CSE 211: Discrete Mathematics **(Due: 30/10/22)**

Homework #1

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Course Policy: Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- It is not a group homework. Do not share your answers to anyone in any circumstance. Any cheating means at least -100 for both sides.
- Do not take any information from Internet.
- No late homework will be accepted.
- For any questions about the homework, send an email to bkarakas2018@gtu.edu.tr
- Use LaTeX. You can work on the tex file shared with you in the assignment document.
- Submit both the tex and pdf files into Homework1. Name of the files should be "*SurnameName_Id.tex*" and "*SurnameName_Id.pdf*".

Problem 1: Sets (3+3+3+3+3=15 points)

Which of the following sets are equal? Show your work step by step.

(a) {t : t is a root of x^2 − 6x + 8 = 0}

(b) {y : y is a real number in the closed interval [2, 3]}

(c) {4, 2, 5, 4}

(d) {4, 5, 7, 2} - {5, 7}

(e) {q: q is either the number of sides of a rectangle or the number of digits in any integer between 11 and 99}

(Solution)

(a) Root of the set are

$$\begin{aligned} x^2 - 6x + 8 = 0 \quad &\Rightarrow (x-2)(x-4) = 0 \\ &\Rightarrow x = 2 \wedge x = 4 \end{aligned}$$

(b) $\{y : y \text{ is a real number in the closed interval } [2,3]\} = [2,3] \subset \mathbb{R}$

(c) Since it does not matter if an element of a set is listed more than once, {4, 2, 5, 4} = $\{2,4,5\}$

(d) {4, 5, 7, 2} − {5, 7} = $\{2,4\}$

(e) Because the number of sides of a rectangle is 4 and the number of digits in any integer between 11 and 99 is 2,
$\{q : q$ is either the number of sides of a rectangle or the number of digits in any integer between 11 and 99$\} = \{q : q = 2 \vee q = 4\} = \{2,4\}$

In conclusion, the sets in (a), (d) and (e) have same elements $\{2, 4\}$. So, they are equal.

Problem 2: Cardinality of Sets (2+2+2+2=8 points)

What is the cardinality of each of these sets? Explain your answers.

(a) $\{\emptyset\}$

(b) $\{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$

(d) $\{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}$

(Solution)

(a) $\emptyset \in \{\emptyset\} \Rightarrow$ the cardinality is 1.

(b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$ and $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\} \Rightarrow$ the cardinality is 2

(c) $\emptyset \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ and $\{\emptyset, \{\emptyset\}\} \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\} \Rightarrow$ the cardinality is 2

(d) $\emptyset \in \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}$ and $\{\emptyset, \{\emptyset, \{\emptyset\}\}\} \in \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\} \Rightarrow$ the cardinality is 2

Problem 3: Cartesian Product of Sets (15 points)

Explain why (A × B) × (C × D) and A × (B × C) × D are not the same.

(Solution)

$(A \times B) \times (C \times D) = \{((a, b), (c, d)) : a \in A, b \in B, c \in C \text{ and } d \in D\}$
Elements of the set are ordered pair of ordered pairs.

$A \times (B \times C) \times D = \{(a, (b, c), d) : a \in A, b \in B, c \in C \text{ and } d \in D\}$
Elements of the set are ordered triple with an ordered pair in the middle.

For $a \in A, b \in B, c \in C$ and $d \in D$,

$$((a, b), (c, d)) \neq (a, (b, c), d)$$

That means the elements of these two sets are not the same, so sets also not.

Problem 4: Cartesian Product of Sets in Algorithms (25 points)

Let A, B and C be sets which have different cardinalities. Let (p, q, r) be each triple of A $\times$ B $\times$ C where p $\in$ A, q $\in$ B and r $\in$ C. Design an algorithm which finds all the triples that are satisfying the criteria: p $\leq$ q and q $\geq$ r. Write the pseudo code of the algorithm in your solution.

For example: Let the set A, B and C be as A = { 3, 5, 7 }, B = { 3, 6 } and C = { 4, 6, 9 }. Then the output should be : { (3, 6, 4), (3, 6, 6), (5, 6, 4), (5, 6, 6) }.

(Note: Assume that you have sets of A, B, C as an input argument.)

(Solution)

Algorithm 1: Pseudo Code of Your Algorithm

```
Input: The sets of A, B, C
Output: The set D
for i = 1 to length of A do
    for j = 1 to length of B do
        p = A[i]
        q = B[j]
        if p ≤ q then
            for k = 1 to length of C do
                r = C[k]
                if r ≤ q then
                    Put (p, q, r) into D
                else
            end
        else
    end
end
return D
```

Problem 5: Functions (16 points)

If f and f ∘ g are one-to-one, does it follow that g is one-to-one? Justify your answer.

