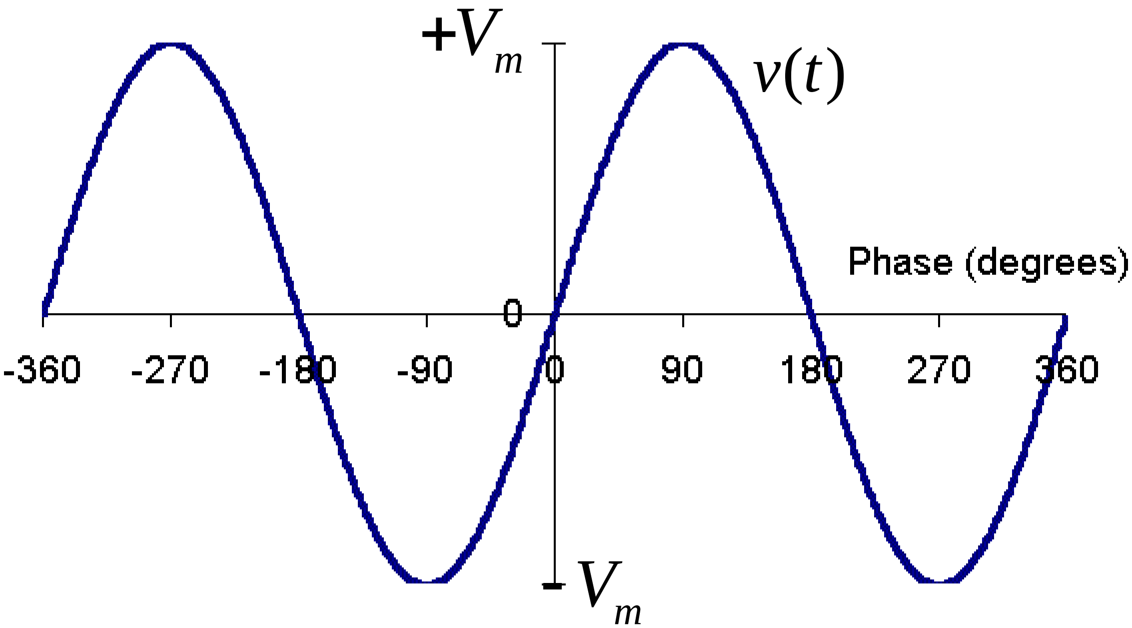
Sinusoidal Steady-State

Week 8-9

Brief Summary

- AC Instead of DC.
- Sinusoidal Inputs Instead of Constant Inputs.
- Can Still Use Techniques we have previously learned.
- Need to Use Complex Numbers
- Everything Else is Almost the Same

Sinusoidal Sources



Sine Wave
Periodic Function

Repeats every 360° or 2π rads or T sec

For a 60Hz sine wave,

$$T = \frac{1}{60}\,\mathrm{sec}$$

$$v(t) = V_m \sin(\omega t)$$

$$\omega = 2\pi f = \frac{2\pi}{T} \quad (rad/s)$$

$$f = 1/T \quad (\mathrm{cycles}/s)$$

ωt is in radians

$V_m \equiv$ Magnitude {sine varies between -1 and 1}

$t =$ time (in seconds)

$\omega =$ Angular Frequency (in rad / s)

$T =$ Period (in seconds)

$f =$ frequency (in $cycles$ / sec $\equiv$ $Hertz$)

Degree/Radian Conversions

$$\boxed{\frac{Deg}{360^o} = \frac{rad}{2\pi}}$$ **Fundamental**

OR $$rad = \left(\frac{Deg}{360}\right)2\pi$$

OR $$rad = \frac{Deg}{180} \cdot \pi$$ **Standard Equation**

Examples

$$2\pi \text{ radians} = 360^o$$

$$\pi \text{ radians} = 180^o$$

General Form of Sine Wave

$$v(t) = V_m \sin(\omega t + \phi)$$

ϕ often given in Degrees, but must be converted to Radians.

$\phi \equiv$ Phase Angle

$$e.g.\ \ v(t) = V_m \sin(5t + 30^\circ)$$

Phase Shift of ϕ between v_1 & v_2

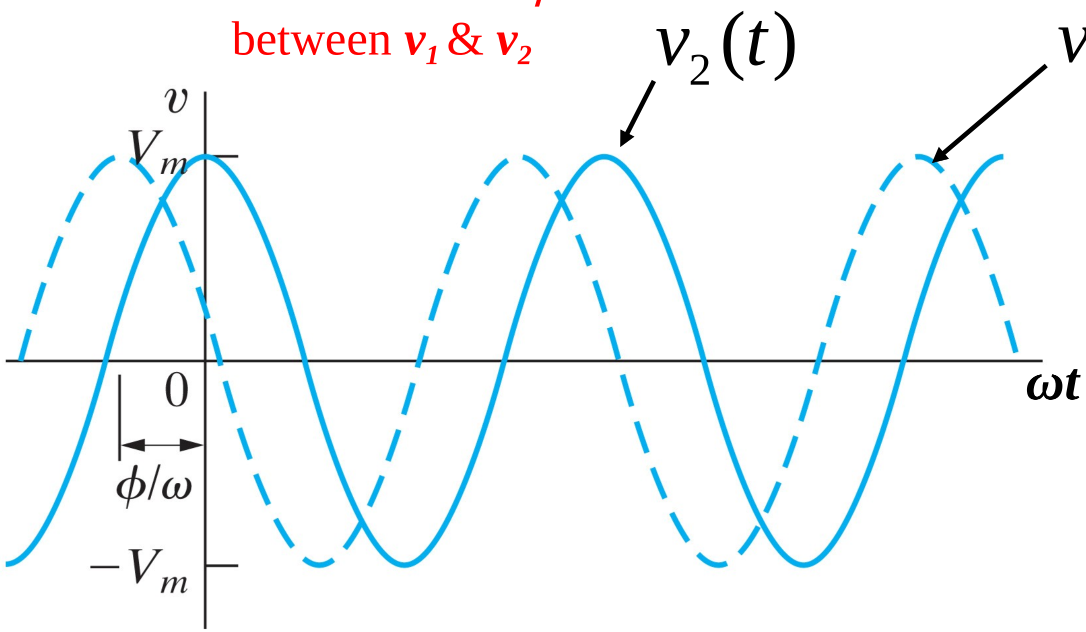


Figure: 09-02

Reducing ϕ to zero shifts $v_1(t)$ (dotted line) ϕ / ω time units to the right

$$v_1(t) = V_m \cos(\omega t + \phi)$$

$$v_2(t) = V_m \cos(\omega t)$$

Root-Mean-Square (RMS)

$$f_{rms} = \sqrt{\frac{1}{T}\int_T f^2(t)dt}$$ **RMS definition**

$$\boxed{V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \sin^2(\omega t + \phi)\, dt}}$$ **RMS Calculation of an AC signal**

(Square Root of the Mean value of the Squared Function)

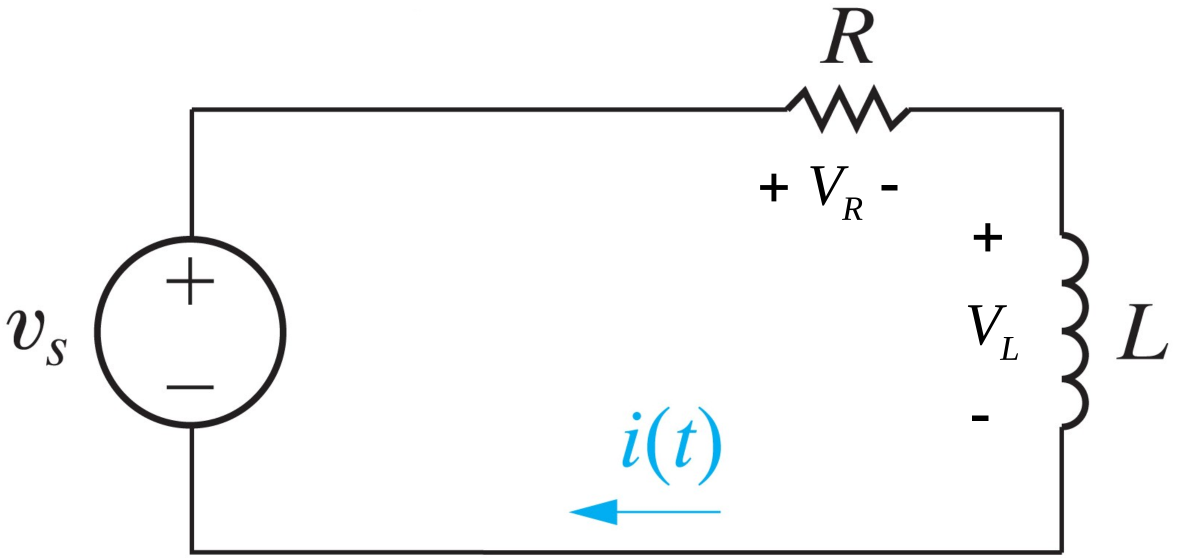
Using $\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$, we can show

$$\boxed{V_{rms} = \frac{V_m}{\sqrt{2}} \quad \text{for a sine wave}}$$ **After integrated over a period and simplifying**

<u>At Power Outlet:</u>

$$V_{rms} = \frac{V_m}{\sqrt{2}} = \frac{311.12}{\sqrt{2}} = 220V$$

Sinusoidal Response: Find *i*(*t*)



$$v_s = V_m \cos(\omega t + \phi)$$

$$V_s = V_R + V_L$$

KVL

$$V_m \cos(\omega t + \phi) = Ri + L\frac{di}{dt}$$

Use Ohm's Law and Inductor Law

Solution of Differential Equation requires Advanced Tools

Sinusoidal Response (Contd.)

$$v_s = V_m \cos(\omega t + \phi)$$ Voltage Input to RL Circuit

Neglects Transients

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t + \phi - \tan^{-1}\left[\frac{\omega L}{R}\right]\right)$$

Steady State Solution

Notes:

1. $i(t)$ has the same form as $V_s(t)$ in steady state
2. $i(t)$ has a different amplitude and different phase when compared to $V_s(t)$
3. Use Phasors to find the Steady State Solution

Phasors Definition

Phasor

A Complex Number Which Contains <u>Amplitude</u> and <u>Phase</u> Information of a Sinusoidal Function.

Does Not Contain Information on frequency (i.e. ω)
Frequency Does Not Change in a Linear Circuit

<u>Phasor exploits Euler's Identity</u>

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \quad \left(j = \sqrt{-1} \right)$$

$$\left.\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin\theta &= -j\frac{1}{2}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned}\right\} \text{Useful identities}$$

Review of Complex Numbers

We can express a complex number in two ways

Rectangular

$$\bar{A} = a + j \cdot b$$

Complex Number — Real Part — Imaginary Part

$$j = \sqrt{-1}$$

$$\mathrm{Re}\left[\bar{A}\right] = a; \quad \mathrm{Im}\left[\bar{A}\right] = b$$

