Ex: Solve $y'' - 3y' - 4y = 0$

Solution: $y'' - 3y' - 4y = 0$: second order, homogenous, constant coeffi.

The characteristic equation: $r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0 \Rightarrow r_1 = 4$

$r_2 = -1$

Hence $y_1 = e^{4x}$, $y_2 = e^{-x}$.

$$W(y_1, y_2) = \begin{vmatrix} e^{4x} & e^{-x} \\ 4e^{4x} & -e^{-x} \end{vmatrix} = -e^{3x} - 4e^{3x} = -5e^{3x} \neq 0$$

thus y_1 and y_2 are linearly independent, y_1 and y_2 are fundamental solutions.

$\Rightarrow y_h = c_1 y_1 + c_2 y_2$

$y_h = c_1 e^{4x} + c_2 e^{-x}$ //

Ex: Solve $y'' - 6y' + 9y = 0$

Solution: $y'' - 6y' + 9y = 0$: second order, homegous, constant coefficient

The characteristic polynomial: $r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0 \quad r_1, r_2 = 3$

Hence $y_1 = e^{3x}$, $y_2 = xe^{3x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + x.3e^{3x} \end{vmatrix} = e^{3x}(e^{3x} + 3xe^{3x}) - 3xe^{6x}$$

$$= e^{6x} + \cancel{3xe^{6x}} - \cancel{3xe^{6x}}$$

$$= e^{6x} \neq 0$$

Hence y_1 and y_2 are linearly independent, y_1 and y_2 are fundamental solutions. $\Rightarrow y_h = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 x e^{3x}$ //

Ex: Solve $y'' + 2y' + 5y = 0$

Solution: $y'' + 2y' + 5y = 0$: second order, homogenous, nonconstant coef.

characteristic polynomial: $r^2 + 2r + 5 = 0$

$$\Delta = 4 - 4.5.1 = -16$$

$$r_1 = \frac{-2 + \sqrt{-16}}{2} = -1 + 2i$$

$$r_2 = \frac{-2 - \sqrt{-16}}{2} = -1 - 2i$$

$\lambda = -1$

$\mu = 2$

$$y_1 = e^{\lambda x} \cos \mu x \qquad\qquad y_1 = e^{-x} \cos 2x$$

$$\Rightarrow$$

$$y_2 = e^{\lambda x} \sin \mu x \qquad\qquad y_2 = e^{-x} \sin 2x$$

$W(y_1, y_2) = \dots \neq 0$. Hence y_1 and y_2 are linearly independent

So y_1 and y_2 are fundamental solutions

Hence $y_h = c_1 y_1 + c_2 y_2$

$$y_h = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$$

$$y_h = e^{-x} (c_1 \cos 2x + c_2 \sin 2x) \;//$$

Ex: Solve IVP: $y'' - 5y' + 6y = 0$, $y(0) = 3$, $y'(0) = 5$

Solution: $y'' - 5y' + 6y = 0$: second order, homogenous, constant coeff.

The characteristic equation: $r^2 - 5r + 6 = 0$

$$(r-3)(r-2) = 0$$

$$r_1 = 3 \; ; \; r_2 = 2$$

Hence $y_1 = e^{3x}$, $y_2 = e^{2x}$

$W(y_1, y_2) = \dots \neq 0$.

Hence y_1 and y_2 are fundamental solutions.

$y_h = c_1 y_1 + c_2 y_2$

$y_h = c_1 e^{3x} + c_2 e^{2x}$ $\qquad (y' = 3c_1 e^{3x} + 2c_2 e^{2x})$

$y(0) = 3 \Rightarrow c_1 e^0 + c_2 e^0 = 3 \Rightarrow c_1 + c_2 = 3$

$y'(0) = 5 \Rightarrow 3c_1 e^0 + 2c_2 e^0 = 5 \Rightarrow 3c_1 + 2c_2 = 5$

$$-c_2 = -4$$

$$c_2 = 4$$

$$c_1 = -1$$

$\Rightarrow y_h = -e^{3x} + 4e^{2x}$ //

Ex: Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

Solution: $y'' - 8y + 16y = 0$: second order, homogenous, constant coeff.

characteristic polyn: $r^2 - 8r + 16 = 0$

$$(r-4)^2 = 0 \qquad r_1, r_2 = 4$$

$y_1 = e^{4x}$, $y_2 = xe^{4x}$

$W(y_1, y_2) \neq 0 \Rightarrow y_1$ and y_2 are fundamental solution

$y_h = c_1 e^{4x} + c_2 x e^{4x}$ //

Ex: Solve $y'' + 4y' + 5y = 0$

Solution: $y'' + 4y' + 5y = 0$: second order, homogenous, non-const. coef.

