Example.
$$C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}, \quad \det C = ?$$

Expand the determinant by the 3rd column:
$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add -2 times the 2nd row to the 1st row:
$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:
$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus
$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36.$$

Example .

$$
\begin{aligned}
& \begin{vmatrix} 4 & 3 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{vmatrix} \\
= \; & 4\begin{vmatrix} 2 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & 2 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} \\
= \; & 4\left(2\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}\right) \\
& -3\left(3\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}\right) \\
& -\left(3\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right) \\
= \; & 4\,(2(-6)) - 3\,(3(-6) + 1(2)) - (-2(2)) \\
= \; & 4
\end{aligned}
$$

Example.

$$
\begin{aligned}
& \begin{vmatrix} 2 & 2 & 5 & 5 \\ 1 & -2 & 4 & 1 \\ -1 & 2 & -2 & -2 \\ -2 & 7 & -3 & 2 \end{vmatrix} \\
= \; & \begin{vmatrix} 0 & 6 & -3 & 3 \\ 1 & -2 & 4 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 3 & 5 & 4 \end{vmatrix} \quad \begin{pmatrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 + 2R_2 \end{pmatrix} \\
= \; & -\begin{vmatrix} 6 & -3 & 3 \\ 0 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} \\
= \; & -3\begin{vmatrix} 2 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} \\
= \; & -3\left(2\begin{vmatrix} -1 & 1 \\ 5 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}\right) \\
= \; & -69
\end{aligned}
$$

Example .

$$\begin{aligned}
& \begin{vmatrix} 2 & 2 & 5 & 5 \\ 1 & -2 & 4 & 1 \\ -1 & 2 & -2 & -2 \\ -2 & 7 & -3 & 2 \end{vmatrix} \\
= & \begin{vmatrix} 2 & 6 & -3 & 3 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -2 & 3 & 5 & 4 \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 + 2C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - C_1 \end{pmatrix} \\
= & -\begin{vmatrix} 6 & -3 & 3 \\ 0 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} \\
= & -\begin{vmatrix} 0 & 0 & 3 \\ 2 & 1 & -1 \\ -5 & 9 & 4 \end{vmatrix} \quad \begin{pmatrix} C_1 \to C_1 - 2C_3 \\ C_2 \to C_2 + C_3 \end{pmatrix} \\
= & -3\begin{vmatrix} 2 & 1 \\ -5 & 9 \end{vmatrix} \\
= & -69
\end{aligned}$$

Example. *Find the inverse of*

$$\mathbf{A} = \begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{pmatrix}$$

Solution:

$$\det(\mathbf{A}) = 4\begin{vmatrix} 6 & 3 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 5 & 6 \\ 3 & 5 \end{vmatrix} = 4(-3) - 3(1) + 2(7) = -1,$$

$$\mathrm{adj}\mathbf{A} = \begin{pmatrix} \begin{vmatrix} 6 & 3 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 6 & 3 \end{vmatrix} \\ -\begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} \\ \begin{vmatrix} 5 & 6 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -3 & 4 & -3 \\ -1 & 2 & -2 \\ 7 & -11 & 9 \end{pmatrix}.$$

Therefore

$$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} -3 & 4 & -3 \\ -1 & 2 & -2 \\ 7 & -11 & 9 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{pmatrix}.$$

Example .

if $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}.

Solution. $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$| A |= 3(-3+4)+3(2-0)+4(-2-0) = 3+6-8 = 1.$

The cofactor of elements of various rows of $| A |$ are

$$\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$$

The cofactor of elements of various rows of $\mid A \mid$ are

$$\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$$

Therefore the matrix formed by the co-factor of $\mid A \mid$ is

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{1} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Example. *Let* $\mathrm{w} = (2, -6, 3)^T \in \mathbb{R}^3$, $\mathrm{v}_1 = (1, -2, -1)^T$ *and* $\mathrm{v}_2 = (3, -5, 4)^T$. *Determine whether* $\mathrm{w} \in \mathrm{span}\{\mathrm{v}_1, \mathrm{v}_2\}$.

Solution: Write

$$c_1 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix},$$

that is

$$\begin{pmatrix} 1 & 3 \\ -2 & -5 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}.$$

The augmented matrix

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ -2 & -5 & -6 \\ -1 & 4 & 3 \end{array}\right)$$

can be reduced by elementary row operations to row echelon form

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 19 \end{array}\right).$$

Since the system is inconsistent, we conclude that $\mathbf{w}$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Example . *Let* $\mathbf{w} = (-7,7,11)^T \in \mathbb{R}^3$, $\mathbf{v}_1 = (1,2,1)^T$, $\mathbf{v}_2 = (-4,-1,2)^T$ *and* $\mathbf{v}_3 = (-3,1,3)^T$. *Express* $\mathbf{w}$ *as a linear combination of* $\mathbf{v}_1$, $\mathbf{v}_2$ *and* $\mathbf{v}_3$.