a and b are "Real" Numbers

jb is an "Imaginary" Number

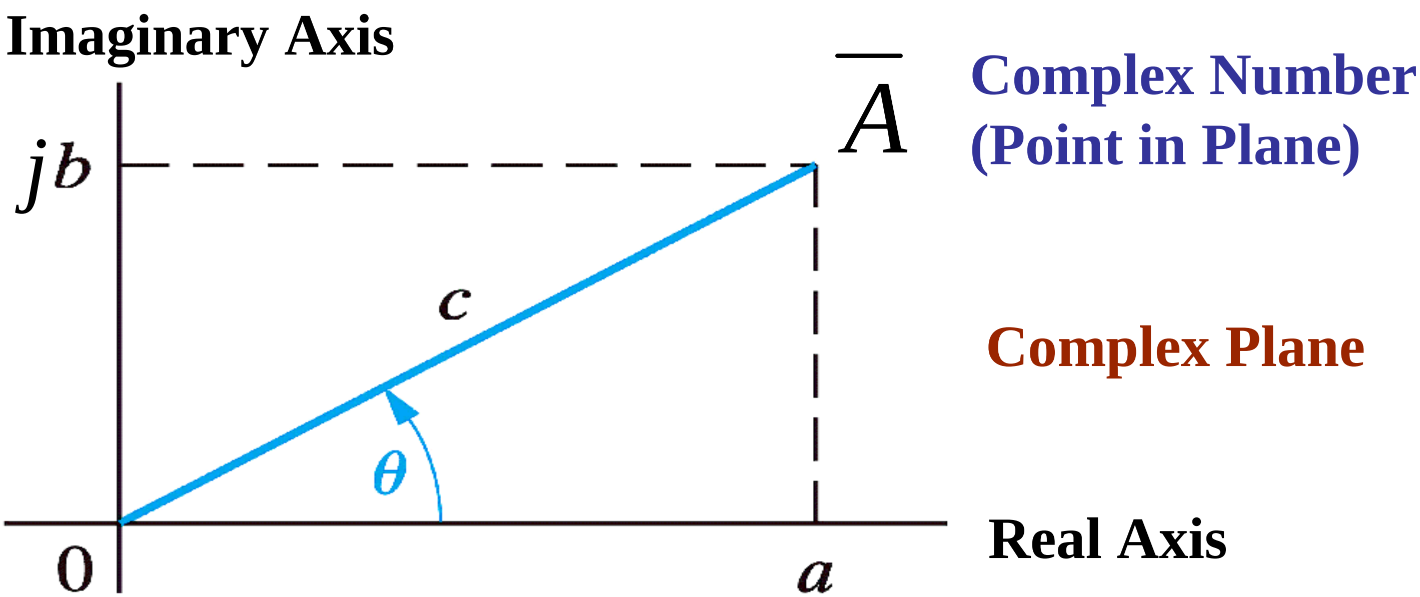
$\bar{A}$ is a "Complex" Number

Polar

$$\bar{A} = ce^{j\theta} = c\angle\theta$$

$$j = \sqrt{-1}$$

Polar $\longleftrightarrow$ Rectangular Conversion



$$\overline{A} = a + jb = c\angle\theta$$

$$c = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\left(b/a\right)$$

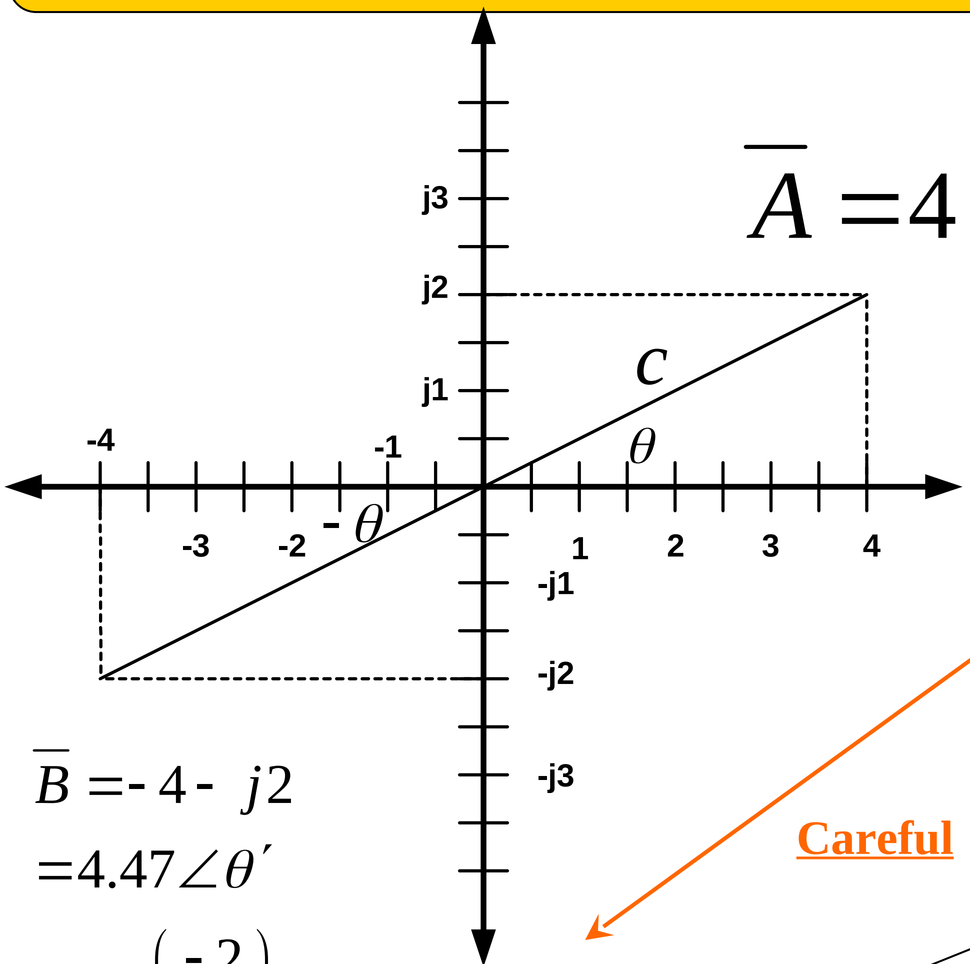
Rectangular → Polar

$$a = c\cos(\theta)$$

$$b = c\sin(\theta)$$

Polar → Rectangular

Two Examples: Rectangular to Polar Conversion



$$\overline{A} = 4 + j2$$

$$c = \sqrt{4^2 + 2^2} = 4.47$$

$$\theta = \tan^{-1}\left(\frac{2}{4}\right) = 26.6^o$$

$$\overline{A} = 4.47\angle 26.6^o \equiv 4.47e^{j26.6}$$

$$\overline{B} = -4 - j2$$

$$= 4.47\angle\theta'$$

$$\tan^{-1}\left(\frac{-2}{-4}\right) = 26.6^\circ \text{ (from x-axis)}$$

$$\theta' = 26.6^\circ + 180^\circ = 206.6^o$$

Careful !

$$\overline{B} = 4.47\angle 206.6^\circ \equiv$$

$$\equiv 4.47\angle 206.6^\circ - 360^\circ$$

$$= 4.47\angle -153.4^\circ$$

Complex Conjugate

Reverse Signs of Imaginary Part

$$\overline{A} = a + jb = c\angle\theta$$

$$\overline{A}^* = a - jb = c\angle -\theta$$

Note: $\overline{A}\cdot\overline{A}^* = (a+jb)(a-jb) = c\angle\theta \cdot c\angle -\theta$ **Can we simplify further?**

$$(a+jb)(a-jb) = a^2 + jba - jba - j^2b^2$$ **Multiply it out**

$$= a^2 - j^2b^2 = a^2 + b^2$$ $j^2 = -1$

$$\overline{A}\cdot\overline{A}^* = a^2 + b^2 = c^2$$

$$c\angle\theta \cdot c\angle -\theta = c^2 \underbrace{\angle(\theta + (-\theta))}_{\text{add angles}} = c^2\angle 0 = c^2$$

Complex Arithmetic

$$\overline{A} = a + jb = c_1\angle\theta_1$$
$$\overline{B} = y + jz = c_2\angle\theta_2$$

Given two complex numbers

$$\overline{A} + \overline{B} = (a + y) + j(b + z)$$ **Add**

$$\overline{A} - \overline{B} = (a - y) + j(b - z)$$ **Subtract**

Must Convert to Rectangular

$$\overline{A} \cdot \overline{B} = (a + jb)\cdot(y + jz) = ay + jby + jza - bz$$

$$\overline{A} \cdot \overline{B} = c_1\angle\theta_1 \cdot c_2\angle\theta_2 = c_1 \cdot c_2\angle(\theta_1 + \theta_2)$$

$$\frac{\overline{A}}{\overline{B}} = \frac{c_1\angle\theta_1}{c_2\angle\theta_2} = \frac{c_1}{c_1}\angle(\theta_1 - \theta_2)$$

$$\frac{\overline{A}}{\overline{B}} = \frac{a + jb}{y + jz}\cdot\frac{y - jz}{y - jz} = \frac{(a + jb)\cdot(y - jz)}{y^2 + z^2}$$

Integer Powers

$$\overline{A}^k = (a + jb)^k = \left[c_1\angle\theta_1\right]^k$$

$$\overline{A}^k = c_1^k\angle(k\cdot\theta_1)$$

Follows from Polar Multiplication

Back to AC Circuits

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \quad \Big\}$$ **Euler Identity**

$$\mathrm{Re}\left[e^{j\theta}\right] = \cos\theta$$

$$\mathrm{Im}\left[e^{j\theta}\right] = \sin\theta$$

$$v = V_m \sin(\omega t + \phi)$$

$$v = V_m \,\mathrm{Im}\left[e^{j(\omega t + \phi)}\right]$$

$$v = \mathrm{Im}\left[V_m e^{j\phi} e^{j\omega t}\right]$$

For steady state

$$\begin{bmatrix} \omega \text{ Does Not Change} \\ \phi \text{ and } V_m \text{ Do Change} \end{bmatrix}$$

Phasor Representation

$$\overline{V} = V_m e^{j\phi} \equiv V_m \angle \phi = P\{V_m \sin(\omega t + \phi)\}$$ **P stands for Phasor**

$$\overline{V} = V_m \angle \phi$$ Polar

$$\overline{V} = V_m \cos\phi + jV_m \sin\phi$$ Rectangular

$$v(t) = V_m \sin(\omega t + \phi)$$ **Time Domain**

$$\overline{V} = V_m \angle \phi$$ **Frequency Domain**

$$\overline{V} = V_m e^{j\phi}$$ **Phasor**

Phasor is a Complex Number

Euler $$e^{j\phi} = \cos\phi + j\sin\phi$$

$$\overline{V} = V_m e^{j\phi} = V_m \cos\phi + jV_m \sin\phi = a + jb = \overline{V}$$

Phasor Representation (Contd.)

$$\overline{V} = V_m e^{j\phi} \equiv V_m \angle \phi$$

Frequency Domain

$$v(t) = V_m \sin(\omega t + \phi)$$

Time Domain

Example

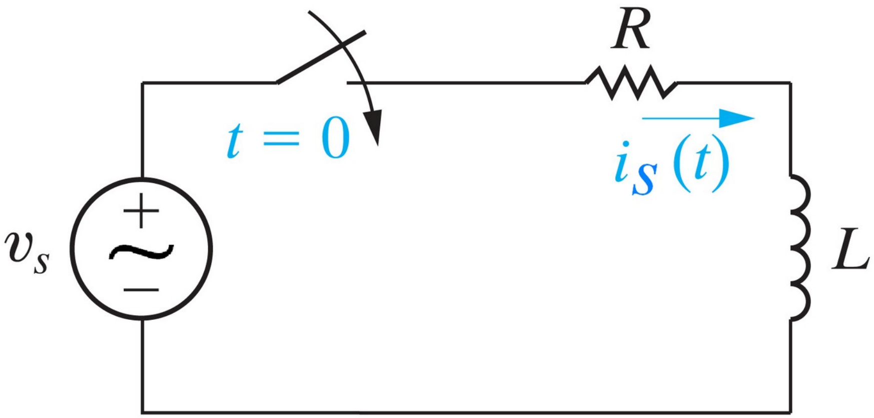
$$\overline{V} = 215\angle -112^o$$

Frequency Domain

$$v(t) = 215\sin\left(\omega t - 112^o\right)$$

Time Domain

Apply Concepts to AC RL Circuit



$$v_s(t) = V_m \sin(\omega t + \phi)$$

$$i_s(t) = I \sin(\omega t + \beta)$$

Find Steady-State Solution

Note v_s and i_s are Not Necessarily in Phase

1 $$v_s = R i_s + L\frac{di_s}{dt}$$ **KVL**

2 $$v_s = \mathrm{Im}\left\{V_m e^{j\phi} e^{j\omega t}\right\} \qquad i_s = \mathrm{Im}\left\{I e^{j\beta} e^{j\omega t}\right\}$$ **Euler Representation**

3 $$\mathrm{Im}\left\{V_m e^{j\phi} e^{j\omega t}\right\} = R\,\mathrm{Im}\left\{I e^{j\beta} e^{j\omega t}\right\} + L\frac{d}{dt}\mathrm{Im}\left\{I e^{j\beta} e^{j\omega t}\right\}$$ **Substitute 2 into 1**

4 $$\mathrm{Im}\left\{V_m e^{j\phi} e^{j\omega t}\right\} = R\,\mathrm{Im}\left\{I e^{j\beta} e^{j\omega t}\right\} + Lj\omega\,\mathrm{Im}\left\{I e^{j\beta} e^{j\omega t}\right\}$$

5 $$\mathrm{Im}\left\{V_m e^{j\phi} e^{j\omega t}\right\} = (R + j\omega L)\,\mathrm{Im}\left\{I e^{j\beta} e^{j\omega t}\right\}$$ **Apply phasor notation to 5**

Ohm's Law-like Relationship 6 $$V_m\angle\phi = (R + j\omega L)\, I\angle\beta$$

Time Domain Solution to the Circuit

$$\bar{I} = I\angle\beta = \frac{V_m\angle\phi}{R + j\omega L}$$

Impedance Law

$$\bar{I} = \frac{V_m\angle\phi}{\sqrt{R^2+\omega^2L^2}\angle\tan^{-1}\left(\frac{\omega L}{R}\right)}$$

Convert denominator to polar form

$$\bar{I} = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\angle\left(\phi - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

Divide magnitudes
Subtract angles

$$\boxed{i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t + \phi - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)}$$

Because V_m was defined to be a sine function

Time Domain Solution to the Circuit
(Contd.)

Cosine Input $\left\{ V_m \cos(\omega t + \phi) = Ri + L\dfrac{di}{dt} \right\}$ **Differential Equation**

Cosine Output $\left\{ i(t) = \dfrac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t + \phi - \tan^{-1}\left(\dfrac{\omega L}{R}\right)\right) \right\}$ **Steady State Solution**

Sine Input $\left\{ V_m \sin(\omega t + \phi) = Ri + L\dfrac{di}{dt} \right\}$ **Differential Equation**

Sine Output $\left\{ i(t) = \dfrac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\omega t + \phi - \tan^{-1}\left(\dfrac{\omega L}{R}\right)\right) \right\}$ **Steady State Solution**

Impedance Functions

$$\bar{V}_s = \bar{I}_s Z_{eq}$$

Ohm's Law for Phasors

$$Z_{eq} = R + j\omega L$$

Impedance Function

$$Z_{eq} = Z_R + Z_L$$

Passive elements have a steady state impedance function

$$\underbrace{\boxed{Z_R = R}}_{\textbf{Resistor}} \qquad \underbrace{\boxed{Z_L = j\omega L}}_{\textbf{Inductor}} \qquad \underbrace{\boxed{Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}}}_{\textbf{Capacitor}}$$

Essentials of Phasors

$$\bar{A} = a + jb = c\angle\theta$$

$$c = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$a = c\cos(\theta) \qquad b = c\sin(\theta)$$

$$\sin(\omega t) \equiv \cos(\omega t - 90^\circ)$$

Important Trig Formulae

$$v(t) = V_m Cos(\omega t + \phi)$$

$$\bar{V} = V_m\angle\phi$$

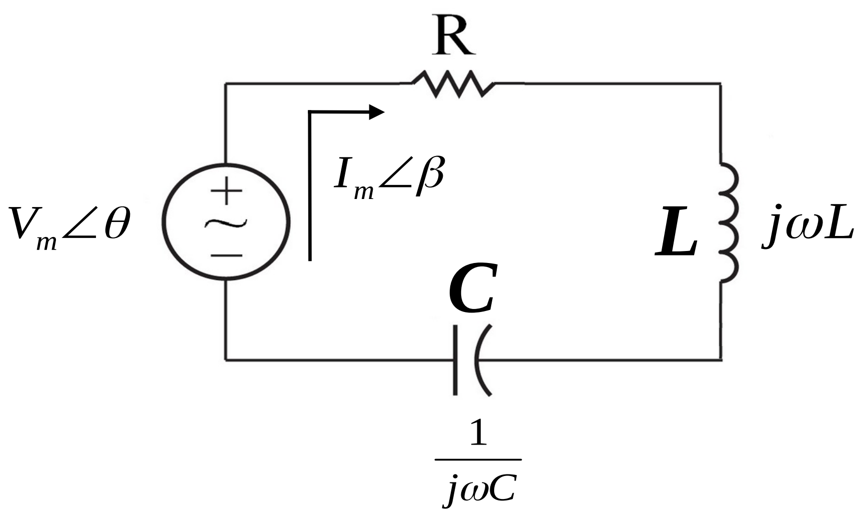
$$I(t) = I_m Cos(\omega t + \beta)$$

$$\bar{I} = I_m\angle\beta$$

Phasors Notation

Impedance for 2nd – order circuit

$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$



$$\left.\begin{aligned} V_m\angle\theta &= I_m\angle\beta\left[Z_R + Z_L + Z_C\right] \\ V_m\angle\theta &= I_m\angle\beta \cdot Z_{eq} \end{aligned}\right\} \text{KVL}$$

$$Z_{eq} = R + j\omega L + \frac{1}{j\omega C}$$

$$\boxed{\bar{I} = I_m\angle\beta = \frac{V_m\angle\theta}{R + j\omega L + \frac{1}{j\omega C}}} \Bigg\} \text{Solve for } \bar{I}$$

Assume Cosine function for consistency with the book

$$v(t) = V_m \cos(\omega t + \phi)$$

$$\bar{V} = V_m \angle \phi$$

Cosine Input

Phasor Representation

$$v(t) = V_m \sin(\omega t + \phi) \equiv V_m \cos(\omega t + \phi - 90)$$

Sine Input

$$\bar{V} = V_m \angle (\phi - 90)$$

Phasor Representation

Note the Trig formula often used

$$\bar{A} = a + jb = c\angle\theta$$

$$c = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$a = c\cos(\theta) \qquad b = c\sin(\theta)$$

$$\left[\sin(\omega t) = \cos(\omega t - 90)\right]$$

Drill Exercise

$a)\quad v(t) = 170\cos\left(377t - 40^\circ\right)\ (V)$ **Time Function**

$\boxed{\bar{V} = 170\angle -40^\circ\ (V)}$ **Phasor Representation**

$b)\quad i(t) = 10\sin\left(1000t + 20^\circ\right)\ (A)$ **Time Function**

$\bar{I} = 10\angle\left(20^\circ - 90^\circ\right)\ (A)$ **sin(ωt) = cos(ωt – 90º)**

