Example. Find a general solution of the equation

$$y^{(5)}+6y'''+9y'=0.$$

Solution. The characteristic equation looks like

$$\lambda^5+6\lambda^3+9\lambda=0.$$

We have

$$\lambda^5+6\lambda^3+9\lambda=\lambda\left(\lambda^4+6\lambda^2+9\right)=\lambda\left(\lambda^2+3\right)^2=0.$$

This implies that the characteristic equation has a simple real root $\lambda_1=0$ and complex conjugate roots $\lambda_{2,3}=\pm\sqrt{3}i$ of multiplicity 2.

this differential equation has the linearly independent solutions:

$$y_1=e^{0\cdot x}=1,\ y_2=\sin\sqrt{3}x,\ y_3=x\sin\sqrt{3}x,$$

$$y_4=\cos\sqrt{3}x,\ y_5=x\cos\sqrt{3}x.$$

A general solution of this differential equation has the form

$$y(x)=C_1+C_2\sin\sqrt{3}x+C_3x\sin\sqrt{3}x+C_4\cos\sqrt{3}x+C_5x\cos\sqrt{3}x.$$

Example. $y''-4y'+5y=0$

$\lambda^2-4\lambda+5=0$

$\lambda_{1,2}=2\pm\sqrt{4-5}=2\pm i$

$y=C_1e^{(2+i)x}+C_2e^{(2-i)x}=C_1e^{2x}e^{ix}+C_2e^{2x}e^{-ix}$

$=C_1e^{2x}(\cos x+i\sin x)+C_2e^{2x}(\cos x-i\sin x)=e^{2x}(C_1\cos x+C_2\sin x)$

$y=e^{2x}(C_1\cos x+C_2\sin x)$

Example. $y^{(IV)} - y = 0$

$\lambda^4 - 1 = 0$

$\lambda^4 = 1$

$\lambda_{1,2} = \pm 1$

$\lambda_{3,4} = \pm i$

$y = C_1 e^x + C_2 e^{-x} + C_3 \sin x + C_4 \cos x$

Example.

Find the general solution of

$$\frac{d^4 y}{dx^4} - 4\frac{d^3 y}{dx^3} + 14\frac{d^2 y}{dx^2} - 20\frac{dy}{dx} + 25y = 0.$$

The auxiliary equation is

$$m^4 - 4m^3 + 14m^2 - 20m + 25 = 0.$$

They are

$$1 + 2i, \qquad 1 - 2i, \qquad 1 + 2i, \qquad 1 - 2i.$$

$$y = e^x[(c_1 + c_2 x)\sin 2x + (c_3 + c_4 x)\cos 2x]$$

or

$$y = c_1 e^x \sin 2x + c_2 x e^x \sin 2x + c_3 e^x \cos 2x + c_4 x e^x \cos 2x.$$

Example. $y^{V}-10y'''+9y'=0$

$$\lambda^5-10\lambda^3+9\lambda=0$$

$$\lambda(\lambda^4-10\lambda^2+9)=0$$

$$\lambda(\lambda^2-1)(\lambda^2-9)=0$$

$$\lambda_1=0$$

$$\lambda_{2,3}=\pm 1$$

$$\lambda_{4,5}=\pm 3$$

$$y(x)=C_1+C_2e^{x}+C_3e^{-x}+C_4e^{3x}+C_5e^{-3x}$$

Example. $y^{(6)}+64y=0.$

$$\lambda^6+64=0. \qquad \lambda=\sqrt[6]{-64}.$$

$$\lambda_k=2\left(\cos\frac{\pi+2\pi k}{6}+i\sin\frac{\pi+2\pi k}{6}\right), \quad k=0,1,2,3,4,5.$$

$$\lambda_0=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\sqrt{3}+i, \quad \lambda_1=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2i,$$

$$\lambda_2=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=-\sqrt{3}+i, \quad \lambda_3=2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=-\sqrt{3}-i,$$

$$\lambda_4=2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-2i, \quad \lambda_5=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=\sqrt{3}-i.$$

