DEFINITION

Let $f_1, f_2, \ldots, f_n$ be n real functions each of which has an $(n-1)$st derivative on a real interval $a \le x \le b$. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

in which primes denote derivatives, is called the Wronskian *of these n functions. We observe that $W(f_1, f_2, \ldots, f_n)$ is itself a real function defined on $a \le x \le b$. Its value at x is denoted by $W(f_1, f_2, \ldots, f_n)(x)$ or by $W[f_1(x), f_2(x), \ldots, f_n(x)]$.*

THEOREM 4.

The n solutions $f_1, f_2, \ldots, f_n$ of the nth-order homogeneous linear differential equation (2) are linearly independent on $a \le x \le b$ if and only if the Wronskian of $f_1, f_2, \ldots, f_n$ is different from zero for some x on the interval $a \le x \le b$.

We have further:

THEOREM 5.

The Wronskian of n solutions $f_1, f_2, \ldots, f_n$ of (2) is either identically zero on $a \le x \le b$ or else is never zero on $a \le x \le b$.

In the case of the general *second*-order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \tag{4}$$

the Wronskian of two solutions f_1 and f_2 is the second-order determinant

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix} = f_1 f'_2 - f'_1 f_2.$$

By Theorem 4, two solutions f_1 and f_2 of (4) are linearly independent on $a \le x \le b$ if and only if their Wronskian is different from zero for some x on $a \le x \le b$; and by Theorem 5, this Wronskian is either always zero or never zero on $a \le x \le b$. Thus if $W[f_1(x), f_2(x)] \neq 0$ on $a \le x \le b$, solutions f_1 and f_2 of (4) are linearly independent on $a \le x \le b$ and the general solution of (4) can be written as the linear combination

$$c_1 f_1(x) + c_2 f_2(x),$$

where c_1 and c_2 are arbitrary constants.

Example.

We apply Theorem 4 to show that the solutions $\sin x$ and $\cos x$ of

$$\frac{d^2y}{dx^2} + y = 0$$

are linearly independent. We find that

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

for all real x. Thus, since $W(\sin x, \cos x) \neq 0$ for all real x, we conclude that $\sin x$ and $\cos x$ are indeed linearly independent solutions of the given differential equation on every real interval.

Example.

The solutions e^x, e^{-x}, and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

are linearly independent on every real interval, for

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0$$

for all real x.

Example.

$y'' - y = 0$

$y_1 = e^x, \quad y_2 = e^{-x}.$

$$W[y_1, y_2] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0.$$

$y = C_1 e^x + C_2 e^{-x}.$

Reduction of Order

THEOREM 6.

Hypothesis. *Let f be a nontrivial solution of the nth-order homogeneous linear differential equation*

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0. \tag{2}$$

Conclusion. *The transformation $y = f(x)v$ reduces Equation (2) to an $(n-1)$st-order homogeneous linear differential equation in the dependent variable $w = dv/dx$.*

we shall now investigate the second-order case in detail. Suppose f is a *known* nontrivial solution of the second-order homogeneous linear equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0.$$

Let us make the transformation

$$y = f(x)v,$$

where f is the *known* solution and v is a function of x that will be determined. Then, differentiating, we obtain

$$\frac{dy}{dx} = f(x)\frac{dv}{dx} + f'(x)v,$$

$$\frac{d^2y}{dx^2} = f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v.$$

$$a_0(x)\left[f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v\right] + a_1(x)\left[f(x)\frac{dv}{dx} + f'(x)v\right] + a_2(x)f(x)v = 0$$

or

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx}$$
$$+ [a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)]v = 0.$$

Since f is a solution, the coefficient of v is zero, and so the last equation reduces to

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx} = 0.$$

Letting $w = dv/dx$, this becomes

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0.$$

This is a *first*-order homogeneous linear differential equation in the dependent variable w. The equation is separable; thus assuming $f(x) \neq 0$ and $a_0(x) \neq 0$, we may write

$$\frac{dw}{w} = -\left[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}\right] dx.$$

