Example.

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4 \sin x - 2 \cos x.$$

The corresponding homogeneous equation is

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 0,$$

and the complementary function is

$$y_c = c_1 + c_2 x + c_3 \sin x + c_4 \cos x.$$

The nonhomogeneous term is the linear combination

$$3x^2 + 4 \sin x - 2 \cos x$$

of the three UC functions given by

$$x^2, \quad \sin x, \quad \text{and} \quad \cos x.$$

1. Form the UC set for each of these three functions. These sets are, respectively,

$$S_1 = \{x^2, x, 1\},$$

$$S_2 = \{\sin x, \cos x\},$$

$$S_3 = \{\cos x, \sin x\}.$$

2. Observe that S_2 and S_3 are identical and so we retain only one of them, leaving the two sets

$$S_1 = \{x^2, x, 1\}, \qquad S_2 = \{\sin x, \cos x\}.$$

3. Now observe that $S_1 = \{x^2, x, 1\}$ includes 1 and x, which, as the complementary function shows, are both solutions of the corresponding homogeneous differential equation. Thus we multiply each member of the set S_1 by x^2 to obtain the revised set

$$S_1' = \{x^4, x^3, x^2\},$$

none of whose members are solutions of the homogeneous differential equation. We observe that multiplication by x instead of x^2 would not be sufficient, since the resulting set would be $(x^3, x^2, x\}$, which still includes the homogeneous solution x. Turning to the set S_2, observe that both of its members, $\sin x$ and $\cos x$, are also solutions of the homogeneous differential equation. Hence we replace S_2 by the revised set

$$S_2' = \{x \sin x, x \cos x\}.$$

4. None of the original UC sets remain here. They have been replaced by the revised sets S_1' and S_2' containing the five elements

$$x^4, \qquad x^3, \qquad x^2, \qquad x \sin x, \qquad x \cos x.$$

We form a linear combination of these,

$$Ax^4 + Bx^3 + Cx^2 + Dx \sin x + Ex \cos x,$$

with undetermined coefficients A, B, C, D, E.

5. We now take this as our particular solution

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx \sin x + Ex \cos x.$$

Then

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + Dx \cos x + D \sin x - Ex \sin x + E \cos x,$$

$$y_p'' = 12Ax^2 + 6Bx + 2C - Dx \sin x + 2D \cos x - Ex \cos x - 2E \sin x,$$

$$y_p''' = 24Ax + 6B - Dx \cos x - 3D \sin x + Ex \sin x - 3E \cos x,$$

$$y_p^{(iv)} = 24A + Dx \sin x - 4D \cos x + Ex \cos x + 4E \sin x.$$

Substituting into the differential equation, we obtain

$$\begin{aligned}24A + Dx\sin x - 4D\cos x + Ex\cos x + 4E\sin x + 12Ax^2 + 6Bx + 2C \\ -Dx\sin x + 2D\cos x - Ex\cos x - 2E\sin x \\ = 3x^2 + 4\sin x - 2\cos x.\end{aligned}$$

Equating coefficients, we find

$$\begin{aligned}24A + 2C &= 0 \\ 6B &= 0 \\ 12A &= 3 \\ -2D &= -2 \\ 2E &= 4.\end{aligned}$$

Hence $A = \frac{1}{4}$, $B = 0$, $C = -3$, $D = 1$, $E = 2$, and the particular integral is

$$y_p = \tfrac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x.$$

The general solution is

$$\begin{aligned}y &= y_c + y_p \\ &= c_1 + c_2x + c_3\sin x + c_4\cos x + \tfrac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x.\end{aligned}$$

Example. An Initial-Value Problem

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x,$$

$$y(0) = 2,$$

$$y'(0) = 4.$$

By Theorem 1, this problem has a unique solution, defined for all x, $-\infty < x < \infty$; let us proceed to find it. The general solution of the differential equation is

$$y = c_1e^{3x} + c_2e^{-x} - \tfrac{1}{2}e^x + 2\sin x - \cos x.$$

From this, we have

$$\frac{dy}{dx} = 3c_1 e^{3x} - c_2 e^{-x} - \tfrac{1}{2}e^x + 2\cos x + \sin x.$$

