LINEAR EQUATIONS

DEFINITION

A first-order ordinary differential equation is linear *in the dependent variable* y *and the independent variable* x *if it is, or can be, written in the form*

$$\frac{dy}{dx} + P(x)y = Q(x).$$

For example, the equation

$$x\frac{dy}{dx} + (x+1)y = x^3$$

is a first-order linear differential equation, for it can be written as

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = x^2,$$

where $P(x) = 1 + (1/x)$ and $Q(x) = x^2$.

Let us write equation in the form

$$[P(x)y - Q(x)]\,dx + dy = 0.$$

or

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

where

$$M(x, y) = P(x)y - Q(x) \quad \text{and} \quad N(x, y) = 1.$$

Since

$$\frac{\partial M(x, y)}{\partial y} = P(x) \quad \text{and} \quad \frac{\partial N(x, y)}{\partial x} = 0,$$

equation is *not* exact unless $P(x) = 0$.
Equation possesses an integrating factor that depends on x only and may easily be found. Let us proceed to find it. Let us multiply equation by $\mu(x)$, obtaining

$$[\mu(x)P(x)y - \mu(x)Q(x)]\,dx + \mu(x)\,dy = 0.$$

By definition, $\mu(x)$ is an integrating factor if and only if

$$\frac{\partial}{\partial y}[\mu(x)P(x)y - \mu(x)Q(x)] = \frac{\partial}{\partial x}[\mu(x)].$$

This condition reduces to

$$\mu(x)P(x) = \frac{d}{dx}[\mu(x)].$$

This differential equation is separable; separating the variables, we have

$$\frac{d\mu}{\mu} = P(x)\,dx.$$

Integrating, we obtain the particular solution

$$\ln|\mu| = \int P(x)\,dx$$

or

$$\mu = e^{\int P(x)\,dx},$$

where it is clear that $\mu > 0$.

$$e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}P(x)y = Q(x)e^{\int P(x)\,dx},$$

which is precisely

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = Q(x)e^{\int P(x)\,dx}.$$

Integrating this we obtain the solution in the form

$$e^{\int P(x)\,dx}y = \int e^{\int P(x)\,dx}Q(x)\,dx + c,$$

where c is an arbitrary constant.

THEOREM.

The linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has an integrating factor of the form

$$e^{\int P(x)\,dx}.$$

A one-parameter family of solutions of this equation is

$$ye^{\int P(x)\,dx} = \int e^{\int P(x)\,dx}Q(x)\,dx + c;$$

that is,

$$y = e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx}Q(x)\,dx + c\right].$$

We consider several examples.

Example.

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}.$$

Here

$$P(x) = \frac{2x+1}{x}$$

and hence an integrating factor is

$$\exp\left[\int P(x)\,dx\right] = \exp\left[\int\left(\frac{2x+1}{x}\right)dx\right] = \exp(2x + \ln|x|)$$

$$= \exp(2x)\exp(\ln|x|) = x\exp(2x).$$

Thus, we obtain

$$xe^{2x}\frac{dy}{dx} + e^{2x}(2x+1)y = x$$

or

$$\frac{d}{dx}(xe^{2x}y) = x.$$

Integrating, we obtain the solutions

$$xe^{2x}y = \frac{x^2}{2} + c$$

or

$$y = \tfrac{1}{2}xe^{-2x} + \frac{c}{x}e^{-2x},$$

where c is an arbitrary constant.

Example.

Solve the linear ODE

$$y' - y = e^{2x}$$

Solution.

Here

$$\mu(x) = e^{\int p\,dx} = e^{\int -1\,dx} = e^{-x}$$

$$y(x) = e^{x}\left(\int e^{-x}e^{2x}dx + c\right) = e^{x}(e^{x}+c) = ce^{x} + e^{2x}.$$

$$(y' - y)e^{-x} = (ye^{-x})' = e^{2x}e^{-x} = e^{x}.$$

Integrating on both sides, you obtain the same result as before;

$$ye^{-x} = e^{x} + c, \qquad \text{hence} \qquad y = e^{2x} + ce^{x}$$

Example.

