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Ex: Solve $y'' - 4y' - 12y = 2t^3 - t + 3$ with undetermined coefficient method.

Solution: $y_g = y_h + y_p$ (second order, non-homogeneous)

y_h: characteristic equation: $r^2 - 4r - 12 = 0$

$$(r-6)(r+2) = 0$$

$$r_1 = 6 \, , \, r_2 = -2$$

$y_1 = e^{6t}$

$y_2 = e^{-2t}$ ⟩ Lets check Wronskian: $W(y_1, y_2) \neq 0$. So they are fundamental solutions.

$y_h = c_1 y_1 + c_2 y_2$

$y_h = c_1 e^{6t} + c_2 e^{-2t}$ //

same degree

y_p: $y'' - 4y' - 12y = 2t^3 - t + 3$

Hence y has degree 3.

$y_p = At^3 + Bt^2 + Ct + D$

$y_p' = 3At^2 + 2Bt + C$

$y_p'' = 6At + 2B$

$y'' - 4y' - 12y = 2t^3 - t + 3 \Rightarrow 6At + 2B - 4(3At^2 + 2Bt + C) - 12(At^3 + Bt^2 + Ct + D) = 2t^3 - t + 3$

$\Rightarrow A = -\frac{1}{6} \, , \, B = \frac{1}{6} \, , \, C = -\frac{1}{9} \, , \, D = -\frac{5}{27}$

$y_p = -\frac{t^3}{6} + \frac{t^2}{6} - \frac{t}{9} - \frac{5}{27} \Rightarrow y_g = y_h + y_p$

$y_g = c_1 e^{6t} + c_2 e^{-2t} - \frac{t^3}{6} + \frac{t^2}{6} - \frac{t}{9} - \frac{5}{27}$ //

Ex: Solve $y''' + 3y'' - 10y' = x - 3$ using the method of undetermined coefficient.

Solution: $y_g = y_h + y_p$ (higher order, non homogeneous)

y_h: characteristic equ: $r^3 + 3r^2 - 10r = 0$

$r(r^2 + 3r - 10) = 0$

$r(r-2)(r+5) = 0$ $\quad r_1 = 0$

$r_2 = 2$

$r_3 = -5$

$$\left.\begin{aligned} y_1 &= e^{0x} = 1 \\ y_2 &= e^{2x} \\ y_3 &= e^{-5x} \end{aligned}\right\} W(y_1, y_2, y_3) \neq 0.$$

$y_h = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 + c_2 e^{2x} + c_3 e^{-5x}$ //

y_p: $y''' + 3y'' - 10y' = x - 3$ same degree

$\Rightarrow y'$ has degree 1 $\Rightarrow y$ has degree 2

$$\begin{cases} y_p = Ax^2 + Bx + C \\ y_p' = 2Ax + B \\ y_p'' = 2A \\ y_p''' = 0 \end{cases}$$

$y''' + 3y'' - 10y' = x - 3 \Rightarrow 0 + 3.2A - 10(2Ax + B) = x - 3$

$\Rightarrow A = -\frac{5}{100}$, $B = \frac{27}{100}$, $C = 0$

$y_p = -\frac{5}{100}x^2 + \frac{27}{100}$

$y_g = y_h + y_p = c_1 + c_2 e^{2x} + c_3 e^{-5x} - \frac{5}{100}x^2 + \frac{27}{100}$ //

__Ex__: Solve $y'' + 2y' + 5y = 5x^2 + 12$ with the method of undetermined coefficient.

__Solution__: $y_g = y_h + y_p$ (second order, non-homogeneous)

__y_h__: characteristic equ: $r^2 + 2r + 5 = 0$

$$\Delta = b^2 - 4ac = 4 - 4.5.1 = -16$$

$$r_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-2 + \sqrt{-16}}{2} = -1 + 2i$$

$$r_2 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-2 - \sqrt{-16}}{2} = -1 - 2i$$