characteristic equ. : $r^2 + 4r + 5 = 0$

$$\Delta = 16 - 4.5.1 = -4$$

$r_1 = \frac{-4 + \sqrt{-4}}{2} = -2 + i$

$r_2 = \frac{-4 - \sqrt{-4}}{2} = -2 - i$

$\lambda = -2$

$\mu = 1$

$y_1 = e^{\lambda x} \cos \mu x = e^{-2x} \cos x$

$y_2 = e^{\lambda x} \sin \mu x = e^{-2x} \sin x$

$W(y_1, y_2) \neq 0$, Hence y_1 and y_2 are fundamental solutions

$y_h = c_1 y_1 + c_2 y_2$

$y_h = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$

$= e^{-2x} (c_1 \cos x + c_2 \sin x)$ //

Ex: $y^{(3)} + 25y' = 0$

Solution: $y^{(3)} + 25y' = 0$: higher order, homogenous, nonconstant coef.

characteristic eq: $r^3 + 25r = 0$

$r(r^2 + 25) = 0$ → $\lambda = 0$, $\mu = 5$

$r_1 = 0 \quad r_2 = 5i \quad r_3 = -5i$

$y_1 = e^{0x} = 1$

$y_2 = e^{0x} \cos 5x = \cos 5x$

$y_3 = e^{0x} \sin 5x = \sin 5x$

$y_h = c_1 y_1 + c_2 y_2 + c_3 y_3$

$y_h = c_1 + c_2 \cos 5x + c_3 \sin 5x$ //

Ex: Solve $y^{(5)} - 3y^{(4)} + 3y^{(3)} - y^{(2)} = 0$

Solution: $y^{(5)} - 3y^{(4)} + 3y^{(3)} - y^{(2)} = 0$: higher order, homogenous, constant coeff.

characteristic equ: $r^5 - 3r^4 + 3r^3 - r^2 = 0$

$r^2(r^3 - 3r^2 - 3r - 1) = 0$

$r^2(r-1)^3 = 0$

$r_1 = r_2 = 0 \qquad r_3 = r_4 = r_5 = 1$

$y_1 = e^{0x} = 1$

$y_2 = xe^{0x} = x$

$y_3 = e^x$

$y_4 = xe^x$

$y_5 = x^2e^x$

$y_h = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 + c_5 y_5$

$y_h = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x$ //

Ex: Solve $y^{(5)} - 6y^{(4)} + 9y^{(3)} = 0$

Solution: $y^{(5)} - 6y^{(4)} + 9y^{(3)} = 0$: higher order, homogeneous, constant coef.

characteristic equ: $r^5 - 6r^4 + 9r^3 = 0$

$$r^3(r^2 - 6r + 9) = 0$$

$$r^3(r-3)(r-3) = 0$$

$$r_1 = r_2 = r_3 = 0 \qquad r_4 = r_5 = 3$$

$\Rightarrow y_1 = e^{0x} = 1$

$y_2 = xe^{0x} = x$

$y_3 = x^2e^{0x} = x^2$

$y_4 = e^{3x} = e^{3x}$

$y_5 = xe^{3x} = xe^{3x}$

$\Rightarrow y_h = c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + c_5y_5$

$y_h = c_1 + c_2x + c_3x^2 + c_4e^{3x} + c_5xe^{3x}$ //

Ex: Solve $y^{(4)} - y = 0$

Solution: $y^{(4)} - y = 0$: higher order, homogeneous, constant coeffic.

characteristic polyn: $r^4 - 1 = 0$

$$(r^2-1)(r^2+1) = 0$$

$$(r-1)(r+1)(r^2+1) = 0$$

$r_1 = 1 \qquad r_3 = i$

$r_2 = -1 \qquad r_4 = -i$ $\Big\}$ $\lambda = 0$, $\mu = 1$

$y_1 = e^x$

$y_2 = e^{-x}$

$y_3 = e^{0x}\cos x = \cos x$

$y_4 = e^{0x}\sin x = \sin x$

$y_h = c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4$

$y_h = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$ //

Ex: $y''' + y'' - 6y' + 4y = 0$

Solution: $y''' + y'' - 6y' + 4y = 0$: higher order, homogenous, constant coeff.

characteristic equation : $r^3 + r^2 - 6r + 4 = 0$

if $r = 0$ then $4 = 0$ ↯

if $r = 1$ then $0 = 0$ ✓ $(r-1)$

$$
\begin{array}{r|l}
r^3 + r^2 - 6r + 4 & r - 1 \\
\hline
-\;\; r^3 - r^2 \qquad\qquad & r^2 + 2r - 4 \\
\hline
2r^2 - 6r + 4 & \\
-\;\; 2r^2 - 2r \qquad & \\
\hline
-4r + 4 & \\
-4r + 4 & \\
\hline
0 &
\end{array}
$$

