Classification: Logistic Regression from Data

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LECTURE 3

Slides adapted from Emily Fox

Reminder: Logistic Regression

$$P(Y=0|X) = \frac{1}{1+\exp\left[\beta_0 + \sum_i \beta_i X_i\right]} \quad (1)$$

$$P(Y=1|X) = \frac{\exp\left[\beta_0 + \sum_i \beta_i X_i\right]}{1+\exp\left[\beta_0 + \sum_i \beta_i X_i\right]} \quad (2)$$

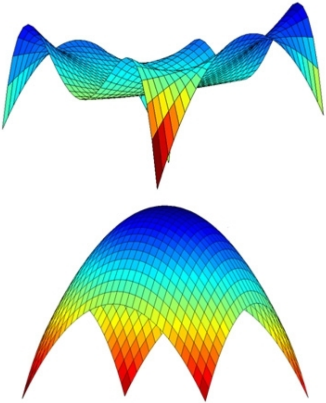
- Discriminative prediction: $p(y|x)$
- Classification uses: ad placement, spam detection
- What we didn't talk about is how to learn β from data

Logistic Regression: Objective Function

$$\mathscr{L} \equiv \ln p(Y \,|\, X, \beta) = \sum_j \ln p(y^{(j)} \,|\, x^{(j)}, \beta) \tag{3}$$

$$= \sum_j y^{(j)} \left(\beta_0 + \sum_i \beta_i x_i^{(j)} \right) - \ln \left[1 + \exp \left(\beta_0 + \sum_i \beta_i x_i^{(j)} \right) \right] \tag{4}$$

Convexity



- Convex function
- Doesn't matter where you start, if you "walk up" objective

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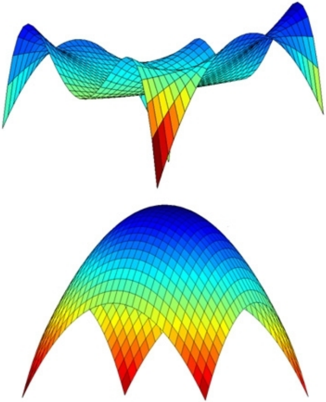

- Convex function
- Doesn't matter where you start, if you "walk up" objective
- Gradient!

Gradient for Logistic Regression

Gradient

$$\nabla_\beta \mathcal{L}(\vec{\beta}) = \left[\frac{\partial \mathcal{L}(\vec{\beta})}{\partial \beta_0}, \dots, \frac{\partial \mathcal{L}(\vec{\beta})}{\partial \beta_n} \right] \tag{5}$$

Update

$$\Delta\beta \equiv \eta \nabla_\beta \mathcal{L}(\vec{\beta}) \tag{6}$$

$$\beta_i \leftarrow \beta_i' + \eta \frac{\partial \mathcal{L}(\vec{\beta})}{\partial \beta_i} \tag{7}$$

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Why are we adding? What would we do if we wanted to do **descent**?

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η: step size, must be greater than zero

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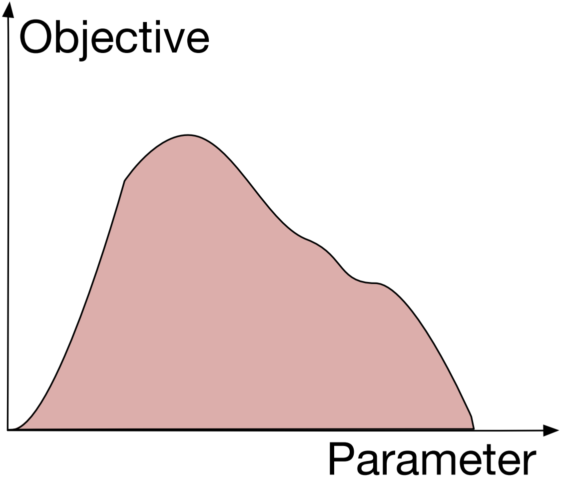
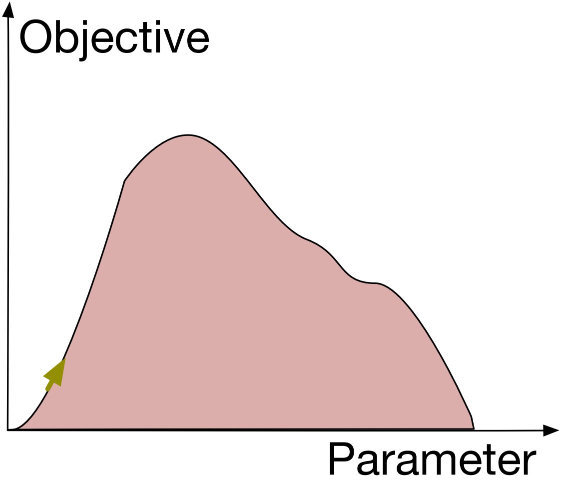
NB: Conjugate gradient is usually better, but harder to implement

