Chapter 8

RLC Circuits

Parallel RLC Circuits   Natural Response

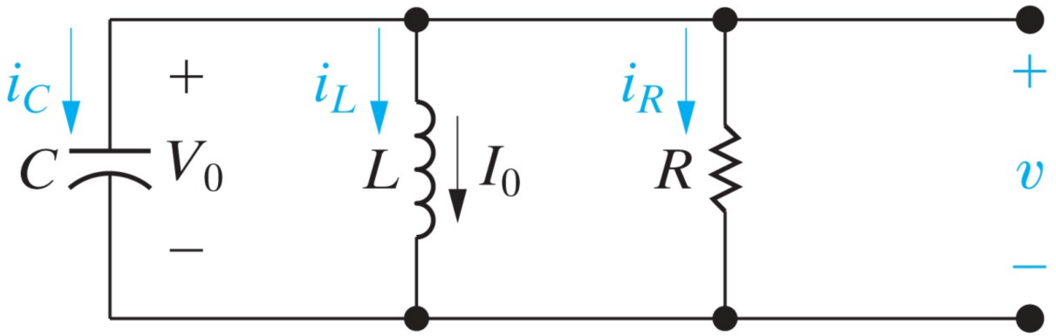


Figure: 08-01

$V_0, I_0 \equiv$ **Initial Voltage and Current**

***v* is same for all elements**

Write equation in terms of *v*, by Σ*i* leaving top node

$$i_c + i_L + i_R = 0$$

KCL

$$C\frac{dv}{dt} + \left[\frac{1}{L}\int_0^t v\,dt + I_0\right] + \frac{v}{R} = 0$$

Substitute *V/I* Relationships for each element

How do we deal with the Integral?

Differentiate the equation

Parallel RLC Circuits (Contd.)

$$C\frac{d^2v}{dt^2}+\left[\frac{v}{L}+0\right]+\frac{1}{R}\frac{dv}{dt}=0$$

Obtain a second order differential equation

$$\boxed{\frac{d^2v}{dt^2}+\left(\frac{1}{RC}\right)\frac{dv}{dt}+\frac{v}{LC}=0}$$

After simplification

2nd – order, ordinary differential equation with constant coefficients.

Solving 2nd – Order Circuits

1. **Need a function whose 1st and 2nd – order derivatives are of the same form as the function.**

2. **1st – Order circuits were exponential.**

$$v = Ae^{st} \quad \text{where } A \text{ and } s \text{ are constants}$$

Assume:

- **Assumed solution leads to the concept of the "<u>Characteristic Equation</u>".**

- **Plug assumed solution into** $\dfrac{d^2v}{dt^2} + \left(\dfrac{1}{RC}\right)\dfrac{dv}{dt} + \dfrac{v}{LC} = 0$

$$\left. As^2e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0 \right\} \text{After plugging in the solution}$$

$$\left. \boxed{Ae^{st}\left[s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC}\right] = 0} \right\} \text{After simplification}$$

When is the above equation satisfied?

Solving 2nd – Order Circuits (Contd.)

When is

$$Ae^{st}\left[s^2+\left(\frac{1}{RC}\right)s+\frac{1}{LC}\right]=0$$

1. $st \to -\infty \Rightarrow v(t)=0$
2. $A=0 \Rightarrow v(t)=0$

Trivial, not useful solutions

3. $$\boxed{s^2+\left(\frac{1}{RC}\right)s+\frac{1}{LC}=0}$$

Quadratic Characteristic Equation

General Procedure for 2nd – Order Circuits

$$a\frac{d^2\chi}{dt^2}+b\frac{d\chi}{dt}+c\chi=0$$

***a*, *b*, *c* are constants**

$$as^2+bs+c=0$$

Characteristic Equation

$$\chi(t)=Ae^{st}$$

Solution

Solving 2nd – Order Circuits (Contd.)

Quadratic Equation Review:

$$a\chi^2 + b\chi + c = 0$$

a, b, c are constants

$$\chi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Roots of quadratic equation

For the Characteristic Equation of the RLC Circuit

$$s_{1,2} = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - 4(1)\frac{1}{LC}}}{2(1)}$$

After using quadratic formula

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

After simplification

Solving 2nd – Order Circuits (Contd.)

Two Solutions for *s* ⇨ **Two solutions for $v(t)$** $v(t) = Ae^{st}$

$$v_1(t) = A_1 e^{s_1 t} \quad \text{and} \quad v_2(t) = A_2 e^{s_2 t}$$

are both solutions

It can be shown that $v = v_1 + v_2$ **is also a solution**

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

General Solution for Parallel RLC

Initial Conditions ⇨ A_1 **and** A_2

Use initial conditions to solve for A_1 and A_2
Need two initial conditions

Parallel RLC Solution for Voltage

Natural Response

1. $$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \Big\} \; \boxed{\text{Solution}}$$

2. $$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad \Big\} \; \boxed{\text{Characteristic roots}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad \Big\} \; \boxed{\text{Complex Frequency}}$$

3. $$\alpha = \frac{1}{2RC} \quad \Big\} \; \boxed{\text{Neper Frequency}}$$

4. $$\omega_0 = \frac{1}{\sqrt{LC}} \quad \Big\} \; \boxed{\text{Resonant Frequency}}$$

Parallel RLC Solution for Voltage (Contd.)

$$\frac{1}{RC} \equiv \frac{1}{\Omega F} \equiv \frac{1}{\frac{v}{A} \cdot \frac{C}{v}} = \frac{A}{C} = \frac{C/s}{C} = \frac{1}{s}$$

Units for frequency "1/seconds"

Neper Frequency

α in units of $\frac{rad}{\sec}$ — rad are dimensionless

$f = \frac{cycles}{\sec}$ — $cycles$ are dimensionless

$\omega = 2\pi f \ \frac{rad}{s} \equiv$ Angular Frequency

3 Possible Outcomes for Roots

1. $\omega_0^2 < \alpha^2 \Rightarrow s_{1,2}$ Real — **Overdamped**
2. $\omega_0^2 > \alpha^2 \Rightarrow s_{1,2}$ Complex — **Underdamped**
3. $\omega_0^2 = \alpha^2 \Rightarrow s_{1,2} = -\alpha$ Repeated — **Critically damped**

Overdamped Voltage Response

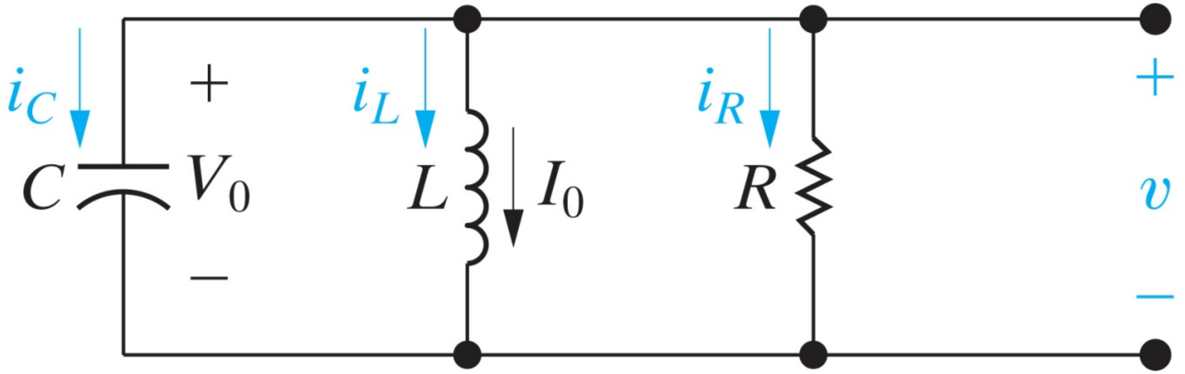


Figure: 08-01

$$\omega_0^2 < \alpha^2$$

Roots are real

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad s_1,\ s_2 \text{ are roots}$$

Initial Conditions ⟹ ***A₁* and *A₂***

$$v(0^+) \text{ and } \frac{dv(0^+)}{dt}$$

Need two initial conditions

$v(0^+)$ and $\frac{dv(0^+)}{dt}$ **can be used to find A_1 and A_2**

$$v(0^+) = A_1 + A_2 \equiv V_0$$

Voltage across Capacitor

$$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2 \equiv \frac{1}{C} i_C(0^+)$$

Current in the Capacitor

Overdamped Voltage Response (Contd.)

