Example.

Solve the equation

$$(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0.$$

The equation is separable; separating the variables by dividing by x^3y^4, we obtain

$$\frac{(x-4)\,dx}{x^3} - \frac{(y^2-3)\,dy}{y^4} = 0$$

or

$$(x^{-2} - 4x^{-3})\,dx - (y^{-2} - 3y^{-4})\,dy = 0.$$

Integrating, we have the one-parameter family of solutions

$$-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c,$$

where c is the arbitrary constant.

In dividing by x^3y^4 in the separation process, we assumed that $x^3 \neq 0$ and $y^4 \neq 0$. We now consider the solution $y = 0$ of $y^4 = 0$. It is not a member of the one-parameter family of solutions which we obtained. However, writing the original differential equation of the problem in the derivative form

$$\frac{dy}{dx} = \frac{(x-4)y^4}{x^3(y^2-3)},$$

it is obvious that $y = 0$ *is* a solution of the original equation. We conclude that it is a solution which was lost in the separation process.

Example. $\dfrac{dy}{dx}=\dfrac{1}{x}$

$(0, \infty)$, $(-\infty, 0)$.

$x_0>0$

$$y=\int_{x_0}^{x}\frac{dx}{x}+y_0=\ln\frac{x}{x_0}+y_0;$$

$x_0<0$

$$y=\int_{x_0}^{x}\frac{dx}{x}+y_0=\ln|x|-\ln|x_0|+y_0=\ln\frac{x}{x_0}+y_0$$

Example. $\left(\dfrac{dy}{dx}\right)^2-x=0.$

$$\frac{dy}{dx}=+\sqrt{x},\quad \frac{dy}{dx}=-\sqrt{x},\quad 0\leqslant x<\infty.$$

$$y=+\frac{2}{3}x^{\frac{3}{2}}+C,\quad y=-\frac{2}{3}x^{\frac{3}{2}}+C.$$

Example. Solve the IVP $\sin(x)\,dx + y\,dy = 0$, where $y(0) = 1$.

$$\begin{aligned} y\,dy &= -\sin(x)\,dx, \\ \int y\,dy &= \int -\sin(x)\,dx, \\ \frac{y^2}{2} &= \cos(x) + C_1, \\ y &= \sqrt{2\cos(x) + C_2}, \end{aligned}$$

where C_1 is an arbitrary constant and $C_2 = 2C_1$. Considering $y(0) = 1$, we have

$$1 = \sqrt{2 + C_2} \Longrightarrow 1 = 2 + C_2 \Longrightarrow C_2 = -1.$$

Therefore, $y = \sqrt{2\cos(x) - 1}$ on the domain $(-\pi/3, \pi/3)$, since we need $\cos(x) \geq 1/2$ and $\cos(\pm\pi/3) = 1/2$.

Example. $\frac{dy}{dx} = y^2$.

$$dx = \frac{dy}{y^2};$$

$$x = \int_{y_0}^{y} \frac{dy}{y^2} + x_0, \quad x - x_0 = -\frac{1}{y} + \frac{1}{y_0},$$

$$y = \frac{1}{\frac{1}{y_0} - x + x_0}.$$

$$y = 0.$$

$$y = \frac{y_0}{1 - y_0(x - x_0)}.$$

Example. $x(y^2-1)\,dx+y(x^2-1)\,dy=0.$

(y^2-1) (x^2-1);

$$\frac{x\,dx}{x^2-1}+\frac{y\,dy}{y^2-1}=0.$$

$$\ln|x^2-1|+\ln|y^2-1|=\ln|C|,$$

$$(x^2-1)(y^2-1)=C$$

Example.

