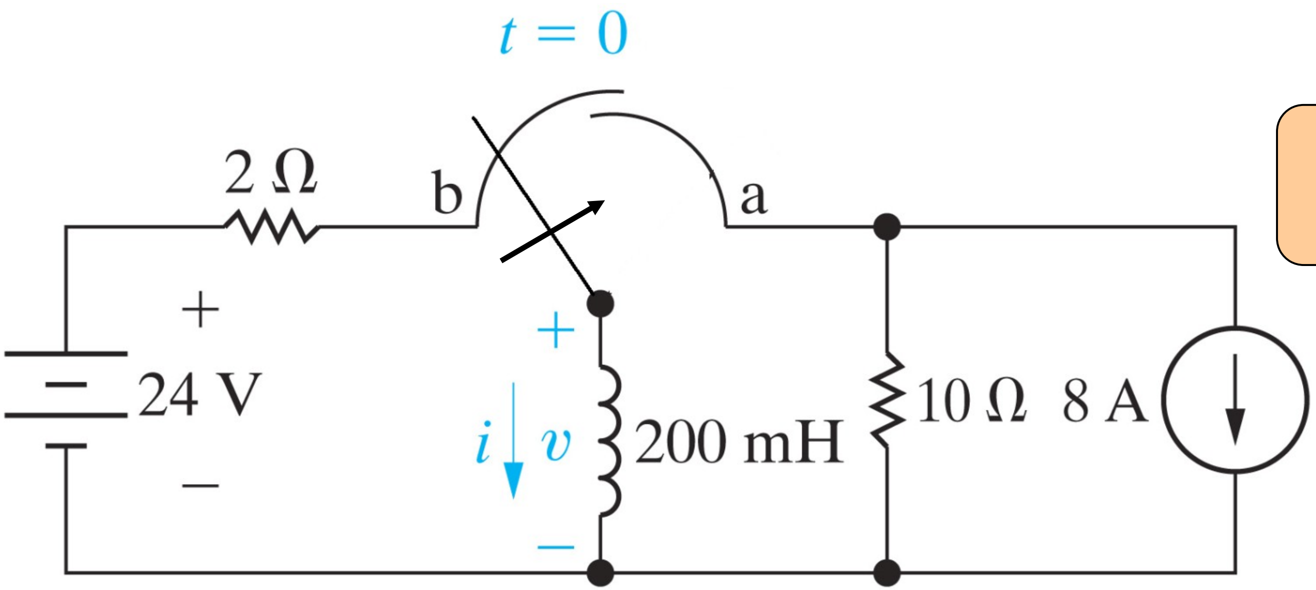
1) $i(0^-) = 0 \quad \therefore V_R = 0 \quad \therefore v(t) = V_s$

All voltage dropped across L, since none across R.

2) $i(t)$ Increases toward $\frac{V_s}{R}$, since L is short in steady state.

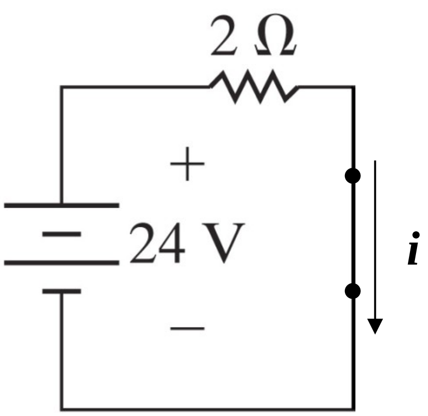
$v(t)$ Decays toward zero, and all voltage is across R in Steady State.

Example: Find $v(t)$ for $t \geq 0$



a) $t < 0$: Steady State;

L is a short.



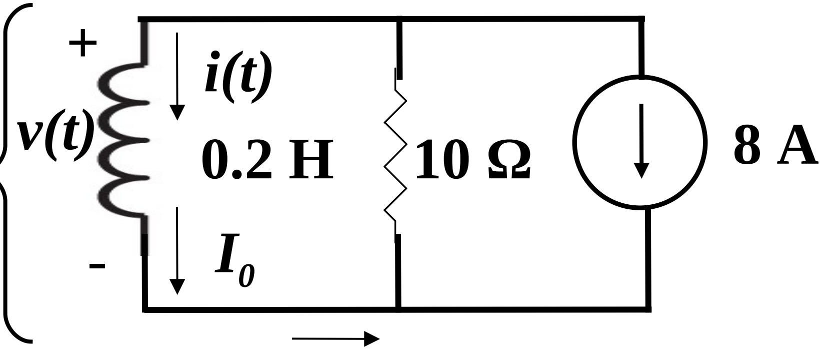
$$i(0^-) = i(0^+) = I_0 = \frac{24(V)}{2\Omega}$$ **Ohm's Law**

$$i(0^+) = 12\,(A)$$

$v(0^-) = 0$ because L is a short

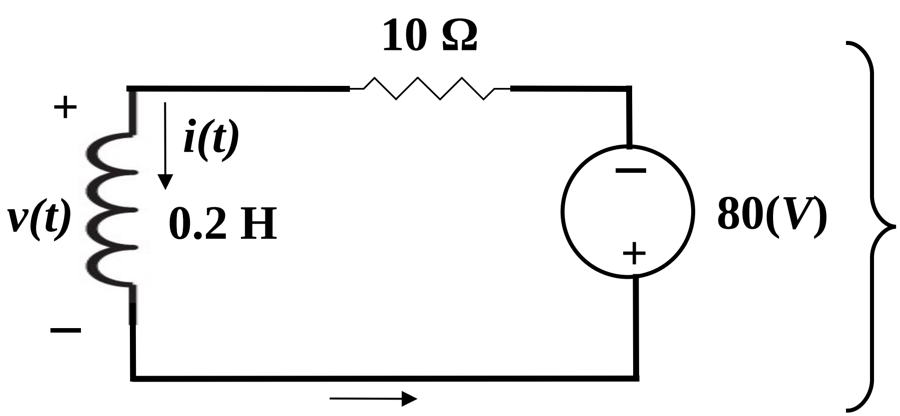
Example (Contd.)

Redraw circuit for $t=0^+$



$I_0=12\ (A)$

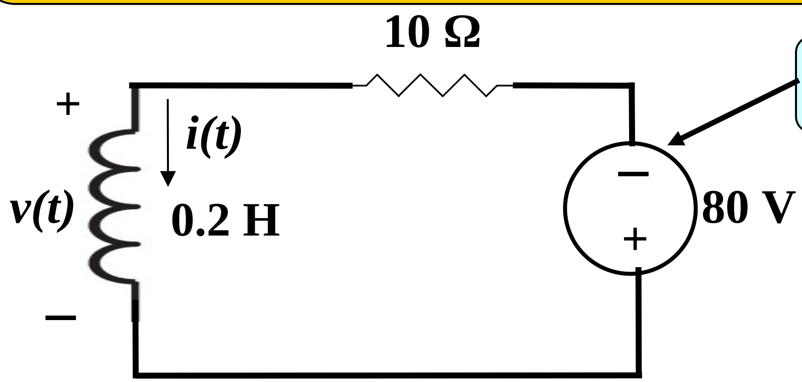
Thevenin Equivalent



$$8(A)\times 10(\Omega)=80(V)$$

$I_0=12\ (A)$

Example (Contd.)