Solution: Write

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ 11 \end{pmatrix},$$

that is

$$\begin{pmatrix} 1 & -4 & -3 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ 11 \end{pmatrix}.$$

The augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{array}\right)$$

has reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system has more than one solution. For example we can write

$$\mathbf{w} = 5\mathbf{v}_1 + 3\mathbf{v}_2,$$

or

$$\mathbf{w} = 3\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3.$$

Theorem. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix. The set of solutions to the system* $\mathbf{A}\mathbf{x} = \mathbf{0}$ *form a vector subspace of* $\mathbb{R}^n$*. The dimension of the solution space equals to the number of free variables.*

Example. *Find a basis for the solution space of the system*

$$\begin{cases} 3x_1 + 6x_2 - x_3 - 5x_4 + 5x_5 = 0 \\ 2x_1 + 4x_2 - x_3 - 3x_4 + 2x_5 = 0 \\ 3x_1 + 6x_2 - 2x_3 - 4x_4 + x_5 = 0. \end{cases}$$

Solution: The coefficient matrix $\mathbf{A}$ reduces to the row echelon form

$$\begin{pmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The leading variables are x_1, x_3. The free variables are x_2, x_4, x_5. The set $\{(-2,1,0,0,0)^T, (2,0,1,1,0)^T, (-3,0,-4,0,1)^T\}$ constitutes a basis for the solution space of the system.

Example. *Let* $\mathbf{v}_1 = (1, 2, 2, 1)^T$, $\mathbf{v}_2 = (2, 3, 4, 1)^T$, $\mathbf{v}_3 = (3, 8, 7, 5)^T$ *be vectors in* $\mathbb{R}^4$. *Write the equation* $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ *as the system*

$$\begin{cases} c_1 &+& 2c_2 &+& 3c_3 &=& 0 \\ 2c_1 &+& 3c_2 &+& 8c_3 &=& 0 \\ 2c_1 &+& 4c_2 &+& 7c_3 &=& 0 \\ c_1 &+& c_2 &+& 5c_3 &=& 0 \end{cases}.$$

The augmented matrix of the system reduces to the row echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus the only solution is $c_1 = c_2 = c_3 = 0$. *Therefore* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *are linearly independent.*

Row and column spaces

Definition. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix.*

1. *The* **null space** $\mathrm{Null}(\mathbf{A})$ *of* $\mathbf{A}$ *is the solution space to* $\mathbf{Ax} = \mathbf{0}$. *In other words,* $\mathrm{Null}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R} : \mathbf{Ax} = \mathbf{0}.\}$.

2. *The* **row space** $\mathrm{Row}(\mathbf{A})$ *of* $\mathbf{A}$ *is the vector subspace of* $\mathbb{R}^n$ *spanned by the* m *row vectors of* $\mathbf{A}$.

3. *The* **column space** $\mathrm{Col}(\mathbf{A})$ *of* $\mathbf{A}$ *is the vector subspace of* $\mathbb{R}^m$ *spanned by the* n *column vectors of* $\mathbf{A}$.

Example. Find a spanning set for the null space of the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution: The general solution of the equation $\mathbf{Ax} = \mathbf{0}$ is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, and x_2, x_4, and x_5 are free variables. We can now decompose any vector in $\mathbb{R}^5$ into a linear combination of vectors where weights are the free variables. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}.$$

Notice that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are linearly independent since the weights are free variables. Thus, $Nul\mathbf{A} = Span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Example . *Find a basis for the null space* Null($\mathbf{A}$), *a basis for the row space* Row($\mathbf{A}$) *and a basis for the column space* Col($\mathbf{A}$) *where*

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 & 2 & 1 \\ 2 & -4 & 8 & 3 & 10 \\ 3 & -6 & 10 & 6 & 5 \\ 2 & -4 & 7 & 4 & 4 \end{pmatrix}.$$

Solution: The reduced row echelon form of $\mathbf{A}$ is

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

1. the set $\{(2,1,0,0,0)^T, (-3,0,-2,4,1)^T\}$ constitutes a basis for Null($\mathbf{A}$).

2. the set $\{(1,-2,0,0,3), (0,0,1,0,2), (0,0,0,1,-4)\}$ constitutes a basis for Row($\mathbf{A}$).

3. the 1st, 3rd and 4th columns contain leading entries. Therefore the set $\{(1,2,3,2)^T, (3,8,10,7)^T, (2,3,6,4)^T\}$ constitutes a basis for Col($\mathbf{A}$).

Theorem. *Let* $\mathbf{E}$ *be a row echelon form. Then*

1. *The set of vectors obtained by setting one free variable equal to 1 and other free variables to be zero constitutes a basis for* Null($\mathbf{E}$).