(Solution)

Let $f : B \to C$ and $g : A \to B$. Then $fog : A \to C$.

f is one-to-one $\iff$ $\forall b_1, b_2 \in B, f(b_1) = f(b_2) \Rightarrow b1 = b2$

fog is one-to-one $\iff$ $\forall a_1, a_2 \in A, fog(a_1) = fog(a_2) \Rightarrow a_1 = a_2$

$$
\begin{array}{llll}
(1) & fog(a_1) = fog(a_2) & \Rightarrow f(g(a_1)) = f(g(a_2)) & \text{by definition of composition} \\
 & & \Rightarrow g(a_1) = g(a_2) & \text{since } f \text{ is } 1-1 \\
(2) & fog(a_1) = fog(a_2) & \Rightarrow a_1 = a_2 & \text{since } fog \text{ is } 1-1
\end{array}
$$

Is g one-to-one? In other words, for all $a_1, a_2 \in A$, are a_1 and a_2 equal when $g(a_1) = g(a_2)$?

Assume g is not $1-1$. That means,

$\exists a_1, a_2 \in A, g(a_1) = g(a_2)$ but $a_1 \neq a_2$.

However, by (1) and (2), both $g(a_1) = g(a_2)$ and $a_1 = a_2$ are true for all $a_1, a_2 \in A$. So, there is a contradiction, which means our assumption is wrong. Therefore, g is one-to-one.

Problem 6: Functions (7+7+7=21 points)

Determine whether the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(a) $f(m, n) = 2m - n$

(b) $f(m, n) = m^2 - n^2$

(c) $f(m, n) =\mid m \mid - \mid n \mid$

(Solution)

f is onto $\iff \forall z \in \mathbb{Z}, \quad z = f(x, y) \quad \therefore \exists (x, y) \in \mathbb{Z} \times \mathbb{Z}$

(a) $f(m, n) = 2m - n$

Let $m \in \mathbb{Z}$.

$$
\begin{array}{lcll}
z = f(m, n) \in \mathbb{Z} & \Rightarrow & z = 2m - n \in \mathbb{Z} & \\
 & \Rightarrow & n = 2m - z \in \mathbb{Z} & \text{since } m, z \in \mathbb{Z} \\
 & \Rightarrow & \text{for } (x, y) = (m, 2m - z), z = f(x, y) & \\
 & \Rightarrow & f \text{ is onto} &
\end{array}
$$

(b) $f(m,n) = m^2 - n^2$

$$\begin{aligned} z = f(m,n) \in \mathbb{Z} &\Rightarrow \quad z = m^2 - n^2 \in \mathbb{Z} \\ &\Rightarrow \quad z = (m-n)(m+n) \in \mathbb{Z} \\ &\Rightarrow \quad (m-n),(m+n) \in \mathbb{Z} \quad \oplus \quad (m-n),(m+n) \notin \mathbb{Z} \end{aligned}$$

So, check both $(m-n),(m+n) \in \mathbb{Z}$ and $(m-n),(m+n) \notin \mathbb{Z}$ situations:

If $(m-n),(m+n) \notin \mathbb{Z}$, then either $m \in \mathbb{Z}$ and $n \notin \mathbb{Z}$ or $m \notin \mathbb{Z}$ and $n \in \mathbb{Z}$. Hence, there is no $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ such that $z = f(m,n)$.

If $(m-n),(m+n) \in \mathbb{Z}$, is f onto?

Let $z = 2 \in \mathbb{Z}$. Then, there are 4 cases: $z = 1 * 2$, $z = 2 * 1$, $z = (-1) * (-2)$ and $z = (-2) * (-1)$.

1. $$\begin{aligned} z = 1 * 2 &\Rightarrow \quad m - n = 1 \quad \wedge \quad m + n = 2 \\ &\Rightarrow \quad m = 3/2 \quad \wedge \quad n = 1/2 \\ &\Rightarrow \quad m, n \notin \mathbb{Z} \end{aligned}$$

2. $$\begin{aligned} z = 2 * 1 &\Rightarrow \quad m - n = 2 \quad \wedge \quad m + n = 1 \\ &\Rightarrow \quad m = 3/2 \quad \wedge \quad n = -1/2 \\ &\Rightarrow \quad m, n \notin \mathbb{Z} \end{aligned}$$

