Department of Computer Science
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Maximum Likelihood

Introduction to Data Science Algorithms

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Exponential Distribution

You observe $x_1, x_2, \dots x_N$. What is the MLE for the parameter θ?

$$f_\theta(x) = \lambda \exp\{-\lambda x\} \mathbb{1}[x > 0] \tag{1}$$

$(\lambda > 0)$

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$(\lambda > 0)$

$$\ell = N \log \lambda - \sum_i \lambda x_i \quad (2)$$

$$0 = \frac{N}{\lambda} - \sum_i x_i \quad (3)$$

$$\lambda = \frac{N}{\sum_i x_i} \quad (4)$$

$$(5)$$

Uniform Distribution

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Maximum at $\theta = \max x_i$. (But biased down: needs to be adjusted up.)

More complicated: Poisson

- We have the following data

Number Marriagies	Age
0	12
0	50
2	30
2	36
6	92

Assuming a model

Let's assume that the number of marriages comes from a Poisson distribution whose parameter is a function of age

$$\lambda_i = \lambda_0 \text{age}_i \tag{8}$$

- Likelihood
- Log-likelihood
- Gradient λ_0
- MLE

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$$p(x_i) = \frac{\exp\{\lambda_0 \text{age}\} (\lambda_0 \text{age})^x}{x!} \tag{9}$$

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- Log-likelihood

$$\ell = \sum_i \log\left[\frac{\exp\left\{\lambda_0 \text{age}_i\right\} (\lambda_0 \text{age}_i)^{x_i}}{x_i!}\right] \tag{10}$$

$$= \sum_i \log\left(\exp\left\{-\lambda_0 \text{age}_i\right\} \lambda_0 \text{age}_i\right) - \log x_i! \tag{11}$$

$$= \left[-\lambda_0 \sum \text{age}_i + \sum x_i \log(\lambda_0 \text{age}_i) - \sum \log x_i!\right] \tag{12}$$

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- Gradient λ_0
- MLE

Age of Marriage

- Gradient λ_0
- λ_0 MLE

Age of Marriage

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$$\frac{\partial \ell}{\partial \lambda_0} = -\sum_i \text{age}_i + \sum_i x_i \frac{\partial \log \lambda_0 \text{age}_i}{\partial \lambda_0} \tag{11}$$

$$= \sum_i \text{age}_i + \sum_i x_i \frac{\text{age}_i}{\lambda_0 \text{age}_i} \tag{12}$$

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- λ_0 MLE

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$$\lambda_0 = \frac{\sum_i x_i}{\sum_i \text{age}_i} \tag{14}$$

Our dataset

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- λ_0?
- Expected number of marriages for someone 22 years old? Most likely number of marriages?

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  - $\mathbb{E}_{\lambda_0}[X] = 1.0$

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- λ_0: 0.044
- Expected number of marriages for someone 22 years old? Most likely number of marriages?
    - $\mathbb{E}_{\lambda_0}[X] = 1.0$
    - Two modes: 0, 1

Now we're getting somewhere!

- Data Science = Reverse of Probabilities
- Building models from data
- Making predictions, refining models

Quiz

- Poisson distribution

$$f(x) = \frac{\exp\{\lambda\}(\lambda)^x}{x!} \tag{15}$$

- Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$