Note: *V* and *I* polarities "opposite" to general circuit

$$i(t) = -\frac{V_s}{R} + \left(I_0 - \left[-\frac{V_s}{R} \right] \right) e^{-\frac{t}{\tau}}$$

Analytic expression for *i(t)*

$$i(t) = -\frac{80}{10} + \left(12 + \left[\frac{80}{10} \right] \right) e^{-\frac{t}{0.02}}$$

Plug in numbers

$$i(t) = (-8 + 20e^{-50t})\,(A) \qquad t \geq 0$$

Time Constant

$$\tau = \frac{L}{R} = \frac{0.2(H)}{10(\Omega)} = 0.02(s)$$

$$\tau = 20(ms)$$

$$v(t) = L\frac{di}{dt} = 0.2[0 + 20(-50)e^{-50t}] = -200e^{-50t}(V) \qquad t \geq 0$$

Find *v(t)*

Example: Plots (Contd.)

Find t' when $i(t') = 0$

$$0 = -8 + 20e^{-50t'}$$

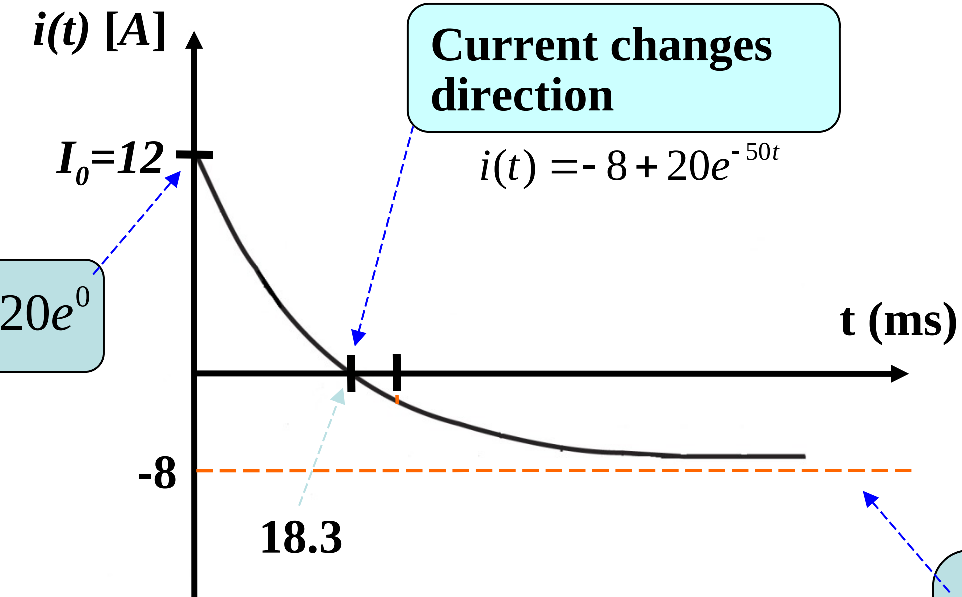
$$\ln\left(\frac{8}{20}\right) = -50t'$$

