THE AUGMENTED MATRIX FOR A SYSTEM OF LINEAR EQUATIONS

Example

Write the augmented matrix for the system: $\begin{cases} 3x + 2y + z = 0 \\ -2x - z = 3 \end{cases}$

Insert "1"s and "0"s to clarify coefficients.

$$\begin{cases} \ \ 3x + 2y + 1z = 0 \\ -2x + 0y - 1z = 3 \end{cases}$$

Write the augmented matrix:

$$\begin{array}{cccc} & \text{Coefficients of} & & \text{Right} \\ x & y & z & \text{sides} \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ -2 & 0 & -1 & 3 \end{array}\right]$$

$$\underbrace{\underline{\text{Coefficient matrix}} \quad \underline{\text{Right-hand side (RHS)}}}_{\text{Augmented matrix}}$$

ELEMENTARY ROW OPERATIONS

Equivalent system have the same solution set.

1) Row Reordering

Example

Consider the system: $\begin{cases} 3x - y = 1 \\ x + y = 4 \end{cases}$

If we switch (i.e., interchange) the two equations, then the solution set is not disturbed:

$$\begin{cases} x + y = 4 \\ 3x - y = 1 \end{cases}$$

This suggests that, when we solve a system using augmented matrices,

We can switch any two rows.

Before:

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 3 & -1 & 1 \\ 1 & 1 & 4 \end{array}\right]$$

Here, we switch rows R_1 and R_2, which we denote by: $R_1 \leftrightarrow R_2$

After:

$$\begin{array}{c} \text{new } R_1 \\ \text{new } R_2 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & -1 & 1 \end{array}\right]$$

In general, we can reorder the rows of an augmented matrix in any order.

2) Row Rescaling

Example

Consider the system: $$\begin{cases} \frac{1}{2}x + \frac{1}{2}y = 3 \\ \qquad\quad y = 4 \end{cases}$$

If we multiply "through" both sides of the first equation by 2, then we obtain an equivalent equation and, overall, an equivalent system:

$$\begin{cases} x + y = 6 \\ \quad\; y = 4 \end{cases}$$

This suggests that, when we solve a system using augmented matrices,

We can multiply (or divide) "through" a row by any nonzero constant.

Before:
$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1/2 & 1/2 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

Here, we multiply through R_1 by 2, which we denote by: $R_1 \leftarrow 2 \cdot R_1$, or $\left(\text{new } R_1\right) \leftarrow 2 \cdot \left(\text{old } R_1\right)$

After:
$$\begin{array}{c} \text{new } R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 4 \end{array}\right]$$

3) <u>Row Replacement</u>

When we solve a system using augmented matrices, …

We can add a multiple of one row to another row.

<u>Example</u>

Consider the system: $\begin{cases} x + 3y = 3 \\ -2x + 5y = 16 \end{cases}$

Before:
$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & 3 & 3 \\ \mathbf{-2} & 5 & 16 \end{array}\right]$$

<u>Note</u>: We will sometimes boldface items for purposes of clarity.

It turns out that we want to add twice the first row to the second row, because we want to replace the "$\mathbf{-2}$" with a "0."

We denote this by:

$$R_2 \leftarrow R_2 + 2 \cdot R_1, \text{ or } \left(\text{new } R_2\right) \leftarrow \left(\text{old } R_2\right) + 2 \cdot R_1$$

old R_2	-2	5		16
$+2\cdot R_1$	2	6		6
new R_2	**0**	**11**		**22**

After:

$$\begin{array}{r} \text{old } R_1 \\ \text{new } R_2 \end{array}\left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 11 & 22 \end{array}\right]$$

If matrix B is obtained from matrix A after applying one or more Elementary Row Operations, then we call A and B <u>row-equivalent matrices</u>, and we write $A \sim B$.

