Ex: (seperable) Solve $y' = \frac{3x^2+4x-4}{2y-4}$, $y(1) = 3$.

Solution: $y' = \frac{3x^2+4x-4}{2y-4}$

$$\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$$

$$(2y-4)\,dy = (3x^2+4x-4)\,dx$$

$$\int (2y-4)\,dy = \int (3x^2+4x-4)\,dx$$

$$\frac{\cancel{2}y^2}{\cancel{2}} - 4y + c_0 = \frac{3x^3}{3} + \frac{4x^2}{2} - 4x + c_1$$

$$y^2 - 4y + \cancel{c_0} = x^3 + 2x^2 - 4 + \cancel{c_1}$$

$$y^2 - 4y = x^3 + 2x^2 - 4 + c_2$$

$y(1) = 3$: $9 - \cancel{4}.\cancel{3} = \cancel{1} + \cancel{2} - 4 + c_2$

$$-3 = -1 + c_2$$

$$c_2 = -2$$

$\Rightarrow$ $y^2 - 4y = x^3 + 2x^2 - 2$ //

Ex (seperable) $y' = \frac{xy^3}{\sqrt{1+x^2}}$

Solution: $y' = \frac{xy^3}{\sqrt{1+x^2}}$

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

$$\frac{dy}{y^3} = \frac{x}{\sqrt{1+x^2}}dx$$

$$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}} dx$$

Let $1+x^2 = u$

$2x\,dx = du$

$x\,dx = \frac{du}{2}$

$$\int y^{-3} dy = \int \frac{du}{2} \cdot \frac{1}{\sqrt{u}}$$

$$\int y^{-3} dy = \frac{1}{2} \int \frac{du}{\sqrt{u}}$$

$$\frac{y^{-2}}{-2} + c_0 = \frac{1}{2} \int u^{-1/2} du$$

$$\frac{y^{-2}}{-2} + c_0 = \cancel{\frac{1}{2}} \cdot \frac{u^{1/2}}{\cancel{1/2}} + c_1$$

$$\frac{1}{-2y^2} = \sqrt{1+x^2} + c_2$$

$$\frac{1}{y^2} = -2\sqrt{1+x^2} - 2c_2 \quad := c_3$$

$$y^2 = \frac{1}{-2\sqrt{1+x^2} + c_3}$$

//

Ex: Solve $y''' = 60x^2 + 4x$

Solution: $y''' = 60x^2 + 4x$

$$\frac{d^3y}{dx^3} = 60x^2 + 4x$$

$$\int d^3y = \int (60x^2 + 4x)\, dx^3$$

$$d^2y = \left(\frac{60x^3}{3} + \frac{4x^2}{2} + c_1\right) dx^2$$

$$\int d^2y = \int (20x^3 + 2x^2 + c_1)\, dx^2$$

$$dy = \left(\frac{20x^4}{4} + \frac{2x^3}{3} + c_1 x + c_2\right) dx$$

$$\int dy = \int \left(5x^4 + \frac{2}{3}x^3 + c_1 x + c_2\right) dx$$

$$y = x^5 + \frac{2}{3.4}x^4 + \overset{c_4}{\frac{c_1}{2}}x^2 + c_2 x + c_3$$

$$y = x^5 + \frac{2}{12}x^4 + c_4 x^2 + c_2 x + c_3$$

//

Ex: Solve $y' + y\cos x = 0$

Solution: $y' + y\cos x = 0 \Rightarrow \frac{dy}{dx} + y\cos x = 0$

$$dx\left(\frac{dy}{dx} + y\cos x\right) = dx\,0$$

$$\frac{dy + y\cos x\,dx}{y} = \frac{0}{y}$$

$$\frac{dy}{y} + \cos x\,dx = 0$$

$$\frac{dy}{y} = -\cos x\,dx$$

$$\int \frac{dy}{y} = \int -\cos x\,dx$$

$$\ln|y| + c_0 = -\sin x + c_1$$

$$\ln|y| = -\sin x + c_2$$

$$y = e^{-\sin x + c_2}$$

$$y = e^{-\sin x} \cdot e^{c_2}\ c_3$$

$$y = c_3 e^{-\sin x} \quad //$$

Ex: (exact) Solve $(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$

Solution: $\underbrace{(2xy - \sec^2 x)}_{M}dx + \underbrace{(x^2 + 2y)}_{N}dy = 0$

if $M_y = N_x$ then it is exact.

