Ex : Solve the IVP : $2x^2y'' + 3xy' - 15y = 0$, $y(1) = 0$, $y'(1) = 1$

Solution: $2x^2y'' + 3xy' - 15y = 0$: second order, cauchy - Euler, homoge.

$ax^2y'' + bxy' + cy = \cdots$

$y_g = y_h$

y_h: $\boxed{y = x^r}$

$y = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1)x^{r-2} = (r^2 - r)x^{r-2}$

$2x^2y'' + 3xy' - 15y = 0$: $2x^2(r^2 - r)x^{r-2} + 3x r x^{r-1} - 15x^r = 0$

$\Rightarrow 2x^r(r^2 - r) + 3x^r . r - 15x^r = 0$

$x^r(2r^2 - 2r + 3r - 15) = 0$

$x^r\underbrace{(2r^2 + r - 15)}_{\text{characteristic polyn.}} = 0$

$2r^2 + r - 15 = (2r - 5)(r + 3) = 0 \quad \Rightarrow r_1 = \frac{5}{2}, \; r_2 = -3$

Hence $y_1 = x^{5/2}$, $y_2 = x^{-3}$

$\Rightarrow y_g = c_1 y_1 + c_2 y_2 = c_1 x^{5/2} + c_2 x^{-3}$ // $\quad (y' = \frac{5}{2} c_1 x^{3/2} - 3c_2 x^{-4})$

IVP: $y(1) = 0 \Rightarrow$ $\begin{cases} 0 = c_1 . 1^{5/2} + c_2 . 1^{-3} \\ 1 = \frac{5}{2} . c_1 . 1^{3/2} - 3 . c_2 . 1^{-4} \end{cases}$

$y'(1) = 1 \Rightarrow$

$\Rightarrow c_1 = \frac{2}{11}, \; c_2 = -\frac{2}{11} \quad \Rightarrow y_g = \frac{2}{11} x^{5/2} - \frac{2}{11} x^{-3}$ //

Ex: Find the general solution of the equation $x^2y'' - 7xy' + 16y = 0$

Solution: $x^2y'' - 7xy' + 16y = 0$ is second order, homogenous, Cauchy - Euler

$y_g = y_h$

y_h: $y = x^r$

$y = x^r$

$y' = r x^{r-1}$

$y' = r(r-1)x^{r-2} = (r^2 - r)x^{r-2}$

$x^2y'' - 7xy' + 16y = 0$: $x^2 \cdot (r^2 - r)x^{r-2} - 7x\, r x^{r-1} + 16x^r = 0$

$x^r(r^2 - r) - x^r \cdot 7r + 16x^r = 0$

$x^r(r^2 - r - 7r + 16) = 0$

$x^r(\underbrace{r^2 - 8r + 16}_{\text{characteristic equation}}) = 0$

$r^2 - 8r + 16 = 0 \Rightarrow (r-4)^2 = 0 \quad r_{1,2} = 4$

Hence $y_1 = x^4$ // $y_2 = x^4 \cdot \ln x$ //

Hence $y_g = c_1y_1 + c_2y_2$

$y_g = c_1x^4 + c_2x^4\ln x$ //

Ex : Find the solution of the following diff. equ.

$x^2y'' + 3xy' + 4y = 0$

Solution: $x^2y'' + 3xy' + 4y = 0$: second order, homogenous, Cauchy - Euler

$y_g = y_h$

$y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2} = (r^2 - r)x^{r-2}$

$x^2y'' + 3xy' + 4y = 0$: $x^2(r^2-r)x^{r-2} + 3xrx^{r-1} + 4x^r = 0$

$x^r(r^2-r) + x^r 3r + x^r . 4 = 0$

$x^r(r^2 - r + 3r + 4) = 0$

$x^r \underbrace{(r^2 + 2r + 4)}_{\text{characteristic equ.}} = 0$

$r^2 + 2r + 4 = 0 \Rightarrow \Delta = 4 - 4.1.4 = -12$

$r_1 = -1 + \sqrt{3}i \qquad r_2 = -1 - \sqrt{3}i \qquad \lambda = -1, \ \mu = \sqrt{3}$

$y_1 = x^{-1}\cos(\sqrt{3}\ln x) \qquad (x^\lambda \cos(\mu \ln x))$

$y_2 = x^{-1}\sin(\sqrt{3}\ln x) \qquad (x^\lambda \sin(\mu \ln x))$

$y_g = y_h = c_1y_1 + c_2y_2 = c_1x^{-1}\cos(\sqrt{3}\ln x) + c_2x^{-1}\sin(\sqrt{3}\ln x)$ //

