Determinants. Inverse Matrices.

The *determinant* of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the number $ad - cb$.
The above sentence is abbreviated as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example. Find the determinant of each matrix.

$$\text{(a) } A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \qquad \text{(b) } B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \qquad \text{(c) } C = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}$$

Solution. (a) $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$

(b) $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$

(c) $|C| = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$

Example.

$$\det\begin{pmatrix}4 & -2\\ 1 & -3\end{pmatrix} = 4(-3) - 1(-2) = -12 + 2 = -10$$

The determinant of a 3×3 matrix can be found using the formula

$$\det\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = a\det\begin{pmatrix}e & f\\ h & i\end{pmatrix} - b\det\begin{pmatrix}d & f\\ g & i\end{pmatrix} + c\det\begin{pmatrix}d & e\\ g & h\end{pmatrix}$$

Example.

$$\det\begin{pmatrix}2 & -1 & 0\\ 0 & 3 & -2\\ 1 & 0 & 1\end{pmatrix} = 2\det\begin{pmatrix}3 & -2\\ 0 & 1\end{pmatrix} - (-1)\det\begin{pmatrix}0 & -2\\ 1 & 1\end{pmatrix} + 0\det\begin{pmatrix}0 & 3\\ 1 & 0\end{pmatrix}$$

$$= 2[3\cdot 1 - 0(-2)] + [0\cdot 1 - 1(-2)] + 0 = 2\cdot 3 + 2 = 8$$

2×2 inverses

Suppose that the determinant of the 2×2 matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

does not equal 0. Then the matrix has an inverse, and it can be found using the formula

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

Notice that in the above formula we are allowed to divide by the determinant since we are assuming that it's not 0.

Example. To find

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1}$$

first check that

$$\det \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = 3 \cdot 2 - 1 \cdot 5 = 1$$

Then

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{1} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

The Determinant of a Matrix:

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified, operations, which is characteristic of the matrix. The determinants are defined only for square matrices. It is denoted by det A or |A| for a square matrix A.

The determinant of the (2 x 2) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$= a_{11}\, a_{22} - a_{12}\, a_{21}$$

Example. If $A = \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$ find $|A|$

Solution:

$$|A| = \begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} = 9 - (-2) = 9 + 2 = 11$$

The determinant of the (3 x 3) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ denoted by } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is given as, det A = $|A|$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Note: Each determinant in the sum is the determinant of a submatrix of A obtained by deleting a particular row and column of A.

These determinants are called minors. We take the sign + or −, according to $(-1)^{i+j} a_{ij}$, where i and j represent row and column.

Minor and Cofactor of Element.

The minor M_{ij} of the element a_{ij} in a given determinant is the determinant of order (n – 1 x n – 1) obtained by deleting the ith row and jth column of A_{nxn}.

For example in the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

The minor of the element a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of the element a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

The minor of the element a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

The scalars $C_{ij} = (-1)^{i+j} M_{ij}$ are called the cofactor of the element a_{ij} of the matrix A.

Note: The value of the determinant in equation (1) can also be found by its minor elements or cofactors, as

$$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \quad \text{Or} \quad a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Hence the det A is the sum of the elements of any row or column multiplied by their corresponding cofactors.

The value of the determinant can be found by expanding it from any row or column.

Example. If $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix}$

find det A by expansion about (a) the first row (b) the first column.

Solution (a)

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3(4 + 6) - 2(0 + 2) + 1\ (0 - 1)$$

$$= 30 - 4 - 1$$

$$|A| = 25$$

(b)

$$|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3\ (4 + 6) + 1\ (-4 - 1)$$

$$= 30 - 5$$

$$|A| = 25$$

If A is a square matrix, then the **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} is given by

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown in the diagram below.

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

As you can see, the minors and cofactors of a matrix can differ only in sign. To obtain the cofactors of a matrix, first find the minors and then apply the checkerboard pattern of $+$'s and $-$'s shown below.

