EXAMPLE 3. Let $[A]$ be the matrix of a linear transformation $A: \mathcal{V} \to \mathcal{V}$ with respect to a basis $\mathcal{B}$ in $\mathcal{V}$, and set $[A]^0 = [I]$ where $[I]$ is the appropriate identity matrix. Then the various powers $[A]^k$, k a nonnegative integer, are all defined, and are simply the powers of A with respect to $\mathcal{B}$. This in turn implies that the matrix of a polynomial

$$p(A) = a_k A^k + a_{k-1} A^{k-1} + \cdots + a_1 A + a_0 I$$

in A is

$$p([A]) = a_k [A]^k + a_{k-1} [A]^{k-1} + \cdots + a_1 [A] + a_0 [I].$$

Such expressions are called polynomials in $[A]$, and are defined whenever $[A]$ is a square matrix. In particular, if

$$m_A(x) = x^k - a_{k-1} x^{k-1} - \cdots - a_1 x - a_0$$

is the minimum polynomial of A, then

$$m_A([A]) = [A]^k - a_{k-1}[A]^{k-1} - \cdots - a_1[A] - a_0[I] = [O],$$

and it follows that $[A]$ is a root of $m_A(x)$. For obvious reasons this polynomial is also called the *minimum polynomial* of the matrix $[A]$.

Definition 2–8 is simply a coordinatized version of the multiplication of linear transformations. This allows us to assert (without proof) that *matrix multiplication is associative, and is distributive over addition;* that is,

$$[(\alpha_{ik})(\beta_{kj})](\gamma_{jl}) = (\alpha_{ik})[(\beta_{kj})(\gamma_{jl})] \tag{2–35}$$

and

$$\begin{aligned}(\alpha_{ik})[(\beta_{kj}) + (\gamma_{kj})] &= (\alpha_{ik})(\beta_{kj}) + (\alpha_{ik})(\gamma_{kj}),\\ [(\alpha_{ik}) + (\beta_{ik})](\gamma_{kj}) &= (\alpha_{ik})(\gamma_{kj}) + (\beta_{ik})(\gamma_{kj})\end{aligned} \tag{2–36}$$

whenever the sums and products appearing in these equations are defined.

Matrix multiplication. By far the most important operation involving matrices is *matrix multiplication*, the process of multiplying one matrix by another. The first step in defining matrix multiplication is to recall the definition of the dot product of two vectors. Let **r** and **c** be two *n*-vectors. Writing **r** as a 1 x *n* row matrix and **c** as an *n* x 1 column matrix, the dot product of **r** and **c** is

$$\mathbf{r}\cdot\mathbf{c} = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$$

Note that in order for the dot product of **r** and **c** to be defined, both must contain the same number of entries. Also, the order in which these matrices are written in this product is important here: The row vector comes first, the column vector second.

Now, for the final step: How are two general matrices multiplied? First, in order to form the product *AB, the number of columns of A must match the number of rows of B*; if this condition does not hold, then the product *AB* is not defined. This criterion follows from the restriction stated above for multiplying a row matrix **r** by a column matrix **c**, namely that the number of entries in **r** must match the number of entries in **c**. If *A* is *m* x *n* and *B* is *n* x *p*, then the product *AB* is defined, and the size of the product matrix *AB* will be *m* x *p*. The following diagram is helpful in determining if a matrix product is defined, and if so, the dimensions of the product:

	A	$\cdot$	B	$=$	AB
dimensions of matrix:	$m \times n$		$n \times p$		$m \times p$

these must match

> The dot product of row i in A and column j in B gives the (i, j) entry of AB

Example: Given the two matrices

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 3 & -1 & 5 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

determine which matrix product, *AB* or *BA*, is defined and evaluate it.

Since *A* is 2 x 3 and *B* is 3 x 4, the product *AB*, in that order, is defined, and the size of the product matrix *AB* will be 2 x 4. The product *BA* is *not* defined, since the first factor (*B*) has 4 columns but the second factor (*A*) has only 2 rows. The number of columns of the first matrix must match the number of rows of the second matrix in order for their product to be defined.

