Let us call this problem *Trade*. *Trade* is in NP, since a pair of subsets A_i and A_j can serve as a certificate. We can verify that

- $\sum_{a \in A_j} v_i(a) > \sum_{a \in A_i} v_i(a)$
- $\sum_{a \in A_i} v_j(a) > \sum_{a \in A_j} v_j(a)$

by summing up the valuation of each person, which is clearly polynomial time doable.

We will prove *Trade* is NP-complete by reduction from *Subset Sum* problem. Suppose we have an instance of *Subset Sum*, that is, n integers $w_1, w_2, \ldots, w_n$, and another integer W. The goal is to find a subset $S \subseteq w_1, w_2, \ldots, w_n$, s.t. $\sum_{w_k \in S} w_k = W$.

Construct an instance of *Trade* as follows: there are $n+1$ objects $a_1, a_2, \ldots, a_{n+1}$, p_i possesses objects $a_1, a_2, \ldots, a_n$, and p_j possesses a_{n+1}. The first n objects are corresponding to the n numbers in *Subset Sum* problem, and the valuations for them are $v_i(a_k) = v_j(a_k) = w_k \ \ (k = 1, 2, \ldots, n)$. The valuations of the last object are $v_i(a_{n+1}) = W + 1$, and $v_j(a_{n+1}) = W - 1$.

Since p_j only possesses a single object a_{n+1}, so if there exist A_i and A_j, then A_j must be $\{a_{n+1}\}$. A_i will be a subset of $\{a_1, a_2, \ldots, a_n\}$.

Now we will prove that if there is a subset of numbers that sums up to W, if and only if there is a subset $A_i \subseteq \{a_1, a_2, \ldots, a_n\}$, together with $A_j = \{a_{n+1}\}$, satisfying the two inequalities stated before.

If there is a subset S for *Subset Sum* problem, s.t. $\sum_{k \in S} w_k = W$, then we let

$$A_i = \{a_k : w_k \in S\}$$

According to our construction:

$$\sum_{a \in A_j} v_i(a) = v_i(a_{n+1}) = W + 1 > W = \sum_{a \in A_i} v_i(a)$$

and

$$\sum_{a \in A_i} v_j(a) = W > W - 1 = v_j(a_{n_1}) = \sum_{a \in A_j} v_j(a)$$

so A_i and A_j satisfy the requirement of $Trade$ problem.

If there is a subset $A_i \subseteq \{a_1, a_2, \ldots, a_n\}$, and $A_j = \{a_{n+1}\}$ satisfying the two inequalities, then we will let

$$S = \{w_k : a_k \in A_i\}$$

According to our construction, $\sum_{a \in A_i} v_j(a) > W - 1$, $\sum_{a \in A_i} v_i(a) < W + 1$, and $\sum_{w_k \in S} w_k = \sum_{a \in A_i} v_i(a) = \sum_{a \in A_i} v_j(a)$, so $\sum_{w_k \in S} w_k = W$.

[1] ex508.142.105