Ex: Let
$$\begin{cases} 2x_1+4x_2-x_3-2x_4+2x_5=6 \\ x_1+3x_2+2x_3-7x_4+3x_5=9 \\ 5x_1+8x_2-7x_3+6x_4+x_5=4 \end{cases}$$
be a linear system. Solve this system with Gauss method.

$$Ax=b: \begin{bmatrix} 2 & 4 & -1 & -2 & 2 \\ 1 & 3 & 2 & -7 & 3 \\ 5 & 8 & -7 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 4 \end{bmatrix}$$

Augmented matrix: $\left[\begin{array}{ccccc|c} 2 & 4 & -1 & -2 & 2 & 6 \\ 1 & 3 & 2 & -7 & 3 & 9 \\ 5 & 8 & -7 & 6 & 1 & 4 \end{array}\right]$

$$\left[\begin{array}{ccccc|c} 2 & 4 & -1 & -2 & 2 & 6 \\ 1 & 3 & 2 & -7 & 3 & 9 \\ 5 & 8 & -7 & 6 & 1 & 4 \end{array}\right] \xrightarrow[-2r_3+5r_1\to r_3]{-2r_2+r_1\to r_2} \left[\begin{array}{ccccc|c} 2 & 4 & -1 & -2 & 2 & 6 \\ 0 & -2 & -5 & 12 & -4 & -12 \\ 0 & 4 & 9 & -22 & 8 & 22 \end{array}\right] \xrightarrow{2r_2+r_3\to r_3}$$

$$\left[\begin{array}{ccccc|c} 2 & 4 & -1 & -2 & 2 & 6 \\ 0 & -2 & -5 & 12 & -4 & -12 \\ 0 & 0 & -1 & 2 & 0 & -2 \end{array}\right] \xrightarrow[\substack{-\frac{1}{2}r_2\to r_2 \\ -r_3\to r_3}]{\frac{1}{2}r_1\to r_1} \left[\begin{array}{ccccc|c} 1 & 2 & -1/2 & -1 & 1 & 3 \\ 0 & 1 & 5/2 & -6 & 2 & 6 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{array}\right] \text{: row echelon form}$$

Let $x_4=s$ // and $x_5=t$ //

Then $x_3-2x_4=2 \Rightarrow x_3-2s=2 \Rightarrow x_3=2s+2$ //

$x_2+\frac{5}{2}x_3-6x_4+2x_5=6 \Rightarrow x_2+\frac{5}{2}(2s+2)-6s+2t=6$

$x_2=s-2t+1$ //

$x_1+2(s-2t+1)-\frac{1}{2}(2s+2)-s+t=3 \Rightarrow x_1=3t+2$ //

Ex: Let $\begin{cases} x_1 + 2x_2 + 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 - x_3 = 3 \end{cases}$ be a lineer system. Solve this system

with Gauss elimination.

Solution: Gauss elimination: (augmented matrix → row echelon form)

Lets write this system as a form $Ax = b$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

The augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & 1 & 3 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & 1 & 3 \end{array}\right] \xrightarrow[-2r_1+r_2 \to r_2]{-3r_1+r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \xrightarrow[-\frac{1}{5}r_2 \to r_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\xrightarrow{6r_2+r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \xrightarrow[-\frac{1}{4}r_3 \to r_3]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$ this is row echelon form

$$\begin{cases} x_1 + 2x_2 + x_3 = 9 \\ x_2 + x_3 = 2 \\ x_3 = 3 \end{cases} \Rightarrow \begin{array}{l} x_2 = -1 \\ x_3 = 3 \\ x_1 = 2 \end{array}$$ //

Ex: Consider the linear system

$$\begin{cases} x+y+2z=-1 \\ x-2y+z=-5 \\ 3x+y+z=3 \end{cases}$$

Find all solutions, if any exists, by using the Gauss elimination

Solution: $Ax=b$: $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -1 \\ -5 \\ 3 \end{bmatrix}$.

Augmented matrix : $\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right] \xrightarrow[-r_1+r_2\to r_2]{-3r_1+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & -5 & 6 \end{array}\right]$.