Solve the initial value problem

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

Here

$$\mu(x) = e^{\int \tan x\,dx} = e^{\ln|\sec x|} = \sec x,$$

From this

$$\mu(x)\,y(x) = \int (\sec x)(2\sin x\cos x)dx + c = 2\int \sin x\,dx + c$$

and the general solution of our equation is

$$y(x) = \cos x\left(2\int \sin x\,dx + c\right) = c\cos x - 2\cos^2 x.$$

From this and the initial condition, $1 = c\cdot 1 - 2\cdot 1^2$, thus $c = 3$ and the solution of our initial value problem is $y = 3\cos x - 2\cos^2 x$.

Example.
Find the solution of the following IVP

$$t^3 y' = t^2 y + 5, \qquad y(1) = 1.$$

Firstly, we need to find the solution of the differntial equation

$$\frac{dy}{dt} = \frac{y}{t} + \frac{5}{t^3}.$$

set

$$\mu = e^{-\int \frac{dt}{t}} = e^{-\ln(t)} = \frac{1}{t}.$$

$$\frac{1}{t}\frac{dy}{dt} - \frac{y}{t^2} = \frac{5}{t^4}.$$

Thus,

$$d(\frac{y}{t}) = \frac{5}{t^4}dt.$$

integrate the two sides

$$\frac{y}{t} = \frac{-25}{t^5} + c,$$

Thus,

$$1 = y(1)/1 = -25/1 + c,$$

which means $c = 26$, therefore, the solution of the IVP takes the form

$$y = \frac{-25}{t^4} + 26t.$$

Example.

Solve the initial-value problem that consists of the differential equation

$$(x^2+1)\frac{dy}{dx}+4xy=x$$

and the initial condition

$$y(2)=1.$$

We divide by x^2+1 to obtain

$$\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{x}{x^2+1}.$$

Equation is in the standard form, where

$$P(x)=\frac{4x}{x^2+1}.$$

An integrating factor is

$$\exp\left[\int P(x)\,dx\right]=\exp\left(\int\frac{4x\,dx}{x^2+1}\right)=\exp[\ln(x^2+1)^2]=(x^2+1)^2.$$

$$(x^2+1)^2\frac{dy}{dx}+4x(x^2+1)y=x(x^2+1)$$

or

$$\frac{d}{dx}[(x^2+1)^2y]=x^3+x.$$

We now integrate to obtain a one-parameter family of solutions in the form

$$(x^2+1)^2y=\frac{x^4}{4}+\frac{x^2}{2}+c.$$

Applying the initial condition, we have

$$25=6+c.$$

Thus $c = 19$ and the solution of the initial-value problem under consideration is

$$(x^2 + 1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19.$$

Example. Suppose we are given the DE

$$\frac{dy}{dx} + \frac{1}{x} y = x^2, \;\; x > 0$$

with initial condition

$$y(1) = 2.$$

- First calculate the integrating factor $\mu(x)$:

$$\mu(x) = \exp\left(\int p(x)\, dx\right) = \exp\left(\int \frac{1}{x}\, dx\right) = \exp(\log x) = x.$$

- Now substitute into (2.3)

$$y(x) = \frac{1}{x} \int x \cdot x^2\, dx + \frac{c}{x} = \frac{1}{x} \cdot \frac{x^4}{4} + \frac{c}{x} = \frac{x^3}{4} + \frac{c}{x}.$$

- Impose the initial condition $y(1) = 2$:

Example .

Consider the differential equation

$$y^2\, dx + (3xy - 1)\, dy = 0.$$

Solving for dy/dx, this becomes

$$\frac{dy}{dx} = \frac{y^2}{1 - 3xy},$$

which is clearly *not* linear in y. Equation is *not* exact, separable, or homogeneous. Let us now regard x as the dependent variable and y as the independent variable. With this interpretation, we now write equation in the derivative form

$$\frac{dx}{dy} = \frac{1 - 3xy}{y^2}$$

or

$$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2}.$$

Equation $\dfrac{dx}{dy} + P(y)x = Q(y)$ is *linear in x*.