$\Big\} \lambda = -1, \quad \mu = 2$

$y_1 = e^{\lambda x} \cos \mu x$

$y_2 = e^{\lambda x} \sin \mu x$

$\Rightarrow y_1 = e^{-x} \cos 2x$

$\quad\;\; y_2 = e^{-x} \sin 2x$ $\Big\} \Rightarrow y_h = c_1 y_1 + c_2 y_2$

$$y_h = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x \,//$$

same degree

__y_p__: $y'' + 2y + 5y = 5x^2 + 12$

$\Rightarrow y$ has degree 2

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$\Big\} \Rightarrow$ $y'' + 2y' + 5y = 5x^2 + 12$

$2A + 2(2Ax + B) + 5(Ax^2 + Bx + C) = 5x^2 + 12$

$\Rightarrow A = 1, \quad B = -\frac{4}{5}, \quad C = \frac{58}{25}$

$$\Rightarrow y_p = x^2 - \frac{4}{5}x + \frac{58}{25}$$

$$y_g = y_h + y_p = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x + x^2 - \frac{4}{5}x + \frac{58}{25} \,//$$

Ex: Solve $y'' - 4y' - 12y = 3e^{5x}$ with undetermined coeffient method

Solution: $y_g = y_h + y_p$ (second order, non-homogenous)

y_h: characteristic equation: $r^2 - 4r - 12 = 0$

$$(r-6)(r+2) = 0$$

$$r_1 = 6 \qquad r_2 = -2$$

$$\left.\begin{aligned} y_1 &= e^{6x} \\ y_2 &= e^{-2x} \end{aligned}\right\} W(y_1, y_2) \neq 0 \Rightarrow y_h = c_1 e^{6x} + c_2 e^{-2x}$$

y_p: $y'' - 4y' - 12y = \underline{\underline{3e^{5x}}}$

$y_p = Ae^{5x}$ (e^{5x} is not in y_h so $y_p = Ae^{5x}$)

$y_p' = 5Ae^{5x}$

$y_p'' = 25Ae^{5x}$

$y'' - 4y' - 12y = 3e^{5x}$: $25Ae^{5x} - 4(5Ae^{5x}) - 12Ae^{5x} = 3e^{5x}$

$$\Rightarrow A = -\frac{3}{7}$$

$y_p = -\frac{3}{7}e^{5x}$

$y_g = y_h + y_p = c_1 e^{6x} + c_2 e^{-2x} - \frac{3}{7}e^{5x}$ //

Ex: Solve $y'' - 3y' + 2y = 5e^{2x}$ with undetermined coeff. method

Solution: $y_g = y_h + y_p$ (second order, non-homogeneous)

y_h: characteristic equation: $r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r_1 = 2 \quad r_2 = 1$

$y_1 = e^{2x}$

$y_2 = e^{x}$ $\Big\rangle$ $W(y_1, y_2) \neq 0$

$y_h = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{x}$ //

y_p: $y'' - 3y' + 2y = 5e^{2x}$

e^{2x} is in y_h? Yes so $y_p = Axe^{2x}$

$y_p = Axe^{2x}$

$y_p' = Ae^{2x} + Ax.2e^{2x} = Ae^{2x} + 2Ae^{2x}$

$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 2Ax.2e^{2x} = 4Ae^{2x} + 4Axe^{2x}$

$\Rightarrow y'' - 3y' + 2y = 5e^{2x}$: $(4Ae^{2x} + 4Axe^{2x}) - 3(Ae^{2x} + 2Ae^{2x})$

$+ 2Axe^{2x} = 5e^{2x}$

$\Rightarrow A = 5$

$\Rightarrow y_p = 5xe^{2x}$

$\Rightarrow y_g = y_h + y_p = c_1 e^{2x} + c_2 e^{x} + 5xe^{2x}$ //

Ex: Solve $y^{(4)} + y^{(3)} = \sin 2x$ with the method of undetermined coeffi.

