Ex: Check that $f_1 = e^{6x}$ and $f_2 = e^{-2x}$ are Linearly independent; or not.

Solution: $f_1 = e^{6x}$, $f_2 = e^{-2x}$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{6x} & e^{-2x} \\ 6e^{6x} & -2e^{-2x} \end{vmatrix}$$

$$= e^{6x} \cdot (-2)e^{-2x} - 6e^{6x} \cdot e^{-2x}$$

$$= -2e^{4x} - 6e^{4x} = -8e^{4x} \neq 0$$

Hence they are linearly independants.

Ex: Check that $f_1 = e^{2x}$ and $f_2 = x^2$ are linearly independant or not.

Solution: $f_1 = e^{2x}$, $f_2 = x^2$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & x^2 \\ 2e^{2x} & 2x \end{vmatrix} = e^{2x} \cdot 2x - 2e^{2x} \cdot x^2$$

$$= 2xe^{2x} - 2x^2e^{2x}$$

$$= \underbrace{2e^{2x}}_{\neq 0} \cdot x(1-x).$$

if $x = 0$ or $x = 1$ then f_1 and f_2 are linearly dependant

if $x \neq 0, 1$ then f_1 and f_2 are linearly independant

Ex: Check that $1, e^{2x}$ and e^{-5x} are linearly independent or not.

Solution: $f_1 := 1$, $f_2 := e^{2x}$ and $f_3 := e^{-5x}$

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} & e^{-5x} \\ 0 & 2e^{2x} & -5e^{-5x} \\ 0 & 4e^{2x} & 25e^{-5x} \end{vmatrix}$$

$$\begin{matrix} 1 & e^{2x} & e^{-5x} \\ 0 & 2e^{2x} & -5e^{-5x} \end{matrix}$$

$$= 2e^{2x} \cdot 25e^{-5x} + 0 + 0 - \left(-5e^{-5x} \cdot 4e^{2x} + 0\right)$$

$$= 50e^{-3x} - \left(-20e^{-3x}\right)$$

$$= 50e^{-3x} + 20e^{-3x}$$

$$= 70e^{-3x} \neq 0$$

Hence $1, e^{2x}$ and e^{-5x} are linearly independent.

Ex : Consider $y'' - 9y = 0$

1) Show that e^{3t} and e^{-3t} are linearly independant

2) Show that they are a solution of the equation

3) Write y_g.

Solution : 1) Lets call $f_1 = e^{3t}$ and $f_2 = e^{-3t}$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{3t} & e^{-3t} \\ 3e^{3t} & -3e^{-3t} \end{vmatrix} = e^{3t} \cdot (-3)e^{-3t} - 3e^{3t} e^{-3t}$$

$$= -3e^0 - 3e^0 = -6 \neq 0$$

So they are linearly independant

2) For e^{3t} :

$$\left.\begin{aligned} y &= e^{3t} \\ y' &= 3e^{3t} \\ y'' &= 9e^{3t} \end{aligned}\right\} \quad \begin{aligned} &y'' - 9y = 0 \\ &9e^{3t} - 9e^{3t} \stackrel{?}{=} 0 \\ &0 = 0 \checkmark \end{aligned}$$

For e^{-3t} :

$$\left.\begin{aligned} y &= e^{-3t} \\ y' &= -3e^{-3t} \\ y'' &= 9e^{-3t} \end{aligned}\right\} \quad \begin{aligned} &y'' - 9y = 0 \\ &9e^{-3t} - 9e^{-3t} \stackrel{?}{=} 0 \\ &0 = 0 \checkmark \end{aligned}$$

Hence e^{3t} and e^{-3t} are the solutions of the equation

3) Write y_g:

$$y_g = c_1 y_1 + c_2 y_2$$

Here $y_1 = e^{3t}$

$y_2 = e^{-3t}$

$$y_g = c_1 e^{3t} + c_2 e^{-3t}$$

//

Ex: Consider $y''' - 3y'' - 10y' + 24y = 0$

1) Show that e^{2x}, e^{-3x} and e^{4x} are Linearly independant

2) Show that they are a solution of the equation

3) Write y_g.