Thus an integrating factor is

$$\exp\left[\int P(y)\,dy\right] = \exp\left(\int \frac{3}{y}\,dy\right) = \exp(\ln|y|^3) = y^3.$$

Multiplying by y^3 we obtain

$$y^3\frac{dx}{dy} + 3y^2x = y$$

or

$$\frac{d}{dy}[y^3x] = y.$$

Integrating, we find the solutions in the form

$$y^3x = \frac{y^2}{2} + c$$

where c is an arbitrary constant.

BERNOULLI EQUATIONS

DEFINITION

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a Bernoulli differential equation.

We observe that if $n = 0$ or 1, then the Bernoulli equation is actually a linear equation and is therefore readily solvable as such. However, in the general case in which $n \neq 0$ or 1, this simple situation does not hold and we must proceed in a different manner. We now state and prove Theorem, which gives a method of solution in the general case.

THEOREM.

Suppose $n \neq 0$ or 1. Then the transformation $v = y^{1-n}$ reduces the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

to a linear equation in v.

Proof. We first multiply *equation* by y^{-n}, thereby expressing it in the equivalent form

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

If we let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

and *equation* transforms into

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

or, equivalently,

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

Letting

$$P_1(x) = (1-n)P(x)$$

and

$$Q_1(x) = (1-n)Q(x),$$

this may be written

$$\frac{dv}{dx} + P_1(x)v = Q_1(x).$$

which is linear in v.

Example.

$$\frac{dy}{dx} + y = xy^3.$$

This is a Bernoulli differential equation, where $n = 3$. We first multiply the equation through by y^{-3}, thereby expressing it in the equivalent form

$$y^{-3}\frac{dy}{dx} + y^{-2} = x.$$

If we let $v = y^{1-n} = y^{-2}$, then $dv/dx = -2y^{-3}(dy/dx)$ and the preceding differential equation transforms into the linear equation

$$-\frac{1}{2}\frac{dv}{dx} + v = x.$$

Writing this linear equation in the standard form

$$\frac{dv}{dx} - 2v = -2x,$$

we see that an integrating factor for this equation is

$$e^{\int P(x)\,dx} = e^{-\int 2\,dx} = e^{-2x}.$$

Thus we obtain

$$e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = -2xe^{-2x}$$

or

$$\frac{d}{dx}(e^{-2x}v) = -2xe^{-2x}.$$

Integrating, we find

$$e^{-2x}v = \tfrac{1}{2}e^{-2x}(2x + 1) + c,$$
$$v = x + \tfrac{1}{2} + ce^{2x},$$

where c is an arbitrary constant. But

$$v = \frac{1}{y^2}.$$

Thus we obtain the solutions in the form

$$\frac{1}{y^2} = x + \tfrac{1}{2} + ce^{2x}.$$

Note. Consider the equation

$$\frac{df(y)dy}{dy\,dx} + P(x)f(y) = Q(x),$$

where f is a known function of y. Letting $v = f(y)$, we have

$$\frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{df(y)}{dy}\frac{dy}{dx},$$

and equation becomes

$$\frac{dv}{dx} + P(x)v = Q(x),$$

which is linear in v. We now observe that the Bernoulli differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is a special case of

$$\frac{df(y)dy}{dy\,dx} + P(x)f(y) = Q(x),$$

Writing Bernoulli differential equation in the form

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

and then multiplying through by $(1 - n)$, we have

$$(1 - n)y^{-n}\frac{dy}{dx} + P_1(x)y^{1-n} = Q_1(x),$$

where $P_1(x) = (1 - n)P(x)$ and $Q_1(x) = (1 - n)Q(x)$. This is of the form

$$\frac{df(y)dy}{dy\,dx} + P(x)f(y) = Q(x),$$

where $f(y) = y^{1-n}$; letting $v = y^{1-n}$, it becomes

$$\frac{dv}{dx} + P_1(x)v = Q_1(x),$$

which is linear in v.

