Conditional Probability

Introduction to Data Science Algorithms
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SLIDES ADAPTED FROM DAVE BLEI AND LAUREN HANNAH

Administrivia

- Autograder
- Office Hours

Context

- Data science is often worried about “if-then” questions
  - If my e-mail looks like this, is it spam?
  - If I buy this stock, will my portfolio improve?
- Since data science uses the language of probabilities, we need *conditional* probabilities (continuing probability intro)
- Also need to **combine** distributions

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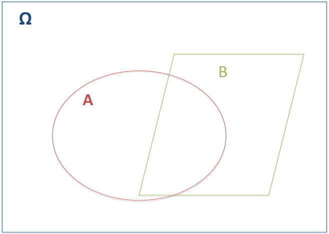

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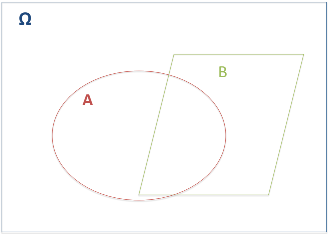

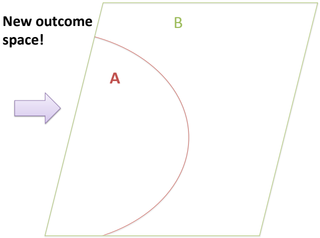

Independence (Reminder)

Random variables X and Y are independent if and only if $P(X = x, Y = y) = P(X = x)P(Y = y)$. How does this interact with conditional probabilities?

Conditional probabilities equal unconditional probabilities with independence:

- $P(X = x \mid Y) = P(X = x)$
- *Knowing Y tells us nothing about X*

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$$= \frac{1}{9}$$

Combining Distributions

- Somtimes distributions you have aren't what you need
  - Conditional $\longrightarrow$ joint (chain)
  - Reverse conditional direction (Bayes')

The chain rule

- The definition of conditional probability lets us derive the *chain rule*, which let's us define the joint distribution as a product of conditionals:

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- For example, let Y be a disease and X be a symptom. We may know $P(X|Y)$ and $P(Y)$ from data. Use the chain rule to obtain the probability of having the disease and the symptom.
- In general, for any set of N variables

$$
P(X_1,\ldots,X_N) = \prod_{n=1}^{N} P(X_n|X_1,\ldots,X_{n-1})
$$

Bayes' Rule

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1. Start with $P(A|B)$
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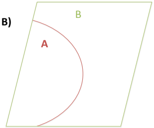

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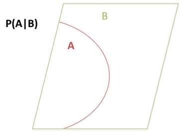

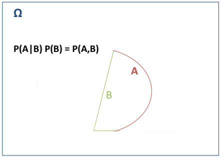

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