Solution: $y_g = y_h + y_p$ (higher order, non-homogeneous)

y_h: characteristic polyn: $r^4 + r^3 = 0$

$$r^3(r+1) = 0$$

$$r_{1,2,3} = 0 \qquad r_4 = -1$$

$y_1 = e^{0x} = 1$

$y_2 = x e^{0x} = x$

$y_3 = x^2 e^{0x} = x^2$ $\qquad$ $y_h = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$ //

$y_4 = e^{-x} = e^{-x}$

y_p: $y^{(4)} + y^{(3)} = \sin 2x$

$$y_p = A \sin 2x + B \cos 2x$$

$$y_p' = 2A \cos 2x - 2B \sin 2x$$

$$y_p'' = -4A \sin 2x - 4B \cos 2x$$

$$y_p''' = -8A \cos 2x + 8B \sin 2x$$

$$y_p^{(4)} = 16A \sin 2x + 16B \cos 2x$$

$$16A \sin 2x + 16B \cos 2x + (-8A \cos 2x + 8B \sin 2x) = \sin 2x$$

$$\sin 2x (16 - 8B) + \cos 2x (16B - 8A) = \sin 2x$$

$$A = \frac{1}{20}, \quad B = \frac{1}{40}$$

$$y_p = \frac{1}{20} \sin 2x + \frac{1}{40} \cos 2x$$

$$y_h = y_g + y_p = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + \frac{1}{20} \sin 2x + \frac{1}{40} \cos 2x \ //$$

Ex: Solve $y''' - 5y'' - y' + 5y = 10t - 63e^{-2t} + 29\sin t$ with method of undetermined coeffi.

Solution: $y_g = y_h + y_p$ (higher order, non-homogeneous)

y_h: characteristic eq. $r^3 - 5r^2 - r + 5 = 0$

$$r^2(r-5) - (r-5) = 0$$

$$(r-5)(r^2-1) = 0$$

$$(r-5)(r-1)(r+1) = 0$$

$$\left.\begin{array}{l} r_1 = 5 \\ r_2 = 1 \\ r_3 = -1 \end{array}\right\} \Rightarrow \begin{array}{l} y_1 = e^{5t} \\ y_2 = e^{t} \\ y_3 = e^{-t} \end{array}$$

$$y_h = c_1 e^{5t} + c_2 e^{t} + c_3 e^{-t}$$

y_p: $y''' - 5y'' - y' + 5y = 10t - 63e^{-2t} + 29\sin t$

$$y_p = (At + B) + (Ce^{-2t}) + (D\sin 2t + E\cos 2t)$$

$$y_p' = A - 2Ce^{-2t} + 2D\cos 2t - 2E\sin 2t$$

$$y_p'' = 4Ce^{-2t} - 4D\sin 2t - 4E\cos 2t$$

$$y_p''' = -8Ce^{-2t} - 8D\cos 2t + 8E\sin 2t$$

$$\Rightarrow (-8Ce^{-2t} - 8D\cos 2t + 8E\sin 2t) - 5(4Ce^{-2t} - 4D\sin 2t - 4E\cos 2t)$$

$$- (A - 2Ce^{-2t} + 2D\cos 2t - 2E\sin 2t) + 5(At + B) + (Ce^{-2t})$$

$$+ (D\sin 2t + E\cos 2t) = 10t - 63e^{-2t} + 29\sin 2t$$

$\Rightarrow A = 2$

$B = \frac{2}{5}$

$C = 3$

$D = 1$

$E = \frac{2}{5}$

$$y_p = 2t + \frac{2}{5} + 3e^{-2t} + \sin 2t + \frac{2}{5} \cos 2t$$

$$y_g = y_h + y_p$$

$$\Rightarrow y_g = c_1 e^{5t} + c_2 e^{t} + c_3 e^{-t} + 2t + \frac{2}{5} + 3e^{-2t} + \sin 2t + \frac{2}{5} \cos 2t$$

Ex: $y'' + y = \frac{1}{\sin x}$ (variation of parameter)

