Decentralized Position Regulation for a Series of Time-Varying Mass-Spring-Damper Systems Via Solutions of Riccati Equations

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Abstract. In this study, we introduce a decentralized control strategy for regulating the position of a series of mass-spring-damper systems characterized by parameters that change over time. We start with establishing a dynamic model for each interconnected subsystem, leading to a time-varying model as the outcome. Given the focus on optimal optimization challenges within time-varying systems, it is essential to determine the solution to the time-varying Riccati differential equation in advance. We derive a mathematical proof that the entire system is exponentially stable, ensuring a predefined level of stability. To validate our approach, computer simulations were performed on a three-mass system that is interconnected and experiences time-varying parameters. The simulation outcomes indicate that the positions of all masses revert to their initial states at an exponential rate, thereby confirming the practicality of the control method we have proposed.

Keywords: decentralized regulation, time-varying system, Riccati equation

1 Introduction

Most conventional systems are characterized by differential equations, with mass-spring-damper systems being a common example of second-order systems. When masses are linked by springs and dampers in a series, the dynamics of each mass are influenced by adjacent masses, excluding the first and last ones. As the number of connected masses grows, the computational demands and transfer costs escalate significantly. This challenge can be mitigated by implementing decentralized control strategies to alleviate these burdens. Furthermore, the parameters of these interconnected systems may not remain constant but instead vary over time, necessitating optimization methods distinct from those used for linear time-invariant (LTI) systems. There is an abundance of research dedicated to decentralized control designs [1-5].

For decentralized control of systems with time-varying characteristics, a minimum-energy approach was proposed in [6] to address time-varying coverage control issues, transforming the coverage cost into a constrained optimization problem amenable to decentralized solutions. In [7], a novel approach was introduced for managing agents within a time-varying context, incorporating new nodes and branches in the graph, using machine learning techniques. The work in [8] focused on decentralized algorithms for coordinating the motion of autonomous vehicles to counteract the impacts of time-varying parameters. Additionally, a decentralized linear time-varying model predictive control strategy for unmanned aerial vehicles was put forward in [9].

In this paper, we introduce a decentralized stabilization control method for linear time-varying interconnected systems. Utilizing optimal control techniques based on a performance index, we have developed a decentralized time-varying feedback law tailored to subsystems. Given the time-varying nature of the system under consideration, time-varying Riccati differential equations must be resolved. The backward Euler's method is employed to derive solutions from a sufficiently large time point back to the initial time, using estimated terminal values. Once the initial conditions are established, these Riccati equations are integrated into the overall system for implementation. The stability of the system is rigorously demonstrated to be exponentially stable with a predetermined level of stability. It is crucial to note that if the interconnected systems adhere to the specified structural constraints, there will be no limitations on the intensity of interactions among the subsystems, and the prescribed exponential stability of the entire system will be maintained.

2 The model Description

We consider a series of mass-spring-damper systems containing s masses, as shown in Figure 1.

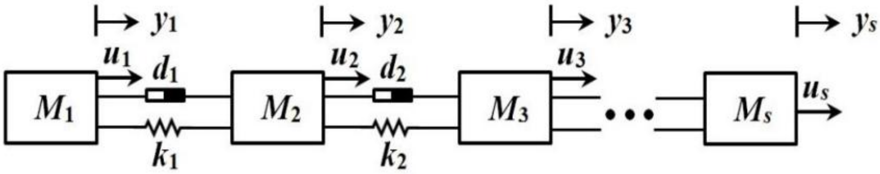

Fig. 1. A series of mass-spring-damper systems.

