Classification: VC Dimension

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LECTURE 7

Slides adapted from Rob Schapire

Recap

- Rademacher complexity provides nice guarantees

$$R(h) \leq \hat{R}(h) + \mathscr{R}_m(H) + \mathcal{O}\left(\sqrt{\frac{\log \frac{1}{\delta}}{2m}}\right) \tag{1}$$

- But in practice hard to compute for real hypothesis classes
- Is there a relationship with simpler combinatorial measures?

Growth Function

Define the **growth function** $\Pi_H : \mathbb{N} \rightarrow \mathbb{N}$ for a hypothesis set H as:

$$\forall m \in \mathbb{N}, \Pi_H(m) \equiv \max_{\{x_1,\ldots,x_m\} \in X} \left| \{(h(x_1), \ldots, h(x_m) : h \in H\} \right| \tag{2}$$

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i.e., the number of ways m points can be classified using H.

Rademacher Complexity vs. Growth Function

If G is a function taking values in $\{-1, +1\}$, then

$$\mathscr{R}_m(G) \leq \sqrt{\frac{2 \ln \Pi_G(m)}{m}} \tag{3}$$

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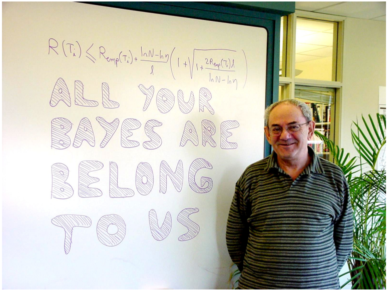
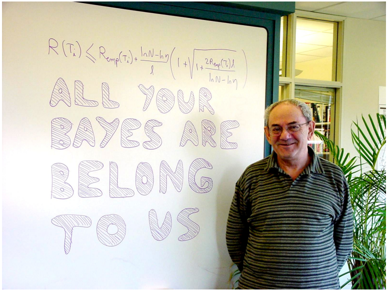
You'll prove this

Vapnik-Chervonenkis Dimension



$$\mathrm{VC}(H) \equiv \max\{m : \Pi_H(m) = 2^m\} \tag{4}$$

Vapnik-Chervonenkis Dimension



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The size of the largest set that can be fully shattered by H.

VC Dimension for Hypotheses

- Need upper and lower bounds
- Lower bound: example
- Upper bound: Prove that no set of $d+1$ points can be shattered by H (harder)

Intervals

What is the VC dimension of $[a, b]$ intervals on the real line.

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- What about two points?

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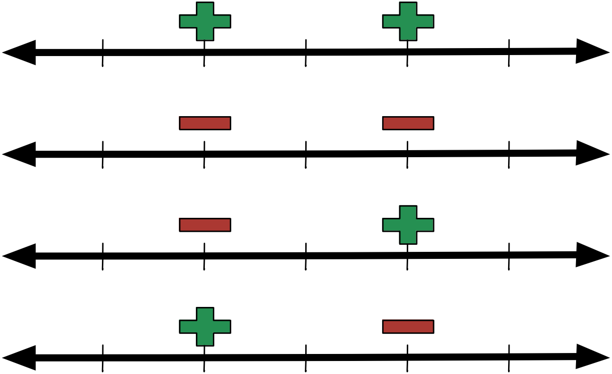

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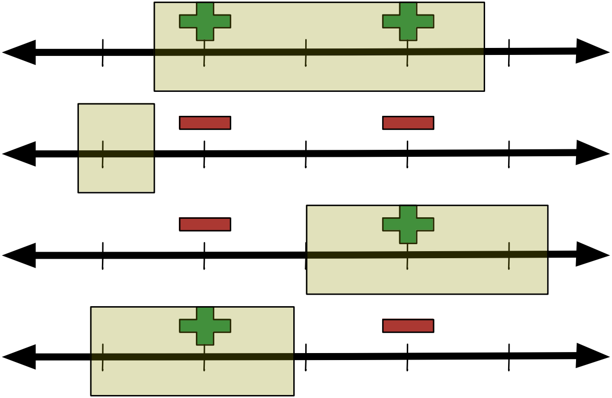

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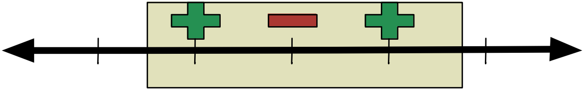

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- Two points can be perfectly classified, so VC dimension ≥ 2
- What about three points?
- Three points cannot be shattered
- Thus, VC dimension of intervals is 2

Sine Functions

- Consider hypothesis that classifies points on a line as either being above or below a sine wave

$$\{t \rightarrow \sin(\omega x) : \omega \in \mathbb{R}\} \tag{5}$$

- Can you shatter three points?

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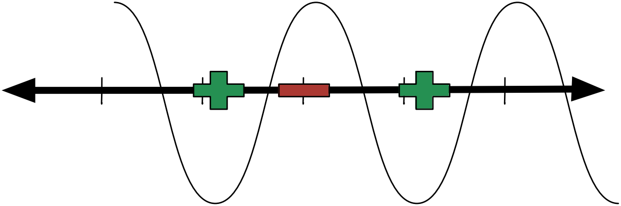


Sine Functions

- Consider hypothesis that classifies points on a line as either being above or below a sine wave

$$\{t \rightarrow \sin(\omega x) : \omega \in \mathbb{R}\} \tag{5}$$

- Can you shatter four points?