Solution: second order, constant coefficient, non-homogenous

We will use variation of parameter

$y_g = y_h + y_p$

y_h: characteristic equation: $r^2 + 1 = 0$

$$r^2 = -1$$

$$r = \mp i < \begin{matrix} \lambda = 0 \\ \mu = 1 \end{matrix}$$

$$y_1 = e^{0x} \cos x = \cos x$$

$$y_2 = e^{0x} \sin x = \sin x$$

$$y_h = c_1 \cos x + c_2 \sin x$$

y_p: replace the constants c_1 and c_2 in y_h by functions $u_1(x)$ and $u_2(x)$, respectively

$$y_p = u_1 \cos x + c_2 \sin x \quad \text{(assumption)}$$

then we have
$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(x) \qquad g(x) = \frac{1}{\sin x}$$

$$\Rightarrow u_1' \cos x + u_2' \sin x = 0$$
$$-u_1' \sin x + u_2' \cos x = \frac{1}{\sin x}$$

Solve this system with Cramer rule.

$$\Delta = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\Delta u_1' = \begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin x} & \cos x \end{vmatrix} = 0 - \frac{1}{\sin x} \sin x = -1$$

$$\Delta u_2' = \begin{vmatrix} \cos x & 0 \\ \sin x & \frac{1}{\sin x} \end{vmatrix} = \cos x \cdot \frac{1}{\sin x} - 0 = \frac{\cos x}{\sin x}$$

$$\Rightarrow u_1' = \frac{\Delta u_1'}{\Delta} = -\frac{1}{1} = -1 \Rightarrow u_1' = -1 \Rightarrow u_1 = -x$$

$$u_2' = \frac{\Delta u_2'}{\Delta} = \frac{\cos x / \sin x}{1} = \frac{\cos x}{\sin x} \Rightarrow u_2 = \ln|\sin x|$$

$$y_p = u_1 \cos x + u_2 \sin x$$

$$= -x \cos x + \ln|\sin x| \sin x$$

$$y_g = y_h + y_p$$

$$= c_1 \cos x + c_2 \sin x + (-x \cos x + \ln|\sin x| \sin x|)$$

Ex: $y'' + 4y' + 5y = e^{-2x} \cdot \sec x$ (variation of parameter)

Solution: second order, constant coeff., non-homogenous

we will use variation of parameter

$y_g = y_h + y_p$

y_h: characteristic equation: $r^2 + 4r + 5 = 0$

$$\Delta = 16 - 4 \cdot 5 \cdot 1 = -4$$

$$r_1 = \frac{-4 + \sqrt{-4}}{2} = -2 + i$$

$$r_2 = \frac{-4 - \sqrt{-4}}{2} = -2 - i$$

$\lambda = -2$
$\mu = 1$

$y_1 = e^{-2x} \cos x$

$y_2 = e^{-2x} \sin x$

$$y_h = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

y_p: replace the constants c_1 and c_2 in y_h by functions $u_1(x)$ and $u_2(x)$, respectively

then we have $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = g(x)$ where $g(x) = e^{-2x} \sec x$

$$\Rightarrow \quad u_1' e^{-2x} \cos x + u_2' e^{-2x} \sin x = 0$$

$$u_1' (-2e^{-2x} \cos x - e^{-2x} \sin x) + u_2' (-2e^{-2x} \sin x + e^{-2x} \cos x) = e^{-2x} \sec x$$

Solve this system (use cramer rule)

$$\Delta = \begin{vmatrix} e^{-2x}\cos x & e^{-2x}\sin x \\ -2e^{-2x}\cos x - e^{-2x}\sin x & -2e^{-2x}\sin x + e^{-2x}\cos x \end{vmatrix}$$

$$= e^{-4x}(2\cos x\sin x + \cos^2 x + 2\cos x\sin x + \sin^2 x) = e^{-4x} //$$

$$\Delta u_1' = \begin{vmatrix} 0 & e^{-2x}\sin x \\ e^{-2x}\sec x & -2e^{-2x}\sin x + e^{-2x}\cos x \end{vmatrix}$$

