Ex (seperable equation) Solve $y' = xy^2 + 2xy$

Solution: $y' = xy^2 + 2xy$

$$\frac{dy}{dx} = x(y^2 + 2y)$$

$$\frac{dy}{y^2+2y} = x\,dx$$

$$\int \frac{dy}{\underbrace{y^2+2y}_{=y(y+2)}} = \int x\,dx$$

$$\frac{A}{y} + \frac{B}{y+2} = \frac{1}{y(y+2)}$$

$A = 1/2 \quad B = -1/2$

$$\frac{1/2}{y} - \frac{1/2}{y+2}$$

$$\int \left(\frac{1/2}{y} - \frac{1/2}{y+2} \right) dy = \int x\,dx$$

$$\frac{1}{2} \int \left(\frac{1}{y} - \frac{1}{y+2} \right) dy = \frac{x^2}{2} + c_1$$

$$\cancel{\frac{1}{2}} \left(\ln|y| - \ln|y+2| \right) + \cancel{c_2} = \cancel{\frac{x^2}{2}} + c_1$$

$$\ln|y| - \ln|y+2| = x^2 + c_3$$

$$\ln\left| \frac{y}{y+2} \right| = x^2 + c_3$$

$$\frac{y}{y+2} = e^{x^2 + c_3}$$

$$\frac{y}{y+2} = e^{x^2} \cdot \underbrace{e^{c_3}}_{c}$$

$$\frac{y}{y+2} = c\,e^{x^2} \quad //$$

Ex (seperable equation) Solve $4xy\,dx + (x^2+1)dy = 0$

Solution: $$\frac{4xy\,dx}{y(x^2+1)} + \frac{(x^2+1)\,dy}{y(x^2+1)} = \frac{0}{y(x^2+1)}$$

$$\Rightarrow \frac{4x}{x^2+1}dx + \frac{1}{y}dy = 0$$

$$\int \frac{4x}{x^2+1}dx + \int \frac{1}{y}dy = \int 0$$

Let $x^2+1 = u$

$2x\,dx = du$

$4x\,dx = 2du$

$$\Rightarrow \int \frac{2du}{u} + \int \frac{1}{y}dy = \int 0$$

$$2\ln|u| + \ln|y| = \ln|c_1|$$

$$2\ln|x^2+1| + \ln|y| = \ln|c_1|$$

$$\ln|y| = \ln|c_1| - \ln|(x^2+1)^2|$$

$$\ln|y| = \ln\left|\frac{c_1}{(x^2+1)^2}\right|$$

$$y = \frac{c_1}{(x^2+1)^2}$$ //

Ex (seperable equation) Solve $x(1+y^2)+y(1+x^2)y'=0$

Solution: $\dfrac{x(1+y^2)+y(1+x^2)y'}{(1+x^2)(1+y^2)} = \dfrac{0}{(1+x^2)(1+y^2)}$

$$\Rightarrow \frac{x}{1+x^2} + \frac{y}{1+y^2}\frac{dy}{dx} = 0$$

$$\Rightarrow dx\left(\frac{x}{1+x^2} + \frac{y}{1+y^2}\frac{dy}{dx}\right) = dx(0)$$

$$\Rightarrow \frac{x}{1+x^2}dx + \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \int \frac{x}{1+x^2}dx + \int \frac{y}{1+y^2}dy = \int 0$$

Let $1+x^2=u$ $\quad$ $1+y^2=v$

$2x\,dx=du$ $\quad$ $2y\,dy=dv$

$x\,dx=\dfrac{du}{2}$ $\quad$ $y\,dy=\dfrac{dv}{2}$

$$\Rightarrow \int \frac{1}{2}du\cdot\frac{1}{u} + \int \frac{1}{2}dv\cdot\frac{1}{v} = \int 0$$

