EXAMPLE Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus is a basis for a subspace W of $\mathbb{R}^4$. Construct an orthogonal basis for W.

SOLUTION

Step 1. Let $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span}\{\mathbf{x}_1\} = \text{Span}\{\mathbf{v}_1\}$.

Step 2. Let $\mathbf{v}_2$ be the vector produced by subtracting from $\mathbf{x}_2$ its projection onto the subspace W_1. That is, let

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 \\
&= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \qquad \text{Since } \mathbf{v}_1 = \mathbf{x}_1 \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}
\end{aligned}$$

$\mathbf{v}_2$ is the component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$, and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the subspace W_2 spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$.

Step 2′ (optional). If appropriate, scale $\mathbf{v}_2$ to simplify later computations. Since $\mathbf{v}_2$ has fractional entries, it is convenient to scale it by a factor of 4 and replace $\{\mathbf{v}_1, \mathbf{v}_2\}$ by the orthogonal basis

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 3. Let $\mathbf{v}_3$ be the vector produced by subtracting from $\mathbf{x}_3$ its projection onto the subspace W_2. Use the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2'\}$ to compute this projection onto W_2:

$$\operatorname{proj}_{W_2} \mathbf{x}_3 = \underbrace{\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1}_{\substack{\text{Projection of} \\ \mathbf{x}_3 \text{ onto } \mathbf{v}_1}} + \underbrace{\frac{\mathbf{x}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_2'}_{\substack{\text{Projection of} \\ \mathbf{x}_3 \text{ onto } \mathbf{v}_2'}} = \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Then $\mathbf{v}_3$ is the component of $\mathbf{x}_3$ orthogonal to W_2, namely,

$$\mathbf{v}_3 = \mathbf{x}_3 - \operatorname{proj}_{W_2} \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Observe that $\mathbf{v}_3$ is in W, because $\mathbf{x}_3$ and $\operatorname{proj}_{W_2}\mathbf{x}_3$ are both in W. Thus $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3\}$ is an orthogonal set of nonzero vectors and hence a linearly independent set in W. Note that W is three-dimensional since it was defined by a basis of three vectors. Hence, $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3\}$ is an orthogonal basis for W.

QR Factorization of Matrices

If an $m \times n$ matrix A has linearly independent columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$, then applying the Gram–Schmidt process (with normalizations) to $\mathbf{x}_1, \ldots, \mathbf{x}_n$ amounts to *factoring* A, as described in the next theorem.

THEOREM 12

The QR Factorization

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

PROOF The columns of A form a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for Col A. Construct an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $W = \text{Col } A$ with property (1) in Theorem 11. This basis may be constructed by the Gram–Schmidt process or some other means. Let

$$Q = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n\,]$$

For $k = 1, \ldots, n$, $\mathbf{x}_k$ is in Span $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ = Span $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. So there are constants, $r_{1k}, \ldots, r_{kk}$, such that

$$\mathbf{x}_k = r_{1k}\mathbf{u}_1 + \cdots + r_{kk}\mathbf{u}_k + 0\cdot\mathbf{u}_{k+1} + \cdots + 0\cdot\mathbf{u}_n$$

We may assume that $r_{kk} \geq 0$. (If $r_{kk} < 0$, multiply both r_{kk} and $\mathbf{u}_k$ by -1.) This shows that $\mathbf{x}_k$ is a linear combination of the columns of Q using as weights the entries in the vector

$$\mathbf{r}_k = \begin{bmatrix} r_{1k} \\ \vdots \\ r_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

That is, $\mathbf{x}_k = Q\mathbf{r}_k$ for $k = 1, \ldots, n$. Let $R = [\,\mathbf{r}_1 \ \cdots \ \mathbf{r}_n\,]$. Then

$$A = [\,\mathbf{x}_1 \ \cdots \ \mathbf{x}_n\,] = [\,Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_n\,] = QR$$

The fact that R is invertible follows easily from the fact that the columns of A are linearly independent. Since R is clearly upper triangular, its nonnegative diagonal entries must be positive. ■

EXAMPLE Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

SOLUTION The columns of A are the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ in the last Example. An orthogonal basis for Col $A = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ was found in that example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

To simplify the arithmetic that follows, scale $\mathbf{v}_3$ by letting $\mathbf{v}_3' = 3\mathbf{v}_3$. Then normalize the three vectors to obtain $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, and use these vectors as the columns of Q:

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

By construction, the first k columns of Q are an orthonormal basis of Span $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$. From the proof of Theorem 12, $A = QR$ for some R. To find R, observe that $Q^TQ = I$, because the columns of Q are orthonormal. Hence

$$Q^TA = Q^T(QR) = IR = R$$

and

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

Systems of Linear Equations

Gaussian elimination.

Gaussian elimination is the process of using elementary row operations to transform the augmented matrix of a system of equation into row echelon form. A matrix is in row echelon form if each row begins with more zeros than the row above it or the row is all zeros.

EXAMPLE Solve the system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \qquad &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

SOLUTION.