From Figure 1, M_i is the mass (kg), y_i is the position (m), k_i is the spring constant (N/m), d_i is the damping coefficient (N·s/m) and u_i is the input (N). In this configuration, there are s masses, s positions, s inputs, $(s-1)$ dampers, and $(s-1)$ springs. Now, we derive the dynamic equations of this interconnected system by Newton's law. First, the dynamic equation for the first mass is described by

$$M_1\ddot{y}_1 = d_1(\dot{y}_2 - \dot{y}_1) + k_1(y_2 - y_1) + u_1 \tag{1}$$

then

$$M_1\ddot{y}_1 + d_1\dot{y}_1 + k_1 y_1 = d_1\dot{y}_2 + k_1 y_1 + u_1 \tag{2}$$

Next, we derive the dynamic equation of the second mass as

$$M_2\ddot{y}_2 = d_2(\dot{y}_3 - \dot{y}_2) + k_2(y_3 - y_2) - d_1(\dot{y}_2 - \dot{y}_1) + k_1(y_2 - y_1) + u_2 \tag{3}$$

After some simplifications, we obtain

$$M_2\ddot{y}_2 + (d_1 + d_2)\dot{y}_2 + (k_1 + k_2)y_2 = d_2\dot{y}_3 + k_2 y_3 + d_1\dot{y}_1 + k_1 y_1 + u_2 \tag{4}$$

When (4) is generalized to the i-th mass where i = 2, 3, ..., s–1, the generalized equation is

$$M_i\ddot{y}_i + (d_{i-1} + d_i)\dot{y}_i + (k_{i-1} + k_i)y_i = d_{i-1}\dot{y}_{i-1} + k_{i-1}y_{i-1} + d_i\dot{y}_{i+1} + k_i y_{i+1} + u_i \tag{5}$$

For the last mass, the equation is

$$M_s\ddot{y}_s + d_{s-1}\dot{y}_s + k_{s-1}y_s = d_{s-1}\dot{y}_{s-1} + k_{s-1}y_{s-1} + u_s \tag{6}$$

Until now, we have obtained three types of differential equations of the masses and these equations are transformed into state-space expressions. Let $x_{i1} = y_i$ and $x_{i2} = \dot{y}_i$, then

$$\begin{aligned}\dot{x}_i = \begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -\dfrac{k_{i-1}+k_i}{M_i} & -\dfrac{d_{i-1}+d_i}{M_i}\end{bmatrix}x_i + \begin{bmatrix}0\\ \dfrac{1}{M_i}\end{bmatrix}\left(u_i + \begin{bmatrix}k_{i-1} & d_{i-1}\end{bmatrix}\begin{bmatrix}x_{(i-1)1}\\ \dot{x}_{(i-1)2}\end{bmatrix} + \begin{bmatrix}k_i & d_i\end{bmatrix}\begin{bmatrix}x_{(i+1)1}\\ \dot{x}_{(i+1)2}\end{bmatrix}\right)\\ &= A_i x_i + B_i\left(u_i + \begin{bmatrix}k_{i-1} & d_{i-1}\end{bmatrix}x_{i-1} + \begin{bmatrix}k_i & d_i\end{bmatrix}x_{i+1}\right),\ i = 1,\ 2,\ ...,\ s\end{aligned} \tag{7}$$

where $d_0 = k_0 = d_s = k_s = 0$. We can rewrite (6) as

$$\dot{x}_i = A_i x_i + B_i\left(u_i + \sum_{j=1}^{s} q_{ij}x_j\right) \tag{8}$$

For the parameter q_{ij}, we have the following descriptions.

(1) When i = 1, then $q_{12} = [k_1\ \ d_1]$; $q_{1j} = [0\ \ 0]$ while $j \neq 2$.

(2) When i = 2, 3, ..., s – 1, then $q_{i(i-1)} = [k_{i-1}\ \ d_{i-1}]$; $q_{i(i+1)} = [k_i\ \ d_i]$; otherwise $q_{ij} = [0\ \ 0]$.

(3) When $i = s$, then $q_{s(s-1)} = [k_{s-1}\ \ d_{s-1}]$; otherwise $q_{sj} = [0\ \ 0]$.

Now if the coefficients of the system are time-varying, (8) becomes

$$\dot{x}_i(t) = A_i(t)x_i(t) + B_i(t)\left(u_i(t) + \sum_{j=1}^{s} q_{ij}(t)x_j(t)\right) \tag{9}$$

The following are some statements and assumptions.

Assumption 1: Each subsystem's time-varying parameters $A_i(t)$ and $B_i(t)$ are assumed to have suitable dimensions and are considered to be uniformly piece-wise controllable.