$M_y = \frac{\partial M}{\partial y} = 2x$

they are equal so it is exact.

$N_x = \frac{\partial N}{\partial x} = 2x$

the general solution is $\varphi(x, y) = c$.

We know $\varphi_y = N$ and $\varphi_x = M$

If $\varphi_y = N$, $\varphi_y = x^2 + 2y$ then

$\int \frac{\partial \varphi}{\partial y} = \int x^2 + 2y \, dy \Rightarrow \varphi = x^2 y + y^2 + h(x)$

And $\varphi_x = M$. $\frac{\partial \varphi}{\partial x} = 2xy + h'(x) = M = 2xy - \sec^2 x$

$\Rightarrow h'(x) = -\sec^2 x \Rightarrow h(x) = -\tan x + c_0$

Hence $\varphi = x^2 y + y^2 + h(x)$

$= x^2 y + y^2 - \tan x + c_0$

The general solution is $\varphi(x, y) = c \Rightarrow$

$x^2 y + y^2 - \tan x + \cancel{c_0} = \cancel{c}$

$x^2 y + y^2 - \tan x = c_1$ //

Ex (exact) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0.$

Solution: $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$

$$\underbrace{(2xy - 9x^2)}_{M}dx + \underbrace{(2y + x^2 + 1)}_{N}dy = 0$$

if $M_y = N_x$ then it is exact

$M_y = \frac{\partial M}{\partial y} = 2x$

they are equal so it is exact

$N_x = \frac{\partial N}{\partial x} = 2x$

The general solution is $\varphi(x, y) = c.$

We know $\varphi_y = N$ and $\varphi_x = M$

If $\varphi_y = N$, $\varphi_y = 2y + x^2 + 1 = N \Rightarrow$

$\int \frac{\partial \varphi}{\partial y} = \int (2y + x^2 + 1)dy \Rightarrow \varphi = y^2 + x^2 y + y + h(x)$

And $\varphi_x = M$ $\quad \frac{\partial \varphi}{\partial x} = \cancel{2xy} + h'(x) = M = \cancel{2xy} - 9x^2$

$h'(x) = -9x^2 \Rightarrow h(x) = -3x^3 + c_0$

Hence $\varphi = y^2 + x^2 y + y - 3x^3 + c_0$

The general solution is $\varphi(x,y) = c \Rightarrow$

$y^2 + x^2 y + y - 3x^5 + c_0 = c$

$y^2 + x^2 y + y - 3x^3 = c_1$ //

Ex (integration factor, exact) Solve $(3x^2y+2xy+y^3)\,dx+(x^2+y^2)\,dy=0$

Solution: $\underbrace{(3x^2y+2xy+y^3)}_{M}\,dx+\underbrace{(x^2+y^2)}_{N}\,dy=0$

if $M_y=N_x$ then it is exact

$M_y=\dfrac{\partial M}{\partial y}=3x^2+2x+3y^2$

$N_x=\dfrac{\partial N}{\partial x}=2x$

they are NOT equal so it is not exact.

lets try to find integration factor.