$\boxed{\bar{I} = 10\angle -70^\circ\ (A)}$ **Phasor Representation**

$c)\quad v(t) = 300\cos\left(20{,}000\pi t + 45^\circ\right) - 100\sin\left(20{,}000\pi t + 30^\circ\right)\ (mV)$ **Time Function**

$\bar{V} = 300\angle 45^\circ - 100\angle\left(30^\circ - 90^\circ\right)$ **Phasor Representation**

$\bar{V} = 300\angle 45^\circ - 100\angle -60^\circ$ **Need to simplify**

$300\angle 45^\circ \Downarrow a_1 + b_1 j$

$$\left\{\begin{aligned} a_1 &= 300\cos\left(45^\circ\right) = 212.13 \\ b_1 &= 300\sin\left(45^\circ\right) = 212.13 \end{aligned}\right.$$

$$\left.\begin{aligned} a_2 &= 100\cos\left(-60^\circ\right) = 50 \\ b_2 &= 100\sin\left(-60^\circ\right) = -86.60 \end{aligned}\right\}$$

$100\angle -60^\circ \Downarrow a_2 + b_2 j$

Drill Exercise (Contd.)

$$\left.\begin{array}{l}\bar{V} = (212.13 + j212.13) - (50 - j86.60)\end{array}\right\}\ \text{Substitute and add}$$

$$\left.\bar{V} = 162.13 + j298.73\right\}\ \text{Rectangular form}$$

$$\left.\begin{array}{l} c = \sqrt{(162.13)^2 + (298.73)^2} = 339.89 \\ \theta = \tan^{-1}(298.73/162.13) = 61.51^\circ \end{array}\right\}\ \text{Convert to polar}$$

$$\left.\boxed{\bar{V} = 339.89\angle 61.51^\circ \quad (mV)}\right\}\ \text{Phasor Representation}$$

$$\left.\boxed{v(t) = 339.89\ \cos(20{,}000\pi t + 61.51^\circ) \quad (mV)}\right\}\ \text{Time Function}$$

Drill Exercise (Contd.)

$$d)\quad \bar{V} = \left(60 + j30 + 100\angle -28^\circ\right)\ (V)$$ **Find Phasor**

$$100\angle -28^\circ = 88.295 - j46.947$$ **Convert to Rectangular form**

$$\bar{V} = \left(60 + j30 + 88.295 - j46.947\right)\ (V)$$ **After combining**

$$\bar{V} = \left(148.295 - j16.947\right)$$ **Simplify**

$$\bar{V} = 149.26\angle -6.52^\circ$$ **Convert to Polar**

$$\boxed{v(t) = 149.26\cos\left(\omega t - 6.52^\circ\right)\ (V)}$$ **Time Function**

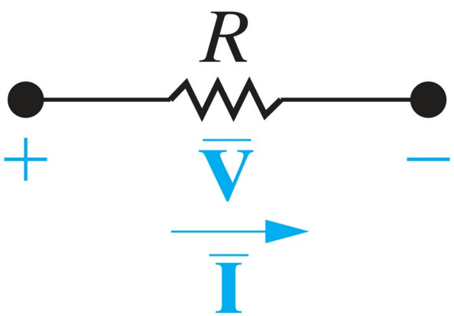
$$\cos\left(\omega t\right) = \sin\left(\omega t + 90^\circ\right)$$ **Trig Identity**

$$v(t) = 149.26\sin\left(\omega t + 90^\circ - 6.52^\circ\right)\ (V)$$

$$\boxed{v(t) = 149.26\sin\left(\omega t + 83.48^\circ\right)(V)}$$ **In terms of a sine function**

Impedance and Reactance (*R*, *L*, and *C*)

Resistor:



Ohm's Law:

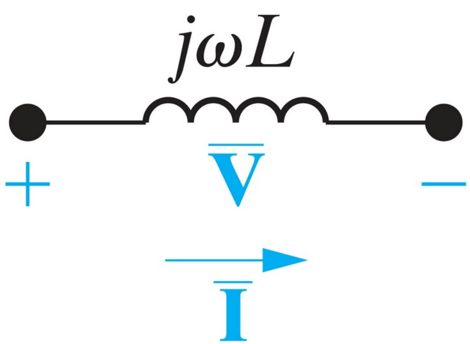
$v = i \cdot R$ Time Domain

$\bar{V} = \bar{I} \cdot R$ Frequency Domain

No Phase Shift

v and ***i*** are in "In – Phase"

Inductor:



For $i(t) = I_m \cos(\omega t + \theta) \Rightarrow \bar{I} = I_m \angle\theta$ } **I_m is Amplitude**

$$v = L\frac{di}{dt} = -L \cdot I_m \cdot \omega \cdot \sin(\omega t + \theta) = -\omega \cdot L \cdot I_m \cdot \sin(\omega t + \theta)$$

$$v = -\omega \cdot L \cdot I_m \cdot \cos(\omega t + \theta - 90^\circ)$$

$$\therefore \bar{V} = -\omega \cdot L \cdot I_m \cdot \angle(\theta - 90^\circ)$$ } **Phasor Representation**

$$\boxed{\bar{V} = -\omega \cdot L \cdot I_m \cdot e^{j(\theta - 90^\circ)} = -\omega \cdot L \cdot I_m \cdot e^{j\theta} e^{-j90^\circ}}$$

Inductor Impedance

$$\bar{V} = -\omega \cdot L \cdot I_m \cdot e^{j\theta} e^{-j90^\circ}$$ **From previous slide**

$$\bar{V} = -\omega \cdot L \cdot I_m \cdot e^{j\theta}\left[-j\right] \qquad e^{-j90^\circ} = \cos\left(90^\circ\right) - j\sin\left(90^\circ\right) = -j$$

$$\bar{V} = \left(j\omega L\right) I_m e^{j\theta} = \left(j\omega L\right) I_m \angle\theta$$ **Simplify**

1 $$\bar{V} = \left(j\omega L\right)\bar{I}$$ Like Ohm's Law

Voltage Leads Current for Inductors

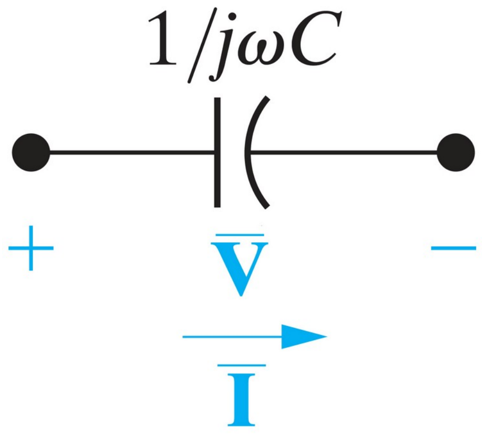
Current Leads Voltage for Capacitors

2 $$\bar{V} = \left(\omega L \angle 90^\circ\right)\left(I_m \angle\theta\right)$$ **Substitute** $j = \angle 90^\circ$ **and** $\bar{I} = I_m \angle\theta$ **into 1**

$$\underbrace{\boxed{\bar{V} = \omega \cdot L \cdot I_m \cdot \angle\left(\theta + 90^\circ\right)}}_{\textbf{Phasor}} \Rightarrow \underbrace{v(t) = \omega \cdot L \cdot I_m \cdot \cos\left(\omega t + \theta + 90^\circ\right)}_{\textbf{Time Domain}}$$ **Voltage Leads Current**

$$i(t) = I_m \cos\left(\omega t + \theta\right)$$ **Original definition of $i(t)$**

Capacitor



Use Similar Arguments

$$\bar{V} = \left(\frac{1}{j\omega C} \right) \bar{I} \quad \Big\} \text{Like Ohm's Law}$$

$$\bar{V} = \left(\frac{-j}{\omega C} \right) \bar{I} \quad \Big\} \text{Use } \frac{1}{j} = -j$$

Current Leads Voltage

Impedance Summary

$$\bar{V}_R = \bar{I}(R)$$

$$\bar{V}_L = \bar{I}(j\omega L)$$

$$\bar{V}_C = \bar{I}\left(\frac{1}{j\omega C}\right)$$

All Look Like A
Form Of
“Ohm’s” Law

$\bar{V} = \bar{I}Z$ where

$$Z_R = R$$

$$Z_L = j\omega L$$

$$Z_C = \left(\frac{1}{j\omega C}\right)$$

<u>Note:</u>

1. ***Z*** **is a Complex Number**
2. ***Z*** **is <u>NOT</u> a Phasor**

<u>Reactance</u>: Imaginary Part of Impedance

ωL for Inductors $\boxed{Z_L = j\omega L}$

$-\dfrac{1}{\omega C}$ for Capacitors $\boxed{Z_C = -\dfrac{j}{\omega C}}$

0 for Resistors $\boxed{Z_R = R}$

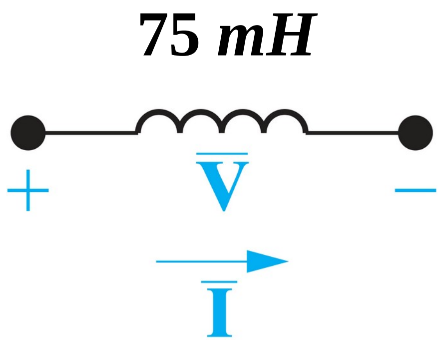
$$\boxed{Z_{Total} = \text{Resistance} + j\ \text{Reactance}}$$

<u>Notes:</u>

So we can think of these "Impedances" as "Analogous to Resistance".

Ohm's Law, KVL, KCL, all behave Mathematically as if ***Z***'s were ***R***'s .

Example: Find *v(t)*



$$i(t) = 4\cos\left(\overset{\omega}{40{,}000}\,t - 38^\circ\right)\ (mA)$$ **Time Domain**

$$\bar{I} = 4\angle -38^\circ\ (mA)$$ **Phasor**

$a)$ Reactance $= \omega L = 40{,}000\left(75 \times 10^{-3}\right)$

$$\omega L = \boxed{3000\ (\Omega)\quad \text{Positive Number}}$$

$b)$ Impedance $= Z_L = j\omega L = \boxed{j3000\,(\Omega)\quad \text{Complex Number}}$

Example (Contd.)

$$c)\quad \bar{V} = \bar{I}\cdot j\omega L = \left(4\times 10^{-3}\angle -38^{\circ}\right)\cdot j40{,}000\cdot 0.075$$ Use Ohm's Law

$$= \left(4\times 10^{-3}\angle -38^{\circ}\right)\cdot j3000$$

$$= \left(12\angle -38^{\circ}\right)\cdot j$$

$$= \left(12\angle -38^{\circ}\right)\cdot\left(\angle 90^{\circ}\right)$$ j = 90º

$$\bar{V} = 12\angle 52^{\circ}\ \ (V)$$ Phasor

$$d)\quad v(t) = 12\cos\left(\omega t + 52^{\circ}\right)\ \ (V)$$ Time Domain

$$v(t) = 12\cos\left(40{,}000t + 52^{\circ}\right)\ \ (V)$$ Plug in ω

Kirchhoff's Laws in the Frequency Domain

$$\left.v_1(t) + v_2(t) + \ldots + v_n(t) = 0\right\}$$ **KVL**

$$\bar{V}_1 + \bar{V}_2 + \ldots + \bar{V}_n = 0$$ Phasor Version

$$\left.i_1(t) + i_2(t) + \ldots + i_n(t) = 0\right\}$$ **KCL**

$$\bar{I}_1 + \bar{I}_2 + \ldots + \bar{I}_n = 0$$ Phasor Version

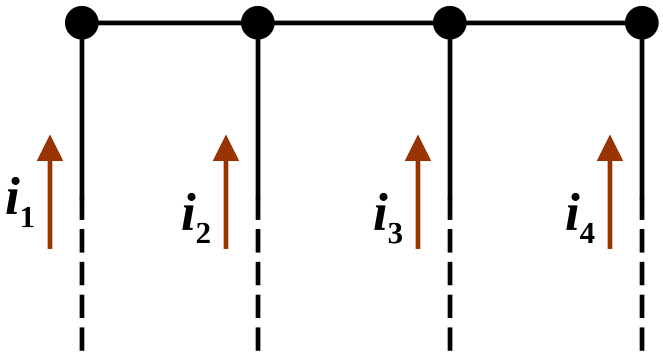
KVL and KCL along with $\bar{V} = \bar{I}\ \bar{Z}$

Forms the Basis of Frequency Domain (AC) Analysis

We can sse same techniques as in Chap. 1 - 4

but we must manipulate complex numbers.

Example: Find $i_4(t)$



$$i_1(t) = 100\cos\left(\omega t + 25^\circ\right)\ A$$
$$i_2(t) = 100\cos\left(\omega t + 145^\circ\right)\ A$$
$$i_3(t) = 100\cos\left(\omega t - 95^\circ\right)\ A$$

KCL: $i_4 = -\left(i_1 + i_2 + i_3\right)$ $\quad \bar{I}_4 = -\left(\bar{I}_1 + \bar{I}_2 + \bar{I}_3\right)$

$\bar{I}_1 = 100\angle 25^\circ = 90.63 + j42.26$ **Convert**

$\bar{I}_2 = 100\angle 145^\circ = -81.92 + j57.36$ **Convert**

$\bar{I}_3 = 100\angle -95^\circ = -8.71 - j99.62$ **Convert**

$\bar{I}_1 + \bar{I}_2 + \bar{I}_3 = 0 - j0$ **Add together**

$\bar{I}_4 = 0 + j0\ \text{A} = 0$

$$i_4(t) = 0\ (A)$$

Series and Parallel Combinations

Series: $$Z_{tot} = Z_1 + Z_2 + \ldots + Z_n$$ **Add**

Parallel: $$\frac{1}{Z_{tot}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_n}$$ **Add reciprocals**

$$n = 2 \Rightarrow Z_{tot} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Admittance Makes Parallel Combinations Easier

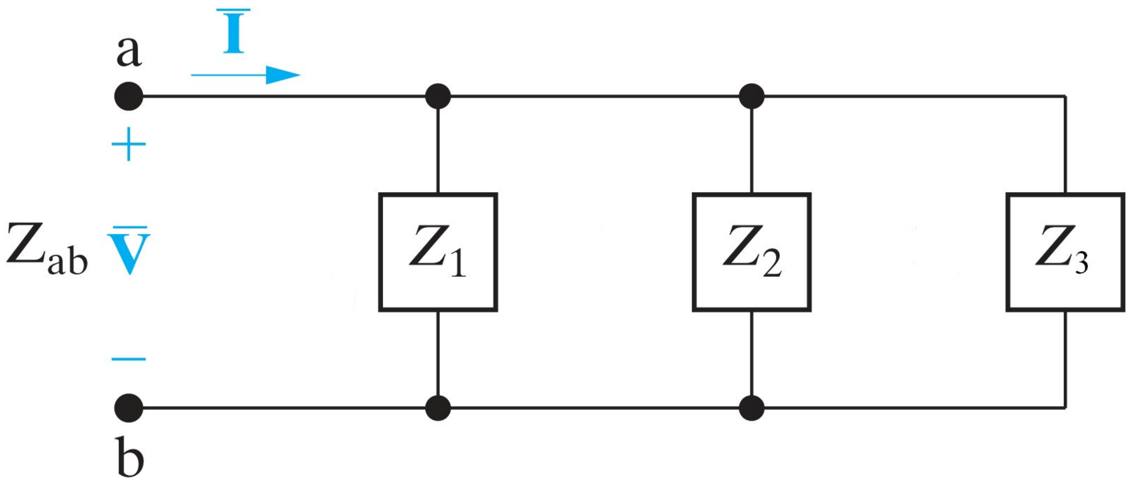
$$Y = \left(\frac{1}{Z}\right) = G + jB$$

$G \equiv$ Conductance

$B \equiv$ Susceptance

Series and Parallel Combinations

(Contd.)