Taking the dot product of row 1 in *A* and column 1 in *B* gives the (1, 1) entry in *AB*. Since

$$\begin{bmatrix} 1 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = (1)(1) + (0)(-2) + (-3)(0) = 1$$

the (1, 1) entry in *AB* is 1:

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 3 & -1 & 5 \\ 0 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{1} & & & \\ & & & \end{bmatrix}$$

The dot product of row 1 in *A* and column 2 in *B* gives the (1, 2) entry in *AB*,

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 3 & -1 & 5 \\ 0 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \boxed{3} & & \\ & & & \end{bmatrix}$$

Similarly,

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 3 & -1 & 5 \\ 0 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & \boxed{-2} & \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 3 & -1 & 5 \\ 0 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 & \boxed{-2} \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 3 & -1 & 5 \\ 0 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 & -2 \\ \boxed{-10} & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 3 & -1 & 5 \\ 0 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 & -2 \\ -10 & \boxed{11} & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ \boxed{-2 \quad 4 \quad 1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \boxed{4} & 1 \\ -2 & 3 & \boxed{-1} & 5 \\ 0 & -1 & \boxed{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 & -2 \\ -10 & 11 & \boxed{-10} & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ \boxed{-2 \quad 4 \quad 1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 & \boxed{1} \\ -2 & 3 & -1 & \boxed{5} \\ 0 & -1 & 2 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 & -2 \\ -10 & 11 & -10 & \boxed{19} \end{bmatrix}$$

Example.

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The product AB is defined since A is of size 2×3 and B is of size 3×2; moreover, AB is of size 2×2. We have

$$AB = \begin{bmatrix} 0 \cdot 2 + 1 \cdot 1 + 0 \cdot 1 & 0 \cdot 0 + 1 \cdot 2 + 0 \cdot 1 \\ 2 \cdot 2 + 3 \cdot 1 + 1 \cdot 1 & 2 \cdot 0 + 3 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix}.$$

Note that in this case the product BA is also defined (since B has the same number of columns as A has rows). The product BA is of size 3×3:

$$BA = \begin{bmatrix} 2 \cdot 0 + 0 \cdot 2 & 2 \cdot 1 + 0 \cdot 3 & 2 \cdot 0 + 0 \cdot 1 \\ 1 \cdot 0 + 2 \cdot 2 & 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 0 + 2 \cdot 1 \\ 1 \cdot 0 + 1 \cdot 2 & 1 \cdot 1 + 1 \cdot 3 & 1 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 7 & 2 \\ 2 & 4 & 1 \end{bmatrix}.$$

Example: Although matrix multiplication is not always commutative, it *is* always *associative*. That is, if *A, B*, and *C* are any three matrices such that the product *(AB)C* is defined, then the product *A(BC)* is also defined, and

$$\boxed{(AB)C = A(BC)}$$

Verify the associative law for the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 0 \\ 4 & 1 & -6 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 0 \\ 4 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 7 & 5 & -12 \\ -4 & -1 & 6 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 7 & 5 & -12 \\ -4 & -1 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 55 \\ -27 \end{bmatrix}$$

$$BC = \begin{bmatrix} -1 & 3 & 0 \\ 4 & 1 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 27 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 27 \end{bmatrix} = \begin{bmatrix} 55 \\ -27 \end{bmatrix}$$

Therefore, *(AB)C = A(BC).*

Definition. *Let* $\mathbf{A} = [a_{ij}]$ *be an* $m \times n$ *matrix. Then the* **transpose** *of* $\mathbf{A}$ *is the* $n \times m$ *matrix defined by interchanging rows and columns and is denoted by* $\mathbf{A}^T$, *i.e.,*

$$[\mathbf{A}^T]_{ji} = a_{ij} \text{ for } 1 \le j \le n, 1 \le i \le m.$$

Example.