$$\lambda_0 = \sqrt{3} + i \qquad y_1 = e^{\sqrt{3}x}\cos x$$

$$\lambda_5 = \sqrt{3} - i \qquad y_2 = e^{\sqrt{3}x}\sin x$$

$$\lambda_1 = 2i \qquad y_3 = \cos 2x$$

$$\lambda_4 = -2i \qquad y_4 = \sin 2x$$

$$\lambda_2 = -\sqrt{3} + i \qquad y_5 = e^{-\sqrt{3}x}\cos x$$

$$\lambda_3 = -\sqrt{3} - i \qquad y_6 = e^{-\sqrt{3}x}\sin x\,.$$

$$y = (c_1\cos x + c_2\sin x)e^{\sqrt{3}x} + c_3\cos 2x + c_4\sin 2x +$$
$$+ (c_5\cos x + c_6\sin x)e^{-\sqrt{3}x}.$$

Example. $y^{(6)} + 8y^{(4)} + 16y'' = 0.$

$$\lambda^6 + 8\lambda^4 + 16\lambda^2 = 0.$$

$$\lambda_1 = 0,\ \lambda_2 = 2i,\ \ \lambda_3 = -2i.$$

$$y_1 = 1,\ y_2 = x,\ y_3 = \cos 2x,\ y_4 = x\cos 2x,$$

$$y_5 = \sin 2x,\ y_6 = x\sin 2x$$

$$y = c_1 + c_2x + (c_3 + c_4x)\cos 2x + (c_5 + c_6x)\sin 2x.$$

Example. $y^{IV} + 4y'' + 3y = 0$

$\lambda^4 + 4\lambda^2 + 3 = 0$
$\lambda^2(\lambda^2 + 1) + 3(\lambda^2 + 1) = 0$
$(\lambda^2 + 3)(\lambda^2 + 1) = 0$
$\lambda_1 = \lambda_2 = \pm i;\ \lambda_3 = \lambda_4 = \pm\sqrt{3}i$
$y(x) = C_1\cos x + C_2\sin x + C_3\cos\sqrt{3}x + C_4\sin\sqrt{3}x$

An Initial-Value Problem

We now apply the results concerning the general solution of a homogeneous linear equation with constant coefficients to an initial-value problem involving such an equation.

Example.

Solve the initial-value problem

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0,$$

$$y(0) = -3,$$

$$y'(0) = -1.$$

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x).$$

From this, we find

$$\frac{dy}{dx} = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x].$$

We now apply the initial conditions.

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x) \qquad y(0) = -3$$

$$-3 = e^0(c_1 \sin 0 + c_2 \cos 0),$$

$$\frac{dy}{dx} = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x] \; ,$$

$$y'(0) = -1, \quad c_2 = -3.$$

$$-1 = e^0[(3c_1 - 4c_2)\sin 0 + (4c_1 + 3c_2)\cos 0],$$

which reduces to

$$4c_1 + 3c_2 = -1.$$

$c_1 = 2, \qquad c_2 = -3.$

$y = e^{3x}(2 \sin 4x - 3 \cos 4x).$

Example. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0.$

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

One also needs to evaluate the first derivative

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Example. Find a solution of the differential equation

$$y'' - 4y' + 29y = 0$$

satisfying the initial conditions $y(0) = 1$, $y'(0) = 7$.

Solution. We write the characteristic equation

$$\lambda^2 - 4\lambda + 29 = 0.$$

The values

$$\lambda_1 = 2 + 5i, \quad \lambda_2 = 2 - 5i$$

are the roots of this equation. Then a general solution of the equation has the form

$$y = e^{2x}(C_1 \cos 5x + C_2 \sin 5x).$$

Now we use the initial conditions. For this we find

$$y' = 2e^{2x}(C_1 \cos 5x + C_2 \sin 5x) + e^{2x}(-5C_1 \sin 5x + 5C_2 \cos 5x).$$

To search for constants C_1 and C_2 we have a system of the equations:

$$\begin{cases} C_1 = 1, \\ 2C_1 + 5C_2 = 7. \end{cases}$$

Hence $C_1 = 1, C_2 = 1$.