Applying the initial conditions we have

$$2 = c_1 e^0 + c_2 e^0 - \tfrac{1}{2}e^0 + 2\sin 0 - \cos 0,$$

$$4 = 3c_1 e^0 - c_2 e^0 - \tfrac{1}{2}e^0 + 2\cos 0 + \sin 0.$$

These equations simplify at once to the following:

$$c_1 + c_2 = \tfrac{7}{2}, \qquad 3c_1 - c_2 = \tfrac{5}{2}.$$

From these two equations we obtain

$$c_1 = \tfrac{3}{2}, \qquad c_2 = 2.$$

$$y = \tfrac{3}{2}e^{3x} + 2e^{-x} - \tfrac{1}{2}e^x + 2\sin x - \cos x.$$

VARIATION OF PARAMETERS

We seek a method of finding a particular integral that applies in all cases (including variable coefficients) in which the complementary function is known. Such a method is the method of *variation of parameters*, which we now consider.

We shall develop this method in connection with the general second-order linear differential equation with variable coefficients

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x). \tag{1}$$

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \tag{2}$$

Then the complementary function of Equation (1) is

$$c_1 y_1(x) + c_2 y_2(x),$$

where y_1 and y_2 are linearly independent solutions of (2) and c_1 and c_2 are arbitrary constants. The procedure in the method of variation of parameters is to replace the arbitrary constants c_1 and c_2 in the complementary function by respective *functions* v_1 and v_2 which will be determined so that the resulting function, which is defined by

$$v_1(x)y_1(x) + v_2(x)y_2(x), \tag{3}$$

will be a particular integral of Equation (1) (hence the name, *variation* of parameters).

We have at our disposal the *two functions* v_1 and v_2 with which to satisfy the *one condition* that (3) be a solution of (1). Since we have *two* functions but only *one* condition on them, we are thus free to impose a second condition, provided this second condition does not violate the first one. We shall see when and how to impose this additional condition as we proceed.

We thus assume a solution of the form (3) and write

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x). \tag{4}$$

Differentiating (4), we have

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x) + v_1'(x)y_1(x) + v_2'(x)y_2(x), \tag{5}$$

where we use primes to denote differentiations. At this point we impose the aforementioned second condition; we simplify y_p' by demanding that

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0. \tag{6}$$

With this condition imposed, (5) reduces to

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x). \tag{7}$$

Now differentiating (7), we obtain

$$y_p''(x) = v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x). \tag{8}$$

We now impose the basic condition that (4) be a solution of Equation (1). Thus we substitute (4), (7), and (8) for y, dy/dx, and d^2y/dx^2, respectively, in Equation (1) and obtain the identity

$$a_0(x)[v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x)]$$
$$+ a_1(x)[v_1(x)y_1'(x) + v_2(x)y_2'(x)] + a_2(x)[v_1(x)y_1(x) + v_2(x)y_2(x)] = F(x).$$

This can be written as

$$v_1(x)[a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x)]$$
$$+ v_2(x)[a_0(x)y_2''(x) + a_1(x)y_2'(x) + a_2(x)y_2(x)]$$
$$+ a_0(x)[v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = F(x). \tag{9}$$

Since y_1 and y_2 are solutions of the corresponding homogeneous differential equation (2), the expressions in the first two brackets in (9) are identically zero. This leaves merely

$$v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = \frac{F(x)}{a_0(x)}. \tag{10}$$

