Path Selection is in NP, since we can be shown a set of k paths from among $P_1, \ldots, P_c$ and check in polynomial time that no two of them share any nodes.

Now, we claim that *3-Dimensional Matching* $\leq_P$ *Path Selection*. For consider an instance of *3-Dimensional Matching* with sets X, Y, and Z, each of size n, and ordered triples $T_1, \ldots, T_m$ from $X \times Y \times Z$. We construct a directed graph $G = (V, E)$ on the node set $X \cup Y \cup Z$. For each triple $T_i = (x_i, y_j, z_k)$, we add edges (x_i, y_j) and (y_j, z_k) to G. Finally, for each $i = 1, 2, \ldots, m$, we define a path P_i that passes through the nodes $\{x_i, y_j, z_k\}$, where again $T_i = (x_i, y_j, z_k)$. Note that by our definition of the edges, each P_i is a valid path in G. Also, the reduction takes polynomial time.

Now we claim that there are n paths among $P_1, \ldots, P_m$ sharing no nodes if and only if there exist n disjoint triples among $T_1, \ldots, T_m$. For if there do exist n paths sharing no nodes, then the corresponding triples must each contain a different element from X, a different element from Y, and a different element from Z — they form a perfect three-dimensional matching. Conversely, if there exist n disjoint triples, then the corresponding paths will have no nodes in common.

Since *Path Selection* is in NP, and we can reduce an NP-complete problem to it, it must be NP-complete.

(Other direct reductions are from Set Packing and from Independent Set.)

[1] ex529.979.546