$$\frac{M_y-N_x}{N}=\frac{3x^2+\cancel{2x}+3y^2-\cancel{2x}}{x^2+y^2}=\frac{3\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}=3$$ (it is depens only x) ✓

$$\mu=e^{\int\frac{M_y-N_x}{N}dx}=e^{\int 3dx}=e^{3x}$$ //

$$\underbrace{e^{3x}(3x^2y+2xy+y^3)}_{M^*}\,dx+\underbrace{e^{3x}(x^2+y^2)}_{N^*}\,dy=0$$

if $M^*_y=N^*_x$ then it is exact

$M^*_y=\dfrac{\partial M^*}{\partial y}=3x^2e^{3x}+2xe^{3x}+3y^2e^{3x}$

$N^*_x=\dfrac{\partial N^*}{\partial x}=3e^{3x}(x^2+y^2)+e^{3x}.2x$

$=3e^{3x}x^2+3e^{3x}y^2+e^{3x}.2x$

they are equal so it is exact

HOMEWORK !! Solve this exact dif. equ.

Ex (lineer) Solve $\frac{dy}{dx} + \frac{1}{2}y = \frac{1}{2}e^{x/3}$

Solution: $\frac{dy}{dx} + \frac{1}{2}y = \frac{1}{2}e^{x/3}$

$$y' + \frac{1}{2}y = \frac{1}{2}e^{x/3} \qquad (y' + p(x)y = g(x))$$

It is lineer dif. equation

$$\mu = e^{\int p(x)dx} = e^{\int \frac{1}{2}dx} = e^{\frac{x}{2}}.$$

The general solution is

$$y = \frac{1}{\mu}\left(\int \mu\, g(x)\,dx + c\right)$$

$$y = \frac{1}{e^{x/2}}\left(\int e^{x/2} \cdot \frac{1}{2} e^{x/3}\,dx + c\right)$$

$$y = \frac{1}{2e^{x/2}}\left(\int e^{5x/6}\,dx + c\right)$$

$$y = \frac{1}{2e^{x/2}} \cdot \left(\frac{6}{5} \cdot e^{5x/6} + c\right)$$

$$y = \frac{3}{5e^{x/2}} + \frac{\overset{c_1}{c}}{2e^{x/2}}$$

$$y = \frac{3}{5e^{x/2}} + \frac{c_1}{e^{x/2}} \quad //$$

Ex: (lineer) $xy' + 2y = x^2 - x + 1$, $y(1) = \frac{1}{2}$

Solution: $xy' + 2y = x^2 - x + 1 \Rightarrow$

$$y' + \underbrace{\frac{2}{x}}_{p(x)} y = \underbrace{x - 1 + \frac{1}{x}}_{g(x)} \qquad (y' + p(x)y = g(x))$$

$$\mu = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx} = e^{2\ln x} = x^2$$

the general solution is

$$y = \frac{1}{\mu}\left(\int \mu\, g(x)\, dx + c\right)$$

$$y = \frac{1}{x^2}\left(\int x^2 \cdot \left(x - 1 + \frac{1}{x}\right) dx + c\right)$$

$$y = \frac{1}{x^2}\left(\int (x^3 - x^2 + x)\, dx + c\right)$$

$$y = \frac{1}{x^2} \cdot \left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + c\right)$$

$$y = \frac{x^2}{2} - \frac{x}{3} + \frac{1}{2} + \frac{c}{x^2}$$

$$y(1) = \frac{1}{2} : \cancel{\frac{1}{2}} = \cancel{\frac{1}{2}} - \underset{(2)}{\frac{1}{3}} + \underset{(3)}{\frac{1}{2}} + \frac{c}{1}$$

$$c = -\frac{1}{6}$$

$$\Rightarrow y = \frac{x^2}{2} - \frac{x}{3} + \frac{1}{2} - \frac{1}{6x^2}$$

Ex: (Bernoulli) Solve $y' + \frac{4}{x} y = x^3 y^2$

Solution: $y' + \frac{4}{x} y = x^3 y^2$ $(y' + p(x) y = g(x) y^n)$

$$v = y^{1-n} \Rightarrow v = y^{1-2} \Rightarrow v = y^{-1} \Rightarrow v = \frac{1}{y} \Rightarrow y = \frac{1}{v}$$

$$v' = -1 \cdot y^{-2} \cdot y'$$

$$\Rightarrow y' = -v' \cdot y^2$$

$y' + \frac{4}{x} y = x^3 y^2$. (put y')

$$-v' \cancel{y^2} + \frac{4}{x} y = x^3 \cancel{y^2}$$

$-v' y + \frac{4}{x} = x^3 y$ (put $y = \frac{1}{v}$)

