# ME 322 Exercise 5 Report - Mechanical Yaw-rate Sensor Design

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**Modified:** 6/3/2025

**Background:**

This script is a lab report for ME322. This lab exercise is split into two parts, Part 1 and Part 2. Part 1 defines given spring constant values and assumes that the dampers are disconnected ($b_1 = b_2 = 0$). In Part 1 the linear graph and normal tree for the a yaw-rate sensor is found and used to develop a state-space model at steady-state conditions. The ouput of the system are the tilt angle of the flywheel, $\theta_y$, and the estimated the yaw-rate, $\hat{\Omega}_z$, of the aircraft such that $\hat{\Omega}_z$ is completely defined in terms of spring deflection. Part 2 defines the same system, however, with the dampers connected again. The linear graph and normal tree for the yaw-sensor are redrawn to include this damper. After symbolically calculating, A, B, C, D the transfer function can be found for the estimated the yaw-rate, $\hat{\Omega}_z$. The relationship between natural frequency and the damping ratio, zeta, to the input spring constants and damping constants was then found. Using this relationship the spring constants and damping constants for any given natural frequency and/or damping constant can be found. Part 2 calculates the spring constant and damping constants required for a damping ratio corresponding to a maximum overshoot percentage of 3% and a corresponding margin of 2% settling time of 0.01 sec. The natural frequency and damping ratio are then recalculated based on the system found using these calculated spring constants and damping constants to confirm that they match the values they were designed too.

**Required Files:**

- ME322_Exercise_5_Matlab.mlx

**References:**

[1] ME322_2254_Yaw_Sensor.pdf

[2] ME322_2254_Lab_Manual.pdf

[3] ME318_formula_sheet.pdf

**Table of Contents**

## Problem Statement

Consider Figure 1 which depicts a mechanical sensor to accurately measure the yaw of an aircraft in a level turn.

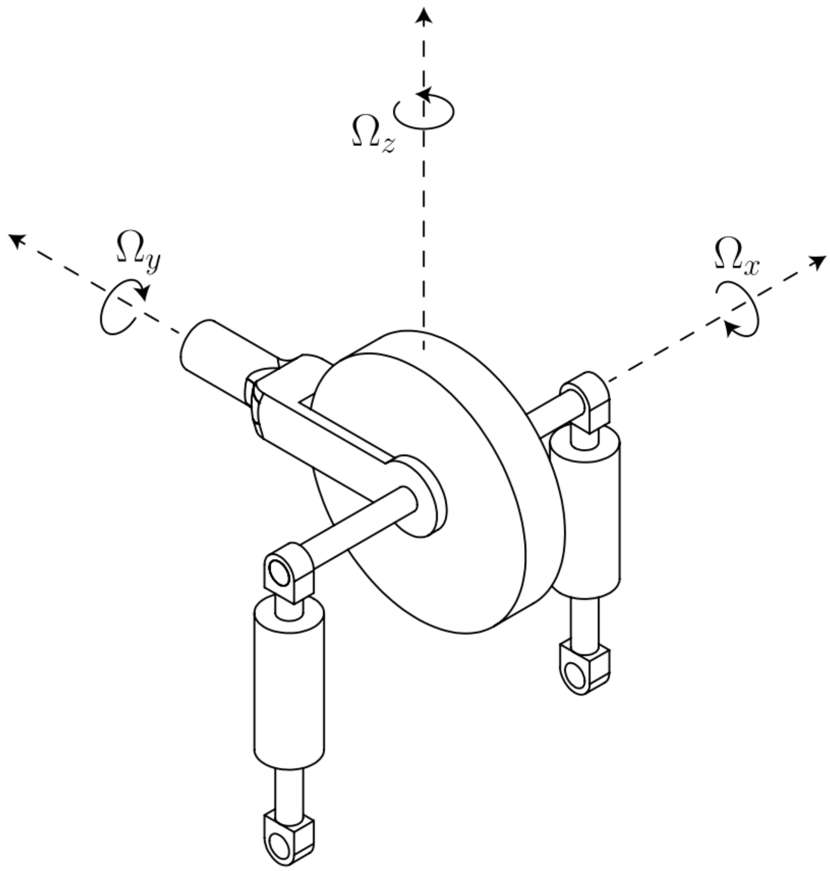



**Figure 1:** Mechanical yaw-rate sensing apparatus diagram [1].

For the system shown in Figure 1 consider the gimballed flywheel spinning at a constant angular velocity of $\Omega_x = 10000\,[\mathrm{RPM}]$. The flywheel has a diameter, $D = 40\,[\mathrm{mm}]$, and a mass, $m = 200\,[\mathrm{g}]$. The struts supporting each end of the flywheel each consist of a spring, $k_1 = k_2$, and a damper, $b_1 = b_2$, in parallel. Each strut is $r = 50\,[\mathrm{mm}]$ away from the center joint. Estimate the yaw-rate, $\widehat{\Omega}_z$, of the aircraft based on the angle of deflection of the flywheel assuming the aircraft is performing a level turn at a constant yaw-rate of, $\Omega_z$ [1].

