Linear Differential Equations with Constant Coefficients.

We shall be concerned with the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are real constants. We shall show that the general solution of this equation can be found explicitly.

More precisely, we shall seek solutions of the form $y = e^{mx}$, where the constant m will be chosen such that e^{mx} *does* satisfy the equation. Assuming then that $y = e^{mx}$ is a solution for certain m, we have:

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2 y}{dx^2} = m^2 e^{mx}, \quad \frac{d^n y}{dx^n} = m^n e^{mx}.$$

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

$$e^{mx}(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) = 0.$$

Since $e^{mx} \neq 0$, we obtain the polynomial equation in the unknown m:

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0.$$

This equation is called the *auxiliary equation* or the *characteristic equation* of the given differential equation.

$$y_1 = e^{k_1 x}, \quad y_2 = e^{k_2 x}, \quad \ldots, \quad y_n = e^{k_n x}.$$

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} =$$

$$= \begin{vmatrix} e^{k_1x} & e^{k_2x} & \ldots & e^{k_nx} \\ k_1e^{k_1x} & k_2e^{k_2x} & \ldots & k_ne^{k_nx} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k_1^{n-1}e^{k_1x} & k_2^{n-1}e^{k_2x} & \ldots & k_n^{n-1}e^{k_nx} \end{vmatrix} =$$

$$= e^{(k_1+k_2+\ldots+k_n)x} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ k_1 & k_2 & \ldots & k_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k_1^{n-1} & k_2^{n-1} & \ldots & k_n^{n-1} \end{vmatrix}.$$

$$\begin{aligned} &(k_1 - k_2)(k_1 - k_3) \ldots (k_1 - k_n) \times \\ &\quad \times (k_2 - k_3) \ldots (k_2 - k_n) \times \\ &\quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ &\qquad \times (k_{n-1} - k_n) \end{aligned}$$

$$y = C_1e^{k_1x} + C_2e^{k_2x} + \ldots + C_ne^{k_nx}.$$

Case 1. Distinct Real Roots

Suppose the roots of the *characteristic equation* are the n distinct real numbers $m_1, m_2, \ldots, m_n$.

THEOREM 1.

Consider the nth-order homogeneous linear differential equation with constant coefficients. If the characteristic equation has the n distinct real roots $m_1, m_2, \ldots, m_n$, then the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x},$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example. $$y'' - y = 0.$$

$$k^2 - 1 = 0.$$

$$k_1 = 1, \quad k_2 = -1.$$

$$y_1 = e^{x}, \quad y_2 = e^{-x}.$$

$$y = C_1 e^{x} + C_2 e^{-x}.$$

Example.
Consider the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0.$$

Hence

$$(m-1)(m-2) = 0, \qquad m_1 = 1, \qquad m_2 = 2.$$

The roots are real and distinct. Thus e^x and e^{2x} are solutions and the general solution may be written

$$y = c_1e^x + c_2e^{2x}.$$

We verify that e^x and e^{2x} are indeed linearly independent. Their Wronskian is

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0.$$

Thus we are assured of their linear independence.

Example.
Consider the differential equation

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0.$$

The auxiliary equation is

$$m^3 - 4m^2 + m + 6 = 0.$$

We observe that $m = -1$ is a root of this equation. By synthetic division we obtain the factorization

$$(m+1)(m^2-5m+6)=0$$

or

$$(m+1)(m-2)(m-3)=0.$$

Thus the roots are the distinct real numbers

$$m_1=-1, \qquad m_2=2, \qquad m_3=3,$$

and the general solution is

$$y=c_1e^{-x}+c_2e^{2x}+c_3e^{3x}.$$

Example. $y^{(5)}-10y'''+9y'=0.$

$$\lambda^5-10\lambda^3+9\lambda=0.$$

$$\lambda_1=0,\ \lambda_2=-1,\ \lambda_3=1,\ \lambda_4=-3,\ \lambda_5=3.$$

$$y_1=1,\ y_2=e^{-x},\ y_3=e^{x},\ y_4=e^{-3x},\ y_5=e^{3x}$$

$$y=c_1+c_2e^{-x}+c_3e^{x}+c_4e^{-3x}+c_5e^{3x}.$$

Example. $2y''-5y'+2y=0$

$$2\lambda^2-5\lambda+2=0$$

$$D=25-16=9$$

$$\lambda_{1,2}=\frac{5\pm\sqrt{9}}{4}=\frac{5\pm3}{4}=\frac{1}{2};2$$

$$y=C_1e^{\frac{1}{2}x}+C_2e^{2x}$$

Case 2. Repeated Real Roots

We shall begin our study of this case by considering a simple example.

