Example. $xy' + y^2 - 2y = 4x^2 - 2x$.

$$y = ax + b$$

$$ax + a^2x^2 + 2abx + b^2 - 2ax - 2b \equiv 4x^2 - 2x$$

$$\begin{cases} a^2 = 4 \\ a + 2ab - 2a = -2 \\ b^2 - 2b = 0 \end{cases} \qquad a = 2,\ b = 0$$

$$y_1 = 2x$$

$$y = 2x + \frac{1}{z} \qquad y' = 2 - \frac{z'}{z^2}$$

$$-\frac{2z'x}{z^2} = -\frac{4x}{z} - \frac{1}{z^2} + \frac{2}{z} \qquad z' + \left(\frac{1}{x} - 2\right)z = \frac{1}{2x}$$

$$z = -\frac{1}{4x} + \frac{C}{x}e^{2x} \qquad z = \frac{1}{y - 2x},$$

$$\frac{1}{y - 2x} = \frac{4Ce^{2x} - 1}{4x} \qquad y = 2x + \frac{4x}{Ce^{2x} - 1}.$$

Method of Integrating Factors.

Example.
Find the general solution to

$$(3x^2y^2 - 3y^2)dx + (2x^3y - 6xy + 3y^2)dy = 0.$$

Step 1: Check to see if $M_y = N_x$.

$$M = 3x^2y^2 - 3y^2,\ N = 2x^3y - 6xy + 3y^2$$

$$M_y = 6x^2y - 6y,\ N_x = 6x^2y - 6y$$

So, it is exact.
Then,

$$f = \int M dx + g(y) = x^3y^2 - 3xy^2 + g(y)$$

$$f_y = N = 2x^3y - 6xy + 3y^2 = 2x^3y - 6xy + g'(y)$$

$$3y^2 = g'(y),\ g(y) = y^3$$

So, we get

$$x^3y^2 - 3xy^2 + y^3 = C$$

as the general solution.

Suppose the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is *not* exact and that $\mu(x, y)$ is an integrating factor of it. Then the equation

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0$$

is exact. Now using the criterion for exactness, the equation is exact if and only if

$$\frac{\partial}{\partial y}[\mu(x, y)M(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)N(x, y)].$$

This condition reduces to

$$N(x, y)\frac{\partial \mu(x, y)}{\partial x} - M(x, y)\frac{\partial \mu(x, y)}{\partial y} = \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right]\mu(x, y).$$

Here M and N are known functions of x and y, but μ is an unknown function of x and y that we are trying to determine. Thus we write the preceding condition in the form

$$N(x, y)\frac{\partial \mu}{\partial x} - M(x, y)\frac{\partial \mu}{\partial y} = \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right]\mu.$$

This equation is a partial differential equation for the general integrating factor μ, and we are in no position to attempt to solve such an equation. Let us instead attempt to determine integrating factors of certain special types. But what special types might we consider? Let us recall that the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

always possesses the integrating factor $e^{\int P(x)\,dx}$, which depends only upon x. Perhaps other equations also have integrating factors that depend only upon x. We therefore multiply the equation by $\mu(x)$, where μ depends upon x alone. We obtain

$$\mu(x)M(x, y)\,dx + \mu(x)N(x, y)\,dy = 0.$$

This is exact if and only if

$$\frac{\partial}{\partial y}[\mu(x)M(x, y)] = \frac{\partial}{\partial x}[\mu(x)N(x, y)].$$

Now M and N are known functions of both x and y, but here the intergrating factor μ depends only upon x. Thus the above condition reduces to

$$\mu(x)\frac{\partial M(x, y)}{\partial y} = \mu(x)\frac{\partial N(x, y)}{\partial x} + N(x, y)\frac{d\mu(x)}{dx}$$

or

$$\frac{d\mu(x)}{\mu(x)} = \frac{1}{N(x, y)}\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right]dx.$$

If

$$\frac{1}{N(x, y)}\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right]$$

involves the variable y, this equation then involves two dependent variables and we again have difficulties. However, if

$$\frac{1}{N(x, y)}\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right]$$

depends upon x only, the equation is a separated ordinary equation in the single independent variable x and the single dependent variable μ. In this case we may integrate to obtain the integrating factor

$$\mu(x) = \exp\left\{\int \frac{1}{N(x, y)}\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right] dx\right\}.$$

In like manner, if

$$\frac{1}{M(x, y)}\left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}\right]$$

depends upon y only, then we may obtain an integrating factor that depends only on y.

