Kronecker-Capelli theorem. *A system of linear equations* (1) *is consistent if and only if the rank of the augmented matrix* $\bar{A}$ *is equal to the rank of the matrix* A.

Example. Solve the system

$$\left.\begin{aligned} 4x_1 + x_2 - 2x_3 + x_4 &= 3, \\ x_1 - 2x_2 - x_3 + 2x_4 &= 2, \\ 2x_1 + 5x_2 \qquad - x_4 &= -1, \\ 3x_1 + 3x_2 - x_3 - 3x_4 &= 1 \end{aligned}\right\}$$

Although the number of equations is equal to the number of unknowns, the determinant of the system is zero and, therefore, Cramer's rule is not applicable. The rank of the coefficient matrix is equal to three—in the upper right corner of this matrix is a nonzero third-order minor. The rank of the augmented matrix is also three, so the system is consistent. Considering only the first three equations and taking the unknown x_1 as free, we obtain the general solution in the form

$$x_2 = -\frac{1}{5} - \frac{2}{5}x_1 \quad x_3 = -\frac{8}{5} + \frac{9}{5}x_1, \quad x_4 = 0$$

Example. Suppose we have a system consisting of $n+1$ equations in n unknowns. The augmented matrix $\bar{A}$ of this system is a square matrix of order $n+1$. If our system is consistent, then, by the Kronecker-Capelli theorem, the determinant of $\bar{A}$ must be zero.

Thus, let there be a system

$$\left.\begin{aligned} x_1 - 8x_2 &= 3, \\ 2x_1 + x_2 &= 1, \\ 4x_1 + 7x_2 &= -4 \end{aligned}\right\}$$

The determinant of the coefficients and the constant terms of these equations is different from zero:

$$\begin{vmatrix} 1 & -8 & 3 \\ 2 & 1 & 1 \\ 4 & 7 & -4 \end{vmatrix} = -77$$

The system is therefore inconsistent.

Example. Investigate a system of algebraic equations

$$\left.\begin{aligned} 0x_1 - 1x_2 + 3x_3 + 0x_4 &= 2 \\ 2x_1 - 4x_2 + 1x_3 + 5x_4 &= 3 \\ -4x_1 + 5x_2 + 7x_3 - 10x_4 &= 0 \\ -2x_1 + 1x_2 + 8x_3 - 5x_4 &= 3 \end{aligned}\right\}$$

(without solving the system) using the Kronecker-Capelli theorem.

Solution.

We write the system in the form

$$\begin{vmatrix} 0 & -1 & 3 & 0 \\ 2 & -4 & 1 & 5 \\ -4 & 5 & 7 & -10 \\ -2 & 1 & 8 & -5 \end{vmatrix} = \begin{vmatrix} 2 \\ 3 \\ 0 \\ 3 \end{vmatrix}$$

For the convenience of calculations, we interchange the lines:

$$\begin{vmatrix} 0 & -1 & 3 & 0 \\ -2 & 1 & 8 & -5 \\ 2 & -4 & 1 & 5 \\ -4 & 5 & 7 & -10 \end{vmatrix} = \begin{vmatrix} 2 \\ 3 \\ 3 \\ 0 \end{vmatrix}$$

Let's add the 3rd row to the 2nd:

$$\begin{vmatrix} 0 & -1 & 3 & 0 \\ 0 & -3 & 9 & 0 \\ 2 & -4 & 1 & 5 \\ -4 & 5 & 7 & -10 \end{vmatrix} = \begin{vmatrix} 2 \\ 6 \\ 3 \\ 0 \end{vmatrix}$$

Multiply the 3rd row by (2). Let's add the 4th row to the 3rd:

$$\begin{vmatrix} 0 & -1 & 3 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & -3 & 9 & 0 \\ -4 & 5 & 7 & -10 \end{vmatrix} = \begin{vmatrix} 2 \\ 6 \\ 6 \\ 0 \end{vmatrix}$$

Multiply the 1st row by (3). Multiply the 2nd row by (-1). Let's add the 2nd row to the 1st:

$$\left|\begin{array}{cccc|c} 0 & 0 & 0 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & -3 & 9 & 0 \\ -4 & 5 & 7 & -10 \end{array}\right| = \left|\begin{array}{c} 0 \\ 6 \\ 6 \\ 0 \end{array}\right|$$

Multiply the 3rd row by (-1). Let's add the 3rd row to the 2nd:

$$\left|\begin{array}{cccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 9 & 0 \\ -4 & 5 & 7 & -10 \end{array}\right| = \left|\begin{array}{c} 0 \\ 0 \\ 6 \\ 0 \end{array}\right|$$

This corresponds to the system:

$-3x_2 + 9x_3 = 6$
$-4x_1 + 5x_2 + 7x_3 - 10x_4 = 0$

We take x_1 and x_2 as basic variables. Then the free ones are x_3 and x_4. The rank of the main matrix is 2. The rank of the augmented matrix is also 2. The system is consistent and has an infinite number of solutions.