1. $\begin{pmatrix} 2 & 0 & 5 \\ 4 & -1 & 7 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ 5 & 7 \end{pmatrix}$ *2.* $\begin{pmatrix} 7 & -2 & 6 \\ 1 & 2 & 3 \\ 5 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 7 & 1 & 5 \\ -2 & 2 & 0 \\ 6 & 3 & 4 \end{pmatrix}$

Theorem.(Properties of transpose). *For any* $m \times n$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$,

1. $(\mathbf{A}^T)^T = \mathbf{A}$;
2. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$;
3. $(c\mathbf{A})^T = c\mathbf{A}^T$;
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Example. For the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 3 & 0 \\ 4 & 1 & -6 \end{bmatrix}$$

verify the equation $(AB)^T = B^T A^T$.

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 0 \\ 4 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 7 & 5 & -12 \\ -4 & -1 & 6 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 7 & -4 \\ 5 & -1 \\ -12 & 6 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -1 & 4 \\ 3 & 1 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 5 & -1 \\ -12 & 6 \end{bmatrix}$$

$$\boxed{(AB)^T = B^T A^T}$$

If a matrix has an inverse, it is said to be **invertible**.

$$AB = BA = I$$

B is called the (multiplicative) **inverse** of A and denoted A^{-1}

Example: If

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$

Since

$$AA^{-1} = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$A^{-1}A = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Although every nonzero real number has an inverse, *there exist nonzero matrices that have no inverse*.

Example: Show that the nonzero matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

has no inverse.

If this matrix had an inverse, then

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ would equal } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for some values of *a, b, c*, and *d*. However, since the second row of *A* is a zero row, you can see that the second row of the product must also be a zero row:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & \boxed{0} \end{bmatrix}$$

Example: Given that

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

verify the equation $(AB)^{-1} = B^{-1}A^{-1}$.

First, compute *AB*:

$$AB = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 8 & -13 \\ 21 & -34 \end{bmatrix}$$

Next, compute $B^{-1}A^{-1}$:

$$B^{-1}A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -34 & 13 \\ -21 & 8 \end{bmatrix}$$

Now, since the product of *AB* and $B^{-1}A^{-1}$ is *I*,

$$(AB)(B^{-1}A^{-1}) = \begin{bmatrix} 8 & -13 \\ 21 & -34 \end{bmatrix}\begin{bmatrix} -34 & 13 \\ -21 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$B^{-1}A^{-1}$ is indeed the inverse of *AB*. In fact, the equation

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

holds true for *any* invertible square matrices of the same size. This says that if *A* and *B* are invertible matrices of the same size, then their product *AB* is also invertible, and the inverse of the product is equal to the product of the inverses *in the reverse order*.

Since

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

it follows that $(AB)^{-1} = B^{-1}A^{-1}$, as desired.

Example. If

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

prove that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

for every positive integer *n*.

A few preliminary calculations illustrate that the given formula does hold true:

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

However, to establish that the formula holds for *all* positive integers *n*, a general proof must be given. This will be done here using *the principle of mathematical induction*, which reads as follows. Let *P(n)* denote a proposition concerning a positive integer *n*. If it can be shown that

$$P(1) \text{ is true}$$

and

$$P(n) \text{ is true} \quad \Rightarrow \quad P(n+1) \text{ is true}$$

then the statement *P(n)* is valid for *all* positive integers *n*. In the present case, the statement *P(n)* is the assertion

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Because $A^1 = A$, the statement *P*(1) is certainly true, since

$$A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Now, assuming that *P(n)* is true, that is, assuming

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

it is now necessary to establish the validity of the statement *P*(*n* + 1), which is

$$A^{n+1} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

But this statement does indeed hold, because

$$A^{n+1} = A^n \cdot A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

By the principle of mathematical induction, the proof is complete.

Example. Show that $T: R^2 \to R^2$, defined by

$$T((x_1, x_2)) = (x_1 + x_2, x_1 - x_2 + 1)$$

is not linear

Solution $T(\theta) = T((0,0)) = (0,1) \neq \theta$. Therefore, T is not linear.

Example. Let $T: R^2 \to R^1$ be defined by $T((x_1, x_2)) = {x_1}^2 + {x_2}^2$

Show that T is not linear even though $T(\theta) = \theta$.