2. *The set of non-zero rows constitutes a basis for* Row($\mathbf{E}$).

3. *The set of columns associated with lead variables constitutes a basis for* Col($\mathbf{E}$)

Definition. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix. The dimension of*

1. *the solution space of* $\mathbf{Ax} = \mathbf{0}$ *is called the* **nullity** *of* $\mathbf{A}$.
2. *the row space is called the* **row rank** *of* $\mathbf{A}$.
3. *the column space is called the* **column rank** *of* $\mathbf{A}$.

Theorem. *Let* $\mathbf{A}$ *be a matrix.*

1. *The nullity of* $\mathbf{A}$ *is equal to the number of free variables.*
2. *The row rank of* $\mathbf{A}$ *is equal to the number of lead variables.*
3. *The column rank of* $\mathbf{A}$ *is equal to the number of lead variables.*

Now we can state two important theorems for general matrices.

Theorem. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix. Then the row rank of* $\mathbf{A}$ *is equal to the column rank of* $\mathbf{A}$.

The common value of the row and column rank of the matrix $\mathbf{A}$ is called the **rank** of $\mathbf{A}$ and is denoted by rank(A). The nullity of A is denoted by nullity(A). The rank and nullity of a matrix is related in the following way.

Theorem. (Rank-Nullity Theorem). *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix. Then*

$$\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) = n$$

where $\operatorname{rank}(\mathbf{A})$ *and* $\operatorname{nullity}(\mathbf{A})$ *are the rank and nullity of* $\mathbf{A}$ *respectively.*

Example .

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 3 & 2 \\ 4 & 6 & 8 & 8 \end{bmatrix} \xrightarrow[\text{to change rows 2,3,4}]{\text{use row 1}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 3 & 0 & 1 \\ 2 & 3 & 0 & -1 \\ 4 & 6 & 0 & 0 \end{bmatrix} \xrightarrow[\text{row 2 by 0.5}]{\text{multiply}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3/2 & 0 & 1/2 \\ 2 & 3 & 0 & -1 \\ 4 & 6 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{to change rows 3,4}]{\text{use row 2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow[\text{to change 4}]{\text{use row 3}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{rows}]{\text{interchange}} \begin{bmatrix} 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{A}$$

Hence,

$$\operatorname{rank}\{A\} = \operatorname{rank}\{\tilde{A}\} = 3,$$

$$\operatorname{nullity}\{A\} = \operatorname{nullity}\{\tilde{A}\} = 1$$

Defintion : Let A be an $m \times n$ matrix. The order of the largest square submatrix of A whose determinant has a non-zero value is called the **'rank'** of the matrix A. The rank of the zero matrix is defiend to be zero.

It is clear from the definition that the rank of a square matrix is r if and only if A has a square submatrix of order r with nonzero determinant, and all square sub matrices of large size have determinant zero.

Example. Find the rank of the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 3 & 1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution : Since A is a square matrix, A is itself a square submatrix of A.

Also, $|A| = \begin{vmatrix} 0 & 1 & -1 \\ 3 & 1 & 2 \\ 2 & 4 & 6 \end{vmatrix}$

$= -1(18-4) + (-1)(12-2)$

$= -24 \neq 0$

Hence, rank of A is 3.

Example. Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution : Here, $|A| = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix} = 0$

So, rank of A cannot be 3.

Now $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a square submatrix of A such that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

rank of A = 2.

Example. Determine the rank of matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

Solution : We first reduce matrix A to triangular form by elementary row operations.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 5 & 3 & 14 & 4 \end{bmatrix} \qquad \text{(by } R_1 \leftrightarrow R_2\text{)}$$

$$\sim \begin{bmatrix} 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & 4 & 4 \end{bmatrix} \qquad \text{(by } R_3 \to R_3 - 5R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & 4 \end{bmatrix} \qquad \text{(by } R_3 \to R_3 - 8R_2\text{)}$$

We have thus reduced A to triangular form. The reduced matrix has a square

submatrix $\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{bmatrix}$ with non zero

determinant $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{vmatrix} = 1 \times 1 \times (-12) = -12.$

So rank of reduced matrix is 3. Hence rank of A = 3.

Example. Reduce the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 5 \end{bmatrix}$$

to triangular form and hence determine its rank.

Solution : Let us first reduce A to triangular form by using elementary row operations.

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 \\ 4 & 7 & -4 & -3 \\ 6 & 9 & -5 & 2 \\ 0 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & -6 & 4 & 14 \\ 0 & -9 & 6 & 15 \end{bmatrix} \quad (\text{by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1)$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & -4 \\ 0 & -3 & 2 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{by } R_3 \to R_3 - 2R_2,\ R_4 \to R_4 - 3R_2)$$

$$= B$$

Clearly, rank of B cannot B 4; as $|B| = 0$.