Example
Solve the IBP

$$y' + ty = \frac{t}{y^3}, \qquad y(1) = 2.$$

This is a Bernoulli equation with $n = -3$, so we let

$$z = y^{1-n} = y^4.$$

Thus $z' = 4y^3y'$, which leads to $y' = z'/(4y^3)$.
Therefore,

$$\frac{z'}{4y^3} + ty = \frac{t}{y^3}.$$

Thus

$$z' + 4tz = 4t, \quad z(1) = y(1)^4 = 2^4 = 16.$$

The integrating factor is $\mu = e^{\int 4tdt} = e^{2t^2}$.
Multiplying the equation by μ, it follows that

$$e^{2t^2}dz + 4te^{2t^2}z = 4te^{2t^2}.$$

Thus

$$d(e^{2t^2}z) = 4te^{2t^2}dt.$$

Integrate the two sides to get

$$e^{2t^2}z = e^{2t^2} + c,$$

Thus

$$z = 1 + \frac{c}{e^{2t^2}}$$

Therefore,

$$z(1) = 16 = 1 + \frac{c}{e^2}.$$

i.e.

$$c = 15e^2,$$

so that

$$z = 1 + \frac{15e^2}{e^{2t^2}}.$$

It follows that

$$y = z^{\frac{1}{4}} = (1 + \frac{15e^2}{e^{2t^2}})^{\frac{1}{4}}.$$

Example. Find every nonzero solution of the differential equation

$$y' = y + 2\,y^5.$$

Solution: This is a Bernoulli equation for $n = 5$. Divide the equation by y^5,

$$\frac{y'}{y^5} = \frac{1}{y^4} + 2.$$

Introduce the function $v = 1/y^4$ and its derivative $v' = -4(y'/y^5)$, into the differential equation above,

$$-\frac{v'}{4} = v + 2 \quad \Rightarrow \quad v' = -4\,v - 8 \quad \Rightarrow \quad v' + 4\,v = -8.$$

The last equation is a linear differential equation for the function v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{4t}$, then

$$\big(e^{4t} v\big)' = -8\,e^{4t} \quad \Rightarrow \quad e^{4t} v = -\frac{8}{4}\,e^{4t} + c.$$

We obtain that $v = c\,e^{-4t} - 2$. Since $v = 1/y^4$,

$$\frac{1}{y^4} = c\,e^{-4t} - 2 \quad \Rightarrow \quad y(t) = \pm\frac{1}{\big(c\,e^{-4t} - 2\big)^{1/4}}.$$

Example. Given any constants a_0, b_0, find every solution of the differential equation

$$y' = a_0\,y + b_0\,y^3.$$

Solution: This is a Bernoulli equation with $n = 3$. Divide the equation by y^3,

$$\frac{y'}{y^3} = \frac{a_0}{y^2} + b_0.$$

Introduce the function $v = 1/y^2$ and its derivative $v' = -2(y'/y^3)$, into the differential equation above,

$$-\frac{v'}{2} = a_0 v + b_0 \quad \Rightarrow \quad v' = -2a_0 v - 2b_0 \quad \Rightarrow \quad v' + 2a_0 v = -2b_0.$$

The last equation is a linear differential equation for v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{2a_0 t}$,

$$\left(e^{2a_0t}v\right)' = -2b_0\, e^{2a_0t} \quad \Rightarrow \quad e^{2a_0t}v = -\frac{b_0}{a_0}\, e^{2a_0t} + c$$

We obtain that $v = c\,e^{-2a_0t} - \dfrac{b_0}{a_0}$. Since $v = 1/y^2$,

$$\frac{1}{y^2} = c\,e^{-2a_0t} - \frac{b_0}{a_0} \quad \Rightarrow \quad y(t) = \pm\frac{1}{\left(c\,e^{-2a_0t} - \frac{b_0}{a_0}\right)^{1/2}}.$$

Example. Find every solution of the equation $t\,y' = 3y + t^5\,y^{1/3}$.