The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix} \qquad R_4 + (-1)R_1 \rightarrow R_4$$

Now that the first column is in the desired form, you should change the second column as shown below.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix} \qquad R_4 + (6)R_2 \rightarrow R_4$$

To write the third column in proper form, multiply the third row by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix} \qquad \left(\tfrac{1}{3}\right)R_3 \rightarrow R_3$$

Similarly, to write the fourth column in proper form, you should multiply the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \qquad (-\tfrac{1}{13})R_4 \to R_4$$

The matrix is now in row-echelon form, and the corresponding system of linear equations is as shown below.

$$\begin{aligned} x_1 + 2x_2 - x_3 \phantom{{}-2x_4} &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

$$x_1 = -1, \qquad x_2 = 2, \qquad x_3 = 1, \qquad x_4 = 3.$$

EXAMPLE Solve the system.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

SOLUTION. The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

Apply Gaussian elimination to the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_2 + (-1)R_1 \to R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_4 + (-3)R_1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_3 + R_2 \rightarrow R_3$$

Note that the third row of this matrix consists of all zeros except for the last entry. This means that the original system of linear equations is *inconsistent.* You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

Because the third "equation" is a false statement, the system has no solution.

EXAMPLE Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

SOLUTION. The augmented matrix of the system of linear equations is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}.$$

where can verify that the reduced row-echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

The corresponding system of equations is

$$\begin{aligned} x_1 \quad\quad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter t to represent the *nonleading* variable x_3, you have

$$x_1 = 2 - 5t, \qquad x_2 = -1 + 3t, \qquad x_3 = t, \qquad \text{where } t \text{ is any real number.}$$

EXAMPLE.

Use Gauss elimination to solve the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \phantom{{}- 2x_4} + 2x_5 \phantom{{}- 3x_6} &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \phantom{{}+ 4x_5} + 15x_6 &= 0 \\ 2x_1 + 6x_2 \phantom{{}- 5x_3} + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

SOLUTION. The augmented matrix for the given homogeneous system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

The reduced row echelon form is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= 0 \end{aligned}$$

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

Note that the trivial solution results when $r = s = t = 0$.

THEOREM Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, *then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Proof If $\det(A) \neq 0$, then A is invertible and $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. Therefore,

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\mathbf{b} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Multiplying the matrices out gives

$$\mathbf{x} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

The entry in the jth row of $\mathbf{x}$ is therefore

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)}$$

Now let

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

Since A_j differs from A only in the jth column, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the jth column of A. The cofactor expansion of $\det(A_j)$ along the jth column is therefore

$$\det(A_j) = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$$

This result gives

$$x_j = \frac{\det(A_j)}{\det(A)}$$

EXAMPLE. Use Cramer's rule to solve

$$\begin{cases} x + 2y - z = -4 \\ x + 4y - 2z = -6 \\ 2x + 3y + z = 3. \end{cases}$$

SOLUTION. Because

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad \text{and} \quad z = \frac{D_z}{D},$$

we need to set up and evaluate four determinants.

Step 1 Set up the determinants.

1. D, the determinant in all three denominators, consists of the x-, y-, and z-coefficients.

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

2. D_x, the determinant in the numerator for x, is obtained by replacing the x-coefficients in D, 1, 1, and 2, with the constants on the right sides of the equations, −4, −6, and 3.

$$D_x = \begin{vmatrix} -4 & 2 & -1 \\ -6 & 4 & -2 \\ 3 & 3 & 1 \end{vmatrix}$$

3. D_y, the determinant in the numerator for y, is obtained by replacing the y-coefficients in D, 2, 4, and 3, with the constants on the right sides of the equations, −4, −6, and 3.

$$D_y = \begin{vmatrix} 1 & -4 & -1 \\ 1 & -6 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

4. D_z, the determinant in the numerator for z, is obtained by replacing the z-coefficients in D, −1, −2, and 1, with the constants on the right sides of the equations, −4, −6, and 3.

$$D_z = \begin{vmatrix} 1 & 2 & -4 \\ 1 & 4 & -6 \\ 2 & 3 & 3 \end{vmatrix}$$

Step 2 **Evaluate the four determinants.**

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$$

$$= 1(4+6) - 1(2+3) + 2(-4+4)$$

$$= 1(10) - 1(5) + 2(0) = 5$$

Using the same technique to evaluate each determinant, we obtain

$$D_x = -10, \quad D_y = 5, \quad \text{and} \quad D_z = 20.$$

Step 3 **Substitute these four values and solve the system.**

$$x = \frac{D_x}{D} = \frac{-10}{5} = -2$$

$$y = \frac{D_y}{D} = \frac{5}{5} = 1$$

$$z = \frac{D_z}{D} = \frac{20}{5} = 4$$

The solution $(-2, 1, 4)$ can be checked by substitution into the original three equations. The solution set is $\{(-2, 1, 4)\}$.