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

Example. Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution. To find the minor M_{11}, delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12}, delete the first row and second column.

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = 4 \\ M_{21} = 2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = 5 & M_{32} = -3 & M_{33} = -6. \end{array}$$

Now, to find the cofactors, combine the checkerboard pattern of signs with these minors to obtain

$$\begin{array}{lll} C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\ C_{31} = 5 & C_{32} = 3 & C_{33} = -6. \end{array}$$

If A is a square matrix (of order 2 or greater), then the determinant of A is the sum of the entries in the first row of A multiplied by their cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdot\cdot\cdot + a_{1n}C_{1n}.$$

Example. Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}.$$

Solution.

By inspecting this matrix, you can see that three of the entries in the third column are zeros. You can eliminate some of the work in the expansion by using the third column.

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because C_{23}, C_{33}, and C_{43} have zero coefficients, you need only find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

Expanding by cofactors in the second row yields

$$C_{13} = (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + 2(1)(-4) + 3(-1)(-7) = 13.$$

You obtain $|A| = 3(13) = 39.$

Upper Triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

To find the determinant of a triangular matrix, simply form the product of the entries on the main diagonal. It is easy to see that this procedure is valid for triangular matrices of order 2 or 3. For instance, the determinant of

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

can be found by expanding by the third row to obtain

$$|A| = 0(-1)^{3+1}\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + 0(-1)^{3+2}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + 3(-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

$$= 3(1)(-2) = -6,$$

which is the product of the entries on the main diagonal.

THEOREM.

If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdots a_{nn}.$$

PROOF

We can use *mathematical induction** to prove this theorem for the case in which A is an upper triangular matrix. The case in which A is lower triangular can be proven similarly. If A has order 1, then $A = [a_{11}]$ and the determinant is $|A| = a_{11}$. Assuming the theorem is true for any upper triangular matrix of order $k - 1$, consider an upper triangular matrix A of order k. Expanding by the kth row, you obtain

$$|A| = 0C_{k1} + 0C_{k2} + \cdot \cdot \cdot + 0C_{k(k-1)} + a_{kk}C_{kk} = a_{kk}C_{kk}.$$

Now, note that $C_{kk} = (-1)^{2k}M_{kk} = M_{kk}$, where M_{kk} is the determinant of the upper triangular matrix formed by deleting the kth row and kth column of A. Because this matrix is of order $k - 1$, you can apply the induction assumption to write

$$|A| = a_{kk}M_{kk} = a_{kk}(a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{k-1\,k-1}) = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{kk}.$$

The following properties of determinants are frequently useful in their evaluation:

1. Interchanging the corresponding rows and columns of a determinant does not change its value (i.e., $|A| = |A'|$). For example, consider a determinant

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \ \ldots \ (2)$$

Now again consider

$$|B| \quad = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expand it by first column

$$|B| \quad = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) \; + c_1(a_2b_3 - a_3b_2)$$

which is same as equation (2)

$$\text{so} \qquad |B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{or} \qquad |B| = |A|$$

2. If two rows or two columns of a determinant are interchanged, the sign of the determinant is changed but its absolute value is unchanged.

   For example if

$$|A| \quad = \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

   Consider the determinant,

$$|B| \quad = \quad \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

   expand by second row,

$$|B| = -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$
$$= -(a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$$

The term in the bracket is same as the equation (2)

So $$|B| = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Or $$|B| = -|A|$$

3. If every element of a row or column of a determinant is zero, the value of the determinant is zero. For example

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= 0(b_2c_3 - b_3c_2) - 0(a_2c_3 - a_3c_2) + 0(a_2b_3 - a_3b_3)$$
$$|A| = 0$$

4. If two rows or columns of a determinant are identical, the value of the determinant is zero. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= a_1(b_1c_3 - b_3c_1) - b_1(a_1c_3 - a_3c_1) + c_1(a_1b_3 - a_3b_1)$$
$$= a_1b_1c_3 - a_1b_3c_1 - a_1b_1c_3 + a_3b_1c_1 + a_1b_3c_1 - a_3b_1c_1$$
$$|A| = 0$$

