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Optimized Model Reduction Using Reducibility Matrix

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***Abstract*— Model reduction techniques capture the attention of researchers and engineers due to their computational advantages. This work proposes a reducible matrix model reduction method to provide a more structured approach for reducing dynamical systems, preserving preselected modal parameters. Preserving the eigenvalues and their corresponding eigenvectors results in a more practical and accurate physical representation of the system.**

Broyden–Fletcher–Goldfarb–Shanno method is used to optimize a coupling matrix resultant from applying the reducibility matrix on the system, and the final norm error was used as criteria to find the minimum possible order for the reduced system. The proposed model reduction algorithm is demonstrated on a Cantilever Beam. The reduced system's accuracy is comparable to that of an ANSYS-generated full-scale system, and it was a better representation of the complete system than a similarly generated ANSYS system with fewer used elements.

Keywords- Model Reduction; Optimization; Reducibility Matrix; Vibration; Reduction Algorithm; Reduced Order Controller

I. INTRODUCTION

Nowadays, dynamical systems tend to be larger and more complex, requiring high computational analysis resources to deal with them and more challenges in their control to ensure that the system works as intended. One way to resolve such issues is to produce an equivalent reduced-order system.

The dimension of complex structures' models may easily reach an order of several thousand variables, which gives importance to the reduction method. Early mechanical-based model reduction techniques converted the stiffness matrix into master and slave degrees of freedom (DOF). Then, they used a transformation matrix to reduce the system to only the master DOFs. The proposed transformations neglected the inertia forces, resulting in an approximate transformation called static condensation [1] [2]. Some researchers formulated an exact reduction method using another approach, including the dynamical effects, using the transformation matrix with the full Finite Element (FE) model and the active subset [3] [4].

Modal-based reduction methods, recommended for high-order systems, focus on the dominant slow modes, omitting the system's fast modes [5] [6]. While a generalized Ritz vector approach replaces the eigenvectors to counter the needed computations of the eigenvalues [7].

Large and complex systems can benefit from the Component Mode Synthesis (CMS) based methods, which combine the previous three groups. This method divides the system into subsystems and then uses Ritz vectors to establish a transformation matrix to reduce the system's order [8].

Model reduction techniques that utilize matrices decomposition-based methods, such as Singular Value Decomposition (SVD) [9], do not preserve the systems' or structures' natural frequencies and mode shapes, making them too far from the actual physical system connectivity from engineering points of view.

Some of the proposed systems' model reduction techniques are more concerned with control, enabling the design and use of reduced order controllers. This helps when the computational burdens are high, mainly when used in real-time control and diagnosis, because the system model is frequently evaluated. Such techniques focus on the observability and controllability properties of the dynamical systems. They utilize subspace concepts and approaches that try to retain some selected mode shapes and omit the rest without using the decomposition of eigenvalues [10].

More recent model reduction methods use balanced realization and specific sparsity patterns to reduce the order of a nonlinear model. They gain results that are close to the full-order system. This class of model reduction techniques is linked to an FE model, using a Bayesian model updating approach based on a stochastic simulation method [11] [12].

There are numerous model reduction techniques in the literature with many engineering applications in various fields, as in [12] [13][14]. However, this work proposes a novel method that enables engineers to retain the structures or the systems' pre-selected frequencies (Eigenvalues) and their corresponding mode shapes (Eigenvectors). This technique depends on the reducibility matrix introduced earlier by the authors, which retains the complete system's eigenvalues while presenting a coupling term for tuning the reduced model. The proposed reducibility matrix, with the preservation of the structures' frequencies and mode shapes, produces physically more realistic equivalent dynamical systems, with a dynamical response close to the original system and preserving as much as possible of the physical characteristics of the original model [15].

These equivalent reduced models can be used instead of the full system in the analysis, control, and computer simulations with minimum computational efforts.

II. PRELIMINARY BACKGROUND

A. *What is Model Reduction?*

Reducing a dynamical system while preserving its physical characteristics is becoming increasingly important. This allows us to use less computational effort to arrive at a reasonable representation of a large-scale system's response.

