THE METHOD OF UNDETERMINED COEFFICIENTS

We now consider the (nonhomogeneous) differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x),$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants but where the nonhomogeneous term F is (in general) a nonconstant function of x. Recall that the general solution of this equation may be written

$$y = y_c + y_p,$$

where y_c is the *complementary function*, that is, the general solution of the corresponding homogeneous equation (with F replaced by 0), and y_p is a *particular integral*. We consider methods of determining a particular integral.

We consider first the method of *undetermined coefficients*. Mathematically speaking, the class of functions F to which this method applies is actually quite restricted; but this mathematically narrow class includes functions of frequent occurrence and considerable importance in various physical applications.

Example.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x}$$

We proceed to seek a particular solution y_p; but what type of function might be a possible candidate for such a particular solution? This differential equation requires a solution which is such that its second derivative, minus twice its first derivative, minus three times the solution itself, add up to twice the exponential function e^{4x}. Since the derivatives of e^{4x} are constant multiples of e^{4x}, it seems reasonable that the desired particular solution might also be a constant multiple of e^{4x}. Thus we assume a particular solution of the form

$$y_p = Ae^{4x},$$

where A is a constant (undetermined coefficient) to be determined such that y_p *is* a solution. Differentiating y_p, we obtain

$$y_p' = 4Ae^{4x} \quad \text{and} \quad y_p'' = 16Ae^{4x}.$$

Then substituting into the equation, we obtain

$$16Ae^{4x} - 2(4Ae^{4x}) - 3Ae^{4x} = 2e^{4x}$$

or

$$5Ae^{4x} = 2e^{4x}.$$

So, we obtain the equation

$$5A = 2,$$

from which we determine the previously undetermined coefficient

$$A = \tfrac{2}{5}.$$

Thus, we obtain the particular solution

$$y_p = \tfrac{2}{5}e^{4x}.$$

Now consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{3x}.$$

We would now assume a particular solution of the form

$$y_p = Ae^{3x}.$$

Then , we obtain

$$y_p' = 3Ae^{3x} \quad \text{and} \quad y_p'' = 9Ae^{3x}.$$

Then substituting into equation , we obtain

$$9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 2e^{3x}$$

or

$$0 \cdot Ae^{3x} = 2e^{3x}.$$

or simply

$$0 = 2e^{3x},$$

which does not hold for any real x. This impossible situation tells us that there is no particular solution of the assumed form.

What is the difference in these two similar cases?

The answer to this is found by examining the solutions of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

which is the homogeneous equation corresponding to both equations. The auxiliary equation is $m^2 - 2m - 3 = 0$ with roots 3 and -1; and so

$$e^{3x} \quad \text{and} \quad e^{-x}$$

are (linearly independent) solutions of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

For, since Ae^{3x} satisfies the *homogeneous* equation, it reduces

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y$$

to 0, *not* $2e^{3x}$.

Now that we have considered what caused the difficulty in attempting to obtain a particular solution of the form Ae^{3x}, we naturally ask what form of solution should we seek? Recall that in the case of a double root m for an auxiliary equation, a solution linearly independent of the basic solution e^{mx} was xe^{mx}.

While this in itself tells us nothing about the situation at hand, it might suggest that we seek a particular solution of the form

$$y_p = Axe^{3x}.$$

Differentiating, we obtain

$$y_p' = 3Axe^{3x} + Ae^{3x}, \qquad y_p'' = 9Axe^{3x} + 6Ae^{3x}.$$

Then substituting into the equation, we obtain

$$(9Axe^{3x} + 6Ae^{3x}) - 2(3Axe^{3x} + Ae^{3x}) - 3Axe^{3x} = 2e^{3x}$$

or

$$(9A - 6A - 3A)xe^{3x} + 4Ae^{3x} = 2e^{3x}.$$

or simply

$$0xe^{3x} + 4Ae^{3x} = 2e^{3x}.$$

Equating coefficients, we obtain the equation

$$4A = 2,$$

from which we determine the previously undetermined coefficient

$$A = \tfrac{1}{2}.$$

So, we obtain the particular solution

$$y_p = \tfrac{1}{2}xe^{3x}.$$

We summarize the results of this example. The differential equations

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x}$$

and

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{3x}$$

each have the same corresponding homogeneous equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0.$$