Example.

Consider the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

or

$$(m - 3)^2 = 0.$$

The roots of this equation are

$$m_1 = 3, \qquad m_2 = 3$$

(real but *not* distinct).

Corresponding to the root m_1 we have the solution e^{3x}, and corresponding to m_2 we have the *same* solution e^{3x}. The linear combination $c_1e^{3x} + c_2e^{3x}$ of these "two" solutions is clearly *not* the general solution of the differential equation, for it is *not* a linear combination of *two linearly independent* solutions. Indeed we may write the combination $c_1e^{3x} + c_2e^{3x}$ as simply c_0e^{3x}, where $c_0 = c_1 + c_2$; and clearly $y = c_0e^{3x}$, involving *one* arbitrary constant, is not the general solution of the given *second*-order equation.

We must find a linearly independent solution; but how shall we proceed to do so? Since we already know the one solution e^{3x}, we may reduce the order. We let $y = e^{3x}v$, where v is to be determined. Then we have

$$\frac{dy}{dx} = e^{3x}\frac{dv}{dx} + 3e^{3x}v,$$

$$\frac{d^2y}{dx^2} = e^{3x}\frac{d^2v}{dx^2} + 6e^{3x}\frac{dv}{dx} + 9e^{3x}v.$$

$$\left(e^{3x}\frac{d^2v}{dx^2} + 6e^{3x}\frac{dv}{dx} + 9e^{3x}v\right) - 6\left(e^{3x}\frac{dv}{dx} + 3e^{3x}v\right) + 9e^{3x}v = 0$$

or

$$e^{3x}\frac{d^2v}{dx^2} = 0.$$

Letting $w = dv/dx$, we have the first-order equation

$$e^{3x}\frac{dw}{dx} = 0$$

or simply

$$\frac{dw}{dx} = 0.$$

The solutions of this first-order equation are simply $w = c$, where c is an arbitrary constant. Choosing the particular solution $w = 1$ and recalling that $dv/dx = w$, we find

$$v(x) = x + c_0,$$

and $v(x)e^{3x} = (x + c_0)e^{3x}$ is a solution of the given second-order equation. Choosing $c_0 = 0$ we obtain the solution $y = xe^{3x}$, and thus corresponding to the *double* root 3 we find the linearly independent solutions e^{3x} and xe^{3x}.

Thus the general solution may be written

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

or

$$y = (c_1 + c_2 x)e^{3x}.$$

With this example as a guide, let us return to the general nth-order equation. If the auxiliary equation has the *double* real root m, we would surely expect that e^{mx} and xe^{mx} would be the corresponding linearly independent solutions. This is indeed the case. Specifically, suppose the roots are the double real root m and the $(n-2)$ distinct real roots $m_1, m_2, \ldots, m_{n-2}$.

Then linearly independent solutions of the equation are

$$e^{mx}, xe^{mx}, e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_{n-2} x},$$

and the general solution may be written

$$y = c_1 e^{mx} + c_2 x e^{mx} + c_3 e^{m_1 x} + c_4 e^{m_2 x} + \cdots + c_n e^{m_{n-2} x}$$

or

$$y = (c_1 + c_2 x)e^{mx} + c_3 e^{m_1 x} + c_4 e^{m_2 x} + \cdots + c_n e^{m_{n-2} x}.$$

In like manner, if the auxiliary equation has the triple real root m, corresponding linearly independent solutions are

$$e^{mx}, xe^{mx}, \quad \text{and} \quad x^2 e^{mx}.$$

The corresponding part of the general solution may be written

$$(c_1 + c_2 x + c_3 x^2)e^{mx}.$$

Proceeding further in like manner, we summarize Case 2 in the following theorem:

THEOREM 2.