5. If every element of a row or column of a determinant is multiplied by the same constant K, the value of the determinant is multiplied by that constant. For example if,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a determinant, $|B| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$|B| = ka_1(b_2c_3 - b_3c_3) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$$
$$= k(a_1(b_2c_3 - b_3c_3) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$$

So $\quad |B| = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Or $\quad |B| = K|A|$

6. The value of a determinant is not changed if each element of any row or of any column is added to (or subtracted from) a constant multiple of the corresponding element of another row or column. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a matrix,

$$|B| = \begin{vmatrix} a_1+ka_2 & b_1+kb_2 & c_1+kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= (a_1+ka_2)(b_2c_3-b_3c_2)-(b_1+kb_2)(a_2c_3-a_3c_2)+(c_1+kc_2)(a_2b_3-a_3b_2)$

$= [a_1(b_2c_3-b_3c_2) -b_1(a_2c_3-a_3c_2) +c_1(a_2b_3-a_3b_2)]$

$= [ka_2(b_2c_3-b_3c_2) -kb_2(a_2c_3-a_3c_2) +kc_2(a_2b_3-a_3b_2)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0)$$ because row 1st and 2nd are identical

$|B| = |A|$

7. The determinant of a diagonal matrix is equal to the product of its diagonal elements. For example, if

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$= 2(-15-0)-(0-0)+0(0-0)$

$= 30$, which is the product of diagonal elements.

i.e., $2(-5)3 = -30$

8. The determinant of the product of two matrices is equal to the product of the determinants of the two matrices, that is |AB| = |A||B|. for example, if

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$\text{Then } AB = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

$$\begin{aligned}
|AB| &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) \\
&\quad - (a_{11}b_{12} + a_{12}b_{22} - a_{11}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \\
&= a_{11}b_{11}\, a_{21}b_{12} + a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} \\
&\quad + a_{12}b_{21}\, a_{22}b_{22} - a_{11}b_{12}\, a_{21}b_{11} - a_{11}b_{12}\, a_{22}b_{21} \\
&\quad - a_{12}b_{22}\, a_{21}b_{11} - a_{12}b_{22}\, a_{22}b_{21} \\
|AB| &= a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}\, a_{21}b_{12} - a_{11}b_{12}\, a_{22}b_{21} \\
&\quad - a_{12}b_{22}\, a_{21}b_{11} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (A)
\end{aligned}$$

and $|A| = a_{11}a_{22} - a_{12}a_{21}$

$$\begin{aligned}
|B| &= b_{11}b_{22} - b_{12}b_{21} \\
|A|\,|B| &= a_{11}b_{11}\, a_{22}\, b_{22} + a_{12}b_{21}\, a_{21}\, b_{12} - a_{11}b_{12}\, a_{22}\, b_{21} \\
&\quad - a_{12}b_{22}\, a_{21}\, b_{11} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (B)
\end{aligned}$$

R.H.S of equations (A) and (B) are equal, so

$$|AB| = |A|\,|B|$$

9. The determinant in which each element in any row, or column, consists of two terms, then the determinant can be expressed as the sum of two other determinants

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 \\ a_2 + \alpha_2 & b_2 & c_2 \\ a_3 + \alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Expand by first column.