Solution We have $T(\theta) = T((0,0)) = 0^2 + 0^2 = 0$, which is the zero of $\boldsymbol{R}^1$. This allows no conclusion; the definition of linearity must be used. To check additivity we calculate

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) &= T((x_1, x_2) + (y_1, y_2)) \\
&= T((x_1 + y_1, x_2 + y_2)) = (x_1 + y_2)^2 + (x_2 + y_2)^2 \\
&= {x_1}^2 + 2x_1y_1 + {y_1}^2 + {x_2}^2 + 2x_2y_2 + {y_2}^2
\end{aligned}$$

and

$$T(\mathbf{x}) + T(\mathbf{y}) = T((x_1, x_2)) + T((y_1, y_2)) = {x_1}^2 + {x_2}^2 + {y_1}^2 + {y_2}^2$$

Since $T(\mathbf{x}+\mathbf{y}) \neq T(\mathbf{x}) + T(\mathbf{y})$, we know that T is not linear.

Example . Find $\ker T$, where T is defined by $T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 - x_3)$.

Solution Since $\ker T = \{\mathbf{x} | T(\mathbf{x}) = \theta\}$, we must solve $T((x_1, x_2, x_3)) = (0, 0)$, that is,

$$(x_1 + x_2, x_2 - x_3) = (0, 0)$$

The resulting equations are

$$\begin{aligned}
x_1 + x_2 &= 0 \\
x_2 - x_3 &= 0
\end{aligned}$$

which have solution $(-k, k, k)$. Therefore

$$\ker T = \{\mathbf{v} \in E^3 | \mathbf{v} = k(-1, 1, 1)\} = \operatorname{span}\{(-1, 1, 1)\}$$

Example . Calculate $\eta(T)$ for the linear transformation $T\colon \boldsymbol{R}^3 \to \boldsymbol{R}^2$ defined by

$$T((a, b, c)) = (a + 2b + c, -a + 3b + c)$$

Find a basis for $\ker T$.

Solution We must find the set of all vectors (a, b, c) in $\boldsymbol{R}^3$ that $T(a, b, c) = (0, 0)$. That is, the equation

$$\begin{pmatrix} a+2b+c \\ -a+3b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

must be solved. The solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -k \\ -2k \\ 5k \end{pmatrix}$$

and $\ker T = \operatorname{span}\{(-1,-2,5)\}$. Therefore $\dim(\ker T) = 1$. A basis is $\{(-1,-2,5)\}$.

Example . Define $T: \boldsymbol{R}^3 \to \boldsymbol{R}^3$ by $T((a,b,c)) = (a-b+c, 2a+b-c, -a-2b+2c)$. Determine range T and dim(range T). Find two vectors in range T and two vectors not in range T. Find a basis for range T. Find $\ker T$.

Solution Let $\mathbf{y} = (y_1, y_2, y_3)$ be in range T. Thus $\mathbf{y} = T((a,b,c))$ for some vector (a,b,c) in E^3. That is, the equation $\mathbf{y} = T((a,b,c))$ **must be consistent**. We reduce the equations and see what conditions the consistency forces. The equations are

$$\begin{aligned} a - \ b + \ c &= y_1 \\ 2a + \ b - \ c &= y_2 \\ -a - 2b + 2c &= y_3 \end{aligned}$$

and they reduce to

$$\begin{aligned} a - b + \ c &= \ y_1 \\ 3b - 3c &= \ y_2 - 2y_1 \\ 0 &= -y_1 + y_2 + y_3 \end{aligned}$$

So if $\mathbf{y} = (y_1, y_2, y_3)$ is to be the range T, then $-y_1 + y_2 + y_3 = 0$. That is,

$$\text{range } T = \{(y_1, y_2, y_3) | y_1 = y_2 + y_3\}$$

The condition on y_1, y_2, and y_3 gives a criterion for inclusion in range T. Some vectors in range T are $(-2, -1, -1)$ and $(0, -1, 1)$. Some vectors not in range T are (1,1,1) and (1,0,0). The dimension of range T is 2, since the equation $-y_1 + y_2 + y_3 = 0$ allows the assignment of arbitrary values to any **two** of the values of y_k.