This homogeneous equation has linearly independent solutions

$$e^{3x} \quad \text{and} \quad e^{-x},$$

and so the complementary function is

$$y_c = c_1e^{3x} + c_2e^{-x}.$$

The right member $2e^{4x}$ of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x}$$

is *not* a solution of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0,$$

and the attempted particular solution

$$y_p = Ae^{4x}$$

suggested by this right member did indeed lead to a particular solution of this assumed form, namely, $y_p = \frac{2}{5}e^{4x}$. On the other hand, the right member $2e^{3x}$ of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{3x}$$

is a solution of the corresponding homogeneous equation [with $c_1 = 2$ and $c_2 = 0$], and the attempted particular solution

$$y_p = Ae^{3x}$$

suggested by this right member *failed* to lead to a particular solution of this form. However, in this case, the revised attempted particular solution,

$$y_p = Axe^{3x},$$

led to a particular solution of this assumed form, namely, $y_p = \frac{1}{2}xe^{3x}$.

The general solutions are, respectively,

$$y = c_1e^{3x} + c_2e^{-x} + \frac{2}{5}e^{4x}$$

and

$$y = c_1e^{3x} + c_2e^{-x} + \frac{1}{2}xe^{3x}.$$

The preceding example illustrates a particular case of the method of undetermined coefficients. It suggests that in some cases the assumed particular solution y_p corresponding to a nonhomogeneous term in the differential equation is of the same type as that nonhomogeneous term, whereas in other cases the assumed y_p ought to be some sort of modification of that nonhomogeneous term. It turns out that this is essentially the case. We now proceed to present the method systematically.

B. The Method

We begin by introducing certain preliminary definitions.

DEFINITION

We shall call a function a UC function *if it is* either (*1*) *a function defined by one of the following:*

(*i*) x^n, *where n is a positive integer or zero,*
(*ii*) e^{ax}, *where a is a constant* $\neq 0$,
(*iii*) $\sin(bx + c)$, *where b and c are constants,* $b \neq 0$,
(*iv*) $\cos(bx + c)$, *where b and c are constants,* $b \neq 0$,

or (*2*) *a function defined as a finite product of two or more functions of these four types.*

Example.

Examples of UC functions of the four basic types (*i*), (*ii*), (*iii*), (*iv*) of the preceeding definition are those defined respectively by

$$x^3, \qquad e^{-2x}, \qquad \sin(3x/2), \qquad \cos(2x + \pi/4).$$

Examples of UC functions defined as finite products of two or more of these four basic types are those defined respectively by

$$x^2e^{3x}, \qquad x\cos 2x, \qquad e^{5x}\sin 3x,$$

$$\sin 2x \cos 3x, \qquad x^3e^{4x}\sin 5x.$$

The method of undetermined coefficients applies when the nonhomogeneous function F in the differential equation is a finite linear combination of UC functions.

Observe that given a UC function f, each successive derivative of f is either itself a constant multiple of a UC function or else a linear combination of UC functions.

DEFINITION

Consider a UC function f. The set of functions consisting of f itself and all linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations will be called the UC set *of f.*

Example.

The function f defined for all real x by $f(x) = x^3$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 3x^2, \qquad f''(x) = 6x, \qquad f'''(x) = 6 = 6 \cdot 1, \qquad f^{(n)}(x) = 0 \quad \text{for} \quad n > 3.$$

The linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations are those given by

$$x^2, \qquad x, \qquad 1.$$

Thus the *UC set* of x^3 is the set $S = \{x^3, x^2, x, 1\}$.

Example.

The function f defined for all real x by $f(x) = \sin 2x$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 2\cos 2x, \qquad f''(x) = -4\sin 2x, \qquad \ldots.$$

The only linearly independent UC function of which the successive derivatives of f are constant multiples or linear combinations is that given by $\cos 2x$. Thus the *UC set* of $\sin 2x$ is the set $S = \{\sin 2x, \cos 2x\}$.

These and similar examples of the four basic types of UC functions lead to the results listed as numbers 1, 2, and 3 of Table 1.

