Ex: Let $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$ be a linear system.

Solve this system.

Solution: this linear system can be written in the following way:

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 0 \end{array}\right]$$

Hence

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 0 \end{array}\right] \xrightarrow{r_1 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow{-3r_1 + r_3 \to r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \xrightarrow{-10r_2 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array}\right] \xrightarrow{-\frac{1}{52} r_3 \to r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-r_2 \to r_2]{r_2 + r_1 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[5r_3 + r_2 \to r_2]{-7r_3 + r_1 \to r_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{array}$$

Recall (row-echelon form for a matrix)

1) if there are any "all_0" rows, then they must be at the bottom of the matrix.

2) every row must have a "1" (called a "leading 1") as its lefmost non-0 entry.

3) The leading 1's must "flow down and to the right"

More precisely: the leading 1 of a row must be in a column to the right of the leading 1's of all higher rows.

Ex: Consider the linear system

$$x + y + 2z = -1$$

$$x - 2y + z = -5$$

$$3x + y + z = 3$$

Find all solutions, if any exists,

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right]$$: augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right] \xrightarrow[-r_1+r_2\to r_2]{-3r_1+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & -5 & 6 \end{array}\right] \xrightarrow{-\frac{2}{3}r_2+r_3\to r_3}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -\frac{13}{3} \end{bmatrix} \xrightarrow{-\frac{3}{13} r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{-\frac{1}{3} r_2 \to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{-\frac{1}{3} r_3 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{-r_2 + r_1 \to r_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{-r_3 + r_1 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Hence $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Hence $x = 1$

$y = 2$

$z = -2$ //

<u>Ex</u>: Let $\begin{cases} x_1 + 2x_2 + 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 - x_3 = 3 \end{cases}$ be a linear system.

Solve this system ~~with Gauss method~~

Lets write this system as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & 1 & 3 \end{array}\right] \xrightarrow[-2r_1+r_2\to r_2]{-3r_1+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \xrightarrow[-\frac{1}{5}r_2\to r_2]{}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right] \xrightarrow{6r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \xrightarrow{-\frac{1}{4}r_3\to r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x_3 = 3 //$

$x_2 + x_3 = 2 \Rightarrow x_2 = -1 //$

$x_1 + 2x_2 + x_3 = 9 \Rightarrow x_1 = 2 //$

Ex: Find a 3×1 matrix X with entries not all zero such that $AX = 3X$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

Solution: Let the required 3×1 matrix be $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

We have $AX = 3X$ which implies $(A - 3I_3)X = 0$ since

$AX = 3X \Rightarrow AX - 3X = 0$

$(A - 3I_3)X = 0$

Now $(A - 3I_3) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 4 \\ 4 & -4 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix}$

Hence we need to solve the following linear system with the augmented matrix $(A - 3I_3)$

$$\begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}r_1 \to r_1} \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right] \xrightarrow[-4r_1 + r_3 \to r_3]{r_2 - r_1 \to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 + r_2 \to r_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $\begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} a + \frac{1}{4}c = 0 \\ b - \frac{1}{4}c = 0 \end{cases}$$

$\Rightarrow a = r \quad (r \neq 0)$

$b = -r$

$c = -4r$

Hence $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} r \\ -r \\ -4r \end{bmatrix}$ //

Ex: Find an equation relating a, b and c so that the linear system

$$\begin{cases} x+2y-3z=a \\ 2x+3y+3z=b \\ 5x+9y-6z=c \end{cases}$$

is consistent for any values of a, b and c that satisfy that equation.

The augmented matrix of the linear system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right] \xrightarrow[-5r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right] \xrightarrow{-r_2\to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 1 & -9 & b-2a \\ 0 & -1 & 9 & c-5a \end{array}\right] \xrightarrow{r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 1 & -9 & 2a-b \\ 0 & 0 & 0 & c-b-3a \end{array}\right]$$

$$\Rightarrow \begin{cases} x+2y-3z=a & (1) \\ y-9z=2a-b & (2) \\ 0=c-b-3a & (3) \end{cases}$$

From the equation (3) we conclude that in order to get the solution for x, y, z the value of $(c-b-3a)$ must be equal to 0.

