Ex: Let $L: \mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation and

$$S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_4} \right\}$$ be a basis for $\mathbb{R}^4$.

Suppose that $L(v_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $L(v_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $L(v_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$L(v_4) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Find $L\left(\begin{bmatrix} 3 \\ -5 \\ -5 \\ 0 \end{bmatrix}\right)$.

Solution: Lets call $v := \begin{bmatrix} 3 \\ -5 \\ -5 \\ 0 \end{bmatrix}$.

$v \in \mathbb{R}^4$, v is a linear combination of v_1, v_2, v_3, v_4 since

$S = \{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$.

$\Rightarrow v = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$

$$\Rightarrow v = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ -5 \\ 0 \end{bmatrix}$$

$$\Rightarrow v = \begin{bmatrix} c_1 + c_4 \\ c_2 + 2c_3 \\ c_1 - c_2 + 2c_3 \\ 2c_2 + c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ -5 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} c_1 + c_4 = 3 \\ c_2 + 2c_3 = -5 \\ c_1 - c_2 + 2c_3 = -5 \\ 2c_2 + c_3 + c_4 = 0 \end{cases} \Rightarrow \begin{matrix} c_1 = 2 \\ c_2 = 1 \\ c_3 = -3 \\ c_4 = 1 \end{matrix}$$

$\Rightarrow v = 2v_1 + v_2 + (-3)v_3 + v_4 \Rightarrow L(v) = L(2v_1 + v_2 + (-3)v_3 + v_4)$

$\Rightarrow L(v) = 2L(v_1) + L(v_2) - 3L(v_3) + L(v_4)$

$$\Rightarrow L(v) = L\left(\begin{bmatrix} 3 \\ -5 \\ -5 \\ 0 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 3 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 7 \end{bmatrix} //$$

Ex: Let $A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

a) Find adj(A).

Solution: $\operatorname{adj}(A) = C^T$ where $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$ (cofactor matrix)

$M_{11} = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2 \Rightarrow C_{11} = (-1)^{1+1} \cdot M_{11} = 2 //$

$M_{12} = \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} = -1 \Rightarrow C_{12} = (-1)^{1+1} \cdot M_{12} = 1 //$

$M_{13} = \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} = -4 \Rightarrow C_{13} = (-1)^{1+3} \cdot M_{13} = -4 //$

$M_{21} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = -7 \Rightarrow C_{21} = (-1)^{2+1} \cdot M_{21} = 7 //$

In the same way

$C_{22} = -7 //$, $C_{23} = 7 //$ $C_{31} = -6 //$ $C_{32} = -3 //$ $C_{33} = 5 //$

Hence $\operatorname{adj}(A) = C^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} 2 & 7 & -6 \\ 1 & -7 & -3 \\ -4 & 7 & 5 \end{bmatrix}$ //

b) Compute det(A)

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \det(A) = \cancel{(-1)^{2+1}} \cdot \cancel{(-1)} \cdot \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} + (-1)^{2+2} \cdot 2 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + (-1)^{2+3} \cdot 0 \cdot \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$$

$$(1 - (-6)) + 2 \cdot (2 - 9) + 0$$

$$7 + (-14) = -7 \,//$$

Ex: Given $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$. Find A^{-1}.

Solution: $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$

$$\det(A) = 2 \begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} - 4 \begin{vmatrix} 7 & 5 \\ 1 & 4 \end{vmatrix} - 6 \begin{vmatrix} 7 & 3 \\ 1 & -2 \end{vmatrix}$$

$= 2 \cdot 22 - 4 \cdot 23 - 6 \cdot (-17) = 54 \,// \neq 0$ thus the inverse matrix exists.

