Department of Computer Science
UNIVERSITY OF COLORADO **BOULDER**

Mathematical Foundations

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Slides adapted from Dave Blei and Lauren Hannah

By the end of today . . .

- You'll be able to apply the concepts of distributions, independence, and conditional probabilities
- You'll be able to derive joint, marginal, and conditional probabilites from each other
- You'll be able to compute expectations and entropies

Outline

Preface: Why make us do this?

- Probabilities are the language we use to describe data
- A reasonable (but geeky) definition of nlp is how to get probabilities we care about from text
- Later classes will be about how to do this for different probability models of text
- But first, we need key definitions of probability (and it makes more sense to do it all at once)

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- So pay attention!

The Statistical Revolution in NLP

- Speech recognition
- Machine translation
- Part of speech tagging
- Parsing

Solution?

They share the same solution: probabilistic models.

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BS

Eugene Charniak refers to the time before statistics in nlp as "BS"; and nothing actually worked.

Random variable

- Probability is about *random variables.*
- A random variable is any “probabilistic” outcome.
- For example,
    - The flip of a coin
    - The height of someone chosen randomly from a population
- We’ll see that it’s sometimes useful to think of quantities that are not strictly probabilistic as random variables.
    - The temperature on 11/12/2013
    - The temperature on 03/04/1905
    - The number of times “streetlight” appears in a document

Random variable

- Random variables take on values in a *sample space*.
- They can be *discrete* or *continuous*:
  - Coin flip: $\{H, T\}$
  - Height: positive real values $(0, \infty)$
  - Temperature: real values $(-\infty, \infty)$
  - Number of words in a document: Positive integers $\{1, 2, \dots\}$
- We call the outcomes *events*.
- Denote the random variable with a capital letter; denote a realization of the random variable with a lower case letter.
- E.g., X is a coin flip, x is the value (H or T) of that coin flip.

Discrete distribution

- A discrete distribution assigns a probability to every event in the sample space
- For example, if X is an (unfair) coin, then

$$\begin{aligned} P(X = H) &= 0.7 \\ P(X = T) &= 0.3 \end{aligned}$$

- And probabilities have to be greater than 0
- Probabilities of disjunctions are sums over part of the space. E.g., the probability that a die is bigger than 3:

$$P(D > 3) = P(D = 4) + P(D = 5) + P(D = 6)$$

- The probabilities over the entire space must sum to one

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Events

An *event* is a set of outcomes to which a probability is assigned

- drawing a black card from a deck of cards
- drawing a King of Hearts

Intersections and unions:

- Intersection: drawing a red and a King

$$P(A \cap B) \tag{1}$$

- Union: drawing a spade or a King

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{2}$$

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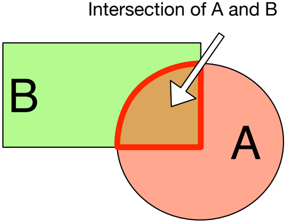

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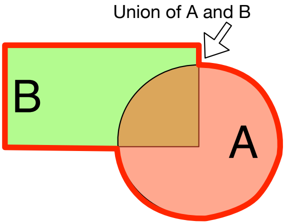


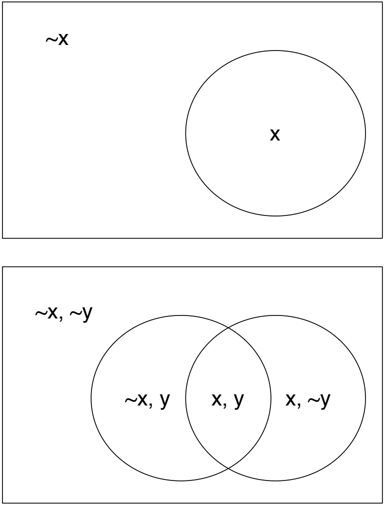
Joint distribution

- Typically, we consider collections of random variables.
- The joint distribution is a distribution over the configuration of all the random variables in the ensemble.
- For example, imagine flipping 4 coins. The joint distribution is over the space of all possible outcomes of the four coins.

$$
\begin{aligned}
P(HHHH) &= 0.0625 \\
P(HHHT) &= 0.0625 \\
P(HHTH) &= 0.0625 \\
&\dots
\end{aligned}
$$

- You can think of it as a single random variable with 16 values.