Ex: Find y_g for $x^2y'' - 4xy + 4y = x^2$

Solution: $x^2y'' - 4xy + 4y = x^2$: second order, Cauchy - Euler, non homogenous

$y_g = y_h + y_p$

y_h: $\begin{cases} y = x^r \\ y' = rx^{r-1} \\ y'' = r(r-1)x^{r-2} = (r^2-r)x^{r-2} \end{cases}$

$x^2y'' - 4xy + 4y = 0$: $x^2(r^2-r)x^{r-2} - 4xrx^{r-1} + 4x^r = 0$

$x^r(r^2-r) - x^r . 4r + x^r . 4 = 0$

$x^r(r^2 - r - 4r + 4) = 0$

$x^r(\underbrace{r^2 - 5r + 4}_{\text{characteristic eq}}) = 0$

$r^2 - 5r + 4 = (r-4)(r-1) = 0$ $\quad r_1 = 4$

$r_2 = 1$

$\left.\begin{array}{l} y_1 = x^4 \\ y_2 = x \end{array}\right\rangle \Rightarrow y_h = c_1x^4 + c_2x$ //

y_p: we will use the method of variation of parameter

replace the constant c_1 and c_2 in y_h by function

$u_1(x)$ and $u_2(x)$, respectively

$y_p = u_1x^4 + u_2x$ (assumption) then we have

$u_1'y_1 + u_2'y_2 = 0$

$u_1'y_1' + u_2'y_2' = g(x)$ $\quad g(x) = x^2$

$\Rightarrow \quad u_1' x^4 + u_2' x = 0$

$u_1' \cdot 4x^3 + u_2' \cdot 1 = x^2$

Solve this system with cramer's rule

$$\Delta = \begin{vmatrix} x^4 & x \\ 4x^3 & 1 \end{vmatrix} = x^4 - 4x^4 = -3x^4$$

$$\Delta u_1' = \begin{vmatrix} 0 & x \\ x^2 & 1 \end{vmatrix} = -x^3$$

$$\Delta u_2' = \begin{vmatrix} x^4 & 0 \\ 4x^3 & x^2 \end{vmatrix} = x^6$$

$$u_1' = \frac{\Delta u_1'}{\Delta} = \frac{-x^3}{-3x^4} = \frac{1}{3}x^{-1} \Rightarrow u_1 = \int \frac{1}{3}x^{-1}\,dx = \frac{1}{3}\ln x$$

$$u_2 = \frac{\Delta u_2'}{\Delta} = \frac{x^6}{-3x^4} = -\frac{1}{3}x^2 \Rightarrow u_2 = \int -\frac{1}{3}x^2 = -\frac{1}{9}x^3$$

$$y_p = u_1 x^4 + u_2 x = \frac{1}{3}\ln x \cdot x^4 - \frac{1}{3} \cdot x^3 \cdot x$$

$$y_g = y_h + y_p = c_1 x^4 + c_2 x + \frac{x^4}{3}\ln x - \frac{x^4}{3} \quad //$$

Ex: Solve $x^2y'' - xy' - 3y = 2x^2$

Solution: $x^2y'' - xy' - 3y = 2x^2$: Cauchy-Euler, second order, non-homog.

$$y_g = y_h + y_p$$

y_h : $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2} = (r^2 - r)x^{r-2}$

$$x^2y'' - xy' - 3y = 0 : x^2(r^2-r)x^{r-2} - x\,r\,x^{r-1} - 3x^r = 0$$

$$x^r(r^2 - r - r - 3) = 0$$

$$x^r(\underbrace{r^2 - 2r - 3}_{\text{charact. eq.}}) = 0$$

$$(r-3)(r+1) = 0 \qquad r_1 = 3$$
$$r_2 = -1$$

$y_1 = x^3$ and $y_2 = x^{-1}$

$$y_h = c_1x^3 + c_2x^{-1}$$

y_p : undetermined coeff. method

$x^2y'' - xy' - 3y = 2x^2$ same degre $(x^2$ is not in $y_h)$

$$\begin{cases} y_p = Ax^2 \\ y_p' = 2Ax \\ y_p'' = 2A \end{cases} \Rightarrow \begin{aligned} x^2y'' - xy' - 3y &= 2x^2 \\ x^2.2A - x.2Ax - 3.Ax^2 &= 2x^2 \\ A &= -\tfrac{2}{3} \end{aligned}$$

$$\Rightarrow y_p = -\frac{2}{3}x^2$$

$$y_g = y_h + y_p = c_1x^3 + c_2x^{-1} - \frac{2}{3}x^2 \quad //$$

Recall: $L\{f(t)\} = \int_0^{\infty} f(t).e^{-st}\, dt$

Ex: If $f(t)=2$ then $L\{f(t)\}$? (Laplace transformation of f)

