MECHANICAL VIBRATIONS [ME 318]
FORMULA SHEET

SPRING-MASS SYSTEM

$$\boxed{m\ddot{x} + kx = 0 \quad (x_0 = x(0)\ ;\ v_0 = \dot{x}(0))}$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

$$\boxed{x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin\left(\omega_n t + \tan^{-1}\frac{\omega_n x_0}{v_0}\right)}$$

SPRING-MASS-DAMPER SYSTEM

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0 \quad x_0 = x(0)\ ;\ v_0 = \dot{x}(0)}$$

$$\boxed{\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}}$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}} \quad \& \quad \zeta = \frac{c}{2\sqrt{mk}}}$$

$$\boxed{\begin{aligned} \lambda_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} \quad \text{for } \zeta \geq 1 \\ \lambda_{1,2} &= -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \quad \text{for } \zeta < 1 \end{aligned}}$$

ω_d

UNDERDAMPED MOTION

$$\boxed{x(t) = A e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}$$

$$\boxed{A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}} \qquad \phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)}$$

SMALL ANGLE APPROXIMATION

FOR EULER'S LAW

$$\boxed{\sin\theta \approx \theta \quad ; \quad \cos\theta \approx 1}$$

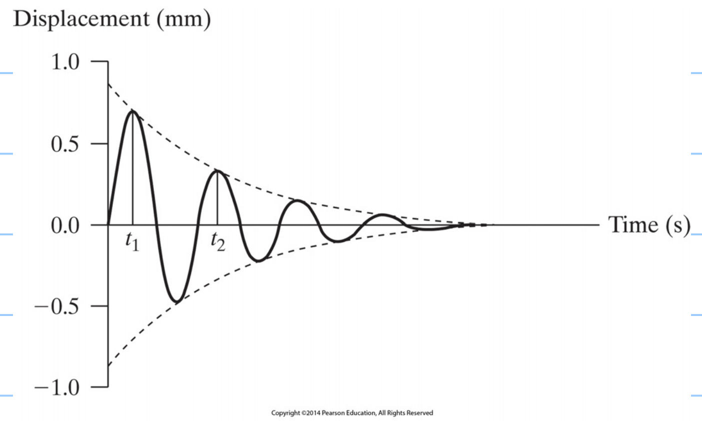
FOR RAYLEIGH'S METHOD

$$\boxed{\sin\theta \approx \theta \quad ; \quad \cos\theta \approx 1 - \frac{\theta^2}{2}}$$

Rayleigh Method Formulation

$$\boxed{T_{max} = U_{max} - U_{min}}$$

Log Decrement



$$\boxed{\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}}$$

$$\boxed{\delta = \frac{1}{n} \ln\left(\frac{x_1}{x_{n+1}}\right)}$$

Harmonic Excitation of Undamped Systems

$$\boxed{m\ddot{x} + kx = F_0 \cos \omega t}$$

$$\boxed{\ddot{x} + \omega_n^2 x = f_0 \cos \omega t}$$

$$\boxed{x(t) = \left(\frac{v_0}{\omega_n}\right) \sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right) \cos \omega_n t + \left(\frac{f_0}{\omega_n^2 - \omega^2}\right) \cos \omega t}$$

$$\boxed{x(t) = \left[\frac{2f_0}{\omega_n^2 - \omega^2} \sin\left(\frac{\omega_n - \omega}{2}\right)t\right] \sin\left(\frac{\omega_n + \omega}{2}\right)t}$$

$$\boxed{x(t) = \frac{v_0}{\omega} \sin\omega t + x_0 \cos\omega t + \frac{f_0}{2\omega} t \sin\omega t}$$

HARMONIC EXCITATION OF DAMPED SYSTEMS

$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t) = F_0 \cos\omega t}$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos\omega t}$$

$$x(t) = \left(\frac{x_0 - X\cos\theta}{\sin\phi}\right) e^{-\zeta\omega_n t} *$$

$$* \sin\left[\omega_d t + \tan^{-1}\frac{\omega_d(x_0 - X\cos\theta)}{(v_0 + (x_0 - X\cos\theta)\zeta\omega_n - \omega X \sin\theta)}\right]$$

$$+ \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos\left(\omega t - \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)}\right]\right)$$