<u>Example</u>

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 7 & 8 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 7 & 8 & 9 \\ 1 & 2 & 3 \end{array}\right]$$

<u>Example</u>

Solve the system: $\begin{cases} 4x - \ \ y = 13 \\ \ \ x - 2y = \ \ 5 \end{cases}$

<u>Solution</u>

Step 1) Write the augmented matrix.

You may first want to insert "1"s and "0"s where appropriate.

$$\begin{cases} 4x - 1y = 13 \\ 1x - 2y = \ \ 5 \end{cases}$$

$$\begin{array}{r} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} \mathbf{4} & -1 & 13 \\ 1 & -2 & 5 \end{array}\right]$$

Step 2) Use Elementary Row Operations **until we obtain the desired form:**

$$\left[\begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array}\right]$$

We want a "1" to replace the "**4**" in the upper left.
Dividing through R_1 by 4 will do it, but we will then end up with fractions. Sometimes, we can't avoid fractions. Here, we can.
Instead, let's switch the rows.

$$R_1 \leftrightarrow R_2$$

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & -2 & 5 \\ \mathbf{4} & -1 & 13 \end{array}\right]$$

We now want a "0" to replace the "**4**" in the bottom left.
Remember, we generally want to "correct" columns from left to right, so we will attack the position containing the -1 later.
We cannot multiply through a row by 0.
Instead, we will use a row replacement ERO that exploits the "1" in the upper left to "kill off" the "**4**." This really represents the <u>elimination</u> of the x term in what is now the second equation in our system.

$$\left(\text{new } R_2\right) \leftarrow \left(\text{old } R_2\right) + \left(-4\right) \cdot R_1$$

The notation above is really unnecessary if you show the work below:

old R_2	4	-1		13
$+\left(-4\right) \cdot R_1$	-4	8		-20
new R_2	**0**	**7**		$\mathbf{-7}$

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & \mathbf{7} & -7 \end{array}\right]$$

We want a "1" to replace the "**7**."

We will divide through R_2 by 7, or, equivalently, we will multiply through R_2 by $\frac{1}{7}$: $\quad R_2 \leftarrow \frac{1}{7} \cdot R_2$, or

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & \mathbf{7} & -7 \end{array}\right] \begin{array}{c} \\ \leftarrow \div 7 \end{array}$$

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -1 \end{array}\right]$$

We now have our desired form.

Step 3) Write the new system.

You may want to write down the variables on top of their corresponding columns.

$$\begin{array}{ccc} x & y & \\ \multicolumn{3}{c}{\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -1 \end{array}\right]} \end{array}$$

$$\begin{cases} x - 2y = \ \ 5 \\ \qquad\ \ y = -1 \end{cases} \uparrow$$

Step 4) Use back-substitution.

We start at the bottom, where we immediately find that $\boldsymbol{y = -1}$.

We then work our way up the system, plugging in values for unknowns along the way whenever we know them.

$$
\begin{aligned}
x - 2y &= 5 \\
x - 2(-1) &= 5 \\
x + 2 &= 5 \\
\boldsymbol{x} &= \mathbf{3}
\end{aligned}
$$

Step 5) Write the solution.

The solution set is: $\{(3,-1)\}$.

Step 6) Check.

Given system: $\begin{cases} 4x - y = 13 \\ x - 2y = 5 \end{cases}$

$$\begin{cases} 4(3) - (-1) = 13 \\ (3) - 2(-1) = 5 \end{cases}$$

$$\begin{cases} 13 = 13 \\ 5 = 5 \end{cases}$$

Our solution checks out.

<u>Example</u>

Solve the system: $\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$

<u>Solution</u>

Step 1) Write the augmented matrix.