Thus integrating, we obtain

$$\ln |w| = -\ln[f(x)]^2 - \int \frac{a_1(x)}{a_0(x)}\, dx + \ln |c|$$

or

$$w = \frac{c \exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]}{[f(x)]^2}.$$

This is the general solution of

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0;$$

choosing the particular solution for which $c = 1$, recalling that $dv/dx = w$, and integrating again, we now obtain

$$v = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]}{[f(x)]^2}\, dx.$$

Finally, from $y = f(x)v$, we obtain

$$y = f(x)\int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]}{[f(x)]^2}\, dx.$$

This function, which we shall henceforth denote by g, is actually a solution of the original second-order equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0.$$

Furthermore, this new solution g and the original known solution f are linearly independent, since

$$W(f, g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} f(x) & f(x)v \\ f'(x) & f(x)v' + f'(x)v \end{vmatrix}$$

$$= [f(x)]^2 v' = \exp\left[-\int \frac{a_1(x)}{a_0(x)}\,dx\right] \neq 0.$$

Thus the linear combination $c_1 f + c_2 g$ is the general solution. We now summarize this discussion in the following theorem.

THEOREM 7.

Hypothesis. *Let f be a nontrivial solution of the second-order homogeneous linear differential equation*

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \tag{1}$$

Conclusion 1. *The transformation $y = f(x)v$ reduces Equation (1) to the first-order homogeneous linear differential equation*

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0 \tag{2}$$

in the dependent variable w, where $w = dv/dx$.

Conclusion 2. *The particular solution*

$$w = \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\,dx\right]}{[f(x)]^2}$$

of Equation (2) *gives rise to the function v, where*

$$v(x) = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\,dx\right]}{[f(x)]^2}\,dx.$$

The function g defined by $g(x) = f(x)v(x)$ is then a solution of the second-order equation (1).

Conclusion 3. *The original known solution f and the "new" solution g are linearly independent solutions of* (1) *, and hence the general solution of* (1) *may be expressed as the linear combination*

$$c_1 f + c_2 g.$$

We now illustrate the method of reduction of order by means of the following example.

Example.

Given that $y = x$ is a solution of

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \tag{1}$$

find a linearly independent solution by reducing the order.

Solution. First observe that $y = x$ *does* satisfy Equation (1). Then let

$$y = xv.$$

Then

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{and} \quad \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}.$$

Substituting the expressions for y, dy/dx, and d^2y/dx^2 into Equation (1), we obtain

$$(x^2 + 1)\left(x \frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(x\frac{dv}{dx} + v\right) + 2xv = 0$$

or

$$x(x^2 + 1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0.$$

Letting $w = dv/dx$ we obtain the *first*-order homogeneous linear equation

$$x(x^2 + 1)\frac{dw}{dx} + 2w = 0.$$

Treating this as a separable equation, we obtain

$$\frac{dw}{w} = -\frac{2\,dx}{x(x^2 + 1)}$$

or

$$\frac{dw}{w} = \left(-\frac{2}{x} + \frac{2x}{x^2 + 1}\right)dx.$$

Integrating, we obtain the general solution

$$w = \frac{c(x^2 + 1)}{x^2}.$$

Choosing $c = 1$, we recall that $dv/dx = w$ and integrate to obtain the function v given by

$$v(x) = x - \frac{1}{x}.$$

Now forming $g = fv$, where $f(x)$ denotes the *known* solution x, we obtain the function g defined by

$$g(x) = x\left(x - \frac{1}{x}\right) = x^2 - 1.$$

By Theorem 7 we know that this is the desired linearly independent solution. The general solution of Equation (1) may thus be expressed as the linear combination $c_1 x + c_2(x^2 - 1)$ of the linearly independent solutions f and g. We thus write the general solution of Equation (1) as