Complex Algebra

$$C_3 = \frac{C_1}{C_2} = \frac{x+iy}{u+iv}$$

$$|C_3| = M = \left|\frac{C_1}{C_2}\right| = \left|\frac{x+iy}{u+iv}\right| = \frac{\sqrt{x^2+y^2}}{\sqrt{u^2+v^2}}$$

$$\angle C_3 = \phi = \angle\left(\frac{C_1}{C_2}\right) = \angle\left(\frac{x+iy}{u+iv}\right) = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{v}{u}$$

TABLE 4.1-2 Steady-State Sine Response of a Second-Order Model

Model: $m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$

$$\zeta = \frac{c}{2\sqrt{mk}} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad r = \frac{\omega}{\omega_n}$$

Steady-state response:

$$x_{ss}(t) = \frac{F_o}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\sin(\omega t + \phi)$$

$$\phi = -\tan^{-1}\frac{2\zeta r}{1-r^2}$$

Third quadrant if $1 - r^2 < 0$
Fourth quadrant if $1 - r^2 > 0$
$\phi = -90°$ if $r^2 = 1$

Magnitude ratio: $M = \frac{|X_{ss}|}{F_o} = \frac{1}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = \frac{1}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Dimensionless magnitude ratio: $\frac{X}{\delta_{st}} = kM \qquad \delta_{st} = \frac{F_o}{k}$

Peak frequency: $\omega_p = \omega_n\sqrt{1-2\zeta^2} \qquad 0 \le \zeta \le \frac{1}{\sqrt{2}}$

Peak response: $X_p = \delta_{st}\frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad 0 \le \zeta \le \frac{1}{\sqrt{2}}$

BASE EXCITATION OF SYSTEMS

$$\boxed{m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky}$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega_b Y \cos\omega_b t + \omega_n^2 Y \sin\omega_b t}$$

$$x_{ss}(t) = \omega_n Y \left[\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}\right]^{1/2} \cos(\omega_b t - \theta_1 - \theta_2)$$

where,

$$\theta_1 = \tan^{-1}\frac{2\zeta\omega_n\omega_b}{(\omega_n^2 - \omega_b^2)}$$

&

$$\theta_2 = \tan^{-1}\frac{\omega_n}{2\zeta\omega_b}$$

$$\boxed{\left|x_{ss}(t)\right|_{max} = X = \omega_n Y \left[\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}\right]^{1/2}}$$

$$\boxed{\frac{X}{Y} = T_d = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}}$$

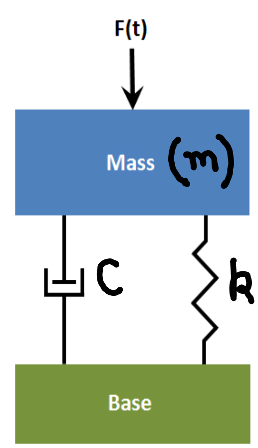
DISPLACEMENT TRANSMISSIBILITY

$$\boxed{F_t = m\omega_b^2\omega_n Y\left[\frac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)+(2\zeta\omega_n\omega_b)^2}\right]^{1/2}\cos(\omega_b t-\theta_1-\theta_2)}$$

$$\boxed{T_f = \frac{F_T}{kY} = r^2\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}}$$