You may first want to insert "1"s and "0"s where appropriate.

$$\begin{cases} 2x + 2y - 1z = 2 \\ 1x - 3y + 1z = -28 \\ -1x + 1y + 0z = 14 \end{cases}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} \mathbf{2} & 2 & -1 & 2 \\ 1 & -3 & 1 & -28 \\ -1 & 1 & 0 & 14 \end{array}\right]$$

Step 2) Use Elementary Row Operations **until we obtain the desired form:**

$$\left[\begin{array}{ccc|c} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{array}\right]$$

We want a "1" to replace the "**2**" in the upper left corner.
Dividing through R_1 by 2 would do it, but we would then end up with a fraction.
Instead, let's switch the first two rows.

$$R_1 \leftrightarrow R_2$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ \mathbf{2} & 2 & -1 & 2 \\ \mathbf{-1} & 1 & 0 & 14 \end{array}\right]$$

We now want to "eliminate down" the first column by using the "1" in the upper left corner to "kill off" the boldfaced entries and turn them into "0"s.

old R_2	2	2	-1		2
$+(-2)\cdot R_1$	-2	6	-2		56
new R_2	**0**	**8**	$\mathbf{-3}$		**58**

old R_3	-1	1	0		14
$+R_1$	1	-3	1		-28
new R_3	**0**	$\mathbf{-2}$	**1**		$\mathbf{-14}$

Now, write down the new matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 8 & -3 & 58 \\ 0 & -2 & 1 & -14 \end{array}\right]$$

The first column has been "corrected."

We will now focus on the second column. We want:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{1} & ? & ? \\ 0 & \mathbf{0} & ? & ? \end{array}\right]$$

Here is our current matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{8} & -3 & 58 \\ 0 & \mathbf{-2} & 1 & -14 \end{array}\right]$$

If we use the "$\mathbf{-2}$" to kill off the "**8**," we can avoid fractions for the time being. Let's first switch R_2 and R_3 so that we don't get confused when we do this. (We're used to eliminating **down** a column.)

$$R_2 \leftrightarrow R_3$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{-2} & 1 & -14 \\ 0 & \mathbf{8} & -3 & 58 \end{array}\right]$$

Step 2) Use Elementary Row Operations **until we obtain the desired form:**

Now, we will use a row replacement ERO to eliminate the "**8**."

old R_3	0	8	-3		58
$+4 \cdot R_2$	0	-8	4		-56
new R_3	**0**	**0**	**1**		**2**

Now, write down the new matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{-2} & 1 & -14 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Once we get a "1" where the "$\mathbf{-2}$" is, we'll have our desired form. We are fortunate that we already have a "1" at the bottom of the third column, so we won't have to "correct" it.

We will divide through R_2 by -2, or, equivalently, we will multiply through R_2 by $-\frac{1}{2}$.

$$R_2 \leftarrow \left(-\frac{1}{2}\right) \cdot R_2, \text{ or}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{-2} & 1 & -14 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} \\ \leftarrow \div(-2) \\ \\ \end{array}$$

We finally obtain a matrix in our desired form:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Step 3) Write the new system.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

$$\begin{cases} x-3y \quad +z=-28 \\ \qquad y-\dfrac{1}{2}z= \quad 7 \quad \uparrow \\ \qquad \qquad z= \quad 2 \quad \uparrow \end{cases}$$

Step 4) Use back-substitution.

We immediately have: $\boldsymbol{z=2}$

Use $z=2$ in the second equation:

$$y-\frac{1}{2}z=7$$

$$y-\frac{1}{2}(2)=7$$

$$y-1=7$$

$$\boldsymbol{y=8}$$

Use $y = 8$ and $z = 2$ in the first equation:

$$\begin{aligned} x - 3y + z &= -28 \\ x - 3(8) + (2) &= -28 \\ x - 24 + 2 &= -28 \\ x - 22 &= -28 \\ \boldsymbol{x} &\boldsymbol{= -6} \end{aligned}$$

Step 5) Write the solution.

The solution set is: $\{(-6, 8, 2)\}$.

Step 6) Check.

Given system: $$\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$$

$$\begin{cases} 2(-6) + 2(8) - (2) = 2 \\ (-6) - 3(8) + (2) = -28 \\ -(-6) + (8) = 14 \end{cases}$$

$$\begin{cases} 2 = 2 \\ -28 = -28 \\ 14 = 14 \end{cases}$$

Our solution checks out.