Find $i_c(0^+)$ $\Rightarrow$ $i_c(0^+) + i_L(0^+) + i_R(0^+) = 0$ } **KCL**

$$i_c(0^+) = -I_0 - \frac{V_0}{R}$$

Initial Capacitor Current in terms of Initial Inductor Current and Initial Voltage across Resistor

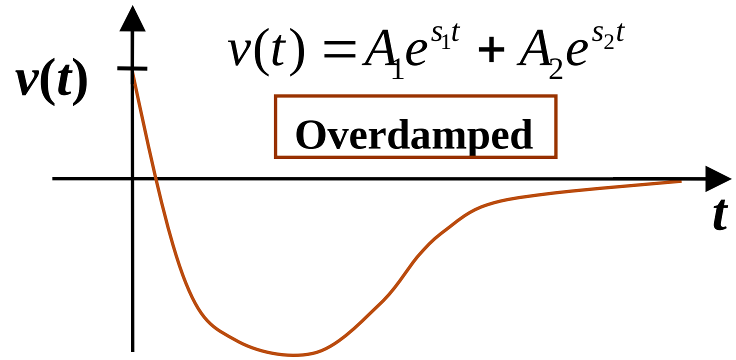
$$\frac{dv(0^+)}{dt} = \frac{1}{C} i_C(0^+) = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)$$

$\frac{dv(0^+)}{dt}$ in terms of I_0 and V_0

Solution Method

1. **Find s_1, s_2 ≡ fcn(R, L, and C)** } **Roots**
2. **Find $v(0^+)$ and $dv(0^+)$/dt** } **Determine Initial Conditions**
3. **Use $v(0+)$ and $dv(0+)$/dt to find A_1, A_2**
4. **Plug s_1, s_2, A_1, A_2 into solution for $v(t)$**

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



Drill Exercise: Find $v(t)$ for $t \geq 0$

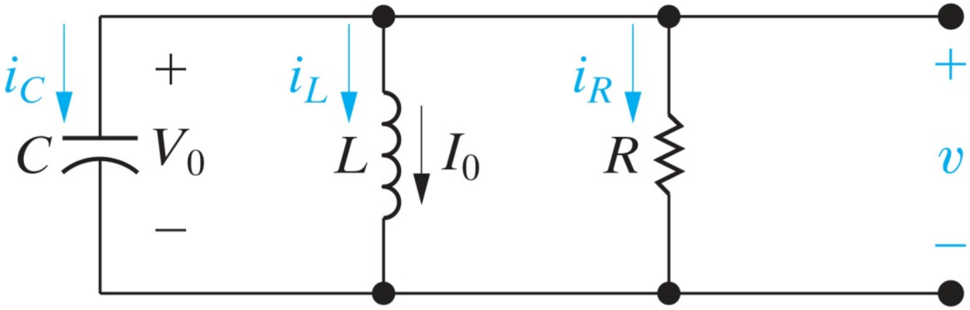


Figure: 08-01

$$R = 400\ \Omega \qquad I_0 = -4\ \text{A}$$
$$L = 50\ \text{mH} \qquad V_0 = 0\ \text{V}$$
$$C = 50\ \text{nF}$$

$$\boxed{\omega_0^2 < \alpha^2}$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2(400)50\times10^{-9}} = 25,000\ \ \text{rad/s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\left(\left(50\times10^{-3}\right)\left(50\times10^{-9}\right)\right)^{1/2}} = 20,000\ \ \text{rad/s}$$

$\therefore$ Overdamped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \Rightarrow \begin{array}{l} s_1 = -10\ \ (Krad/s) \\ s_2 = -40\ \ (Krad/s) \end{array}$$

Roots obtained from quadratic formula

a) $v(0^+) = V_0 = 0$

Given

Drill Exercise (Contd.)

b). Find $\dfrac{dv(0^+)}{dt}$ by using $i_C(0^+)$

$i_L(0^+) = I_0 = -4\text{ A}$ } **Given**

$i_C(0^+) = i_L(0^+) - i_R(0^+)$ } **KCL**

$= -(-4\text{A}) - 0$ } $V_0 = 0 \Rightarrow i_R(0^+) = 0$

$$\boxed{i_C(0^+) = 4\ (A)}$$

$$\frac{dv(0^+)}{dt} = \frac{1}{C} i_C(0^+) = \frac{4\text{A}}{50 \times 10^{-9}\text{ F}}$$

$$\boxed{\frac{dv(0^+)}{dt} = 8 \times 10^7\text{ V/s}}$$

c). Initial Conditions $\Rightarrow A_1$ and A_2

$$v(0^+) = 0 = A_1 + A_2$$

$$\frac{dv(0^+)}{dt} = 8 \times 10^7 = s_1 A_1 + s_2 A_2 = -10{,}000 A_1 - 40{,}000 A_2$$

$$A_1 + A_2 = 0$$

$$(\div 10\text{K}) \Rightarrow \quad A_1 + 4A_2 = -8000$$

$$\Rightarrow \boxed{A_1 = \frac{8000}{3} \qquad A_2 = -\frac{8000}{3}}$$

d). $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

Final Solution

$$\boxed{v(t) = \frac{8000}{3}\left[e^{-10{,}000t} - e^{-40{,}000t}\right] \quad (V)} \quad t \geq 0$$

Underdamped Voltage Response

$\omega_0^2 > \alpha^2$ → **Roots are complex**

$$s_{1,2} = -\alpha \pm \sqrt{-(\omega_0^2 - \alpha^2)}$$

$$= -\alpha \pm \sqrt{-1} \cdot \sqrt{\omega_0^2 - \alpha^2}$$

Factor out the complex number

Let $j = \sqrt{-1}$

$$\begin{bmatrix} s_1 = -\alpha + j\omega_d \\ s_2 = -\alpha - j\omega_d \end{bmatrix} \qquad \begin{bmatrix} \omega_d = \sqrt{\omega_0^2 - \alpha^2} \\ \text{Damped Radian Freq.} \end{bmatrix}$$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$$

Underdamped Solution

Underdamped Voltage Response (Contd.)

Using Euler's Identity → $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$ } **Substitute in expression for *v*(*t*)**

1 $v(t) = e^{-\alpha t}[(A_1 + A_2)\cos(\omega_d t) + j(A_1 - A_2)\sin(\omega_d t)]$ } **After simplifying the expression for *v*(*t*)**

2 **Let** $B_1 = A_1 + A_2$ **and** $B_2 = j(A_1 - A_2)$

3 $v(t) = e^{-\alpha t}[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)]$ } **Real Number Solution**

$B_1 = A_1 + A_2$

$B_2 = j(A_1 - A_2)$

} **Both are real numbers because** $A_1 = A_2^*$ **Complex Conjugate**

Find the equations for the Initial Conditions

$v(0^+) = V_0 = B_1$ } **Use 3**

$$\frac{dv(0^+)}{dt} = \frac{1}{C}i_C(0^+) = -\alpha B_1 + \omega_d B_2$$ } **Use 3**

<u>After some Algebra</u> → **See next slide**

Find Constants A_1 and A_2

General Solution

$$v(t) = e^{-\alpha t}[(A_1 + A_2)\cos(\omega_d t) + j(A_1 - A_2)\sin(\omega_d t)]$$

Initial Conditions

$$v(0^+) = V_0 = A_1 + A_2 \qquad \textbf{(1)}$$

$$\frac{dv(t)}{dt} = -\alpha e^{-\alpha t}(A_1 + A_2)\cos(\omega_d t) - e^{-\alpha t}\omega_d(A_1 + A_2)\sin(\omega_d t)$$

$$-\alpha e^{-\alpha t} j(A_1 - A_2)\sin(\omega_d t) + e^{-\alpha t} j\omega_d(A_1 - A_2)\cos(\omega_d t)$$

$$\frac{dv(0^+)}{dt} = \frac{1}{C} i_C(0^+) = -\alpha(A_1 + A_2) + j\omega_d(A_1 - A_2) \qquad \textbf{(2)}$$

$$\text{Let } K = \frac{1}{C} i_C(0^+) \equiv \text{constant}$$

(Contd.)

Find Constants A_1 and A_2 (Contd.)

$$A_1 + A_2 = V_0 \quad (3)$$

$$-\alpha(A_1 + A_2) + j\omega_d(A_1 - A_2) = K \quad (4)$$

Two equations for IC's

From (3) $A_2 = V_0 - A_1$ **(5)**

$$A_1 - A_2 = A_1 - V_0 + A_1 = 2A_1 - V_0 \quad (6)$$

Subtract (5) from A_1

Insert (3) and (6) into (4)

$$-\alpha(V_0) + j\omega_d(2A_1 - V_0) = K \quad (7)$$

$$2A_1 - V_0 = \frac{K + \alpha V_0}{j\omega_d} = -j\left(\frac{K + \alpha V_0}{\omega_d}\right) \quad (8)$$

Solve for A_1

$$2A_1 = V_0 - j\frac{K + \alpha V_0}{\omega_d}$$

Simplify

$$A_1 = \frac{1}{2}\left(V_0 - j\frac{K + \alpha V_0}{\omega_d}\right) \quad (9)$$

Find Constants A_1 and A_2 (Contd.)

