



Kernel Functions for Support Vector Machines

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LECTURE 11

Statistics Professors HATE Him!



Doctor's discovery revealed the secret to learning any problem with just 10 training samples. Watch this shocking video and learn how rapidly you can find a solution to your learning problems using this one sneaky kernel trick! Free from overfitting!

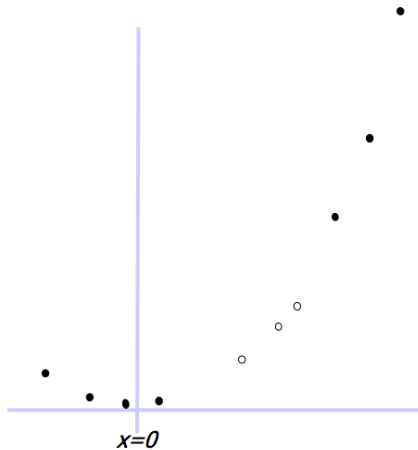
<http://www.oneweirdkerneltrick.com>

Slides adapted from Jerry Zhu

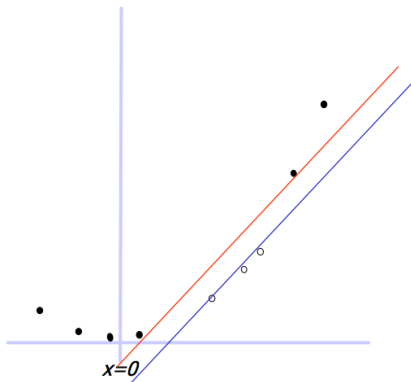
Can you solve this with linear separator?



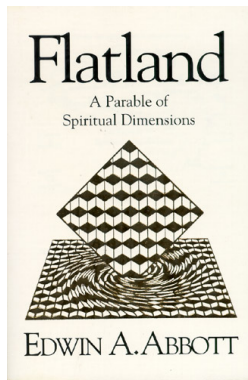
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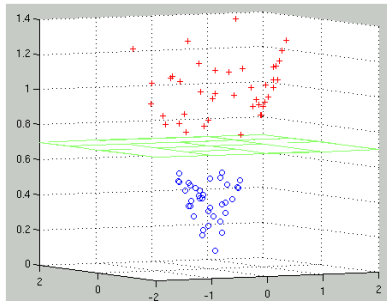
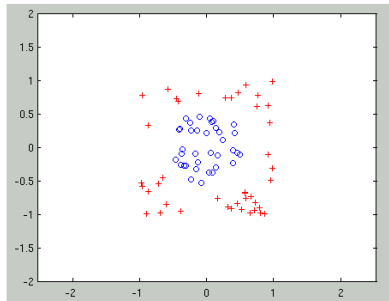
Adding another dimension



Behold yon miserable creature. That Point is a Being like ourselves, but confined to the non-dimensional Gulf. He is himself his own World, his own Universe; of any other than himself he can form no conception; he knows not Length, nor Breadth, nor Height, for he has had no experience of them; he has no cognizance even of the number Two; nor has he a thought of Plurality, for he is himself his One and All, being really Nothing. Yet mark his perfect self-contentment, and hence learn this lesson, that to be self-contented is to be vile and ignorant, and that to aspire is better than to be blindly and impotently happy.

Problems get easier in higher dimensions

$$(x_1, x_2) \Rightarrow (x_1, x_2, \sqrt{x_1^2 + x_2^2})$$



What's special about SVMs?

$$\max_{\vec{\alpha}} \sum_{i=1}^m \alpha_i - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j (\vec{x}_i \cdot \vec{x}_j) \quad (1)$$

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- Kernels!

What's a kernel?