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

DEFINITION

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For examples of differential equations we list the following:

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0,$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t,$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v,$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

DEFINITION

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

DEFINITION

The order of the highest ordered derivative involved in a differential equation is called the order *of the differential equation.*

Proceeding with our study of ordinary differential equations, we now introduce the important concept of *linearity* applied to such equations. This concept will enable us to classify these equations still further.

DEFINITION

A linear ordinary differential equation *of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in, the form*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x),$$

where a_0 is not identically zero.

The following ordinary differential equations are both linear. In each case y is the dependent variable. Observe that y and its various derivatives occur to the first degree only and that no products of y and/or any of its derivatives are present.

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y = 0,$$

$$\frac{d^4 y}{dx^4} + x^2\frac{d^3 y}{dx^3} + x^3\frac{dy}{dx} = xe^x.$$

DEFINITION

A nonlinear ordinary differential equation *is an ordinary differential equation that is not linear.*

The following ordinary differential equations are all nonlinear:

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0,$$

$$\frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0,$$

$$\frac{d^2 y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0.$$

We now consider the concept of a solution of the nth-order ordinary differential equation.

DEFINITION

Consider the nth-order ordinary differential equation

$$F\left[x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right] = 0, \tag{1}$$

where F is a real function of its $(n + 2)$ *arguments* $x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}$.

1. Let f be a real function defined for all x in a real interval I and having an nth derivative (and hence also all lower ordered derivatives) for all $x \in I$. *The function f is called an* explicit solution *of the differential equation (1) on I if it fulfills the following two requirements:*

$$F[x, f(x), f'(x), \ldots, f^{(n)}(x)] \tag{A}$$

is defined for all $x \in I$, *and*

$$F[x, f(x), f'(x), \ldots, f^{(n)}(x)] = 0 \tag{B}$$

for all $x \in I$. *That is, the substitution of* $f(x)$ *and its various derivations for y and its corresponding derivatives, respectively, in (1) reduces (1) to an identity on I.*

2. A relation $g(x, y) = 0$ *is called an* implicit solution *of (1) if this relation defines at least one real function f of the variable x on an interval I such that this function is an explicit solution of (1) on this interval.*

3. Both explicit solutions and implicit solutions will usually be called simply solutions.

Example 1.

The function f defined for all real x by

$$f(x) = 2 \sin x + 3 \cos x$$

is an explicit solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0 \tag{2}$$

for all real x. First note that f is defined and has a second derivative for all real x. Next observe that

$$f'(x) = 2 \cos x - 3 \sin x,$$

$$f''(x) = -2 \sin x - 3 \cos x.$$

Upon substituting $f''(x)$ for d^2y/dx^2 and $f(x)$ for y in the differential equation (2), it reduces to the identity

$$(-2 \sin x - 3 \cos x) + (2 \sin x + 3 \cos x) = 0,$$

which holds for all real x. Thus the function f is an explicit solution of the differential equation (2) for all real x.

Example.

The relation

$$x^2 + y^2 - 25 = 0 \tag{3}$$

is an implicit solution of the differential equation

$$x + y\frac{dy}{dx} = 0 \tag{4}$$

on the interval I defined by $-5 < x < 5$. For the relation (3) defines the two real functions f_1 and f_2 given by

$$f_1(x) = \sqrt{25 - x^2}$$

and

$$f_2(x) = -\sqrt{25 - x^2},$$

respectively, for all real $x \in I$, and both of these functions are explicit solutions of the differential equations (4) on I.

Let us illustrate this for the function f_1. Since

$$f_1(x) = \sqrt{25 - x^2},$$

we see that

$$f'_1(x) = \frac{-x}{\sqrt{25 - x^2}}$$

for all real $x \in I$. Substituting $f_1(x)$ for y and $f'_1(x)$ for dy/dx in (4), we obtain the identity

$$x + (\sqrt{25 - x^2})\left(\frac{-x}{\sqrt{25 - x^2}}\right) = 0 \quad \text{or} \quad x - x = 0,$$

which holds for all real $x \in I$. Thus the function f_1 is an explicit solution of (4) on the interval I.