We will use $A^{-1} = \dfrac{1}{\det A} \cdot \operatorname{adj}(A)$

$M_{11} = \begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 22$

$M_{12} = \begin{vmatrix} 7 & 5 \\ 1 & 4 \end{vmatrix} = 23$

$M_{13} = \begin{vmatrix} 7 & 3 \\ 1 & -2 \end{vmatrix} = -17$

$M_{21} = \begin{vmatrix} 4 & -6 \\ -2 & 4 \end{vmatrix} = 4$

$M_{22} = \begin{vmatrix} 2 & -6 \\ 1 & 4 \end{vmatrix} = 14$

$M_{23} = \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -8$

$M_{31} = \begin{vmatrix} 4 & -6 \\ 3 & 5 \end{vmatrix} = 38$

$M_{32} = \begin{vmatrix} 2 & -6 \\ 7 & 5 \end{vmatrix} = 52$

$M_{33} = \begin{vmatrix} 2 & 4 \\ 7 & 3 \end{vmatrix} = -22$

$\Rightarrow C_{11} = 22$

$C_{12} = -23$

$C_{13} = -17$

$C_{21} = -4$

$C_{22} = 14$

$C_{23} = 8$

$C_{31} = -38$

$C_{32} = -52$

$C_{33} = -22$

$\Rightarrow C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$\operatorname{adj}(A) = C^T = \begin{bmatrix} 22 & -4 & 38 \\ -23 & 14 & -52 \\ -17 & 8 & -22 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \frac{1}{54} \cdot \begin{bmatrix} 22 & -4 & 38 \\ -23 & 14 & -52 \\ -17 & 8 & -22 \end{bmatrix} \,//$$

Ex: Solve the following linear system:

$$\begin{cases} 3x-2y+z=-6 \\ 4x-3y+3z=7 \\ 2x+y-z=-9 \end{cases}$$

Solution: We can write this linear system as

$$\begin{bmatrix} 3 & -2 & 1 \\ 4 & -3 & 3 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -6 \\ 7 \\ -9 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & -6 \\ 4 & -3 & 3 & 7 \\ 2 & 1 & -1 & -9 \end{array}\right]$$

Hence

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & -6 \\ 4 & -3 & 3 & 7 \\ 2 & 1 & -1 & -9 \end{array}\right] \xrightarrow[2r_3-r_2\to r_3]{r_2-r_1\to r_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 4 & -3 & 3 & 7 \\ 0 & 5 & -5 & -25 \end{array}\right] \xrightarrow[\frac{1}{5}r_3\to r_3]{r_2-4r_1\to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 0 & -1 & 5 & 45 \\ 0 & 1 & -1 & -5 \end{array}\right] \xrightarrow[r_3+r_2\to r_3]{r_1+r_3\to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & -1 & 5 & 45 \\ 0 & 0 & 4 & 40 \end{array}\right] \xrightarrow[\frac{1}{4}r_3\to r_3]{-r_2\to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -5 & -45 \\ 0 & 0 & 1 & 10 \end{array}\right] \xrightarrow[r_2+5r_3\to r_2]{r_1-r_3\to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

Hence

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2 \\ 5 \\ 10 \end{bmatrix} \Rightarrow \begin{array}{l} x=-2 \\ y=5 \\ z=10 \end{array}$$

//

Ex: Find $\begin{vmatrix} 4 & -2 & 3 & -5 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

$$\begin{vmatrix} 4 & -2 & 3 & -5 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 0\cdot\ldots + 0\cdot\ldots + 0\cdot\ldots + 2\cdot\underbrace{(-1)^{4+4}}_{=1}\cdot\begin{vmatrix} 4 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= 2\cdot\left(0\ldots + 0 + \ldots + (-1)\cdot\cancel{1\cdot(-1)^{3+3}}\cdot\begin{vmatrix} 4 & -2 \\ 0 & 1 \end{vmatrix}\right)$$

$$= 2\cdot(-1\cdot(4-0)) = -8 \,//$$

Ex: Find $\begin{vmatrix} t-1 & -1 & 2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{vmatrix}$

$$\begin{vmatrix} t-1 & -1 & 2 \\ 0 & t-2 & 2 \\ 0 & 0 & t-3 \end{vmatrix} = (t-3)\cdot(-1)^{3+3}\cdot\begin{vmatrix} t-1 & -1 \\ 0 & t-2 \end{vmatrix}$$

$$= (t-3)\cdot(t-1)\cdot(t-2) \,//$$

Ex: Find the rank of $\begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix}$.

