Recall $y' + p(x)y = g(x)$

$\mu = e^{\int p(x)dx}$

integration factor

$\Rightarrow y = \frac{1}{\mu}\left(\int \mu\, g(x)\,dx + c\right)$

Ex: (Lineer equation) Solve $(4+x^2)\frac{dy}{dx} + 2xy = 4x$

Solution: $(4+x^2)y' + 2xy = 4x$

$\Rightarrow y' + \frac{2x}{4+x^2}y = \frac{4x}{4+x^2}$ $\quad (y' + p(x)y = g(x))$

Here $p(x) = \frac{2x}{4+x^2}$, $g(x) = \frac{4x}{4+x^2}$

$\mu = e^{\int p(x)dx} = e^{\int \frac{2x}{4+x^2}dx} = e^{\int \frac{du}{u}} = e^{\ln u} = u = (4+x^2)$

Use the formula: $y = \frac{1}{\mu}\left(\int \mu \cdot g(x)\,dx + c\right)$

$= \frac{1}{4+x^2}\left(\int \cancel{(4+x^2)} \cdot \frac{4x}{\cancel{4+x^2}}\,dx + c\right)$

$y = \frac{1}{4+x^2} \cdot \left(\frac{\cancel{4}x^2}{\cancel{2}} + c\right) = \frac{2x^2}{4+x^2} + \frac{c}{4+x^2}$ //

Ex : (Linear equ.) Solve $\frac{dy}{dx} + \frac{1}{2}y = \frac{1}{2}e^{x/3}$

Solution: $y' + \frac{1}{2}y = \frac{1}{2}e^{x/3}$ $\quad (y' + p(x)y = g(x))$

$$\mu = e^{\int p(x)dx} = e^{\int \frac{1}{2}dx} = e^{\frac{x}{2}} //$$

Thus the general solution is

$$y = \frac{1}{\mu}\left(\int \mu \cdot g(x)dx + c\right)$$

$$y = \frac{1}{e^{x/2}}\left(\int e^{x/2} \cdot \frac{1}{2} e^{x/3} dx + c\right)$$

$$y = \frac{1}{2e^{x/2}} \cdot \left(\int e^{5x/6} dx + c\right)$$

$$y = \frac{1}{2e^{x/2}} \cdot \left(\frac{6}{5}e^{5x/6} + c\right)$$

$$y = \frac{3}{5}e^{x/3} + ce^{-x/2} //$$

Ex (linear equ) Solve $xy' + 2y = 4x^2$, $y(1) = 2$, $x \neq 0$

Solution: $\frac{xy' + 2y}{x} = \frac{4x^2}{x}$

$\Rightarrow y' + \frac{2}{x} y = 4x \qquad (y' + p(x) y = g(x))$

$$N = e^{\int p(x) dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

The general solution is

$$y = \frac{1}{N} \left(\int N \, g(x) dx + c \right)$$

$$y = \frac{1}{x^2} \left(\int x^2 . 4x \, dx + c \right)$$

$$y = \frac{1}{x^2} \cdot \left(\frac{4x^4}{4} + c \right)$$

$$y = x^2 + \frac{c}{x^2}$$

$y(1) = 2 \Rightarrow \quad 2 = 1 + \frac{c}{1} \Rightarrow c = 1$

$$\Rightarrow y = x^2 + \frac{1}{x^2} \; //$$

Ex (linear equ) Solve $y'\sin x - y\cos x = \frac{\sin^2 x}{x^2}$

Solution:

$$\frac{y'\sin x - y\cos x}{\sin x} = \frac{\frac{\sin^2 x}{x^2}}{\sin x}$$

$$y' - \frac{\cos x}{\sin x} y = \frac{\sin x}{x^2} \qquad (y' + p(x)y = g(x))$$

$$\mu = e^{\int p(x)dx} = e^{\int -\frac{\cos x}{\sin x}dx} \qquad \begin{array}{l}\sin x = u\\ \cos x dx = du\end{array}$$

$$= e^{\int -\frac{du}{u}} = e^{-\ln u} = u^{-1} = (\sin x)^{-1} = \frac{1}{\sin x}$$

