Ex: Find the characteristic polynomial of the following matrices

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

Solution: We find $(xI_3 - A)$

$$(xI_3 - A) = \left(x \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix} \right) = \begin{bmatrix} x-1 & -2 & -1 \\ 0 & x-1 & -2 \\ 1 & -3 & x-2 \end{bmatrix}$$

The characteristic polynomial of A is

$$p(x) = \det(xI_3 - A)$$

$$= \begin{vmatrix} x-1 & -2 & -1 \\ 0 & x-1 & -2 \\ 1 & -3 & x-2 \end{vmatrix}$$

$$= 0.(-1)^{2+1} \cdot \begin{vmatrix} -2 & -1 \\ -3 & x-2 \end{vmatrix} + (x-1).(-1)^{2+2} \begin{vmatrix} x-1 & -1 \\ 1 & x-2 \end{vmatrix} + (-2)(-1)^{2+3} \cdot \begin{vmatrix} x-1 & -2 \\ 1 & -3 \end{vmatrix}$$

$$= 0 + (x-1).\big((x-1)(x-2)+1\big) + 2.\big((x-1).(-3)+2\big)$$

$$= (x-1)(x^2-3x+3) + 2(-3x+5)$$

$$= \cancel{x^3} - 3\cancel{x^2} + 3x - \cancel{x^2} + \cancel{3x} - 3 - \cancel{6x} + 10$$

$$= x^3 - 4x^2 + 7$$

Hence $p(x) = x^3 - 4x^2 + 7$ //

Ex: Find the characteristic polynomial of

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: We find $(x I_3 - A)$

$$(x I_3 - A) = \left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \right) = \begin{bmatrix} x-4 & 1 & -3 \\ 0 & x-2 & -1 \\ 0 & 0 & x-3 \end{bmatrix}$$

The characteristic polynomial of A is

$$p(x) = \det(x I_3 - A)$$

$$= \begin{vmatrix} x-4 & 1 & -3 \\ 0 & x-2 & -1 \\ 0 & 0 & x-3 \end{vmatrix}$$

$$= 0 \dots + 0 \dots + (x-3)(-1)^{3+3} \begin{vmatrix} x-4 & 1 \\ 0 & x-2 \end{vmatrix}$$

$$= (x-3) \cdot (x-4)(x-2)$$

$$= (x-3)(x^2 - 6x + 8)$$

$$p(x) = x^3 - 9x^2 + 26x - 24$$

Ex: Find the characteristic polynomial and eigenvalues of

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution: The characteristic polynomial of A is

$$p(x) = \det(xI_3 - A) = \det\left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix} \right)$$

$$= \begin{vmatrix} x-2 & +2 & -3 \\ 0 & x-3 & 2 \\ 0 & 1 & x-2 \end{vmatrix} = 0 .. + 1.(-1)^{3+2} \begin{vmatrix} x-2 & -3 \\ 0 & 2 \end{vmatrix} + (x-2).(-1)^{3+3} \begin{vmatrix} x-2 & 2 \\ 0 & x-3 \end{vmatrix}$$

$$= -\left((x-2).2\right) + (x-2)\left((x-2)(x-3)\right)$$

$$= -2x + 4 + (x-2)\left(x^2 - 5x + 6\right)$$

$$= -2x + 4 + \cancel{x^3} - \cancel{5x^2} + \cancel{6x} - 2x^2 + \cancel{10x} - 12$$

$$= x^3 - 7x^2 + 14x - 8$$

Hence the characteristic polynomial of A is

$$p(x) = x^3 - 7x^2 + 14x - 8$$

The roots of $p(x)$ are the eigenvalues of A

Let $x = 0$ be a root of $p(x)$: $0 - 0 + 0 - 8 = 0$ ↯ So $x = 0$ is not root

Let $x = 1$ be a root of $p(x)$: $1 - 7 + 14 - 8 = 0$ ✓

Hence $x = 1$ is a root of $p(x)$

$$
\begin{array}{l|l}
x^3 - 7x^2 + 14x - 8 & x - 1 \\
\hline
x^3 - x^2 & x^2 - 6x + 8 \\
\hline
-6x^2 + 14x - 8 & \\
-6x^2 + 6x & \\
\hline
8x - 8 & \\
8x - 8 & \\
\hline
0 &
\end{array}
$$

$$x^2 - 6x + 8 = (x-4)(x-2)$$

Hence $x^3 - 7x^2 + 14x - 8 = (x-1)(x-4)(x-2)$

The roots : $(x-1)(x-4)(x-2) = 0$

$$\left.\begin{array}{l} x = 1 \\ x = 4 \\ x = 2 \end{array}\right\} \text{ are the eigenvalues of } A$$