$$y = c_1 x + c_2(x^2 - 1).$$

Liouville-Ostrogradsky formula

$$y^{(n)} + p_1 y^{(n-1)} + \dots + p_n y = 0,$$

$$W[y_1, y_2, \dots, y_n, y] \equiv \begin{vmatrix} y_1 & y_2 & \dots & y_n & y \\ y_1' & y_2' & \dots & y_n' & y' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n)} & y_2^{(n)} & \dots & y_n^{(n)} & y^{(n)} \end{vmatrix} = 0.$$

$$y^{(n)}\begin{vmatrix} y_1 & y_2 & \dots y_n \\ y_1' & y_2' & \dots y_n' \\ \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots y_n^{(n-1)} \end{vmatrix} -$$

$$- y^{(n-1)}\begin{vmatrix} y_1 & y_2 & \dots y_n \\ y_1' & y_2' & \dots y_n' \\ \dots & \dots & \dots \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \dots y_n^{(n)} \end{vmatrix} + \dots$$

$$\dots + (-1)^n y \begin{vmatrix} y_1' & y_2' & \dots y_n' \\ y_1'' & y_2'' & \dots y_n'' \\ \dots & \dots & \dots \\ y_1^{(n)} & y_2^{(n)} & \dots y_n^{(n)} \end{vmatrix} = 0.$$

$$p_1 = -\frac{\begin{vmatrix} y_1 & y_2 & \dots y_n \\ y_1' & y_2' & \dots y_n' \\ \dots & \dots & \dots \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \dots y_n^{(n)} \end{vmatrix}}{W[y_1, y_2, \dots, y_n]}.$$

$$p_1 = -\frac{W'(x)}{W(x)}, \qquad W(x) = Ce^{-\int\limits_{x_0}^{x} p_1\,dx}$$

$$x = x_0, \qquad W(x) = W(x_0)\,e^{-\int\limits_{x_0}^{x} p_1\,dx}.$$

$$y'' + p_1 y' + p_2 y = 0,$$

$$W[y_1, y] = \begin{vmatrix} y_1 & y \\ y_1' & y' \end{vmatrix} = Ce^{-\int p_1\,dx}.$$

$$y_1 y' - y_1' y = Ce^{-\int p_1\,dx};$$

$$\frac{d}{dx}\left(\frac{y}{y_1}\right) = \frac{1}{y_1^2}\,Ce^{-\int p_1\,dx},$$

$$y = y_1\left\{\int \frac{Ce^{-\int p_1\,dx}}{y_1^2}\,dx + C'\right\}.$$

EXAMPLE.

$$(1 - x^2)\,y'' - 2xy' + 2y = 0$$

$$p_1 = \frac{-2x}{1 - x^2}, \qquad y_1 = x.$$

$$y = x\left\{\int \frac{Ce^{\int \frac{2x\,dx}{1-x^2}}}{x^2}\,dx + C'\right\} = x\left\{C\int \frac{dx}{x^2(1 - x^2)} + C'\right\} =$$

$$= x\left\{C\int\left[\frac{dx}{x^2} + \frac{1}{2}\,\frac{dx}{1 - x} + \frac{1}{2}\,\frac{dx}{1 + x}\right] + C'\right\} =$$

$$= x\left\{C\left[-\frac{1}{x} + \frac{1}{2}\ln\frac{1 + x}{1 - x}\right] + C'\right\} = C'x + C\left(\frac{1}{2}\,x\ln\frac{1 + x}{1 - x} - 1\right).$$

EXAMPLE. $x, \quad x^2, \quad x^3.$

$$\begin{vmatrix} x & x^2 & x^3 & y \\ 1 & 2x & 3x^2 & y' \\ 0 & 2 & 6x & y'' \\ 0 & 0 & 6 & y''' \end{vmatrix} = 0.$$

$$2x^3y''' - 6x^2y'' + 12xy' - 12y = 0.$$

$W(x) = 2x^3 \neq 0$ $(-\infty, \ 0)$ $(0, \ +\infty).$

$$y''' - \frac{3}{x}y'' + \frac{6}{x^2}y' - \frac{6}{x^3}y = 0.$$

The Nonhomogeneous Equation

We now return briefly to the nonhomogeneous equation

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x). \tag{1}$$

The basic theorem dealing with this equation is the following.