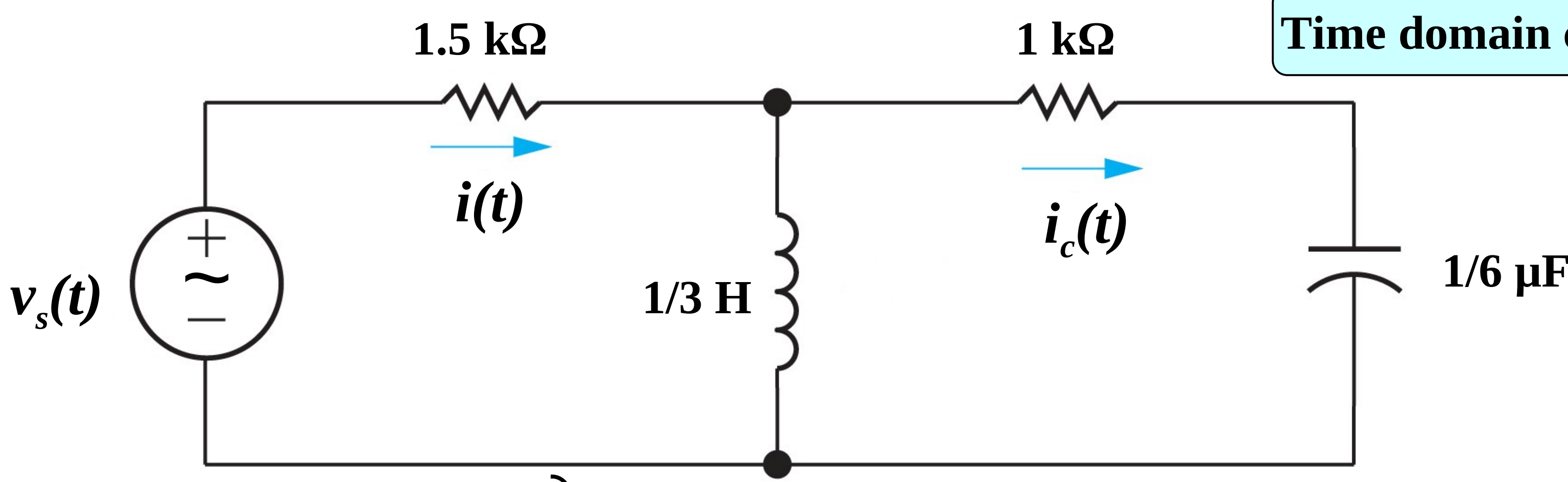
$$Y_{ab} = Y_1 + Y_2 + Y_3 = Y_m \angle \theta$$

Add admittance

$$Z_{ab} = \frac{1}{Y_{ab}} = \frac{1}{Y_m \angle \theta} = \frac{1}{Y_m} \angle -\theta$$

Reciprocate to obtain impedance

Example: RLC Circuit



Time domain circuit

$$v_s = 40\sin(3000t)$$
$$\omega = 3000\ rad/s$$

Given

Find the steady currents $i(t)$ and $i_c(t)$

$$j\omega L = j(3000)\left(\frac{1}{3}\right) = j1\,(k\Omega)$$

Inductor impedance

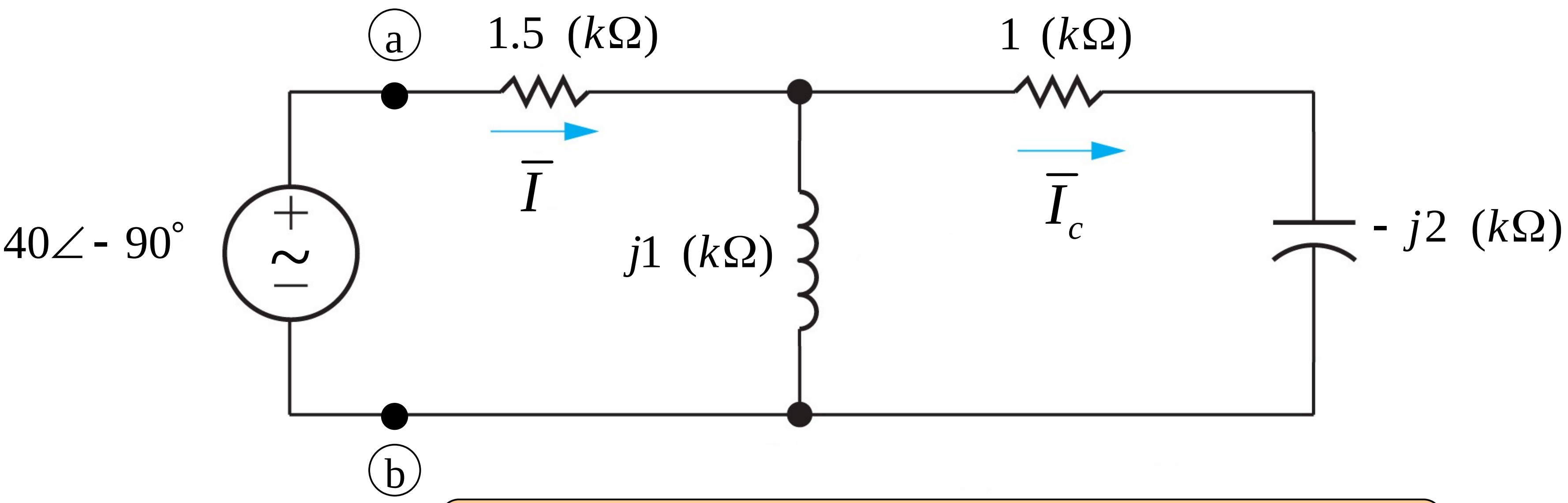
$$\frac{1}{j\omega C} = \frac{-j}{\omega C} = -j\left(\frac{1}{3000}\right)\left(\frac{1}{1/6\times 10^{-6}}\right) = -j2\,(k\Omega)$$

Capacitor impedance

Example (Contd.)

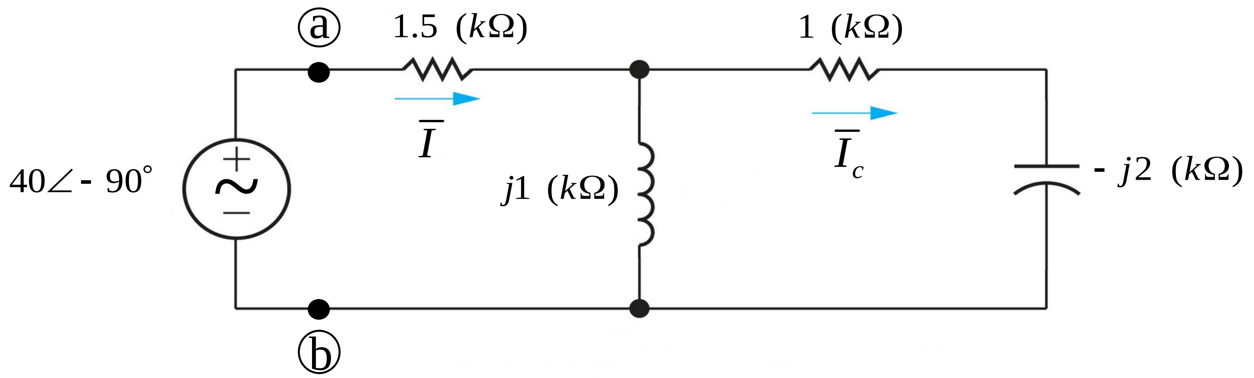
Find the Frequency Domain Circuit

$$\bar{V}_s = 40\angle -90^o \quad \text{since } V_s = 40\underline{\underline{\sin}}3000t$$



Find Impedance at a and b then find $i(t)$ and $i_c(t)$

Example (Contd.)



Find Equivalent Impedance "Seen" by the Source:

$$Z_{eq} = [1.5 + j \,||\, (1 - j2)] \ (k\Omega)$$

Impedance at ⓐ and ⓑ

$$= 1.5 + \frac{j(1-j2)}{j+(1-j2)} = 1.5 + \frac{j+2}{-j+1}$$

$$= 1.5 + \frac{2+j}{1-j}\left(\frac{1+j}{1+j}\right) = 1.5 + \frac{2+2j+j-1}{1-j+j+1}$$

Complex Conjugate

$$= 1.5 + \frac{1+j3}{2} = 2 + j1.5$$

$$\boxed{Z_{eq} = 2.5\angle 36.9^\circ \ (k\Omega)}$$

Convert to Polar

Example: RLC Circuit

Find $\bar{I}$

$$\bar{V}_s = \bar{I} \cdot Z_{eq}$$ Ohm's Law

$$\bar{V}_s = 40\angle -90^o (V)$$
$$Z_{eq} = 2.5\angle 36.9^o (K\Omega)$$

Found this so far

$$\bar{I} = \frac{\bar{V}_s}{Z_{eq}} = \frac{40\angle -90^o}{2.5\angle 36.9^o}\left(\frac{V}{k\Omega}\right)$$

$$\bar{I} = 16\angle -126.9^o (mA)$$ **Phasor**

Plug in numbers

Note: Phase shift between $\bar{I}$ and $\bar{V}$

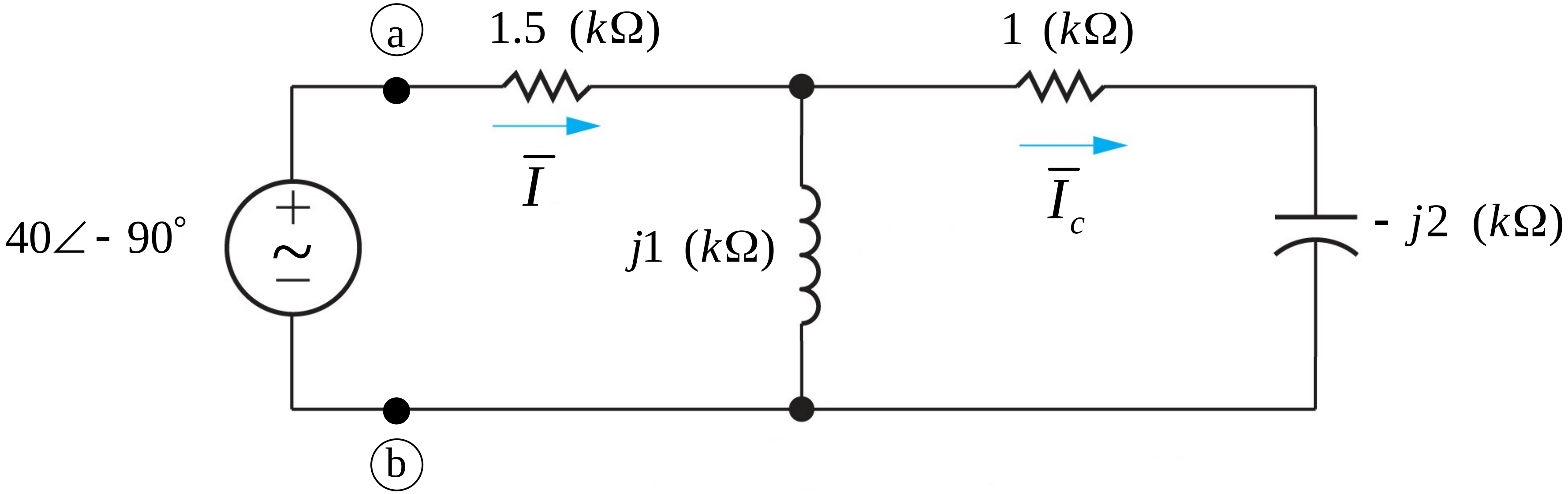
$$\bar{I} = 16\angle -126.90$$
$$\bar{V}_s = 40\angle -90^o$$

- V and I Out of Phase 36.9^o
- Due to Impedance of Circuit

$$i(t) = 16\cos(3000t - 126.9^\circ)\ \ (mA)$$

Time domain

Example: Find $\overline{I}_c$ **(Contd.)**



$$\overline{I} = 16\angle -126.9^\circ$$

From Previous slide

$$\overline{I}_c = \frac{j1}{j1 + 1 - j2}\overline{I}$$

Current Division

Example (Contd.)

$$\bar{I}_c = \frac{j1}{1-j}\cdot\bar{I} = \frac{-1+j}{2}\cdot\bar{I}$$
From Previous Slide