1. Consider the nth-order homogeneous linear differential equation with constant coefficients. If the characteristic equation has the real root m occurring k times, then the part of the general solution corresponding to this k-fold repeated root is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{mx}.$$

2. If, further, the remaining roots of the characteristic equation are the distinct real numbers $m_{k+1}, \ldots, m_n$, then the general solution of the equation is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{mx} + c_{k+1}e^{m_{k+1}x} + \cdots + c_n e^{m_n x}.$$

3. If, however, any of the remaining roots are also repeated, then the parts of the general solution of the equation corresponding to each of these other repeated roots are expressions similar to that corresponding to m in part 1.

We now consider several examples.

Example.

Find the general solution of

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0.$$

The auxiliary equation

$$m^3 - 4m^2 - 3m + 18 = 0$$

has the roots, 3, 3, -2. The general solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{-2x}$$

or

$$y = (c_1 + c_2 x)e^{3x} + c_3 e^{-2x}.$$

Example. $y^{(5)} - 4y^{(4)} + 4y''' = 0.$

$$\lambda^5 - 4\lambda^4 + 4\lambda^3 = 0.$$

$$\lambda_1 = 0, \quad \lambda_2 = 2$$

$$y_1 = 1, y_2 = x, y_3 = x^2, y_4 = e^{2x},$$

$$y = c_1 + c_2 x + c_3 x^2 + (c_4 + c_5 x)e^{2x}.$$

Example.

Find the general solution of

$$\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0.$$

The auxiliary equation is

$$m^4 - 5m^3 + 6m^2 + 4m - 8 = 0,$$

with roots $2, 2, 2, -1$. The part of the general solution corresponding to the three-fold root 2 is

$$y_1 = (c_1 + c_2 x + c_3 x^2)e^{2x}$$

and that corresponding to the simple root -1 is simply

$$y_2 = c_4 e^{-x}.$$

Thus the general solution is $y = y_1 + y_2$, that is,

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} + c_4 e^{-x}.$$

$$L[y] = \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0,$$

$$f(x) = u(x) + iv(x),$$

$$L[u(x) + iv(x)] = L[u(x)] + iL[v(x)].$$

$$L[u(x)] = 0, \qquad L[v(x)] = 0,$$

$$y = e^{\alpha x} \cdot e^{i\beta x} = e^{\alpha x}(\cos \beta x + i \sin \beta x) = e^{\alpha x} \cos \beta x + ie^{\alpha x} \sin \beta x.$$

$$y_1 = e^{\alpha x} \cos \beta x; \quad y_2 = e^{\alpha x} \sin \beta x.$$

$$y = e^{(\alpha - \beta i)x} = e^{\alpha x} \cos \beta x - ie^{\alpha x} \sin \beta x,$$

Case 3. Conjugate Complex Roots

Now suppose that the auxiliary equation has the complex number $a + bi$ (a, b real, $i^2 = -1$, $b \neq 0$) as a nonrepeated root. Then, since the coefficients are real, the conjugate complex number $a - bi$ is also a nonrepeated root. The corresponding part of the general solution is

$$k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x},$$

where k_1 and k_2 are arbitrary constants. The solutions defined by $e^{(a+bi)x}$ and $e^{(a-bi)x}$ are complex functions of the real variable x. It is desirable to replace these by two *real* linearly independent solutions. This can be accomplished by using Euler's formula,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

which holds for all real θ. Using this we have:

$$
\begin{aligned}
k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x} &= k_1 e^{ax} e^{bix} + k_2 e^{ax} e^{-bix} \\
&= e^{ax}[k_1 e^{ibx} + k_2 e^{-ibx}] \\
&= e^{ax}[k_1(\cos bx + i \sin bx) + k_2(\cos bx - i \sin bx)] \\
&= e^{ax}[(k_1 + k_2)\cos bx + i(k_1 - k_2)\sin bx] \\
&= e^{ax}[c_1 \sin bx + c_2 \cos bx],
\end{aligned}
$$

where $c_1 = i(k_1 - k_2)$, $c_2 = k_1 + k_2$ are two new arbitrary constants. Thus the part of the general solution corresponding to the nonrepeated conjugate complex roots $a \pm bi$ is

$$e^{ax}[c_1 \sin bx + c_2 \cos bx].$$

Combining this with the results of Case 2, we have the following theorem covering Case 3.