Sine Functions

- Consider hypothesis that classifies points on a line as either being above or below a sine wave

$$\{t \rightarrow \sin(\omega x) : \omega \in \mathbb{R}\} \tag{5}$$

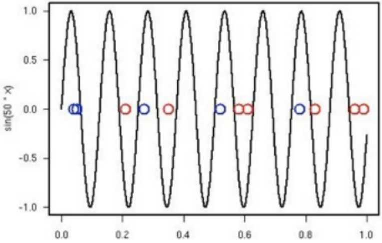
- How many points can you shatter?

Sine Functions

- Consider hypothesis that classifies points on a line as either being above or below a sine wave

$$\{t \rightarrow \sin(\omega x) : \omega \in \mathbb{R}\} \quad (5)$$

- Thus, VC dim of sine on line is ∞



Connecting VC with growth function

VC dimension obviously encodes the complexity of a hypothesis class, but we want to connect that to Rademacher complexity and the growth function so we can prove generalization bounds.

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Theorem

Sauer's Lemma *Let H be a hypothesis set with VC dimension d. Then $\forall m \in \mathbb{N}$*

$$\Pi_H(m) \leq \sum_{i=0}^{d} \binom{m}{i} \equiv \Phi_d(m) \tag{6}$$

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This is good because the sum when multiplied out becomes $\binom{m}{i} = \frac{m \cdot (m-1) \dots}{i!} = \mathcal{O}\left(m^d\right)$. When we plug this into the learning error limits: $\log(\Pi_H(2m)) = \log(\mathcal{O}\left(m^d\right)) = \mathcal{O}\left(d \log m\right)$.

Proof of Sauer's Lemma

Prelim:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1} \quad \text{This comes from Pascal's Triangle}$$

$$\binom{m}{k} = 0 \quad \text{if} \begin{cases} k < 0 \\ k > m \end{cases} \quad \text{This convention is consistent with Pascal's Triangle}$$

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We'll proceed by induction. Our two base cases are:

- If $m = 0$, $\Pi_H(m) = 1$. You have no data, so there's only one (degenerate) labeling
- If $d = 0$, $\Pi_H(m) = 1$. If you can't even shatter a single point, then it's a fixed function

Induction Step

Assume that it holds for all m', d' for which $m' + d' < m + d$. We are given H, $|S| = m$, $S = \langle x_1, \ldots, x_m \rangle$, and d is the VC dimension of H.

Induction Step

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Build two new hypothesis spaces

$\mathcal{H}$							$\mathcal{H}_1$						$\mathcal{H}_2$				
	$\mathbf{x}_1, \dots, \mathbf{x}_m$							$\mathbf{x}_1, \dots, \mathbf{x}_{m-1}$						$\mathbf{x}_1, \dots, \mathbf{x}_{m-1}$			
h1	0	1	1	0	0	$\rightarrow$	h1	0	1	1	0	$\rightarrow$	h1	0	1	1	0
h2	0	1	1	0	1	$\nearrow$											
h3	0	1	1	1	0	$\rightarrow$	h3	0	1	1	1						
h4	1	0	0	1	0	$\rightarrow$	h4	1	0	0	1	$\rightarrow$	h4	1	0	0	1
h5	1	0	0	1	1	$\nearrow$											
h6	1	1	0	0	1	$\rightarrow$	h6	1	1	0	0						

Encodes where the extended set has differences on the first m points.

Bounding Growth Function

$$
\begin{aligned}
|\Pi_H(S)| &= |H_1| + |H_2| && (7) \\
&\leq \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i} && (8) \\
& && (9)
\end{aligned}
$$

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$$\tag{9}$$

We can rewrite this as $\sum_{i=0}^{d} \binom{m-1}{i-1}$ because $\binom{x}{-1} = 0$.

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$$= \sum_{i=0}^{d} \binom{m}{i} \tag{10}$$

$$\tag{11}$$

Pascal's Triangle

Bounding Growth Function

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&= \sum_{i=0}^{d} \binom{m}{i} && (10)\\
&= \Phi_d(m) && (11)
\end{aligned}
$$

Wait a minute . . .

Is this combinatorial expression really $\mathcal{O}\left(m^d\right)$?

$$
\begin{aligned}
\sum_{i=0}^{d}\binom{m}{i} &\leq \sum_{i=0}^{d}\binom{m}{i}\left(\frac{m}{d}\right)^{d-i} \\
&\leq \sum_{i=0}^{m}\binom{m}{i}\left(\frac{m}{d}\right)^{d-i} \\
&= \left(\frac{m}{d}\right)^{d}\sum_{i=0}^{m}\binom{m}{i}\left(\frac{d}{m}\right)^{i} \\
&= \left(\frac{m}{d}\right)^{d}\left(1+\frac{d}{m}\right)^{m} \leq \left(\frac{m}{d}\right)^{d} e^{d}.
\end{aligned}
$$

Generalization Bounds

Combining our previous generalization results with Sauer's lemma, we have that for a hypothesis class H with VC dimension d, for any $\delta > 0$ with probability at least $1-\delta$, for any $h \in H$,

$$R(h) \leq \hat{R}(h) + \sqrt{\frac{2d \log \frac{em}{d}}{m}} + \sqrt{\frac{\log \frac{1}{\delta}}{2m}} \tag{12}$$

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- We're now going to see if we can find an algorithm that has good VC dimension

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- We're now going to see if we can find an algorithm that has good VC dimension
- And works well in practice . . . Support Vector Machines
- In class: more VC dimension examples