Choosing Step Size



Choosing Step Size



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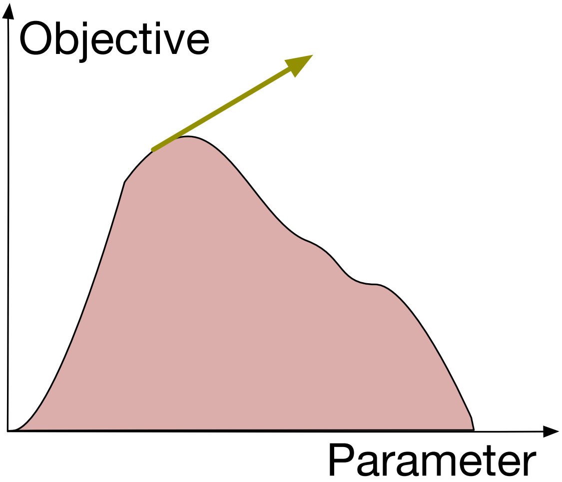

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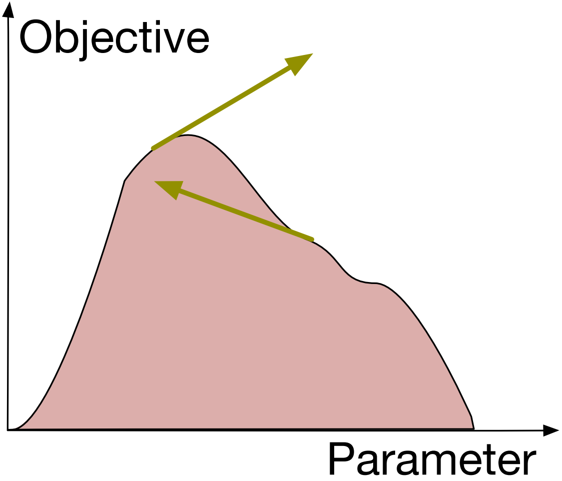

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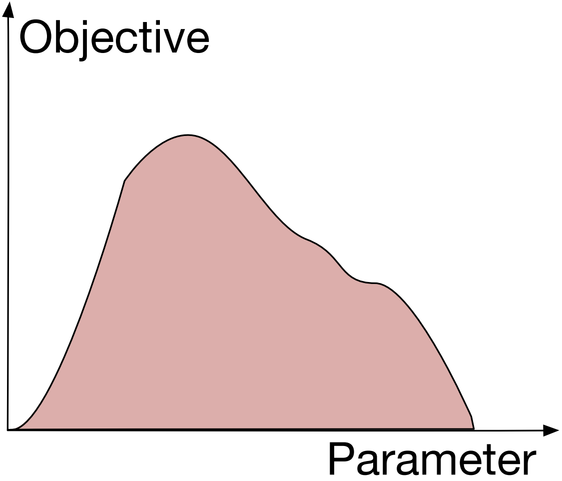

Remaining issues

- When to stop?
- What if β keeps getting bigger?

Regularized Conditional Log Likelihood

Unregularized

$$\beta^* = \arg\max_{\beta} \ln \left[p(y^{(j)} \,|\, x^{(j)}, \beta) \right] \tag{8}$$

Regularized

$$\beta^* = \arg\max_{\beta} \ln \left[p(y^{(j)} \,|\, x^{(j)}, \beta) \right] - \frac{\mu}{2} \sum_i \beta_i^2 \tag{9}$$

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μ is “regularization” parameter that trades off between likelihood and having small parameters

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- Our datasets are big (to fit into memory)
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$$\mathscr{L}(\beta) \equiv \mathbb{E}_x [\nabla \mathscr{L}(\beta, x)] \tag{10}$$

- Average over all observations

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- Average over all observations
- What if we compute an update just from one observation?

Getting to Union Station

Pretend it's a pre-smartphone world and you want to get to Union Station

Stochastic Gradient for Logistic Regression

Given a **single observation** x chosen at random from the dataset,

$$\beta_i \leftarrow \beta_i' + \eta \left(-\mu \beta_i' + x_i \left[y - p(y = 1 \,|\, \vec{x}, \vec{\beta}') \right] \right) \tag{11}$$

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Examples in class.

Proofs about Stochastic Gradient

- Depends on convexity of objective and how close ϵ you want to get to actual answer
- Best bounds depend on changing η over time and **per dimension** (not all features created equal)

In class

- Your questions!
- Working through simple example
- Prepared for logistic regression homework