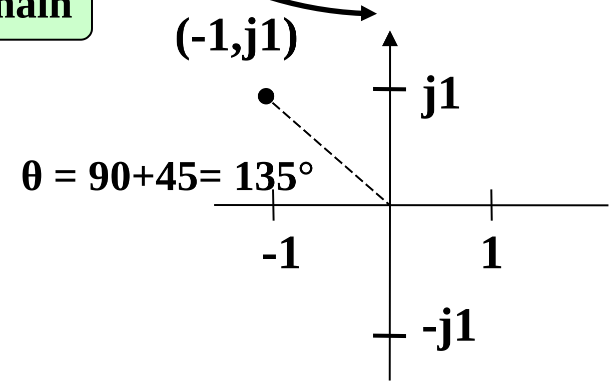
$$\bar{I}_c = 16\angle -126.9^\circ\,(-1+j)\,(1/2) = 8\angle -126.9^\circ\,[-1+j]$$
Plug in numbers

$$\bar{I}_c = 8\angle -126.9^\circ\,\sqrt{2}\,\angle 135^\circ$$
Caution

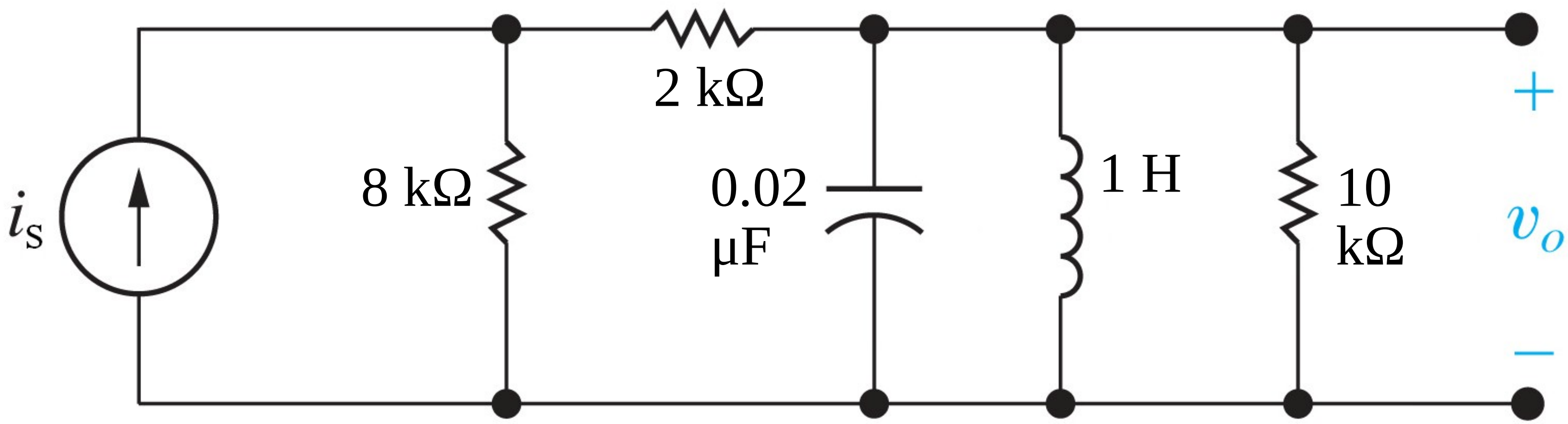
$$\bar{I}_c = 8\sqrt{2}\,\angle(-126.9+135)^\circ\ (mA)$$
Simplify

$$\bar{I}_c = 11.31\angle 8.1^\circ$$
Phasor

$$i_c(t) = 11.31\cos(3000t + 8.1^\circ)\ \ (mA)$$
Time Domain



Example: Find $v_0(t)$



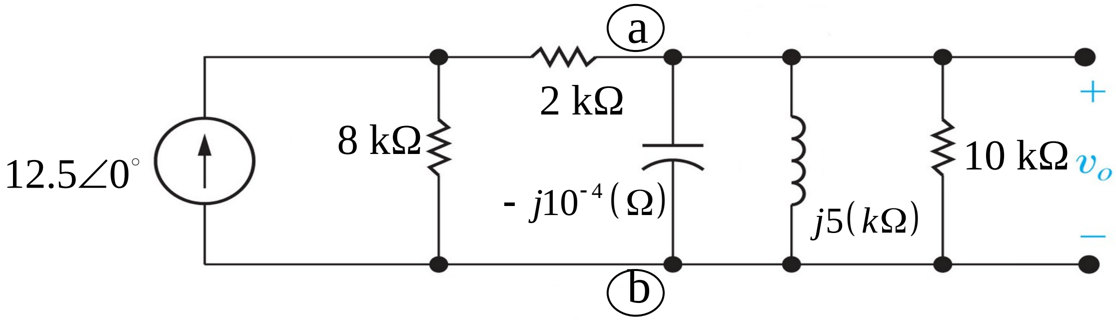
$$i_s = 12.5\cos(5000t)\ (mA) \qquad \omega = 5000\, rad/\sec$$ **Given**

$$Z_L = j\omega L = j(5000)(1) = j5000\ (\Omega)$$ **Impedance**

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{5000}\bullet\frac{1}{0.02\times 10^{-6}} = -j10^4\ (\Omega)$$ **Impedance**

$$\bar{I}_s = 12.5\angle 0^\circ\ \ (mA)$$ **Phasor**

Example (Contd.)



Frequency Domain Circuit

$$\bar{V}_s' = \bar{I}_s \cdot 8k\Omega = (12.5\angle 0^\circ\ mA)(8\times 10^3\,\Omega)$$

$$\bar{V}_s' = 100\angle 0^\circ\ (V)$$

Source transformation at a and b

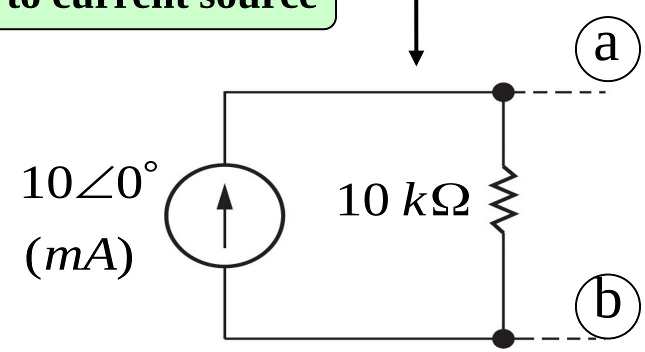
$$R_{eq} = 8k\Omega + 2k\Omega = 10(k\Omega)$$

Series resistors

$$\bar{I}_s' = \frac{\bar{V}_s'}{R_{eq}} = \frac{100\angle 0^\circ}{10k\Omega}$$

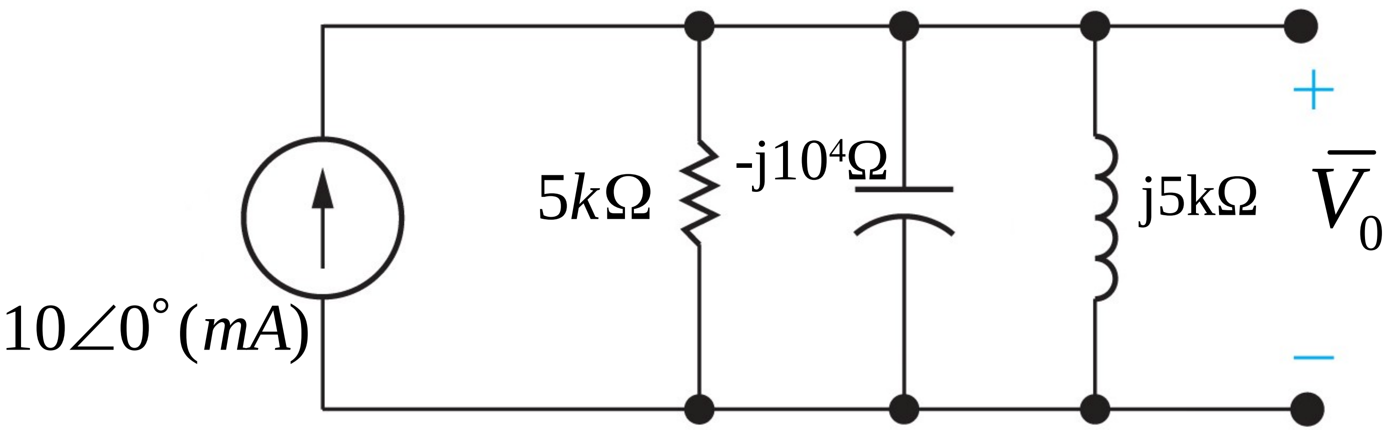
Convert back to current source

$$\bar{I}_s' = 10\angle 0^\circ\ (mA)$$



Example (Contd.)

Frequency Domain Equivalent Circuit



Combine two $10K\Omega$ resistors $10K \parallel 10K = 5(K\Omega)$

$$\bar{V}_0 = \bar{I}.Z_{eq} = \frac{\bar{I}}{Y_{eq}}$$

Use admittance for parallel circuit

$$Y = \frac{1}{Z}$$

$$Y_{eq} = \frac{1}{5K} + \frac{1}{-j10^4} + \frac{1}{j5\times 10^3} = \frac{1}{5K} + j10^{-4} - j0.2\times 10^{-3}$$

Find Y_{eq}

$$Y_{eq} = 2\times 10^{-4} - j10^{-4} = 0.224\angle -26.57^\circ \left(m\Omega^{-1}\right)$$

Simplify

Example (Contd.)

$$Y_{eq} = 2\times10^{-4} - j10^{-4} = 0.224\angle -26.57^{\circ}\left(m\Omega^{-1}\right)$$ From previous slide

$$Z_{eq} = \frac{1}{Y_{eq}} = \frac{1}{0.224\times10^{-3}}\angle 26.57^{o}\left(\Omega\right)$$ Find Z_{eq}

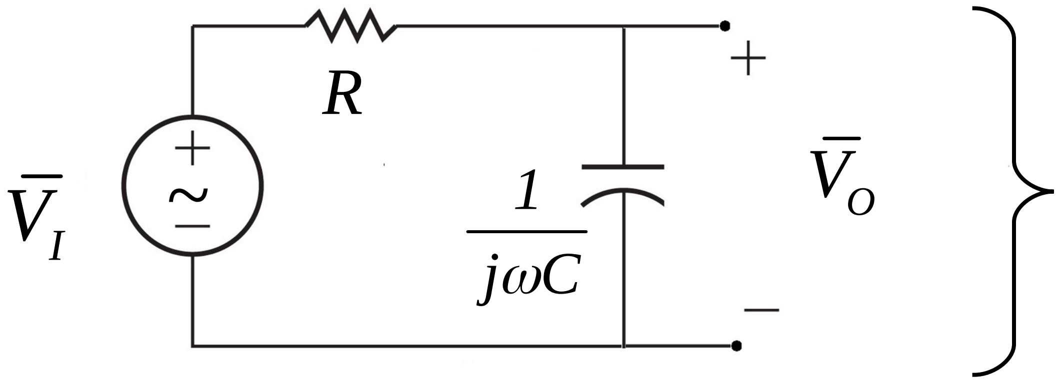
$$\bar{V}_0 = \bar{I}\cdot Z_{eq}$$ Ohm's Law

$$\bar{V}_0 = 10\angle 0^{\circ}(mA)\cdot\left[\frac{1}{0.224\times10^{-3}}\angle 26.57^{\circ}\right](\Omega)$$ Plug in numbers

$$\bar{V}_0 = 44.64\angle 26.57^{\circ}$$ Phasor

$$\boxed{v_0(t) = 44.64\cos(5000t + 26.57^{\circ})\ \ (V)}$$ Time Domain

Example: Frequency Dependent Impedance



Low Pass Filter

Find the Gain $\equiv$ Transfer Function $\equiv \dfrac{\bar{V}_0}{\bar{V}_I} = \bar{A}_v$

$\bar{A}_v \equiv 1$ is the best we can hope for in a Passive Circuit.

Example: Frequency Dependent Impedance (Contd.)

Voltage Division:

$$\bar{V}_0 = \frac{1/\,j\omega C}{1/\,j\omega C + R}\cdot\bar{V}_I = \frac{1/\,\cancel{j\omega C}}{(1+j\omega CR)/\,\cancel{j\omega C}}\cdot\bar{V}_I$$

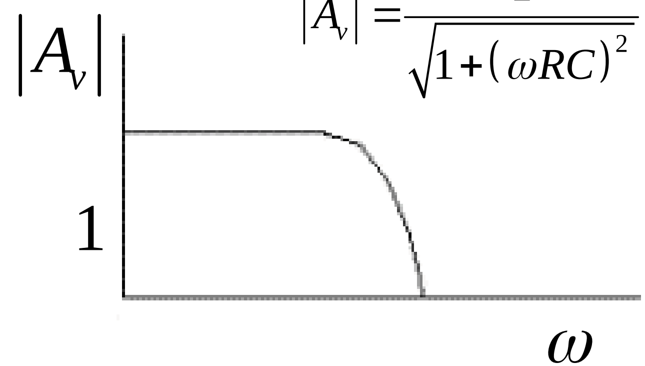
Simplify

$$A_v = \frac{\bar{V}_0}{\bar{V}_I} = \frac{1}{1+j\omega RC}$$

Transfer function

$$\omega = 0 \Rightarrow A_v = 1$$
$$\omega \to \infty \Rightarrow A_v \to 0$$

$$|A_v| = \frac{1}{\sqrt{1+(\omega RC)^2}}$$



Taking Output Across "R" $\Rightarrow$ High Pass Filter

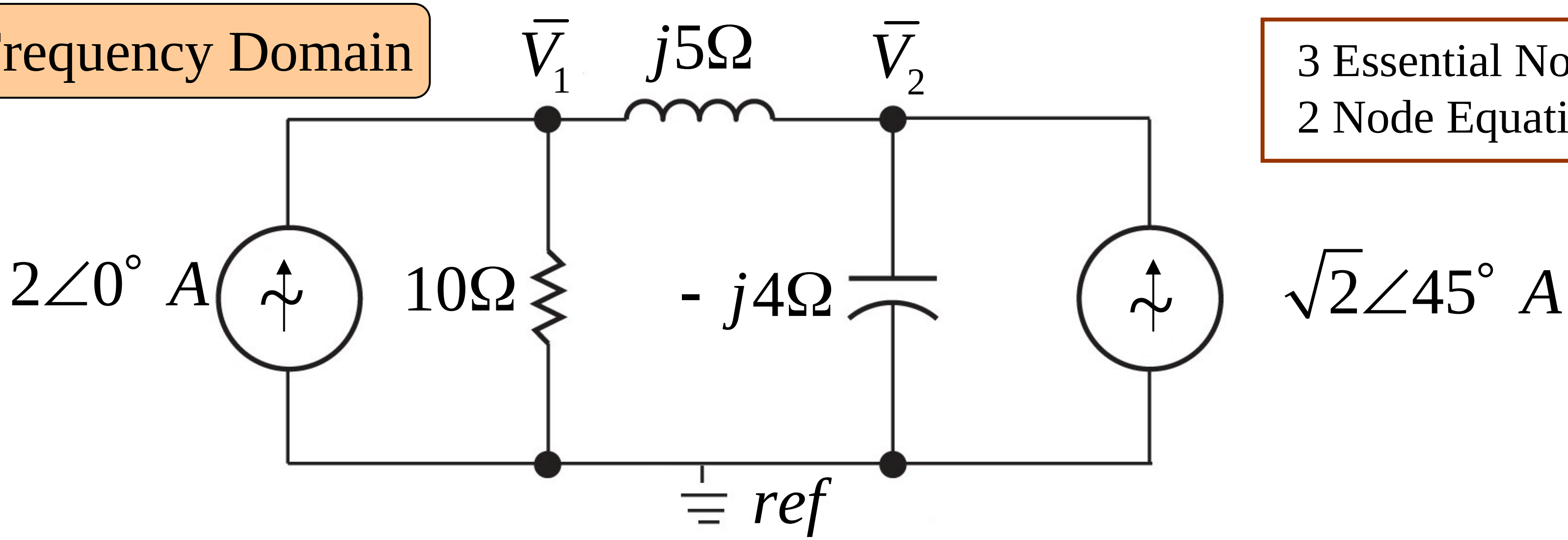
$$A_V = \frac{\bar{V}_R}{\bar{V}_I} = \frac{R}{R + \frac{1}{j\omega C}}$$