Solution: Rewrite the differential equation as

$$y' = \frac{3}{t}\,y + t^4\,y^{1/3}.$$

This is a Bernoulli equation for $n = 1/3$. Divide the equation by $y^{1/3}$,

$$\frac{y'}{y^{1/3}} = \frac{3}{t}\,y^{2/3} + t^4.$$

Define the new unknown function $v = 1/y^{(n-1)}$, that is, $v = y^{2/3}$, compute is derivative, $v' = \dfrac{2}{3}\,\dfrac{y'}{y^{1/3}}$, and introduce them in the differential equation,

$$\frac{3}{2}\,v' = \frac{3}{t}\,v + t^4 \quad \Rightarrow \quad v' - \frac{2}{t}\,v = \frac{2}{3}\,t^4.$$

This is a linear equation for v. Integrate this equation using the integrating factor method. To compute the integrating factor we need to find

$$A(t) = \int \frac{2}{t}\,dt = 2\ln(t) = \ln(t^2).$$

Then, the integrating factor is $\mu(t) = e^{-A(t)}$. In this case we get

$$\mu(t) = e^{-\ln(t^2)} = e^{\ln(t^{-2})} \quad \Rightarrow \quad \mu(t) = \frac{1}{t^2}.$$

Therefore, the equation for v can be written as a total derivative,

$$\frac{1}{t^2}\Big(v' - \frac{2}{t}\,v\Big) = \frac{2}{3}\,t^2 \quad \Rightarrow \quad \Big(\frac{v}{t^2} - \frac{2}{9}\,t^3\Big)' = 0.$$

$$\frac{v}{t^2} - \frac{2}{9}\,t^3 = c \quad \Rightarrow \quad v(t) = t^2\left(c + \frac{2}{9}\,t^3\right) \quad \Rightarrow \quad v(t) = ct^2 + \frac{2}{9}\,t^5.$$

Once v is known we compute the original unknown $y = \pm v^{3/2}$, where the double sign is related to taking the square root. We finally obtain

$$y(t) = \pm\Big(ct^2 + \frac{2}{9}\,t^5\Big)^{3/2}.$$

Riccati equation

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x),$$

$P \equiv 0$ — **Linear Equation**

$R \equiv 0$ — Bernoulli equation

$y = y_1(x)$ **solution**

$$y_1' = Py_1^2 + Qy_1 + R$$

$$y = y_1 + z$$

$$\frac{dy_1}{dx} + \frac{dz}{dx} = Py_1^2 + 2Py_1z + Pz^2 + Qy_1 + Qz + R$$

$$\frac{dz}{dx} = Pz^2 + (2Py_1 + Q)z$$

$$z = \frac{1}{u}, \qquad u = \frac{1}{z} = \frac{1}{y - y_1}$$

$$\frac{du}{dx} + (2Py_1 + Q)u = -P.$$

$$u = C\varphi(x) + \psi(x),$$

$$y = y_1 + \frac{1}{C\varphi(x) + \psi(x)} = \frac{Cy_1\varphi(x) + y_1\psi(x) + 1}{C\varphi(x) + \psi(x)}.$$

Example.

$$\frac{dy}{dx}+ay^2=bx^{\alpha},$$

1) $\alpha=0$; $\frac{dy}{dx}+ay^2=b$;

$$\frac{dy}{b-ay^2}=dx.$$

2) $\alpha=-2$;

$$\frac{dy}{dx}+ay^2=\frac{b}{x^2}.$$

$$y=\frac{1}{z}.$$

$$-\frac{1}{z^2}\cdot\frac{dz}{dx}+\frac{a}{z^2}=\frac{b}{x^2},$$

$$\frac{dz}{dx}=a-b\left(\frac{z}{x}\right)^2.$$

Example.

$$\frac{dy}{dx}=y^2+\frac{1}{2x^2} \qquad y=\frac{1}{z}$$

$$\frac{dz}{dx}=-1-\frac{1}{2}\left(\frac{z}{x}\right)^2.$$

$$z=ux.$$

$$u+x\frac{du}{dx}=-1-\frac{1}{2}u^2, \quad \frac{du}{u^2+2u+2}=-\frac{dx}{2x},$$

$$\frac{du}{1+(u+1)^2}=-\frac{dx}{2x}, \qquad u+1=\tan\left(C-\frac{1}{2}\ln x\right),$$

$$z=x\left[-1+\tan\left(C-\frac{1}{2}\ln z\right)\right]$$

$$y=\frac{1}{x\left[-1+\tan\left(C-\frac{1}{2}\ln x\right)\right]}.$$