Visualizing a joint distribution



Marginalization

If we are given a joint distribution, what if we are only interested in the distribution of one of the variables?

We can compute the distribution of $P(X)$ from $P(X, Y, Z)$ through *marginalization*:

$$\begin{aligned}\sum_y \sum_z P(X, Y = y, Z = z) &= \sum_y \sum_z P(X) P(Y = y, Z = z | X) \\ &= P(X) \sum_y \sum_z P(Y = y, Z = z | X) \\ &= P(X)\end{aligned}$$

Marginalization (from Leyton-Brown)

Joint distribution

temperature (T) and weather (W)

	T=Hot	T=Mild	T=Cold
W=Sunny	.10	.20	.10
W=Cloudy	.05	.35	.20

Marginalization allows us to compute distributions over smaller sets of variables:

- $P(X,Y) = \sum_z P(X,Y,Z=z)$
- Corresponds to summing out a table dimension
- New table still sums to 1

- Marginalize out weather
- Marginalize out temperature

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Conditional Probabilities

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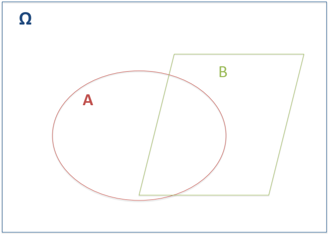

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$$P(A > 3 \cap B + A = 6) =$$
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$$= \frac{1}{9}$$

Outline

The chain rule

- The definition of conditional probability lets us derive the *chain rule*, which let's us define the joint distribution as a product of conditionals:

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- For example, let Y be a disease and X be a symptom. We may know $P(X|Y)$ and $P(Y)$ from data. Use the chain rule to obtain the probability of having the disease and the symptom.
- In general, for any set of N variables

$$P(X_1, \dots, X_N) = \prod_{n=1}^{N} P(X_n | X_1, \dots, X_{n-1})$$

Bayes' Rule

What is the relationship between $P(A|B)$ and $P(B|A)$?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

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2. Change outcome space from B to Ω
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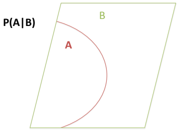

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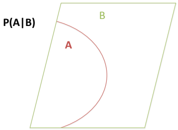

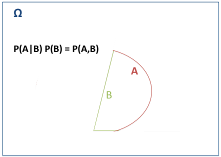

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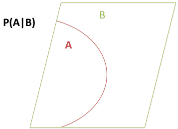

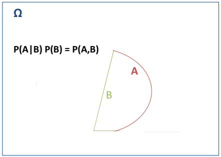

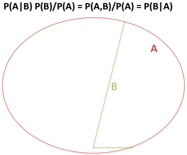

Independence

Random variables X and Y are independent if and only if $P(X = x, Y = y) = P(X = x)P(Y = y)$.

Conditional probabilities equal unconditional probabilities with independence:

- $P(X = x \mid Y) = P(X = x)$
- *Knowing Y tells us nothing about X*

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Mathematical examples:

- If I draw two socks from my (multicolored) laundry, is the color of the first sock independent from the color of the second sock?
- If I flip a coin twice, is the first outcome independent from the second outcome?

Independence

Intuitive Examples:

- Independent:
  - you use a Mac / the Hop bus is on schedule
  - snowfall in the Himalayas / your favorite color is blue

Independence

Intuitive Examples:

- Independent:
  - you use a Mac / the Hop bus is on schedule
  - snowfall in the Himalayas / your favorite color is blue
- Not independent:
  - you vote for Mitt Romney / you are a Republican
  - there is a traffic jam on 25 / the Broncos are playing

Independence

Sometimes we make convenient assumptions.