3. $$\begin{aligned} z = (-1) * (-2) &\Rightarrow \quad m - n = -1 \quad \wedge \quad m + n = -2 \\ &\Rightarrow \quad m = -3/2 \quad \wedge \quad n = -1/2 \\ &\Rightarrow \quad m, n \notin \mathbb{Z} \end{aligned}$$

4. $$\begin{aligned} z = (-2) * (-1) &\Rightarrow \quad m - n = -2 \quad \wedge \quad m + n = -1 \\ &\Rightarrow \quad m = -3/2 \quad \wedge \quad n = 1/2 \\ &\Rightarrow \quad m, n \notin \mathbb{Z} \end{aligned}$$

These cases imply that, for $z = 2 \in \mathbb{Z}$, there is no pair $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ such that $z = f(m,n)$. Hence, f is not onto when $(m-n),(m+n) \in \mathbb{Z}$.

In summary, for both possible situations $(m-n),(m+n) \in \mathbb{Z}$ and $(m-n),(m+n) \notin \mathbb{Z}$, f is not onto.

(c) $f(m,n) = |m| - |n|$

Since there are taking absolute value operation, we should handle it in two cases which are $z = f(m,n) < 0$ and $z = f(m,n) \geq 0$.

1. For $z < 0$, let $m = 0$.

$$\begin{aligned} z = f(m,n) \quad \wedge \quad m = 0 &\iff \quad z = |m| - |n| \quad \wedge \quad m = 0 \\ &\iff \quad z = 0 - |n| \\ &\iff \quad z = -|n| \\ &\iff \quad n = z \quad \vee \quad n = -z \\ &\iff \quad (m,n) = (0,z) \in \mathbb{Z} \times \mathbb{Z} \quad \vee \quad (m,n) = (0,-z) \in \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

We can say that, for $z < 0$, $(m,n) = (0,z) \in \mathbb{Z} \times \mathbb{Z}$ satisfies $z = f(m,n)$.

2. For $z \geq 0$, let $n = 0$.

$$\begin{aligned} z = f(m,n) \quad \wedge \quad n = 0 \quad &\iff \quad z = |m| - |n| \quad \wedge \quad n = 0 \\ &\iff \quad z = |m| - 0 \\ &\iff \quad z = |m| \\ &\iff \quad m = z \quad \vee \quad m = -z \\ &\iff \quad (m,n) = (z,0) \in \mathbb{Z} \times \mathbb{Z} \quad \vee \quad (m,n) = (-z,0) \in \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

We can say that, for $z < 0$, $(m,n) = (z,0) \in \mathbb{Z} \times \mathbb{Z}$ satisfies $z = f(m,n)$.

To conclude, for all $z = f(m,n) \in \mathbb{Z}$, there exists a pair $(m,n) \in \mathbb{Z} \times \mathbb{Z}$, meaning f is onto.

Problem 7: Functions (Bonus 20 points)

Suppose that f is a function from A to B, where A and B are finite sets with $\mid A \mid=\mid B \mid$. Show that f is one-to-one if and only if it is onto.

(Solution)

$f : A \to B$ and $|A| = |B| = n < \infty$.

Show that: f is $1-1 \quad \Longleftrightarrow \quad f$ is onto

$(\Rightarrow)$

f is one-to-one $\quad \Longleftrightarrow \quad \forall a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

Is f onto? In other words, is there any $a \in A$ such that $b = f(a)$ for all $b \in B$?

Assume f is not onto. That means,

$\exists b \in B, \forall a \in A, b \neq f(a)$.

Let say for k different $b \in B$, this proposition is true. Then, just for $n-k$ elements of B, there exists an $a \in A$ such that $b = f(a)$. On the other hand, because f is one-to-one, for each b, at most one a satisfies $b = f(a)$. Hence, only $n-k$ elements of A go to B, that is k elements of A goes nowhere. In this situation f cannot be a function, which is a contradiction. Thus our assumption is wrong. The function f is onto.

$(\Leftarrow)$

f is onto $\quad \Longleftrightarrow \quad \forall b \in B, b \neq f(a) \quad \therefore \exists a \in A$

Is f one-to-one?

Assume f is not one-to-one. That means,

$\exists a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ but $a_1 \neq a_2$.

For these a_1 and a_2, there exists a $b \in B$ such that $b = f(a_1)$ and $b = f(a_2)$. For the other $n-1$ elements of B, there also exists at least one $a \in A$ due to the fact that f is onto. Nevertheless, the remaining $n-2$ elements of A maps to at most $n-2$ elements of B. So, for $(n-1) - n(-2) = 1$ element of B, there is no element in A satisfying $b = f(a)$. This is a contradiction. Our assumption is wrong. Thus, f is one-to-one when it is onto.