

We can conclude that A must be a minimum-cost arborescence. Let r be the designated root vertex in $G = (V, E)$; recall that a set of edges $A \subseteq E$ forms an arborescence if and only if (i) each node other than r has in-degree 1 in (V, A) , and (ii) r has a path to every other node in (V, A) .

We claim that in a directed acyclic graph, any set of edges satisfying (i) must also satisfy (ii). (Note that this is certainly not true in an arbitrary directed graph.) For suppose that A satisfies (i) but not (ii), and let v be a node not reachable from r . Then if we repeatedly follow edges backwards starting at v , we must re-visit a node eventually, and this would be a cycle in G .

Thus, every way of choosing a single incoming edge for each $v \neq r$ yields an arborescence. It follows that an arborescence A has minimum cost if and only, for each $v \neq r$, the edge in A entering v has minimum cost over all edges entering v ; and similarly, an edge (u, v) belongs to a minimum-cost arborescence if and only if it has minimum cost over all edges entering v .

Hence, if we are given an arborescence $A \subseteq E$ with the guarantee that for every $e \in A$, e belongs to *some* minimum-cost arborescence in G , then for each $e = (u, v)$, e has minimum cost over all edges entering v , and hence A is a minimum-cost arborescence.

¹ex910.984.431