Classification: The PAC Learning Framework

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LECTURE 5B

Content Questions

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Quiz!

Admin Questions

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PAC Learnability: Rectangles

Is the hypothesis class of axis-aligned rectangles PAC learnable?

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A. Blumer, A. Ehrenfeucht, D. Haussler, and M.K. Warmuth. Learnability and the Vapnik-Chervonenkis dimension. Journal of the ACM (JACM), 36(4):929?965, 1989

What's the learning algorithm

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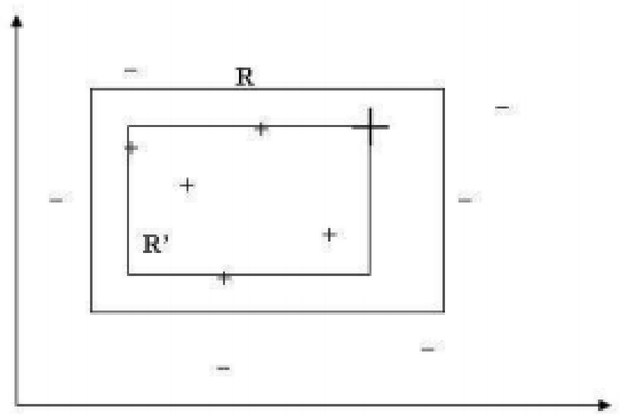

Call this h_S, which we learned from data. $h_s \in c$

Proof

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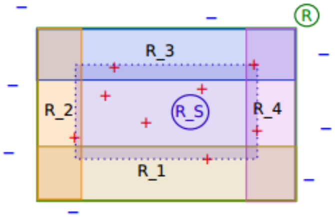

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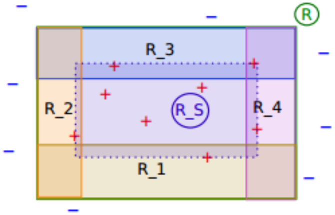

We get a bad h_S only if we have an observation fall in this region. So let's bound this probability.

Bounds

$$
\begin{aligned}
\Pr[\mathit{error}] &= \Pr[\Cup_{i=1}^{4} x \notin R_i] && (1)\\
&\leq \sum_{i=1}^{4} \Pr[x \notin R_i] && (2)\\
&= \sum_{i=1}^{4} (1 - P(R_i))^m && (3)
\end{aligned}
$$

Bounds

$$\Pr[error] = \Pr[\Cup_{i=1}^{4} x \notin R_i] \tag{1}$$

$$\leq \sum_{i=1}^{4} \Pr[x \notin R_i] \tag{2}$$

$$= \sum_{i=1}^{4} (1 - P(R_i))^m \tag{3}$$

If we assume that $P(R_i) \geq \frac{\epsilon}{4}$, then

$$\Pr[error] \leq 4\left(1 - \frac{\epsilon}{4}\right)^m \leq 4 \cdot \exp\left\{-\frac{m\epsilon}{4}\right\} \tag{4}$$

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Solving for m gives

$$m \geq \frac{4 \ln 4/\delta}{\epsilon} \tag{5}$$

Concept Learning

Are Boolean conjunctions PAC learnable? Think of every feature as a Boolean variable; in a given example the variable is given the value 1 if its corresponding feature appears in the examples and 0 otherwise. In this way, if the number of measured features is n the concept is represented as a Boolean function $c : \{0, 1\} \longmapsto \{0, 1\}$. For example we could define a chair as something that has four legs **and** you can sit on **and** is made of wood. Can you learn such a conjunction concept over n variables?

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- After last example, $x_1 \bar{x}_3 \bar{x}_4$

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- Our algorithm can be two specific. It might not say yes when it should.
- We make an error on a literal if we've never seen it before (there are $2n$ literals: $x_1, \bar{x_1}$)

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Let $p(z)$ be the probability that our concept returns a positive example in which literal z is false.

$$R(h) \leq \sum_z p(z) \tag{7}$$

A literal z is bad if $p(z) \geq \frac{\epsilon}{2n}$.

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If h has no bad literals, then h will have error less than ϵ.

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$$m \geq \frac{2n}{\epsilon}\left(\ln 2n + \ln \frac{1}{\delta}\right) \tag{8}$$

3-DNF

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Not efficiently learnable unless P = NP.