Department of Computer Science
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Logistic Regression

Introduction to Data Science Algorithms

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SLIDES ADAPTED FROM WILLIAM COHEN

Gradient for Logistic Regression

To ease notation, let's define

$$\pi_i = \frac{\exp \beta^T x_i}{1 + \exp \beta^T x_i} \tag{1}$$

Our objective function is

$$\ell = \sum_i \log p(y_i \,|\, x_i) = \sum_i \ell_i = \sum_i \begin{cases} \log \pi_i & \text{if } y_i = 1 \\ \log(1 - \pi_i) & \text{if } y_i = 0 \end{cases} \tag{2}$$

Taking the Derivative

Apply chain rule:

$$\frac{\partial \ell}{\partial \beta_j} = \sum_i \frac{\partial \ell_i(\vec{\beta})}{\partial \beta_j} = \sum_i \begin{cases} \frac{1}{\pi_i}\frac{\partial \pi_i}{\partial \beta_j} & \text{if } y_i = 1 \\ \frac{1}{1-\pi_i}\left(-\frac{\partial \pi_i}{\partial \beta_j}\right) & \text{if } y_i = 0 \end{cases} \tag{3}$$

If we plug in the derivative,

$$\frac{\partial \pi_i}{\partial \beta_j} = \pi_i(1-\pi_i)x_j, \tag{4}$$

we can merge these two cases

$$\frac{\partial \ell_i}{\partial \beta_j} = (y_i - \pi_i)x_j. \tag{5}$$

Gradient for Logistic Regression

Gradient

$$\nabla_\beta \ell(\vec{\beta}) = \left[\frac{\partial \ell(\vec{\beta})}{\partial \beta_0}, \ldots, \frac{\partial \ell(\vec{\beta})}{\partial \beta_n} \right] \tag{6}$$

Update

$$\Delta\beta \equiv \eta \nabla_\beta \ell(\vec{\beta}) \tag{7}$$

$$\beta_i' \leftarrow \beta_i + \eta \frac{\partial \ell(\vec{\beta})}{\partial \beta_i} \tag{8}$$

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Why are we adding? What would well do if we wanted to do **descent**?

Gradient for Logistic Regression

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η: step size, must be greater than zero

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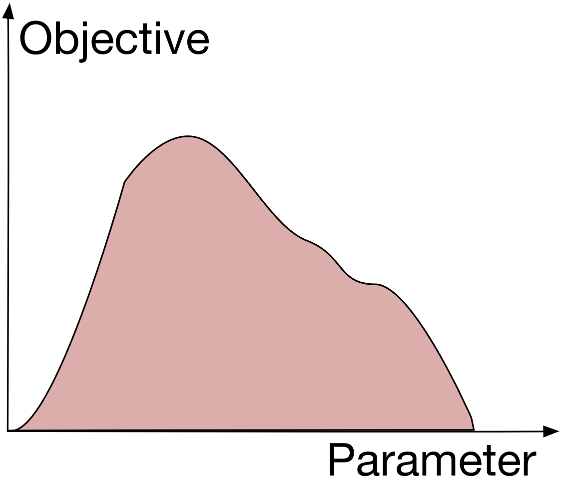
NB: Conjugate gradient is usually better, but harder to implement

Choosing Step Size



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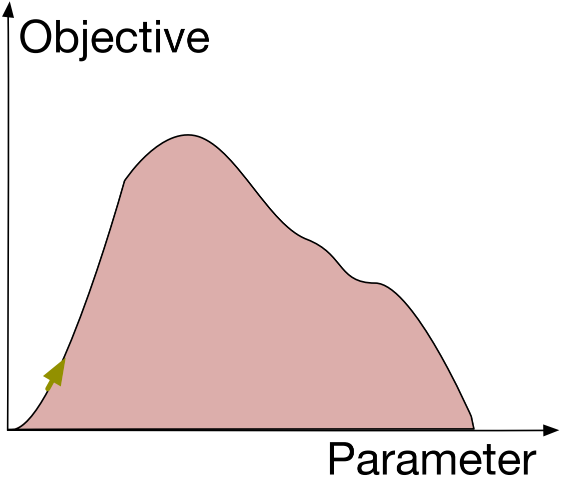