Assumption 2: It is assumed that all system parameter matrices are norm-bounded, using either Euclidean norms or induced Euclidean norms.

Assumption 3: $q_{ij}(t)$ is the time-varying parameter matrix, yielding:

$$\sup_t \left\| q_{ij}(t) \right\| = h_{ij} \geq 0 \tag{10}$$

3 The Controller Design

Now, we will examine the following optimization issue as

$$\min J_i = \int_0^{\infty} \left[\left(e^{\xi t} \boldsymbol{x}_i\right)^T Q_i \left(e^{\xi t} \boldsymbol{x}_i\right) + \left(e^{\xi t} u_i\right)^T R_i \left(e^{\xi t} u_i\right) \right] dt \tag{11}$$

This minimization index is a modified linear optimal regulator (MLQR) approach that can make a stable closed-loop system with a high-level control performance. The time-varying weighting matrices $Q_i(t)$ and $R_i(t)$ are both assumed to be symmetric positive definite matrices. Besides, in our method, the weighting matrices can also be time-varying. The optimal solution $u_i(t)$ is

$$u_i(t) = K_i(t)\boldsymbol{x}_i(t) = -R_i(t)^{-1} B_i(t)^T P_i(t)\boldsymbol{x}_i(t) \tag{12}$$

The above time-varying matrix $P_i(t)$ is the solution of a time-varying Riccati equation,

$$\dot{P}_i(t) = P_i(t)B_i(t)R_i(t)^{-1}B_i(t)^T P_i(t) - (A_i(t)+\xi I_i)^T P_i(t) - P_i(t)(A_i(t)+\xi I_i) - Q_i(t) \tag{13}$$

The parameter I_i in (13) is an identity matrix and $P_i(t)$ is a time-varying, symmetric positive definite matrix. Different from time-invariant cases, we have $\dot{P}_i(t) \neq 0$. In order to determine the feedback input, $P_i(t)$ must be pre-calculated to complete equation (12). From our previous work [10], if

$$\alpha_i \geq \frac{1}{2}\left\{ \left(\sum_{j=1}^{s} \frac{h_{ij}{}^2}{\lambda_{\min}(R_i^{-1})\lambda_{\min}\left(Q_j/s\right)} \right) + 1 \right\} \tag{14}$$

and

$$K_i^*(t) = -\alpha_i R_i(t)^{-1} B_i(t)^T P_i(t) \tag{15}$$

then the global system (8) will be asymptotically stable with a prescribed convergent rate.

4 Computer Simulations

A 3-mass interconnected mass-spring-damper system is considered and the parameters are as follows.

$$M_1 = M_2 = 1 + 0.1\cos 2t;\ M_3 = 1 + 0.2\cos 4t;\ d_1 = d_2 = 0.2 + 0.2\sin t;\ k_1 = k_2 = 1 + 0.2\sin t \quad (16)$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\dfrac{1+0.2\sin t}{1+0.1\cos 2t} & -\dfrac{0.2+0.2\sin t}{1+0.1\cos 2t} \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ \dfrac{1}{1+0.1\cos 2t} \end{bmatrix};$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ -\dfrac{2+0.4\sin t}{1+0.1\cos 2t} & -\dfrac{0.4+0.4\sin t}{1+0.1\cos 2t} \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ -\dfrac{1}{1+0.1\cos 2t} \end{bmatrix} \quad (17)$$
$$A_3 = \begin{bmatrix} 0 & 1 \\ -\dfrac{1+0.2\sin t}{1+0.2\cos 4t} & -\dfrac{0.2+0.2\sin t}{1+0.2\cos 4t} \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ \dfrac{1}{1+0.2\cos 4t} \end{bmatrix}$$

The performance indices are simply chosen as $Q_i = 3I$ and $R_i = 1$, $i = 1, 2, 3$. The prescribed stability index is chosen as $\xi = 1$ and the Riccati Equation (13) becomes

$$\dot{P}_i(t) = P_i(t)B_i(t)B_i(t)^T P_i(t) - (A_i(t) + I_2)^T P_i(t) - P_i(t)(A_i(t) + I_2) - 3I_2 \quad (18)$$

Here we pre-calculate the time-varying values of the solutions for Riccati equations. First, we make a guess for the boundary values of $P_i(t)$, choosing a terminal time as $t_f = 10$ (sec) and

$$P_1(10) = P_2(10) = P_3(10) = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix} \quad (19)$$

The trajectories of $P_1(t)$ through $P_3(t)$ are plotted in Figure 2 through Figure 4 by Euler's backward method with the time step $h = 0.001$. Figure 5 is the determinants of $P_1(t)$ through $P_3(t)$ to show the positive definiteness.