- the values of two dice
- the value of the first die and the sum of the values
- whether it is raining and the number of taxi cabs
- whether it is raining and the amount of time it takes me to hail a cab
- the first two words in a sentence

Outline

Continuous random variables

- We've only used discrete random variables so far (e.g., dice)
- Random variables can be continuous.
- We need a *density* $p(x)$, which *integrates* to one.
  E.g., if $x \in \mathbb{R}$ then
  $$\int_{-\infty}^{\infty} p(x)dx = 1$$
- Probabilities are integrals over smaller intervals. E.g.,
  $$P(X \in (-2.4, 6.5)) = \int_{-2.4}^{6.5} p(x)dx$$
- Notice when we use P, p, X, and x.

The Gaussian distribution

- The Gaussian (or Normal) is a continuous distribution.

$$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

- The density of a point x is proportional to the negative exponentiated half distance to μ scaled by σ^2.
- μ is called the *mean*; σ^2 is called the *variance*.

Outline

Expectation

An *expectation* of a random variable is a weighted average:

$$\mathrm{E}[f(X)] = \sum_{x=1}^{\infty} f(x)\, p(x) \qquad \text{(discrete)}$$

$$= \int_{-\infty}^{\infty} f(x)\, p(x)\, dx \qquad \text{(continuous)}$$

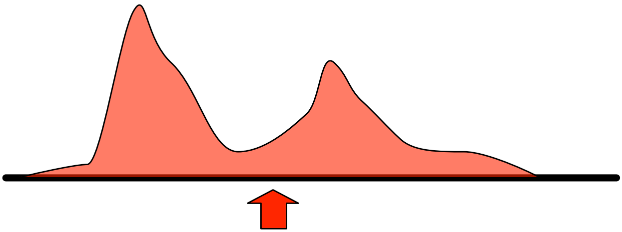
Expectation

Expectations of constants or known values:

- $\mathrm{E}[a] = a$
- $\mathrm{E}[Y \mid Y = y] = y$

Expectation Intuition

- Average or outcome (might not be an event: 2.4 children)
- Center of mass



- "Fair Price" of a wager

Expectation of die / dice

What is the expectation of the roll of die?

Expectation of die / dice

What is the expectation of the roll of die?

One die

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} =$$

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What is the expectation of the sum of two dice?

Expectation of die / dice

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What is the expectation of the sum of two dice?

Two die

$$2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} =$$

Expectation of die / dice

What is the expectation of the roll of die?

One die

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

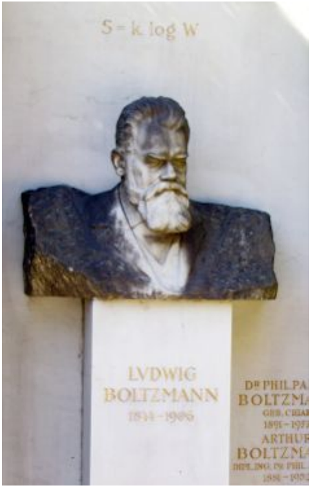
What is the expectation of the sum of two dice?

Two die

$$2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} = 7$$

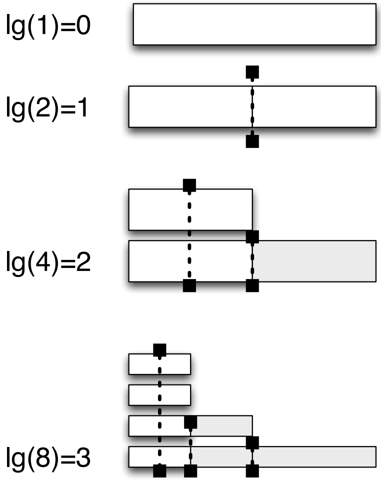
Entropy

- Measure of disorder in a system
- In the real world, entroy in a system tends to increase
- Can also be applied to probabilities:
  - Is one (or a few) outcomes certain (low entropy)
  - Are things equiprobable (high entropy)
- In data science
  - We look for features that allow us to *reduce* entropy (decision trees)
  - All else being equal, we seek models that have *maximum* entropy (Occam's razor)