$$\Rightarrow \quad t' = 18.3\,(ms)$$

Current changes direction

$$i(t) = -8 + 20e^{-50t}$$

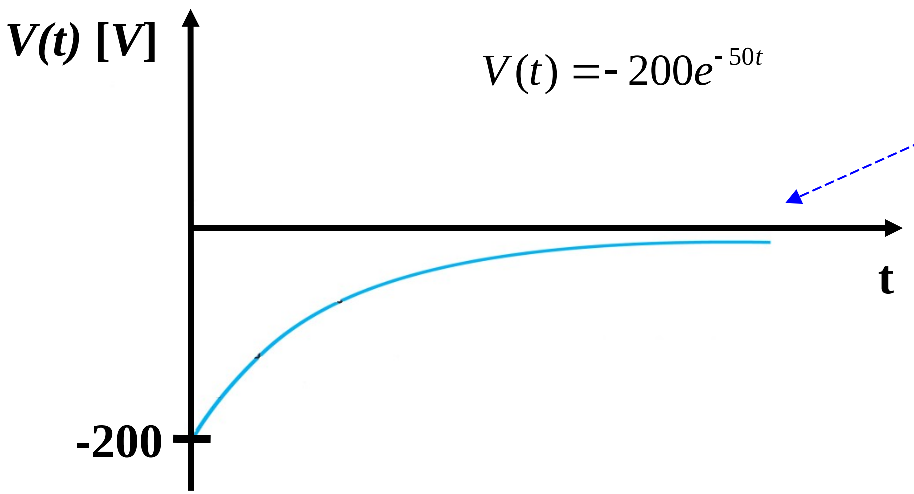
$$-8 + 20e^{0}$$



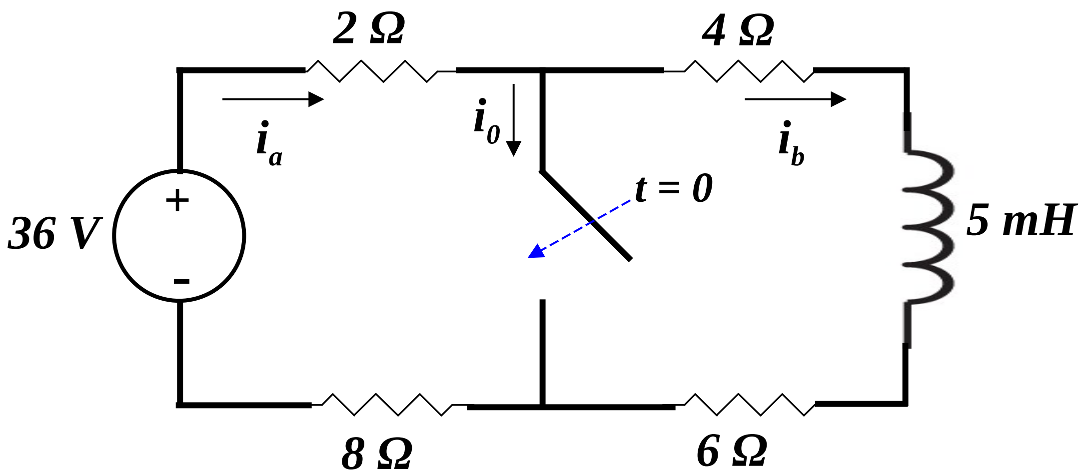
$$i(\infty) = -8\,(A)$$

$$V(\infty) = 0$$

$$V(t) = -200e^{-50t}$$



Example: Find $i_o(t)$ for $t \geq 0$



a) $-\infty \leq t < 0$ Steady State

$i_a = i_b$ } Inductor is Short

$i_a = 36(V)/(2+4+6+8)\Omega$

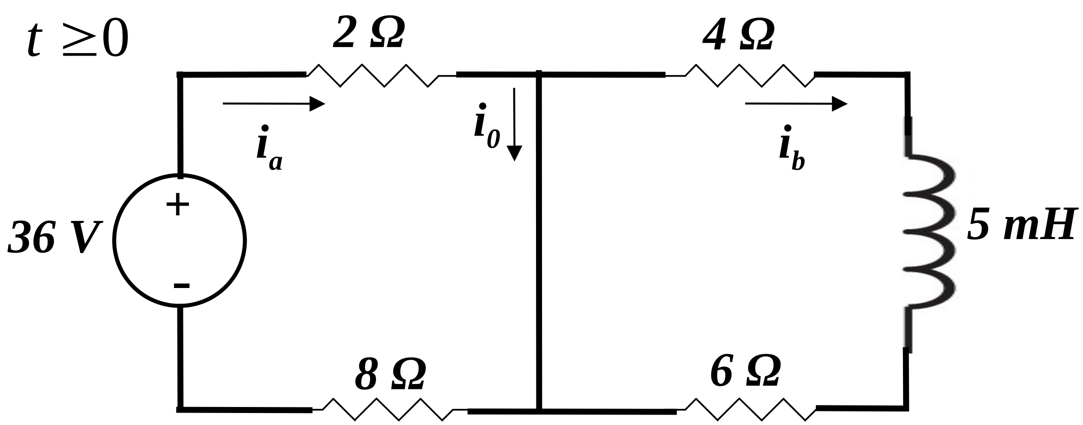
$i_a = 1.8(A)$

} **Ohm's Law**

$\boxed{i_0 = 0}$ $\Rightarrow$ $i_0(0^-) = 0 \quad i_a(0^-) = i_b(0^-) = 1.8(A)$

Example (Contd.)

$t \geq 0$



Current can't change instantly in *L*

Current circulating in both loops

KVL

$$i_a(0) = \frac{36V}{(2+8)\Omega} = 3.6(A)$$

$$i_a(0^-) = 1.8(A) \quad i_a(0) = 3.6(A)$$

$$i_b(0) = i_b(0^-) = 1.8(A)$$

i_b can't change instantly

KCL

$$i_0(t) = i_a(t) - i_b(t)$$

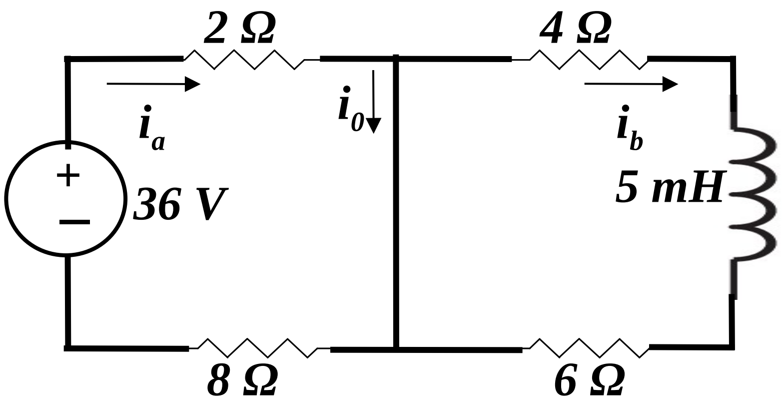
$$i_0(0) = i_a(0) - i_b(0) = 3.6(A) - 1.8(A)$$

$$i_0(0) = 1.8(A)$$

$$i_0(0^-) = 0 \quad i_0(0) = 1.8(A)$$

Example (Contd.)

The right-hand side of the circuit is shorted out for $t \geq 0$



b) RL Circuit Essentially Isolated from the Rest of the Circuit

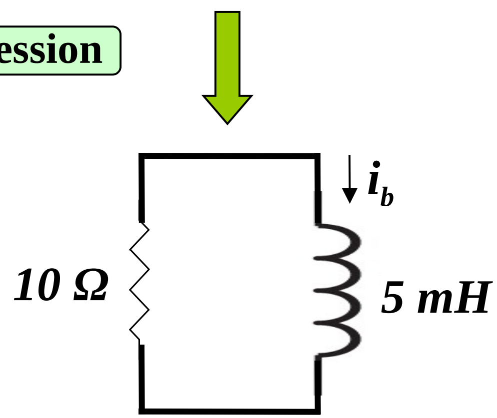
$$i_b(t) = i_b(0)e^{-\frac{t}{\tau}} \qquad i_b(0) = 1.8(A)$$

Use analytic expression

$$\tau = \frac{L}{R} = \frac{5 \times 10^{-3} H}{(4+6)\Omega} = 0.5(ms)$$

Time constant

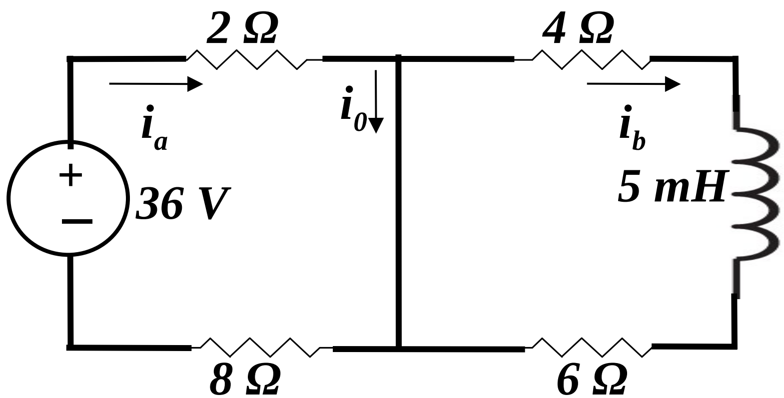
$$i_b(t) = 1.8e^{-2000t}\ (A) \qquad t \geq 0$$



Example (Contd.)