$$\frac{1}{2}\ln|u| + \frac{1}{2}\ln|v| = c_1$$

$$\ln|u| + \ln|v| = c_2$$

$$\ln|1+x^2| + \ln|1+y^2| = c_2$$

$$\ln|(1+x^2)(1+y^2)| = c_2$$

$$(1+x^2)(1+y^2) = e^{c_2}\ c$$

$$1+y^2 = \frac{c}{1+x^2} \Rightarrow y = \sqrt{\frac{c}{1+x^2} - 1}$$ //

Ex : (seperable equation) Solve $e^x dx - (1+e^x) y dy = 0$, $y(0) = 1$

Solution: $\frac{e^x dx}{1+e^x} - \frac{(1+e^x) y dy}{1+e^x} = \frac{0}{1+e^x}$

$$\frac{e^x}{1+e^x} dx - y dy = 0$$

$$\int \frac{e^x}{1+e^x} dx - \int y dy = \int 0$$

Let $1+e^x = u$

$e^x dx = du$

$$\int \frac{du}{u} - \int y dy = \int 0$$

$$\ln|u| - \frac{y^2}{2} = c_1$$

$$\ln|1+e^x| - \frac{y^2}{2} = c_1$$

$$\frac{y^2}{2} = \ln|1+e^x| \overbrace{-c_1}^{+c_2}$$

$$y^2 = 2\ln|1+e^x| + \overbrace{2c_2}^{c_3}$$

$$y^2 = 2\ln|1+e^x| + c_3$$

$y(0) = 1 \Rightarrow 1^2 = 2\ln|1+e^0| + c_3 \Rightarrow c_3 = 1 - 2\ln 2$

$$\Rightarrow y^2 = 2\ln|1+e^x| + 1 - 2\ln 2$$

$$y = \sqrt{2\ln|1+e^x| + 1 - 2\ln 2} \quad //$$

Ex: (exact dif. equation) $\underbrace{(y\cos x + 2xe^{y})}_{M}dx + \underbrace{(\sin x + x^2e^{y} - 1)}_{N}dy = 0$

Solution: If $\underbrace{\frac{\partial M}{\partial y}}_{M_y} = \underbrace{\frac{\partial N}{\partial x}}_{N_x}$ then it is exact dif. equation

$M_y = \cos x + 2xe^{y}$

$N_x = \cos x + 2xe^{y}$ } they are equal so it is exact

The general solution is $\varphi(x,y) = c$

$M \overset{?}{=} y\cos x + 2xe^{y} \Rightarrow \varphi(x,y) = \int (y\cos x + 2xe^{y})dx$

$M = \varphi_x = \frac{d\varphi}{dx}$

$\varphi(x,y) = y\sin x + x^2e^{y} + h(y)$

$N = \sin x + x^2e^{y} - 1$

$N = \varphi_y = \frac{d\varphi}{dy}$

We know that $\varphi = y\sin x + x^2e^{y} + h(y)$

So $\frac{d\varphi}{dy} = \sin x + x^2e^{y} + h'(y)$

$= N = \sin x + x^2e^{y} - 1$

$\Rightarrow h'(y) = -1 \Rightarrow h(y) = -y + c_1$

Hence $\varphi(x,y) = y\sin x + x^2e^{y} - y + c_1$

The general solution is $\varphi(x,y) = c \Rightarrow$

$y\sin x + x^2e^{y} - y + c_1 = c_2$

$y\sin x + x^2e^{y} - y = c$ //

Ex: (exact dif. equation) Solve $(y^2+3)dx+(2xy-4)dy=0$

Solution: $\underbrace{(y^2+3)}_{M}dx + \underbrace{(2xy-4)}_{N}dy = 0$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then it is exact.

$M = y^2+3 \Rightarrow \frac{\partial M}{\partial y} = 2y$

$N = 2xy-4 \Rightarrow \frac{\partial M}{\partial x} = 2y$

they are equal so it is exact dif. equation

The general solution is $\varphi(x,y)=c$

$M = y^2+3 \Rightarrow \varphi(x,y) = \int (y^2+3)\,dx = y^2x+3x+h(y)$

$M = \frac{\partial \varphi}{\partial x} = \varphi_x$

$N = 2xy-4$. We know that $\varphi(x,y) = y^2x+3x+h(y)$

$N = \frac{\partial \varphi}{\partial y} = \varphi_y$

$\frac{\varphi'(x,y)}{\partial y} = 2yx + h'(y) = N = 2xy-4$

$\Rightarrow h'(y) = -4$

$\Rightarrow h(y) = -4y + c_1$

So $\varphi(x,y) = y^2x+3x+h(y) = y^2x+3x-4y+c_1$

The general solution is $y^2x+3x-4y+c_1 = c$

$y^2x+3x-4y = c_2$ //

Ex: (exact dif. equation) Solve the IVP:

$$\underbrace{(2y\sin x\cos x + y^2\sin x)}_{M}dx + \underbrace{(\sin^2 x - 2y\cos x)}_{N}dy = 0 \quad , \; y(0) = 3$$

Solution: if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then it is exact

$\frac{\partial M}{\partial y} = 2\sin x\cos x + 2y\sin x$

they are equal so it is exact.

