INFO-F-524 - Continuous Optimisation

Project 2022

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Introduction

We consider the problem of partitioning the nodes of a graph into K subsets such that the sum of the costs of the edges between the subsets of the partition is minimized. In addition, we impose that each subset contains at most P nodes. This problem has a large number of applications, for example in circuit design or when partitioning CPUs on different boards for parallel architectures.

Formally, we are given a graph $G = (V, E)$ with $n = |V|$ nodes and edge costs $c_e \geq 0$ for all $e \in E$.

Model 1 (adapted from [1])

A natural formulation arises by considering explicitely K subsets of nodes, numbered from 1 to K, and to introduce binary variables z_i^k equal to one if and only if $i \in V$ belongs to subset k, and y_e^k equal to one if and only if edge $e \in E$ joins two nodes in subset k. Minimizing the sum of the costs of the edge between the subsets is equivalent to maximize the sum of the costs of the edges inside the subsets, leading to the following formulation.

$$
\begin{aligned}
\max \quad & \sum_{k=1}^{K} \sum_{e \in E} c_e y_e^k && && (1)\\
\text{s.t.} \quad & \sum_{k=1}^{K} z_i^k = 1 && i \in V && (2)\\
& y_{ij}^k \leq z_i^k && ij \in E,\ k = 1, \ldots, K && (3)\\
& y_{ij}^k \leq z_j^k && ij \in E,\ k = 1, \ldots, K && (4)\\
& 1 \leq \sum_{i \in V} z_i^k \leq P && k = 1, \ldots, K && (5)\\
& y_e^k \in \{0, 1\} && e \in E,\ k = 1, \ldots, K && (6)\\
& z_i^k \in \{0, 1\} && i \in V,\ k = 1, \ldots, K && (7)
\end{aligned}
$$

Because of the assumption $c_e \geq 0$, there always exists an optimal solution with $y_{ij}^k = 1$ if $z_i^k = z_j^k = 1$. Constraints (2) ensure each node is in exactly one subset. Constraints (3) and (4) link y and z variables. Finally, constraints (5) make sure subsets are not empty and contain at most P nodes.

Model 2

One drawback of Model 1 is that it induces a lot of symmetry, because the numbering of the subsets can be changed without modifying the solution. To avoid this problem, we propose a new formulation based on the notion of representatives. First, let us assume that the nodes in V are ordered and numbered, hence w.l.o.g. $V = \{1, \ldots, n\}$.

Instead of numbering the subsets, we choose in each subset the node with the smallest number as representative for that set. We modify the variables as follows: for every pair of nodes k, i with $k \leq i$, we introduce a binary variable z_i^k equal to one if and only if node i belongs to the subset represented by node k, and for each edge $ij \in E$ and each node $k \in V$ such that $k \leq i$ and $k \leq j$, we introduce a variable y_{ij}^k equal to one if both i and j are in the subset represented by k.

For example, if the partition contains subset $\{2, 4, 7\}$, it leads to $z_2^2 = z_4^2 = z_7^2 = 1$. Note that $z_i^i = 1$ if and only if i is representative for a subset.

We can now formulate our problem as:

$$
\begin{aligned}
\max \quad & \sum_{ij \in E} \sum_{k=1}^{\min\{i,j\}} c_{ij} y_{ij}^k & & (8)\\
\text{s.t.} \quad & \sum_{k=1}^{i} z_i^k = 1 & i \in V \quad & (9)\\
& y_{ij}^k \leq z_i^k & ij \in E,\ k = 1, \ldots, \min\{i,j\} \quad & (10)\\
& y_{ij}^k \leq z_j^k & ij \in E,\ k = 1, \ldots, \min\{i,j\} \quad & (11)\\
& \sum_{i > k} z_i^k \leq (P-1) z_k^k & k \in V \quad & (12)\\
& \sum_{k \in V} z_k^k = K & & (13)\\
& y_{ij}^k \in \{0, 1\} & e \in E,\ k = 1, \ldots, \min\{i,j\} \quad & (14)\\
& z_i^k \in \{0, 1\} & i \in V,\ k = 1, \ldots, i \quad & (15)
\end{aligned}
$$

Constraints (9) ensure each node is in exactly one subset (i.e. has exactly one representative). Constraints (10) and (11) link y and z variables. Constraints (12) make sure a node is representative if and only if z_k^k is equal to one, and that each node represents at most $P - 1$ other nodes, hence leading to subsets with at most P nodes. Finally, constraints (13) make sure we have exactly K subsets in the partition.

Instructions

All implementation should be made using either:

- the Pyomo python library python (Documentation and examples);
- or the JuMP modeling library for the Julia language (Example of Lagrangian relaxation code in Julia).

Use GLPK or Gurobi as MIP solver (instructions for Pyomo and Julia).

To Do

1. Implement the two models as Mixed Integer Programs.
2. Implement a Lagrangian relaxation of Model 2, relaxing constraints (9) and (13), and implement a subgradient algorithm to find (near-)optimal multipliers. Note: observe that the resulting subproblem can be decomposed by node, use this property to speed up the subproblem resolution. Bonus point if you can develop an alternative fast solution algorithm for the subproblem without solving a MIP explicitly.
3. Develop a heuristic for finding a feasible solution (a Lagrangian heuristic is preferred, but you can also implement any other method of your choice).
4. In your report:

(i) Detail the Lagrangian relaxation subproblem, your method for solving it and how you update the multipliers, as well as your heuristic.
(ii) Test your codes on various instances from the TSPLIB (Instances, Documentation) for $K = 3, 4, 5$ and different values of P, in particular $P = \lceil \frac{n}{K} \rceil$. You can fix a time limit (e.g. 10 minutes) for each solving method and instance. For each method (Model 1, Model 2 solved as MIP and Lagrangian relaxation combined with your heuristic for Model 2), report the computing time and the best upper and lower bounds found on the value of the optimal solution. The choice of instances tested is up to you, but try to select different sizes of graphs and report any comment you find interesting on the quality of the different methods.

Deadlines

- All the communication about the project should be made by email : bernard.fortz@ulb.be
- Send your group composition (max. 3 students per group) by **April 18**.
- Send your code and report by **June 13**.

References

[1] Ferreira, C.E., Martin, A., de Souza, C.C. et al. Formulations and valid inequalities for the node capacitated graph partitioning problem. Mathematical Programming 74, 247–266 (1996). https://doi.org/10.1007/BF02592198