Find $i_0(t)$ for $t \geq 0$

For $t \geq 0$



$$i_0(t) = i_a(t) - i_b(t)$$ **KCL**

$$i_a = \frac{36(V)}{(2+8)\Omega} = 3.6(A)$$ **Ohm's Law**

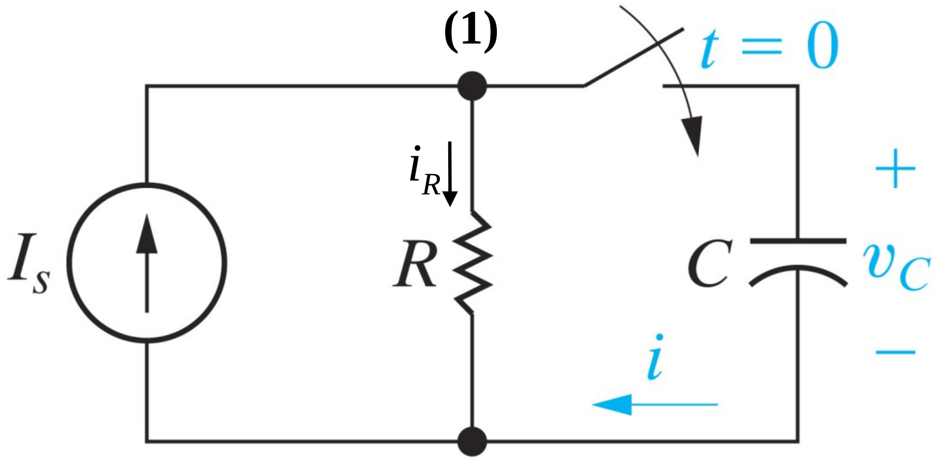
$$i_0(t) = (3.6 - 1.8e^{-2000t})(A) \quad t \geq 0$$ **Plug in values Use i_b from previous slide**

$$i_0(0) = 3.6 - 1.8 = 1.8(A)$$

$$i(0^-) = 0$$

i_o changed instantly

RC Step Response



Find $i(t)$ for $t \geq 0$

Find $v_c(t)$ for $t \geq 0$

$v_c(0^-) = V_0$

$$i_R = \frac{v_c}{R}$$ **Ohm's Law**

$$i = C\frac{dv_c}{dt}$$ **Capacitor equation**

$$I_s = i_R + i$$ **KCL at (1)**

$$I_s = \frac{v_c}{R} + C\frac{dv_c}{dt}$$ **Substitute for i_R, i_c**

$$\frac{I_s}{C} = \frac{v_c}{RC} + \frac{dv_c}{dt}$$ **Divide by C**

Same form as for Inductor

$$V_s = Ri_L + L\frac{di_L}{dt}$$

$$\frac{V_s}{L} = \frac{R}{L}i_L + \frac{di_L}{dt}$$

Comparison between Inductor and Capacitor

Inductor	**Capacitor**
V_s	I_s
L	C
$\tau = \frac{L}{R}$	$\tau = RC$
$i_L(t)$	$v_C(t)$
I_0	V_0

RC Step Response

$$v_c(t) = I_s R + (V_0 - I_s R)e^{-t/RC}; \quad t \geq 0$$

Solution to differential equation

$$v_c(t) = I_s R\left[1 - e^{-t/RC}\right]; \quad t \geq 0$$

$v_c(t)$ when $V_0 = 0$

$$i(t) = C\frac{dv_c}{dt}; \quad t \geq 0$$

Find $i(t)$ from $v_c(t)$

$$i(t) = \left(I_s - \frac{V_0}{R}\right)e^{-t/\tau}; \quad \tau = RC; \quad t \geq 0$$