Aside: Logarithms

- $\lg(x) = b \Leftrightarrow 2^b = x$
- Makes big numbers small
- Way to think about them: cutting a carrot



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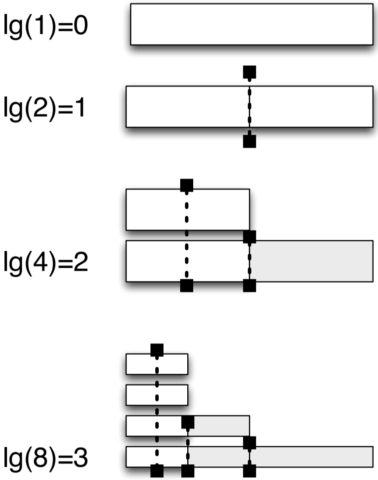
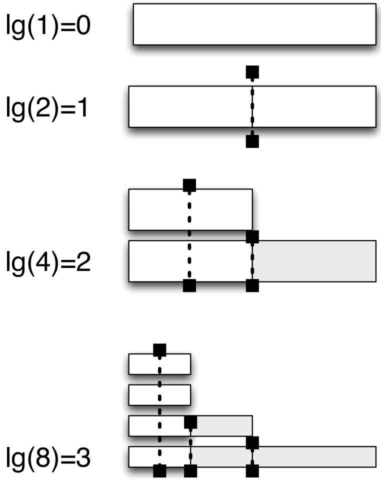


Aside: Logarithms

- $\lg(x) = b \Leftrightarrow 2^b = x$
- Makes big numbers small
- Way to think about them: cutting a carrot
- Negative numbers?
- Non-integers?



Entropy

Entropy is a measure of uncertainty that is associated with the distribution of a random variable:

$$
\begin{aligned}
H(X) &= -\mathrm{E}\left[\lg(p(X))\right] \\
&= -\sum_x p(x) \lg(p(x)) && \text{(discrete)} \\
&= -\int_{-\infty}^{\infty} p(x) \lg(p(x))\, dx && \text{(continuous)}
\end{aligned}
$$

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Does not account for the values of the random variable, only the spread of the distribution.

- $H(X) \geq 0$
- uniform distribution = highest entropy, point mass = lowest
- suppose $P(X=1)=p,\ P(X=0)=1-p$ and $P(Y=100)=p,\ P(Y=0)=1-p$: X and Y have the same entropy

Outline

Independence

Example: two coins, C_1, C_2 with

$$P(H | C_1) = 0.5, \quad P(H | C_2) = 0.3$$

Suppose that I randomly choose a number $Y \in \{1,2\}$ and take coin C_Z. I flip it twice, with results (X_1, X_2)

- are X_1 and X_2 independent?
- what about if I know Y?

Independence

Bayes Rule

There's a test for Boogie Woogie Fever (BWF). The probability of geting a positive test result given that you have BWF is 0.8, and the probability of getting a positive result given that you do not have BWF is 0.01. The overall incidence of BWF is 0.01.

1. What is the marginal probability of getting a positive test result?
2. What is the probability of having BWF given that you got a positive test result?

Bayes Rule

Conditional Probabilities

One coin in a collection of 65 has two heads. The rest are fair. If a coin, chosen at random from the lot and then tossed, turns up heads 6 times in a row, what is the probability that it is the two-headed coin?

Conditional Probabilities

Entropy

What's the entropy of

- One die?
- The sum of two dice?

Entropy

Wrap up

- Probabilities are the language of modern nlp
- You'll need to manipulate probabilities and understand conditioning and independence
- But not next week: deterministic algorithms for morphology