(a) False. A bad example can consist of a single edge $e = (u, v)$. Assume the cost of u is 1 while the cost of v is more than $2c$. The minimum cost of a vertex cover is 1, while the algorithm selects node v with probability $1/2$, and hence has expected cost more than c. Alternately we could have u at most 0 and v at most 1. Now the algorithm's expected cost is $1/2$, while the optimum is 0.

(b) This is true. Let p_e be the probability that edge e is selected by the algorithm. Note that the algorithm, as given by the problem set, does not specify the selection rule of edges. You may select uncovered edges at random, or by smallest index, etc. The probability p_e will of course depend on what selection rule was used. But any selection rule gives rise to such probabilities. Now we need to notice two facts. First that $\sum_{e \in E} p_e$ is exactly, the expected number of nodes selected by the algorithm. This is true, as every time we select an edge e we add one node to the vertex cover.

Next we consider the sum of the probabilities p_e for edges adjacent to a vertex v. Let $\delta(v)$ denote the set of edges adjacent to vertex v, and consider $\sum_{e \in \delta(v)} p_e$. Note that this is exactly the expected number of edges selected that are adjacent to node v. Let $S(v)$ be the random variable indicating the selected edges adjacent to v. We have that $Exp(|S(v)|) = \sum_{e \in \delta(v)} p_e$. We claim that this expectation is at most 2. This is true as each time an edge in $\delta(v)$ is selected, with $1/2$ probability, we use node v cover edge e, and then all edges in $\delta(v)$ are covered, and no more edges in this set will be selected selected. To make this argument precise, let E_i denote the event that at least i edges are selected adjacent to i. Now we have the following inequality for the expected number of edges selected.

$$Exp(|S(v)|) = \sum_i i Prob(E_i - E_{i+1}) = \sum_i Prob(E_i) \leq 1 + \sum_{i>1} 2^{i-1} \leq 2,$$

where the inequality $Prob(E_i) \leq 2^{i-1}$ follows for $I > 1$ as after each edge selected adjacent to v we add v to the vertex cover with probability $1/2$.

Now we are ready to bound the expected size of the vertex cover compared to the optimum. Let S^* be an optimum vertex cover.

$$\sum_e p_e \leq \sum_{v \in S^*} \sum_{e \in \delta(v)} p_e \leq \sum_{v \in S^*} 2 = 2|S^*|,$$

where the first inequality follows as S^* is a vertex cover, and so the second sum must cover each edge e.

[1] ex593.991.129