Solution: Lets call $A := \begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix}$

A is 2x2 matrix so $rank(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = 9 - 15 = -6 \neq 0$$

There is a minor of order 2 which is not zero : $rank(A) = 2$ //

Ex: Find the rank of $\begin{bmatrix} -5 & -7 \\ 5 & 7 \end{bmatrix}$

Solution: Lets call $A := \begin{bmatrix} -5 & -7 \\ 5 & 7 \end{bmatrix}$

A is 2x2 matrix so $rank(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} -5 & -7 \\ 5 & 7 \end{vmatrix} = -35 - (-35) = 0$$

Since the second order minor vanishes, $rank(A) \neq 2$

Consider a first order minor $|-5| = -5 \neq 0$

There is a minor of order 1, which is not zero so $rank(A) = 1$ //

Ex: Find the rank of the matrix $\begin{bmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{bmatrix}$

Solution: Lets call $A := \begin{bmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{bmatrix}$

A is 3×3 matrix so $\text{rank}(A) \leq 3$

Consider the third order minor

$$\begin{vmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{vmatrix} = 6 \neq 0$$

There is a minor of order 3, which is not zero so $\text{rank}(A) = 3$ //

Ex: Find the rank of the matrix $\begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$

Solution: Lets call $A := \begin{bmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{bmatrix}$

A is 3×3 matrix so $\text{rank}(A) \leq 3$

Consider the third order minor

$$\begin{vmatrix} 5 & 3 & 0 \\ 1 & 2 & -4 \\ -2 & -4 & 8 \end{vmatrix} = 0$$

Since the third order minor vanishes, $\text{rank}(A) \neq 3$

Consider a second order minor $\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7 \neq 0$

There is a minor of order 2, which is not zero.

So $\text{rank}(A) = 2$ //

Ex: Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

Solution: Lets call $A := \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$

A is 3×4 matrix so $\text{rank}(A) \leq 3$

Consider the third order minors

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 6 & 3 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -2 \\ 3 & 6 & -7 \end{vmatrix} = 0 \qquad \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 6 & 3 & -7 \end{vmatrix} = 0$$

Since all third order minors vanishes, $\text{rank}(A) \neq 3$

Now, let us consider the second order minors:

Consider one of the second order minors $\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$

There is a minor of order 2 which is not zero

So $\text{rank}(A) = 2$ //

Ex: Find the rank of the matrix $A=\begin{bmatrix}1&2&3\\2&3&4\\3&5&7\end{bmatrix}$

Solution: A is 3x3 matrix so $\mathrm{rank}(A)\leq 3$

Let us transform the matrix A to an echelon form by using elementary operations.

$$\begin{bmatrix}1&2&3\\2&3&4\\3&5&7\end{bmatrix}\xrightarrow[r_3-3r_1\to r_3]{r_2-2r_1\to r_2}\begin{bmatrix}1&2&3\\0&-1&-2\\0&-1&-2\end{bmatrix}\xrightarrow{r_3-r_2\to r_3}\begin{bmatrix}1&2&3\\0&-1&-2\\0&0&0\end{bmatrix}$$

The above matrix is in echelon form.

The number of non zero rows is 2.

Hence $\mathrm{rank}(A)=2$ //

Note: A row having at least one non-zero element is called as non-zero row.

Ex: Find the rank of the matrix $A=\begin{bmatrix}0&1&2&1\\1&2&3&2\\3&1&1&3\end{bmatrix}$

Solution: A is 3x4 matrix so $\mathrm{rank}(A)\leq 3$

Let us transform the matrix A to an echelon form by using elementary operations.

$$\begin{bmatrix}0&1&2&1\\1&2&3&2\\3&1&1&3\end{bmatrix}\xrightarrow{r_1\leftrightarrow r_2}\begin{bmatrix}1&2&3&2\\0&1&2&1\\3&1&1&3\end{bmatrix}\xrightarrow{r_3-3r_1\to r_3}\begin{bmatrix}1&2&3&2\\0&1&2&1\\0&-5&-8&-3\end{bmatrix}$$

$$\xrightarrow{r_3+5r_2\to r_3}\begin{bmatrix}1&2&3&2\\0&1&2&1\\0&0&2&2\end{bmatrix}\xrightarrow{\frac{1}{2}r_3\to r_3}\begin{bmatrix}1&2&3&2\\0&1&2&1\\0&0&1&1\end{bmatrix}$$ This is echelon form

The number of non-zero rows is 3

Hence $\mathrm{rank}(A)=3$ //

Ex: Construct a 2x2 matrix, $A = [a_{ij}]$, whose elements are given by

$$a_{ij} = \frac{(i+j)^2}{2}$$

Solution: Since A is 2x2 matrix, it has 2 rows and 2 columns

So let A be

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Now it is given that $a_{ij} = \frac{(i+j)^2}{2}$

a_{11} : $i=1, j=1 \Rightarrow a_{11} = \frac{(1+1)^2}{2} = 2$

a_{12} : $i=1, j=2 \Rightarrow a_{12} = \frac{(1+2)^2}{2} = \frac{9}{2}$

a_{21} : $i=2 \ j=1 \Rightarrow a_{21} = \frac{(2+1)^2}{2} = \frac{9}{2}$

a_{22} : $i=2, j=2 \Rightarrow a_{22} = \frac{(2+2)^2}{2} = 8$

Hence the required matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 9/2 \\ 9/2 & 8 \end{bmatrix}$$ //

Ex: Let T be a transformation defined by $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-z \end{bmatrix} \quad \text{for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

Show that T is a linear transformation

Solution: Take $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$.