TABLE 1

	UC function	*UC set*
1	x^n	$\{x^n, x^{n-1}, x^{n-2}, \ldots, x, 1\}$
2	e^{ax}	$\{e^{ax}\}$
3	$\sin(bx + c)$ or $\cos(bx + c)$	$\{\sin(bx + c), \cos(bx + c)\}$
4	$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1}e^{ax}, x^{n-2}e^{ax}, \ldots, xe^{ax}, e^{ax}\}$
5	$x^n \sin(bx + c)$ or $x^n \cos(bx + c)$	$\{x^n \sin(bx + c), x^n \cos(bx + c), x^{n-1} \sin(bx + c), x^{n-1} \cos(bx + c), \ldots, x \sin(bx + c), x \cos(bx + c), \sin(bx + c), \cos(bx + c)\}$
6	$e^{ax} \sin(bx + c)$ or $e^{ax} \cos(bx + c)$	$\{e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$
7	$x^n e^{ax} \sin(bx + c)$ or $x^n e^{ax} \cos(bx + c)$	$\{x^n e^{ax} \sin(bx + c), x^n e^{ax} \cos(bx + c), x^{n-1}e^{ax}\sin(bx + c), x^{n-1}e^{ax} \cos(bx + c), \ldots, xe^{ax} \sin(bx + c), xe^{ax} \cos(bx + c), e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$

Example.

The function f defined for all real x by $f(x) = x^2 \sin x$ is the product of the two UC functions defined by x^2 and $\sin x$. Hence f is itself a UC function. Computing derivatives of f, we find

$$f'(x) = 2x \sin x + x^2 \cos x,$$

$$f''(x) = 2 \sin x + 4x \cos x - x^2 \sin x,$$

$$f'''(x) = 6 \cos x - 6x \sin x - x^2 \cos x, \qquad \cdots.$$

No "new" types of functions will occur from further differentiation. Each derivative of f is a linear combination of certain of the six UC functions given by $x^2 \sin x$, $x^2 \cos x$, $x \sin x$, $x \cos x$, $\sin x$, and $\cos x$. Thus the set

$$S = \{x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, \sin x, \cos x\}$$

is the *UC set* of $x^2 \sin x$. Note carefully that x^2, x, and 1 are *not* members of this UC set.

Observe that the UC set of the product $x^2 \sin x$ is the set of all products obtained by multiplying the various members of the UC set $\{x^2, x, 1\}$ of x^2 by the various members of the UC set $\{\sin x, \cos x\}$ of $\sin x$. This observation illustrates the general situation regarding the UC set of a UC function defined as a finite product of two or more UC functions of the four basic types. In particular, suppose h is a UC function defined as the product fg of two basic UC functions f and g. Then the UC set of the product function h is the set of all the products obtained by multiplying the various members of the UC set of f by the various members of the UC set of g.

Example.

The function defined for all real x by $f(x) = x^3 \cos 2x$ is the product of the two UC functions defined by x^3 and $\cos 2x$. Using the result stated in the preceding paragraph, the UC set of this product $x^3 \cos 2x$ is the set of all products obtained by multiplying the various members of the UC set of x^3 by the various members of the UC set of $\cos 2x$. Using the definition of UC set or the appropriate numbers of Table 1, we find that the UC set of x^3 is

$$\{x^3, x^2, x, 1\}$$

and that of $\cos 2x$ is

$$\{\sin 2x, \cos 2x\}.$$

Thus the UC set of the product $x^3 \cos 2x$ is the set of all products of each of x^3, x^2, x, and 1 by each of $\sin 2x$ and $\cos 2x$, and so it is

$$\{x^3 \sin 2x, x^3 \cos 2x, x^2 \sin 2x, x^2 \cos 2x, x \sin 2x, x \cos 2x, \sin 2x, \cos 2x\}.$$

Observe that this can be found directly from Table , with $n = 3, b = 2,$ and $c = 0$.

We now outline the method of undetermined coefficients for finding a particular integral y_p of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x),$$

where F is a finite linear combination

$$F = A_1 u_1 + A_2 u_2 + \cdots + A_m u_m$$

of UC functions $u_1, u_2, \ldots, u_m$, the A_i being known constants. Assuming the complementary function y_c has already been obtained, we proceed as follows:

1. For *each* of the UC functions

$$u_1, \ldots, u_m$$

of which F is a linear combination, form the corresponding UC set, thus obtaining the respective sets

$$S_1, S_2, \ldots, S_m.$$

2. Suppose that one of the UC sets so formed, say S_j, is identical with or completely included in another, say S_k. In this case, we omit the (identical or smaller) set S_j from further consideration (retaining the set S_k).