Proof:

L.H.S $= (a_1+\alpha_1)(b_2c_3-b_3c_2)-(a_2+\alpha_2)(b_1c_3-b_3c_1)+(a_3+\alpha_3)(b_1c_2-b_2c_1)$

$= [(a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)]$

$+ [(\alpha_1(b_2c_3-b_3c_2)-\ \alpha_2(b_1c_2-b_3c_1)+\ \alpha_3(b_1c_2-b_2c_1)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

= R.H.S

Similarly

$$\begin{vmatrix} \alpha_1+a_1 & b_1+\beta_1 & c_1 \\ \alpha_2+a_2 & b_2+\beta_2 & c_2 \\ \alpha_3+a_3 & b_3+\beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

And,

$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2+\gamma_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3+\gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \text{sum of six determinant} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Also

$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Example. Verify that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution:

Multiply row first, second and third by a, b and c respectively, in the L.H.S., then

$$\text{L.H.S} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

Take abc common from 3rd column

$$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Interchange column first and third

$$= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$$

Again interchange column second and third

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$= \text{R.H.S}$

Example. Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b - c)\,(c - a)\,(a - b)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

subtracting row first from second and third row

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

from row second and third taking $(b - a)$ and $(c - a)$ common.

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

expand from first column

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b)$$

Or L.H.S $= (b-c)(c-a)(a-b)(-1)(-1)$

$$= (b-c)(c-a)(a-b) = \text{R.H.S}$$

Example. Without expansion, show that

$$\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -4 & 0 & 2 & 8 \end{vmatrix} = 0$$

Solution:

In the L.H.S Taking 2 common from fourth row, so

$$\text{L.H.S} = 2\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -2 & 0 & 1 & 4 \end{vmatrix}$$

Since rows 2nd and 3rd are identical, so

$$= 2(0) = 0$$

$$\text{L.H.S} = \text{R.H.S}$$

Adjoint of a Matrix.

Let $A = (a_{ij})$ be a square matrix of order n x n and (c_{ij}) is a matrix obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij} then $(c_{ij})^t$ is called the adjoint of A. It is written as adj. A.

For example , if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Cofactor of A are:

$A_{11} = 5$, $A_{12} = -2$, $A_{13} = +1$

$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$

$A_{31} = 3$, $A_{32} = -2$, $A_{33} = 3$

Matrix of cofactors is

$$C = \begin{bmatrix} 5 & -2 & +1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}$$

$$C^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$$

$$\text{Hence adj } A = C^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$$

Note: Adjoint of a 2×2 Matrix:

The adjoint of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by adjA is defined as

$$\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example.

Compute the adjugate of $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix}$ and calculate $A(\operatorname{adj} A)$ and $(\operatorname{adj} A)A$.

Solution.

$$\begin{bmatrix} c_{11}(A) & c_{12}(A) & c_{13}(A) \\ c_{21}(A) & c_{22}(A) & c_{23}(A) \\ c_{31}(A) & c_{32}(A) & c_{33}(A) \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 1 & 5 \\ -6 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ -6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}$$

Then the adjugate of A is the transpose of this cofactor matrix.

$$\operatorname{adj} A = \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}^T = \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

The computation of $A(\operatorname{adj} A)$ gives

$$A(\operatorname{adj} A) = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix} \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I$$

we can verify that also $(\operatorname{adj} A)A = 3I$. Hence, analogy with the 2×2 case would indicate that $\det A = 3$; this is, in fact, the case.

6. Inverse of a Matrix:

If A is a non-singular square matrix , then $A^{-1} = \frac{\text{adj } A}{|A|}$

For example if matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Then $\text{adj } A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence $A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Example. If $A = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ then show that $AB = BA = I$ and therefore, $B = A^{-1}$

Solution:

$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $BA = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence $AB = BA = I$

and therefore $B = A^{-1} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$

Example. Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution:

$$|A| = 0 + 2(-2+3) - 3(-2+3) = 2 - 3$$

$|A| = -1$, Hence solution exists.

Cofactor of A are:

$A_{11} = 0,\quad A_{12} = 1,\quad A_{13} = 1$

$A_{21} = 2,\quad A_{22} = -3,\quad A_{23} = 2$

$A_{31} = 3,\quad A_{32} = -3,\quad A_{33} = 2$

Matrix of transpose of the cofactors is

$$\text{adj } A = C' = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

So

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$