- A function $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ is a kernel over \mathcal{X} .
- This is equivalent to taking the dot product $\langle \phi(x_1), \phi(x_2) \rangle$ for some mapping
- **Mercer's Theorem:** So long as the function is continuous and symmetric, then K admits an expansion of the form

$$K(x, x') = \sum_{n=0}^{\infty} a_n \phi_n(x) \phi_n(x') \quad (2)$$

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- The computational cost is just in computing the kernel

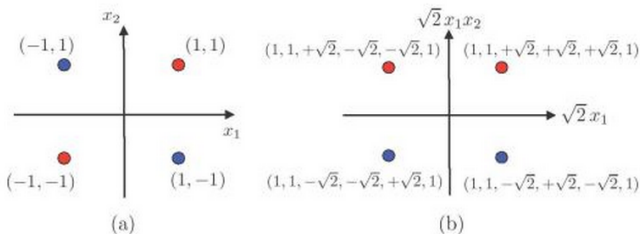
Polynomial Kernel

$$K(x, x') = (x \cdot x' + c)^d \quad (3)$$

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When $d = 2$:



Gaussian Kernel

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(All polynomials!)

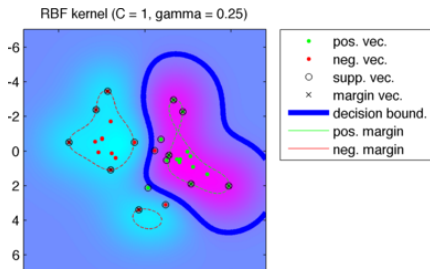
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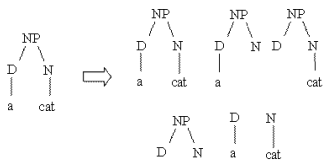
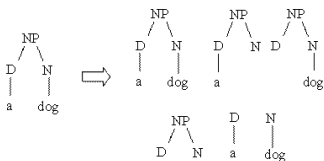


Tree Kernels

- Sometimes we have example x that are hard to express as vectors
- For example sentences “a dog” and “a cat”: internal syntax structure

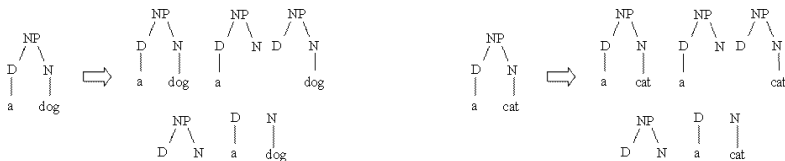
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3/5 structures match, so tree kernel returns .6

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- Rademacher complexity for a kernel with radius Λ and data with radius r : $S \subset \{x : K(x, x) \leq r^2\}$, $H = \{x \mapsto w \cdot \phi(x) : \|w\| \leq \Lambda\}$

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- Proof requires real analysis

Margin learnability

- With probability $1 - \delta$:

$$R(h) \leq \hat{R}_\rho(h) + 2\sqrt{\frac{r^2\Lambda^2/\rho^2}{m}} + \sqrt{\frac{\log \frac{1}{\delta}}{2m}} \quad (7)$$

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- So if you can find a simple kernel representation that induces a margin, use it!
- ...so long as you can handle the computational complexity

How does it effect optimization

- Replace all dot product with kernel evaluations $K(x_1, x_2)$
- Makes computation more expensive, overall structure is the same
- Try linear first!

Recap

Pennsylvania Learners Being Ripped Off By Not Knowing this One Weird Kernel Trick

The President has ordered all learners to minimize their empirical risk. Learn how to minimize your empirical risk for less by following this simple rule.

PICK YOUR VC DIMENSION

1-10	11-50	51-100	100-1000	1001+
				

The image is a meme with a black background. It features a brain scan image with a green frog superimposed on it. Purple arrows point from the text to the brain scan and the frog. Below the brain scan is a row of five cat images, each corresponding to a VC dimension range. The first cat (green frog) is highlighted with a green border.

- This completes our discussion of SVMs
- Workhorse method of machine learning
- Flexible, fast, effective

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The image is a meme featuring a brain scan with a purple and green highlighted area. Below the brain scan is a table with five columns representing different VC dimensions and corresponding images of animals. The first column (1-10) shows a green frog, the second (11-50) shows a Siamese cat, the third (51-100) shows an orange cat, the fourth (100-1000) shows a Siamese cat, and the fifth (1001+) shows a white cat. The text 'PICK YOUR VC DIMENSION' is written above the table. The entire image is overlaid with a large, stylized purple 'X'.

- This completes our discussion of SVMs
- Workhorse method of machine learning
- Flexible, fast, effective
- Kernels: applicable to wide range of data, inner product trick keeps method simple