If we **ever** get a row of the form:

$$0 \quad 0 \quad \cdots \quad 0 \quad \Big| \quad \left(\text{non-0 constant}\right),$$

then STOP! We know at this point that the solution set is $\varnothing$.

Example

Solve the system: $\begin{cases} x + y = 1 \\ x + y = 4 \end{cases}$

Solution

The augmented matrix is:

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 4 \end{array}\right]$$

We can quickly subtract R_1 from R_2. We then obtain:

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

The new R_2 implies that the solution set is $\varnothing$.

Comments: This is because R_2 corresponds to the equation $0 = 3$, which cannot hold true for any pair (x, y).

If we get a row of all "0"s, such as:

$$0 \quad 0 \quad \cdots \quad 0 \quad \Big| \quad 0,$$

then what does that imply?

Example

Solve the system: $\begin{cases} x + y = 4 \\ x + y = 4 \end{cases}$

Solution

The augmented matrix is:

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & 1 & 4 \end{array}\right]$$

We can quickly subtract R_1 from R_2. We then obtain:

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

The corresponding system is then:

$$\begin{cases} x + y = 4 \\ \quad\; 0 = 0 \end{cases}$$

The equation $0 = 0$ is pretty easy to satisfy. All ordered pairs (x, y) satisfy it. In principle, we could delete this equation from the system.

The solution set is:

$$\left\{(x, y) \middle| x + y = 4\right\}$$

The system has infinitely many solutions; they correspond to all of the points on the line $x + y = 4$.

However, a row of all "0"s does **not** automatically imply that the corresponding system has infinitely many solutions.

Example

Consider the augmented matrix:

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Because of R_1, the corresponding system actually has no solution.

ROW-ECHELON FORM FOR A MATRIX

Properties of a Matrix in Row-Echelon Form

1) If there are any "all-0" rows, then they must be at the bottom of the matrix.

Aside from these "all-0" rows,

2) Every row must have a "1" (called a "leading 1") as its leftmost non-0 entry.

3) The "leading 1"s must "flow down and to the right."

More precisely: The "leading 1" of a row must be in a column to the right of the "leading 1"s of all higher rows.

Example

The matrix below is in Row-Echelon Form:

$$\left[\begin{array}{ccccc|c} \mathbf{1} & 3 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 9 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The "leading **1**"s are boldfaced.
The "1" in the upper right corner is **not** a "leading 1."

Example

To solve linear systems, use row operations to make them simpler.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Step 1: $(R3 \to R3 + 4R1)$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Step 2: $(R2 \to \frac{1}{2}R2)$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

Step 3: $(R3 \to R3 + 3R2)$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 4: $(R2 \to R2 + 4R3, \quad R1 \to R1 - R3)$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 5: $(R1 \to R1 + 2R2)$

$$\begin{array}{rrrcr} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the **original** system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcrcrcr} 29 & - & 32 & + & 3 & = & 0 \\ & & 32 & - & 24 & = & 8 \\ -116 & + & 80 & + & 27 & = & -9 \end{array}$$

<u>Example</u>

To solve linear systems, use row operations to make them simpler.

$$\begin{array}{rrrrrcr} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \\ & 2x_2 & -4x_3 & +4x_4 & +2x_5 & = & -6 \end{array}$$

Augmented matrix:

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{array}\right]$$

$$\xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{array}\right]$$

$$\xrightarrow{R2\to R2-R1}\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{array}\right]$$

$$\xrightarrow{R4\to R4-R2}\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R3\to R3-\frac{3}{2}R2}\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R2\to \frac{1}{2}R2]{R1\to \frac{1}{3}R1}\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R2\to R2-R3]{R1\to R1-2R3}\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R1\to R1+3R2}\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{rrrrrrl} x_1 & & -2x_3 & +3x_4 & & = & -24 \\ & x_2 & -2x_3 & +2x_4 & & = & -7 \\ & & & & x_5 & = & 4 \\ & & & & 0 & = & 0 \end{array}$$

$$\begin{cases} x_1 & = 2x_3 - 3x_4 - 24 \\ x_2 & = 2x_3 - 2x_4 - 7 \\ x_3 & \text{free} \\ x_4 & \text{free} \\ x_5 & = 4 \end{cases}$$

The free variables act as parameters.