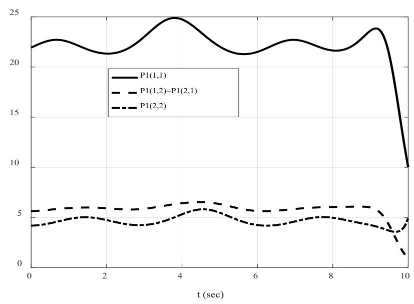

Fig. 2. Trajectories of $P_1(t)$.

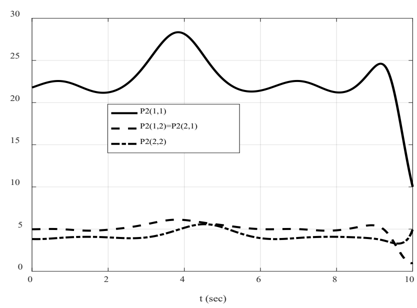

Fig. 3. Trajectories of $P_2(t)$.

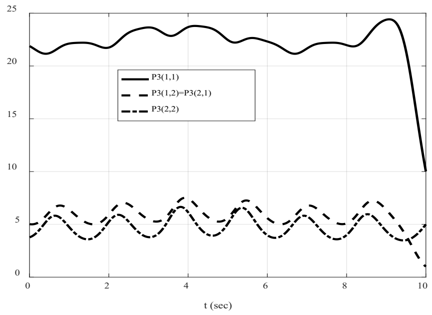

Fig. 4. Trajectories of $P_3(t)$.

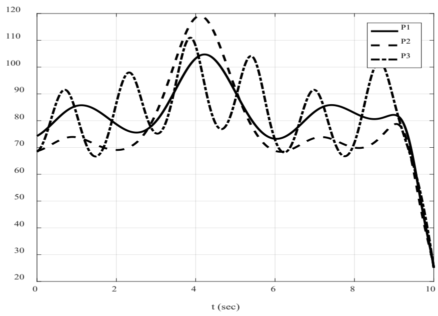

Fig. 5. Determinants of $P_i(t)$.

If the boundary condition is chosen as

$$P_1(10) = P_2(10) = P_3(10) = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} \tag{20}$$

The corresponding figures are plotted in Figure 6 through Figure.9.

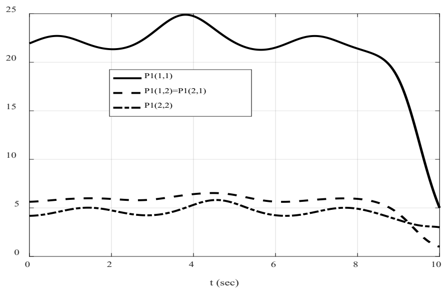

Fig. 6. Trajectories of $P_1(t)$.

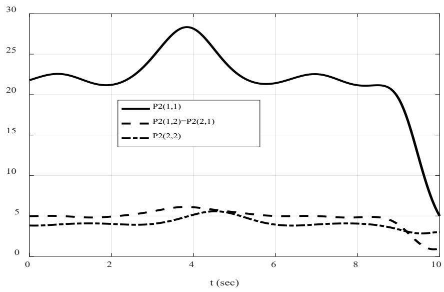

Fig. 7. Trajectories of $P_2(t)$.