To obtain a basis, we can use $(-2, -1, -1)$ and $(0, -1, 1)$ as above, since they are linearly independent in the range and dim(range T) = 2. In fact, any two linearly independent vectors in range T form a basis for range T. The kernel of T is found by setting $y_1 = y_2 = y_3 = 0$ in the linear equations above. We obtain $\ker T = \text{span}\{(0, 1, 1)\}$.

OPERATOR EQUATIONS

Much of the study of linear transformations is given over to devising methods for solving equations of the form

$$Ax = y, \tag{2-37}$$

where y is known, x unknown, and A a linear transformation from $\mathcal{V}_1$ to $\mathcal{V}_2$. Such equations are known under the generic name of *operator equations*. In general, of course, the technique for solving a particular operator equation depends upon the operator involved, and also upon the underlying vector spaces. Nevertheless there are a number of facts concerning such equations which can be proved without using anything other than the linearity of A, and we propose to get them on record here before going on to more specialized topics.

A vector x_0 in $\mathcal{V}_1$ is said to be a *solution* of (2–37) if $A(x_0) = y$. The totality of such vectors is called the *solution set* of the equation. In the special case of a *homogeneous equation*

$$Ax = 0 \tag{2-38}$$

whose right-hand side is zero, we know that this set is a *subspace* of $\mathcal{V}_1$. It is called the *solution space* of the equation. One of the most important properties of operator equations is that the problem of solving a *nonhomogeneous* equation $Ax = y, y \neq 0$, can all but be reduced to that of solving its *associated homogeneous equation* $Ax = 0$. In fact, if x_p is a fixed solution of $Ax = y$, and if x_h is any solution whatever of $Ax = 0$, then $x_p + x_h$ is also a solution of $Ax = y$, since

$$
\begin{aligned}
A(\mathbf{x}_p + \mathbf{x}_h) &= A(\mathbf{x}_p) + A(\mathbf{x}_h) \\
&= \mathbf{y} + \mathbf{0} \\
&= \mathbf{y}.
\end{aligned}
$$

Moreover, *every* solution $\mathbf{x}_0$ of $A\mathbf{x} = \mathbf{y}$ can be written in this form for a suitable $\mathbf{x}_h$, since from

$$
\begin{aligned}
A(\mathbf{x}_0 - \mathbf{x}_p) &= A(\mathbf{x}_0) - A(\mathbf{x}_p) \\
&= \mathbf{y} - \mathbf{y} \\
&= \mathbf{0}
\end{aligned}
$$

it follows that $\mathbf{x}_0 - \mathbf{x}_p = \mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$, and hence that $\mathbf{x}_0 = \mathbf{x}_p + \mathbf{x}_h$, as asserted.

The solution $\mathbf{x}_p$ appearing in this argument is frequently called a *particular solution* of $A\mathbf{x} = \mathbf{y}$, and in these terms we can state the above result as follows:

Theorem 2–9. *If $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{y}$, then the solution set of this equation consists of all vectors of the form $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is an arbitrary solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.*

EXAMPLE 1. Let $A: \mathcal{C}^2(-\infty, \infty) \to \mathcal{C}(-\infty, \infty)$ be the linear transformation $D^2 - I$ introduced in Example 4, Section 2–4. Then the operator equation

$$Ay = 1,$$

y in $\mathcal{C}^2(-\infty, \infty)$, assumes the form

$$\frac{d^2y}{dx^2} - y = 1, \tag{2–39}$$

and its solution set consists of all functions in $\mathcal{C}^2(-\infty, \infty)$ which satisfy this equation on the entire real line. In this case it is obvious that $y = -1$ is one such function. Thus, to complete the solution of (2–39) it suffices to find *all* solutions of the homogeneous equation

$$\frac{d^2y}{dx^2} - y = 0.$$

In Chapter 3 we will prove that the solution space of this equation has the functions e^x and e^{-x} as a basis, and hence as a corollary, that the solution set of (2–39) is the totality of functions in $\mathcal{C}^2(-\infty, \infty)$ of the form

$$y = -1 + c_1e^x + c_2e^{-x},$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 2. Let A be a linear transformation from $\mathcal{R}^2$ to $\mathcal{R}^1$, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the standard basis vectors in $\mathcal{R}^2$, and let $A(\mathbf{e}_1) = \alpha_1$, $A(\mathbf{e}_2) = \alpha_2$, α_1 and α_2 real numbers. Then if $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ is any vector in $\mathcal{R}^2$,