The general solution is

$$y = \frac{1}{\mu}\left(\int \mu \cdot g(x)\,dx + c\right)$$

$$y = \frac{1}{1/\sin x}\left(\int \frac{1}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{x^2}\,dx + c\right)$$

$$y = \sin x\left(\int \frac{1}{x^2}\,dx + c\right)$$

$$y = \sin x\left(\frac{x^{-1}}{-1} + c\right) = \sin x\left(-\frac{1}{x} + c\right) //$$

Ex (Bernoulli) Recall that Bernoulli dif. eq. $y' + p(x)y = g(x)y^n$

$v = y^{1-n}$ $\Rightarrow$ $v' + p'(x)v = g'(x)$: Linear dif. eq.

Solve the dif. eq. $y' + \frac{4}{x}y = x^3 y^2$

Solution: $y' + \frac{4}{x}y = x^3 y^2$ $\quad (y' + p(x)y = g(x)y^n)$

$v = y^{1-n}$, $n = 2$

$v = y^{1-2} \Rightarrow v = y^{-1} \Rightarrow y = \frac{1}{v}$

$\boxed{v = y^{-1}} \Rightarrow \boxed{v' = -1 \cdot y^{-2} \cdot y'} \Rightarrow y' = -y^2 v'$ //

$y' + \frac{4}{x}y = x^3 y^2$

$-\cancel{y^2}v' + \frac{4}{x}\cancel{y} = x^3 \cancel{y^2}$ (put y')

$-yv' + \frac{4}{x} = x^3 y$ (put y) $\quad y = \frac{1}{v}$

$v /$ $-\frac{1}{v}v' + \frac{4}{x} = x^3 \cdot \frac{1}{v}$

$-v' + \frac{4}{x}v = x^3$

$v' - \frac{4}{x}v = -x^3$ $\quad (v' + p'(x)v = g'(x))$

Lineer dif. equ.

$\mu = e^{\int p'(x)dx} = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = x^{-4}$ //

$$v' - \frac{4}{x} v = -x^3 \quad \text{and} \quad \mu = x^{-4}$$

use the formula: $y = \frac{1}{\mu} \cdot \left(\int \mu \cdot g(x) dx + c \right)$

$$v = \frac{1}{x^{-4}} \left(\int x^{-4} \cdot -x^3 \, dx + c \right)$$

$$v = x^4 \cdot \left(\int -x^{-1} \, dx + c \right)$$

$$v = x^4 \left(\int -\frac{1}{x} dx + c \right)$$

$$v = x^4 \left(-\ln|x| + c \right)$$

$$v = -x^4 \ln|x| + cx^4$$

$$\Rightarrow v = y^{-1} \Rightarrow y = \frac{1}{v} \Rightarrow y = \frac{1}{-x^4 \ln|x| + cx^4} \quad //$$

Ex (Bernoulli) Solve $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$

Solution: $y' - \frac{1}{x}y = -\frac{1}{x}y^2$ $\quad (y' + p(x)y = g(x)y^n)$

$v = y^{1-n}$, $n=2 \Rightarrow v = y^{1-2} \Rightarrow v = y^{-1}$, $y = \frac{1}{v}$

$v = y^{-1} \Rightarrow v' = -1\, y^{-2} \cdot y' \Rightarrow y' = -y^2 v'$

our dif. eq. $\quad y' - \frac{1}{x}y = -\frac{1}{x}y^2$. (put y')

$-y^2 v' - \frac{1}{x}y = -\frac{1}{x}y^2$

$\Rightarrow -y v' - \frac{1}{x} = -\frac{1}{x}y$ $\quad$ (put y) $y = \frac{1}{v}$

$\Rightarrow -\frac{1}{v}v' - \frac{1}{x} = -\frac{1}{x} \cdot \frac{1}{v}$ $\quad (\times(-v))$

$\Rightarrow v' + \frac{1}{x}v = \frac{1}{x}$ $\quad (v' + p'(x)v = g'(x))$ linear dif. equ.