Figure 2 shows a gyroscope with torque directions, $T_1$ and $T_2$ labeled along with each respective $\Omega_1$ and $\Omega_2$. The gyroscope is spinning about its $\widehat{e_3}$ axis to match the gyroscope shown in Figure 1 spinning in the x-direction. Based on these defined directions the transducer equation can be found based on the equation, $T = \Omega \times H$ in which $\Omega$ is the angular velocity of any given applied spin and $H$ is the angular momentum in the direction of the current spin (about the $\widehat{e_3}$ axis). Since the gyroscope is spinning only in the about the $\widehat{e_3}$ axis $H = H_3$.

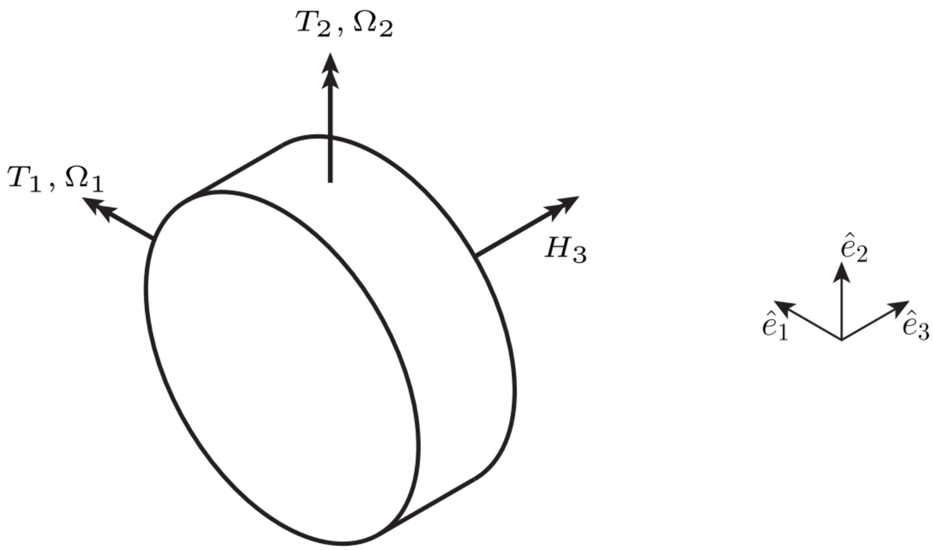



**Figure 2:** Gyroscope spinning about its $\hat{e_3}$ axis [1].

The transducer equation for a gyroscope spinning about its center as shown in Figure 2 is:

$$\begin{bmatrix} \Omega_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{H_3} \\ H_3 & 0 \end{bmatrix} \begin{bmatrix} \Omega_2 \\ T_2 \end{bmatrix} \text{ [1].}$$

This transformer equation is used in the Hand Calculations Section.

## Hand Calculations

## Given

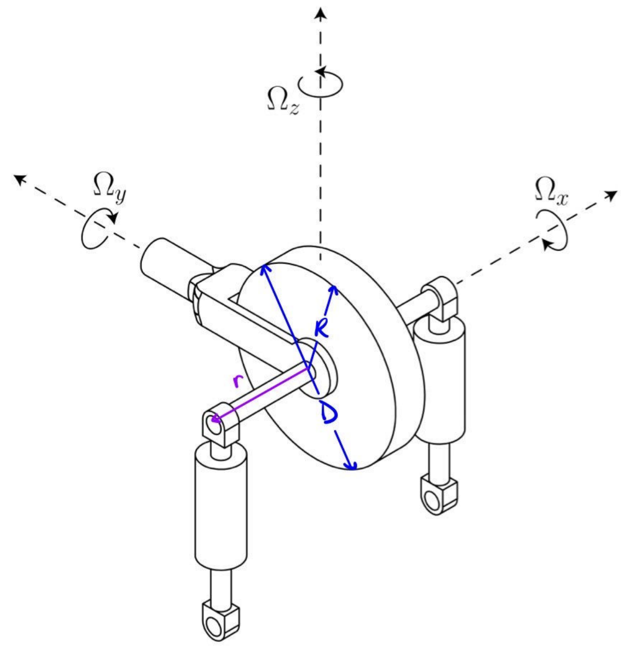
$D = 40 \text{ mm}$
$= 0.04 \text{ m}$

$R = \frac{D}{2} = 0.02 \text{ m}$

$m = 200 \text{ g}$
$= 0.2 \text{ kg}$

$\Omega_x = 10000 \text{ rpm} \times \frac{\frac{2\pi}{60} \frac{\text{rad}}{\text{sec}}}{1 \text{ rpm}}$
$= 1047.2 \text{ rad/sec}$