FORCE TRANSMISSIBILITY

DIRECT EXCITATION

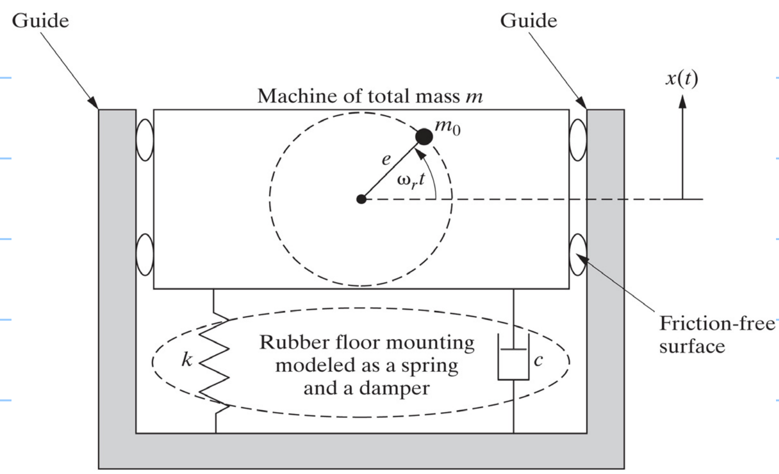


$$\boxed{m\ddot{x}+c\dot{x}+kx = F(t)}$$

$$\boxed{\frac{F_T}{F_0} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}}$$

→ FORCE TRANSMISSIBILITY

Rotating Unbalance In Systems



$$\boxed{m\ddot{x} + c\dot{x} + kx = m_0 e \omega_r^2 \sin(\omega_r t)}$$

$$\boxed{X = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

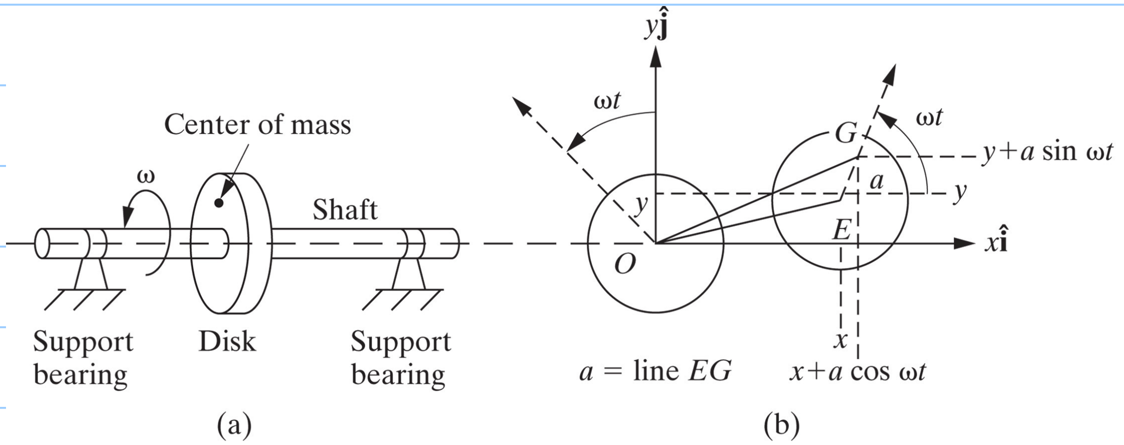
Motion Amplitude

$$\boxed{\frac{F_T}{F_0} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}}$$

Force Transmissibility

$$\boxed{F_T = m_0 \omega_r^2 e \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}}$$

CRITICAL SPEED OF ROTATING DISCS



$$\boxed{\begin{aligned} m\ddot{x} + c\dot{x} + kx &= m\omega^2 a \cos\omega t \\ m\ddot{y} + c\dot{y} + ky &= m\omega^2 a \sin\omega t \end{aligned}}$$

$$\boxed{X = \frac{ar^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

AMPLITUDE OF WHIRL

Laplace Transform Tables

Item no.	Theorem		Name
1.	$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)]$	$= F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Item no.	***f*(*t*)**	***F*(*s*)**
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