Now consider the relation

$$x^2 + y^2 + 25 = 0. \tag{5}$$

Is this also an implicit solution of Equation (4)? Let us differentiate the relation (5) implicitly with respect to x. We obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Substituting this into the differential equation (4), we obtain the *formal* identity

$$x + y\left(-\frac{x}{y}\right) = 0.$$

Thus the relation (5) *formally* satisfies the differential equation (4). Can we conclude from this alone that (5) is an implicit solution of (4)? The answer to this question is "no," for we have no assurance from this that the relation (5) defines any function that is an explicit solution of (4) on any real interval I. All that we have shown is that (5) is a relation between x and y that, upon implicit differentiation and substitution, *formally* reduces the differential equation (4) to a *formal* identity. It is called a *formal* solution; it has the *appearance* of a solution; but that is all that we know about it at this stage of our investigation.

Let us investigate a little further. Solving (5) for y, we find that

$$y = \pm\sqrt{-25 - x^2}.$$

Since this expression yields nonreal values of y for all real values of x, we conclude that the relation (5) does not define any real function on any interval. Thus the relation (5) is not truly an implicit solution but merely a *formal solution* of the differential equation (4).

Example.

Consider the first-order differential equation

$$\frac{dy}{dx} = 2x. \tag{6}$$

The function f_0 defined for all real x by $f_0(x) = x^2$ is a solution of this equation. So also are the functions f_1, f_2, and f_3 defined for all real x by $f_1(x) = x^2 + 1$, $f_2(x) = x^2 + 2$, and $f_3(x) = x^2 + 3$, respectively. In fact, for each real number c, the function f_c defined for all real x by

$$f_c(x) = x^2 + c \tag{7}$$

is a solution of the differential equation (6). In other words, the formula (7) defines an infinite family of functions, one for each real constant c, and every function of this family is a solution of (6). We call the constant c in (7) an *arbitrary constant* or *parameter* and refer to the family of functions defined by (7) as a *one-parameter faimily of solutions* of the differential equation (6). We write this one-parameter family of solutions as

$$y = x^2 + c. \tag{8}$$

We now consider the geometric significance of differential equations and their solutions. We first recall that a real function F may be represented geometrically by a curve $y = F(x)$ in the xy plane and that the value of the derivative of F at x, $F'(x)$, may be interpreted as the slope of the curve $y = F(x)$ at x. Thus the general first-order differential equation

$$\frac{dy}{dx} = f(x, y), \tag{9}$$

where f is a real function, may be interpreted geometrically as defining a slope $f(x, y)$ at every point (x, y) at which the function f is defined. Now assume that the differential equation (9) has a so-called one-parameter family of solutions that can be written in the form

$$y = F(x, c), \tag{10}$$

where c is the arbitrary constant or parameter of the family. The one-parameter family of functions defined by (10) is represented geometrically by a so-called *one-parameter family of curves* in the xy plane, the slopes of which are given by the differential equation (9). These curves, the graphs of the solutions of the differential equation (9), are called the *integral curves* of the differential equation (9).

Example.

Consider again the first-order differential equation

$$\frac{dy}{dx} = 2x \tag{6}$$

This differential equation may be interpreted as defining the slope $2x$ at the point with coordinates (x, y) for every real x. Now, we observe that the differential equation (6) has a one-parameter family of solutions of the form

$$y = x^2 + c, \tag{8}$$

where c is the arbitrary constant or parameter of the family. The one-parameter family of functions defined by (8) is represented geometrically by a one-parameter family of curves in the xy plane, namely, the family of *parabolas* with Equation (8).

INITIAL-VALUE PROBLEMS AND EXISTENCE OF SOLUTIONS

DEFINITION

Consider the first-order differential equation

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

where f is a continuous function of x and y in some domain D of the xy plane; and let (x_0, y_0) be a point of D. The initial-value problem *associated with (1) is to find a solution ϕ of the differential equation (1), defined on some real interval containing x_0, and satisfying the* initial condition

$$\phi(x_0) = y_0.$$

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{dy}{dx} = f(x, y),$$

$$y(x_0) = y_0.$$

Example.

Problem. Find a solution f of the differential equation

$$\frac{dy}{dx} = 2x \tag{2}$$

such that at $x = 1$ this solution f has the value 4.