$\xrightarrow{-\frac{2}{3}r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -\frac{13}{3} & \frac{26}{3} \end{array}\right] \xrightarrow{-\frac{3}{13}r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{-\frac{1}{3}r_2\to r_2}$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$: row echelon form

$$\begin{cases} x+y+2z=-1 \\ y+\frac{1}{3}z=\frac{4}{3} \\ z=-2 \end{cases} \Rightarrow \quad x=1, \quad y=2, \quad z=-2 \,//$$

Ex: Let $\begin{cases} -2x_1 + 3x_2 - x_3 = 1 \\ x_1 + 2x_2 - x_3 = 4 \\ -2x_1 - x_2 + x_3 = -3 \end{cases}$ be a linear system. Solve this system with cramer rule.

Solution: $Ax = b$: $\begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2.(-1)^{1+1} \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + 3.(-1)^{1+2} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + (-1)(-1)^{1+3} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= -2 //$$

$$|A_1| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ 3 & -1 & 1 \end{vmatrix} = -4 \qquad |A_2| = \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & 3 & 1 \end{vmatrix} = -6$$

$$|A_3| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & 3 \end{vmatrix} = -8$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-4}{-2} = 2 //$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-6}{-2} = 3 //$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{-2} = 4 //$$

Ex: Let $\begin{cases} 3x_1 - x_2 - 5x_3 = 3 \\ 4x_1 - 4x_2 - 3x_3 = -4 \\ x_1 - 5x_3 = 2 \end{cases}$ be a Linear system. Solve this system with Cramer rule.

Solution: $Ax = b \Rightarrow \begin{bmatrix} 3 & -1 & -5 \\ 4 & -4 & -3 \\ 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -1 & -5 \\ 4 & -4 & -3 \\ 1 & 0 & -5 \end{vmatrix} = 1.(-1)^{3+1} \begin{vmatrix} -1 & -5 \\ -4 & -3 \end{vmatrix} + 0 \cdots + (-5).(-1)^{3+3} \begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix}$$

$$= (3-20) - 5(-12+4) = -17 + 40 = 23 \neq 0 \checkmark$$

$$|A_1| = \begin{vmatrix} 3 & -1 & -5 \\ -4 & -4 & -3 \\ 2 & 0 & -5 \end{vmatrix} = 46 \qquad |A_2| = \begin{vmatrix} 3 & 3 & -5 \\ 4 & -4 & -3 \\ 1 & 2 & -5 \end{vmatrix} = 69$$

$$|A_3| = \begin{vmatrix} 3 & -1 & 3 \\ 4 & -4 & -4 \\ 1 & 0 & 2 \end{vmatrix} = 0$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{46}{23} = 2 /\!/$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{69}{23} = 3 /\!/$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{0}{23} = 0 /\!/$$

Ex: Let $\begin{cases} -2x_1 - 4x_2 = -6 \\ x_1 + 2x_2 + 3x_3 = -3 \\ 3x_1 + 6x_2 + x_3 = 9 \end{cases}$ be a linear system. Find the solution if there is any.

Solution: $Ax = b$: $\begin{bmatrix} -2 & -4 & 0 \\ 1 & 2 & 3 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}$

Augmented matrix is $\left[\begin{array}{ccc|c} -2 & -4 & 0 & -6 \\ 1 & 2 & 3 & -3 \\ 3 & 6 & 1 & 9 \end{array}\right]$

$$\left[\begin{array}{ccc|c} -2 & -4 & 0 & -6 \\ 1 & 2 & 3 & -3 \\ 3 & 6 & 1 & 9 \end{array}\right] \xrightarrow{-\frac{1}{2}r_1 \to r_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 2 & 3 & -3 \\ 3 & 6 & 1 & 9 \end{array}\right] \xrightarrow[r_3 - 3r_1 \to r_3]{r_2 - r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 9 \end{array}\right] \xrightarrow{r_3 - r_2 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

$$0 \cdot x_1 + 0x_2 + 0 \cdot x_3 = 10$$

$$0 = 10 \ ↯$$

Hence the system is inconsistent.

There is no solution.