Solution: 1) Lets call $f_1 = e^{2x}$

$f_2 = e^{-3x}$

$f_3 = e^{4x}$

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-3x} & e^{4x} \\ 2e^{2x} & -3e^{-3x} & 4e^{4x} \\ 4e^{2x} & 9e^{-3x} & 16e^{4x} \end{vmatrix} = -60e^{3x} \neq 0$$

So they are Linearly independant

2) For e^{2x}:

$y = e^{2x}$

$y' = 2e^{2x}$

$y'' = 4e^{2x}$

$y''' = 8e^{2x}$

$y''' - 3y'' - 10y' + 24y = 0$

$8e^{2x} - \underbrace{3.4}_{12}.e^{2x} - \underbrace{10.2}_{-20}.e^{2x} + 24e^{2x} \stackrel{?}{=} 0$

$32e^{2x} - 32e^{2x} \stackrel{?}{=} 0$

$0 = 0$ ✓

For e^{-3x}:

$y = e^{-3x}$

$y' = -3e^{-3x}$

$y'' = 9e^{-3x}$

$y''' = -27e^{-3x}$

$y''' - 3y'' - 10y' + 24y = 0$

$-27e^{-3x} - \overbrace{3.9e^{-3x} - 10.(-3)e^{-3x}}^{30} +$

$24.e^{-3x} \stackrel{?}{=} 0$

$-54e^{-3x} + 54e^{-3x} \stackrel{?}{=} 0$

$0 = 0$ ✓

For e^{4x}:

$$\left.\begin{aligned} y &= e^{4x} \\ y' &= 4e^{4x} \\ y'' &= 16e^{4x} \\ y''' &= 64e^{4x} \end{aligned}\right\}$$

$$y''' - 3y'' - 10y' + 24y = 0$$

$$64e^{4x} - \underbrace{3 \cdot 16e^{4x}}_{48} - \underbrace{10 \cdot 4e^{4x}}_{-40} + 24 \cdot e^{4x} \stackrel{?}{=} 0$$

$$84e^{4x} - 80e^{4x} \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

Hence e^{2x}, e^{-3x} and e^{4x} are the solutions of the equation

3) Write y_g:

$$y_g = c_1 y_1 + c_2 y_2 + c_3 y_3$$

Here $y_1 = e^{2x}$

$y_2 = e^{-3x}$

$y_3 = e^{4x}$

$$\Rightarrow \quad y_g = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{4x} \quad //$$

Ex: $y_1(x) = e^x$ is a solution of $xy'' - (2x+1)y' + (x+1)y = 0$

Find other solution and general solution.

Solution: $xy'' - (2x+1)y' + (x+1)y = 0$: second order

non-constant coefficient

We will use the method of reduction of order

$y_1(x) = e^x \Rightarrow y_2(x) = \overbrace{v}^{v(x)} \cdot y_1$

$\Rightarrow y_2 = v \cdot e^x$

So $y_2 = ve^x$ is other solution

$y_2 = ve^x$

$y_2' = v'e^x + ve^x$

$y_2'' = v''e^x + v'e^x + v'e^x + ve^x$

$= v''e^x + 2v'e^x + ve^x$

$xy'' - (2x+1)y' + (x+1)y = 0$ put y_2, y_2' and y_2''

$x.(v''e^x + 2v'e^x + ve^x) - (2x+1)(v'e^x + ve^x) + (x+1)(ve^x) = 0$

$xe^x v'' + \cancel{2xe^x v'} + \cancel{xve^x} - \cancel{2xe^x v'} - \cancel{2xve^x} - v'e^x - \cancel{ve^x}$

$+ \cancel{xve^x} + \cancel{ve^x} = 0$

$xe^x v'' - v'e^x = 0$

Lets call $v' = u$; $v'' = u' \Rightarrow x\cancel{e^x}u' - u\cancel{e^x} = 0$

$\Rightarrow xu' - u = 0$

$xu' - u = 0$ where $u' = \frac{du}{dx}$

$x\frac{du}{dx} - u = 0$ (seperable dif. eq.)