Thus the required relation between a, b and c is

$$c-b-3a=0 \quad //$$

Ex: Let $A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$ Find the following minors and cofactors.

a) M_{13} and C_{13}

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix} \Rightarrow M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix}$$

$$\Rightarrow M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 0 \begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= 0 + 0 + 3(4-4) = 0 //$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = 0 //$$

b) M_{23} and C_{23}

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix} \Rightarrow M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix}$$

$$\Rightarrow M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 4 \cdot \begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6 \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= 4 \cdot (\overset{12}{2} - 14) + (-1)(8 - 56) + 6(4 - 4)$$

$$= -48 - 48 + 0$$

$$= -96 //$$

$$C_{23} = (-1)^{2+3} \cdot M_{23} = (-1) \cdot -96 = 96 //$$

c) M_{22} and C_{22}

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -2 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix} \Rightarrow M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix} = 4.\underbrace{\cancel{(-1)^{2+1}}}_{-1} \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + \cancel{0.(-1)^{2+2}} \begin{vmatrix} 4 & 6 \\ 4 & 2 \end{vmatrix} + 14.\cancel{(-1)^{2+3}} \begin{vmatrix} 4 & 1 \\ 4 & 3 \end{vmatrix}$$

$$= -4(\overbrace{2-18}^{-16}) + 0 - 14(\overbrace{12-4}^{8})$$

$$= 64 - 112 = -48$$

$$C_{22} = (-1)^{2+2}.(-48) = -48$$

<u>Ex</u>: Find the determinant of the following matrices:

a) $A = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} \Rightarrow |A| = 3.4 - (-2.5)$

$= 12 + 10 = 22$

b) $B = \begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix} \Rightarrow |B| = \sqrt{2}.\sqrt{3} - 4\sqrt{6} = \sqrt{6} - 4\sqrt{6} = -3\sqrt{6}$

c) $C = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix} \Rightarrow |C| = -2.(-1)^{1+1}. \begin{vmatrix} 5 & -7 \\ 6 & 2 \end{vmatrix} + 1.(-1)^{1+2} \begin{vmatrix} 3 & -7 \\ 1 & 2 \end{vmatrix}$

$$+ 4.(-1)^{1+3} \begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix}$$

$$= -2.(10-(-42)) - 1(6-(-7)) + 4(\overbrace{18-5}^{13})$$

$$= -104 - 13 + 52 = -65$$

Ex: Is $\det(AB) = \det(BA)$? Justify your answer.

Solution: If A and B are square matrix then we know that

$$\det(AB) = \det(A)\det(B)$$

And the multiplication of real numbers is commutative

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

The above is true only for square matrix. If A and B are not square matrices the this result may not be true.

For example take

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times3} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}_{3\times2}$$

then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

So $\det(AB) = 1 \neq 0 = \det(BA)$

Result: If A and B are not square matrix then this result may not be true.

Ex: Verify that det(AB) = det(A) det(B) for the following

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

• $\det(A) = \cancel{0.(-1)^{3+1} \begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix}} + 1.(-1)^{3+2} \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} + \cancel{0.(-1)^{3+3} \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}}$

$= -1.(\overbrace{1-(-6)}^{7}) = -7$ //

• $\det(B) = 1.(-1)^{1+1} \begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} + \cancel{0.(-1)^{1+2} \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}} + 2.(-1)^{1+3} \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}$

$= 1.(-6-5) + 0 + 2.(3-(-4))$

$= -11 + 0 + 14 = 3$ //

• $A.B = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 1 \\ 9 & -5 & 14 \\ 3 & -2 & 5 \end{bmatrix}$

• $\det(AB) = 1.(-1)^{1+1} \begin{vmatrix} -5 & 14 \\ -2 & 5 \end{vmatrix} + 7.(-1)^{1+2} \begin{vmatrix} 9 & 14 \\ 3 & 5 \end{vmatrix} + 1.(-1)^{1+3} \begin{vmatrix} 9 & -5 \\ 3 & -2 \end{vmatrix}$

$= -21$ //

$\Rightarrow \det(AB) = \det(A)\det(B) = -21$ //

Ex: Find all values of t for which

$$\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t+2 & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0$$

Solution: We have $\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t+2 & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0 \cdot \ldots + 0 \ldots + (t+1).(-1)^{3+3} \begin{vmatrix} t-1 & 0 \\ -2 & t+2 \end{vmatrix}$

$$= (t+1).((t-1)(t+2) - 0)$$

$$= (t+1)(t-1)(t+2) = 0$$

$\Rightarrow t = -1, \quad t = 1, \quad t = -2 \Rightarrow t = 1, -1, -2$ //

Ex: $A = \begin{bmatrix} 3 & 5 & 2 \\ -2 & 3 & -4 \\ -5 & 0 & -5 \end{bmatrix}$. Find A^{-1} with the method of cofactor.

$$A^{-1} = \frac{1}{|A|} \cdot adj(A) \text{ where } adj(A) = C_{IJ}^{T}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \qquad A = \begin{bmatrix} 3 & 5 & 2 \\ -2 & 3 & -4 \\ -5 & 0 & -5 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 3 & -4 \\ 0 & -5 \end{vmatrix} = -15 \Rightarrow C_{11} = -15 //$$

$$M_{12} = \begin{vmatrix} -2 & -4 \\ -5 & -5 \end{vmatrix} = -10 \Rightarrow C_{12} = 10 //$$