Ex: Find all values of a for which the resulting system has

- no solution
- a unique solution
- infinitely many solutions

$$\begin{cases} x_1 + x_2 - x_3 = 3 \\ x_1 - x_2 + 3x_3 = 4 \\ x_1 + x_2 + (a^2 - 10)x_3 = a \end{cases}$$

Solution:
$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 1 & -1 & 3 & 4 \\ 1 & 1 & a^2-10 & a \end{array}\right] \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right] \xrightarrow{-\frac{1}{2}r_2 \to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right] \xrightarrow{r_1 - r_2 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7/2 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right]$$

• If $a^2 - 9 \neq 0$ then $a \neq \mp 3$. Then the augmented matrix is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 7/2 \\ 0 & 1 & 2 & -1/2 \\ 0 & 0 & 1 & 1/(a+3) \end{array}\right].$$

The system is consistent but it has no free variable. The system has unique solution.
(if $a \neq \mp 3$ then it has unique solution)

• If $a^2 - 9 = 0$ then $a = \mp 3$.

→ if $a = 3$ then the augmented matrix is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 7/2 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system has infinitely many since

$$x_1 + x_3 = 7/2$$
$$x_2 - 2x_3 = -1/2$$

we have a parametrization here.

(if $a = 3$ then it has infinitely many solution)

→ if $a = -3$ then the augmented matrix is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 7/2 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

The system is inconsistent.

It has no solution

<u>Conclusion:</u>
- if $a \neq \mp 3$ then the system has unique solution
- if $a = 3$ then the system has infinitely many solution
- if $a = -3$ then the system has no solution

Ex: Find conditions that the b's must satisfy for the system to be consistent

$$\begin{cases} x_1 - x_2 + 3x_3 + 2x_4 = b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 = b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 = b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 = b_4 \end{cases}$$

Solution: $Ax = b$: $\begin{bmatrix} 1 & -1 & 3 & 2 \\ -2 & 1 & 5 & 1 \\ -3 & 2 & 2 & -1 \\ 4 & -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \xrightarrow[r_4 - 4r_1 \to r_4]{\substack{r_2 + 2r_1 \to r_2 \\ r_3 + 3r_1 \to r_3}} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & b_2 + 2b_1 \\ 0 & -1 & 11 & 5 & b_3 + 3b_1 \\ 0 & 1 & -11 & -5 & b_4 - 4b_1 \end{array}\right]$$

$$\xrightarrow{-r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -b_2 - 2b_1 \\ 0 & -1 & 11 & 5 & b_3 + 3b_1 \\ 0 & 1 & -11 & -5 & b_4 - 4b_1 \end{array}\right] \xrightarrow[\substack{r_3 + r_2 \to r_3 \\ r_4 - r_2 \to r_4}]{r_1 + r_2 \to r_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -9 & -3 & -b_1 - b_2 \\ 0 & 1 & -11 & -5 & -b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & -2b_1 + b_2 + b_4 \end{array}\right]$$

The system has a solution if and only if b's satisfy the condition (the system is consistent iff)

$$-2b_1 + b_2 + b_4 = 0 \quad \text{AND} \quad b_1 - b_2 + b_3 = 0 \quad //$$

Ex: Find all solutions to the given linear system:

$$2x_1 + x_2 - 2x_3 - 2x_5 = 0$$

$$x_1 - x_3 = 0$$

$$3x_1 + 3x_2 - 4x_3 - x_5 = 0$$

$$x_2 - 2x_4 = 0$$

Solution: $Ax = b$:
$$\begin{bmatrix} 2 & 1 & -2 & 0 & -2 \\ 1 & 0 & -1 & 0 & 0 \\ 3 & 3 & -4 & 0 & -1 \\ 0 & 1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix:
$$\left[\begin{array}{ccccc|c} 2 & 1 & -2 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 3 & 3 & -4 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 2 & 1 & -2 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 3 & 3 & -4 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 2 & 1 & -2 & 0 & -2 & 0 \\ 3 & 3 & -4 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \end{array}\right] \xrightarrow[r_3 - 3r_1 \to r_3]{r_2 - 2r_1 \to r_2}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 3 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \end{array}\right] \xrightarrow[r_4 - r_2 \to r_4]{r_3 - 3r_2 \to r_3} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \xrightarrow[-\frac{1}{2}r_4 \to r_4]{-r_3 \to r_3}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_1 + r_3 \to r_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] //$$