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2-z_1-z_2 \end{bmatrix}$$

$$=$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1 \\ x_1-z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ x_2-z_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2-z_1-z_2 \end{bmatrix}$$

Take $k \in \mathbb{R}$

$$T\left(k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix}\right) = \begin{bmatrix} kx_1+ky_1 \\ kx_1-kz_1 \end{bmatrix} = k\begin{bmatrix} x_1+y_1 \\ x_1-z_1 \end{bmatrix}$$

$$=$$

$$kT\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = k\begin{bmatrix} x_1+y_1 \\ x_1-z_1 \end{bmatrix}$$

Hence T is a linear transformation

Ex: Find the inverse matrix of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{bmatrix} := A$

Solution: We will use the formula $A^{-1} = \frac{1}{|A|} \cdot adj(A)$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{vmatrix} = 1.(-1)^{1+1}. \begin{vmatrix} 5 & 6 \\ 2 & 9 \end{vmatrix} + 2.(-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3.(-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 2 \end{vmatrix}$$

$$= (45-12) + 2.(-1).(36-42) + 3.(14-35)$$

$$= -36$$

$\Rightarrow |A| = -36$

To find the adj(A):

$M_{11} = \begin{vmatrix} 5 & 6 \\ 2 & 9 \end{vmatrix} = 33 \Rightarrow C_{11} = 33$

$M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6 \Rightarrow C_{12} = +6$

In the same way: $C_{13} = -27$, $C_{21} = -12$, $C_{22} = -12$, $C_{23} = 12$

$C_{31} = -3$, $C_{32} = 6$, $C_{33} = -3$

$\Rightarrow$ the cofactor matrix is $\begin{bmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{bmatrix}$

the transpose of this matrix is adj(A)

$\Rightarrow adj(A) = \begin{bmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{bmatrix}$

Hence $A^{-1} = \frac{1}{-36} \begin{bmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{bmatrix}$

Ex: Let $T(v_1, v_2, v_3) = (2v_1 + v_2,\ 2v_2 - 3v_1,\ v_1 - v_3)$

i) Compute $T(-4, 5, 1)$.

Solution: $T(-4,5,1) = (2.(-4)+5\ ,\ 2.5 - 3.(-4)\ ,\ -4-1)$

$$= (-8+5\ ,\ 10+12\ ,\ -5)$$

$$= (-3, 22, -5)$$ //

ii) Compute the preimage of $w = (4, 1, -1)$

Solution: Suppose (v_1, v_2, v_3) is in the preimage of $(4,1,-1)$. Then

$$(2v_1 + v_2,\ 2v_2 - 3v_1,\ v_1 - v_3) = (4, 1, -1)$$

So

$$2v_1 + v_2 = 4$$

$$2v_2 - 3v_1 = 1$$

$$v_1 - v_3 = -1$$

The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 0 & 2 & -3 & 1 \\ 1 & 0 & -1 & -1 \end{array}\right].$$ The row echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5714 \\ 0 & 1 & 0 & 285714 \\ 0 & 0 & 1 & 15714 \end{array}\right].$$ So $v_1 = 5714$

$v_2 = 285714$

$v_3 = 15714$

Hence Preimage$(4, 1, -1) = (5714, 285714, 15714)$ //

Ex: Determine whether the function

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(x,y) = (x^2, y)$ is Linear ?

Solution: take $(x,y), (z,w) \in \mathbb{R}^2$

$$T((x,y)+(z,w)) = T(x+z, y+w) = ((x+z)^2, y+w)$$
$$= (x^2+z^2+2xz, y+w)$$

$$T((x,y)) + T((z,w)) = (x^2, y) + (z^2, w) = (x^2+z^2, y+w)$$

$\neq$

Hence T is not linear transformation