High Pass Filter if we take V_0 across the Resistor

Node Voltage Analysis: Find $v_1(t)$

Frequency Domain

3 Essential Nodes
2 Node Equations



$$(1) \quad -2\angle 0^\circ + \frac{\bar{V}_1}{10} + \frac{\bar{V}_1 - \bar{V}_2}{j5} = 0$$

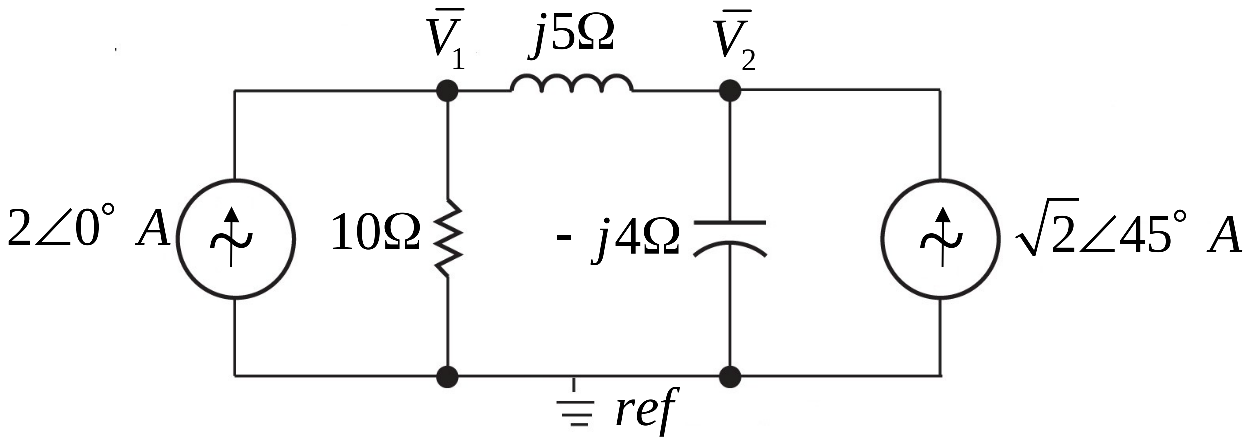
Node equation for $\bar{V}_1$

$$(1) \quad \bar{V}_1 \frac{j5}{10} + \bar{V}_1 - \bar{V}_2 = +2\angle 0^\circ (j5) = 2j5 = j10$$

Simplify

$$(1) \quad \boxed{\bar{V}_1(1 + j0.5) - \bar{V}_2 = j10}$$

Node Voltage Analysis (Contd.)



$$(2)\quad \frac{\bar{V}_2 - \bar{V}_1}{j5} + \frac{\bar{V}_2}{-j4} = \sqrt{2}\angle 45^\circ = 1 + j$$

Node equation for $\bar{V}_2$

$$-j4(\bar{V}_2 - \bar{V}_1) + j5\bar{V}_2 = 20 + j20$$

$$j4(\bar{V}_1) + \bar{V}_2(j5 - j4) = 20 + j20$$

Simplify

$$(2)\quad \bar{V}_1(j4) + \bar{V}_2(j) = 20 + j20$$

$$(2)\quad j\bar{V}_2 = 20 + j20 - \bar{V}_1(j4)$$

Solve for $\bar{V}_2$

$$\bar{V}_2 = \frac{20}{j} + 20 - \bar{V}_1 4$$

Simplify

$$(2)\quad \boxed{\bar{V}_2 = 20 - 4\bar{V}_1 - j20}$$

$$\bar{V}_1(1 + j0.5) - 20 + 4\bar{V}_1 + j20 = j10$$

Plug (2) into (1)

Node Voltage Analysis (Contd.)

$$\bar{V}_1(1 + j0.5 + 4) = j10 - j20 + 20$$ From previous slide

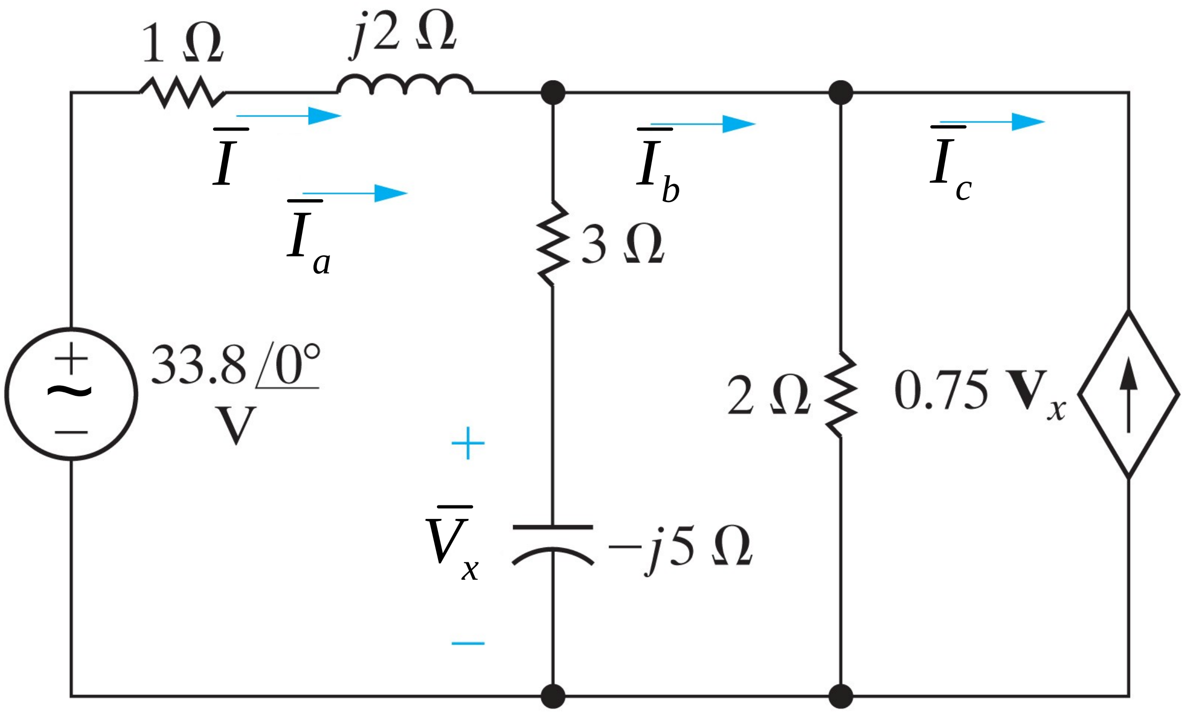
$$\bar{V}_1(5 + j0.5) = 20 - j10$$ Simplify

$$\bar{V}_1 = \frac{20 - j10}{5 + j0.5} = \frac{22.36\angle -26.57^\circ}{5.02\angle 5.71^\circ}$$ Solve for $\bar{V}_1$

$$\bar{V}_1 = 4.45\angle -32.28^\circ$$ Phasor

$$\boxed{v_1(t) = 4.45\cos(\omega t - 32.28^\circ) \ (V)}$$ Time domain

Mesh Analysis: Find $\bar{I}$



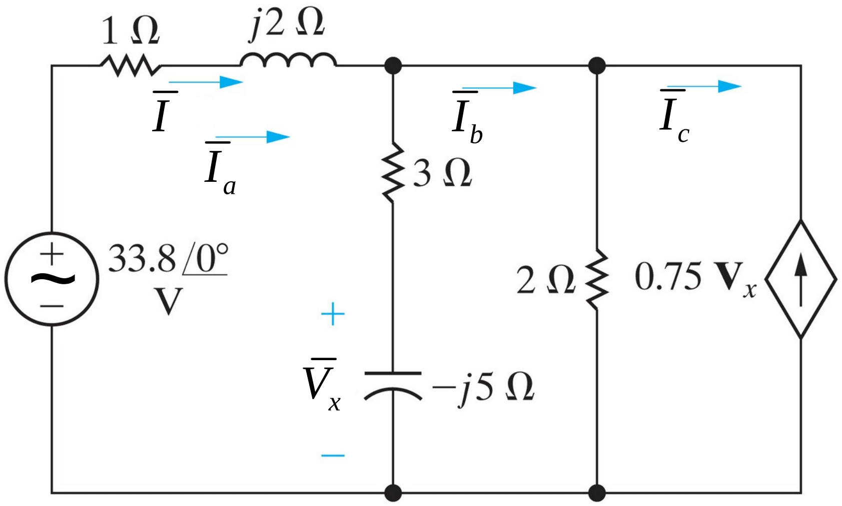
Note $\bar{I} \equiv \bar{I}_a$

$$(a) \quad 33.8\angle 0^\circ = (1 + j2)\bar{I}_a + (3 - j5)\left(\bar{I}_a - \bar{I}_b\right)$$ **Mesh a**

$$(1 + j2 + 3 - j5)\bar{I}_a - (3 - j5)\bar{I}_b = 33.8$$ **Simplify**

$$(a) \quad \boxed{(4 - j3)\bar{I}_a - (3 - j5)\bar{I}_b = 33.8}$$

Mesh Analysis (Contd.)



$$(b)\quad (3-j5)\left(\bar{I}_b-\bar{I}_a\right)+2\left(\bar{I}_b-\bar{I}_c\right)=0$$ **Mesh b**

$$-(3-j5)\bar{I}_a+(3-j5+2)\bar{I}_b-2\bar{I}_c=0$$ **Simplify**

$$(b)\quad \boxed{-(3-j5)\bar{I}_a+(5-j5)\bar{I}_b-2\bar{I}_c=0}$$

$$(c)\quad \bar{I}_c=-0.75\bar{V}_x=-0.75\left[-j5\left(\bar{I}_a-\bar{I}_b\right)\right]$$ **Constraint**

$$(c)\quad \bar{I}_c=j3.75\left(\bar{I}_a-\bar{I}_b\right)$$

$$\boxed{\bar{V}_x=-j5\left(\bar{I}_a-\bar{I}_b\right)}$$

Mesh Analysis (Contd.)

$$-(3-j5)\bar{I}_a + (5-j5)\bar{I}_b - j7.5\left(\bar{I}_a - \bar{I}_b\right) = 0$$ **Substitute (c) into (b)**

$$(b) \quad -(3+j2.5)\bar{I}_a + (5+j2.5)\bar{I}_b = 0$$ **Simplify**

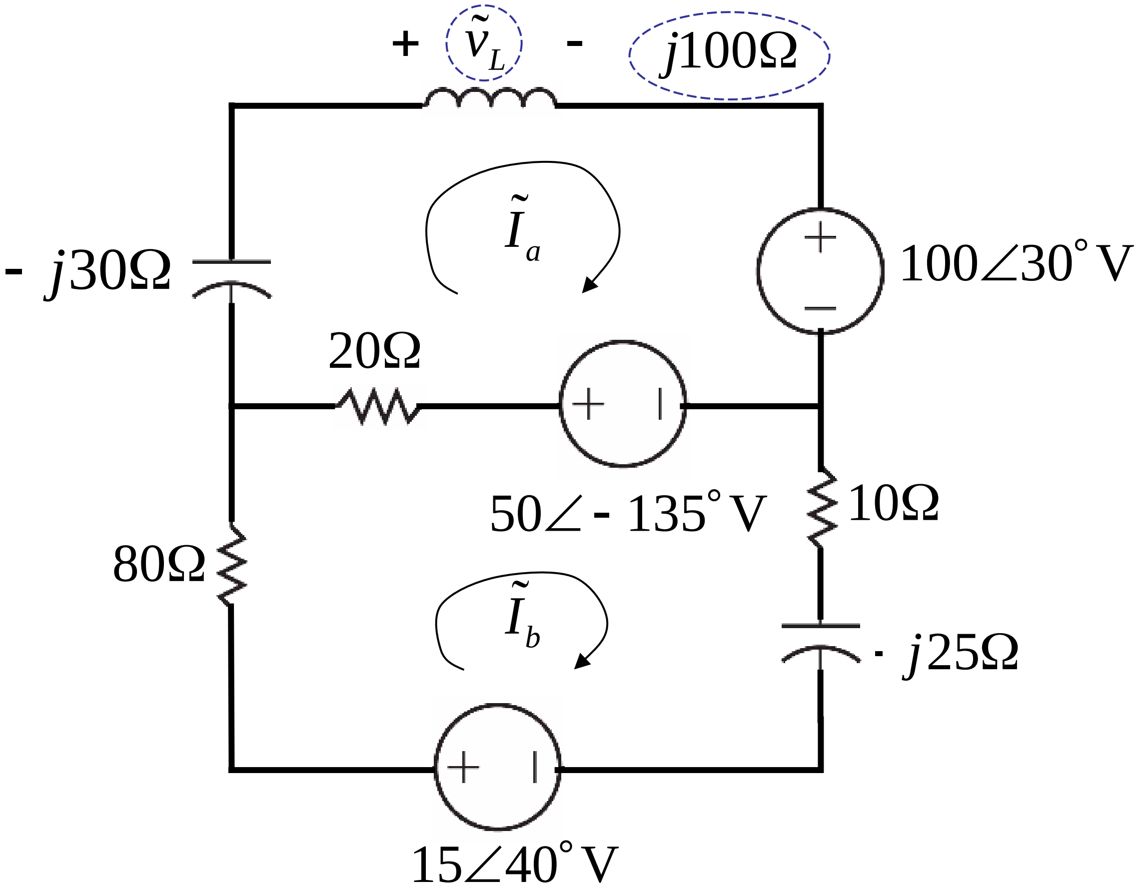
$$(a) \quad (4-j3)\bar{I}_a - (3-j5)\bar{I}_b = 33.8$$

Solve two Equations for two unknowns

Result:

$$\bar{I} = \bar{I}_a = 29.07\angle 3.95^\circ \; (A)$$

Example Using Mesh Current



Find Mesh Currents

$\tilde{I}_a$, $\tilde{I}_b$, **and** $\tilde{v}_L$.

Can You Find L ?