$M_{13} = \begin{vmatrix} -2 & 3 \\ 5 & 0 \end{vmatrix} = -15 \Rightarrow C_{13} = 15$ //

$M_{31} = \begin{vmatrix} 5 & 2 \\ 3 & -4 \end{vmatrix} = -26 \Rightarrow C_{31} = -26$ //

$M_{21} = \begin{vmatrix} 3 & 5 \\ -5 & 0 \end{vmatrix} = -25 \Rightarrow C_{21} = 25$ //

$M_{32} = \begin{vmatrix} 3 & 2 \\ -2 & -4 \end{vmatrix} = -8 \Rightarrow C_{32} = 8$ //

$M_{22} = \begin{vmatrix} 3 & 2 \\ -5 & -5 \end{vmatrix} = -5 \Rightarrow C_{22} = -5$ //

$M_{33} = \begin{vmatrix} 3 & 5 \\ -2 & 3 \end{vmatrix} = 19 \Rightarrow C_{33} = 19$ //

$M_{23} = \begin{vmatrix} 3 & 5 \\ -5 & 0 \end{vmatrix} = 25 \Rightarrow C_{23} = -25$ //

$$C = \begin{bmatrix} -15 & 10 & 15 \\ 25 & -5 & -25 \\ -26 & 8 & 19 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 5 & 2 \\ -2 & 3 & -4 \\ -5 & 0 & -5 \end{bmatrix} \Rightarrow |A| = -5 . (-1)^{1+3} \begin{vmatrix} 5 & 2 \\ 3 & -4 \end{vmatrix} + 0 \,.. + (-5)(-1)^{3+3} \begin{vmatrix} 3 & 5 \\ -2 & 3 \end{vmatrix}$$

$$= -5 . (-20 - 6) + 0 + (-5)(9 + 10)$$

$$= -5 . (-26) - 5 . 19 = -5(-26 + 19)$$

$$= -5 . -7 = 35 \,//$$

$$\neq 0$$

$$A^{-1} = \frac{1}{|A|} . C^T$$

$$= \frac{1}{35} . \begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix} = \begin{bmatrix} -3/7 & 5/7 & -26/35 \\ 2/7 & -1/7 & 8/35 \\ 3/7 & -5/7 & 19/35 \end{bmatrix} \,//$$

Ex: Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find A^{-1}.

Solution: We know that $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$ where $\operatorname{adj}(A) = C^T$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -12 \end{bmatrix}$$

$$\Rightarrow C^T = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -12 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow |A| = 4 \cdot (-1)^{1+1} \cdot \begin{vmatrix} -3 & 0 \\ 0 & 2 \end{vmatrix} = 4 \cdot -6 = -24 \,//$$

Hence $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$

$$= \frac{1}{-24} \cdot \begin{bmatrix} -6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} //$$

Ex: Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & -4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}$ Find the determinant of A.

$$|A| = \cancel{(-1)^{4+1} \cdot 0 \cdot \begin{vmatrix} 0 & 3 & 0 \\ 1 & -4 & -1 \\ 2 & 4 & 0 \end{vmatrix}} + (-1)^{4+2} \cdot 3 \begin{vmatrix} 1 & 3 & 0 \\ 2 & -4 & -1 \\ 3 & 4 & 0 \end{vmatrix} + (-1)^{4+3} \cdot (-1) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 0 \end{vmatrix}$$

$$\cancel{+ (-1)^{4+4} \cdot 0 \cdot \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -4 \\ 3 & 2 & 4 \end{vmatrix}}$$

$$3 \cdot \left[3 \cdot \begin{vmatrix} 3 & 0 \\ -4 & -1 \end{vmatrix} - 4 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} \right] = -15 //$$

$$1 \cdot \left[1 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \right] = 2 //$$

$$\Rightarrow |A| = -15 + 2 = -13 //$$

Ex: Consider the linear system

$$\begin{cases} x+y+2z=1 \\ x-2y+z=-5 \\ 3x+y+z=3 \end{cases}$$

Find the solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right]$$: augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 1 & 3 \end{array}\right] \xrightarrow{-3r_1+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & -2 & 1 & -5 \\ 0 & -2 & -5 & 6 \end{array}\right] \xrightarrow{-r_1+r_2\to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & -5 & 6 \end{array}\right] \xrightarrow{-2/3\, r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -13/3 & 26/3 \end{array}\right] \xrightarrow{-13/3\, r_3\to r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & +1 & -2 \end{array}\right] \xrightarrow{-1/3\, r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

row echelon form

$z=-2$

$y+\frac{1}{3}z=\frac{4}{3} \Rightarrow y=2$

$x+y+2z=-1 \Rightarrow x=1$ //