$$\begin{aligned} A(\mathbf{x}) &= x_1A(\mathbf{e}_1) + x_2A(\mathbf{e}_2) \\ &= \alpha_1x_1 + \alpha_2x_2, \end{aligned}$$

and the operator equation $A\mathbf{x} = \mathbf{0}$ is an abbreviated version of

$$\alpha_1 x_1 + \alpha_2 x_2 = 0. \tag{2-40}$$

Since (2–40) is the equation of the line through the origin in $\mathcal{R}^2$ with slope $-\alpha_1/\alpha_2$, the solution space of the equation $A\mathbf{x} = \mathbf{0}$ is just the set of points in $\mathcal{R}^2$ which comprise that line. In this case the solution set of the nonhomogeneous equation $A\mathbf{x} = \beta$, β a real number, can be interpreted as a translation of the line described by (2–40), as shown in Fig. 2–10.

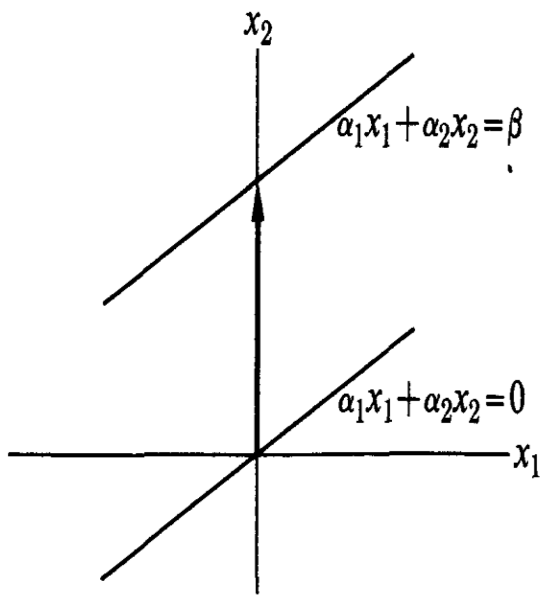

FIGURE 2–10

It goes without saying that in particular instances the solution set of $A\mathbf{x} = \mathbf{y}$ may be empty, in which case the equation has no solutions. In fact, one of the major problems in the study of operator equations (or arbitrary equations for that matter) is to determine conditions under which the equation will have solutions. This is the so-called *existence problem* for operator equations, and theorems which establish such conditions are called *existence theorems.*

Of equal, or even greater importance is the problem of ascertaining when $A\mathbf{x} = \mathbf{y}$ admits *at most one* solution for any given $\mathbf{y}$ in $\mathcal{V}_2$. This problem is known as the *uniqueness problem* for operator equations, and can always be answered by examining the homogeneous equation $A\mathbf{x} = \mathbf{0}$ and using the following theorem.

Theorem 2–10. *An operator equation $A\mathbf{x} = \mathbf{y}$ will have a unique solution (provided it has any solutions at all) if and only if its associated homogeneous equation $A\mathbf{x} = \mathbf{0}$ has no nonzero solutions, i.e., if and only if $\mathfrak{N}(A) = \mathcal{O}$.*

In the case where A admits an inverse of one of the various types discussed in Section 2–4, the equation $A\mathbf{x} = \mathbf{y}$ can be immediately solved. If, for instance, A is invertible, then from $A\mathbf{x} = \mathbf{y}$ we deduce that

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{y},$$

or

$$\mathbf{x} = A^{-1}\mathbf{y},$$

and the solution (which in this case must be unique) has been described in terms of A^{-1}. Similarly, if B is either a right or left inverse for A we find that the solution set of $A\mathbf{x} = \mathbf{y}$ is the set of all $\mathbf{x}$ in $\mathcal{V}_1$ such that $\mathbf{x} = B\mathbf{y}$. This technique for solving an operator equation is known as *inverting the operator*, and is used whenever an explicit formula for an inverse can be deduced from the definition of A.