From (3) $A_2 = V_0 - A_1$ **(10)**

$$A_2 = V_0 - \frac{1}{2}V_0 + j\frac{K + \alpha V_0}{2\omega_d} \quad \textbf{(11)}$$

Substitute (9) into (10)

$$A_2 = \frac{1}{2}V_0 + j\frac{K + \alpha V_0}{2\omega_d} \quad \textbf{(12)}$$

From the examination of (9) and (12), we see that:

$$A_1 = A_2^*$$

Since $A_1 = A_2^*$

Then $B_1 = A_1 + A_2 = V_0 \equiv \text{Real}$

and $B_2 = j(A_1 - A_2) = \dfrac{K + \alpha V_0}{\omega_d} \equiv \text{Real}$

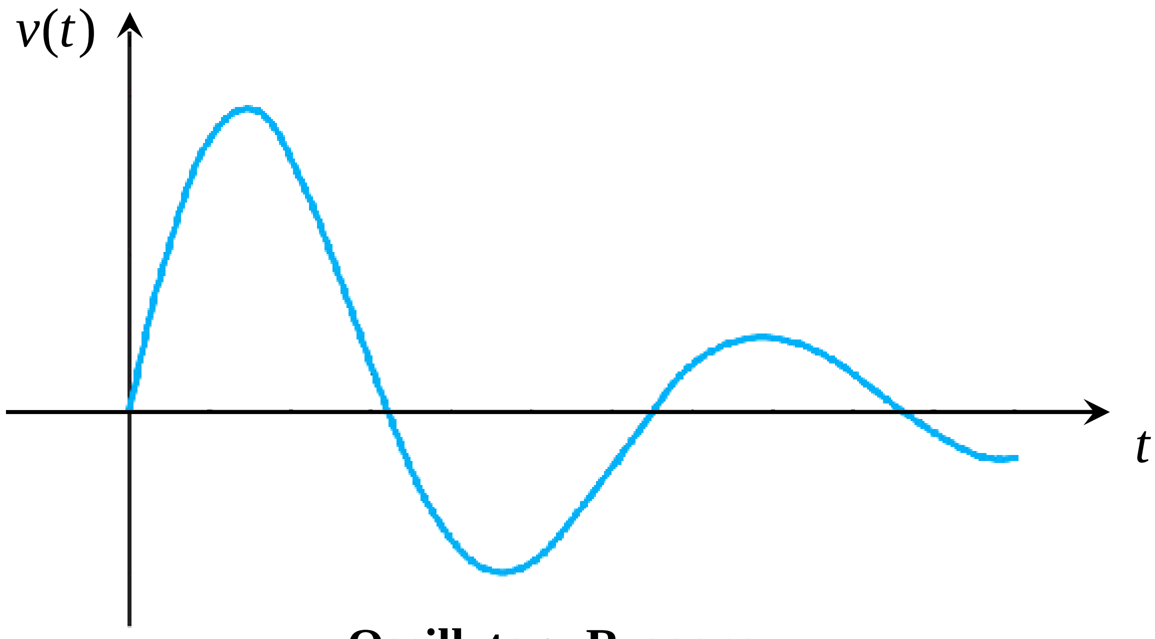
Thus, the constants B_1 **and** B_2 **are real.**

Underdamped Voltage Response (Contd.)

$$\omega_d = \sqrt{\omega_0^{\,2} - \alpha^2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \alpha = \frac{1}{2RC}$$

Frequency parameter related to 2nd – order circuit parameters



Oscillatory Response

Typical Underdamped Response

Critically Damped Voltage Response

$$\omega_0^2 = \alpha^2 \qquad \textbf{OR} \qquad \omega_0 = \alpha$$

On the verge of oscillating

$$s_{1,2} = -\alpha$$

Advanced Solution — **Proof beyond our scope**

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

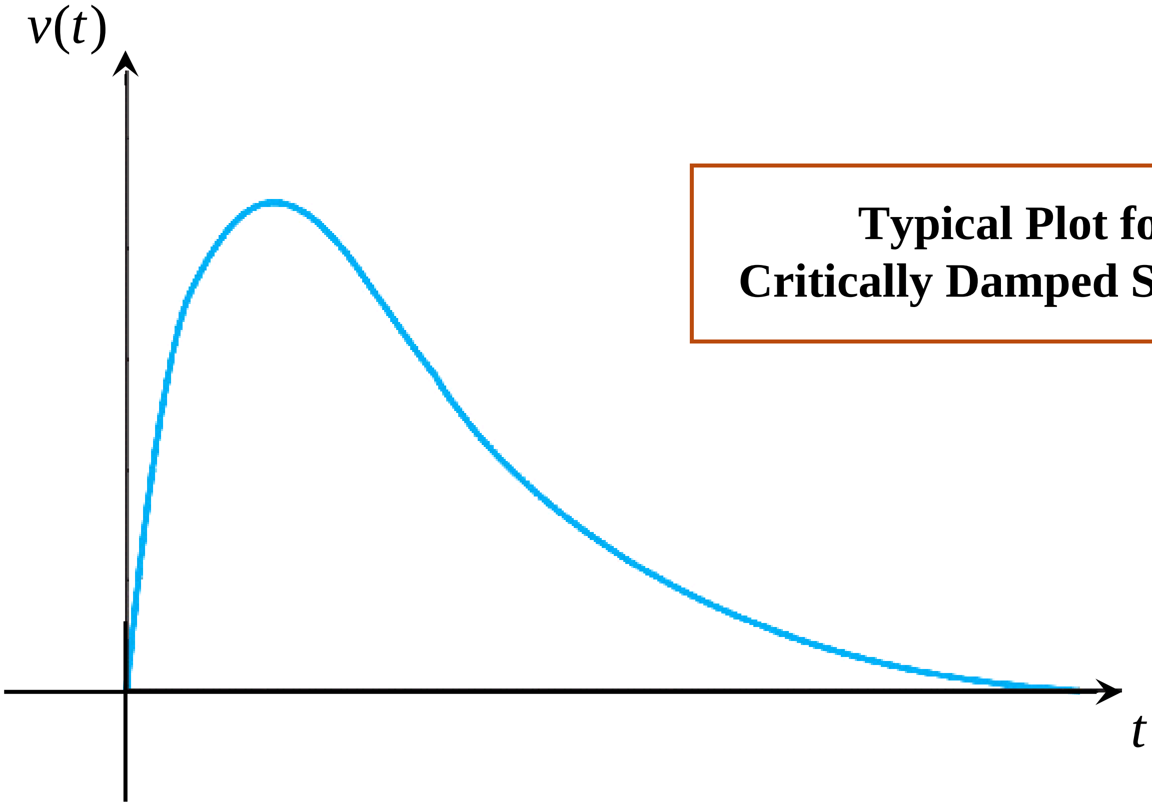
Critically Damped Solution

$$v(0^+) = V_0 = D_2$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2$$

Equations for finding D_1 and D_2 from the Initial Conditions

Critically Damped Voltage Response (contd.)