TABLE 5.2-1 Laplace Transform Pairs

$F(s)$	$f(t), t \geq 0$
1. 1	$\delta(t)$, unit impulse at $t=0$
2. $\dfrac{1}{s}$	$u_s(t)$, unit step
3. $\dfrac{n!}{s^{n+1}}$	t^n
4. $\dfrac{1}{s+a}$	e^{-at}
5. $\dfrac{1}{(s+a)^n}$	$\dfrac{1}{(n-1)!}t^{n-1}e^{-at}$
6. $\dfrac{a}{s(s+a)}$	$1-e^{-at}$
7. $\dfrac{1}{(s+a)(s+b)}$	$\dfrac{1}{b-a}(e^{-at}-e^{-bt})$
8. $\dfrac{s+p}{(s+a)(s+b)}$	$\dfrac{1}{b-a}\left[(p-a)e^{-at}-(p-b)e^{-bt}\right]$
9. $\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{e^{-at}}{(b-a)(c-a)}+\dfrac{e^{-bt}}{(c-b)(a-b)}+\dfrac{e^{-ct}}{(a-c)(b-c)}$
10. $\dfrac{s+p}{(s+a)(s+b)(s+c)}$	$\dfrac{(p-a)e^{-at}}{(b-a)(c-a)}+\dfrac{(p-b)e^{-bt}}{(c-b)(a-b)}+\dfrac{(p-c)e^{-ct}}{(a-c)(b-c)}$
11. $\dfrac{b}{s^2+b^2}$	$\sin bt$
12. $\dfrac{s}{s^2+b^2}$	$\cos bt$
13. $\dfrac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
14. $\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
15. $\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}t \qquad \zeta<1$
16. $\dfrac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$	$1+\dfrac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right) \qquad \zeta<1$
	$\phi=\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}+\pi$ (third quadrant)

RESPONSE TO AN ARBITRARY PERIODIC INPUT

BANDWIDTH OF SECOND ORDER SYSTEMS

CASE 1 : DAMPING RATIO $(0 < \zeta \leq 0.38)$

System Bandwidth $\rightarrow (\omega_{B_1} \leq \omega_{i/p} \leq \omega_{B_2})$ rad/s

$\omega_{B_1} \rightarrow$ Bandwidth lower limit (f_1)

$$\omega_{B_1} = \omega_n \sqrt{1 - 2\zeta^2 - 2\zeta\sqrt{1-\zeta^2}}$$

$\omega_{B_2} \rightarrow$ Bandwidth higher limit (f_2)

$$\omega_{B_2} = \omega_n \sqrt{1 - 2\zeta^2 + 2\zeta\sqrt{1-\zeta^2}}$$

CASE 2: DAMPING RATIO $(0.38 < \zeta < 1)$

System Bandwidth $\rightarrow \quad 0 \leq \omega_{i/p} \leq \omega_B$

$$\boxed{\begin{array}{l} \omega_B \rightarrow \text{Bandwidth higher limit (BW)} \\ \omega_B = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}} \end{array}}$$

Impulse Response Function
[Impulse & Pulse]

Pulse Input $f(t) = M u_s(t) - M u_s(t-T)$

$$\boxed{F(s) = \frac{M}{s}\left[1 - e^{-sT}\right]}$$

Impulse Input $f(t) = A\,\delta(t)$

$$\boxed{F(s) = A}$$

[A = MT (Area under the Pulse)]

Impulse Approximation Criteria

$$\boxed{\frac{1}{\zeta\omega_n} \geq 10T}$$

Taylor Series Approx

$$\boxed{e^A = 1 + A}$$

Multiple Degrees Of Freedom

Cramer's Rule

Consider the linear system

$$\begin{cases} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Assume $a_1b_2 - b_1a_2$ nonzero.