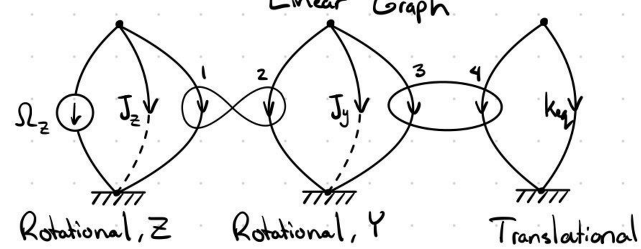
$r = 50 \text{ mm} \Leftarrow$ strut distance
$= 0.05 \text{ m}$




## Part 1

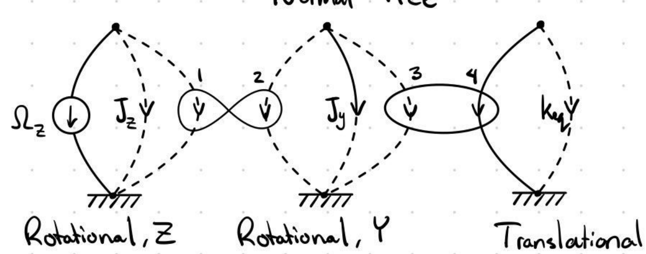
$k_1 = k_2 = 500 \text{ N/m} \quad b_1 = b_2 = 0$

Linear Graph




$k_{eq} = k_1 + k_2$

Normal Tree

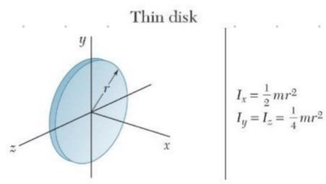



$k_{eq} = k_1 + k_2$

## Transducer Matrices

$$\begin{bmatrix} \Omega_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{H_3} \\ H_3 & 0 \end{bmatrix} \begin{bmatrix} \Omega_2 \\ T_2 \end{bmatrix}$$

$$\begin{bmatrix} \Omega_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & -r \end{bmatrix} \begin{bmatrix} V_4 \\ F_4 \end{bmatrix}$$




$$J_x = \frac{1}{2} m R^2$$
$$= \frac{1}{2}(0.2\,kg)(0.02\,m)^2$$
$$J_x = 4.0 \times 10^{-5}\ kg \cdot m^2$$

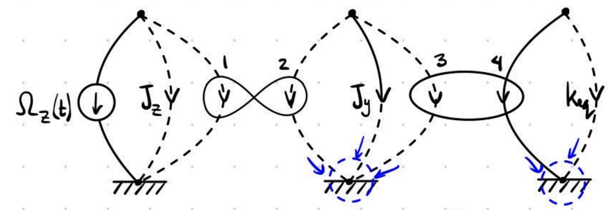
$$J_y = J_z = \frac{1}{4} m R^2$$
$$= \frac{1}{4}(0.2\,kg)(0.02\,m)^2$$
$$J_y = J_z = 2.0 \times 10^{-5}\ kg \cdot m^2$$

$$H_3 = J_x \Omega_x$$

$$H_3 = 0.0419\ kg \cdot m^2 \cdot \frac{rad}{sec}$$

$$k_{eq} = k_1 + k_2 = 1000\ \frac{N}{m}$$

## Normal Tree




**Elemental**

$$T_{J_z} = J_z \dot{\Omega}_{J_z}$$
$$T_{J_y} = J_y \dot{\Omega}_{J_y}$$
$$\dot{F}_{k_{eq}} = k_{eq} V_{k_{eq}}$$

**Transducer**

$$\Omega_1 = -\frac{1}{H_3} T_2$$
$$T_1 = H_3 \Omega_2$$
$$\Omega_3 = \frac{1}{r} V_4$$
$$T_3 = -r F_4$$

**Compatibility**

$$\Omega_{J_z} = \Omega_z(t)$$
$$\Omega_1 = \Omega_z(t)$$
$$\Omega_2 = \Omega_{J_y}$$
$$\Omega_3 = \Omega_{J_y}$$
$$V_{k_{eq}} = V_4$$

**Continuity**

$$T_2 + T_{J_y} + T_3 = 0$$
$$F_4 + F_{k_{eq}} = 0$$

State Variables $\Omega_{J_y}$, $F_{k_{eq}}$

$$\dot{\Omega}_{J_y} = \frac{1}{J_y} T_{J_y}$$
$$= \frac{1}{J_y}(-T_2 - T_3)$$
$$= \frac{1}{J_y}(H_3 \Omega_1 + r F_4)$$
$$= \frac{1}{J_y}(H_3 \Omega_z + r(-F_{k_{eq}}))$$
$$= \frac{1}{J_y}(H_3 \Omega_z - r F_{k_{eq}})$$
$$\dot{\Omega}_{J_y} = -\frac{r}{J_y} F_{k_{eq}} + \frac{H_3}{J_y} \Omega_z$$

$$\dot{F}_{k_{eq}} = k_{eq} V_{k_{eq}}$$
$$= k_{eq} V_4$$
$$= k_{eq} r \Omega_3$$
$$\dot{F}_{k_{eq}} = k_{eq} r \Omega_{J_y}$$