This augmented matrix represents the following linear system:

$$\begin{cases} x_1 - 5x_5 = 0 \\ x_2 - 2x_5 = 0 \\ x_3 - 5x_5 = 0 \\ x_4 - x_5 = 0 \end{cases}$$

It is equivalent to

$$x_1 = 5x_5$$

$$x_2 = 2x_5$$

$$x_3 = 5x_5$$

$$x_4 = x_5 \qquad (x_5 \text{ is free variable})$$

Let $x_5 := r$, a solution to the given linear system is

$$x_1 = 5r$$

$$x_2 = 2r$$

$$x_3 = 5r$$

$$x_4 = r$$

$$x_5 = r \qquad \text{where } r \in \mathbb{R}$$

The given linear system has infinetly many solutions

The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5r \\ 2r \\ 5r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} 5 \\ 2 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}$$

The set of solutions is

$$\left\{ r \begin{bmatrix} 5 \\ 2 \\ 5 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; r \in \mathbb{R} \right\}$$

Ex: Each of the given linear system is in row echelon form. Solve the system

a) $x+y-z+2w=4$

$w=5$

Solution: $Ax=b$: $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Augmented matrix : $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{array}\right]$

We have $w=5$ from second row. Let $z=t$ and $y=s$. From first first row $x+s-t+10=4$. So we have $x=-6-s+t$.

The general solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} -6-s+t \\ s \\ t \\ 5 \end{bmatrix} \quad , \; t,s \in \mathbb{R}$$

//

b) $x-y+z=0$

$y+2z=0$

$z=1$

Solution: $Ax=b$: $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Augmented matrix : $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$

From third row we have $z=1$. From second row, $y+2z=0 \Rightarrow y+2=0$

$y=-2$. From first row $x-y+z=0 \Rightarrow x+2+1=0$. $x=-3$.

The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

//

Ex: Let $f(x) = 2\sin x + 3\cos x$. Show that $f(x)$ is a solution for

$$\frac{d^2y}{dx^2} + y = 0$$

Solution: $f(x) = 2\sin x + 3\cos x \Rightarrow$

$$f'(x) = \frac{dy}{dx} = 2\cos x - 3\sin x$$

$$f''(x) = \frac{d^2y}{dx^2} = -2\sin x - 3\cos x$$

Hence $\frac{d^2y}{dx^2} + y \stackrel{?}{=} 0$: $\cancel{-2\sin x} \cancel{-3\cos x} + \cancel{2\sin x} + \cancel{3\cos x} = 0$ ✓

Hence $f(x)$ is a solution for $\frac{d^2y}{dx^2} + y = 0$ //

Ex: Solve $y'' - 6x^2 = 0$

Solution: $y'' - 6x^2 = 0$

$$\frac{d^2y}{dx^2} - 6x^2 = 0 \Rightarrow \int \frac{d^2y}{dx^2} = \int 6x^2 \, dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x^3}{3} + c_1$$

$$\Rightarrow \int \frac{dy}{dx} = \int (2x^3 + c_1) \, dx$$

$$\Rightarrow y = \frac{2x^4}{4} + c_1 x + c_2$$

$$\Rightarrow y = \frac{x^4}{2} + c_1 x + c_2 \,//$$

Ex: Solve $y' + y\cos x = 0$

Solution: $y' + y\cos x = 0$

$$\frac{dy}{dx} + y\cos x = 0$$

$$\frac{\left(\frac{dy}{dx} + y\cos x\right)}{y} = \frac{0}{y}$$

$$\frac{dy}{dx} \cdot \frac{1}{y} + \cos x = 0$$

$$dx\left(\frac{dy}{dx} \cdot \frac{1}{y} + \cos x\right) = dx(0)$$

$$\frac{dy}{y} + \cos x\, dx = 0$$

$$\int \frac{dy}{y} + \int \cos x\, dx = \int 0$$

$$\ln|y| + \sin x = c_1$$

$$\ln|y| = c_1 - \sin x$$

$$y = e^{c_1 - \sin x}$$

$$y = e^{-\sin x} \cdot e^{c_1} \quad c$$

$$y = c \cdot e^{-\sin x}$$

//