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Generally, all reduction techniques aim to preserve the system's dynamical behavior (i.e., response) while representing it with a lower-order mathematical model. In vibration, dynamics, and control, this is usually done by omitting upper energy level responses, which are less likely to get excited (i.e., keeping the system's dominant poles).

The original Reducibility Matrix approach aimed to retain the slower modes of the systems while omitting the faster modes; this was done by preserving the eigenvalues corresponding to these modes.

B. *Reducibility Matrix Concept*

Definition: A matrix $A \in M_n$ is called reducible for $n \geq 2$, if there exists a permutation matrix $P \in R^{n\times n}$, and there is some integer $r \in Z$ with $1 \leq r \leq n-1$ such that

$$P^T AP = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \tag{1}$$

Where $U \in R^{r\times r}$, $V \in R^{r\times n-r}$, $W \in R^{n-r\times n-r}$, and $0 \in R^{n-r\times r}$ is zero matrix [16].

This concept is the core of the algorithm used in this work. It will be applied as a similarity transformation, and the permutation matrix P will be selected using Linear Matrix Inequalities (LMI) [15].

The (LMI) optimization problem will be cast as follows:

$$\min_P \|P - P_0\| \tag{2}$$

Subjected to $\left\|P^{-1} AP - \tilde{A}\right\| < \varepsilon$

Which is written in LMI equivalent form as

$$min_S \; trace(S) \tag{3}$$

Subjected to $\begin{bmatrix} S & P-P_0 \\ (P-P_0)^T & I \end{bmatrix} > 0$

$$\begin{bmatrix} \varepsilon^2 I & P^{-1}AP - \tilde{A} \\ \left(P^{-1}AP - \tilde{A}\right)^T & I \end{bmatrix} > 0$$

Where S is a symmetric slack matrix.

III. SYSTEMS MODEL REDUCTION

A structured novel strategy for model reduction based on the reducibility matrix will be presented. In this work, we aim to preserve the eigenvalues and eigenvectors of the retained modes, scaling the eigenvectors to match the original eigenvectors, keeping more of the original system's physical properties, and aiming to have a more accurate resemblance to the original system.

One way to achieve this is to cast the problem as an optimization problem as follows:

$$\min_P \|y - y_r\| \tag{4}$$

$$Subjected\ to\text{:}\quad \dot{x}(t) = Ax(t) + Bu$$
$$y = Cx + Du$$

Where the reduced model

$$\dot{x_r} = A_r x_r + B_r u \tag{5}$$
$$y_r = C_r x_r + Du$$
$$\sigma(A_r) \subset \sigma(A)$$

Where $x \in R^n$ and $x_r \in R^m$, $m < n$ are the state vectors for the original (full) and the reduced systems, respectively. On the other hand, y and y_r are the system's output (response) for the complete and reduced system, respectively. Also, the retained eigenvalues must be a subset of the entire system eigenvalues.

Finally, the retained eigenvectors will be scaled eigenvectors corresponding to the retained eigenvalues of the whole system.

Substituting (1) in the state-space system in (4) will result into:

$$\dot{x} = P\tilde{A}P^{-1}x + Bu \tag{6}$$
$$y = Cx + Du$$

Where the new transformed matrix is given as follows:

$$\tilde{A} = \begin{bmatrix} A_r & A_c \\ 0 & A_o \end{bmatrix} \tag{7}$$

Now, by multiplying by P^{-1} from left, it results in:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \tag{8}$$
$$y = \tilde{C}\tilde{x} + Du$$

Where $\tilde{x}(t) = P^{-1}x$, $\tilde{B} = P^{-1}B$, $\tilde{C} = CP$ and $\tilde{x} = \begin{bmatrix} \tilde{x}_r \\ \tilde{x}_o \end{bmatrix}$