This is actually what the basic condition demands. Thus the two imposed conditions require that the functions v_1 and v_2 be chosen such that the system of equations

$$y_1(x)v_1'(x) + y_2(x)v_2'(x) = 0,$$

$$y_1'(x)v_1'(x) + y_2'(x)v_2'(x) = \frac{F(x)}{a_0(x)}, \tag{11}$$

is satisfied. The determinant of coefficients of this system is precisely

$$W[y_1(x), y_2(x)] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous differential equation (2), we know that $W[y_1(x), y_2(x)] \neq 0$. Hence the system (11) has a unique solution. Actually solving this system, we obtain

$$v_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ \dfrac{F(x)}{a_0(x)} & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = -\frac{F(x)y_2(x)}{a_0(x)W[y_1(x), y_2(x)]},$$

$$v_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \dfrac{F(x)}{a_0(x)} \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = \frac{F(x)y_1(x)}{a_0(x)W[y_1(x), y_2(x)]}.$$

Thus we obtain the functions v_1 and v_2 defined by

$$\begin{aligned} v_1(x) &= -\int^x \frac{F(t)y_2(t)\,dt}{a_0(t)W[y_1(t), y_2(t)]}, \\ v_2(x) &= \int^x \frac{F(t)y_1(t)\,dt}{a_0(t)W[y_1(t), y_2(t)]}. \end{aligned} \tag{12}$$

Therefore a particular integral y_p of Equation (1) is defined by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

where v_1 and v_2 are defined by (12).

Examples.

Consider the differential equation

$$\frac{d^2y}{dx^2} + y = \tan x.$$

The complementary function is defined by

$$y_c(x) = c_1 \sin x + c_2 \cos x.$$

We assume $$y_p(x) = v_1(x)\sin x + v_2(x)\cos x,$$

where the functions v_1 and v_2 will be determined such that this is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + y = \tan x.$$

Then

$$y_p'(x) = v_1(x)\cos x - v_2(x)\sin x + v_1'(x)\sin x + v_2'(x)\cos x.$$

We impose the condition

$$v_1'(x)\sin x + v_2'(x)\cos x = 0,$$

leaving $$y_p'(x) = v_1(x)\cos x - v_2(x)\sin x.$$

From this

$$y_p''(x) = -v_1(x)\sin x - v_2(x)\cos x + v_1'(x)\cos x - v_2'(x)\sin x$$

and we obtain

$$v_1'(x)\cos x - v_2'(x)\sin x = \tan x.$$

Thus we have the two equations from which to determine $v_1'(x)$, $v_2'(x)$:

$$v_1'(x)\sin x + v_2'(x)\cos x = 0,$$
$$v_1'(x)\cos x - v_2'(x)\sin x = \tan x.$$

Solving we find:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \tan x}{-1} = \sin x,$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \tan x}{-1} = \frac{-\sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x.$$

Integrating we find:

$$v_1(x) = -\cos x + c_3, \qquad v_2(x) = \sin x - \ln|\sec x + \tan x| + c_4.$$

$$\begin{aligned} y_p(x) &= (-\cos x + c_3)\sin x + (\sin x - \ln|\sec x + \tan x| + c_4)\cos x \\ &= -\sin x \cos x + c_3 \sin x + \sin x \cos x \\ &\quad -\ln|\sec x + \tan x|\,(\cos x) + c_4 \cos x \\ &= c_3 \sin x + c_4 \cos x - (\cos x)(\ln|\sec x + \tan x|). \end{aligned}$$

Since a particular integral is a solution free of arbitrary constants, we may assign any particular values A and B to c_3 and c_4, respectively, and the result will be the particular integral

$$A \sin x + B \cos x - (\cos x)(\ln|\sec x + \tan x|).$$

Thus $y = y_c + y_p$ becomes

$$y = c_1 \sin x + c_2 \cos x + A \sin x + B \cos x - (\cos x)(\ln|\sec x + \tan x|),$$

which we may write as

$$y = C_1 \sin x + C_2 \cos x - (\cos x)(\ln|\sec x + \tan x|),$$

where $C_1 = c_1 + A$, $C_2 = c_2 + B$.

Thus we see that we might as well have chosen the constants c_3 and c_4 both equal to 0, for essentially the same result,

$$y = c_1 \sin x + c_2 \cos x - (\cos x)(\ln|\sec x + \tan x|),$$

would have been obtained. This is the general solution of the equation.