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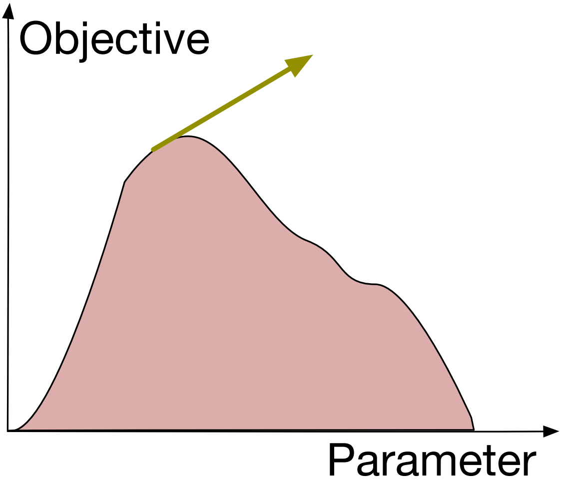

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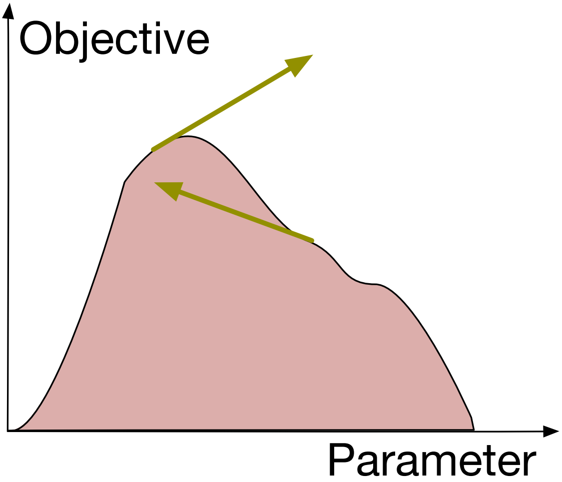

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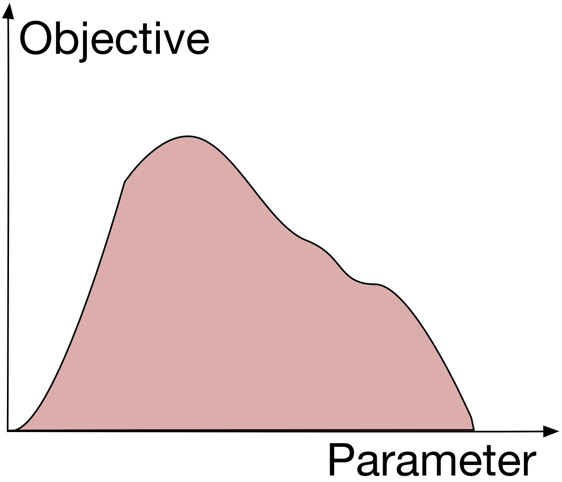

Approximating the Gradient

- Our datasets are big (to fit into memory)
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$$\ell(\beta) \equiv \mathbb{E}_x [\nabla \ell(\beta, x)] \tag{9}$$

- Average over all observations

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- Average over all observations
- What if we compute an update just from one observation?

Getting to Union Station

Pretend it's a pre-smartphone world and you want to get to Union Station

Stochastic Gradient for Logistic Regression

Given a **single observation** x_i chosen at random from the dataset,

$$\beta_j \leftarrow \beta_j' + \eta\left(-\mu\beta_j' + x_{ij}\left[y_i - \pi_i\right]\right) \tag{10}$$

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Examples in class.

Algorithm

1. Initialize a vector B to be all zeros
2. For $t = 1, \dots, T$
   - For each example $\vec{x}_i, y_i$ and feature j:
     - Compute $\pi_i \equiv \Pr(y_i = 1 \,|\, \vec{x}_i)$
     - Set $\beta[j] = \beta[j]' + \lambda(y_i - \pi_i)x_i$
3. Output the parameters $\beta_1, \dots, \beta_d$.

Wrapup

- Logistic Regression: Regression for outputting Probabilities
- Intuitions similar to linear regression
- We'll talk about feature engineering for both next time