3. We now consider in turn each of the UC sets which still remain after Step 2.

Suppose now that one of these UC sets, say S_l, includes one or more members which are solutions of the corresponding homogeneous differential equation. If this is the case, we multiply *each* member of S_l by the lowest positive integral power of x so that the resulting revised set will contain no members that are solutions of the corresponding homogeneous differential equation. We now replace S_l by this revised set, so obtained. Note that here we consider one UC set at a time and perform the indicated multiplication, if needed, only upon the members of the one UC set under consideration at the moment.

4. In general there now remains:
   (i) certain of the original UC sets, which were neither omitted in Step 2 nor needed revision in Step 3, and
   (ii) certain revised sets resulting from the needed revision in Step 3.

Now form a linear combination of *all* of the sets of these two categories, with unknown constant coefficients (*undetermined coefficients*).

5. Determine these unknown coefficients by substituting the linear combination formed in Step 4 into the differential equation and demanding that it identically satisfy the differential equation (that is, that it be a particular solution).

This outline of procedure at once covers all of the various special cases to which the method of undetermined coefficients applies, thereby freeing one from the need of considering separately each of these special cases.

Example .

Consider the two equations

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^x \tag{1}$$

and

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^x \tag{2}$$

The UC set of x^2e^x is

$$S = \{x^2e^x, xe^x, e^x\}.$$

The homogeneous equation corresponding to (1) has linearly independent solutions e^x and e^{2x}, and so the complementary function of (1) is $y_c = c_1e^x + c_2e^{2x}$. Since member e^x of UC set S is a solution of the homogeneous equation corresponding to (1), we multiply each member of UC set S by the lowest positive integral power of x so that the resulting revised set will contain no members that are solutions of the homogeneous equation corresponding to (1). This turns out to be x itself; for the revised set

$$S' = \{x^3e^x, x^2e^x, xe^x\}$$

has no members that satisfy the homogeneous equation corresponding to (1).

The homogeneous equation corresponding to (2) has linearly independent solutions e^x and xe^x, and so the complementary function of (2) is $y_c = c_1 e^x + c_2 xe^x$. Since the two members e^x and xe^x of UC set S are solutions of the homogeneous equation corresponding to (2), we must modify S here also. But now x itself will not do, for we would get S', which still contains xe^x. Thus we must here multiply each member of S by x^2 to obtain the revised set

$$S'' = \{x^4e^x, x^3e^x, x^2e^x\},$$

which has no member that satisfies the homogeneous equation corresponding to (2).

Example.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x.$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

and the complementary function is

$$y_c = c_1e^{3x} + c_2e^{-x}.$$

The nonhomogenous term is the linear combination $2e^x - 10\sin x$ of the two UC functions given by e^x and $\sin x$.

1. Form the UC set for each of these two functions. We find

$$S_1 = \{e^x\},$$

$$S_2 = \{\sin x, \cos x\}.$$

2. Note that neither of these sets is identical with nor included in the other; hence both are retained.

3. Furthermore, by examining the complementary function, we see that none of the functions e^x, $\sin x$, $\cos x$ in either of these sets is a solution of the corresponding homogeneous equation. Hence neither set needs to be revised.

4. Thus the original sets S_1 and S_2 remain intact in this problem, and we form the linear combination

$$Ae^x + B \sin x + C \cos x$$

of the three elements e^x, $\sin x$, $\cos x$ of S_1 and S_2, with the undetermined coefficients A, B, C.