Ex: Find all eigenvalues of $\begin{bmatrix} 4 & 2 & -4 \\ 1 & 5 & -4 \\ 0 & 0 & 6 \end{bmatrix}$

Solution: The characteristic polynomial of A is

$$p(x) = \det(xI_3 - A) = \det\left(x\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 & -4 \\ 1 & 5 & -4 \\ 0 & 0 & 6 \end{bmatrix} \right)$$

$$= \begin{vmatrix} x-4 & -2 & 4 \\ -1 & x-5 & 4 \\ 0 & 0 & x-6 \end{vmatrix} = 0\ldots + 0\ldots + (x-6)\begin{vmatrix} x-4 & -2 \\ -1 & x-5 \end{vmatrix}$$

$$= (x-6)\Big((x-4)(x-5) - 2\Big)$$

$$= (x-6)\left(x^2 - 9x + 20 - 2\right)$$

$$= (x-6)(x^2 - 9x + 18)$$

$= (x-6)(x-6)(x-3)$ is the characteristic polynomial of A

The roots of $p(x)$ are the eigenvalues of A so

$$p(x) = (x-6)(x-6)(x-3)$$

$x = 6$ and $x = 3$ are the eigenvalues of A

Ex: Find the characteristic polynomial, the eigenvalues, ~~eigenvectors~~ and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix}$$

Solution: We find $(xI_3 - A)$.

$$\left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix} \right) = \begin{bmatrix} x-1 & 0 & 0 \\ 1 & x-3 & 0 \\ -3 & -2 & x+2 \end{bmatrix}$$

The characteristic polynomial of A is

$$p(x) = \det(xI_3 - A)$$

$$= \begin{vmatrix} x-1 & 0 & 0 \\ 1 & x-3 & 0 \\ -3 & -2 & x+2 \end{vmatrix}$$

$$= (x-1)(-1)^{1+1} \cdot \begin{vmatrix} x-3 & 0 \\ -2 & x+2 \end{vmatrix}$$

$$= (x-1)(x-3)(x+2)$$

$$= x^3 - 2x^2 - 5x + 6 \,//$$

Hence the characteristic polynomial of A is

$$p(x) = x^3 - 2x^2 - 5x + 6 \,//$$

The roots of $p(x)$ are $1, -2, 3$.

Hence the eigenvalues of A are $1, -2, 3 \,//$

• Now, let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to the eigen-value 1.

Therefore $AX_1 = 1.X_1$ which means $(I_3 - A)X_1 = 0$. We get

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $\begin{cases} x_1 - 2x_2 = 0 \\ -3x_1 - 2x_2 + 3x_3 = 0 \end{cases}$ $\begin{cases} x_1 = 2x_2 \\ x_3 = \frac{8}{3} x_2 \end{cases}$

So $\begin{bmatrix} 2r \\ r \\ 8/3 r \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1

for any non-zero real number r. By putting $r = 1$, we conclude that

$\begin{bmatrix} 2 \\ 1 \\ 8/3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

• Now let $X_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be an eigenvector corresponding to the eigenvalue -2. So $AX_2 = -2X_2$ which means

$(2I_3 + A)X_2 = 0$. We get

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & -5 & 0 \\ -3 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $\begin{cases} -3y_1 = 0 \\ y_1 - 5y_2 = 0 \\ -3y_1 - 2y_2 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 = r \end{cases}$

So $\begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue -2 for any non-zero real number r. By putting $r = 1$, we conclude that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigvector corresponding to the eigenvalue -2.

- Now let $X_3 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ be an eigenvector corresponding to the eigenvalue 3. Therefore $AX_3 = 3X_3$ which means $(3I_3 - A)X_3 = 0$. We get

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & -2 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $\begin{cases} 2z_1 = 0 \\ z_1 = 0 \\ -3z_1 - 2z_2 - 5z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = 0 \\ z_2 = \frac{5}{2} z_3 \end{cases}$

Hence $\begin{bmatrix} 0 \\ 5/2\, r \\ r \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3 for any nonzero number r. By putting $r = 1$, we conclude that $\begin{bmatrix} 0 \\ 5/2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3.

Ex: Find the eigenvectors of $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Solution: The characteristic polynomial of A is

$$p(x) = \det(xI_3 - A) = \det\left(x\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}\right)$$

$$= \begin{vmatrix} x-2 & 0 & -3 \\ 0 & x-1 & 0 \\ 0 & x-1 & x-2 \end{vmatrix} = (x-1).(-1)^{2+2}.\begin{vmatrix} x-2 & -3 \\ 0 & x-2 \end{vmatrix}$$

$$= (x-1)(x-2)(x-2)$$

Hence

$$p(x) = (x-1)(x-2)(x-2)$$

The roots of the characteristic polynomial of A are $x=1$ and $x=2$. Hence the $x=1$ and $x=2$ are the eigenvalues of A.