EXAMPLE. Use Cramer's rule to solve

$$
\begin{aligned}
x_1 + \quad\quad + 2x_3 &= 6 \\
-3x_1 + 4x_2 + 6x_3 &= 30 \\
-x_1 - 2x_2 + 3x_3 &= 8
\end{aligned}
$$

SOLUTION.

$$
A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}
$$

Therefore,

$$
x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},
$$

$$
x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}
$$

EXAMPLE. Use Cramer's rule to solve

$$
\begin{aligned}
5x_1 - 2x_2 + 3x_3 &= 16 \\
2x_1 + 3x_2 - 5x_3 &= 2 \\
4x_1 - 5x_2 + 6x_3 &= 7
\end{aligned}
$$

SOLUTION.

$$
\begin{bmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 7 \end{bmatrix}
$$

The primary determinant $|A| = \begin{vmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{vmatrix} = 5(18 - 25) + 2(12 + 20) + 3(-10 - 12) = -37$

The three special determinants are:

$$|A_1| = \begin{vmatrix} 16 & -2 & 3 \\ 2 & 3 & -5 \\ 7 & -5 & 6 \end{vmatrix} = 16(18 - 25) + 2(12 + 35) + 3(-10 - 21) = -111$$

$$|A_2| = \begin{vmatrix} 5 & 16 & 3 \\ 2 & 2 & -5 \\ 4 & 7 & 6 \end{vmatrix} = 5(12 + 35) - 16(12 + 20) + 3(14 - 8) = -259$$

$$|A_3| = \begin{vmatrix} 5 & -2 & 16 \\ 2 & 3 & 2 \\ 4 & -5 & 7 \end{vmatrix} = 5(21 + 10) + 2(14 - 8) + 16(-10 - 12) = -185$$

Applying Cramer's Rule:

$$x_1 = \frac{|A_1|}{A} = \frac{-111}{-37} = 3$$

$$x_2 = \frac{|A_2|}{A} = \frac{-259}{-37} = 7$$

$$x_3 = \frac{|A_3|}{A} = \frac{-185}{-37} = 5$$

Suppose we have a system of linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n = b_s \end{array}\right\} \quad (1)$$

A system of equations is called consistent if there is at least one set of values for the unknowns that satisfies each equation in the system. In contrast, a linear or non linear equation system is called inconsistent if there is no set of values for the unknowns that satisfies all of the equations.

Kronecker-Capelli theorem. *A system of linear equations* (1) *is consistent if and only if the rank of the augmented matrix* $\bar{A}$ *is equal to the rank of the matrix* A.

Example. Solve the system

$$\left.\begin{array}{r} 5x_1 - x_2 + 2x_3 + x_4 = 7, \\ 2x_1 + x_2 + 4x_3 - 2x_4 = 1, \\ x_1 - 3x_2 - 6x_3 + 5x_4 = 0 \end{array}\right\}$$

The rank of the coefficient matrix is two: the second-order minor in the upper left corner of this matrix is nonzero, but both third-order minors bordering it are zero. The rank of the augmented matrix is three, since

$$\begin{vmatrix} 5 & -1 & 7 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{vmatrix} = -35 \neq 0$$

The system is thus inconsistent.

Example. Solve the system

$$\left.\begin{aligned} 7x_1+3x_2&=2, \\ x_1-2x_2&=-3, \\ 4x_1+9x_2&=11 \end{aligned}\right\}$$

The rank of the coefficient matrix is two, i.e., it is equal to the number of unknowns; the rank of the augmented matrix is also two. Thus, the system is consistent and has a unique solution. The left-hand sides of the first two equations are linearly independent; solving the system of these two equations, we get the values

$$x_1=-\frac{5}{17}, \quad x_2=\frac{23}{17}$$

for the unknowns. It is easy to see that this solution also satisfies the third equation.

Example. Solve the system

$$\left.\begin{aligned} x_1+x_2-2x_3-x_4+x_5&=1, \\ 3x_1-x_2+x_3+4x_4+3x_5&=4, \\ x_1+5x_2-9x_3-8x_4+x_5&=0 \end{aligned}\right\}$$

The system is consistent since the rank of the augmented matrix (like the rank of the matrix of coefficients) is two. The left members of the first and third equations are linearly independent since the coefficients of the unknowns x_1 and x_2 constitute a nonzero minor of order two. Solve the system of these two equations, the unknowns x_3, x_4, x_5 being considered free; transpose them to the right members of the equations and assume that they have been given certain numerical values. Using Cramer's rule, we get

$$x_1=\frac{5}{4}+\frac{1}{4}x_3-\frac{3}{4}x_4-x_5,$$

$$x_2=-\frac{1}{4}+\frac{7}{4}x_3+\frac{7}{4}x_4$$

These equations determine the *general solution* of the given system: assigning arbitrary numerical values to the free unknowns, we obtain all the solutions of our system. Thus, for example, the vectors (2, 5, 3, 0, 0), (3, 5, 2, 1, −2), $\left(0, -\frac{1}{4}, -1, 1, \frac{1}{4}\right)$ and so on are solutions of our system.