$\frac{\partial N}{\partial x} = 2\sin x\cos x + 2y\sin x$

$N = \frac{\partial \varphi}{\partial y} = N_y$

$N = \sin^2 x - 2y\cos x \Rightarrow \varphi(x,y) = \int(\sin^2 x - 2y\cos x)dy$

$= \sin^2 x\, y - y^2\cos x + h(x)$

$M = \frac{\partial \varphi}{\partial x} = M_x$

$M = 2y\sin x\cos x + y^2\sin x$, We know $\varphi(x,y) = \sin^2 x\, y - y^2\cos x + h(x)$

$\frac{\varphi(x,y)}{\partial x} = 2\sin x\cos x . y + y^2\sin x + h'(x) = N$

$= 2y\sin x\cos x + y^2\sin x$

$\Rightarrow h'(x) = 0 \Rightarrow h(x) = c_1$

$\varphi(x,y) = y\sin^2 x - y^2\cos x + c_1$

the general solution is $y\sin^2 x - y^2\cos x = c$

$y(0) = 3 \Rightarrow 3\sin^2 0 - 3.\cos 0 = c \Rightarrow c = -9$

general solution is $y\sin^2 x - y^2\cos x = -9$ //

Ex: (exact dif. equ./integ. factor) Solve $(3xy+y^2)dx+(x^2+xy)dy=0$

Recall if $Mdx+Ndy=0$ is not exact

$$\frac{M}{\partial y}=M_y \qquad \frac{N}{\partial x}=N_x \Rightarrow$$

- $\dfrac{M_y-N_x}{N}$ = depends only x

- $\dfrac{N_x-M_y}{M}$ = depends only y

$$\mu=e^{\int\frac{M_y-N_x}{N}dx} \quad \text{or} \quad \mu=e^{\int\frac{N_x-M_y}{M}dy}$$

Solution: $\underbrace{(3xy+y^2)}_{M}dx+\underbrace{(x^2+xy)}_{N}dy=0$

$M=3xy+y^2 \Rightarrow \dfrac{\partial M}{\partial y}=3x+2y$

they are not equal. So it is not exact

$N=x^2+xy \Rightarrow \dfrac{\partial N}{\partial x}=2x+y$

- $\dfrac{M_y-N_x}{N}=\dfrac{3x+2y-2x-y}{x^2+xy}=\dfrac{x+\cancel{y}}{x(\cancel{x+y})}=\dfrac{1}{x}$ (depends only x)

$$\mu=e^{\int\frac{M_y-N_x}{N}dx}=e^{\int\frac{1}{x}dx}=e^{\ln x}=x \text{ integrating factor}$$

$x/ \quad (3xy+y^2)dx+(x^2+xy)dy=0$

$(3x^2y+y^2x)dx+(x^3+x^2y)dy=0$

$$\underbrace{(3x^2y + y^2x)}_{M}\,dx + \underbrace{(x^3 + x^2y)}_{N}\,dy = 0$$

$$M = 3x^2y + y^2x \Rightarrow \frac{\partial M}{\partial y} = 3x^2 + 2xy$$

$$N = x^3 + x^2y \Rightarrow \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

they are equal so it is exact

$$M = \frac{\partial \varphi}{\partial x} = M_x$$

$$M = 3x^2y + y^2x \Rightarrow \varphi(x,y) = \int (3x^2y + y^2x)\,dx$$

$$= x^3y + \frac{x^2}{2}y^2 + h(y)$$

$$N = \frac{\partial \varphi}{\partial y} = N_y$$

$N = x^3 + x^2y$. We know $\varphi(x,y) = x^3y + \frac{x^2}{2}y^2 + h(y)$

$$\frac{\partial \varphi}{\partial y} = x^3 + \cancel{2}\frac{x^2}{\cancel{2}}y + h'(y) = N = x^3 + x^2y$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = c_1$$

$$\varphi(x,y) = x^3y + \frac{x^2}{2}y^2 + c_1$$

The general solution is $x^3y + \frac{x^2}{2}y^2 = c$ //

Ex (~~seperable~~ homogenous dif. eq.) Solve $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Solution: $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$ $\rightarrow$ $\frac{x^2+y^2}{xy} = \frac{x^2}{xy} + \frac{y^2}{xy}$