Examples.

Consider the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^x.$$

The complementary function is

$$y_c(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

We assume as a particular integral

$$y_p(x) = v_1(x)e^x + v_2(x)e^{2x} + v_3(x)e^{3x}.$$

Since we have *three* functions v_1, v_2, v_3 at our disposal in this case, we can apply three conditions. We have:

$$y_p'(x) = v_1(x)e^x + 2v_2(x)e^{2x} + 3v_3(x)e^{3x} + v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x}.$$

Proceeding in a manner analogous to that of the second-order case, we impose the condition

$$v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x} = 0,$$

leaving

$$y_p'(x) = v_1(x)e^x + 2v_2(x)e^{2x} + 3v_3(x)e^{3x}.$$

Then

$$y_p''(x) = v_1(x)e^x + 4v_2(x)e^{2x} + 9v_3(x)e^{3x} + v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x}.$$

We now impose the condition

$$v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x} = 0,$$

leaving

$$y_p''(x) = v_1(x)e^x + 4v_2(x)e^{2x} + 9v_3(x)e^{3x}.$$

From this,

$$y_p'''(x) = v_1(x)e^x + 8v_2(x)e^{2x} + 27v_3(x)e^{3x} + v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x}.$$

$$\begin{aligned} &v_1(x)e^x + 8v_2(x)e^{2x} + 27v_3(x)e^{3x} + v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x} \\ &\quad - 6v_1(x)e^x - 24v_2(x)e^{2x} - 54v_3(x)e^{3x} + 11v_1(x)e^x + 22v_2(x)e^{2x} + 33v_3(x)e^{3x} \\ &\qquad - 6v_1(x)e^x - 6v_2(x)e^{2x} - 6v_3(x)e^{3x} = e^x \end{aligned}$$

or

$$v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x} = e^x.$$

Thus we have the three equations from which to determine $v_1'(x)$, $v_2'(x)$, $v_3'(x)$:

$$v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x} = 0,$$

$$v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x} = 0,$$

$$v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x} = e^x.$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{e^{6x}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}{e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}} = \frac{1}{2},$$

$$v_2'(x) = \frac{\begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^x & 9e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{-e^{5x}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{2e^{6x}} = -e^{-x},$$

$$v_3'(x) = \frac{\begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^x \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{e^{4x}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{2e^{6x}} = \frac{1}{2}e^{-2x}.$$

We now integrate, choosing all the constants of integration to be zero. We find:

$$v_1(x) = \tfrac{1}{2}x, \qquad v_2(x) = e^{-x}, \qquad v_3(x) = -\tfrac{1}{4}e^{-2x}.$$

Thus

$$y_p(x) = \tfrac{1}{2}xe^x + e^{-x}e^{2x} - \tfrac{1}{4}e^{-2x}e^{3x} = \tfrac{1}{2}xe^x + \tfrac{3}{4}e^x.$$

Thus the general solution is

$$y = y_c + y_p = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + \tfrac{1}{2}xe^x + \tfrac{3}{4}e^x$$

or

$$y = c_1' e^x + c_2 e^{2x} + c_3 e^{3x} + \tfrac{1}{2}xe^x,$$

where $c_1' = c_1 + \frac{3}{4}$.

Examples.

Consider the differential equation

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2 + 1)^2.$$

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

the complementary function of equation

$$y_c(x) = c_1 x + c_2(x^2 - 1).$$

To find a particular integral, we therefore let

$$y_p(x) = v_1(x)x + v_2(x)(x^2 - 1).$$

Then

$$y_p'(x) = v_1(x) \cdot 1 + v_2(x) \cdot 2x + v_1'(x)x + v_2'(x)(x^2 - 1).$$

We impose the condition

$$v_1'(x)x + v_2'(x)(x^2 - 1) = 0,$$

leaving

$$y_p'(x) = v_1(x) \cdot 1 + v_2(x) \cdot 2x.$$

From this, we find

$$y_p''(x) = v_1'(x) + 2v_2(x) + v_2''(x) \cdot 2x.$$

we obtain

$$(x^2 + 1)[v_1'(x) + 2v_2(x) + 2xv_2'(x)] - 2x[v_1(x) + 2xv_2(x)] + 2[v_1(x)x + v_2(x)(x^2 - 1)] = 6(x^2 + 1)^2$$