Solve the initial-value problem that consists of the differential equation

$$x\sin y\,dx+(x^2+1)\cos y\,dy=0$$

and the initial condition

$$y(1)=\frac{\pi}{2}.$$

Separating the variables by dividing by $(x^2+1)\sin y$, we obtain

$$\frac{x}{x^2+1}\,dx+\frac{\cos y}{\sin y}\,dy=0.$$

Thus

$$\int\frac{x\,dx}{x^2+1}+\int\frac{\cos y}{\sin y}\,dy=c_0,$$

where c_0 is an arbitrary constant. Recall that

$$\int \frac{du}{u} = \ln |u| + C \quad \text{and} \quad |u| = \begin{cases} u & \text{if} \quad u \geq 0, \\ -u & \text{if} \quad u \leq 0. \end{cases}$$

Then, carrying out the integrations, we find

$$\tfrac{1}{2} \ln (x^2 + 1) + \ln |\sin y| = c_0.$$

We could leave the family of solutions in this form, but we can put it in a neater form in the following way. Since each term of the left member of this equation involves the logarithm of a function, it would seem reasonable that something might be accomplished by writing the arbitrary constant c_0 in the form $\ln |c_1|$. This we do, obtaining

$$\tfrac{1}{2} \ln(x^2 + 1) + \ln |\sin y| = \ln |c_1|.$$

Multiplying by 2, we have

$$\ln (x^2 + 1) + 2 \ln |\sin y| = 2 \ln |c_1|.$$

Since

$$2 \ln |\sin y| = \ln (\sin y)^2,$$

and

$$2 \ln |c_1| = \ln c_1^2 = \ln c,$$

where

$$c = c_1^2 \geq 0,$$

we now have

$$\ln (x^2 + 1) + \ln \sin^2 y = \ln c.$$

Since $\ln A + \ln B = \ln AB$, this equation may be written

$$\ln (x^2 + 1)\sin^2 y = \ln c.$$

From this we have at once

$$(x^2 + 1)\sin^2 y = c.$$

In dividing by $(x^2 + 1)\sin y$ in the separation process, we assumed that $\sin y \neq 0$. Now consider the solutions of $\sin y = 0$. These are given by $y = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Writing the original differential equation in the derivative form, it is clear that each of these solutions $y = n\pi (n = 0, \pm 1, \pm 2, \ldots)$ of $\sin y = 0$ is a constant solution of the original differential equation. Now, each of these constant solutions $y = n\pi$ is a member of the one-parameter family of solutions for $c = 0$.
Thus none of these solutions was lost in the separation process.

We now apply the initial condition to the family of solutions. We have

$$(1^2 + 1)\sin^2 \frac{\pi}{2} = c$$

and so $c = 2$. Therefore the solution of the initial-value problem under consideration is

$$(x^2 + 1)\sin^2 y = 2.$$

Homogeneous Equations

We now consider a class of differential equations that can be reduced to separable equations by a change of variables.

DEFINITION

The first-order differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be homogeneous if, when written in the derivative form $(dy/dx) = f(x, y)$, there exists a function g such that $f(x, y)$ can be expressed in the form $g(y/x)$.

Example.

The differential equation $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$ is homogeneous. To see this, we first write this equation in the derivative form

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}.$$

Now observing that

$$\frac{3y^2 - x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right),$$

we see that the differential equation under consideration may be written as

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right),$$

in which the right member is of the form $g(y/x)$ for a certain function g.

Before proceeding to the actual solution of homogeneous equations we shall consider slightly different procedure for recognizing such equations. A function F is called *homogeneous of degree n* if $F(tx, ty) = t^n F(x, y)$. This means that if tx and ty are substituted for x and y, respectively, in $F(x, y)$, and if t^n is then factored out, the other factor that remains is the original expression $F(x, y)$ itself. For example, the function F given by $F(x, y) = x^2 + y^2$ is homogeneous of degree 2, since

$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 F(x, y).$$

Now suppose the functions M and N in the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ are both homogeneous of the *same* degree n. Then since $M(tx, ty) = t^n M(x, y)$, if we let $t = 1/x$, we have

$$M\left(\frac{1}{x} \cdot x, \frac{1}{x} \cdot y\right) = \left(\frac{1}{x}\right)^n M(x, y).$$

Clearly this may be written more simply as

$$M\left(1, \frac{y}{x}\right) = \left(\frac{1}{x}\right)^n M(x, y);$$

and from this we at once obtain

$$M(x, y) = \left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right).$$

Likewise, we find

$$N(x, y) = \left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right).$$

Now writing the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ in the form

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)},$$

we find

$$\frac{dy}{dx} = -\frac{\left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)}{\left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)} = -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)}.$$

Clearly the expression on the right is of the form $g(y/x)$, and so the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is homogeneous in the sense of the original definition of homogeneity. Thus we conclude that if M and N in $M(x, y)\,dx + N(x, y)\,dy = 0$ are both homogeneous functions of the same degree n, then the differential equation is a homogeneous differential equation.