$= \frac{x}{y} + \frac{y}{x}$

$v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot 1$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{xy} \Rightarrow \frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$

$\Rightarrow \frac{dv}{dx} \cdot x + \cancel{v} = \frac{1}{v} + \cancel{v}$

$\Rightarrow \frac{dv}{dx} \cdot x = \frac{1}{v} \Rightarrow v \cdot dv = \frac{dx}{x}$

$\Rightarrow \int v\,dv = \int \frac{dx}{v} \Rightarrow \frac{v^2}{2} = \ln x + c$ $\ln c_1$

$\Rightarrow \frac{v^2}{2} = \ln x + \ln c_1 \Rightarrow \frac{1}{2} \cdot \left(\frac{y}{x}\right)^2 = \ln(c_1 \cdot x)$

$\Rightarrow \frac{y^2}{x^2} = 2\ln(c_1 x) \Rightarrow y^2 = x^2 \cdot 2 \cdot \ln(c_1 x)$ //

Ex (homogenous dif. equ.) Solve $\frac{dy}{dx} = \frac{y(x-y)}{x^2}$

Solution: $\frac{dy}{dx} = \frac{y(x-y)}{x^2} \rightarrow \frac{xy - y^2}{x^2} = \frac{xy}{x^2} - \frac{y^2}{x^2} = \frac{y}{x} - \frac{y^2}{x^2}$

Let $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot 1$

$\Rightarrow \frac{dy}{dx} = \frac{y(x-y)}{x^2} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2}$

$\Rightarrow \frac{dv}{dx} \cdot x + \cancel{v} = \cancel{v} - v^2$

$\Rightarrow \frac{dv}{dx} \cdot x = -v^2 \Rightarrow -\frac{dv}{v^2} = \frac{dx}{x}$

$\Rightarrow \int -\frac{dv}{v^2} = \int \frac{dx}{x} \Rightarrow \frac{1}{v} = \ln x + c \quad = \ln c_1$

v^{-2}

$\Rightarrow \frac{1}{v} = \ln(c_1 x) \Rightarrow \frac{1}{y/x} = \ln(c_1 x)$

$\Rightarrow \frac{x}{y} = \ln(c_1 x) \Rightarrow y = \frac{x}{\ln(c_1 x)}$ //

Ex (homogenous dif. eq) Solve $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y}$

Solution: $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y}$

$$\Rightarrow \frac{dy}{dx} = \frac{(y-4x)/x}{(x-y)/x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}} \qquad \text{(homogenous)} \qquad \left(\frac{dy}{dx} = f\left(\frac{y}{x}\right)\right)$$

Let $v := \dfrac{y}{x} \Rightarrow y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}} \Rightarrow x \cdot \frac{dv}{dx} + v = \frac{v-4}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-4}{1-v} - v \Rightarrow x\frac{dv}{dx} = \frac{-4+v^2}{1-v}$$

$$\Rightarrow \frac{x}{dx} = \frac{-4+v^2}{1-v} \cdot \frac{1}{dv} \Rightarrow \frac{dx}{x} = \frac{(1-v)\,dv}{(v^2-4)}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{(1-v)}{(v^2-4)}\,dv \qquad \text{(separable dif. eq.)}$$

$$\frac{1-v}{v^2-4} = \frac{1-v}{(v-2)(v+2)} = \frac{A}{\underset{(v+2)}{v-2}} + \frac{B}{\underset{(v-2)}{v+2}}$$

$$\frac{A\cancel{v}+2A+B\cancel{v}-2B}{(v-2)(v+2)} = \frac{v(A+B)+(2A+2B)}{(v-2)(v+2)}$$

$$2/\ \begin{array}{l} A+B=-1 \\ 2A-2B=4 \end{array} \qquad 4A=-1 \quad A=-1/4 \quad B=-\frac{3}{4}$$

