Problem Set 6

COSC 290 Spring 2018

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1 Problem 1: Equivalence classes

1.1 DLN 8.110

R_1 is an equivalence relationship if and only if it is reflexive, symmetric, and transitive.

- *Reflexivity*: If A is an empty set, then the $A = A = \emptyset$, so $\langle A, A\rangle \in R_1$. If A is not an empty set, then the greatest element in A will equal the greatest element in A, because $A = A$, thus $\langle A, A\rangle \in R_1$. Therefore R_1 is reflexive.
- *Symmetry*: In the case that $A = \emptyset$ and $\langle A, B\rangle \in R_1$, then by the second condition $B = \emptyset$. Therefore, also by the second condition, $\langle B, A\rangle \in R_1$. In the case that $\langle A, B\rangle \in R_1$ and $A \neq \emptyset$, the greatest element in A must equal the greatest element in B. This means that the greatest element of B must equal the largest element in A, and so $\langle B, A\rangle \in R_1$, making R_1 symmetric.
- *Transitivity*: If $\langle A, B\rangle \in R_1$ and $\langle B, C\rangle \in R_1$, and $A = \emptyset$, then $A = B = \emptyset$, and thus $B = C = \emptyset$. Therefore, if A,B, and C are all empty sets, then $A = C = \emptyset$, and so $\langle A, C\rangle \in R_1$. In the case that A is not an empty set, then the greatest element in A must equal the greatest element in B. Since $\langle B, C\rangle \in R_1$, then the greatest element in B must equal the greatest element in C. Thus, the greatest element in A must equal the greatest element in C, therefore $\langle A, C\rangle \in R_1$. This means that the relation must be transitive.

Since the relation R_1 is reflexive, symmetric, and transitive, it is an equivalence relationship. The equivalence clauses are as follows:

- $\{\emptyset\}$
- $\{\{0\}\}$
- $\{\{1\}, \{0, 1\}\}$
- $\{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
- $\{\{3\}, \{0, 3\}, \{1, 3\}, \{2, 3\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}, \{0, 1, 2, 3\}\}$

1.2 DLN 8.111

R_1 is an equivalence relationship if and only if it is reflexive, symmetric, and transitive. $P = \mathcal{P}(\{0, 1, 2, 3\})$.

- *Reflexivity*: The sum of the items in A will always equal the sum of items in A. Thus, for every $A \in P, \langle A, A\rangle \in R_2$. Therefore, the relation is reflexive.
- *Symmetry*: If the sum of items in A equals the sum of items in B, the reverse must also be true. Therefore, for every $A, B \in P$, if $\langle A, B\rangle \in R_2$, then $\langle B, A\rangle \in R_2$. This means that R_2 is symmetric.
- *Transitivity*: If $\langle A, B\rangle \in R_2$, then the sum of elements in A equals the sum of elements in B. If $\langle B, C\rangle \in R_2$, then the sum of elements in B must equal the sum of elements in C. Thus, the sum of elements in all three sets (A, B, and C) must be equal, giving us $\langle A, C\rangle \in R_2$. This means that the relation R_2 is transitive.

Since the relation R_1 is reflexive, symmetric, and transitive, it is an equivalence relationship. The equivalence clauses are as follows:

- $\{\emptyset\}$
- $\{\{0\}\}$
- $\{\{1\}, \{0, 1\}\}$
- $\{\{2\}, \{0, 2\}\}$

- $\{\{3\}, \{0, 3\}, \{1, 2\}, \{0, 1, 2\}\}$
- $\{\{1, 3\}, \{0, 1, 3\}\}$
- $\{\{2, 3\}, \{0, 2, 3\}\}$
- $\{\{1, 2, 3\}, \{0, 1, 2, 3\}\}$

1.3 DLN 8.113

Etc.

2 Problem 2: DLN 8.84

Claim: The claim is that for any relation $R \subseteq A \times A$ that is both irreflexive and transitive, R must also be asymmetric.

Proof. We will prove by assuming the claim is false and showing a contradiction.

- *Given*: Assume a relation $R \subseteq A \times A$ is irreflexive, transitive, and not asymmetric.
- *Want to show*: The assumption leads to a contradiction.

By the definition of asymmetry, since R is not asymmetric there must be one pair $\langle a, b \rangle \in R$ such that $\langle b, a \rangle \in R$. By the definition of transitivity, if $\langle a, b \rangle \in R$, and $\langle b, a \rangle \in R$, then $\langle a, a \rangle \in R$. Thus, if R is not asymmetric and transitive, then we must have $\langle a, a \rangle \in R$.

By the definition of irreflexivity, for every $a \in A, \langle a, a \rangle \notin R$. By the assumption, R is not asymmetric and is transitive, thus $\langle a, a \rangle \in R$, and is also irreflexive, thus for every $a \in A, \langle a, a \rangle \notin R$. These two statements are in direct contradiction, therefore if R is both irreflexive and transitive, it must be asymmetric, proving the claim.

□

3 Problem 3: DLN 8.130

Replace this with your answer to problem 3.