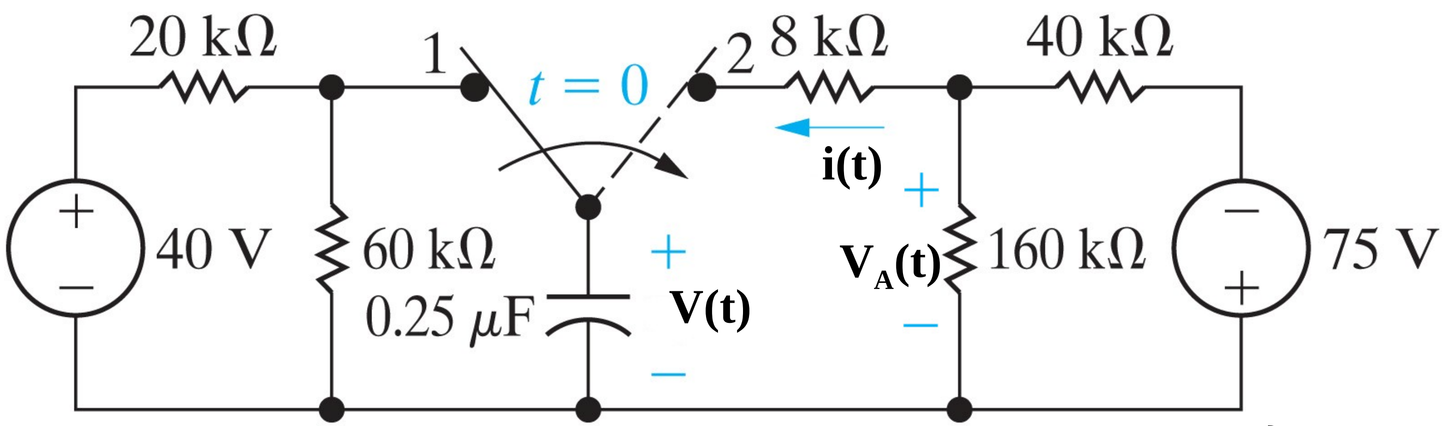
After substitution for $v_c(t)$

$$i(t) = I_s e^{-t/\tau}; \quad t \geq 0$$

$i(t)$ when $V_0 = 0$

Drill Exercise

Find $V_A(t)$ for $t \geq 0$

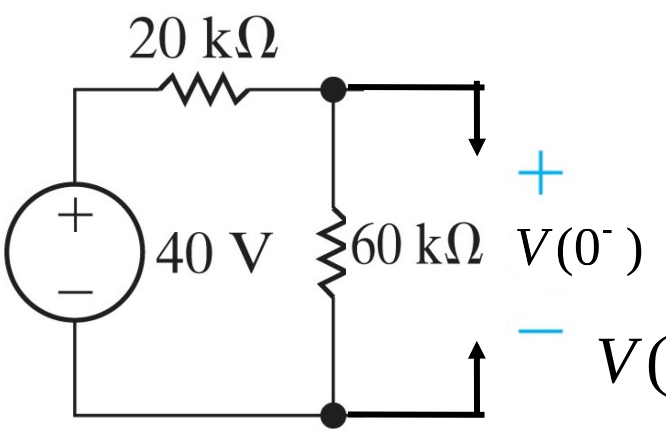


t < 0

$$V_A(0^-) = -\frac{160}{40+160}75 = -60$$

Voltage Division

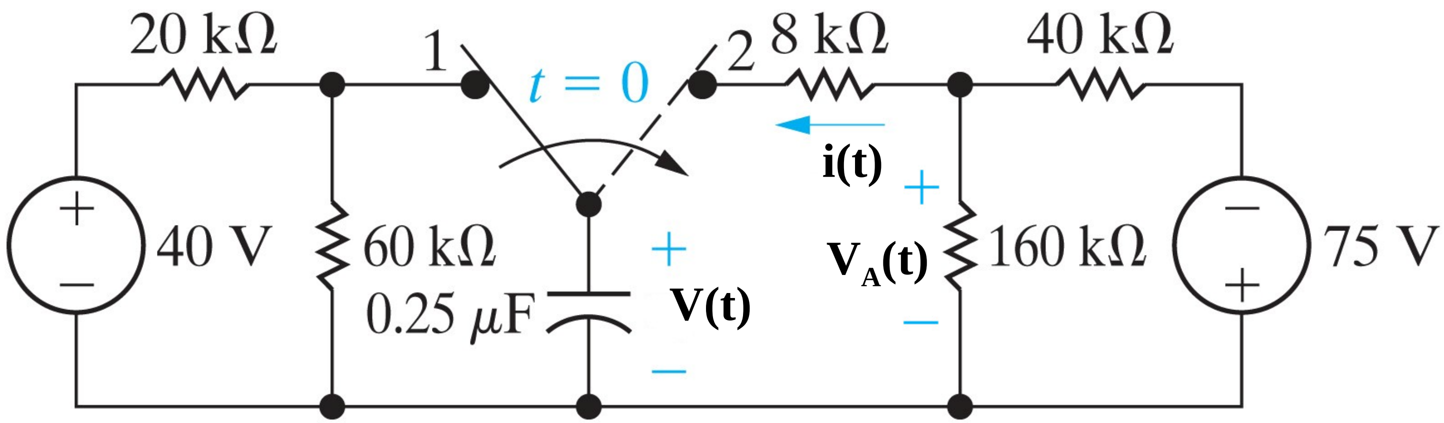
Capacitor is Open In Steady State



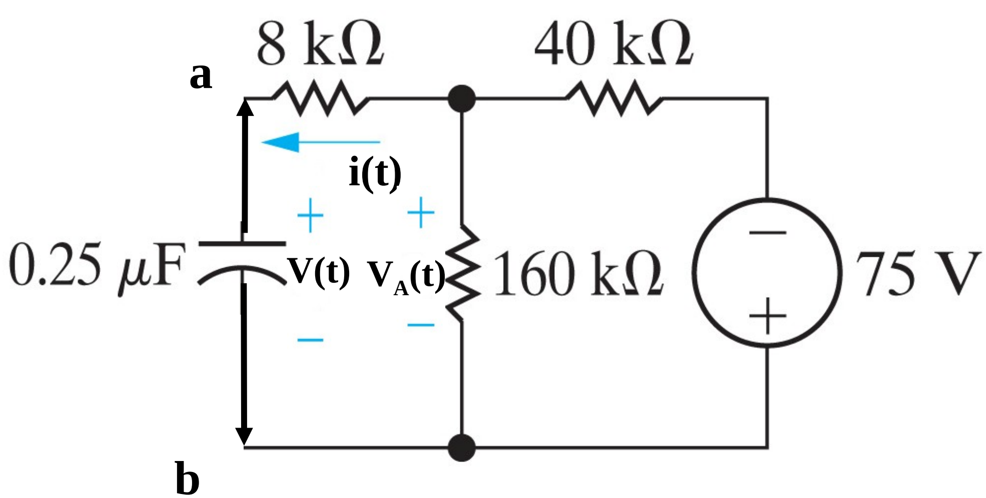
Voltage Division

$$V(0^-) = 40(V)\frac{60}{60+20} \Rightarrow V_0 = 30(V)$$

Drill Exercise (Contd.)



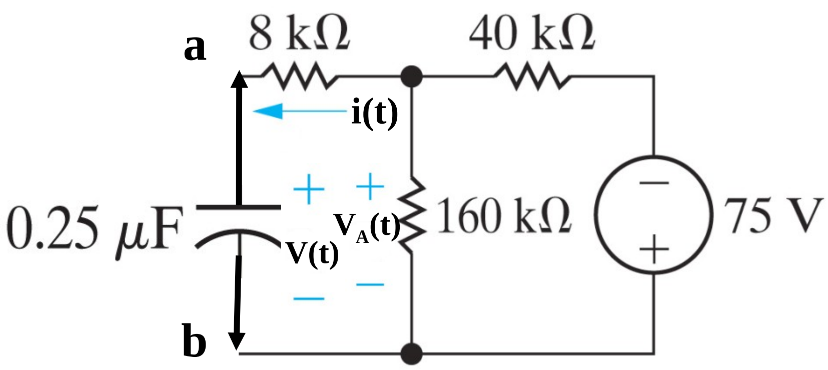
$t \geq 0$



Need Norton Equivalent Between a-b, to use Derived Equations For RC Circuits

Drill Exercise (Contd.)

Short voltage source to obtain R_{Th}



Parallel + Series

$$R_{ab} = R_{Th} = (40K \parallel 160K) + 8K$$

$$R_{Th} = 40\left(K\Omega\right)$$

Open circuit voltage, V_{ab}, to obtain $V_{oc} = V_{Th}$

$8K$ Resistor is open, $\therefore$ not relevant.

$$V_{oc} = V_{ab} \equiv V_{Th} = -75V\left[\frac{160K}{160K + 40K}\right]$$

Voltage Division

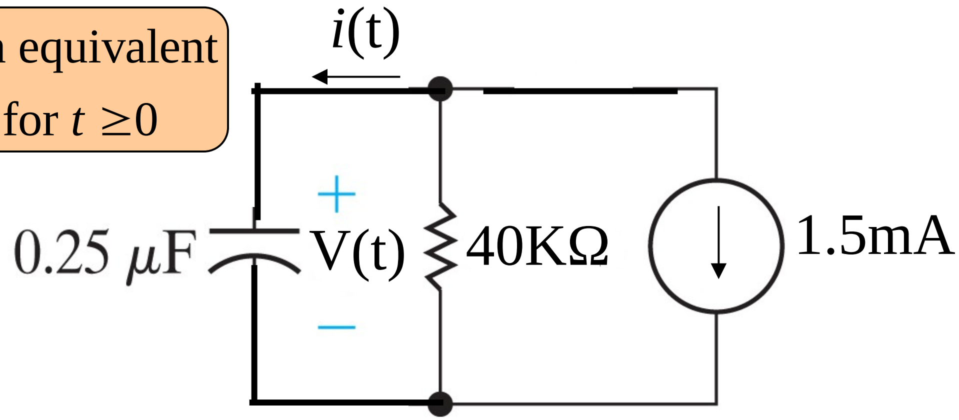
$$\boxed{V_{Th} = -60\left(V\right)}$$

$$\therefore I_N = V_{Th} / R_{Th}$$

$$I_N = -60/40\left(K\right) = \boxed{-1.5\left(mA\right)}$$

Drill Exercise (Contd.)