$x\frac{du}{dx} = u \Rightarrow \frac{du}{u} = \frac{dx}{x}$

$\Rightarrow \int \frac{du}{u} = \int \frac{dx}{x} \Rightarrow \ln u = \ln x + \cancel{c}^{\,0}$

$u = x$.

$u = v' \Rightarrow u = \frac{dv}{dx}$

$\frac{dv}{dx} = x \Rightarrow dv = x\,dx \Rightarrow \int dv = \int x\,dx$

$\Rightarrow v = \frac{x^2}{2} + \cancel{c}^{\,0}$

$y_2 = v \cdot e^x \Rightarrow y_2 = \frac{x^2}{2} \cdot e^x$ is the other solution.

the general solution is $y_g = c_1 y_1 + c_2 y_2$

$y_g = c_1 e^x + c_2 \frac{x^2}{2} e^x$ //

Ex: $y_1(x) = x$ is a solution of $x^2y'' + 2xy' - 2y = 0$. Find other solution and general solution.

Solution: $x^2y'' + 2xy' - 2y = 0$: second order, non-constant coefficient

We will use the method of reduction of order.

$y_1(x) = x \Rightarrow y_2(x) = v \cdot y_1$ → $v(x)$

$\Rightarrow y_2 = v \cdot x$

So $y_2 = v \cdot x$ is other solution.

$y_2 = v\,x$

$y_2' = v' \cdot x + v \cdot 1 = v'x + v$

$y_2'' = v'' \cdot x + v' \cdot 1 + v' = v'' \cdot x + 2v'$

$x^2y'' + 2xy' - 2y = 0$ put y_2, y_2', y_2''

$\Rightarrow x^2 \cdot (v'' \cdot x + 2v') + 2x(v'x + v) - 2 \cdot (vx) = 0$

$x^3v'' + 2x^2v' + 2x^2v' + \cancel{2xv} - \cancel{2xv} = 0$

$x^3v'' + 4x^2v' = 0$

So we obtain $x^3v'' + 4x^2v' = 0$

Lets call $v' = u$; so $v'' = u'$

Hence $x^3u' + 4x^2u = 0$ (separable equ.)

What is $u' = ?$ $\quad u' = \frac{du}{dx}$

$$\Rightarrow x^3 u' + 4x^2 u = 0$$

$$x^3 \frac{du}{dx} + 4x^2 u = 0$$

$$x^3 \frac{du}{dx} = -4x^2 u$$

$$\frac{du}{dx} = \frac{-4x^2}{x^3} u$$

$$\frac{du}{u} = -\frac{4}{x} dx \Rightarrow \int \frac{du}{u} = \int -\frac{4}{x} dx$$

$$\ln u = -4\ln x + \cancel{c}^{\,0} \Rightarrow u = e^{-4\ln x} \Rightarrow u = x^{-4\ln e} \Rightarrow$$

$$u = x^{-4}$$

$$u = v' \Rightarrow v' = x^{-4} \Rightarrow \frac{dv}{dx} = x^{-4}$$

$$\Rightarrow dv = x^{-4} dx \Rightarrow \int dv = \int x^{-4} dx$$

$$\Rightarrow v = \frac{x^{-3}}{-3} + \cancel{c}^{\,0} \Rightarrow v = \frac{x^{-3}}{3}$$

$y_2 = v \cdot x \Rightarrow y_2 = \frac{x^{-3}}{3} \cdot x \Rightarrow y_2 = \frac{x^{-2}}{3}$ is the other solution

the general solution is $y_g = c_1 y_1 + c_2 y_2$

$$y_g = c_1 \cdot x + c_2 \cdot \frac{x^{-2}}{3}$$

Ex: $y_1(t) = t^{-1}$ is a solution of $2t^2y'' + ty' - 3y = 0 \quad (t>0)$

Find other solution and general solution.