estimate $\dot{\Omega}_{J_y} \approx 0 \quad \Omega_{J_y} \approx 0$

$$0 = -\frac{r}{J_y} F_{k_{eq}} + \frac{H_3}{J_y} \hat{\Omega}_z$$
$$\frac{H_3}{J_y} \hat{\Omega}_z = \frac{r}{J_y} F_{k_{eq}}$$
$$H_3 \hat{\Omega}_z = r F_{k_{eq}}$$
$$\hat{\Omega}_z = \frac{r}{H_3} F_{k_{eq}}$$

$$F_{k_{eq}} = \int k_{eq} r \Omega_{J_y} \, dt$$
$$F_{k_{eq}} = k_{eq} r \theta_y$$
$$\theta_y = \frac{1}{k_{eq} r} F_{k_{eq}}$$

## State Equation

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\Omega}_{Jy} \\ \dot{F}_{keq} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & -\frac{r}{J_y} \\ k_{eq} r & 0 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} \Omega_{Jy} \\ F_{keq} \end{bmatrix}} + \overset{B}{\begin{bmatrix} \frac{H_3}{J_y} \\ 0 \end{bmatrix}} \overset{\vec{u}}{\begin{bmatrix} \Omega_z \end{bmatrix}}$$

## Output Equation

$$\vec{y} = \begin{bmatrix} \Theta_y \\ \hat{\Omega}_z \end{bmatrix} = \overset{C}{\begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} \Omega_{Jy} \\ F_{keq} \end{bmatrix}} + \overset{D}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \overset{\vec{u}}{\begin{bmatrix} \Omega_z \end{bmatrix}}$$

$$H(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix} \begin{bmatrix} s & \frac{r}{J_y} \\ -k_{eq} r & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{H_3}{J_y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix} \frac{1}{(s)(s) - \left(\frac{r}{J_y}\right)(-k_{eq} r)} \begin{bmatrix} s & -\frac{r}{J_y} \\ k_{eq} r & s \end{bmatrix} \begin{bmatrix} \frac{H_3}{J_y} \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + \frac{k_{eq} r^2}{J_y}} \begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix} \begin{bmatrix} s \frac{H_3}{J_y} \\ k_{eq} r \frac{H_3}{J_y} \end{bmatrix}$$

$$= \frac{1}{s^2 + \frac{k_{eq} r^2}{J_y}} \begin{bmatrix} \frac{H_3}{J_y} \\ \frac{k_{eq} r^2}{J_y} \end{bmatrix}$$

$$H(s) = \begin{bmatrix} \dfrac{\frac{H_3}{J_y}}{s^2 + \frac{k_{eq} r^2}{J_y}} \\[2ex] \dfrac{\frac{k_{eq} r^2}{J_y}}{s^2 + \frac{k_{eq} r^2}{J_y}} \end{bmatrix}$$

Transfer function of estimated yaw-rate to actual yaw-rate

$$G(s) = \frac{\hat{\Omega}_z(s)}{\Omega_z(s)} = \frac{\frac{k_{eq} r^2}{J_y}}{s^2 + \frac{k_{eq} r^2}{J_y}}$$

At steady state $s = 0$

$$G(0) = \frac{\hat{\Omega}_z(0)}{\Omega_z(0)} = \frac{\cancel{\frac{k_{eq} r^2}{J_y}}}{0^2 + \cancel{\frac{k_{eq} r^2}{J_y}}} = 1$$

As expected steady-state estimated yaw-rate = actual yaw-rate

$\therefore$ transfer function $= 1$ @ steady-state

Find Natural Frequency of system, $\omega_n$:

$$G(s) = \frac{\frac{k_{eq} r^2}{J_y}}{s^2 + \frac{k_{eq} r^2}{J_y}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ from reference [3].

$$\sqrt{\omega_n^2} = \sqrt{\frac{k_{eq} r^2}{J_y}}$$

$$\omega_n = \sqrt{\frac{(1000\,\frac{N}{m})(0.05\,m)^2}{2 \times 10^{-5}\ kg\,m^2}}$$

$$\omega_n = 250\sqrt{2}\ \frac{rad}{s}$$

$$\omega_n \approx 353.6\ \frac{rad}{s}$$

$$2\zeta\omega_n = 0$$

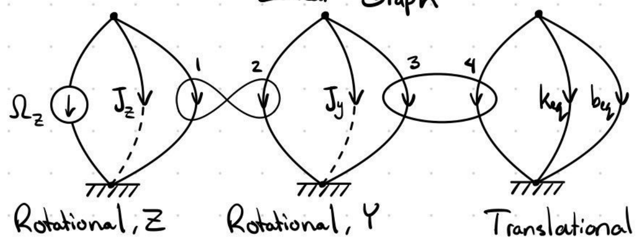
$$\zeta = 0$$

$\downarrow$

as expected for a system without dampers

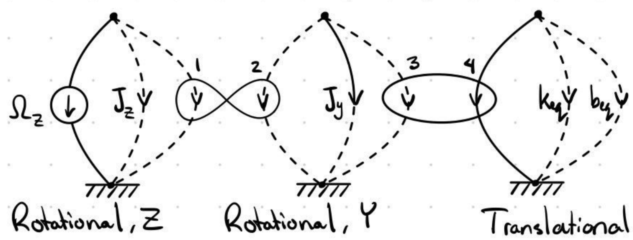
Part 2 $\quad k_1 = k_2 \quad b_1 = b_2$