$$\Rightarrow \int \frac{dx}{x} = \int \left(\frac{-1/4}{v-2} + \frac{-3/4}{v+2} \right) dv$$

$$\Rightarrow \int \frac{dx}{x} = -\frac{1}{4} \int \frac{1}{v-2}\, dv + \frac{3}{4} \int \frac{1}{v+2}\, dV$$

$$\Rightarrow \ln x = -\frac{1}{4} \ln|v-2| - \frac{3}{4} \ln|v+2| + c = \ln c_1$$

$$\Rightarrow \ln x = -\frac{1}{4} \ln\left|\frac{y}{x} - 2\right| - \frac{3}{4} \ln\left|\frac{y}{x} + 2\right| + \ln c_1 \quad //$$

Ex (homogenous dif. eq.) Solve $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$

Solution: $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$

$$\frac{x-y}{x+y} = \frac{(x-y)/x}{(x+y)/x} = \frac{\frac{x}{x} - \frac{y}{x}}{\frac{x}{x} + \frac{y}{x}}$$

$$= \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$$

Let $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} \cdot x + v$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y} \Rightarrow \frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}} \Rightarrow$$

$$\frac{dv}{dx} \cdot x + v = \frac{1-v}{1+v} \Rightarrow \frac{dv}{dx} \cdot x = \frac{1-v}{1+v} - v$$

$$\Rightarrow x \cdot \frac{dv}{dx} = \frac{1-2v-v^2}{1+v} \Rightarrow \int \frac{1+v}{(1-2v-v^2)} dv = \int \frac{dx}{x}$$

$1-2v-v^2 = u$

$-2v-2 = du$

$-2(v+1) = du \Rightarrow v+1 = -\dfrac{du}{2}$

$$\Rightarrow \int -\frac{du}{2} \cdot \frac{1}{u} = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2} \ln|u| = \ln|x| + c$$

$$\Rightarrow \ln|u|^{-1/2} = \ln|x| + \ln|c_1| \quad \ln|c_1 x|$$

$$\Rightarrow (u)^{-1/2} = c_1 x \Rightarrow (1-2v-v^2)^{-1/2} = c_1 x \,//$$

Ex: (seperable eq.) Solve IVP: $y' = \dfrac{3x^2+4x-4}{2y-4}$, $y(1)=3$

Solution: $y' = \dfrac{3x^2+4x-4}{2y-4}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2+4x-4}{2y-4}$

$\Rightarrow (2y-4)dy = (3x^2+4x-4)dx$

$\Rightarrow \int (2y-4)\,dy = \int (3x^2+4x-4)\,dx$

$\Rightarrow y^2 - 4y = x^3 + 2x^2 - 4x + c$

$y(1)=3 \Rightarrow 9 - 12 = 1 + 2 - 4 + c$

$\Rightarrow c = -2$

the general solution is

$y^2 - 4y = x^3 + 2x^2 - 4x - 2$ //

Ex (exact dif. eq.) Solve $(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$

Solution: $\underbrace{(2xy - 9x^2)}_{M}dx + \underbrace{(2y + x^2 + 1)}_{N}dy = 0$

$M = 2xy - 9x^2 \Rightarrow M_y = \frac{\partial M}{\partial y} = 2x$

$N = 2y + x^2 + 1 \Rightarrow N_x = \frac{\partial N}{\partial x} = 2x$

$\Rightarrow$ they are equal so it is exact

$\frac{\partial \varphi}{\partial x} = M = 2xy - 9x^2 \Rightarrow \varphi(x,y) = \int (2xy - 9x^2)dx$

$= x^2y - 3x^3 + h(y)$

$\frac{\partial \varphi}{\partial y} = N = 2y + x^2 + 1$. We know $\varphi(x,y) = x^2y - 3x^3 + h(y)$

$\frac{\partial \varphi}{\partial y} = \cancel{x^2} + h'(y) = N = 2y + \cancel{x^2} + 1$

$\Rightarrow h'(y) = 2y + 1$

$\Rightarrow h(y) = y^2 + y + c_1$

$\varphi(x,y) = c$ is general solution

$\varphi(x,y) = x^2y - 3x^2 + h(y)$

$\varphi(x,y) = x^2y - 3x^2 + y^2 + y + c_1$ //