

(a) For every node v_k that comes later than v_j , i.e. $k > j$, it has probability $\frac{1}{k-1}$ to link to v_j , since v_k chooses from the $k-1$ existing nodes with equal probabilities. For all the nodes coming before v_j , such probability is obviously zero.

So the expected number of incoming links to node v_j is

$$\begin{aligned} \sum_{k=j+1}^n \frac{1}{k-1} &= \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=1}^{j-1} \frac{1}{k} \\ &= H(n-1) - H(j-1) \\ &= \Theta(\ln n) - \Theta(\ln j) \\ &= \Theta\left(\ln \frac{n}{j}\right) \end{aligned}$$

(b) Consider a node v_j , every node v_k with $k > j$ has probability $1 - \frac{1}{k-1}$ not to link to v_j . So if we have random variable X_j s.t.

$$X_j = \begin{cases} 1 & \text{node } v_j \text{ has no in-coming links} \\ 0 & \text{otherwise} \end{cases}$$

then

$$\begin{aligned} \text{Exp}[X_j] &= \text{Pr}[\text{no nodes links to } v_j] \\ &= \prod_{k=j+1}^n \left(1 - \frac{1}{k-1}\right) \\ &= \frac{j-1}{j} \cdot \frac{j}{j+1} \cdot \frac{j+1}{j+2} \cdots \frac{n-2}{n-1} \\ &= \frac{j-1}{n-1} \end{aligned}$$

Therefore, by linearity of expectations, we get the expected number of nodes without in-coming links

$$\sum_{j=1}^n \text{Exp}[X_j] = \sum_{j=1}^n \frac{j-1}{n-1} = \frac{1}{n-1} \sum_{j=1}^n (j-1) = \frac{1}{n-1} \cdot \frac{n(n-1)}{2} = \frac{n}{2}$$