Solution: $L\{f(t)\} = \int_0^{\infty} f(t).e^{-st}\, dt$ (Recall $\int e^{ax+b}\, dx = \frac{e^{ax+b}}{a}$

$$L\{2\} = \int_0^{\infty} 2.e^{-st}\, dt = 2\,\frac{e^{-st}}{-s}\Big|_0^{\infty}$$

$$= \frac{2e^{-\infty}}{-s} - \left(\frac{2e^{0}}{-s}\right) = \cancel{\frac{0}{-s}} - \frac{2}{-s} = \frac{2}{s} \quad \Rightarrow L\{2\} = \frac{2}{s}$$

$e^{\infty} = \infty$

$e^{-\infty} = \frac{1}{\infty} = 0$

① $L\{a\} = \frac{a}{s}$, a: constant

② $L\{t\} = \frac{1}{s^2}$ and $L(at) = aL(t) = \frac{a}{s^2}$

$L(f(t) \mp g(t)) = L(f(t)) \mp L(g(t))$

③ $L(t^n) = \frac{n!}{s^{n+1}}$, $n \in \mathbb{Z}^+$

④ $L\{e^{at}\} = \dfrac{1}{s-a}$

⑤ $L\{\sin(at)\} = \dfrac{a}{s^2+a^2}$

⑥ $L\{\cos(at)\} = \dfrac{s}{s^2+a^2}$

⑦ $L\{e^{at}\cdot t\} = \dfrac{1}{(s-a)^2}$

$L\{e^{at}\cdot t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

$L\{e^{at}\cdot \sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}$

$L\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}$

$L\{y\} = Y(s)$

$L\{y'\} = sY(s) - y(0)$

$L\{y''\} = s^2 Y(s) - sy(0) - y'(0)$

Ex: Solve $y' - 2y = e^{2t}$, $y(0) = 1$ with Laplace transformation

Solution: $y' - 2y = e^{2t}$

$$L\{y' - 2y\} = L\{e^{2t}\}$$

$$L\{y'\} - L\{2y\} = L\{e^{2t}\}$$

$$\underbrace{L\{y'\}} - 2\underbrace{L\{y\}} = L\{e^{2t}\} \qquad \left(L\{e^{at}\} = \frac{1}{s-a}\right)$$

$$\overbrace{sY(s) - y(0)} - \overbrace{2Y(s)} = \frac{1}{s-2}$$

$$sY(s) - 1 - 2Y(s) = \frac{1}{s-2}$$

$$sY(s) - 2Y(s) = \frac{1}{s-2} + 1$$

$$Y(s)(s-2) = \frac{s-1}{s-2}$$

$$Y(s) = \frac{s-1}{(s-2)^2}$$

$$\frac{s-1}{(s-2)^2} = \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$\underbrace{L^{-1}\{Y(s)\}}_{=y} = L^{-1}\left\{\frac{s-1}{(s-2)^2}\right\}$$

$$y = L^{-1}\left\{\underbrace{\frac{1}{s-2}} + \underbrace{\frac{1}{(s-2)^2}}\right\} \qquad \left(L\{e^{at}\cdot t\} = \frac{1}{(s-a)^2}\right)$$

$$y = \overbrace{e^{2t}} + \overbrace{e^{2t}\cdot t} \quad //$$

Ex: Solve $y'' - 2y' - 3y = 0$, $y(0) = 0$, $y'(0) = 2$ with Laplace trans.

Solution: $L\{y'' - 2y' - 3y\} = L\{0\}$

$$L\{y''\} - 2L\{y'\} - 3L\{y\} = 0$$

$$\underline{s^2 Y(s) - s y(0) - y'(0)} - \underline{2(sY(s) - y(0))} - 3Y(s) = 0$$

$$s^2 Y(s) - \cancel{s \cdot 0} - 2 - 2(sY(s) - 0) - 3Y(s) = 0$$

$$s^2 Y(s) - 2sY(s) - 3Y(s) = 2$$

$$Y(s) \cdot (s^2 - 2s - 3) = 2$$

$$Y(s) = \frac{2}{s^2 - 2s - 3}$$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2}{s^2 - 2s - 3}\right\} \qquad \frac{2}{(s-3)(s+1)}$$

$$y = L^{-1}\left\{\frac{\frac{1}{2}}{s-3} + \frac{-\frac{1}{2}}{s+1}\right\}$$

$$y = \frac{1}{2} L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$y = \frac{1}{2} e^{3t} - \frac{1}{2} e^{-t} \quad // \qquad \left(L\{e^{at}\} = \frac{1}{s-a} \right)$$

Ex: Solve $y'' - 4y = 2e^{3t}$, $y(0) = 0, y'(0)$ with Laplace transform.