THEOREM.

If

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{3}$$

is a homogeneous equation, then the change of variables $y = vx$ transforms (3) *into a separable equation in the variables v and x.*

Proof. Since $M(x, y)\,dx + N(x, y)\,dy = 0$ is homogeneous, it may be written in the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

and (3) becomes

$$v + x\frac{dv}{dx} = g(v)$$

or

$$[v - g(v)]\,dx + x\,dv = 0.$$

This equation is separable. Separating the variables we obtain

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0. \tag{4}$$

Thus to solve a homogeneous differential equation of the form (3), we let $y = vx$ and transform the homogeneous equation into a separable equation of the form (4). From this, we have

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c,$$

where c is an arbitrary constant. Letting $F(v)$ denote

$$\int \frac{dv}{v - g(v)}$$

and returning to the original dependent variable y, the solution takes the form

$$F\left(\frac{y}{x}\right) + \ln|x| = c.$$

Example. $y' = (x - y)/x + y$.
This is a homogeneous equation since

$$\frac{x - y}{x + y} = \frac{1 - y/x}{1 + y/x}.$$

Setting $u = y/x$, our equation becomes

$$xu' + u = \frac{1 - u}{1 + u}$$

so that

$$xu' = \frac{1 - u}{1 + u} - u = \frac{1 - 2u - u^2}{1 + u}.$$

Note that the right-hand side is zero if $u = -1 \pm \sqrt{2}$. Separating variables and integrating with respect to x, we get

$$\int \frac{(1 + u)du}{1 - 2u - u^2} = \ln|x| + C_1$$

which in turn gives

$$(-1/2)\ln|1-2u-u^2| = \ln|x| + C_1.$$

Exponentiating, we get

$$\frac{1}{\sqrt{|1-2u-u^2|}} = e^{C_1}|x|.$$

Squaring both sides and taking reciprocals, we get

$$u^2 + 2u - 1 = C/x^2$$

Example. $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$.

$$y = ux, \ \frac{dy}{dx} = u + x\frac{du}{dx};$$

$$u + x\,\frac{du}{dx} = \frac{2u}{1-u^2}, \quad x\,\frac{du}{dx} = \frac{u+u^3}{1-u^2}.$$

$$\frac{dx}{x} + \frac{u^2-1}{u\,(u^2+1)}\,du = 0.$$

$$\ln x + \ln(u^2+1) - \ln u = \ln C, \ \frac{x\,(u^2+1)}{u} = C.$$

$$u = \frac{y}{x}$$

$$x^2 + y^2 = Cy, \ y = 0.$$

Example. $9y\dfrac{dy}{dx} - 18xy + 4x^3 = 0.$

$$y = z^2.$$

$$9z^3\frac{dz}{dx} - 9xz^2 + 2x^3 = 0.$$

$$z = ux:$$

$$9u^3(u\,dx + x\,du) - 9u^2\,dx + 2\,dx = 0,$$

$$\frac{9u^3\,du}{9u^4 - 9u^2 + 2} + \frac{dx}{x} = 0;$$

$u^2 = v;$

$$\frac{9v\,dv}{9v^2 - 9v + 2} + \frac{2\,dx}{x} = 0, \qquad \frac{6\,dv}{3v-2} - \frac{3\,dv}{3v-1} + \frac{2\,dx}{x} = 0,$$

$$\frac{(3v-2)^2x^2}{3v-1} = C,$$

$$\frac{(3u^2-2)^2x^2}{3u^2-1} = C, \qquad \frac{(3z^2-2x^2)^2}{3z^2-x^2} = C, \qquad \frac{(3y-2x^2)^2}{3y-x^2} = C.$$

Example. Solve $xy^2\,dy = \left(x^3 + y^3\right)dx$.