The above system has **infinitely many solutions.**

Because you can pick any value of x_3 and x_4.

Example. To solve linear systems, use row operations to make them simpler.

$$\begin{array}{rcrcrcr} x & - & 2y & - & z & = & 2 \\ 2x & - & y & + & z & = & 4 \\ -x & + & y & - & 2z & = & -4 \end{array}$$

Solution:

Step 1: Write the system of equations in an augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & -2 & -4 \end{array}\right]$$

Step 2: Get a 1 in the first row of the first column

This is already done so we can skip to the next step.

Step 3: Use row 1 to get 0's in the first column of rows 2 and 3

For the second row we can obtain a zero by multiplying row 1 by -2 and adding it to row 2.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & -2 & -4 \end{array}\right] -2R_1 + R_2$$

$$\begin{array}{r} -2\left[\begin{array}{cccc} 1 & -2 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} -2 & 4 & 2 & -4 \end{array}\right] \\ +\left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \end{array}\right] \\ \hline =\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ -1 & 1 & -2 & -4 \end{array}\right]$$

For the third row we can simply add row 1 to row 3.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ -1 & 1 & -2 & -4 \end{array}\right] R_1 + R_3$$

$$\begin{array}{r} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \end{array}\right] \\ +\left[\begin{array}{ccc|c} -1 & 1 & -2 & -4 \end{array}\right] \\ \hline =\left[\begin{array}{ccc|c} 0 & -1 & -3 & -2 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -3 & -2 \end{array}\right]$$

Step 4: Get a 1 in the second row of the second column

To get the 1, we can multiply row 2 by one-third

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -3 & -2 \end{array}\right] \begin{array}{l} \\ \frac{1}{3}R_2 \\ \\ \end{array} \qquad \frac{1}{3}[0 \;\; 3 \;\; 3 \mid 0] = \left[\frac{1}{3}(0) \;\; \frac{1}{3}(3) \;\; \frac{1}{3}(3) \;\middle|\; \frac{1}{3}(0)\right] = [0 \;\; 1 \;\; 1 \mid 0]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -3 & -2 \end{array}\right]$$

Step 5: Use row 2 to get a 0 in the second column of row 3

To make the second column of row 3 a zero, we can add row 2 to row 3

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -3 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_2 + R_3 \end{array} \qquad \begin{array}{r} [0 \;\; 1 \;\; 1 \mid 0] \\ +[0 \;\; -1 \;\; -3 \mid -2] \\ \hline =[0 \;\; 0 \;\; -2 \mid -2] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

Step 6: Get a 1 in the third row of the third column

To make the -2 a 1, we can multiply row 3 by a negative one-half

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right] -\tfrac{1}{2}R_3$$

$$-\frac{1}{2}\left[0 \quad 0 \quad -2 \mid -2\right] = \left[-\frac{1}{2}(0) \quad -\frac{1}{2}(0) \quad -\frac{1}{2}(-2) \;\middle|\; -\frac{1}{2}(-2)\right]$$

$$= \left[0 \quad 0 \quad 1 \mid 1\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Step 7: Change the augmented matrix back into a system of equations

$$\begin{aligned} x - 2y - z &= 2 \\ y + z &= 0 \\ z &= 1 \end{aligned}$$

Step 8: Use back-substitution to solve for the variables

$z = 1$ so we can substitute it into the second equation to find y

$y + z = 0$

$y + 1 = 0$

$y = -1$