Linear Graph




$$k_{eq} = k_1 + k_2 \; ; \; b_{eq} = b_1 + b_2$$

Normal Tree



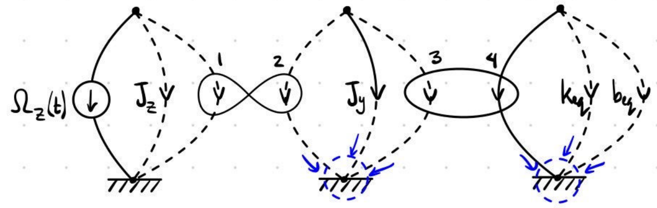

$$k_{eq} = k_1 + k_2 \; ; \; b_{eq} = b_1 + b_2$$

Transducer Matrices

$$\begin{bmatrix} \Omega_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{H_3} \\ H_3 & 0 \end{bmatrix} \begin{bmatrix} \Omega_2 \\ T_2 \end{bmatrix}$$

$$\begin{bmatrix} \Omega_3 \\ T_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & -r \end{bmatrix} \begin{bmatrix} V_4 \\ F_4 \end{bmatrix}$$

## Normal Tree




| Elemental | Transducer | Compatibility | Continuity |
|---|---|---|---|
| $T_{J_z} = J_z \dot{\Omega}_{J_z}$ | $\Omega_1 = -\frac{1}{H_3} T_2$ | $\Omega_{J_z} = \Omega_z(t)$ | $T_2 + T_{J_y} + T_3 = 0$ |
| $T_{J_y} = J_y \dot{\Omega}_{J_y}$ | $T_1 = H_3 \Omega_2$ | $\Omega_1 = \Omega_z(t)$ | $F_4 + F_{keq} + F_{beq} = 0$ |
| $\dot{F}_{keq} = k_{eq} V_{keq}$ | $\Omega_3 = \frac{1}{r} V_4$ | $\Omega_2 = \Omega_{J_y}$ | |
| $F_{beq} = b_{eq} V_{beq}$ | $T_3 = -r F_4$ | $\Omega_3 = \Omega_{J_y}$ | |
| | | $V_{keq} = V_4$ | |
| | | $V_{beq} = V_4$ | |

State Variables $\Omega_{J_y}$ , $F_{keq}$

$$
\begin{aligned}
\dot{\Omega}_{J_y} &= \frac{1}{J_y} T_{J_y} \\
&= \frac{1}{J_y}(-T_2 - T_3) \\
&= \frac{1}{J_y}(H_3 \Omega_1 + r F_4) \\
&= \frac{1}{J_y}(H_3 \Omega_z + r(-F_{keq} - F_{beq})) \\
&= \frac{1}{J_y}(H_3 \Omega_z + r(-F_{keq} - b_{eq} V_{beq})) \\
&= \frac{1}{J_y}(H_3 \Omega_z - r F_{keq} - r b_{eq} V_4) \\
&= \frac{1}{J_y}(H_3 \Omega_z - r F_{keq} - r^2 b_{eq} \Omega_3) \\
&= \frac{1}{J_y}(H_3 \Omega_z - r F_{keq} - r^2 b_{eq} \Omega_{J_y})
\end{aligned}
$$

$$\dot{\Omega}_{J_y} = -\frac{r^2 b_{eq}}{J_y} \Omega_{J_y} - \frac{r}{J_y} F_{keq} + \frac{H_3}{J_y} \Omega_z$$

$$
\begin{aligned}
\dot{F}_{keq} &= k_{eq} V_{keq} \\
&= k_{eq} V_4 \\
&= k_{eq} r \Omega_3
\end{aligned}
$$

$$\dot{F}_{keq} = k_{eq} r \Omega_{J_y}$$

$$F_{keq} = \int k_{eq} r \Omega_{J_y} \, dt$$

$$F_{keq} = k_{eq} r \theta_y$$

$$\theta_y = \frac{1}{k_{eq} r} F_{keq}$$

estimate $\dot{\Omega}_{J_y} \approx 0$ $\Omega_{J_y} \approx 0$

$$0 = -\frac{r^2 b_{eq}}{J_y}(0) - \frac{r}{J_y} F_{keq} + \frac{H_3}{J_y} \hat{\Omega}_z$$

$$\frac{H_3}{J_y} \hat{\Omega}_z = \frac{r}{J_y} F_{keq}$$

$$H_3 \hat{\Omega}_z = r F_{keq}$$

$$\hat{\Omega}_z = \frac{r}{H_3} F_{keq}$$

State Equation

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\Omega}_{J_y} \\ \dot{F}_{k_{eq}} \end{bmatrix} = \overset{A}{\begin{bmatrix} -\frac{r^2 b_{eq}}{J_y} & -\frac{r}{J_y} \\ k_{eq} r & 0 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} \Omega_{J_y} \\ F_{k_{eq}} \end{bmatrix}} + \overset{B}{\begin{bmatrix} \frac{H_3}{J_y} \\ 0 \end{bmatrix}} \overset{\vec{u}}{\begin{bmatrix} \Omega_z \end{bmatrix}}$$