Solution: $2t^2y'' + ty' - 3y = 0$ second order, non-constant coeffi.

We will use the reduction of order

$$y_1(t) = t^{-1} \Rightarrow y_2(t) = \overbrace{V}^{v(t)} \cdot y_1$$

$$y_2 = v \cdot t^{-1}$$

So $y_2 = vt^{-1}$ is the other solution

$$y_2 = vt^{-1}$$

$$y_2' = v't^{-1} + v\cdot(-1)t^{-2} = v't^{-1} - vt^{-2}$$

$$y_2'' = v''t^{-1} + v'\cdot(-1)t^{-2} - (v't^{-2} - 2vt^{-3})$$

$$= v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3}$$

$$= v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

$2t^2y'' + ty' - 3y = 0$ put y_2, y_2', y_2''

$$2t^2\cdot(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + t(v't^{-1} - vt^{-2}) - 3(vt^{-1}) = 0$$

$$2tv'' - 4v' + \cancel{4vt^{-1}} + v' - \cancel{vt^{-1}} - \cancel{3vt^{-1}} = 0$$

$$2tv'' - 3v' = 0$$

We obtain $2tv'' - 3v' = 0$

lets call $v' = u$; so $v'' = u'$

Hence $2tv'' - 3v' = 0$

$\Rightarrow 2tu' - 3u = 0$ (seperable dif. eq.)

$u' = \frac{du}{dt}$

$\Rightarrow 2t \cdot \frac{du}{dt} - 3u = 0$

$2t \frac{du}{dt} = 3u \Rightarrow \frac{du}{3u} = \frac{dt}{2t}$

$\int \frac{du}{3u} = \int \frac{dt}{2t} \Rightarrow \frac{1}{3} \ln u = \frac{1}{2} \ln t + \cancel{c} \to 0$

$2 \ln u = 3 \ln t \Rightarrow \ln u^2 = \ln t^3$

$\Rightarrow u^2 = t^3 \Rightarrow u = t^{3/2}$

$u = ?$ $u = v' = \frac{dv}{dt}$

$\frac{dv}{dt} = t^{3/2} \Rightarrow dv = t^{3/2} dt \Rightarrow \int dv = \int t^{3/2} dt$

$v = t^{5/2} + \cancel{c} \to 0 \Rightarrow v = t^{5/2}$

$y_2 = v \cdot t^{-1} = t^{5/2} \cdot t^{-1} \Rightarrow y_2 = t^{3/2}$ is other solution

general solution : $y_g = c_1 y_1 + c_2 y_2$

$y_g = c_1 t^{-1} + c_2 t^{3/2}$ //

EXTRA

Ex: Solve $y' = \dfrac{xy^3}{\sqrt{1+x^2}}$

Solution: $\dfrac{dy}{dx} = \dfrac{xy^3}{\sqrt{1+x^2}}$

$$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}}\,dx$$

Let $1+x^2 = u$

$2x\,dx = du$

$x\,dx = \dfrac{du}{2}$

$$\int y^{-3}dy = \int \frac{1}{2}\,\frac{du}{\sqrt{u}} \qquad u^{-1/2}$$

$$\frac{y^{-2}}{-2} = \cancel{\frac{1}{2}} \cdot \frac{u^{1/2}}{\cancel{1/2}} + c$$

$$-\frac{y^{-2}}{2} = u^{1/2} + c$$

$$y^{-2} = -2\sqrt{u} - c$$

$$y^{-2} = -2\sqrt{1+x^2} - 2c$$

$$y^2 = \frac{-1}{2\sqrt{1+x^2} - 2c}$$

Ex: Solve $(2xy^2+4)+(2x^2y-6)y'=0$

Solution: $(2xy^2+4)+(2x^2y-6)\dfrac{dy}{dx}=0$

$$\underbrace{(2xy^2+4)dx}_{M}+\underbrace{(2x^2y-6)dy}_{N}=0$$

$M=2xy^2+4 \Rightarrow M_y=\dfrac{\partial M}{\partial y}=4xy$

$N=2x^2y-6 \Rightarrow N_x=\dfrac{\partial N}{\partial x}=4xy$

$M_y = N_x$ so it is exact dif. eq.