Typical Plot for Critically Damped Solution

Summary of Parallel RLC Circuits

Natural Response

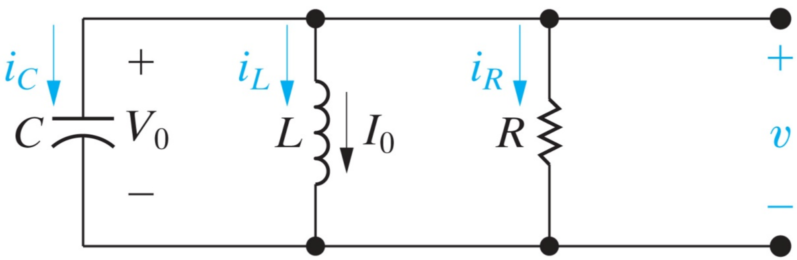


Figure: 08-01

V_0, I_0 ≡ Initial Voltage and Current

v **is same for all elements**

$$v(0^+) = V_0$$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \frac{dv(0^+)}{dt} = \frac{1}{C} i_C(0^+) = \frac{1}{C}\left(-I_0 - \frac{V_0}{R} \right)$$

Overdamped $\omega_0^2 < \alpha^2$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$A_1 = \frac{1}{s_1 - s_2}\left[\frac{dv(0^+)}{dt} - s_2 v(0^+) \right]$$

$$A_2 = \frac{1}{s_2 - s_1}\left[\frac{dv(0^+)}{dt} - s_1 v(0^+) \right]$$

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Underdamped $\omega_0^2 > \alpha^2$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$s_1 = -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d$$

$$B_1 = v(0^+)$$

$$B_2 = \frac{1}{\omega_d}\left[\frac{dv(0^+)}{dt} + \alpha v(0^+) \right]$$

$$V(t) = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t) \right]$$

Critically Damped $\omega_0^2 = \alpha^2$

$$\omega_0 = \alpha$$

$$s_1 = -\alpha$$

$$s_2 = -\alpha$$

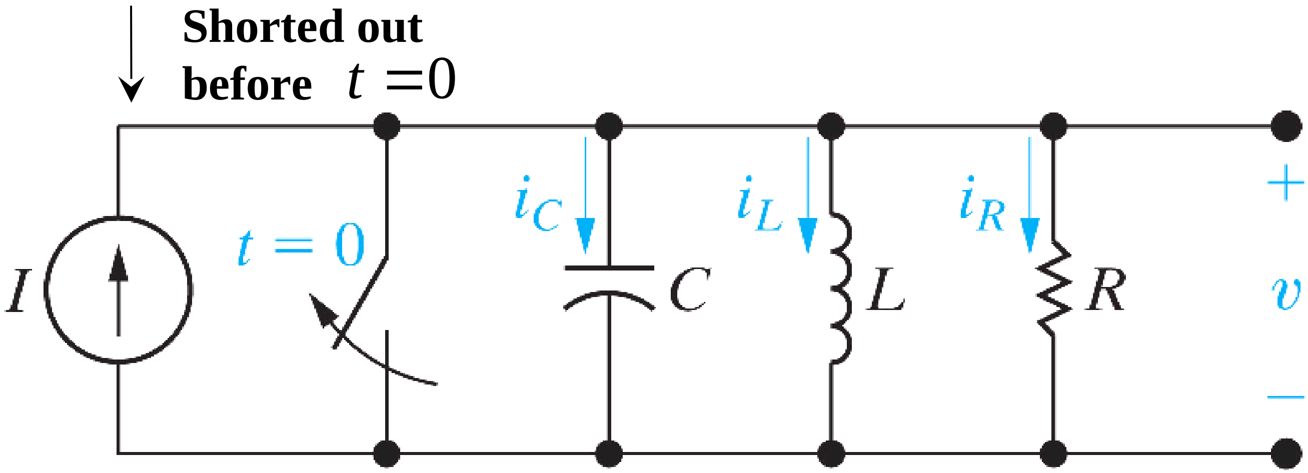
$$D_1 = \alpha D_2 + \frac{dv(0^+)}{dt}$$

$$D_2 = v(0^+)$$

$$V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

Step Response of a Parallel RLC Circuit

Apply a DC current source
Find $i_L(t)$ for $t \geq 0$



1 $$I = i_C + i_L + i_R$$ **KCL**

2 $$i_C = C\frac{dv}{dt} \qquad i_R = \frac{v}{R}$$ **Capacitor Law**
Ohm's Law

3 $$I = C\frac{dv}{dt} + i_L + \frac{v}{R}$$ **Substitute 2 into 1**

Step Response of a Parallel RLC Circuit (Contd.)

4 $$v = L\frac{di_L}{dt} \Rightarrow \frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}$$

Differentiate the Inductor Law

5 $$I = LC\frac{d^2 i_L}{dt^2} + i_L + \frac{L}{R}\frac{di_L}{dt}$$

Substitute 4 into 3

Simplify 5

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{I}{LC}$$

Differential Equation for $i_L(t)$ "forced" by constant source I

Note the similarity to the natural response differential equation

$$\frac{d^2 v}{dt^2} + \left(\frac{1}{RC}\right)\frac{dv}{dt} + \left(\frac{1}{LC}\right)v = 0$$

Form is the same, except for constant.

Step Response of a Parallel RLC Circuit (Contd.)

In general, <u>complete</u> response of a 2nd – order system, with a <u>constant forcing function</u> is given by

<u>Natural response + Forced response</u>

Forced response — **Caused by a constant forcing function**

$$i_L(t) = i_n(t) + I_f(t)$$

Natural response — **Caused by initial conditions**

Coefficients in the analytical solutions are different from the case of zero forcing function.

Step Response of a Parallel RLC Circuit (Contd.)

Three cases: $\alpha = \dfrac{1}{2RC}$ $\omega_0 = \dfrac{1}{\sqrt{LC}}$

① **Overdamped** $\omega_0^2 < \alpha^2$

$$i_L(t) = A_1' e^{s_1 t} + A_2' e^{s_2 t} + \underline{\underline{I_f}}$$

Constant I_f Caused by a constant forcing function

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

② **Underdamped** $\omega_0^2 > \alpha^2$

$$i_L(t) = e^{-\alpha t}[B_1' \cos(\omega_d t) + B_2' \sin(\omega_d t)] + \underline{\underline{I_f}}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \qquad s_{1,2} = -\alpha \pm j\omega_d$$

(Contd.)

Step Response of a Parallel RLC Circuit (Contd.)

③ **Critically damped** $\omega_0^2 = \alpha^2$

$$i_L(t) = D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} + \underline{\underline{I_f}}$$

Final condition:

Capacitor open

Inductor short

$$V(t \to \infty) = 0$$

$$\therefore I_f = i_L(t \to \infty) = I \; \} \text{ current source input}$$

Initial conditions: Used to determine scaling constants.

$$i_L(0^+) = i_L(0^-) = I_0$$

$$\frac{di_L(0^+)}{dt} = \frac{V_0}{L}$$

since $v \equiv v_L = L \dfrac{di_L}{dt}$

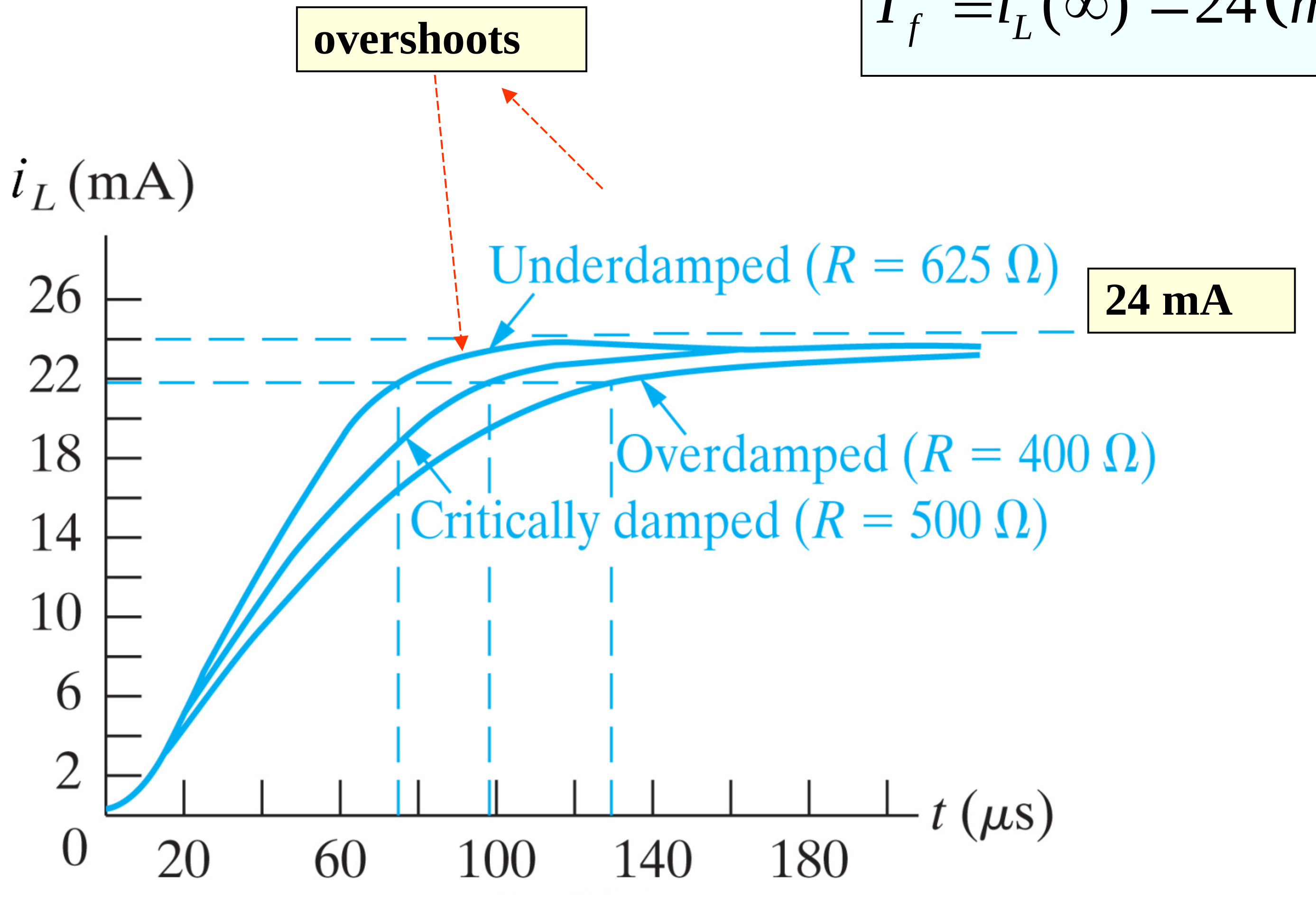
Note:

For initial currents, consider Inductor.