Output Equation

$$\vec{y} = \begin{bmatrix} \Theta_y \\ \hat{\Omega}_z \end{bmatrix} = \overset{C}{\begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} \Omega_{J_y} \\ F_{k_{eq}} \end{bmatrix}} + \overset{D}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \overset{\vec{u}}{\begin{bmatrix} \Omega_z \end{bmatrix}}$$

$$H(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix} \begin{bmatrix} s + \frac{r^2 b_{eq}}{J_y} & \frac{r}{J_y} \\ -k_{eq} r & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{H_3}{J_y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix} \frac{1}{\left(s + \frac{r^2 b_{eq}}{J_y}\right)(s) - \left(\frac{r}{J_y}\right)\left(-k_{eq} r\right)} \begin{bmatrix} s & -\frac{r}{J_y} \\ k_{eq} r & s + \frac{r^2 b_{eq}}{J_y} \end{bmatrix} \begin{bmatrix} \frac{H_3}{J_y} \\ 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + \frac{r^2 b_{eq}}{J_y} s + \frac{k_{eq} r^2}{J_y}} \begin{bmatrix} 0 & \frac{1}{k_{eq} r} \\ 0 & \frac{r}{H_3} \end{bmatrix} \begin{bmatrix} s \frac{H_3}{J_y} \\ k_{eq} r \frac{H_3}{J_y} \end{bmatrix}$$

$$= \frac{1}{s^2 + \frac{r^2 b_{eq}}{J_y} s + \frac{k_{eq} r^2}{J_y}} \begin{bmatrix} \frac{H_3}{J_y} \\ \frac{k_{eq} r^2}{J_y} \end{bmatrix}$$

$$H(s) = \begin{bmatrix} \dfrac{\frac{H_3}{J_y}}{s^2 + \frac{r^2 b_{eq}}{J_y} s + \frac{k_{eq} r^2}{J_y}} \\ \dfrac{\frac{k_{eq} r^2}{J_y}}{s^2 + \frac{r^2 b_{eq}}{J_y} s + \frac{k_{eq} r^2}{J_y}} \end{bmatrix}$$

Transfer function of estimated yaw-rate to actual yaw-rate

$$G(s) = \frac{\hat{\Omega}_z(s)}{\Omega_z(s)} = \frac{\frac{K_{ey} r^2}{J_y}}{s^2 + \frac{r^2 b_q}{J_y} s + \frac{K_{ey} r^2}{J_y}}$$

At steady state $s = 0$

$$G(0) = \frac{\hat{\Omega}_z(0)}{\Omega_z(0)} = \frac{\frac{K_{ey} r^2}{\cancel{J_y}}}{\cancel{0^2 + \frac{r^2 b_q}{J_y} 0} + \frac{K_{ey} r^2}{\cancel{J_y}}} = 1$$

As expected steady-state estimated yaw-rate = actual yaw-rate

$\therefore$ transfer function = 1 @ steady-state

Calculating $k_1, k_2, b_1, b_2$ corresponding to a
Maximum Overshoot, $M_p = 3\%$
2% settling time, $t_s = 0.01$ sec

The maximum overshoot percentage can be computed as

$$M_p = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}, \tag{C.7}$$

from reference [2].

$$0.03 = e^{\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

$$\ln(0.03) = \frac{\zeta\pi}{\sqrt{1-\zeta^2}}$$

$$\sqrt{1-\zeta^2}\,\ln(0.03) = \zeta\pi$$

$$(1-\zeta^2)\ln(0.03)^2 = \zeta^2\pi^2$$

$$\ln(3)^2 - \zeta^2\ln(0.03)^2 - \zeta^2\pi^2 = 0$$

$$\zeta^2(\ln(0.03)^2 + \pi^2) = \ln(0.03)^2$$

$$\zeta = \pm\sqrt{\frac{\ln(0.03)^2}{\ln(0.03)^2 + \pi^2}}$$

$$\cancel{\zeta = \frac{\ln(0.03)}{\sqrt{\ln(0.03)^2 + \pi^2}}}$$

$$\zeta = \frac{-\ln(0.03)}{\sqrt{\ln(0.03)^2 + \pi^2}}$$

$$\zeta = 0.7448$$

For a margin of ±2%, the settling time can be computed as

$$t_s = \frac{4}{\zeta \omega_n}. \quad \text{(C.4)}$$

from reference [2].