We summarize these observations in the following theorem.

THEOREM.

Consider the differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0.$$

If

$$\frac{1}{N(x, y)}\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right]$$

depends upon x only, then

$$\exp\left\{\int \frac{1}{N(x, y)}\left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x}\right] dx\right\}$$

is an integrating factor of the equation. If

$$\frac{1}{M(x, y)}\left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}\right]$$

depends upon y only, then

$$\exp\left\{\int \frac{1}{M(x, y)}\left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}\right] dy\right\}$$

is an integrating factor of the equation.

Example:
Consider the equation

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

$$M = 3xy + y^2, \;\; N = x^2 + xy$$

Step 1: Check if exact

$$M_y - N_x = 3x + 2y - 2x - y = x + y$$

So, not exact.
Step 2: Compute

$$\frac{M_y - N_x}{N} = \frac{x + y}{x^2 + xy} = \frac{1}{x}$$

So, get integrating factor of the form

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

So,

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

is exact.

$$f_x = M = 3x^2y + xy^2, f = x^3y + x^2y^2/2 + g(y)$$

$$f(x, y) = x^3y + x^2y^2/2 = C$$

is the general solution.

Example.

Consider the differential equation

$$(2x^2 + y)\, dx + (x^2y - x)\, dy = 0.$$

Let us first observe that this equation is *not* exact, separable, homogeneous, linear, or Bernoulli. Let us then see if Theorem applies. Here $M(x, y) = 2x^2 + y$, and $N(x, y) = x^2y - x$

$$\frac{1}{x^2y - x}[1 - (2xy - 1)] = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x}.$$

This depends upon x only, and so

$$\exp\left(-\int \frac{2}{x}\, dx\right) = \exp(-2\ln|x|) = \frac{1}{x^2}$$

is an integrating factor of the equation. Multiplying by this integrating factor, we obtain the equation

$$\left(2+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0.$$

We may verify that this equation is indeed exact and that the solution is

$$2x+\frac{y^2}{2}-\frac{y}{x}=c.$$

Example. $(1-x^2y)\,dx+x^2(y-x)\,dy=0.$

$M=1-x^2y,\ N=x^2y-x^3$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-x^2-(2xy-3x^2)=2x(x-y)\neq 0,$$

$$\frac{d\ln\mu}{dx}=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{2x(x-y)}{x^2(y-x)}=-\frac{2}{x}$$

$$\frac{d\ln\mu}{dx}=-\frac{2}{x}\qquad d\ln\mu=-\frac{2\,dx}{x}\qquad \mu=\frac{1}{x^2}$$

$$\left(\frac{1}{x^2}-y\right)dx+(y-x)\,dy=0,\qquad \frac{dx}{x^2}+y\,dy-(y\,dx+x\,dy)=0.$$

$$d\left(-\frac{1}{x}\right)+\frac{1}{2}d(y^2)-d(xy)=0,\qquad -\frac{1}{x}+\frac{1}{2}y^2-xy=C.$$