$-v' \cdot \frac{1}{v} + \frac{4}{x} = x^3 \cdot \frac{1}{v}$ $(\cdot (-v))$

$v' - \frac{4}{x} v = -x^3$ (Lineer: $v' + p(x) y = g(x)$)

$$\mu = e^{\int p(x) dx} = e^{\int -\frac{4}{x} dx} = e^{-4 \int \frac{1}{x} dx} = e^{-4 \ln x} = x^{-4}$$

$$\Rightarrow v = \frac{1}{\mu} \left(\int \mu g(x) dx + c \right)$$

$$= \frac{1}{x^{-4}} \left(\int x^{-4} \cdot (-x^3) dx + c \right) = \frac{1}{x^{-4}} \left(\int -x^{-1} dx + c \right)$$

$$= x^4 (-\ln x + c) = -x^4 \ln x + c x^4 \Rightarrow v = -x^4 \ln x + c x^4$$

$$y = \frac{1}{v} \Rightarrow y = \frac{1}{-x^4 \ln x + c x^4}$$

Ex (Riccati) Solve $y' + xy - y^2 = 1$, $y_1(x) = x$

Solution: $y' + xy - y^2 = 1$ $\quad (y' + P(x)y + Q(x)y^2 = R(x)$: Riccati$)$

$$\boxed{y = y_1(x) + z}$$

$y = x + z \Rightarrow y' = 1 + z'$

$y' + xy - y^2 = 1$: $\quad 1 + z' + x(x+z) - (x+z)^2 = 1$

$$\cancel{1} + z' + \cancel{x^2} + \cancel{xz} - \cancel{x^2} - z^2 - \cancel{2xz} = \cancel{1}$$

$$z' + (-x)z = z^2 \quad (\text{Bernoulli}: z' + p(x)z = g(x)z^n)$$

$n = 2$

$$z' + (-x)z = z^2$$

$v = z^{1-n} \Rightarrow v = z^{1-2} \Rightarrow v = z^{-1} \Rightarrow v = \frac{1}{z} \Rightarrow z = \frac{1}{v}$

$v' = -1 \cdot z^{-2} \cdot z'$

$\Rightarrow z' = -v' z^2$

$z' + (-x)z = z^2$ $\quad$ (put z')

$-v' z^2 + (-x)z = z^2$

$-v' z - x = z$ $\quad$ (put z)

$-v' \cdot \frac{1}{v} - x = \frac{1}{v}$ $\quad (\cdot (-v))$

$v' + x \cdot v = -1$ $\quad$ (linear: $v' + p(x)v = g(x)$)

$$\mu = e^{\int p(x)dx} = e^{\int x dx} = e^{x^2/2}$$

$$v = \frac{1}{\mu}\left(\int \mu\, g(x)\,dx + c\right)$$

$$= \frac{1}{e^{x^2/2}} \cdot \left(\int e^{x^2/2} \cdot -1 \cdot dx + c\right)$$

$$v = \frac{-1}{e^{x^2/2}} \cdot \left(\int e^{x^2/2}\, dx + c\right)$$

N.B.: $\int e^{x^2/2}\,dx = \underbrace{\sqrt{\frac{\pi}{2}}\, \mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right)}_{\text{lets call } A} + c_0$

$$v = -\frac{1}{e^{x^2/2}} \cdot (A + c)$$

$$v = \frac{1}{z} \Rightarrow z = \frac{1}{v} = \frac{1}{-\frac{1}{e^{x^2/2}} \cdot (A + c)}$$

$$y = x + z = x + \frac{1}{-\frac{1}{e^{x^2/2}} \cdot (A + c)}$$

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