$$0.01 = \frac{4}{\zeta \omega_n}$$

$$\zeta \omega_n = 400$$

$$\omega_n = \frac{400}{\zeta}$$

$$\omega_n = \frac{400}{0.7448}$$

$$\omega_n = 537.1 \frac{rad}{sec}$$

Transfer function of estimated yaw-rate to actual yaw-rate

$$G(s) = \frac{\frac{k_{eq} r^2}{J_y}}{s^2 + \frac{r^2 b_{eq}}{J_y} s + \frac{k_{eq} r^2}{J_y}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = \frac{k_{eq} r^2}{J_y}$$

$$k_{eq} = \frac{J_y \omega_n^2}{r^2}$$

$$= \frac{(2\times10^{-5}\,kg\,m^2)(537.1\,\frac{rad}{sec})^2}{(0.05\,m)^2}$$

$$k_{eq} = 2307.4\,\frac{N}{m}$$

$$2\zeta\omega_n = \frac{r^2 b_{eq}}{J_y}$$

$$b_{eq} = \frac{2 J_y \zeta \omega_n}{r^2}$$

$$b_{eq} = \frac{2(2\times10^{-5}\,kgm^2)(0.7448)(537.1\,\frac{rad}{sec})}{(0.05m)^2}$$

$$b_{eq} = 6.4\,\frac{Ns}{m}$$

## Analysis

This section will define simulation parameters and confirm the hand calculations based on simulation parameters.

```
% Clear workspace of previous MATLAB calculations
```

```
clear;

% Variable Definition
D = 0.04; % [m]          Flywheel Diameter
R = D/2;  % [m]          Flywheel Radius
m = 0.2;  % [kg]         Flywheel Mass
r = 0.05; % [m]          Distance to Each Strut from center joint
omegax = 10000; % [rpm]  Flywheel Constant Velocity
omegax = omegax*(2*pi)/60; % [rad/sec]
J_x = (1/2)*m*R^2 % [kg*m^2]
```

```
J_x =
4.0000e-05
```

```
J_y = (1/4)*m*R^2 % [kg*m^2]
```

```
J_y =
2.0000e-05
```

```
J_z = J_y; % [kg*m^2]
H_3 = J_x*omegax % [kg*m^2*rad/sec]
```

```
H_3 =
0.0419
```

## Part 1

This section calculates the equivalent spring constant based on the given k_1 and k_2. This section also uses the A, B, C, and D symbolically calculated matrices from the Hand Calculations Section to recalculate the transfer function matrix and calculated natural frequency, wn, and damping ratio, zeta.

```
% Given k_1 = k_2 = 500 [N/m] and b_1 = b_2 = 0
% k_eq = k_1 + k_2 and there is no damper in the system
k_eq = 500 + 500; % [N/m] Equivalent spring constant for both struts

% Calculate A, B, C, D with k_eq:
A = [   0       -r/J_y
      k_eq*r       0   ];

B = [ H_3/J_y
        0    ];

C = [ 0    1/k_eq*r
      0     r/H_3   ];

D = [ 0
      0 ];

sys = ss(A, B, C, D)
```

```
sys =

  A =
           x1     x2
```

```
    x1      0  -2500
    x2     50      0

  B =
          u1
    x1  2094
    x2     0

  C =
           x1     x2
    y1      0  5e-05
    y2      0  1.194

  D =
        u1
    y1   0
    y2   0

Continuous-time state-space model.
Model Properties
```

```
tf(sys)
```

```
ans =

  From input to output...
           5.236
   1:  -------------
       s^2 + 1.25e05

          1.25e05
   2:  -------------
       s^2 + 1.25e05

Continuous-time transfer function.
Model Properties
```

```
[wn_1, zeta_1] = damp(sys)
```

```
wn_1 = 2×1
  353.5534
  353.5534
zeta_1 = 2×1
     0
     0
```