Norton equivalent circuit for $t \geq 0$



Time Constant

$$\tau = RC = 40K\Omega[0.25\mu F]$$

$$\tau = 10(ms)$$

$$V(t) = I_s R + (V_0 - I_s R)e^{-t/\tau}; \quad t \geq 0$$

Analytic solution

$$V(t) = (-1.5mA)\,40K\Omega + (30 - [-1.5\,mA]\,40\,K\Omega)e^{-t/0.01}$$

Plug in numbers

$$= -60 + (30 + 60)e^{-100t}$$

Simplify

$$\boxed{V(t) = (-60 + 90e^{-100t})(V)}$$

Note

V(0) = 30 (*V*)

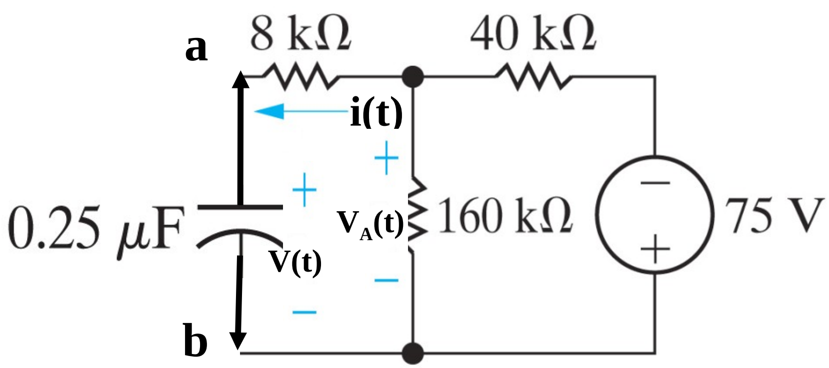
V(∞) = –60 (*V*)

$$i(t) = C\frac{dV}{dt} = 0.25\mu F[-100(90)e^{-100t}]$$

Find *i(t)* by using capacitor formula

$$\boxed{i(t) = -2.25e^{-100t}\ (mA); \qquad t \geq 0}$$

Drill Exercise (Contd.)



Find $V_A(t)$ for $t \geq 0^+$ } **This was the original problem statement**

$$V_A(t) = 8K \cdot i(t) + V(t)$$ } **KVL**

$$V_A(t) = 8000\left[-2.25e^{-100t}(mA)\right] + \left[-60 + 90e^{-100t}\right]$$

Substitute for $i(t)$ and $V(t)$ from previous slide

$$V_A(t) = -18e^{-100t} - 60 + 90e^{-100t}$$ } **Simplify**

$$\boxed{V_A(t) = \left(72e^{-100t} - 60\right)(V) \qquad t \geq 0}$$

$V_A(0) = 12$ } **From the above expression**

$V_A\left(0^-\right) = -60$ } **From the original circuit**

$V_A(t)$ Changes Instantly at $t=0$, so $V_A(t)$ is discontinuous at $t=0$.

General Solution for RL & RC Circuits

Differential Equations are of the same form

	Inductor	Capacitor
Natural	$\frac{di_L}{dt} + \frac{R}{L} i_L = 0$	$\frac{dv_c}{dt} + \frac{v_c}{RC} = 0$
Step	$\frac{di_L}{dt} + \frac{R}{L} i_L = \frac{V_s}{L}$	$\frac{dv_c}{dt} + \frac{v_c}{RC} = \frac{I_s}{C}$
Time-Constant	$\frac{R}{L} = \frac{1}{\tau_{RL}}$	$\frac{1}{RC} = \frac{1}{\tau_{RC}}$

General Solution for RL & RC Circuits
(Contd.)

- Let $x(t)$ be the Unknown Quantity.

$$i_L, \qquad v_L, \qquad i_c, \qquad v_c$$

$x(t)$ could be any of these signals

- Let K be a Constant.

$$V_s/L, \qquad I_s/C, \qquad 0$$

K could be any of these constants

$$\boxed{\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = K}$$

General form of Differential Equations

- Final Value x_f as $t \to \infty$

Steady State

$$\left.\frac{dx(t)}{dt}\right|_{t\to\infty} = 0 \Rightarrow \frac{x_f}{\tau} = K \qquad \boxed{\therefore x_f = K\tau}$$

Solve General Differential Equation

$$\frac{dx}{dt} = K - \frac{x}{\tau} = \frac{K\tau - x}{\tau} = \frac{x_f - x}{\tau} = -\frac{\left(x - x_f\right)}{\tau}$$

Rearrange differential equation where $x_f = K\tau$

$$\frac{dx}{x - x_f} = -\frac{1}{\tau}dt$$

Separate variables

$$\int \frac{dx}{x - x_f} = -\frac{1}{\tau}\int dt + Z$$

Integrate both sides

Arbitrary Constant

$$* \quad \ln\left(x - x_f\right) = -\frac{t}{\tau} + Z$$

Find anti-derivative

Solve General Differential Equation (Contd.)

* $\ln\left(x - x_f\right) = -\dfrac{t}{\tau} + Z$ Find anti-derivative

1 $\ln\left[x(t_0) - x_f\right] = -\dfrac{t_0}{\tau} + Z$ Substitute $t = t_0$ into * to find Z

2 $Z = \ln\left[x(t_0) - x_f\right] + \dfrac{t_0}{\tau}$ Solve for Z

3 $\ln\left(x - x_f\right) = -\dfrac{t}{\tau} + \left(\dfrac{t_0}{\tau} + \ln\left[x(t_0) - x_f\right]\right)$ Substitute 2 into *

4 $\ln\left[\dfrac{x - x_f}{x(t_0) - x_f}\right] = -\dfrac{t - t_0}{\tau}$ Simplify 3