Example. $\left(1-\frac{x}{y}\right)dx+\left(2xy+\frac{x}{y}+\frac{x^2}{y^2}\right)dy=0.$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=\frac{x}{y^2}-2y-\frac{1}{y}-\frac{2x}{y^2}=-\left(2y+\frac{1}{y}+\frac{x}{y^2}\right)$$

$$\frac{d\ln\mu}{dx}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=-\frac{1}{x},\qquad \mu=\frac{1}{x}$$

$$\left(\frac{1}{x}-\frac{1}{y}\right)dx+\left(2y+\frac{1}{y}+\frac{x}{y^2}\right)dy=0.$$

$$\frac{dx}{x}+\left(\frac{1}{y}+2y\right)dy-\frac{ydx-xdy}{y^2}=0,$$

$$d\left(\ln|x|+\ln|y|+y^2-\frac{x}{y}\right)=0,\quad \ln|x|+\ln|y|+y^2-\frac{x}{y}=C.$$

Example. $(x^2+y^2+x)dx+y\,dy=0.$

$$P(x,y)=x^2+y^2+x,\quad Q(x,y)=y$$

$$\frac{\partial P}{\partial y}=2y\neq\frac{\partial Q}{\partial x}=0,$$

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=2,$$

$$\frac{1}{\mu}\frac{d\mu}{dx}=2\quad\Rightarrow\quad\mu(x)=C_1e^{2x}.$$

$$C_1 = 1, \quad \mu(x) = e^{2x}$$

$$e^{2x}(x^2 + y^2 + x)\,dx + e^{2x} y\,dy = 0.$$

$U(x, y)$:

$$\frac{\partial U}{\partial x} = e^{2x}(x^2 + y^2 + x) \;\Rightarrow\; U(x, y) = \int e^{2x}(x^2 + y^2 + x)\,dx + \varphi(y).$$

$$U(x, y) = \frac{1}{2} e^{2x}(x^2 + y^2) + \varphi(y)$$

$$\frac{\partial U}{\partial y} = e^{2x} y + \varphi'(y) = Q(x, y) = e^{2x} y \;\Rightarrow\; \varphi'(y) = 0 \;\Rightarrow\; \varphi(y) = C_1.$$

$$C_1 = 0 \;\Rightarrow\; \quad U(x, y) = \frac{1}{2} e^{2x}(x^2 + y^2),$$

$$\frac{1}{2} e^{2x}(x^2 + y^2) = C \quad \Rightarrow \quad e^{2x}(x^2 + y^2) = C.$$

Example.

$$\left(2xy + x^2 y + \frac{y^3}{3}\right) dx + (x^2 + y^2)\,dy = 0.$$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2x + x^2 + y^2 - 2x}{x^2 + y^2} = 1.$$

$$\frac{d \ln \mu}{dx} = 1, \quad \mu = e^x.$$

$$e^x\left(2xy + x^2 y + \frac{y^3}{3}\right) dx + e^x (x^2 + y^2)\,dy = 0$$

$$U = \int e^x\left(2xy + x^2y + \frac{y^3}{3}\right)dx + \varphi(y) =$$

$$= y\int e^x(2x + x^2)\,dx + \frac{y^3}{3}e^x + \varphi(y) = ye^x\left(x^2 + \frac{y^2}{3}\right) + \varphi(y).$$

$$e^x(x^2 + y^2) + \varphi'(y) = e^x(x^2 + y^2),$$

$$\varphi'(y) = 0,$$

$$ye^x\left(x^2 + \frac{y^2}{3}\right) = C.$$

Example. $2x^2 + y + (x^2y - x)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2}{x}$$

so that there is an integrating factor μ which is a function of x only which satisfies $\mu' = -2\mu/x$. Hence $\mu = 1/x^2$ is an integrating factor and $2 + y/x^2 + (y - 1/x)y' = 0$ is an exact equation whose general solution is $2x - y/x + y^2/2 = C$ or $2x^2 - y + xy^2/2 = Cx$.