For initial voltage, consider Capacitor.

Step Response of Parallel RLC Circuit

$$I_f \equiv i_L(\infty) = 24(mA)$$



Study Suggestions:

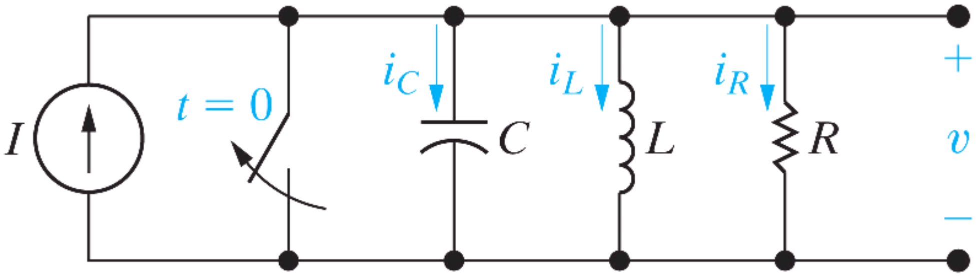
Make neat page or two of notes with equations for natural response and step response, for parallel RLC circuits.

Do the same for series RLC circuits.

Read all the examples

Do all the drill exercises.

Example: Find $v(t)$ for $t \geq 0$



Given

$R = 250\ (\Omega)$ $\quad I_0 = 0.5\ (A) = i_L(0)$

$L = 0.32\ (H)$ $\quad V_0 = 80\ (V) = v_c(0)$

$C = 2\ \ (\mu F)$ $\quad I = -1.5(A)$

Current source switched in at $t = 0$

$$\alpha = \frac{1}{2RC} = \frac{1}{2(250)(2\times 10^{-6})} = 1000\ \frac{rad}{s} = \alpha$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.32(2\times 10^{-6})}} = 1250\frac{rad}{s} = \omega_0$$

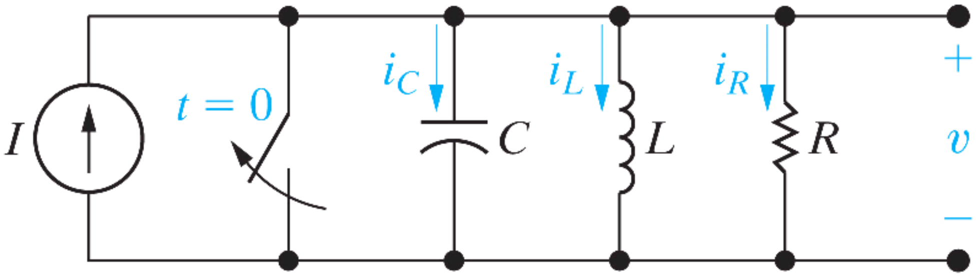
Find α and ω_0

$\omega_0^2 > \alpha^2 \quad \Rightarrow \quad$ Underdamped response

Note $i_L(\infty) = I = I_f$

All steady state current is in inductor

Example (Contd.)



Given

$R = 250\ (\Omega)$ $I_0 = 0.5\ (A) = i_L(0)$

$L = 0.32\ (H)$ $V_0 = 80\ (V) = v_c(0)$

$C = 2\ (\mu F)$ $I = -1.5(A)$

Solution is of the form

Underdamped Solution

Find $\alpha, \omega_d, B_1', B_2'$

$a)$ $i_L(t) = e^{-\alpha t}[B_1'\cos(\omega_d t) + B_2'\sin(\omega_d t)] + I_f$

$$\boxed{I_f = I = -1.5\ (A)}$$

$b)$ $s_{1,2} = -\alpha + j\omega_d$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{1250^2 - 1000^2} = 750$ } **Calculate the roots of the characteristic equation**

$$\boxed{s_{1,2} = -1000 \pm j750\ \frac{rad}{\sec}}$$ } **Underdamped Solution**

Find Initial Conditions

$c)$ $v_L = L\frac{di}{dt} \Rightarrow \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{V_0}{L} = \frac{80(V)}{0.32(H)} = 250\left(A/s\right)$

$i_L(0) = 0.5(A)$

$\frac{di_L(0^+)}{dt} = 250\left(A/s\right)$

Example (Contd.)

$d)$ Solution has the form

$$i_L(t) = e^{-1000t}\left[B_1'\cos 750t + B_2'\sin 750t\right] - 1.5$$

$$i_L(0^+) = B_1' - 1.5 \Rightarrow i_L(0^+) = 0.5(A) \Rightarrow B_1' = 2(A)$$

$e)$ Differentiate $i_L(t)$ and substitute $t = 0^+$ to find B_2'

$$\frac{di_L(t)}{dt} = -1000e^{-1000t}\left[B_1'\cos 750t + B_2'\sin 750t\right] + e^{-1000t}\left[-750B_1'\sin 750t + 750B_2'\cos 750t\right]$$

$$\frac{di_L(0^+)}{dt} = -1000B_1' + 750\mathrm{B}_2' \quad \boxed{\begin{array}{c}\textbf{Use}\\ \begin{cases}\dfrac{di_L(0^+)}{dt} = 250(A/s)\\ B_1' = 2(A)\end{cases}\end{array}} \Rightarrow B_2' = 3(A)$$

$f)$ Solution has the final form

$$i_L(t) = e^{-1000t}\left[2\cos 750t + 3\sin 750t\right] - 1.5(A)$$

Example (Contd.)

$$g)\quad i_L(t) = e^{-1000t}[2\cos(750t) + 3\sin(750t)] - 1.5 \ \ (A)$$

Note: $i_L(0^+) = 2 - 1.5 = 0.5\ A \equiv I_0$

$i_L(\infty) = -1.5\,(A) = I_f \ \equiv I$

Perform as a Check

$h)$ We know $i_L(t)$ and $v(t)$ is the same for each element.