$\mu = e^{\int p'(x)\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x^{\ln e} = x$ //

$\Rightarrow \mu = x$ and $v' + \frac{1}{x}v = \frac{1}{x}$

use the formula: $y = \frac{1}{\mu} \cdot \left(\int \mu \cdot g(x) dx + c \right)$

$v = \frac{1}{x} \left(\int \cancel{x} \cdot \frac{1}{\cancel{x}} dx + c \right)$

$v = \frac{1}{x} (x + c)$

$v = 1 + \frac{c}{x}$

$v = 1 + cx^{-1}$ $\qquad v = y^{-1} \Rightarrow v = \frac{1}{y} \Rightarrow y = \frac{1}{v}$

$y = \frac{1}{1 + cx^{-1}}$ //

Ex (Bernoulli) Solve the IVP: $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$, $y(1) = 2$

Solution: $y' + \frac{1}{2x} y = x \cdot y^{-3}$ (Bernoulli: $y' + p(x)y = g(x)y^n$)

$v = y^{1-n}$, $n = -3 \Rightarrow v = y^4$ //

$v' = 4y^3 y' \Rightarrow y' = \frac{v'}{4y^3}$ //

$y' + \frac{1}{2x} y = x y^{-3}$ (put y')

$\frac{v'}{4y^3} + \frac{1}{2x} y = x y^{-3}$ $(x (y^3))$

$\frac{v'}{4} + \frac{1}{2x} y^4 = x$ (put y) $v = y^4 \Rightarrow y = \sqrt[4]{v}$

$\frac{v'}{4} + \frac{1}{2x} \cdot (\sqrt[4]{v})^4 = x$

$\frac{v'}{4} + \frac{1}{2x} v = x$

$v' + \frac{2}{x} v = 4x$ $(v' + p'(x)v = g'(x))$ linear dif. equ.

$\mu = e^{\int p(x)dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

So $v' + \frac{2}{x} v = 4x$ and $\mu = x^2$

$\Rightarrow$ use the formula: $y = \frac{1}{\mu} \cdot \left(\int \mu \, g(x) \, dx + c \right)$

$$v = \frac{1}{x^2} \left(\int \overbrace{x^2 \cdot 4x}^{4x^3} \, dx + c \right)$$

$$v = \frac{1}{x^2} \cdot \left(\frac{\cancel{4} x^4}{\cancel{4}} + c \right)$$

$$v = x^2 + \frac{c}{x^2}$$

$$v = y^4 \Rightarrow y = \sqrt[4]{v}$$

$$\Rightarrow y = \sqrt[4]{x^2 + \frac{c}{x^2}} \quad //$$

We know that $y(1) = 2$: $2 = \sqrt[4]{1^2 + \frac{c}{1}}$

$$1 + \frac{c}{1} = 16 \Rightarrow c = 15$$

Hence $y = \sqrt[4]{x^2 + \frac{15}{x^2}}$ //

Ex (Bernoulli) $y' - \frac{3}{x}y = x^4 y^{1/3}$

Solution: $y' - \frac{3}{x}y = x^4 y^{1/3}$ $\quad (y' + p(x)y = g(x)y^n)$

$v = y^{1-n}$, $n = \frac{1}{3} \Rightarrow v = y^{1-\frac{1}{3}} \Rightarrow v = y^{2/3}$ //

