Numerical Methods for Integration

Objectives:

By the end of these notes, you should be able to:

- ☐ Explain the concept of numerical analysis and why it is used.
- ☐ Apply Euler's method to integrate functions.
- ☐ Write a high order differential equation using the state space representation.
- ☐ Solve a second order differential equation numerically.

Use these objectives as checkboxes. If you feel like you have achieved these objectives, you should be good to go. Otherwise, please look at the extra example problems or get in touch with me for assistance.

Function for class

```
v0 = 3;
T_final = 1;
lineThickness = 1;
deltaT = 0.2;
v = [ ];
Time = [ ];
v(1) = v0;
i = 1;
for t = 0 : deltaT : T_final
    dvdt = 3/v(i);
    v(i+1) = v(i) + deltaT * dvdt;
    Time(i) = t;
    i = i+1;
end
plot(Time, v(1:end-1), 'lineWidth', lineThickness)
hold on
```

Background

As we are going through this course, you will notice that we will analyze systems at 1 state (when studying kinematics or Newton's Second Law) or after the transition between one state and the next (Energy Methods and Impulse-Momentum Methods). This is usually out of convenience, or because the state studied is of particular importance.

In more practical applications, it is more useful to study the system for a long duration of time, as the system is transitioning between many states. In doing so, it is convenient to solve the problem symbolically and then produce the time response of the system using a computer.

Another reason for using computers to solve engineering problems, especially for dynamics, is that the response of the system studied is almost always modeled as a differential equation. In some cases, these differential equations are (relatively easily) solvable. In others, the system may be nonlinear and the solution to these differential equations may not be possible analytically (through closed form solutions and mathematical trick). In such case, there is no better option to choose but to use a computer to solve the differential equations.

For computers to solve equations, they need methods that deal with numbers direction, instead of symbols, characters and letters. These methods are called **'Numerical Methods'**. While numerical methods are used to solve so many problems that are unsolvable analytically, we will only introduce one small concept in this field: **numerical integration**.

Numerical Integration

Imagine that we are analyzing a particle moving in one dimension x due to the influence of an external force $F = 3m/v$, where m is the mass of the particle and v is its speed (an adaptive force). We want to find:

a. The velocity of the particle as a function of time, for $t \in [0,1]$.
b. The position of the particle as a function of time $t \in [0,1]$.

We know that the particle initially was traveling with a speed v_0 and it was at position x_0.

$\Sigma F = ma \Rightarrow 3\cancel{m}/v = \cancel{m}a \Rightarrow \boxed{a = \frac{3}{v}}$

initial conditions: $v(0) = v_0$, $x(0) = x_0$

$\frac{dv}{dt} = \frac{3}{v} \Rightarrow \int_{v_0}^{v(t)} v\,dv = \int_0^t 3\,dt$ (speration of variables)

$\Rightarrow \frac{1}{2}v^2 \Big|_{v_0}^{v(t)} = 3t \Big|_0^t \Rightarrow \frac{1}{2}v^2(t) - \frac{1}{2}v_0^2 = 3t$

$\Rightarrow v^2(t) = 6t - v_0^2 \Longrightarrow \boxed{v(t) = \sqrt{6t + v_0^2}}$

same method: $\boxed{x(t) = x_0 + \frac{1}{9}(6t + v_0^2)^{3/2}}$

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Numerical methods for solving for velocity:

Let's say we cannot integrate the function

we still known that $\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$

if Δt is small: $\frac{dv}{dt} \approx \frac{v(t+\Delta t) - v(t)}{\Delta t}$

$\Rightarrow \boxed{v(t+\Delta t) = v(t) + \frac{dv}{dt}\Delta t}$ Euler's integration method

In our example: $v(t+\Delta t) = v(t) + \frac{3\Delta t}{v(t)}$

iteration:

t	$v(t)$	$v(t+\Delta t)$
0	1	$1 + \frac{3(0.1)}{1} = 1.3$
0.1	1.3	$1.3 + \frac{(3)(0.1)}{1.3} = 1.531$
⋮	⋮	⋮