Example. $3y(x+1)dx + x(3x+4)dy = 0 \qquad \mu(x,y) = x^a y^b$.

$$3x^a y^{b+1}(x+1)dx + x^{a+1}y^b(3x+4)dy = 0,$$

$$\frac{\partial\left(3x^a y^{b+1}(x+1)\right)}{\partial y} = \frac{\partial\left(x^{a+1}y^b(3x+4)\right)}{\partial x}$$

$$3(b+1)x^a y^b(x+1) = \left(3(a+2)x^{a+1} + 4(a+1)x^a\right)y^b$$

$$3(b+1)(x+1) = 3(a+2)x + 4(a+1)$$

$$\begin{cases} 3(b+1) = 3(a+2) \\ 3(b+1) = 4(a+1) \end{cases} : \qquad a = 2,\ b = 3.$$

$$3x^2y^4(x+1)dx + x^3y^3(3x+4)dy = 0$$

Example. $(x^2 - \sin^2 y)dx + x\sin 2y\,dy = 0$.

$$M(x,y) = x^2 - \sin^2 y, \quad N(x,y) = x\sin 2y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\sin y\cos y - \sin 2y}{x\sin 2y} = -\frac{2}{x}.$$

$$\frac{d\ln\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x(x-y)}{x^2(y-x)} = -\frac{2}{x}$$

$$\frac{d\ln\mu}{dx} = -\frac{2}{x}, \quad d\ln\mu = -\frac{2\,dx}{x}, \quad \mu = \frac{1}{x^2}$$

$$\left(1-\frac{\sin^2 y}{x^2}\right)dx+\frac{\sin 2y}{x}dy=0$$

$$dx+\frac{x\sin 2y\,dy-\sin^2 y\,dx}{x^2}=0.$$

$$d\left(x+\frac{\sin^2 y}{x}\right)=0,\ x+\frac{\sin^2 y}{x}=C,\ x^2+\sin^2 y=Cx.$$

Example.

$$\frac{dy}{dx}-y\tan x=\cos x$$

$$\mu(x)=e^{-\int \tan x\,dx}=\cos x$$

$$\cos x\,dy-y\sin x\,dx-\cos^2 x\,dx=0,$$

$$y\cos x-\int\cos^2 x\,dx=C,$$

$$y\cos x-\frac{x}{2}-\frac{1}{2}\sin x\cos x=C$$

Example. *Solve the differential equation*

$$(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$$

Solution

$M(x,y) = 3xy + y^2$, $N(x,y) = x^2 + xy$

$M_y = 3x + 2y$ and $N_x = 2x + y$ Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, therefore the differential equation is not exact.

$$M\mu_y - N\mu_x = \mu(N_x - M_y).$$

Let μ be a function of x alone, that is, $\mu = \mu(x)$.

$$\xi(x) = \frac{M_y - N_x}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}$$

$$\mu(x) = e^{\int \xi(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x\,.$$

Multiplying $\mu(x) = x$ through the differential equation we obtain:

$$(3x^2y + xy^2) + (x^3 + x^2y)\frac{dy}{dx} = 0$$

which is a new equation in same form i.e.: $M(x,y) + N(x,y)y' = 0$

$$M(x,y) = (3x^2y + xy^2); \;\; N(x,y) = (x^3 + x^2y)$$

$$M_y = 3x^2 + 2xy; \;\; N_x = 3x^2 + 2xy$$

which is exact. Let $f_x = 3x^2y + xy^2$. Integrating:

$$f(x,y) = \int (3x^2y + xy^2)dx = x^3y + \frac{1}{2}x^2y^2 + \psi(y)$$

Find the derivative w.r.t y and equating with $N(x,y)$ we obtain

$$\implies f_y = x^3 + x^2y + \psi'(y) = x^3 + x^2y$$

$$\psi'(y) = 0 \implies \psi(y) = c.$$

Finally $x^3y + \frac{1}{2}x^2y^2 = c$

Example. $y + (2x - ye^y)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y}$$

so that there is an integrating factor which is a function of y only which satisfies $\mu' = 1/y$. Hence y is an integrating factor and $y^2 + (2xy - y^2e^y)y' = 0$ is an exact equation with general solution $xy^2 + (-y^2 + 2y - 2)e^y = C$.