$v' = \frac{2}{3}y^{-1/3} \cdot y' \Rightarrow y' = \frac{3}{2}y^{1/3}v'$ //

$y' - \frac{3}{x}y = x^4 y^{1/3}$

First multiplying with $y^{-1/3}$ both sides

$y^{-1/3}(y' - \frac{3}{x}y) = y^{-1/3}(x^4 y^{1/3})$

$y^{-1/3}y' - \frac{3}{x}y^{2/3} = x^4$ $\quad$ (put y')

$\cancel{y^{-1/3}} \cdot \frac{3}{2}\cancel{y^{1/3}}v' - \frac{3}{x}y^{2/3} = x^4$

$\frac{3}{2}v' - \frac{3}{x}y^{2/3} = x^4$ $\quad$ (put y)

$\frac{3}{2}v' - \frac{3}{x} \cdot v = x^4$ $\quad$ $(\cdot \frac{2}{3})$

$v' - \frac{2}{x}v = \frac{2}{3}x^4$ $\quad$ $(v' + p'(x)v = g(x))$ Linear dif. eq.

$\mu = e^{\int p'(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$ //

Hence $\mu = x^{-2}$ and $v' - \frac{2}{x}v = \frac{2}{3}x^4$

use the formula: $y = \frac{1}{\mu}\left(\int \mu\, g(x)\,dx + c\right)$

$$v = \frac{1}{x^{-2}}\left(\int x^{-2}\cdot\frac{2}{3}x^4\,dx + c\right)$$

$$v = x^2\left(\int \frac{2}{3}x^2\,dx + c\right)$$

$$v = x^2\cdot\left(\frac{2}{3}\cdot\frac{x^3}{3} + c\right)$$

$$v = \frac{2}{9}x^5 + cx^2$$

$$v = y^{2/3} \Rightarrow y^{2/3} = \frac{2}{9}x^5 + cx^2$$

$$y = \mp\sqrt[3]{\left(\frac{2}{9}x^5 + cx^2\right)^2} \quad //$$

Ex (Riccati) Recall that : $y' = A(x)y^2 + B(x)y + C(x)$: Riccatti

$y_1(x)$ is given

then $y = v + y_1$ $\quad (y = v(x) + y_1(x))$

$y' = v' + y_1'$ //

Bernoulli

Linear dif. eq.

$y' = y^2 - \frac{y}{x} - \frac{1}{x^2}$, $x > 0$, $y_1(x) = \frac{1}{x}$ and $y(1) = 2$. Solve this dif. equ.

Solution: $y' = y^2 - \frac{1}{x}y - \frac{1}{x^2}$

$(y' = A(x)y^2 + B(x)y + C(x))$

$y = v + y_1 \Rightarrow y = v + \frac{1}{x}$ // $\Rightarrow y' = v' - \frac{1}{x^2}$ //

$y' = y^2 - \frac{1}{x}y - \frac{1}{x^2} \Rightarrow$

$$v' - \cancel{\frac{1}{x^2}} = \left(v + \frac{1}{x}\right)^2 - \frac{1}{x}\cdot\left(v + \frac{1}{x}\right) - \cancel{\frac{1}{x^2}}$$

$$v' = v^2 + \cancel{\frac{1}{x^2}} + \frac{2v}{x} - \frac{v}{x} - \cancel{\frac{1}{x^2}}$$

$$v' = v^2 + \frac{v}{x} \Rightarrow v' - \frac{1}{x}v = v^2$$

(Bernoulli : $v' + p(x)v = g(x)v^n$)

$v' - \frac{1}{x} v = v^2$ (Bernoulli)

$v = y^{1-n}$, $n = 2$ Hence $w = v^{1-n}$, $n = 2$

$w = v^{1-2} \Rightarrow w = v^{-1}$ // $\Rightarrow w = \frac{1}{v} \Rightarrow v = \frac{1}{w}$

$w' = -1 v^{-2} . v' \Rightarrow v' = -v^2 w'$

$v' - \frac{1}{x} v = v^2$ (put v')

$-v^2 w' - \frac{1}{x} v = v^2$

$-v w' - \frac{1}{x} = v$ (put v)

$-\frac{1}{w} w' - \frac{1}{x} = \frac{1}{w}$ $(\cdot (-w))$

$w' + \frac{1}{x} w = -1$ (linear dif. eq.)