## Part 2

This section calculates the spring constants, k_1 and k_2, and the damping constants, b_1 and b_2, corresponding to the given settling time and maximum overshoot. This section also uses the A, B, C, and D symbolically calculated matrices from the Hand Calculations Section to recalculate the transfer function matrix and calculated natural frequency, wn, and damping ratio, zeta to ensure the calculated spring constants and damping constants result in a system with the expected wn and zeta.

```
% Calculating k_1, k_2, b_1, b_2 corresponding to 2% settling time of
% t_s = 0.01 [sec] and a maximum overshoot, M_p = 3 [%]

t_s = 0.01; % [sec]
M_p = 0.03; % [-]
```

```
zeta = -log(M_p)/sqrt(log(M_p)^2 + pi^2) % [-]
```

```
zeta =
0.7448
```

```
wn = 4/(t_s*zeta) % [rad/sec]
```

```
wn =
537.0544
```

```
k_eq = (J_y*wn^2)/r^2 % [N/m] Equivalent spring constant for both struts
```

```
k_eq =
2.3074e+03
```

```
k_1 = k_eq/2 % [N/m]
```

```
k_1 =
1.1537e+03
```

```
k_2 = k_1    % [N/m]
```

```
k_2 =
1.1537e+03
```

```
b_eq = 2*J_y*zeta*wn/r^2 % [N*s/m] Equivalent damping constant for both struts
```

```
b_eq =
6.4000
```

```
b_1 = b_eq/2 % [N*s/m]
```

```
b_1 =
3.2000
```

```
b_2 = b_1    % [N*s/m]
```

```
b_2 =
3.2000
```

```
% Recalculate A, B, C, D with new k_eq and b_eq:
A = [ -(r^2*b_eq)/J_y     -r/J_y
           k_eq*r            0   ];

B = [ H_3/J_y
         0    ];

C = [ 0    1/k_eq*r
      0     r/H_3   ];

D = [ 0
      0 ];

sys = ss(A, B, C, D)
```

```
sys =
```

```
  A =
          x1     x2
   x1   -800  -2500
   x2  115.4      0

  B =
         u1
   x1  2094
   x2     0

  C =
              x1         x2
   y1          0  2.167e-05
   y2          0      1.194

  D =
       u1
   y1   0
   y2   0

Continuous-time state-space model.
Model Properties
```

```
tf(sys)
```

```
ans =

  From input to output...
              5.236
   1:  ----------------------
       s^2 + 800 s + 2.884e05

             2.884e05
   2:  ----------------------
       s^2 + 800 s + 2.884e05

Continuous-time transfer function.
Model Properties
```

```
[wn_2, zeta_2] = damp(sys)
```

```
wn_2 = 2×1
  537.0544
  537.0544
zeta_2 = 2×1
    0.7448
    0.7448
```

## Discussion Questions

### Question 1

*Are there any remaining assumptions other than those mentioned in this assignment?*

The additional assumptions used, but not explicitly stated in this assignment, are as follows: the system is frictionless, it has uniform material properties, the gyrator can be approximated as a thin disk, and the state equations model the system at steady state. This means the angular acceleration and velocity in both roll and pitch are zero (although the angle itself is not, as it is an output of the equations).

We assume the system is frictionless because the shaft must be supported by some type of bearing, which is not included in the model. The assumption of uniform material properties ensures that the center of rotation of the thin disk coincides with its gravitational center. Finally, we assume the flywheel is spinning at such a high speed that any angular velocity or acceleration caused by yawing of the plane is negligible at steady state, and therefore has an insignificant effect on the overall system dynamics.

**Question 2**

*Consider what may happen if the angle of deflection is large; what can be done to guarantee that the maximum deflection is less than some acceptable limit?*

If the deflection angle is found to be too large, two straightforward adjustments can bring it back within a reasonable range. The first is to increase the spring constants of both springs, which limits the gyroscope's rotation under the same applied torque, resulting in smaller spring deflections for the same turns. The second is to increase the strut distance relative to the center joint. This greater distance reduces the force each spring must exert to counteract the applied torques because less force is needed from each spring for the same torque.

Either adjustment or a combination of both allows the gyroscope to be rated up to a certain angle of turn, ensuring that the applied torque does not exceed a set limit and that the deflection remains within acceptable bounds. To prevent  spring breakage if this limit is exceeded, stoppers could be added to ensure the springs do not stretch beyond the point of permanent deformation. However, these stoppers can cause inaccuratacies in yaw readings unless they are equipped to measure the force applied to them. However, this is generally not recommended because these stoppers should only engage when the system operates outside the intended design parameters.

**Question 3**

*From this model would it be possible to determine the reaction torque on the clevis supporting the flywheel? If so, what would the expression be?*

Yes, the reaction torque on the clevis supporting the flywheel is $T_1$ in the transformer equation shown in the Problem Statement Section.

$$\begin{bmatrix} \Omega_1 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{H_3} \\ H_3 & 0 \end{bmatrix} \begin{bmatrix} \Omega_2 \\ T_2 \end{bmatrix} \text{ [1].}$$

The equation corresponding to $T_1$ is,

$$T_1 = H_3 \Omega_2$$

$T_1$ represents the torque cause by angular momentum, $H_3$, in respond to angular velocity in the z-direction (change of angle in yaw).

**Question 4**

*If this sensor were to be implemented in an actual aircraft what parameters would need to be measured in order to compute the estimated yaw-rate?*

The two inputs to the system, $\Omega_{J_y}$ and $F_{k_{eq}}$, since the roll angle, $\theta_y$, and estimated yaw-rate, $\widehat{\Omega_z}$, both rely on the magnitude of these values to calculate them. The estimated yaw-rate, $\widehat{\Omega_z}$, is also dependent upon the angular momentum in the x-direction, $H_3$, which relies upon $\Omega_x$ to calculate. All constant parameters of the system must also be known, such as the spring constants, damper constants, strut distance, and the moment of inertia in each direction to compute the estimated yaw-rate.