$$x(t) = x_f + \left[x(t_0) - x_f\right] e^{\frac{-(t - t_0)}{\tau}}$$

Solve 4 for $x(t)$

First Order DE for RC or RL Circuits

$$\frac{dx}{dt} + \frac{x(t)}{\tau} = K$$

Differential Equation

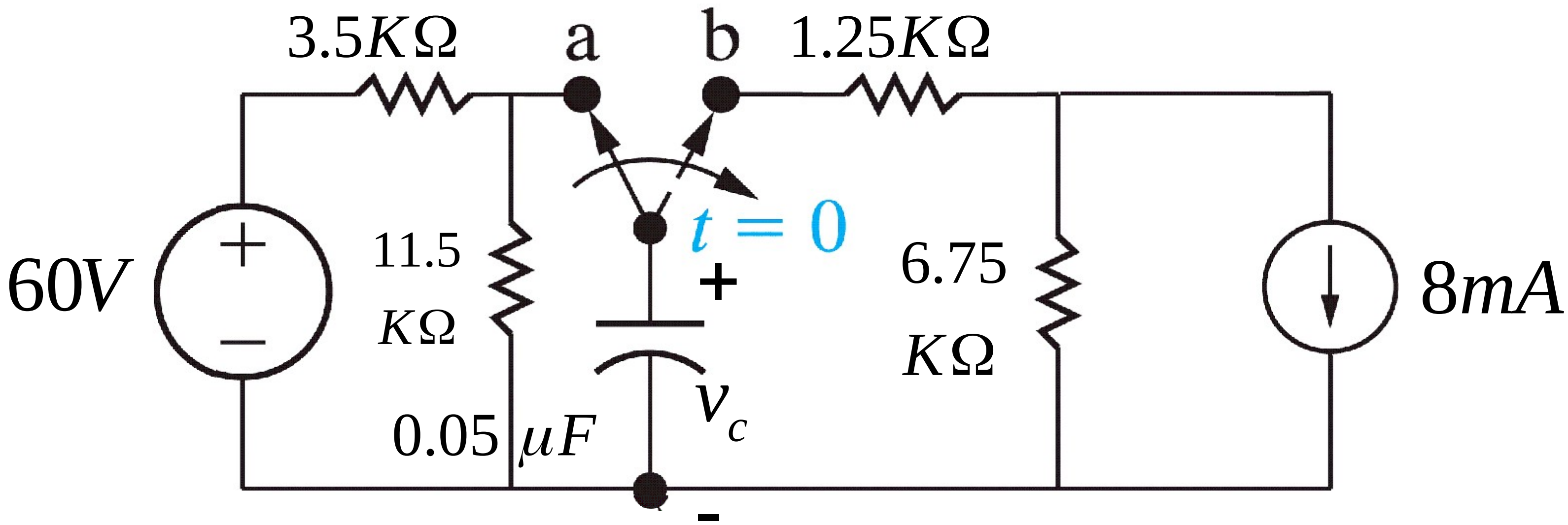
$$x(t) = x_f + \left[x(t_0) - x_f \right] e^{-\frac{(t - t_0)}{\tau}}$$

Solution where

$x_f = K\tau$

1. Initial time; t_0
1. Initial Value; $x(t_0)$
1. Final Value; x_f
1. Time Constant; τ

Example: Find $v_c(t)$



a) $t < 0$ **Steady State; Capacitor is Open**

Note V_c can't change Instantly

Initial Voltage

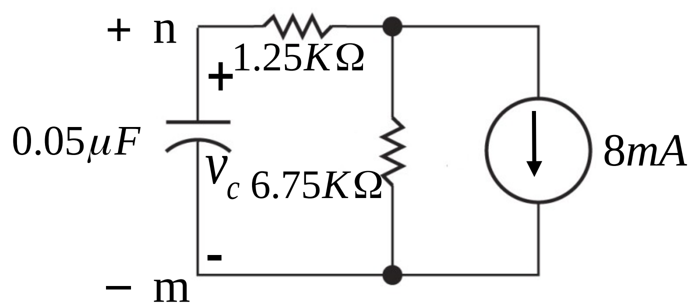
$$v_c(0^-) = v_c(0) = V_{11.5K\Omega} = 60\frac{11.5K\Omega}{(3.5+11.5)K\Omega}$$

Voltage Division

$$\boxed{v_c(0) = 46(V)}$$

Example (Contd.)

b) Find $v_c(t)$ $\quad t \geq 0$



Find Norton Equivalent between n and m

$$R_{Th} = (1.25 + 6.75)K = 8K$$

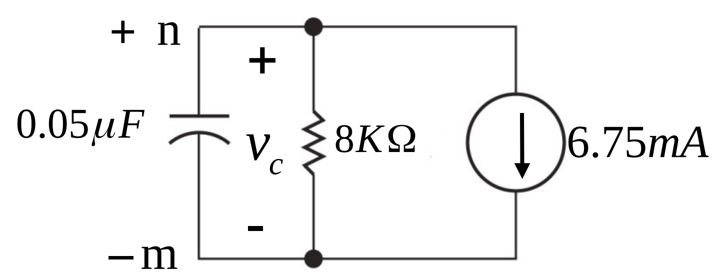
Add series resistor

$$V_{Th} = -8mA(6.75K\Omega) = -54(V)$$

Ohm's Law

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{-54(V)}{8(K\Omega)} = -6.75(mA)$$

Short Circuit Current



Norton Equivalent

$$v_{c(final)} = -(6.75mA)(8K\Omega) = -54(V)$$

Capacitor is open in steady state

c) $\tau = R_{Th}C = 8K\Omega(0.05\mu F)$ $\quad \boxed{\tau = 400(\mu s)}$

Find time constant

Example (Contd.)

d) General Solution

$$v_c(t) = v_{c(final)} + \left[v_c(t_0) - v_{c(final)} \right] e^{-\frac{(t-t_0)}{\tau}}$$

Note
$t_0 = 0$

$$\therefore v_c(t) = -54 + \left[46 - (-54) \right] e^{-\frac{t}{400\mu s}} \ (V)$$

$$= -54 + 100 e^{-2500t} \ (V)$$

Plug in values then simplify

How long until $v_c(t) = 0$?