$$\therefore v(t) = L\frac{di_L}{dt}$$

Differentiate $i_L(t)$ of (g) and multiply by L

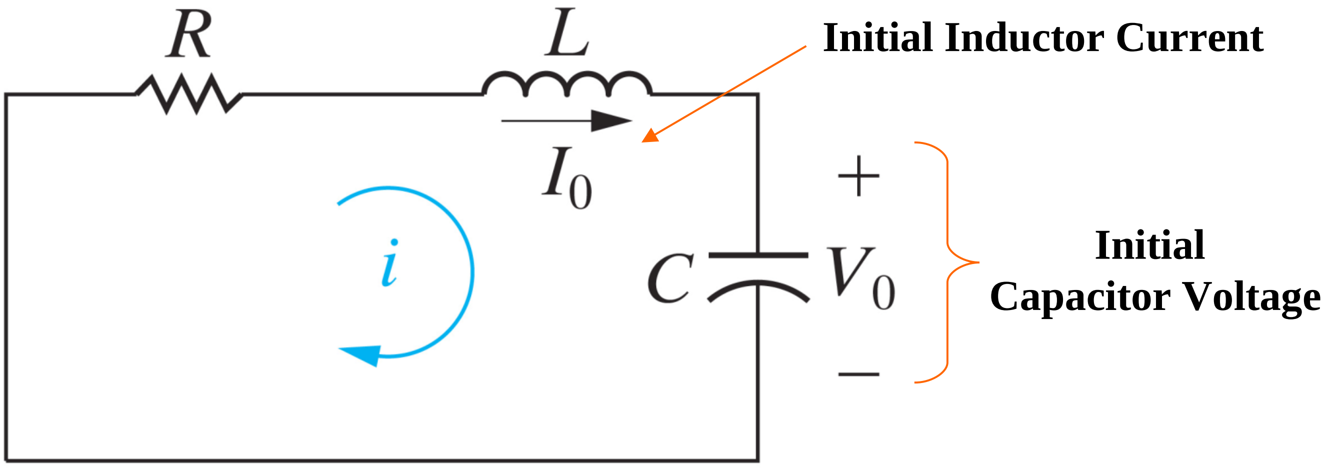
Final Solution

$$v(t) = 80e^{-1000t}[\cos(750t) - 18\sin(750t)] \ \ (V)$$

Note: $v(0) = 80 = V_0$

After Simplification

Series RLC Circuits



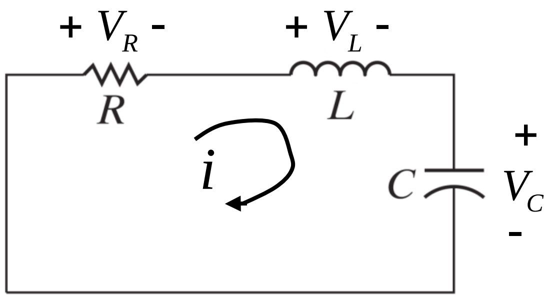
Similar to Parallel RLC Analysis

Find Natural Response for $i(t)$ by

obtaining Differential Equation in terms of $i(t)$.

Note that $i(t)$ is the same for all elements.

Series RLC Circuits (Contd.)



$$V_R + V_L + V_C = 0$$ KVL

$$Ri + L\frac{di}{dt} + \left[\frac{1}{C}\int_{t_0}^{t} i(t)dt + V_0\right] = 0$$ Substitute Ohm's Law Capacitor Law Inductor Law

Differentiate to get rid of integral

$$\Rightarrow \quad \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

Characteristic Equation

$$s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC} = 0$$

Same form as for parallel RLC

Series RLC Circuits (Contd.)

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$(\omega_0^2 < \alpha^2 \Rightarrow \text{ overdamped})$$

$$i(t) = B_1 e^{-\alpha t} \cos(\omega \alpha t) + B_2 e^{-\alpha t} \sin(\omega \alpha t)$$

$$(\omega_0^2 < \alpha^2 \Rightarrow \text{ underdamped})$$

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$(\omega_0^2 = \alpha^2 \Rightarrow \text{ critically damped})$$

Solution of Differential Equation

where: $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$\alpha = R / 2L$ } Differs from $\|$ RLC

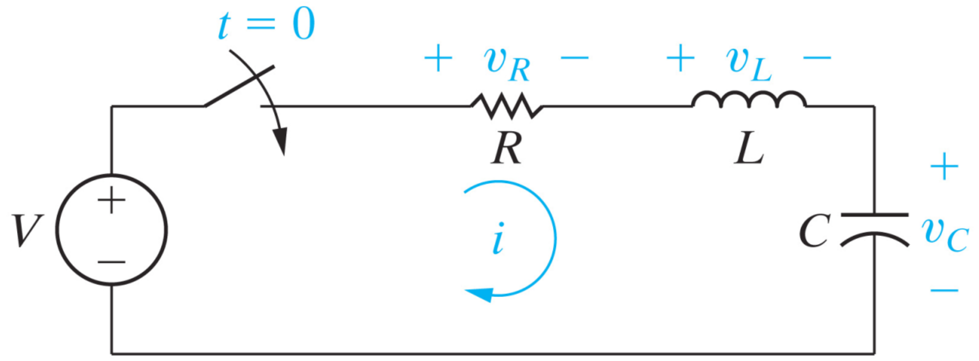
$\omega_0 = 1 / \sqrt{LC}$ } Same as $\|$ RLC

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Series RLC Circuits: Step Response (Contd.)

Apply a DC Voltage Source

Focus on v_C since it is continuous



Similar to Parallel RLC analysis

$$v_R + v_L + v_C = V$$ **KVL**

(1) $$Ri + L\frac{di}{dt} + v_C = V$$ **Substitute Ohm's Law and inductor law**

Series RLC Circuits (Contd.)

Differentiate capacitor law

$$i = C\frac{dv_c}{dt} \Rightarrow \frac{di}{dt} = C\frac{d^2v_c}{dt^2} \qquad (2)$$

$$Ri + L\frac{di}{dt} + v_C = V \qquad (1)$$

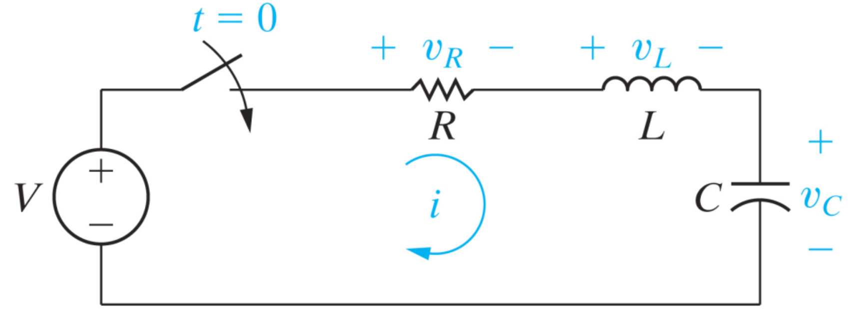
From the previous slide

$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_C}{LC} = \frac{V}{LC}$$

Substitute (2) into (1)

Same form as for $\|$ RLC circuit

Series RLC Circuits: Step response for $v_c(t)$



$$V_f = V_{cap(final)} = V$$ $\boxed{C \text{ - open; } L \text{ - short}}$ } **In steady state**

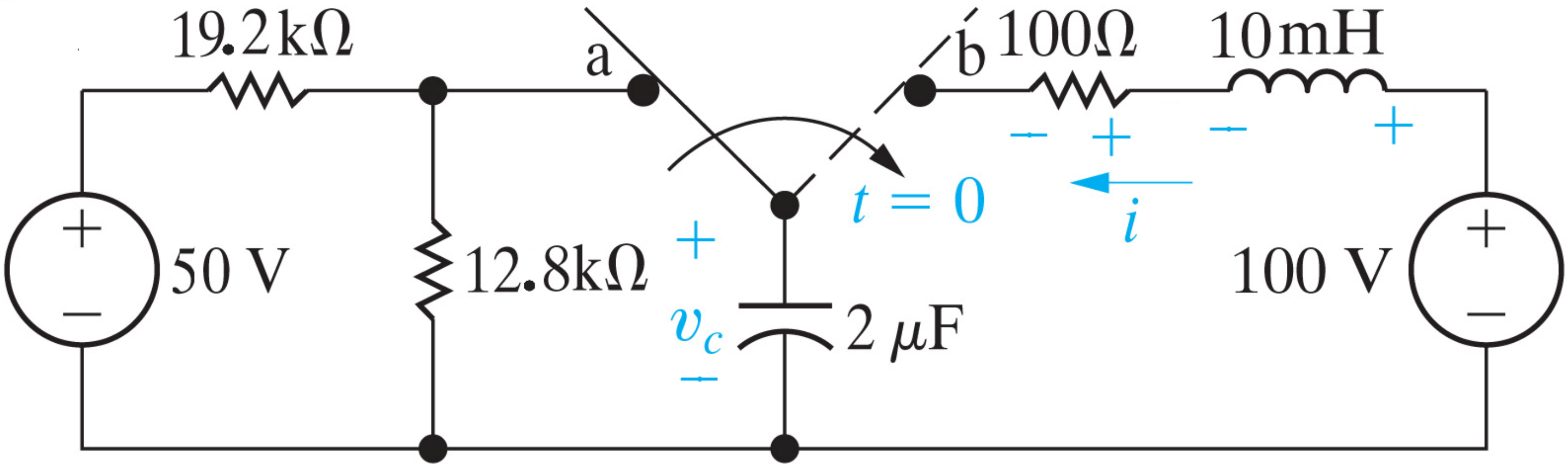
$$v_C(t) = A_1' e^{s_1 t} + A_2' e^{s_2 t} + V_f$$ overdamped

$$v_C(t) = e^{-\alpha t}[B_1' \cos(\omega_d t) + B_2' \sin(\omega_d t)] + V_f$$ underdamped

$$v_C(t) = D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} + V_f$$ critically damped

Need
$s_1, s_2,$
$\alpha, A_1', A_2',$
$B_1', B_2',$
D_1', D_2', ω_d

Example

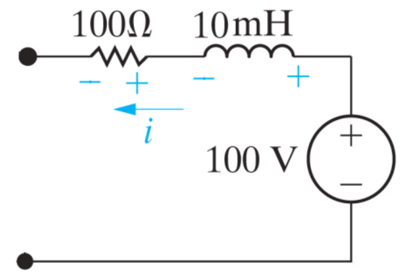


Find $i(t)$ for $t \geq 0$

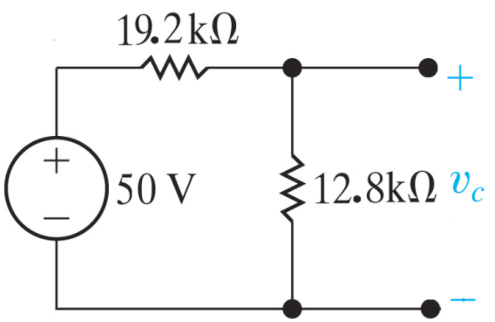
Example: Find Initial Conditions (Contd.)