No

$$Z_L = j\omega L = j100$$

$$L = \frac{j100}{j\omega}$$

Side Note

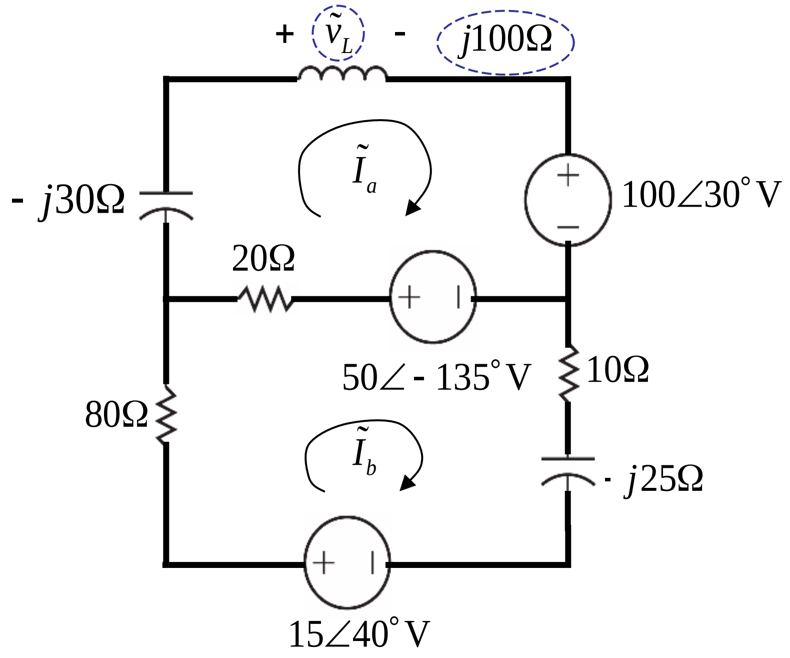
Need

$$L = \frac{100}{\omega} \qquad f = \frac{\omega}{2\pi}$$

Note: $$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

$$Z_L = j\omega L$$

Example Using Mesh Current (Contd.)



Mesh a

a $$\tilde{I}_a(-j30) + \tilde{I}_a(j100) + 100\angle 30^\circ - 50\angle -135^\circ + (\tilde{I}_a - \tilde{I}_b)(20) = 0$$

b $$\tilde{I}_b(80) + (\tilde{I}_b - \tilde{I}_a)(20) + 50\angle -135^\circ + \tilde{I}_b(10 - j25) - 15\angle 40^\circ = 0$$

Mesh b

Example Using Mesh Current (Contd.)

(a) $$\tilde{I}_a(-j30 + j100 + 20) + \tilde{I}_b(-20) = (50\angle -135^\circ - 100\angle 30^\circ)$$

$$(20 + j70)\tilde{I}_a - 20\tilde{I}_b = 148.9\angle -145^\circ$$ **Convert to Rectangular to Add**

(b) $$-20\tilde{I}_a + (80 + 20 + 10 - j25)\tilde{I}_b = (15\angle 40^\circ - 50\angle -135^\circ)$$

$$-20\tilde{I}_a + (110 - j25)\tilde{I}_b = 64.96\angle 43.85^\circ$$ **Convert to Rectangular to Add**

Matrix Form

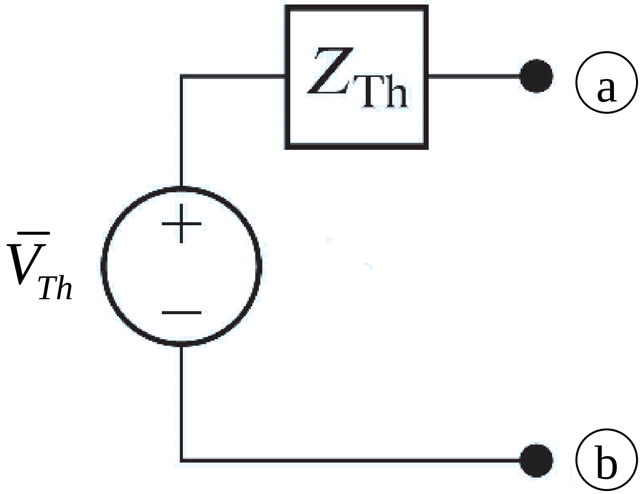
$$\begin{bmatrix} (20 + j70) & -20 \\ -20 & (110 - j25) \end{bmatrix} \begin{bmatrix} \tilde{I}_a \\ \tilde{I}_b \end{bmatrix} = \begin{bmatrix} 148.9\angle -145^\circ \\ 64.96\angle 43.85^\circ \end{bmatrix}$$

Two equations Two unknowns

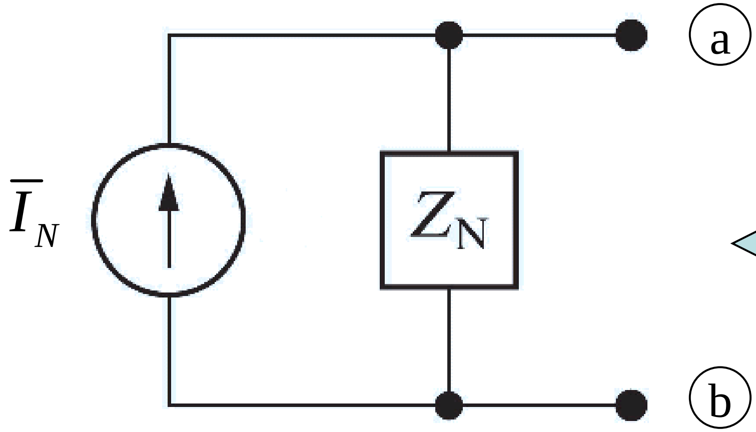
$$\rightarrow \tilde{I}_a = 1.943\angle -136.7^\circ\,(A) \quad \tilde{I}_b = 0.6535\angle 88.2^\circ\,(A)$$ **Solution**

$$\tilde{v}_L = \tilde{I}_a Z_L = \underbrace{(1.943\angle 136.7^\circ)}_{\tilde{I}_a}\underbrace{(j100)}_{Z_L} = 194.3\angle -133^\circ\,(V)$$ **Ohm’s Law**

Thevenin and Norton Equivalent Circuits

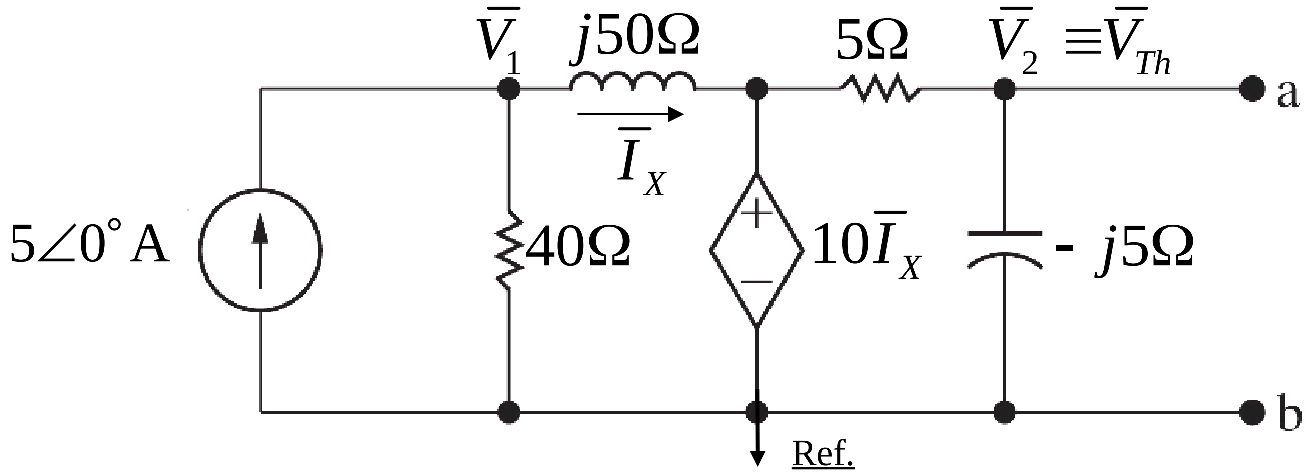


Thevenin Equivalent



Norton Equivalent

Example: Find Thevenin Equivalent



$\overline{V}_{Th} = \overline{V}_{oc} = \overline{V}_{ab}$

Use Node Voltages ; only 2 Unknown: $\overline{V}_1$, and $\overline{V}_{Th}$

Write equation for $\overline{I}_x$ in terms of $\overline{V}_1$ and $\overline{V}_{Th}$

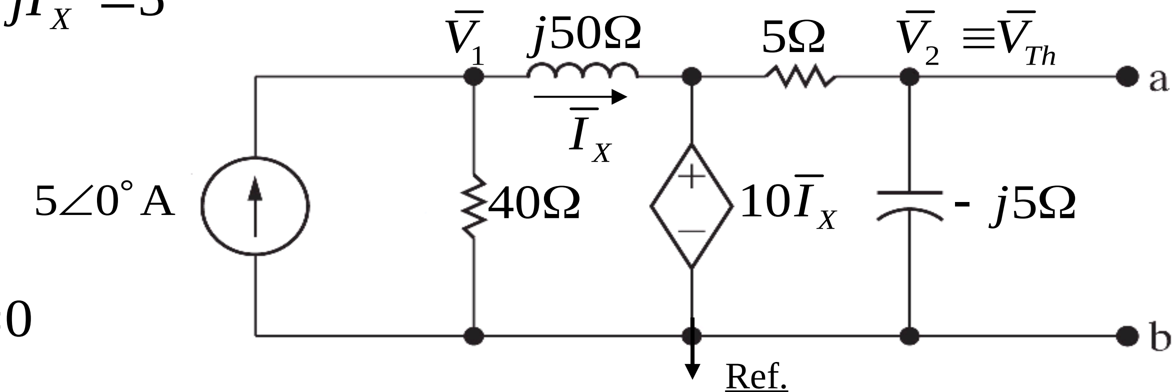
Example (Contd.)

Node $\bar{V}_1$ $$-5\angle 0^\circ + \frac{\bar{V}_1}{40} + \frac{\bar{V}_1 - 10\bar{I}_X}{j50} = 0$$

(1) $$\bar{V}_1[0.025 - j0.02] + 0.2j\bar{I}_X = 5$$

Node $\bar{V}_{Th}$ $$\frac{\bar{V}_{Th} - 10\bar{I}_X}{5} + \frac{\bar{V}_{Th}}{-j5} = 0$$

(2) $$\bar{V}_{Th}[0.2 + j0.2] - 2\bar{I}_X = 0$$



$$j50\bar{I}_X = \bar{V}_1 - 10\bar{I}_X$$ **Find another equation for $\bar{I}_X$**

$$\bar{I}_X[j50 + 10] = \bar{V}_1$$ **Solve for $\bar{I}_X$**

$$\bar{I}_X = \frac{\bar{V}_1}{10 + j50}$$

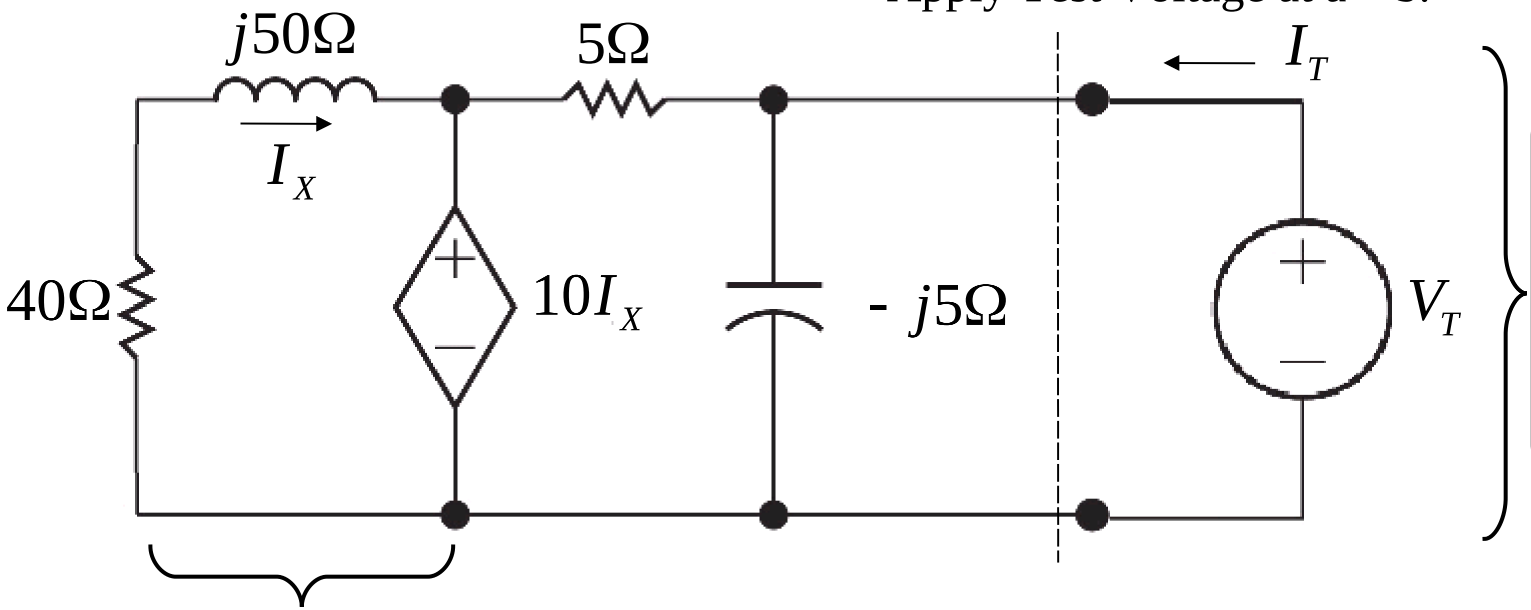
Substitute into (1) and (2)

Solve for $\bar{V}_{Th} = 20\angle -90^\circ (V)$

Example (Contd.)

Find Z_{Th}

- Open Independent Current Source
- Apply Test Voltage at a – b.



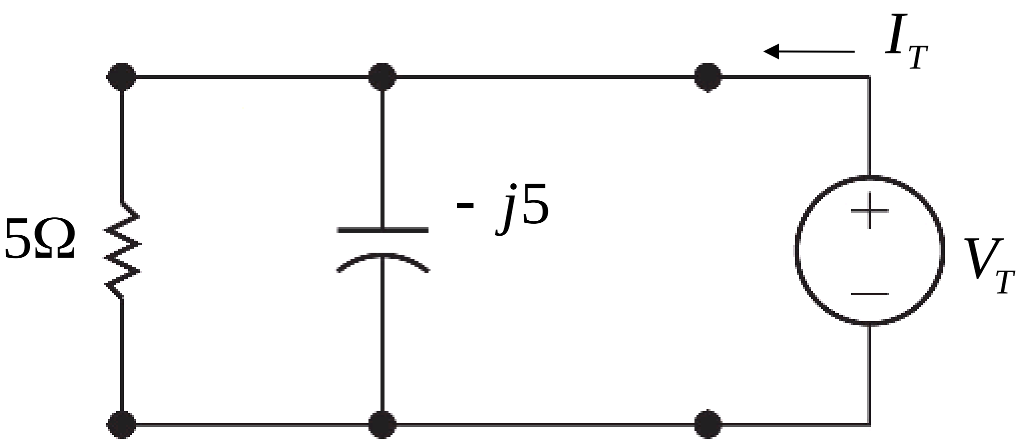
Redraw circuit with 5 amp source open

Write an equation for I_X

$$10\bar{I}_X = -\bar{I}_X\left[40 + j50\right]$$ **KVL**

$$I_X = 0 \Rightarrow 10I_X = 0$$

Example (Contd.)