Hence, a partitioned system can be written as follows:

$$\begin{bmatrix} \dot{\tilde{x}}_r \\ \dot{\tilde{x}}_o \end{bmatrix} = \begin{bmatrix} A_r & A_c \\ 0 & A_o \end{bmatrix}\begin{bmatrix} \tilde{x}_r \\ \tilde{x}_o \end{bmatrix} + \begin{bmatrix} B_r \\ B_o \end{bmatrix} u \tag{9}$$
$$\tilde{y}_r = \begin{bmatrix} C_r & C_o \end{bmatrix}\begin{bmatrix} \tilde{x}_r \\ \tilde{x}_o \end{bmatrix} + Du$$

Where r indicates the retained eigenvalues/vectors pairs, o is omitted eigenvalues/vector pairs, A_r a Matrix contains the retained eigenvalues/vectors pairs, A_c is Coupling matrix, A_o Matrix containing the omitted eigenvalues/vectors pairs, and $\tilde{x}_r$ is the retained states, $\tilde{x}_o$ is the omitted states. Expanding the equations results in the following coupled equations:

$$\dot{\tilde{x}}_r = A_r\tilde{x}_r + A_c\tilde{x}_o + B_r u \tag{10}$$
$$\dot{\tilde{x}}_o = A_o\tilde{x}_o + B_o u$$

Using static condensation with $\dot{\tilde{x}}_o = 0$, the previous equations become:

$$\tilde{x}_o = -A_o^{-1}B_o u \tag{11}$$
$$\dot{\tilde{x}}_r = A_r\tilde{x}_r + (-A_c A_o^{-1}B_o + B_r)u$$

This will result in some deviation between the models' response due to the static condensation; however, properly tuning the coupling matrix Ac will help us to overcome this problem. An optimization procedure is needed to optimize the selection of the entries for the coupling of the matrix, *and the* optimization problem will be cast as follows:

$$\min_{A_c} \|y - y_r\| \qquad (12)$$

$$Subject\ to: \dot{\tilde{x}}_r = A_r\tilde{x}_r + (-A_c A_o^{-1} B_o + B_r)u$$

Where, $y = Cx$, is the full-order response, $y_r = C_r\tilde{x}_r - C_o A_o^{-1} B_o$, is the reduced-order response. Fig. (1) shows a block diagram for (11), which will be used later in the MATLAB simulation.

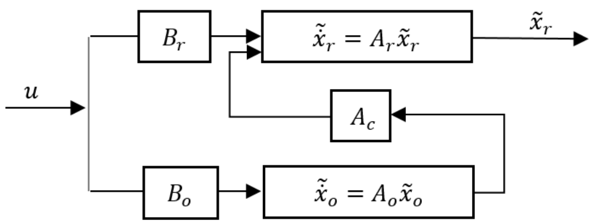

Figure 1. Matlab Model Reduction Diagram

An optimization technique is required to find the optimum coupling matrix A_c for the reduction method; we will be using (the BFGS) method to solve the optimization problem represented in equation (32).

IV. Optimization using (BFGS) method

In this part, we show a numerical-based solution to this optimization problem using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) method.

The flow chart in Fig. (2) illustrates the iterative algorithm that utilizes (BFGS) to provide an optimal coupling matrix. The algorithm starts with an initial guess for the coupling matrix, and the transformation matrix P is computed using (LMI) according to (3). After that, the response of both systems, the full and the reduced, will be calculated and compared under a step input using MATLAB. The convergence is monitored using the Euclidean norm; if the comparison result is acceptable, we have reached the optimum coupling matrix; otherwise, the search will continue.

V. Numerical application

The proposed structured method for formulating a minimum reduced-order system of a full-order dynamic system is verified and validated using computer simulations of a cantilevered beam structure, shown in Fig. (3) and Table (1), which has the beam properties. MATLAB® will be used as the primary tool to implement the proposed model reduction algorithm. All the dynamical model's data are acquired using ANSYS® software.

A. Cantilever Beam

Consider a cantilevered beam with a rectangular cross-sectional area, as in Fig. (2). The beam will be analyzed using the Finite Elements method partitioned into ten equal elements. The beam FE mathematical model is given as:

$$M\ddot{x} + C\dot{x} + Kx = f(t) \qquad (13)$$

Where M is mass, C is damping, and K is stiffness matrices acquired using ANSYS with Table I material properties.