THEOREM 3.

1. Consider the nth-order homogeneous linear differential equation with constant coefficients. If the characteristic equation has the conjugate complex roots $a + bi$ and $a - bi$, neither repeated, then the corresponding part of the general solution of may be written

$$y = e^{ax}(c_1 \sin bx + c_2 \cos bx).$$

2. If, however, $a + bi$ and $a - bi$ are each k-fold roots of the characteristic equation, then the corresponding part of the general solution may be written

$$
\begin{aligned}
y = e^{ax}[(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})\sin bx \\
+ (c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots + c_{2k} x^{k-1})\cos bx].
\end{aligned}
$$

We now give several examples.

Example.

Find the general solution of

$$\frac{d^2y}{dx^2} + y = 0.$$

$m^2 + 1 = 0 \qquad m = \pm i$

These are the pure imaginary complex numbers $a \pm bi$, where $a = 0$, $b = 1$. The general solution is thus

$$y = e^{0x}(c_1 \sin 1 \cdot x + c_2 \cos 1 \cdot x),$$

which is simply

$$y = c_1 \sin x + c_2 \cos x.$$

Example. $\dfrac{d^2x}{dt^2} + a^2x = 0$

$$k^2 + a^2 = 0, \qquad k = \pm ai.$$

$$x_1 = e^{iat}, \quad x_2 = e^{-iat},$$

$$x_1 = \cos at, \quad x_2 = \sin at.$$

$$x = C_1 \cos at + C_2 \sin at.$$

Example. $\dfrac{d^2x}{dt^2} + 2n\dfrac{dx}{dt} + a^2x = 0 \quad (n > 0).$

$n < a.$

$$k_1 = -n + i\sqrt{a^2 - n^2},$$

$$k_2 = -n - i\sqrt{a^2 - n^2},$$

$$x_1 = e^{-nt} \cos \sqrt{a^2 - n^2}\, t,$$

$$x_2 = e^{-nt} \sin \sqrt{a^2 - n^2}\, t,$$

$$x = e^{-nt}\left(C_1 \cos \sqrt{a^2 - n^2}\, t + C_2 \sin \sqrt{a^2 - n^2}\, t\right),$$

$$x = Ae^{-nt} \sin\left(\sqrt{a^2 - n^2}\, t + \alpha\right)$$

Example. $y''' + y = 0.$

$$k^3 + 1 = 0,$$

$$k_1 = -1, \ k_2 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

$$y = C_1 e^{-x} + e^{\frac{x}{2}}\left(C_2 \cos x\frac{\sqrt{3}}{2} + C_3 \sin x\frac{\sqrt{3}}{2}\right).$$

Example.

Find the general solution of

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0.$$

The auxiliary equation is $m^2 - 6m + 25 = 0$. Solving it, we find

$$m = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.$$

Here the roots are the conjugate complex numbers $a \pm bi$, where $a = 3$, $b = 4$. The general solution may be written

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x).$$

Example. $y''' - y'' - y' + y = 0.$

$$k^3 - k^2 - k + 1 = 0,$$

$$k_1 = k_2 = 1, \quad k_3 = -1.$$

$$y = e^x (C_1 + C_2 x) + C_3 e^{-x}.$$

Example. $y^{IV} + 8y'' + 16y = 0.$

$$k^4 + 8k^2 + 16 = 0, \quad (k^2 + 4)^2 = 0.$$

$$k_1 = k_2 = 2i; \quad k_3 = k_4 = -2i.$$

$$y = (C_1 + C_2 x) \cos 2x + (C_3 + C_4 x) \sin 2x.$$