

(a) Suppose for simplicity that n is a perfect square. We drop the first jar from heights that are multiples of \sqrt{n} (i.e. from $\sqrt{n}, 2\sqrt{n}, 3\sqrt{n}, \dots$) until it breaks.

If we drop it from the top rung and it survives, then we're also done. Otherwise, suppose it breaks from height $j\sqrt{n}$. Then we know the highest safe rung is between $(j-1)\sqrt{n}$ and $j\sqrt{n}$, so we drop the second jar from rung $1 + (j-1)\sqrt{n}$ on upward, going up by one each time.

In this way, we drop each of the two jars at most \sqrt{n} times, for a total of at most $2\sqrt{n}$. If n is not a perfect square, then we drop the first jar from heights that are multiples of $\lfloor \sqrt{n} \rfloor$, and then apply the above rule for the second jar. In this way, we drop the first jar at most $2\sqrt{n}$ times (quite an overestimate if n is reasonably large) and the second jar at most \sqrt{n} times, still obtaining a bound of $O(\sqrt{n})$.

(b) We claim by induction that $f_k(n) \leq 2kn^{1/k}$. We begin by dropping the first jar from heights that are multiples of $\lfloor n^{(k-1)/k} \rfloor$. In this way, we drop the first jar at most $2n/n^{(k-1)/k} = 2n^{1/k}$ times, and thus narrow the set of possible rungs down to an interval of length at most $n^{(k-1)/k}$.

We then apply the strategy for $k-1$ jars recursively. By induction it uses at most $2(k-1)(n^{(k-1)/k})^{1/(k-1)} = 2(k-1)n^{1/k}$ drops. Adding in the $\leq 2n^{1/k}$ drops made using the first jar, we get a bound of $2kn^{1/k}$, completing the induction step.

¹ex291.532.145