Redraw circuit with dependent source shorted

KCL

$$I_T = \frac{V_T}{5} + \frac{V_T}{-j5}$$

Solve for I_T

$$I_T = V_T\left[\frac{1}{5} + \frac{1}{-j5}\right]$$

$$I_T = V_T\left[\frac{-j5+5}{-j25}\right]$$

Solve for Z_{Th}

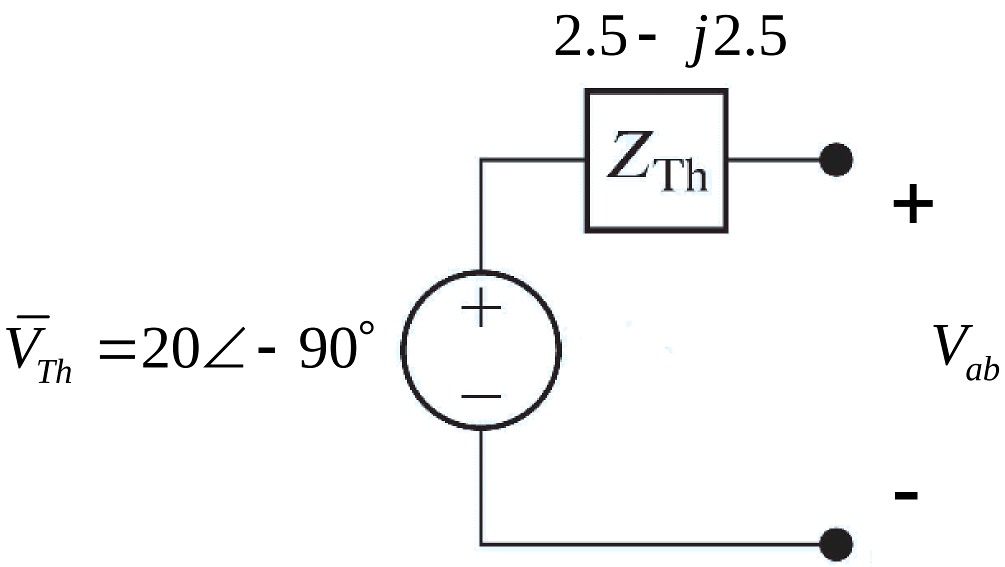
$$Z_{Th} = \frac{V_T}{I_T} = \frac{-j25}{5-j5} = \frac{-j5}{1-j}$$

Simplify

$$Z_{Th} = \frac{5}{+j(1-j)} = \frac{5}{j+1}\cdot\left(\frac{j-1}{j-1}\right)$$

$$Z_{Th} = \frac{5(j-1)}{-2} = 2.5 - j2.5(\Omega)$$

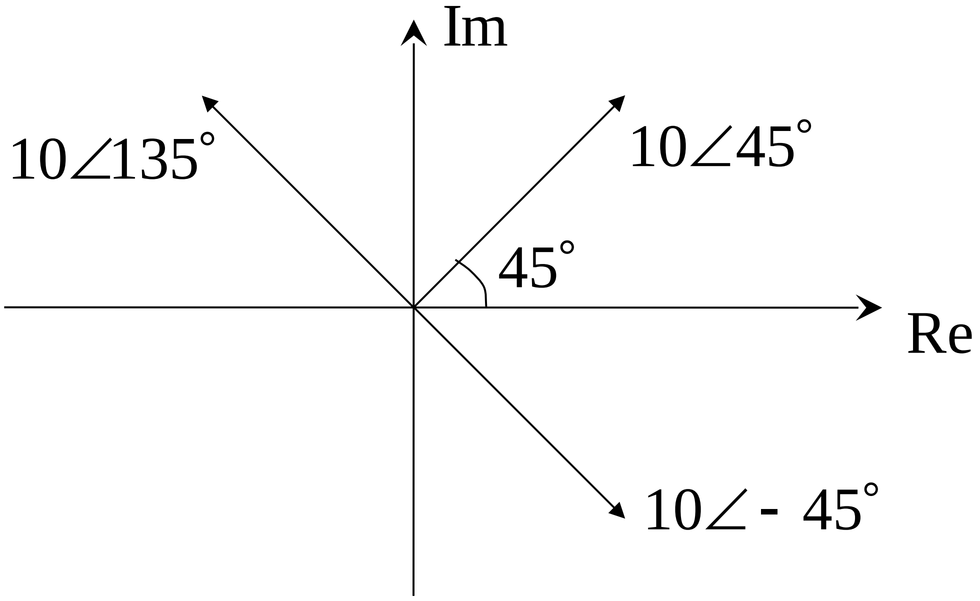
Example (Contd.)



Thevenin Equivalent

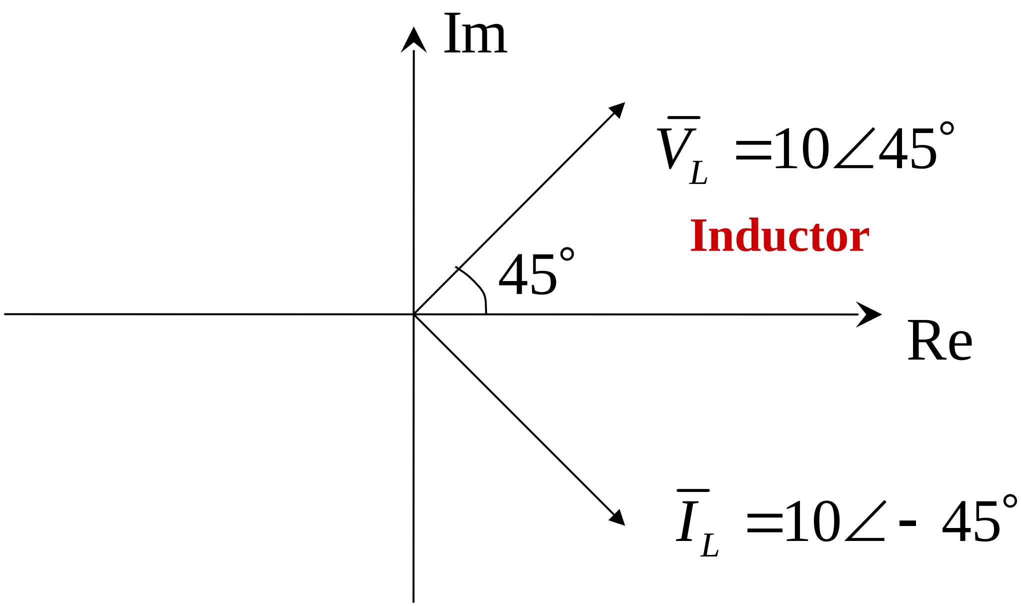
Phasor Diagrams **"For Visualization"**

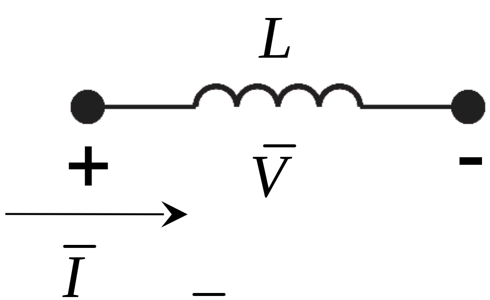
Magnitude∠Phase Angle "Graphs"



Can then plot $\bar{I}$ and $\bar{V}$ on Complex Planes

Phasor Diagrams (Contd.)





$\bar{I} = I_m\angle -45^\circ$

$\bar{V} = V_m\angle 45^\circ$

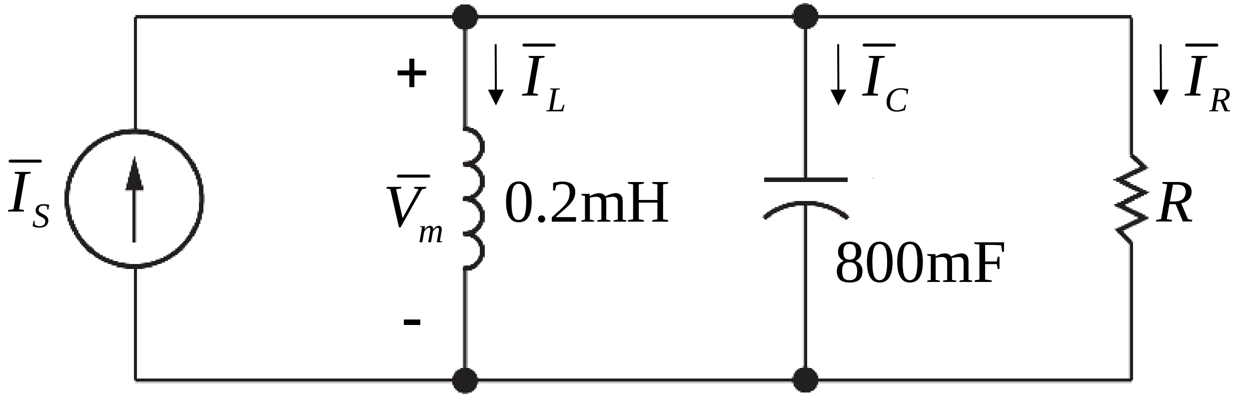
If $\bar{V} = V_m\angle 45^\circ$

$$\bar{I} = \frac{\bar{V}}{j\omega L} = \frac{V_m}{\omega L}\angle 45^\circ(-j) = \frac{V_m}{\omega L}\angle -45^\circ$$

I_m

Voltage "Leads" Current.

Example



① Assume

$$\omega = 5000\, rad/s$$

② Assume

$$\bar{V}_m = V_m \angle 0^\circ$$

Choose R such that $\bar{I}_R$ lags $\bar{I}_S$ by $\angle 45^\circ$

$$\bar{I}_S = \bar{I}_L + \bar{I}_C + \bar{I}_R \Big\} \text{ KCL}$$

Construct Phasor Diagram of $\bar{I}_S$ by plotting $\bar{I}_L$, $\bar{I}_C$, and $\bar{I}_R$

① $$\bar{I}_L = \frac{\bar{V}_m}{Z_L} = \frac{V_m \angle 0^\circ}{j\omega L} = \frac{V_m \angle 0^\circ}{j(5000)(0.2 \times 10^{-3})} \Bigg\} \text{ Ohm's Law}$$

$$\bar{I}_L = V_m \angle -90^\circ \equiv -jV_m$$

Example (Contd.)

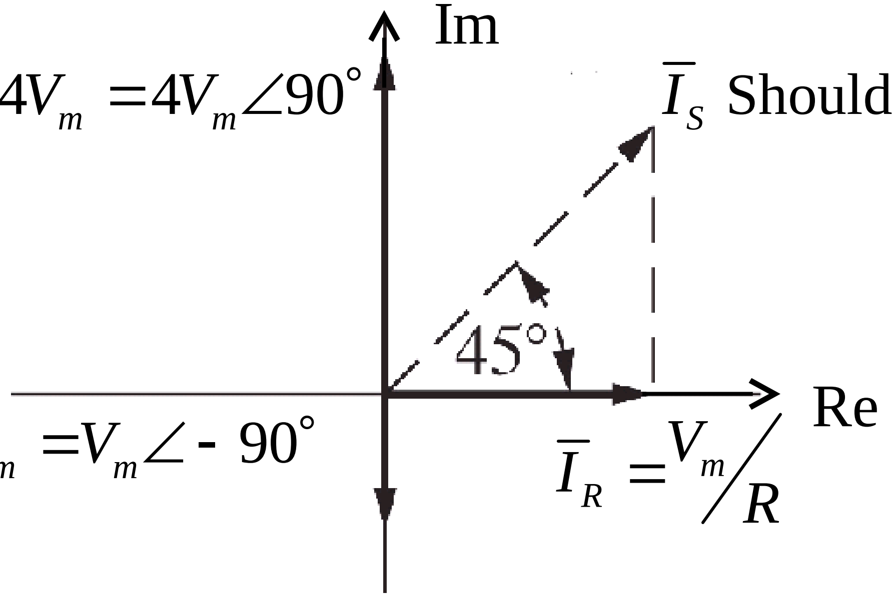
$$② \quad \bar{I}_C = \frac{\bar{V}_m}{Z_L} = \frac{V_m \angle 0^\circ}{\dfrac{1}{j\omega C}} = j(5000)(800 \times 10^{-6}) V_m \angle 0^\circ$$ **Ohm's Law**

$$= j4V_m \angle 0^\circ \qquad \bar{I}_C = 4V_m \angle 90^\circ = j4V_m$$

$$③ \quad \bar{I}_R = \frac{V_m \angle 0^\circ}{R} = \frac{V_m}{R} \angle 0^\circ$$ **Ohm's Law**

$$\bar{I}_C = j4V_m = 4V_m \angle 90^\circ$$

$\bar{I}_S$ Should Lead $\bar{I}_R$ by $\angle 45^\circ$



So What's R ?

$$\bar{I}_L = -jV_m = V_m \angle -90^\circ$$

$$\bar{I}_R = V_m / R$$

Example (Contd.)

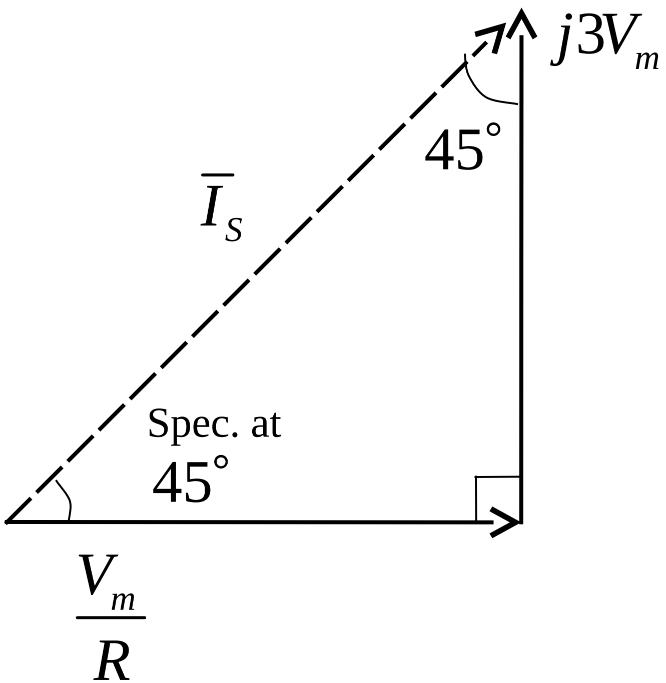
$$\overline{I}_s = \overline{I}_L + \overline{I}_C + \overline{I}_R$$

KCL

$$\overline{I}_s = \frac{V_m}{R} + j3V_m$$

Plug in values

Construct $\overline{I}_S$ by "Vector" or Phasor Sum



Isosceles Triangle ⇨ 2 Sides Are Equal

$$|\mathrm{Re}| = |\mathrm{Im}|$$

$$\left|\frac{V_m}{R}\right| = \left|j3V_m\right|$$

$$\frac{V_m}{R} = 3V_m \Rightarrow R = \frac{1}{3}(\Omega)$$