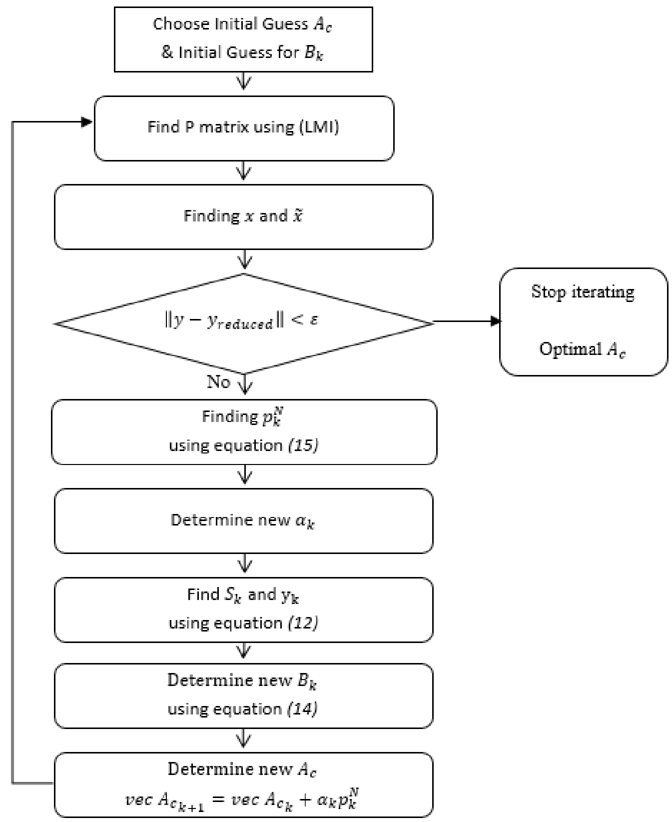

Figure 2. Optimization Algorithm using BFGS

Fig. (4) shows the step response for the proposed method's reduced-order model response (i.e., reduced-2) compared to the response of the full-order (i.e., full-10) system. The reduced order system seems to track the full order trajectory. In contrast, the reaction of fewer elements directly generated in the ANSYS model deviates from the targeted response (i.e., full-2). The reduced order system eigenvalues are a subset of the full order system full set eigenvalues. Furthermore, the reduced system eigenvalues corresponding eigenvectors are physically more meaningful.

Fig. (5) depicts the full-order system eigenvalues and their corresponding eigenvectors. Also, Fig. (6) shows the retained eigenvalues as a select subset from the original system, and the retained corresponding eigenvectors are scaled versions of the original system's eigenvectors. These eigenvectors are physically interpreted as the beam’s mode shapes, which are the beam deformations at these frequencies.

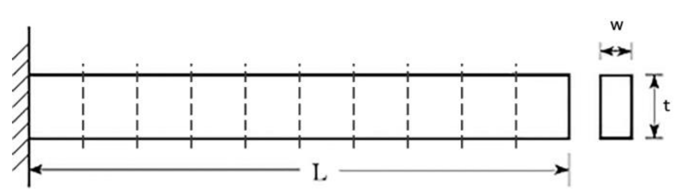

Figure 3. Ten Elements Cantilevered Beam

Table 1. Cantilever beam properties

Thickness (t)	Width (w)	Length (L)	Young's Modulus	Poisson Ratio	Density (ρ)
$0.25m$	$0.25m$	$4m$	$2 \times 10^{11}\ N/m^2$	0.32	$7850\ kg/m^3$