or

$$(x^2 + 1)[v_1'(x) + 2xv_2'(x)] = 6(x^2 + 1)^2.$$

Thus we have the two equations from which to determine $v_1'(x)$ and $v_2'(x)$; that is, $v_1'(x)$ and $v_2'(x)$ satisfy the system

$$v_1'(x)x + v_2'(x)[x^2 - 1] = 0,$$
$$v_1'(x) + v_2'(x)[2x] = 6(x^2 + 1).$$

Solving this system, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & x^2 - 1 \\ 6(x^2 + 1) & 2x \end{vmatrix}}{\begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix}} = \frac{-6(x^2 + 1)(x^2 - 1)}{x^2 + 1} = -6(x^2 - 1),$$

$$v_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & 6(x^2 + 1) \end{vmatrix}}{\begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix}} = \frac{6x(x^2 + 1)}{x^2 + 1} = 6x.$$

Integrating, we obtain

$$v_1(x) = -2x^3 + 6x, \qquad v_2(x) = 3x^2,$$

where we have chosen both constants of integration to be zero.

$$\begin{aligned} y_p(x) &= (-2x^3 + 6x)x + 3x^2(x^2 - 1) \\ &= x^4 + 3x^2. \end{aligned}$$

Therefore the general solution may be expressed in the form

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 x + c_2(x^2 - 1) + x^4 + 3x^2. \end{aligned}$$

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \ldots + p_{n-1} y' + p_n y = f(x),$$

$$y^{(n)} + p_1 y^{(n-1)} + \ldots + p_{n-1} y' + p_n y = 0.$$

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n,$$

$$y_1 \frac{dC_1}{dx} + y_2 \frac{dC_2}{dx} + \ldots + y_n \frac{dC_n}{dx} = 0.$$

$$y_1' \frac{dC_1}{dx} + y_2' \frac{dC_2}{dx} + \ldots + y_n' \frac{dC_n}{dx} = 0,$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$y_1^{(n-2)} \frac{dC_1}{dx} + y_2^{(n-2)} \frac{dC_2}{dx} + \ldots + y_n^{(n-2)} \frac{dC_n}{dx} = 0,$$

$$y_1^{(n-1)} \frac{dC_1}{dx} + y_2^{(n-1)} \frac{dC_2}{dx} + \ldots + y_n^{(n-1)} \frac{dC_n}{dx} = f(x).$$

$$\frac{dC_i}{dx} = \varphi_i(x),$$

$$C_i = \int \varphi_i(x)\, dx + \gamma_i$$

$$y = \gamma_1 y_1 + \gamma_2 y_2 + \ldots + \gamma_n y_n + \sum_{i=1}^{n} y_i \int \varphi_i(x)\, dx.$$

Example. $xy'' - y' = x^2.$

$$xy'' - y' = 0, \quad \frac{y''}{y'} = \frac{1}{x},$$

$$y' = Ax, \quad y = \frac{A}{2}x^2 + B; \quad y = C_1 + C_2 x^2;$$

$$1 \cdot \frac{dC_1}{dx} + x^2 \frac{dC_2}{dx} = 0,$$

$$0 \cdot \frac{dC_1}{dx} + 2x \frac{dC_2}{dx} = x.$$

$$\frac{dC_2}{dx} = \frac{1}{2}, \quad C_2 = \frac{x}{2} + \gamma_2, \quad \frac{dC_1}{dx} = -\frac{x^2}{2}, \quad C_1 = -\frac{x^3}{6} + \gamma_1.$$

$$y = \gamma_1 + \gamma_2 x^2 + \frac{x^3}{3}$$