$a)$ $i(0^-) = 0$

$\boxed{i(0^-) = i(0^+) = 0}$ ⟵ **Since i_L can't change instantaneously**



$b)$ Since capacitor is open at $t = 0^-$



$$\left.v_C(0^-) = \left(\frac{12.8K}{12.8K + 19.2K}\right)50V = 20\ (V)\right\}$$ **Voltage Division**

$\boxed{v_C(0^-) = v_C(0^+) = 20(V)}$ ⟵ **Since v_c can't change instantaneously**

Example: Find Parameters (Contd.)

c)

$$\alpha = \frac{R}{2L} = \frac{100}{2\left(10\times10^{-3}\right)} = \boxed{500\ rad/s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\left[\left(10\times10^{-3}\right)\left(2\times10^{-6}\right)\right]^{1/2}} = \boxed{7071.07\ rad/s}$$

Find α and ω_0

$$\boxed{\omega_0 > \alpha^2 \quad \Rightarrow \quad Underdamped}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(7071.07)^2 - (5000)^2}$$

$$\omega_d = \boxed{5000\ rad/s}$$

Find ω_d

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\omega_d$$

$$\boxed{s_{1,2} = -5000 \pm j5000}$$

Find Characteristic equation roots

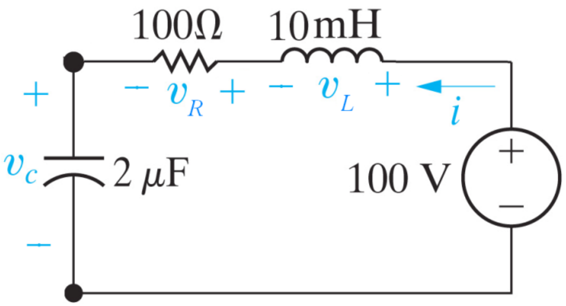
Example: Find $i(t)$ (Contd.)

d)

$$i(t) = e^{-\alpha t}\left[B_1' \cos(\omega_d t) + B_2' \sin(\omega_d t)\right] + I_f$$

General solution for Underdamped system

$\boxed{I_f = 0}$ since eventually capacitor becomes open



e)

We know α, ω_d, and I_f

We must use $i(0^+)$ and $\dfrac{di(0^+)}{dt}$ to find B_1' and B_2'

f)

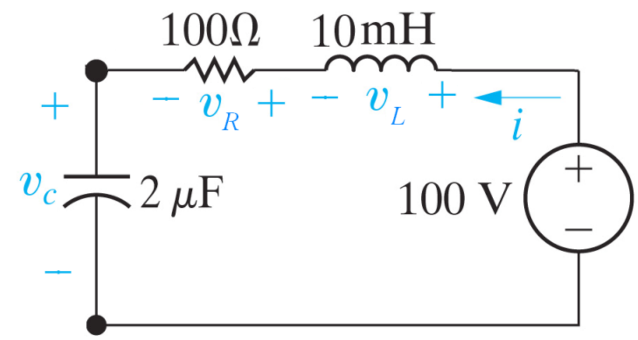
Find $i(0^+)$

$i(0^+) = 0$ **From a previous slide**

Example: Find Initial Conditions (Contd.)

***g)* Find** $\dfrac{di(0^+)}{dt}$

Circuit at $t = 0^+$



Need $\dfrac{di(0^+)}{dt}$ $\overset{\text{USE}}{\Rightarrow}$ $v_L(0^+) = L\dfrac{di(0^+)}{dt}$

From a previous slide

KVL $\left\{ 100V = v_L + v_R + v_C \right.$

$i(0^+) = 0 \Rightarrow v_R(0^+) = 0$ **Ohm's Law**

$$\boxed{100 = v_L(0^+) + v_C(0^+)}$$

$v_c(0^+) = 20$ from the previous slide

$$100 = L\frac{di(0^+)}{dt} + 20$$

Substitute inductor law

$$\frac{di(0^+)}{dt} = \frac{100 - 20}{L} = \frac{80}{10 \times 10^{-3}}$$

Solve for $\dfrac{di(0^+)}{dt}$

$$\boxed{\frac{di(0^+)}{dt} = 8000\ (A/s)}$$

Example: Find $i(t)$ (Contd.)

h) Find B_1' and B_2' in general solution

$$i(t) = e^{-\alpha t}\left[B_1'\cos(\omega_d t) + B_2'\sin(\omega_d t)\right] + I_f$$

General Solution

$$i(0^+) = 0$$

From a previous slide

$$0 = B_1' + I_f$$

Substitute $t = 0^+$

$$B_1' = 0$$

Since $I_f = 0$

$$\frac{di(t)}{dt} = -\alpha e^{-\alpha t}\left[B_1'\cos(\omega_d t) + B_2'\sin(\omega_d t)\right] + e^{-\alpha t}\left[-B_1'\omega_d\sin(\omega_d t) + B_2'\omega_d\cos(\omega_d t)\right]$$