$$100 e^{-2500t} = 54$$

Plug in $v_c(t)=0$ and solve for t

$$\therefore t = \frac{1}{2500} \ln\left(\frac{100}{54}\right)$$

$$t = 246.47(\mu s)$$

Side Note

$$v_c(0) = +46(V)$$

$$v_c(\infty) = -54(V)$$

$\therefore v_c(t)$ **Crosses Time Axis**

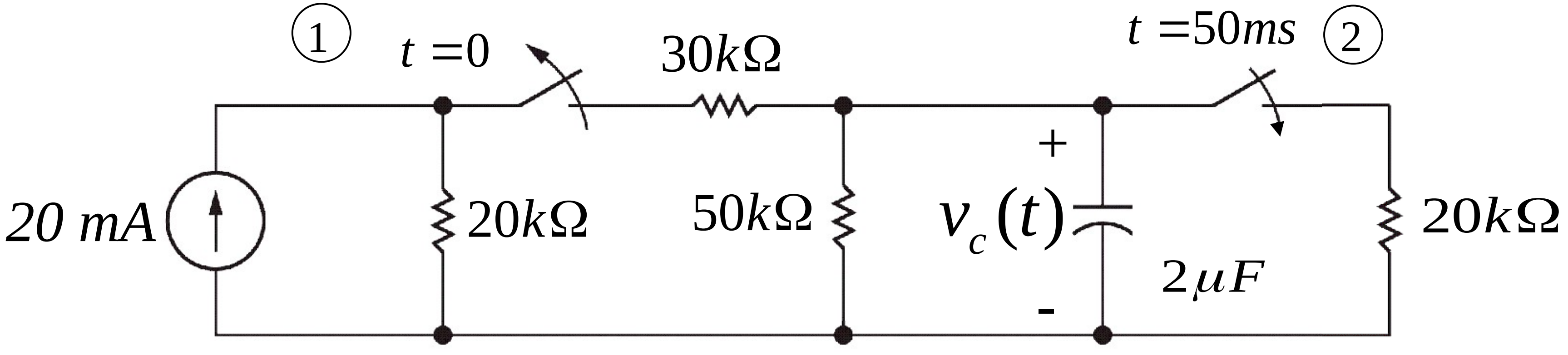
Sequential Switching

When Switching Occurs More Than Once

Must Find Initial Values at each event, we then solve for

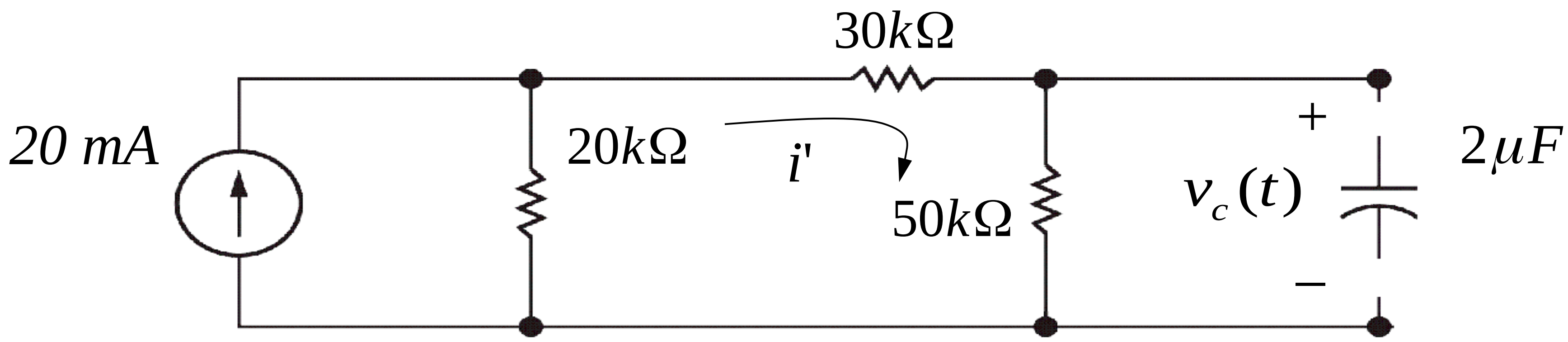
- Inductive Currents
- Capacitive Voltages

Example: Find $v_c(t)$



Switch 1 Has Been Closed
Switch 2 Has Been Open } Long Time

Example (Contd.) For $t<0$ Redraw Circuit



Capacitor is Open in Steady State

$$v_c(0^-) \equiv v_c(0) = v_{50K\Omega}(0^-)$$

Voltage can't be changed Instantly across capacitor

$$i' = 20mA\frac{20K\Omega}{(20+30+50)K\Omega} = 4(mA)$$

Current Division

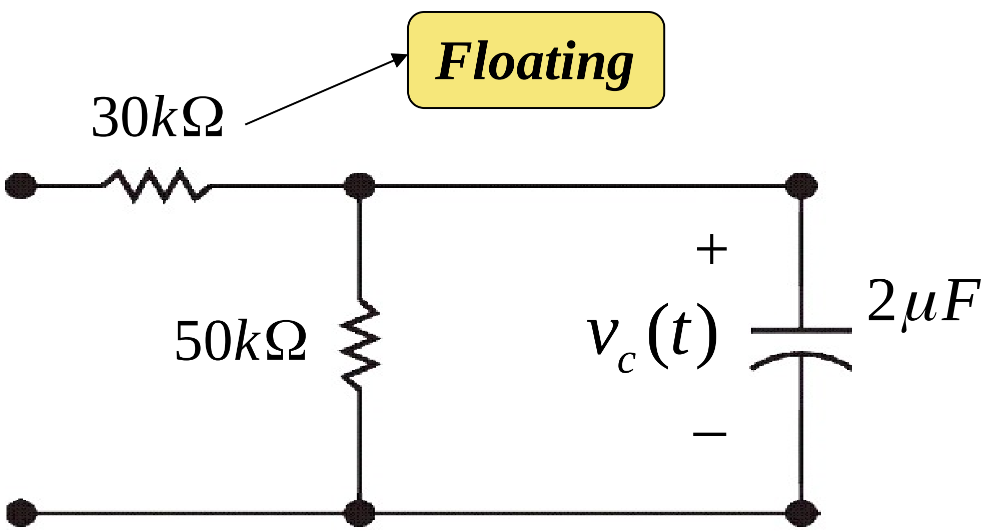
$$v_c(0^-) = 50K\Omega(i') = 50K(4mA)$$

Ohm's Law

$$v_c(0) = 200(V)$$

Example: Find $v_c(t)$

For $0 \le t < 50(ms)$, $v_c(0) = 200$ **Switch 1 Opens**



Find Time Constant

$$\tau = RC = 50K\Omega(2\mu F)$$

$$\tau = 0.1s$$

$30K\Omega$ Not Involved

Use the Natural Response formula

$$v_c(t) = v_c(0)e^{-t/\tau} = 200e^{-t/0.1}$$

$$v_c(t) = 200e^{-10(t)}(V)$$

$$0 \le t < 50ms$$

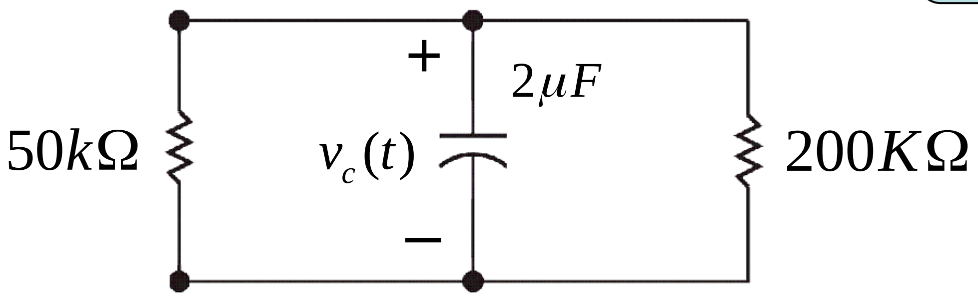
Example (Contd.)

b) $t \geq 50ms$, Switch 2 Closes

$$v_c(t) = 200e^{-10t} \text{ for } 0 \leq t \leq 50ms$$

$$v_c(50ms) = 200e^{-10(50ms)}$$

$$v_c(50ms) = 121.31(V)$$



$$R_{eq} = 50K \| 200K = 40(K\Omega)$$

$$\tau = R_{eq}C = 40K\Omega(2\mu F) = 80(ms)$$

$$v_c(t) = v_{c(50ms)}e^{-\frac{(t-t_0)}{\tau}} = 121.31e^{-\frac{(t-50ms)}{80ms}}$$

Use the
Natural Response formula
Note t_0 = 50 ms

$$v_c(t) = 121.31e^{-12.5(t-0.05)}(V) \qquad t \geq 50ms$$