$r^3 + r^2 - 6r + 4 = (r-1)(r^2 + 2r - 4) = 0$

$r_1 = 1$

$\Delta = 4 - 4 \cdot (-4) = 20$

$r_2 = \dfrac{-2 + \sqrt{20}}{2} = -1 + \sqrt{5}$

$r_3 = \dfrac{-2 - \sqrt{20}}{2} = -1 - \sqrt{5}$

$r_1 = 1 \Rightarrow y_1 = e^x$

$r_2 = -1 + \sqrt{5} \Rightarrow y_2 = e^{-x} \cos\sqrt{5}x$

$r_3 = -1 - \sqrt{5} \Rightarrow y_3 = e^{-x} \sin\sqrt{5}x$

$y_h = c_1 e^x + c_2 e^{-x} \cos\sqrt{5}x + c_3 e^{-x} \sin\sqrt{5}x$

$y_h = c_1 e^x + e^{-x}(c_2 \cos\sqrt{5}x + c_3 \sin\sqrt{5}x)$ //

Ex: Solve $2y''' - 6y'' - 5y' + 15y = 0$

Solution: $2y''' - 6y'' - 5y' + 15y = 0$: higher order, homogenous, constant coef.

characteristic equation: $2r^3 - 6r^2 - 5r + 15 = 0$

$$2r^2(r-3) - 5(r-3) = 0$$

$$(r-3)(2r^2-5) = 0$$

$r_1 = 3$

$$2r^2 - 5 = 0$$
$$2r^2 = 5$$
$$r^2 = \frac{5}{2}$$
$$r = \mp \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}.\cancel{\sqrt{2}}}{\sqrt{2}.\cancel{\sqrt{2}}} = \mp \frac{\sqrt{10}}{2}$$

$r_2 = \frac{\sqrt{10}}{2}$ $\quad$ $r_3 = -\frac{\sqrt{10}}{2}$

$\Rightarrow y_1 = e^{3x}$

$y_2 = e^{\frac{\sqrt{10}}{2}x}$

$y_3 = e^{-\frac{\sqrt{10}}{2}x}$

$$\Rightarrow y_h = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 e^{3x} + c_2 e^{\frac{\sqrt{10}}{2}x} + c_3 e^{-\frac{\sqrt{10}}{2}x}$$

Ex: Solve $y''' + 8y'' + 12y' = 0$: higher order, homogenous, constant coef.

Solution: characteristic poly: $r^3 + 8r^2 + 12r = 0$

$$r(r^2 + 8r + 12) = 0$$

$$r(r+6)(r+2) = 0 \Rightarrow \begin{array}{l} r_1 = 0 \\ r_2 = -6 \\ r_3 = -2 \end{array}$$

$y_1 = e^{0x} = 1$

$y_2 = e^{-6x}$

$y_3 = e^{-2x}$

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$y = c_1 + c_2 e^{-6x} + c_3 e^{-2x}$$

Ex: Solve $y^{(4)}+y'''-7y''-y'+6y=0$. : higher order, homogenous, constant coef.

Solution: characteristic poly: $r^4+r^3-7r^2-r+6=0$

$r=0 \Rightarrow 0+0-0-0+6=0$ ↯ so $r\neq 0$

$r=1 \Rightarrow \cancel{1}+\cancel{1}-7-\cancel{1}+6=0$ ✓ so $r=1$ $(r-1)$

$$
\begin{array}{rl|l}
 & r^4+r^3-7r^2-r+6 & r-1 \\
- & r^4-r^3 & r^3+2r^2-5r-6 \\
\hline
 & 2r^3-7r^2-r+6 & \\
- & 2r^3-2r^2 & \\
\hline
 & -5r^2-r+6 & \\
- & -5r^2+5r & \\
\hline
 & -6r+6 & \\
 & -6r+6 & \\
\hline
 & 0 &
\end{array}
$$

r^3+2r^2-5r-6

$r=0 \Rightarrow 0+0-0-6=0$ ↯ so $r=0$

$r=1 \Rightarrow 1+2-5-6=0$ ↯ so $r\neq 1$

$r=-1 \Rightarrow \cancel{-1}+\cancel{2}+\cancel{5}\cancel{-6}=0$ ✓ so $r=-1$ $(r+1)$

$$
\begin{array}{rl|l}
 & r^3+2r^2-5r-6 & r+1 \\
- & r^3+r^2 & r^2+r-6 \\
\hline
 & r^2-5r-6 & \rightarrow (r+3)(r-2) \\
- & r^2+r & \\
\hline
 & -6r-6 & \\
 & -6r-6 & \\
\hline
 & 0 &
\end{array}
$$

$\Rightarrow r^4+r^3-7r^2-r+6=(r-1)(r+1)(r+3)(r-2)$

$r_1=1 \quad r_2=-1 \quad r_3=-3 \quad r_4=2$

$y_1 = e^{x}$

$y_2 = e^{-x}$

$y_3 = e^{-3x}$

$y_4 = e^{2x}$

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-3x} + c_4 e^{2x}$$ //