$\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ //

use the formula : $w = \frac{1}{\mu} \cdot \left(\int \mu \cdot g(x) dx + c \right)$

$w = \frac{1}{x} \left(\int x \cdot -1 \, dx + c \right) = \frac{1}{x} \cdot \left(-\frac{x^2}{2} + c \right) = -\frac{x}{2} + \frac{c}{x}$ //

$w = -\frac{x}{2} + \frac{c}{x}$ $\left(w = v^{-1} \Rightarrow v = \frac{1}{w} \right)$

$v = \frac{1}{-\frac{x}{2} + \frac{c}{x}}$ //

Ricatti : $y = v + y_1$

$y = \frac{1}{-\frac{x}{2} + \frac{c}{x}} + \frac{1}{x}$ //

And we know $y(1) = 2$ $\left(y = \dfrac{1}{-\frac{x}{2} + \frac{c}{x}} + \dfrac{1}{x} \right)$

$$2 = \frac{1}{-\frac{1}{2} + \frac{c}{1}} + 1 \Rightarrow c = \frac{3}{2}$$

$$\Rightarrow y = \frac{1}{-\frac{x}{2} + \frac{3}{2x}} + \frac{1}{x} \;//$$

Ex: (Riccati) $y' + 3y - y^2 = 2$, $y_1(x) = 1$. Solve this dif. equ.

Solution: $y' + 3y - y^2 = 2$

$y' = y^2 - 3y + 2$ $\quad (y' = \underset{1}{A(x)}y^2 + \underset{-3}{B(x)}y + \underset{2}{C(x)})$

$y = v + y_1 \Rightarrow y = v + 1 \Rightarrow y' = v'$

$y' = y^2 - 3y + 2 \Rightarrow v' = (v+1)^2 - 3(v+1) + 2$

$v' = v^2 + \cancel{1} + 2v - 3v - \cancel{3} + \cancel{2}$

$v' = v^2 - v$

$v' + v = v^2$ $\quad$ (Bernoulli : $v' + \underset{1}{p(x)}v = \underset{1}{g(x)}v^n$)

$\Rightarrow v' + v = v^2$ (Bernoulli with $n = 2$)

$v = y^{1-n}$, $n = 2$ $\Rightarrow w = v^{1-n}$, $n = 2$

$w = v^{1-2} \Rightarrow w = v^{-1} \Rightarrow w = \frac{1}{v} \Rightarrow v = \frac{1}{w}$

$w' = -1\, v^{-2} \cdot v' \Rightarrow v' = -v^2 \cdot w'$

$v' + v = v^2$ (put v')

$-v^2 w' + v = v^2$

$-v w' + 1 = v$ (put v)

$$-\frac{1}{w}w' + 1 = \frac{1}{w} \quad (\cdot(-w))$$

$$w' - w = 1 \quad \text{(lineer dif. equ.)}$$

$$\mu = e^{\int -1\,dx} = e^{-x}$$

use the formule: $w = \frac{1}{\mu}\left(\int \mu\, g(x)\,dx + c\right)$

$$w = \frac{1}{e^{-x}}\left(\int e^{-x} \cdot 1 \cdot dx + c\right)$$

$$w = \frac{1}{\cancel{e^{-x}}} \cdot \left(-\cancel{e^{-x}} + c\right)$$

$$w = -1 + ce^{x} \quad \left(w = v^{-1} \Rightarrow v = \frac{1}{w}\right)$$

$$v = \frac{1}{-1 + ce^{x}}$$

Riccatti: $y = v + y_1$

$$y = \frac{1}{-1+ce^{x}} + 1 \quad //$$