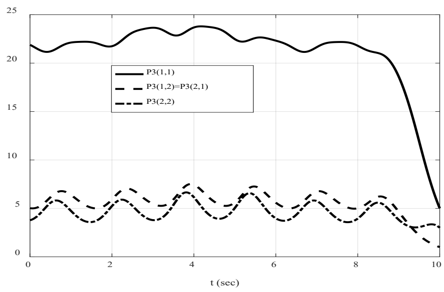

Fig. 8. Trajectories of $P_3(t)$.

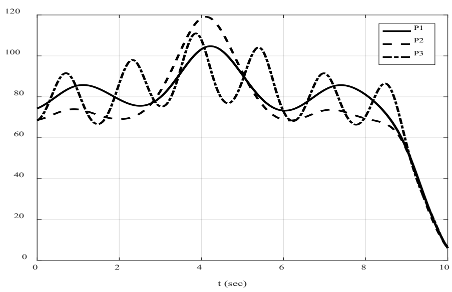

Fig. 9. Determinants of $P_i(t)$.

Though the boundary conditions are different, they are the same at the time interval $t \in [0, 6]$ and also positive definite. Now, the interconnection parameter matrix is

$$\left[q_{ij}\right]=\begin{bmatrix} 0 & 0 & 1+0.2\sin t & 0.2+0.2\sin t & 0 & 0 \\ 1+0.2\sin t & 0.2+0.2\sin t & 0 & 0 & 1+0.2\sin t & 0.2+0.2\sin t \\ 0 & 0 & 1+0.2\sin t & 0.2+0.2\sin t & 0 & 0 \end{bmatrix} \quad (21)$$

By the control laws in (15), the responses of the masses with the initial condition [2 2 –2 2 –2] are plotted in Figure 10 through Figure 12. These figures also show the trajectories of masses without control input in which the responses are oscillating. Besides, in the captions, the subscript c denotes responses with control, and subscript n denotes responses without control.

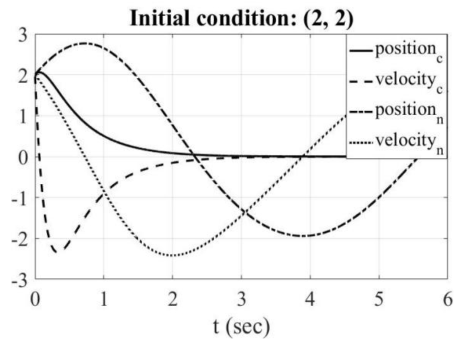

Fig. 10. Trajectories of Mass-1.

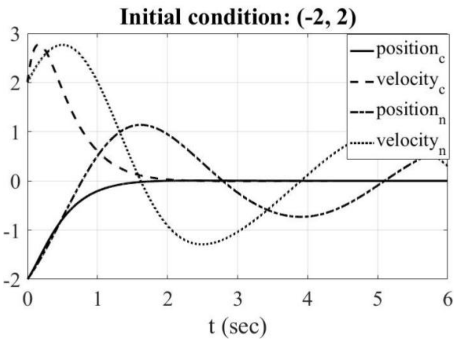

Fig. 11. Trajectories of Mass-2.

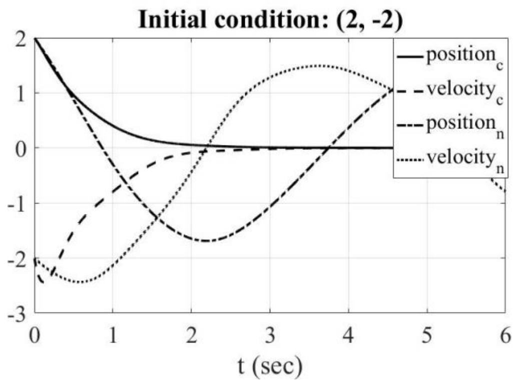

Fig. 12. Trajectories of Mass-3.

5 Conclusion

The mass-spring-damper system is very common in the application field, and many systems can have similar properties. In this paper, we first make an interconnection model and then derive a decentralized regulation design to make each mass approach its setup state exponentially. The solutions for the time-varying Riccati equations are solved in advance, and these results are used to generate the decentralized control inputs. Computer simulations of a 3-mass system are also conducted. Future works are extending this approach to robotic systems, water treatment plants, and multi-machine power systems.

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