Also, $\begin{bmatrix} 2 & 5 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ is a square submatrix

of order 3 of B and $\begin{vmatrix} 2 & 5 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 2 \times (-3) \times 4 = -24 \neq 0$

So, rank of matrix B is 3.
Hence rank of matrix A = 3.

Example

Find the rank of the matrix

$$\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$$

Solution.

Let A= $\begin{pmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{pmatrix}$.

Order of A is 3x3, so that *rank A* ≤ 3.

Consider the third order minör

$$\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 0$$

Since the third order minor vanishes, *rank A* ≠ 3.

Consider a second order minor

$$\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$$

There is a minor of order 2, which is not zero.

Hence, *rank A*=2.

Example

Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$$

Solution.

Let A= $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$.

rank A ≤ 3.

Consider the third order minors

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 0$$

Since the third order minor vanishes, *rank A* $\neq 3$.

Now, let us consider the second order minors.

$$\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$$

There is a minor of order 2 which is not zero.

Hence, *rank A* =2.

Example

Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

Solution.

$$\text{Let } A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}. \qquad \textit{rank } A \le 3.$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \qquad R_3 \to R_3 - R_2$$

The number of non zero rows is 2. So that, rank of *A is* 2.

Example

Find the rank of the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

Solution.

$$\text{Let } A=\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}. \qquad \textit{rank } A \le 3.$$

Let us transform the matrix A to an echelon form

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix} \qquad R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \qquad R_3 \to R_3 + 5R_2$$

The number of non zero rows is 3. So that, rank of A *is* 3.

Definition. The **Vandermonde determinant** is the determinant of the following matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix},$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Equivalently,

$$V = (a_{ij})_{1 \le i,j \le n}, \text{ where } a_{ij} = x_i^{j-1}.$$

Examples.

$$\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1.$$

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & 0 \\ 1 & x_2 & x_2^2 - x_1x_2 \\ 1 & x_3 & x_3^2 - x_1x_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1x_2 \\ 1 & x_3 - x_1 & x_3^2 - x_1x_3 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1x_2 \\ x_3 - x_1 & x_3^2 - x_1x_3 \end{vmatrix}$$

$$= (x_2 - x_1) \begin{vmatrix} 1 & x_2 \\ x_3 - x_1 & x_3^2 - x_1x_3 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

Note that some authors define the transpose of this matrix as the Vandermonde matrix .

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \dots & x_n^{n-1} \end{pmatrix}.$$

Theorem. We have

$$V(x_1, \dots, x_n) =$$
$$(x_2 - x_1)(x_3 - x_2)(x_3 - x_1) \cdots (x_n - x_{n-1}) = \prod_{i<j}(x_j - x_i).$$

Proof. Let us subtract, for each $i = n-1, n-2, \dots, 1$, the row i times x_1 from the row $i+1$. Combining rows does not change the determinant, so we conclude that $V(x_1, \dots, x_n)$ is equal to the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \dots & x_n - x_1 \\ 0 & x_2^2 - x_1x_2 & x_3^2 - x_1x_3 & \dots & x_n^2 - x_1x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-1} - x_1x_2^{n-2} & x_3^{n-1} - x_1x_3^{n-2} & \dots & x_n^{n-1} - x_1x_n^{n-2} \end{pmatrix}.$$

Let us expand the determinant

$$\det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \dots & x_n - x_1 \\ 0 & x_2^2 - x_1x_2 & x_3^2 - x_1x_3 & \dots & x_n^2 - x_1x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-1} - x_1x_2^{n-2} & x_3^{n-1} - x_1x_3^{n-2} & \dots & x_n^{n-1} - x_1x_n^{n-2} \end{pmatrix}$$

along the first column, the result is

$$\det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 & \dots & x_n - x_1 \\ x_2^2 - x_1 x_2 & x_3^2 - x_1 x_3 & \dots & x_n^2 - x_1 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-1} - x_1 x_2^{n-2} & x_3^{n-1} - x_1 x_3^{n-2} & \dots & x_n^{n-1} - x_1 x_n^{n-2} \end{pmatrix}.$$

We note that the k-th column of the determinant

$$\det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 & \dots & x_n - x_1 \\ x_2^2 - x_1 x_2 & x_3^2 - x_1 x_3 & \dots & x_n^2 - x_1 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-1} - x_1 x_2^{n-2} & x_3^{n-1} - x_1 x_3^{n-2} & \dots & x_n^{n-1} - x_1 x_n^{n-2} \end{pmatrix}$$

is divisible by $x_{k+1} - x_1$, so it is equal to

$$(x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_2 & x_3 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} & x_3^{n-2} & \dots & x_n^{n-2} \end{pmatrix},$$

so we encounter a smaller Vandermonde determinant, and can proceed by induction. □