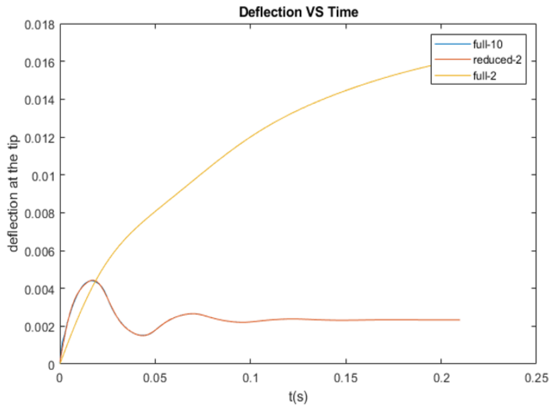

Figure 4. System's Step Response

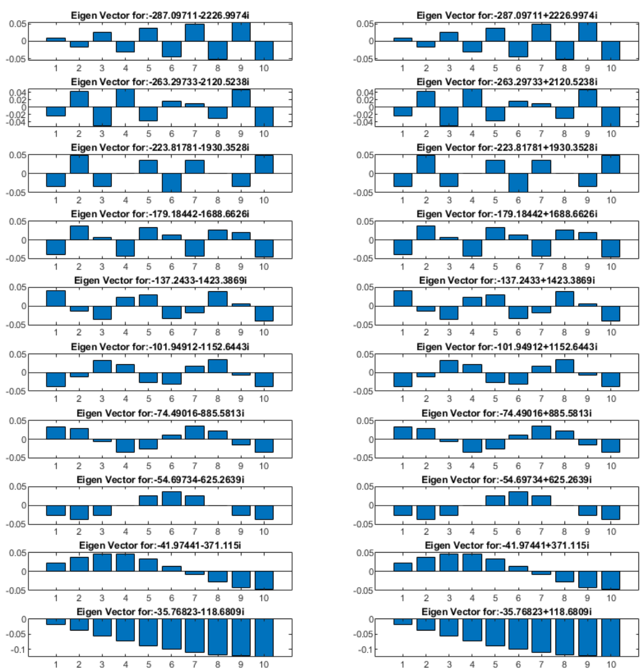

Figure 5. Full Order System, first five modes

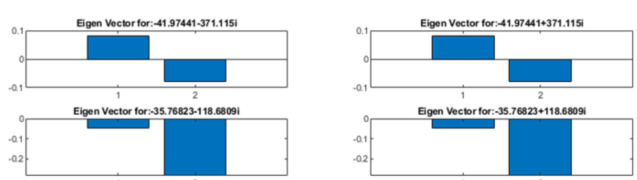

Figure 6. Reduced Order System Frequencies and Modeshapes

From the frequency In the response function in Fig. (7), the reduced system retains the first two modes of the original system, as this is the system's capacity with such an order. Based on the results obtained and the graphs, the proposed method has successfully reduced the cantilever beam system to a lower-order system with high accuracy.

VI. Conclusions

A novel model reduction problem solution is successfully formulated and cast as an optimization problem with minimum reducible order using LMI for the similarity terms. Furthermore, the BFGS method is used to solve the optimization problem by finding the required coupling matrix.

A cantilever beam application was selected to verify and validate the proposed method. The simulated results showed that the achieved step tracking of response for the reduced system compared to the ANSYS full-scale system is satisfactory.

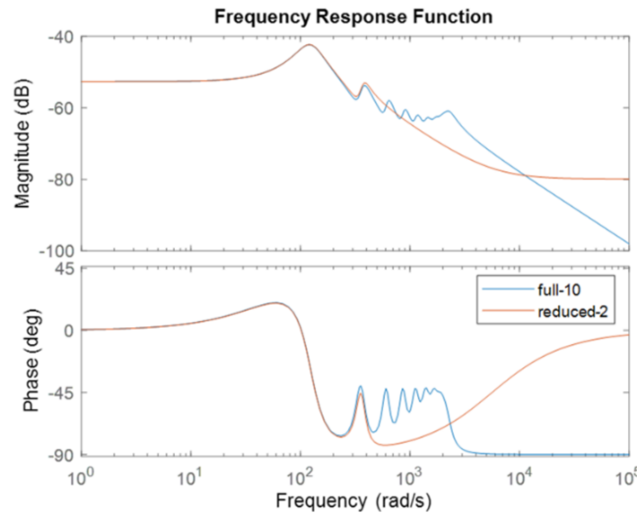

Figure 7. Frequency Response Function (BODE)

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