Markov Inequality

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Proof I:

$$a\mathbb{I}_{X\geq a} \leq X$$

If $X \geq a$, $a\mathbb{I}_{X\geq a} \leq X \to a \leq X$
If $X < a$, $a\mathbb{I}_{X\geq a} \leq X \to 0 \leq X$
Take expectation on both side:

$$aE[\mathbb{I}_{X\geq a}] \leq E[X] \to aP(X \geq a) \leq E[X] \to P(X \geq a) \leq \frac{E[X]}{a}$$

Proof II:

$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{a} xf(x)\,dx + \int_{a}^{\infty} xf(x)\,dx \geq \int_{a}^{\infty} xf(x)\,dx \geq \int_{a}^{\infty} af(x)\,dx$$
$$= a\int_{a}^{\infty} f(x)\,dx = aP(X \geq a) \to P(X \geq a) \leq \frac{E[X]}{a}$$