5. We determine these unknown coefficients by substituting the linear combination formed in Step 4 into the differential equation and demanding that it satisfy the differential equation identically. That is, we take

$$y_p = Ae^x + B \sin x + C \cos x$$

as a particular solution. Then

$$y_p' = Ae^x + B \cos x - C \sin x,$$

$$y_p'' = Ae^x - B \sin x - C \cos x.$$

Actually substituting, we find

$$(Ae^x - B \sin x - C \cos x) - 2(Ae^x + B \cos x - C \sin x) - 3(Ae^x + B \sin x + C \cos x) = 2e^x - 10 \sin x$$

or

$$-4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10 \sin x.$$

Since the solution is to satisfy the differential equation identically for *all* x on some real interval, this relation must be an identity for all such x and hence the coefficients of like terms on both sides must be respectively equal. Equating coefficients of these like terms, we obtain the equations

$$-4A = 2, \qquad -4B + 2C = -10, \qquad -4C - 2B = 0.$$

From these equations, we find that

$$A = -\tfrac{1}{2}, \qquad B = 2, \qquad C = -1,$$

and hence we obtain the particular integral

$$y_p = -\tfrac{1}{2}e^x + 2\sin x - \cos x.$$

Thus the general solution of the differential equation under consideration is

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{-x} - \tfrac{1}{2}e^x + 2\sin x - \cos x.$$

Example.

$$\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}.$$

The corresponding homogeneous equation is

$$\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

and the complementary function is

$$y_c = c_1 e^x + c_2 e^{2x}.$$

The nonhomogeneous term is the linear combination

$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

of the four UC functions given by x^2, e^x, xe^x, and e^{3x}.

1. Form the UC set for each of these functions. We have

$$S_1 = \{x^2, x, 1\},$$
$$S_2 = \{e^x\},$$
$$S_3 = \{xe^x, e^x\},$$
$$S_4 = \{e^{3x}\}.$$

2. We note that S_2 is completely included in S_3, so S_2 is omitted from further consideration, leaving the three sets

$$S_1 = \{x^2, x, 1\} \qquad S_3 = \{xe^x, e^x\}, \qquad S_4 = \{e^{3x}\}.$$

3. We now observe that $S_3 = \{xe^x, e^x\}$ includes e^x, which is included in the complementary function and so is a solution of the corresponding homogeneous differential equation. Thus we multiply *each* member of S_3 by x to obtain the revised family

$$S_3' = \{x^2e^x, xe^x\},$$

which contains no members that are solutions of the corresponding homogeneous equation.

4. Thus there remain the original UC sets

$$S_1 = \{x^2, x, 1\}$$

and

$$S_4 = \{e^{3x}\}$$

and the revised set

$$S_3' = \{x^2e^x, xe^x\}.$$

These contain the six elements

$$x^2, \quad x, \quad 1, \quad e^{3x}, \quad x^2e^x, \quad xe^x.$$

We form the linear combination

$$Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x$$

of these six elements.

5. Thus we take as our particular solution,

$$y_p = Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x.$$

From this, we have

$$y_p' = 2Ax + B + 3De^{3x} + Ex^2e^x + 2Exe^x + Fxe^x + Fe^x,$$

$$y_p'' = 2A + 9De^{3x} + Ex^2e^x + 4Exe^x + 2Ee^x + Fxe^x + 2Fe^x.$$

We substitute y_p, y'_p, y''_p into the differential equation for y, dy/dx, d^2y/dx^2, respectively, to obtain:

$$\begin{aligned} 2A + 9De^{3x} + Ex^2e^x + (4E + F)xe^x + (2E + 2F)e^x \\ - 3[2Ax + B + 3De^{3x} + Ex^2e^x + (2E + F)xe^x + Fe^x] \\ + 2(Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x) \\ = 2x^2 + e^x + 2xe^x + 4e^{3x}, \end{aligned}$$

or

$$\begin{aligned} (2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 + 2De^{3x} + (-2E)xe^x + (2E - F)e^x \\ = 2x^2 + e^x + 2xe^x + 4e^{3x}. \end{aligned}$$

Equating coefficients of like terms, we have:

$$\begin{aligned} 2A - 3B + 2C &= 0, \\ 2B - 6A &= 0, \\ 2A &= 2, \\ 2D &= 4, \\ -2E &= 2, \\ 2E - F &= 1. \end{aligned}$$

From this $A = 1, B = 3, C = \frac{7}{2}, D = 2, E = -1, F = -3$, and so the particular integral is

$$y_p = x^2 + 3x + \tfrac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$

The general solution is therefore

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + x^2 + 3x + \tfrac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$