Take time derivative

$$\frac{di(0^+)}{dt} = -\alpha B_1' + \omega_d B_2' = 8000$$

Substitute $t = 0^+$ and use value for $\frac{di(0^+)}{dt}$

$$B_1' = 0,\ \alpha = 500,\ \omega_d = 5000 \Rightarrow B_2' = 1.6(A)$$

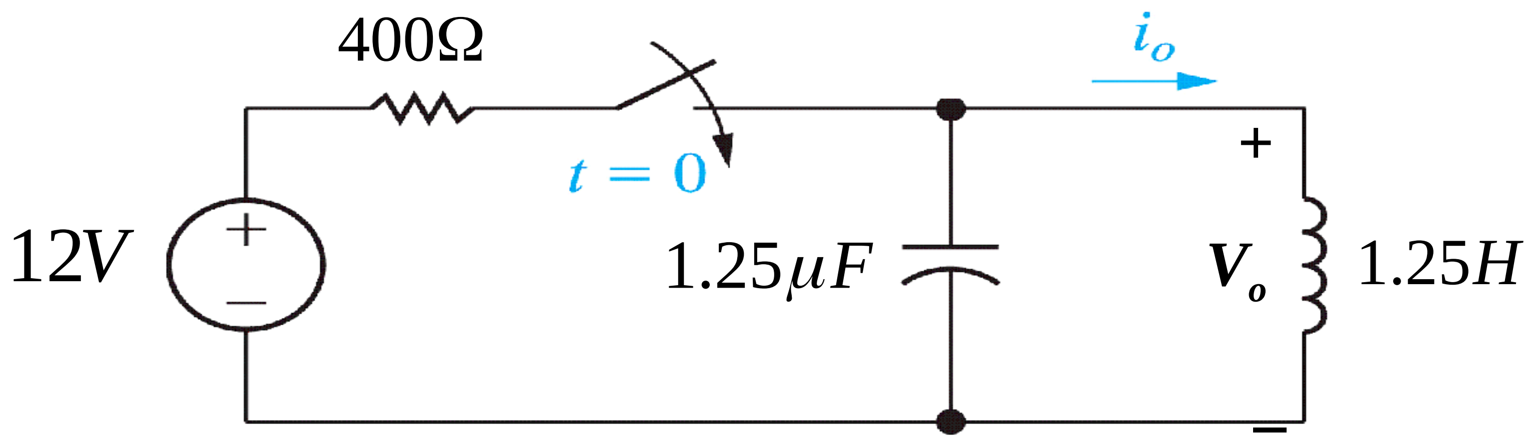
Final Solution

$$i(t) = 1.6e^{-5000t}\sin(5000t)(A) \qquad t \geq 0$$

Check

$$i(0) = 0, i(\infty) = 0$$

Example: Find $V_0(t)$ for $t \geq 0$



No initial energy stored in the circuit

Given in the problem

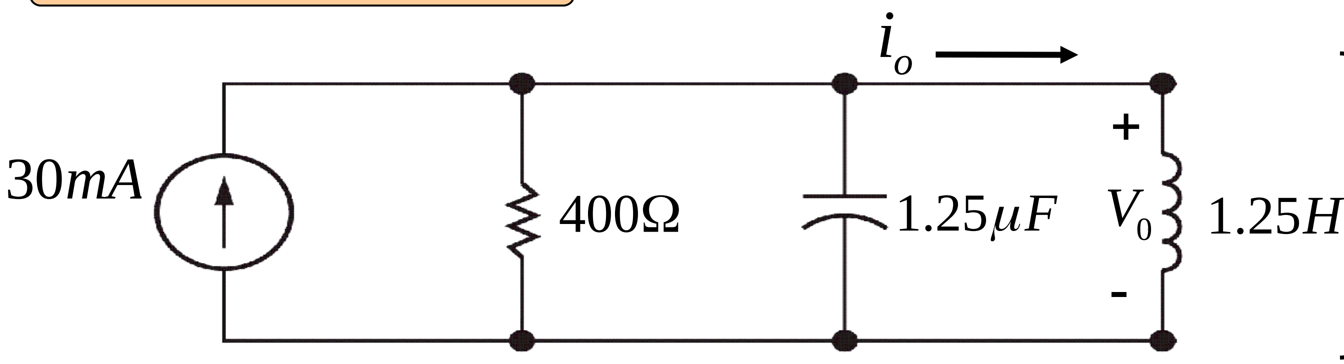
$$i_0(0^-) = i_0(0^+) = 0$$

$$V_0(0^-) = V_0(0^+) = 0$$

Properties of inductors and capacitors

Example: Use Source Transformation (Contd.)

Redraw the circuit for $t \geq 0$



Source Transformation

$$30(mA) = \frac{12(V)}{400(\Omega)}$$

Parallel *RLC*

$$\alpha = \frac{1}{2RC} = \frac{1}{2(400)(1.25\times10^{-6})} = 1000$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1.25(1.25\times10^{-6})}} = \sqrt{64\times10^4} = 800$$

Find α and ω_0

$\alpha > \omega_0$ *overdamped*

Example (Contd.)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1000 \pm 600$$

$$s_1 = -400, \qquad s_2 = -1600$$

Calculate the roots of the Characteristic equation

$$V_o(t) = V_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Overdamped solution has this form

$V_f = 0$ ***C* open, *L* short, all current in *L***

$V_o = 0$ as $t \to \infty$

$$V_o(t) = A_1 e^{-400t} + A_2 e^{-1600t}$$

Use initial conditions to find A_1 and A_2

Now must find $V_0(0^+)$ and $\dfrac{dV_0(0^+)}{dt}$ to solve for A_1 and A_2

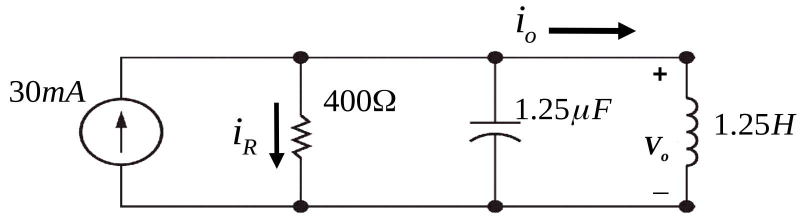
Example (Contd.)

Find $V_0(0^+) \Rightarrow V_0(0^+) = 0$ } **Given in the problem**

Find $\dfrac{dV_0(0^+)}{dt}$ using the circuit

$V_o(0) = V_o(0^+) = 0$ } **Voltage across capacitor can't change instantly**

$i_R(0^+) = V_o/R = 0$ } **Ohm's Law**



$30mA = i_R(0^+) + i_C(0^+) + i_o(0^+)$ } **KCL**

$30mA = i_C(0^+)$ } since $i_R(0^+) = i_o(0^+) = 0$

$$\frac{dV_o(0^+)}{dt} = \frac{1}{C} i_C$$ } **Capacitor Law**

$$\frac{dV_o(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{30mA}{1.25\mu F} = 24{,}000 V/s$$ } **Plug in numbers**

Example (Contd.)

Find A_1 and A_2

$$V_0(t) = A_1 e^{-400t} + A_2 e^{-1600t}$$ Solution so far

1 $$V_o(0^+) = \boxed{0 = A_1 + A_2}$$ Plug in $t = 0^+$

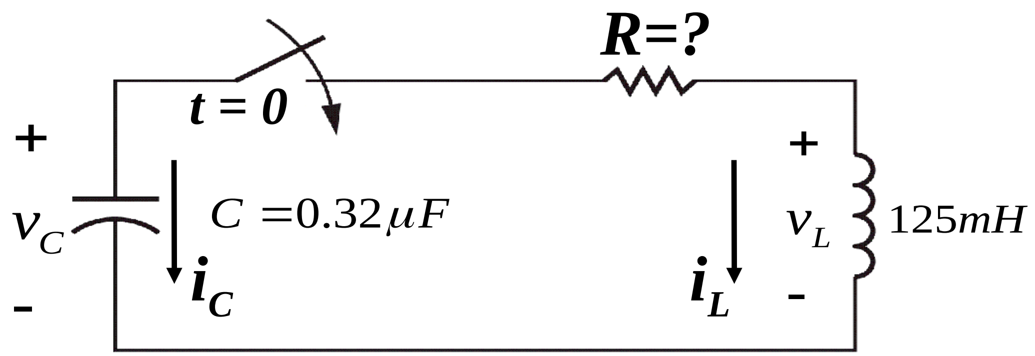
2 $$\frac{dV_0(0^+)}{dt} = 24,000 = -400A_1 - 1600A_2$$ Differentiate $v_0(t)$ and substitute $t = 0^+$

$$\Rightarrow A_1 = 20V \qquad A_2 = -20V$$ Solve 1 and 2 for 2 unknowns

Final Solution

$$\boxed{V_o(t) = 20\left[e^{-400t} - e^{-1600t}\right](V)}$$

Example: Design R so that $v_c(t)$ is critically damped



Given

$$v_C(0^+) = 15(V)$$
$$i_L(0^+) = 6(mA)$$

a) **Critical Damping:** $\alpha = \omega_0$

$$Series \rightarrow \frac{R}{2L} = \frac{1}{\sqrt{LC}} \Rightarrow R = 1250(\Omega)$$

b) $i_L(0^+) = i_L(0^-) = 6(mA)$ } **Current cannot change instantly**

Series *RLC* Current same in all components

Example (Contd.)

c) $\alpha = \frac{R}{2L} = \frac{1250}{2(0.125)} = 5000 \frac{rad}{s}$ Calculate α

d) $v_c(t) = D_1 t e^{-5000t} + D_2 e^{-5000t}$ **General expression for critically damped system**

e) We must find $v_C(0^+)$ and $\frac{dv_C(0^+)}{dt}$ to solve for D_1 and D_2

f) $v_C(0^+) = 15(V)$ **Given in the problem**

g) $i_C(t) = C\frac{dv_C(t)}{dt}$ **Capacitor Law**

so $\frac{dv_C(0^+)}{dt} = \frac{1}{C} i_C(0^+) = \frac{-i_L(0^+)}{32(\mu F)}$ **From Circuit** $i_C = -i_L$

so $\frac{dv_C(0^+)}{dt} = -18,750(A/F)$ **Substitute $i_L(0^+) = 6(mA)$**

Example (Contd.)

h) Find D_1 and D_2

$$v_C(t) = D_1 t e^{-5000t} + D_2 e^{-5000t}$$ **General Solution**

$$v_C(0^+) = 15(V) \qquad \frac{dv_C(0^+)}{dt} = -18{,}750\,(A/F)$$ **From previous slide**

$$v_C(0^+) = 15(V) = D_2$$ **Substitute *t* = 0⁺**

$$\frac{dv_C(t)}{dt} = D_1 e^{-5000t} - 5000tD_1 e^{-5000t} - 5000D_2 e^{-5000t}$$ **Time derivative of $v_C(t)$**

$$\frac{dv_C(0^+)}{dt} = -18{,}750 = D_1 - 5000D_2$$ **Substitute *t* = 0⁺**

$$D_1 = 56{,}250\,(V/s)$$ **Substitute D_2 = 15**

$$v_C(t) = 56{,}250e^{-5000t} + 15e^{-5000t}$$ **Final Solution**