The general solution is $\varphi(x,y)=c$

We know $\varphi_x=M$ and $\varphi_y=N$

$\varphi_x=\dfrac{\partial \varphi}{\partial x}=M=2xy^2+4 \Rightarrow \varphi(x,y)=\int(2xy^2+4)dx$

$$=x^2y^2+4x+h(y)$$

Lets find $h(y)$

$\varphi_y=\dfrac{\partial \varphi}{\partial y}=N=\dfrac{\partial(x^2y^2+4x+h(y))}{\partial y}=\cancel{2x^2y}+h'(y)=N=\cancel{2x^2y}-6$

$h'(y)=-6 \Rightarrow h(y)=-6y+c_1$

general solution is $\varphi(x,y)=c$

$$x^2y^2+4x-6y+c_1=c$$

$$x^2y^2+4x-6y=c_2$$

Ex: Solve $\dfrac{3x+8}{x^2+5x+6}\,dx - \dfrac{4y}{y^2+4}\,dy = 0$

Solution: $\dfrac{3x+8}{x^2+5x+6} = \dfrac{3x+8}{(x+3)(x+2)} = \dfrac{A}{(x+3)} + \dfrac{B}{(x+2)}$

$(x+2) \qquad (x+3)$

$= \dfrac{Ax+2A+Bx+3B}{(x+3)(x+2)} \Rightarrow \begin{array}{l} A+B=3 \\ 2A+3B=8 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=2 \end{array}$

$\Rightarrow \dfrac{3x+8}{x^2+5x+6} = \dfrac{1}{x+3} + \dfrac{2}{x+2}$

$\Rightarrow \left(\dfrac{1}{x+3} + \dfrac{2}{x+2}\right)dx - \dfrac{4y}{y^2+4}\,dy = 0$

$\int \dfrac{1}{x+3}\,dx + \int \dfrac{2}{x+2}\,dy - \int \dfrac{4y}{y^2+4}\,dy = 0$

$y^2+4=u$

$2y\,dy = du$

$\ln|x+3| + 2\ln|x+2| + c_1 - 2\int \dfrac{du}{u} = 0$

$\ln|x+3| + 2\ln|x+2| + \cancel{c_1} - 2\ln u + \cancel{c_2} = 0$

$\ln|x+3| + 2\ln|x+2| - 2\ln|y^2+4| + c_3 = 0$

$\ln\left|\dfrac{(x+3)(x+2)^2}{(y^2+4)^2}\right| = -c_3 \quad \ln c_4$

$\dfrac{(x+3)(x+2)^2}{(y^2+4)^2} = c_4 \Rightarrow (y^2+4)^2 = \dfrac{c_4}{(x+3)(x+2)^2}$

$y^2 = \sqrt{\dfrac{c_4}{(x+3)(x+2)^2}} - 4$ //

Ex: Solve IVP $y' = \dfrac{3x^2+4x-4}{2y-4}$, $y(1)=3$

Solution: $y' = \dfrac{3x^2+4x-4}{2y-4}$

$$\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$$

$$(2y-4)dy = (3x^2+4x-4)\,dx$$

$$\int (2y-4)\,dy = \int (3x^2+4x-4)\,dx$$

$$y^2-4y+c_1 = x^3+2x^2-4x+c_2$$

$$y^2-4y = x^3+2x^2-4x+c_3$$

$$y(1)=3 \Rightarrow 9-\cancel{12} = \cancel{1}+\cancel{2}\cancel{-4}+c_3$$

(with -11 written above $\cancel{12}$ and -1 above $\cancel{1}+\cancel{2}\cancel{-4}$)

$$c_3 = -2$$

$\Rightarrow$ solution is

$$y^2-4y = x^3+2x^2-4x-2$$