$$y = e^{2x} \cos 5x + e^{2x} \sin 5x$$

Example Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution: $\Rightarrow (D^2 - 8D + 15)y = 0$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3,5$

$c_1e^{3x} + c_2e^{5x}$

$\Rightarrow y = c_1e^{3x} + c_2e^{5x}$

Example Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Solution: $\Rightarrow (D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$

May be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

$c_1e^{x} + c_2e^{2x} + c_3e^{3x}$

$\Rightarrow y = c_1e^{x} + c_2e^{2x} + c_3e^{3x}$

Example Solve $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$

May be rewritten as

$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$

$\Rightarrow m^3(m-1) - 9m^2(m-1) + 26m(m-1) - 24(m-1) = 0$

$\Rightarrow (m-1)(m^3 - 9m^2 + 26m - 24) = 0$

May be rewritten as

$(m-1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$

$\Rightarrow (m-1)[m^2(m-2) - 7m(m-2) + 12(m-2)] = 0$

$\Rightarrow (m-1)(m^2 - 7m + 12)(m-2) = 0$

$\Rightarrow (m-1)(m-3)(m-4)(m-2) = 0$

$\Rightarrow m = 1,2,3,4$

$c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Example Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = 0$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m+1)^2 = 0$

$\Rightarrow \ m = 0, -1, -1$

$c_1 + \ (c_2 + c_3 x)e^{-x}$

$\Rightarrow y \ = \ \ c_1 + \ (c_2 + c_3 x)e^{-x}$

Example Solve the differential equation: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + \ y = 0$

Solution: $\Rightarrow (D^4 - 2D^2 + 1)y = 0$

Auxiliary equation is: $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2 - 1)^2 = 0$

$\Rightarrow (m+1)^2(m-1)^2 = 0$

$\Rightarrow \ m = -1, -1, 1, 1$

$(c_1 + c_2 x)e^{-x} + \ (c_3 + c_4 x)e^{x}$

$\Rightarrow y \ = \ \ (c_1 + c_2 x)e^{-x} + \ (c_3 + c_4 x)e^{x}$

Example Solve the differential equation: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

Solution: $\Rightarrow (D^3 - 2D + 4)y = 0$

Auxiliary equation is: $m^3 - 2m + 4 = 0$

May be rewritten as

$m^3 + 2m^2 - 2m^2 - 4m + 2m + 4 = 0$

$\Rightarrow m^2(m + 2) - 2m(m + 2) + 2(m + 2) = 0$

$\Rightarrow (m + 2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm i$

$c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

$\Rightarrow y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Example Solve the differential equation: $(D^2 - 2D + 5)^2 y = 0$

Solution: Auxiliary equation is: $(m^2 - 2m + 5)^2$

$\Rightarrow m = 1 \pm 2i, 1 \pm 2i$

$e^x[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

$\Rightarrow y = e^x[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x]$

Example Solve the differential equation: $(D^2 + 4)^3 y = 0$

Solution: Auxiliary equation is: $(m^2 + 4)^3$

$\Rightarrow \ m = \pm 2i, \pm 2i, \pm 2i$

$(c_1 + c_2 x + c_3\, x^2) \cos 2x + \ (c_4 + c_5 x + c_6 x^2) \sin 2x$

$\Rightarrow y \ = \ \ (c_1 + c_2 x + c_3\, x^2) \cos 2x + \ (c_4 + c_5 x + c_6 x^2) \sin 2x$

Example. $y^V + 8y''' + 16y' = 0$

$\lambda^5 + 8\lambda^3 + 16\lambda = 0$
$\lambda(\lambda^4 + 8\lambda^2 + 16) = 0$
$\lambda(\lambda^2 + 4)^2 = 0$
$\lambda_1 = 0;\ \lambda_2 = \lambda_3 = 2i;\ \lambda_4 = \lambda_5 = -2i$
$y(x) = C_1 + (C_2 + C_3 x)\cos 2x + (C_4 + C_5 x)\sin 2x$