THEOREM 8.

Hypothesis

(1) Let v be any solution of the given (nonhomogeneous) nth-order linear differential equation (1).

(2) Let u be any solution of the corresponding homogeneous equation

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0. \tag{2}$$

Conclusion. *Then $u + v$ is also a solution of the given (nonhomogeneous) equation (1).*

Example.

Observe that $y = x$ is a solution of the nonhomogeneous equation

$$\frac{d^2y}{dx^2} + y = x.$$

and that $y = \sin x$ is a solution of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Then by Theorem 8 the sum

$$\sin x + x$$

is also a solution of the given nonhomogeneous equation

$$\frac{d^2y}{dx^2} + y = x.$$

Now let us apply Theorem 8 in the case where v is a given solution y_p of the nonhomogeneous equation (1) involving no arbitrary constants, and u is the general solution

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

of the corresponding homogeneous equation (2). Then by this theorem,

$$y_c + y_p$$

is also a solution of the nonhomogeneous equation (1), and it is a solution involving n arbitrary constants $c_1, c_2, \ldots, c_n$. Concerning the significance of such a solution, we now state the following result.

THEOREM 9.
Hypothesis

(1) Let y_p be a given solution of the nth-order nonhomogeneous linear equation (1) involving no arbitrary constants.
(2) Let

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

be the general solution of the corresponding homogeneous equation (2).

Conclusion. *Then every solution ϕ of the nth-order nonhomogeneous equation (1) can be expressed in the form*

$$y_c + y_p,$$

that is,

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$$

for suitable choice of the n arbitrary constants $c_1, c_2, \ldots, c_n$.

This result suggests that we call a solution of Equation (1) of the form $y_c + y_p$, a *general solution* of (1), in accordance with the following definition:

DEFINITION
Consider the nth-order (nonhomogeneous) linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \tag{1}$$

and the corresponding homogeneous equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0. \tag{2}$$

1. *The general solution of (2) is called the* complementary function *of Equation (1). We shall denote this by y_c.*

2. *Any particular solution of (1) involving no arbitrary constants is called a* particular integral *of (1). We shall denote this by y_p.*
3. *The solution $y_c + y_p$ of (1), where y_c is the complementary function and y_p is a particular integral of (1), is called the* general solution *of (1).*

Thus to find the general solution of (1), we need merely find:

1. *The complementary function*, that is, a "general" linear combination of n linearly independent solutions of the corresponding homogeneous equation (2); and
2. *A particular integral*, that is, any particular solution of (1) involving no arbitrary constants.

Example.

Consider the differential equation

$$\frac{d^2y}{dx^2} + y = x.$$

The complementary function is the general solution

$$y_c = c_1 \sin x + c_2 \cos x$$

of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0.$$

A particular integral is given by

$$y_p = x.$$

Thus the general solution of the given equation may be written

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + x.$$

THEOREM 10.

Hypothesis

1. Let f_1 be a particular integral of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_1(x).$$

2. Let f_2 be a particular integral of

$$a_o(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_2(x).$$

Conclusion. *Then $k_1 f_1 + k_2 f_2$ is a particular integral of*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = k_1 F_1(x) + k_2 F_2(x),$$

where k_1 and k_2 are constants.

Example.

Suppose we seek a particular integral of

$$\frac{d^2 y}{dx^2} + y = 3x + 5\tan x.$$

We may then consider the two equations

$$\frac{d^2 y}{dx^2} + y = x \tag{1}$$

and

$$\frac{d^2 y}{dx^2} + y = \tan x. \tag{2}$$

We have already noted that a particular integral of Equation (1) is given by

$$y = x.$$

Further, we can verify (by direct substitution) that a particular integral of Equation (2) is given by

$$y = -(\cos x)\ln|\sec x + \tan x|.$$

Therefore, applying Theorem 10, a particular integral of Equation (2) is

$$y = 3x - 5(\cos x)\ln|\sec x + \tan x|.$$