$$= -e^{-4x}\sec x\sin x //$$

$$\Delta u_2' = \begin{vmatrix} e^{-2x}\cos x & 0 \\ -2e^{-2x}\cos x - e^{-2x}\sin x & e^{-2x}\sec x \end{vmatrix}$$

$$= e^{-4x}\cos x\sec x = e^{-4x}\cos x \cdot \frac{1}{\cos x} = e^{-4x} //$$

$$u_1' = \frac{\Delta u_1'}{\Delta} = \frac{-e^{-4x}\sec x\sin x}{e^{-4x}} = -\frac{1}{\cos x}\sin x = -\frac{\sin x}{\cos x}$$

$$\Rightarrow u_1' = -\frac{\sin x}{\cos x} \Rightarrow u_1 = \ln|\cos x|$$

$$u_2' = \frac{\Delta u_2'}{\Delta} = \frac{e^{-4x}}{e^{-4x}} = 1 \Rightarrow u_2' = 1 \Rightarrow u_2 = x$$

$$y_p = u_1 e^{-2x}\cos x + u_2 e^{-2x}\sin x$$

$$= \ln|\cos x| e^{-2x}\cos x + x e^{-2x}\sin x$$

$$y_g = y_h + y_p = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x + \ln|\cos x| e^{-2x}\cos x + x e^{-2x}\sin x //$$

Ex: $y'' + 6y' + 9y = \dfrac{e^{-3x}}{x^3}$ (variation of parameter)

Solution: second order, constant coef. nonhomogenous

we will use the variation of parameter

y_h: characteristic equ: $r^2 + 6r + 9 = 0$

$(r+3)^2 = 0 \Rightarrow r_1, r_2 = -3$

$y_1 = e^{-3x}$

$y_2 = x e^{-3x}$

$\Rightarrow y_h = c_1 e^{-3x} + c_2 x e^{-3x}$ //

y_p: replace the constants c_1 and c_2 in y_h by functions $u_1(x)$ and $u_2(x)$, respectively

$y_p = u_1 e^{-3x} + u_2 . x . e^{-3x}$

then we have $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = g(x)$ where $g(x) = \dfrac{e^{-3x}}{x^3}$

$\Rightarrow u_1' e^{-3x} + u_2' x e^{-3x} = 0$

$u_1' (-3e^{-3x}) + u_2' (e^{-3x} + x . (-3) e^{-3x}) = \dfrac{e^{-3x}}{x^3}$

Solve this system (use cramer rule)

$$\Delta = \begin{vmatrix} e^{-3x} & x e^{-3x} \\ -3e^{-3x} & e^{-3x} - 3x e^{-3x} \end{vmatrix} = e^{-6x}(1 - 3x + 3x)$$

$$= e^{-6x}$$

$$\Delta u_1' = \begin{vmatrix} 0 & x e^{-3x} \\ e^{-3x} x^{-3} & e^{-3x} - 3x e^{-3x} \end{vmatrix} = -e^{-6x} x^{-2}$$

$$\Delta u_2' = \begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & e^{-3x} x^{-3} \end{vmatrix} = e^{-6x} x^{-3}$$

$$u_1' = \frac{\Delta u_1'}{\Delta} = \frac{-e^{-6x} x^{-2}}{e^{-6x}} = -x^{-2}$$

$$u_1' = -x^{-2} \Rightarrow u_1 = \frac{1}{x}$$

$$u_2' = \frac{\Delta u_2'}{\Delta} = \frac{e^{-6x} x^{-3}}{e^{-6x}} = x^{-3}$$

$$u_2' = x^{-3} \Rightarrow u_2 = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$$

$$y_p = u_1 e^{-3x} + u_2 x e^{-3x}$$

$$= \frac{1}{x} e^{-3x} - \frac{1}{2x^2} x e^{-3x}$$

$$y_g = y_h + y_p = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{e^{-3x}}{x} - \frac{e^{-3x}}{2x}$$