$\Delta t = 0.1$ s

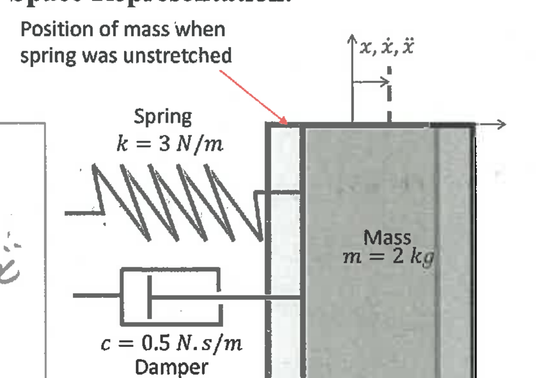
lets say: $v_0 = 1$

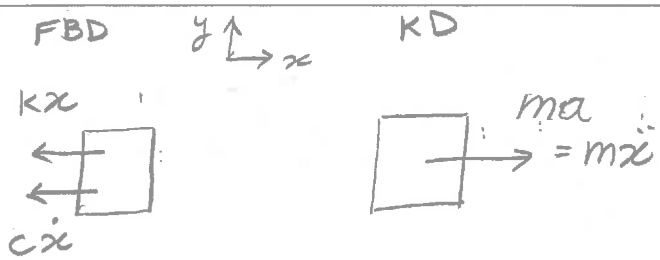
Second Order Differential Equations and State Space Representation

In dynamics, it is very common having to work with second order differential equations. Most of the tools we have (including the one shown above) only work on first order differential equations.

Luckily, there is a trick to convert second order differential equations into a system of linear differential equations. This trick is the use of the **State Space Representation**.

First, we derive the equation of motion from Newton's Second Law:





$\Sigma F_x = ma_x \Rightarrow m\ddot{x} = -kx - c\dot{x}$

$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = 0}$

second order DE

Now, we put the system in state space form as follows:

Ex: $m=2$, $c=0.5$, $k=3$ $\Rightarrow$ $2\ddot{x}+0.5\dot{x}+3x=0$

$\Rightarrow$ $\ddot{x}=-0.25\dot{x}-1.5x$

lets define $x_1(t)=x(t)$

① — $x_2(t)=\dot{x}_1(t)$ By definition

② — $\dot{x}_2(t)=-0.25x_2(t)-1.5x_1(t)$ From system

$$\begin{Bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{Bmatrix}=\begin{bmatrix}0 & 1\\ -1.5 & -0.25\end{bmatrix}\begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix} \quad — ③$$

And finally, we setup the state space equation to be solved numerically using the Euler Method as follows:

Noting that $\dot{x}_1(t)\approx\dfrac{x_1(t+Dt)-x_1(t)}{Dt}$ — ④

$\dot{x}_2(t)\approx\dfrac{x_2(t+Dt)-x_2(t)}{Dt}$ — ⑤

Plug ④ & ⑤ into ③

Multiply by Dt & manipulate a bit

$$\begin{Bmatrix}\dfrac{x_1(t+Dt)-x_1(t)}{Dt}\\ \dfrac{x_2(t+Dt)-x_2(t)}{Dt}\end{Bmatrix}=\begin{bmatrix}0 & 1\\ -1.5 & -0.25\end{bmatrix}\begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix}$$

$$\begin{Bmatrix}x_1(t+Dt)\\ x_2(t+Dt)\end{Bmatrix}=\begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix}+\Delta t\begin{bmatrix}0 & 1\\ -1.5 & -0.25\end{bmatrix}\begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix}$$

References:

1. Prof Eileen Rossman Lecture Notes.
2. Griffiths, D. V., & Smith, I. M. (2006). Numerical methods for engineers. CRC press.

Runge-kutta $y(t+Dt)=y(t)+\frac{1}{6}(k_1+2k_2+2k_3+k_4)Dt$

f is $\frac{d}{dt}$ function

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$k_1=f(t,y(t))$

$k_2=f(t+\frac{Dt}{2},\ y(t)+\frac{Dt}{2}k_1)$

$k_3=f(t+\frac{Dt}{2},\ y(t)+\frac{Dt}{2}k_2)$

$k_4=f(t+Dt,\ y(t)+Dt\,k_3)$