- Now let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be eigenvector to the eigenvalue 1

So $AX_1 = 1.X_1$ which means $(I_3 - A)X_1 = 0$. We get

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -x_1 - 3x_3 = 0 \\ -x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -x_3 \\ -x_1 = 3x_3 \end{cases} \Rightarrow \begin{array}{l} x_1 = -3r \\ x_2 = -r \\ x_3 = r \end{array}$$

$\begin{bmatrix} -3r \\ -r \\ r \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1. By putting

$r=1$ we obtain $\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ //

- Now let $X_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be eigenvector corresponding to the eigenvalue 3

So $AX_2 = 2X_2$ which means $(2I_3 - A) = 0$. We get

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 = r \end{cases}$$

$\begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3. By putting

$r=1$ we obtain $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ //

Ex: Find the dot product (inner product) of $u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}$.

Solution: $\underset{\substack{\downarrow \\ \text{dot product}}}{u_1 \cdot u_2} = u_1^T \cdot u_2 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} = 1.4 + 2.(-5) + 3.6$

$= 4 - 10 + \cancel{18}^{\,8} = 12$ //

Ex: Find the length of the vector $u = (2, 4, -1)$.

Solution: $\underset{\substack{\swarrow \\ \text{length of } u}}{\|u\|} = \sqrt{2^2 + 4^2 + (-1)^2} = \sqrt{4 + 16 + 1} = \sqrt{21}$ //

Ex: Let there be two vectors $\|u\| = 4$ and $\|v\| = 2$ and $\theta = 60^\circ$. (θ is the angle between u and v). Find their dot product.

Solution: We know $u \cdot v = \|u\| . \|v\| . \cos\theta$

$= 4 . 2 . \cos 60$

$= 4 . 2 . \frac{1}{2}$

$= 4$ //

Ex: Find the angle θ between the vectors $u = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

Solution: $u = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \|u\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$

$v = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \Rightarrow \|v\| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$

$u \cdot v = u^T . v = [2 \ 2 \ -1] \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 4 - 2 - 2 = 0$

dot product

We know that $u \cdot v = \|u\| . \|v\| . \cos\theta$

$0 = 3 . 3 . \cos\theta$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = 90°$ //

Ex: Given two vectors $u = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$. Find the distance between u and v.

Solution: $d(u, v) = \|u - v\| = \left\| \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\|$

$= \left\| \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + (-2)^2 + 1^2}$

$= \sqrt{1 + 4 + 1} = \sqrt{6}$ //

Ex: Is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal set ?

Solution: Label the vectors u_1, u_2 and u_3 respectively. Then

$$u_1 \bullet u_2 = u_1^T \cdot u_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 + (-1) + 0 = 0 \,//$$

$$u_1 \bullet u_3 = u_1^T \cdot u_3 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 0 = 0 \,//$$

$$u_2 \bullet u_3 = u_2^T \cdot u_3 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 0 = 0 \,//$$

Hence $u_1 \bullet u_2 = 0$, $u_1 \bullet u_3 = 0$ and $u_2 \bullet u_3 = 0$.

Therefore $\{u_1, u_2, u_3\}$ is an orthogonal set //

Ex: The set $S = \left\{ \overset{u_1}{\overset{\shortparallel}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}}, \overset{u_2}{\overset{\shortparallel}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}}, \overset{u_3}{\overset{\shortparallel}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}} \right\}$ is ~~linear combination of~~ a orthogonal basis for $\mathbb{R}^3$. Express the vector $y = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of the vectors of S.

Solution: Compute

$$y \bullet u_1 = \begin{bmatrix} 3 & 7 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 3 - 7 + 4 = -4$$

$$y \bullet u_2 = \begin{bmatrix} 3 & 7 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 3 + 7 + 0 = 10$$

$$y \bullet u_3 = \begin{bmatrix} 3 & 7 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 4 = 4$$

$$u_1 \cdot u_1 = [1 \ -1 \ 0]\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 + 1 + 0 = 2$$

$$u_2 \cdot u_2 = [1 \ 1 \ 0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 + 1 + 0 = 2$$

$$u_3 \cdot u_3 = [0 \ 0 \ 1]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 1 = 1$$

We know

$$y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} \cdot u_3$$

$$y = \frac{-4}{2} u_1 + \frac{10}{2} u_2 + \frac{4}{1} u_3$$

$$y = -2u_1 + 5u_2 + 4u_3 \ //$$