Let $y = vx$. Then $dy = v\,dx + x\,dv$, and our equation becomes

$$xv^2x^2\,(v\,dx + x\,dv) = \left(x^3 + v^3x^2\right)dx,$$
$$x^3v^3\,dx + x^4v^2\,dv = x^3\,dx + v^3x^3\,dx.$$

Therefore, $x = 0$ or $v^2\,dv = dx/x$. So we have

$$\frac{v^3}{3} = \ln(|x|) + C = \ln(|x|) + \ln(|A|) = \ln(|Ax|) = \ln(Ax)\,.$$

where the sign of A is the opposite of the sign of x. Therefore, the general solution is $y = x\left(3\ln(Ax)\right)^{1/3}$, where A is a nonzero constant. Every $A > 0$ yields a solution on the domain $(0,\infty)$; every $A < 0$ yields a solution on $(-\infty, 0)$. In addition, there is the solution $y = 0$ on the domain $\mathbb{R}$.

Example.

The equation

$$(y + \sqrt{x^2 + y^2})\,dx - x\,dy = 0$$

is homogeneous. When written in the form

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x},$$

the right member may be expressed as

$$\frac{y}{x} \pm \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}}$$

or

$$\frac{y}{x} \pm \sqrt{1 + \left(\frac{y}{x}\right)^2},$$

depending on the sign of x. This is obviously of the form $g(y/x)$.

Example.
Solve the equation

$$(x^2 - 3y^2)\,dx + 2xy\,dy = 0.$$

We have already observed that this equation is homogeneous. Writing it in the form

$$\frac{dy}{dx} = -\frac{x}{2y} + \frac{3y}{2x}$$

and letting $y = vx$, we obtain

$$v + x\frac{dv}{dx} = -\frac{1}{2v} + \frac{3v}{2},$$

or

$$x\frac{dv}{dx} = -\frac{1}{2v} + \frac{v}{2},$$

or, finally,

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v}.$$

This equation is separable. Separating the variables, we obtain

$$\frac{2v\,dv}{v^2 - 1} = \frac{dx}{x}.$$

Integrating, we find

$$\ln|v^2 - 1| = \ln|x| + \ln|c|,$$

and hence

$$|v^2 - 1| = |cx|,$$

where c is an arbitrary constant. We observe that no solutions were lost in the separation process. Now, replacing v by y/x we obtain the solutions in the form

$$\left|\frac{y^2}{x^2} - 1\right| = |cx|$$

or

$$|y^2 - x^2| = |cx|x^2.$$

If $y \geq x \geq 0$, then this may be expressed somewhat more simply as

$$y^2 - x^2 = cx^3.$$

Example.

Solve the initial-value problem

$$(y + \sqrt{x^2 + y^2})\,dx - x\,dy = 0,$$
$$y(1) = 0.$$

We have seen that the differential equation is homogeneous. As before, we write it in the form

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

Since the initial x value is 1, we consider $x > 0$ and take $x = \sqrt{x^2}$ and obtain

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}.$$

We let $y = vx$ and obtain

$$v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

or

$$x\frac{dv}{dx} = \sqrt{1 + v^2}.$$

Separating variables, we find

$$\frac{dv}{\sqrt{v^2 + 1}} = \frac{dx}{x}.$$

Using tables, we perform the required integrations to obtain

$$\ln |v + \sqrt{v^2 + 1}| = \ln |x| + \ln |c|,$$

or

$$v + \sqrt{v^2 + 1} = cx.$$

Now replacing v by y/x, we obtain the general solution of the differential equation in the form

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} = cx$$

or

$$y + \sqrt{x^2 + y^2} = cx^2.$$

The initial condition requires that $y = 0$ when $x = 1$. This gives $c = 1$ and hence

$$y + \sqrt{x^2 + y^2} = x^2,$$

from which it follows that

$$y = \tfrac{1}{2}(x^2 - 1).$$