Explanation. First let us be certain that we thoroughly understand this problem. We seek a real function f which fulfills the two following requirements:

1. The function f must satisfy the differential equation (2). That is, the function f must be such that $f'(x) = 2x$ for all real x in a real interval I.
2. The function f must have the value 4 at $x = 1$. That is, the function f must be such that $f(1) = 4$.

Notation. The stated problem may be expressed in the following somewhat abbreviated notation:

$$\frac{dy}{dx} = 2x,$$

$$y(1) = 4.$$

In this notation we may regard y as representing the desired solution. Then the differential equation itself obviously represents requirement 1, and the statement $y(1) = 4$ stands for requirement 2. More specifically, the notation $y(1) = 4$ states that the desired solution y must have the value 4 at $x = 1$; that is, $y = 4$ at $x = 1$.

Solution. We observed that the differential equation (2) has a one-parameter family of solutions which we write as

$$y = x^2 + c, \tag{3}$$

where c is an arbitrary constant, and that each of these solutions satisfies requirement 1. Let us now attempt to determine the constant c so that (3) satisfies requirement 2, that is, $y = 4$ at $x = 1$. Substituting $x = 1$, $y = 4$ into (3), we obtain $4 = 1 + c$, and hence $c = 3$. Now substituting the value $c = 3$ thus determined back into (3), we obtain

$$y = x^2 + 3,$$

which is indeed a solution of the differential equation (2), which has the value 4 at $x = 1$. In other words, the function f defined by

$$f(x) = x^2 + 3,$$

satisfies both of the requirements set forth in the problem.

Example.

Solve the initial-value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \tag{4}$$

$$y(3) = 4, \tag{5}$$

given that the differential equation (4) has a one-parameter family of solutions which may be written in the form

$$x^2 + y^2 = c^2. \tag{6}$$

Thus the pair of values (3, 4) must satisfy the relation (6). Substituting $x = 3$ and $y = 4$ into (6), we find

$$9 + 16 = c^2 \quad \text{or} \quad c^2 = 25.$$

Now substituting this value of c^2 into (6), we have

$$x^2 + y^2 = 25.$$

Solving this for y, we obtain

$$y = \pm\sqrt{25 - x^2}.$$

Obviously the positive sign must be chosen to give y the value $+4$ at $x = 3$. Thus the function f defined by

$$f(x) = \sqrt{25 - x^2}, \qquad -5 < x < 5,$$

is the solution of the problem. In the usual abbreviated notation, we write this solution as $y = \sqrt{25 - x^2}$.

THEOREM. BASIC EXISTENCE AND UNIQUENESS THEOREM

Hypothesis. *Consider the differential equation*

$$\frac{dy}{dx} = f(x, y), \tag{7}$$

where

1. The function f is a continuous function of x and y in some domain D of the xy plane, and

2. The partial derivative $\partial f/\partial y$ is also a continuous function of x and y in D; and let (x_0, y_0) be a point in D.

Conclusion. *There exists a unique solution ϕ of the differential equation (7), defined on some interval $|x - x_0| \leq h$, where h is sufficiently small, that satisfies the condition*

$$\phi(x_0) = y_0. \tag{8}$$

Example. Let us consider the initial-value problem

$$\frac{dy}{dx} = y^{1/3},$$

$$y(0) = 0.$$

One may verify that the functions f_1 and f_2 defined, respectively, by

$$f_1(x) = 0 \quad \text{for all real } x;$$

and

$$f_2(x) = \left(\tfrac{2}{3}x\right)^{3/2}, \quad x \geq 0; \qquad f_2(x) = 0, \quad x \leq 0;$$

are *both* solutions of this initial-value problem!

In order to ensure uniqueness, some additional requirement must certainly be imposed.

Example.

Consider the initial-value problem

$$\frac{dy}{dx} = x^2 + y^2,$$

$$y(1) = 3.$$

Let us apply Theorem. We first check the hypothesis. Here $f(x, y) = x^2 + y^2$ and $\dfrac{\partial f(x, y)}{\partial y} = 2y$. Both of the functions f and $\partial f/\partial y$ are continuous in every domain D of the xy plane. The initial condition $y(1) = 3$ means that $x_0 = 1$ and $y_0 = 3$, and the point $(1, 3)$ certainly lies in some such domain D. Thus all hypotheses are satisfied and the conclusion holds. That is, there is a unique solution ϕ of the differential equation $dy/dx = x^2 + y^2$, defined on some interval $|x - 1| \leq h$ about $x_0 = 1$, which satisfies that initial condition, that is, which is such that $\phi(1) = 3$.