Solution: $y'' - 4y = 2e^{3t}$

$$\mathcal{L}\{y'' - 4y\} = \mathcal{L}\{2e^{3t}\}$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y\} = 2\mathcal{L}\{e^{3t}\}$$

$$s^2 Y(s) - \underbrace{s y(0)}_{0} - \underbrace{y'(0)}_{0} - 4Y(s) = 2 \cdot \frac{1}{s-3}$$

$$s^2 Y(s) - 4Y(s) = \frac{2}{s-3}$$

$$Y(s)(s^2 - 4) = \frac{2}{s-3}$$

$$Y(s) = \frac{2}{(s^2-4)(s-3)} \qquad \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s^2-4)(s-3)}\right\} \qquad = \frac{-5/4}{s-2} + \frac{1/4}{s+2} + \frac{1}{s-3}$$

$$y = \mathcal{L}^{-1}\left\{\frac{-\frac{5}{4}}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$y = -\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$y = -\frac{5}{4}e^{2t} + \frac{1}{4}e^{-2t} + e^{3t} \quad // \qquad \left(\mathcal{L}\{e^{at}\} = \frac{1}{s-a}\right)$$

Example: Solve $2x^2y'' + 3xy' - y = 0$

Solution: $2x^2y'' + 3xy' - y = 0$: second order, homogenous, Cauchy-Euler

$y_g = y_h$

y_h: $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2} = (r^2-r)x^{r-2}$

$2x^2y'' + 3xy' - y = 0$: $2x^2(r^2-r)x^{r-2} + 3x\,r x^{r-1} - x^r = 0$

$x^r . 2(r^2-r) + x^{r-1} . 3xr - x^r = 0$

$x^r(2r^2 - 2r + 3r - 1) = 0$

$x^r\underline{(2r^2 + r - 1)} = 0$

$(2r-1)(r+1) = 0 \quad r_1 = \frac{1}{2} \quad r_2 = -1$

$y_1 = x^{1/2}$ and $y_2 = x^{-1}$

$y_h = y_g = c_1x^{1/2} + c_2x^{-1}$ //

Ex: Solve $x^2y'' + 5xy' + 4y = 0$

Solution: $x^2y'' + 5xy' + 4y = 0$: second order, homogenous, Cauchy-Euler

$y_g = y_h$

y_h: $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2} = (r^2-r)x^{r-2}$

$x^2y'' + 5xy' + 4y = 0 \Rightarrow x^2(r^2-r)x^{r-2} + 5xrx^{r-1} + 4x^r = 0$

$x^r(r^2 - r + 5r + 4) = 0$

$x^r\underline{(r^2 + 4r + 4)} = 0$

$(r+2)^2 \Rightarrow r_{1,2} = -2$

$y_1 = x^{-2}$, $y_2 = x^{-2}.\ln x$

$y_h = y_g = c_1x^{-2} + c_2x^{-2}.\ln x$ //

Ex: Solve $x^2y'' + xy' + y = 0$

Solution: $x^2y'' + xy' + y = 0$ second order, Cauchy-Euler, homogeneous

$y_g = y_h$

y_h: $\begin{cases} y = x^r \\ y' = rx^{r-1} \\ y'' = r(r-1)x^{r-2} = (r^2-r)x^{r-2} \end{cases}$

$x^2y'' + xy' + y = 0$: $x^2 \cdot (r^2-r)x^{r-2} + x r x^{r-1} + x^r = 0$

$x^r(r^2 - r + r + 1) = 0$

$x^r\underbrace{(r^2+1)} = 0$

$r = \mp i \qquad \lambda = 0 \quad \mu = 1$

$y_1 = x^0 \cos(1 . \ln x) = \cos(\ln x)$

$y_2 = x^0 \sin(1 . \ln x) = \sin(\ln x)$

$y_h = y_g = c_1 \cos(\ln x) + c_2 \sin(\ln x)$ //