Kernel Functions for Support Vector Machines

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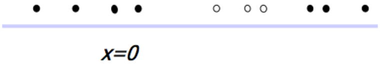
LECTURE 11

Slides adapted from Jerry Zhu

Can you solve this with linear separator?



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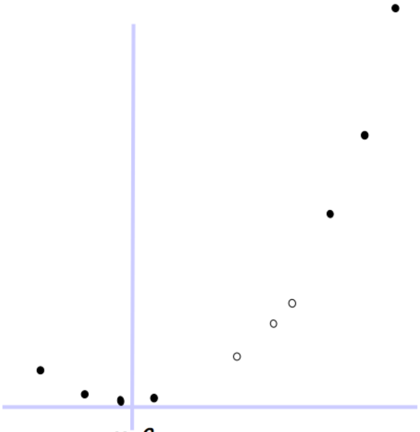

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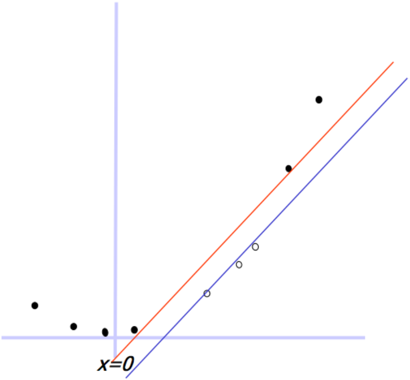
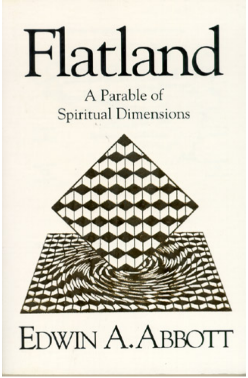


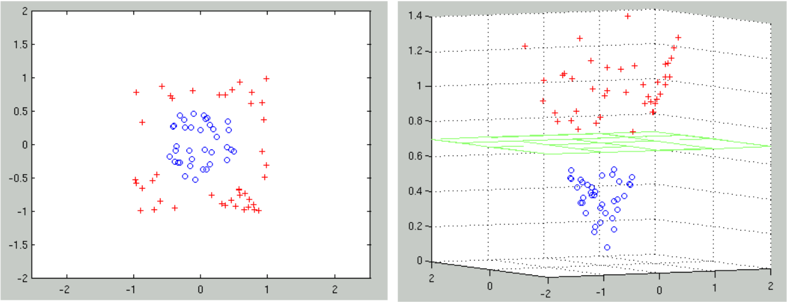
Adding another dimension



Behold yon miserable creature. That Point is a Being like ourselves, but confined to the non-dimensional Gulf. He is himself his own World, his own Universe; of any other than himself he can form no conception; he knows not Length, nor Breadth, nor Height, for he has had no experience of them; he has no cognizance even of the number Two; nor has he a thought of Plurality, for he is himself his One and All, being really Nothing. Yet mark his perfect self-contentment, and hence learn this lesson, that to be self-contented is to be vile and ignorant, and that to aspire is better than to be blindly and impotently happy.

Problems get easier in higher dimensions

$$(x_1, x_2) \Rightarrow (x_1, x_2, \sqrt{x_1^2 + x_2^2})$$



What's special about SVMs?

$$\max_{\vec{\alpha}} \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{i=1}^{m} \alpha_i \alpha_j y_i y_j (\vec{x}_i \cdot \vec{x}_j) \tag{1}$$

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- Kernels!

What's a kernel?

- A function $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ is a kernel over $\mathcal{X}$.
- This is equivalent to taking the dot product $\langle \phi(x_1), \phi(x_2) \rangle$ for some mapping
- **Mercer's Theorem**: So long as the function is continuous and symmetric, then K admits an expansion of the form

$$K(x, x') = \sum_{n=0}^{\infty} a_n \phi_n(x) \phi_n(x') \tag{2}$$

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- The computational cost is just in computing the kernel

Polynomial Kernel

$$K(x, x') = (x \cdot x' + c)^d \tag{3}$$

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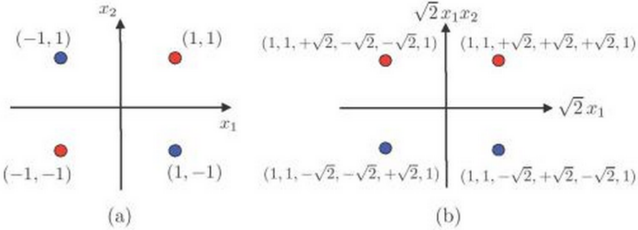
When $d = 2$:



Gaussian Kernel

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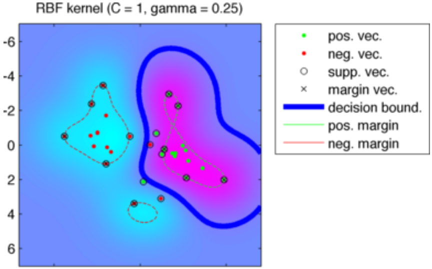

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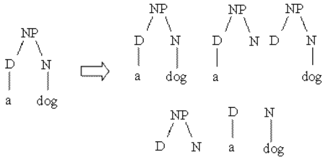

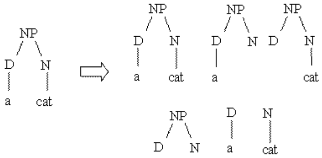

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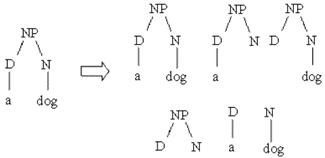

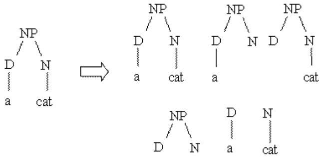

3/5 structures match, so tree kernel returns .6

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- Rademacher complexity for a kernel with radius Λ and data with radius r: $S \subset \{x : K(x,x) \leq r^2\}$, $H = \{x \mapsto w \cdot \phi(x) : \|w\| \leq \Lambda\}$

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- Proof requires real analysis

Margin learnability

- With probability $1 - \delta$:

$$R(h) \leq \hat{R}_\rho(h) + 2\sqrt{\frac{r^2\Lambda^2/\rho^2}{m}} + \sqrt{\frac{\log \frac{1}{\delta}}{2m}} \tag{7}$$

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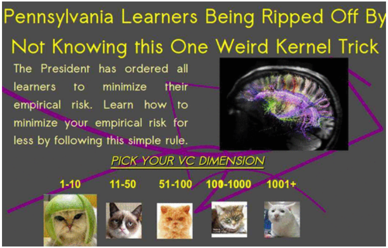
- So if you can find a simple kernel representation that induces a margin, use it!
- ...so long as you can handle the computational complexity

How does it effect optimization

- Replace all dot product with kernel evaluations $K(x_1, x_2)$
- Makes computation more expensive, overall structure is the same
- Try linear first!

Recap



- This completes our discussion of SVMs
- Workhorse method of machine learning
- Flexible, fast, effective

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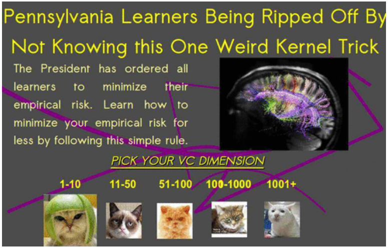

- This completes our discussion of SVMs
- Workhorse method of machine learning
- Flexible, fast, effective
- Kernels: applicable to wide range of data, inner product trick keeps method simple