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}.$$

Partial Fraction Expansion:

Case ⓐ : Real-Distinct Roots

$$X(s) = \frac{C_1}{s+r_1} + \frac{C_2}{s+r_2} + \cdots + \frac{C_n}{s+r_n}$$

$$\boxed{C_i = \lim_{s \to -r_i} \left[X(s)(s+r_i) \right]} \qquad i = 1, 2, \ldots, n$$

case ⓑ Real-Repeated Roots

$$X(s) = \frac{C_1}{(s+r_1)^p} + \frac{C_2}{(s+r_1)^{p-1}} + \cdots + \frac{C_p}{s+r_1} + \frac{C_{p+1}}{s+r_{p+1}} + \cdots + \frac{C_n}{s+r_n}$$

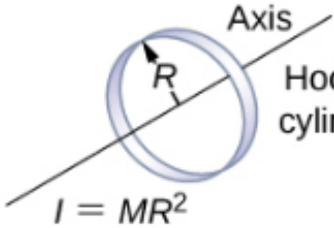
$$\boxed{C_i = \lim_{s \to -r_i} \left\{ \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[X(s)(s+r_1)^p \right] \right\}} \qquad i = 1, 2, 3, \ldots p$$

case ⓒ : Complex-Conjugate Roots

$$[s+(\sigma+j\omega)][s+(\sigma-j\omega)] \text{ as } (s+a)^2 + b^2$$

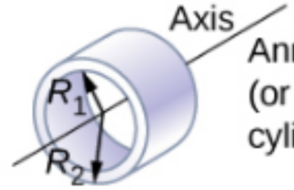
$$X(s) = \frac{C_1 (s+a)}{(s+a)^2 + b^2} + \frac{C_2\, b}{(s+a)^2 + b^2} + \frac{C_3}{s+r_3} + \cdots + \frac{C_n}{s+r_n}$$

MASS MOMENT OF INERTIA TABLE



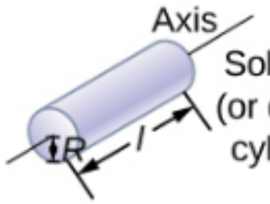
Hoop about cylinder axis

$I = MR^2$



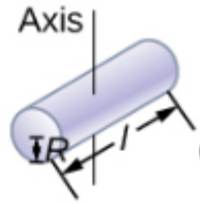
Annular cylinder (or ring) about cylinder axis

$I = \frac{M}{2}(R_1^2 + R_2^2)$



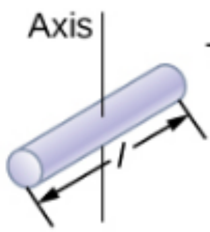
Solid cylinder (or disk) about cylinder axis

$I = \frac{MR^2}{2}$



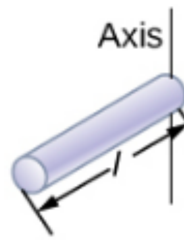
Solid cylinder (or disk) about central diameter

$I = \frac{MR^2}{4} + \frac{Ml^2}{12}$



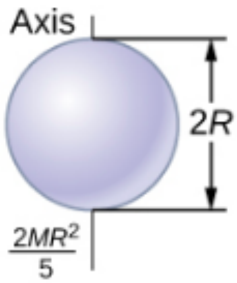
Thin rod about axis through center ⊥ to length

$I = \frac{Ml^2}{12}$



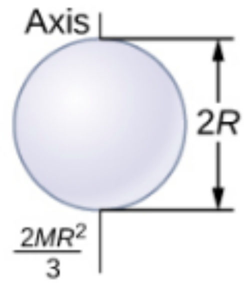
Thin rod about axis through one end ⊥ to length

$I = \frac{Ml^2}{3}$



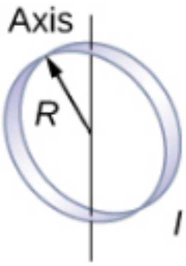
Solid sphere about any diameter

$I = \frac{2MR^2}{5}$



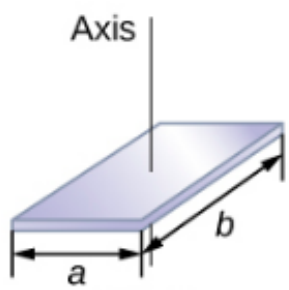
Thin spherical shell about any diameter

$I = \frac{2MR^2}{3}$



Hoop about any diameter

$I = \frac{MR^2}{2}$



Slab about ⊥ axis through center

$I = \frac{M(a^2 + b^2)}{12}$