Problem Set 4

COSC 290 Spring 2018

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1 Problem 1: Direct proof and proof by contrapositive

1.1 DLN 4.45

Claim: A x $B = B$ x $A \iff (A = \varnothing) \vee (B = \varnothing) \vee (A = B)$

This proof will be subdivided into two smaller proofs. The first being a direct proof of the $\implies$ and the second a contrapositive of $\Longleftarrow$. If both are proved true, then the overall claim must be true by mutual implication.

Proof 1 will directly show that A x $B = B$ x $A \implies (A = \varnothing) \vee (B = \varnothing) \vee (A = B)$.
Given that A x $B = B$ x A is true.
Want to show that $(A = \varnothing) \vee (B = \varnothing) \vee (A = B)$ is true.

This proof can exist in two cases: either one of A or B contains no elements and is an empty set, or both sets contain at least one element and are equal. The product of A x B produces the Cartesian product of the sets, defined by:

$$A \text{ x } B = P\{(a, b)|a \in A, b \in B\}$$

If either A or B is equal to $\emptyset$, the Cartesian products A x B and B x A will also be $\emptyset$, as there cannot be any (a, b) pair as defined above, because either A, B or both contain no elements to begin with.
The other way in which A and B can exist is if neither are equal to $\emptyset$, and thus contain at least one element. In this situation, A must equal B as both the Cartesian products of A and B are given as equal. This is because if A x $B = B$ x A, then every (a, b) pair must exactly equal one (b, a) pair, without exception. If A and B contained different elements, there would exist (a, b) pairs that would not match any (b, a) pairs. Thus, given A x $B = B$ x A, A and B must either exist such that at least one of them is equal to $\emptyset$, or they must be equal.

Proof 2 will show that A x $B = B$ x $A \Longleftarrow (A = \varnothing) \vee (B = \varnothing) \vee (A = B)$ by the contrapositive. The contrapositive claim is: $\neg(A \text{ x } B = B \text{ x } A) \implies \neg(A = \varnothing) \wedge \neg(B = \varnothing) \wedge \neg(A = B)$
Given that $\neg(A \text{ x } B = B \text{ x } A)$ is true.
Want to show that $\neg(A = \varnothing) \wedge \neg(B = \varnothing) \wedge \neg(A = B)$ is true.

In this case, neither A nor B can equal $\emptyset$, as the Cartesian products of A and B are known to not be equal. Per the definition of Cartesian products shown in proof 1, if either of the operands in a Cartesian product are equal to $\emptyset$, the result will also be equal to $\emptyset$. As it is given that $\neg(A \text{ x } B = B \text{ x } A)$, then neither A or B can be $\emptyset$. As such, $\neg(A = \varnothing) \wedge \neg(B = \varnothing)$ must be true.
It is also not possible that A is equal to B. If A and B were equal, there would be no distinction between elements $a \in A$ and elements $b \in B$, as we would have $A \equiv B$. Thus, there would be no difference between the ordered pairs (a, b) and (b, a), and consequently no difference between the products A x B and B x A. As such, $\neg(A = B)$ must be true, thus proving our contrapositive claim, and subsequently the original claim of **Proof 2**.

As **Proof 1** proves the $\implies$ aspect of the original claim, and **Proof 2** proves the $\longleftarrow$ side of it, then the original claim A x $B = B$ x $A \iff (A = \varnothing) \vee (B = \varnothing) \vee (A = B)$ must be true by mutual implication.

2 Problem 2: Proof by contradiction

2.1 DLN 4.60

Claim: For any array $A[1...n]$, A contains at most one strictly majority element.
Proof by contradiction: Assume the claim is false, and that there are *two* distinct elements x and y in A such that both x and y are strictly majority elements.

The definition of a strictly majority element is:

$$|i \in \{1, 2, ..., n\} : A[i] = x| > \frac{n}{2}$$

In natural language, this states that the cardinality of set X, where X contains all elements i in A such that $A[i] = x$, is more than half of the cardinality of set A where n is that cardinality, meaning more than half the elements are equal to x. As we assume both x and y are strictly majority elements, then there exists two sets X and Y such that each contain all elements equal to x and y respectively, in the same manner defined above. By the definition of strictly majority, it must be true that:

$$|X| > \frac{n}{2}$$

$$|Y| > \frac{n}{2}$$

As x and y are distinct elements, there are no elements $A[i]$ such that $A[i] = x \wedge A[i] = y$, thus $A \cap B = \emptyset$, meaning each element in X and Y is unique to the set it belongs to. Therefore:

$$|X| + |Y| > \frac{n}{2} + \frac{n}{2}$$
$$|X| + |Y| > 2(\frac{n}{2})$$
$$|X| + |Y| > n$$

This statement is inherently contradictory, as it states that the sum of the cardinality of two unique subsets of n is greater than the cardinality of n. In the case that every element in A was in either X or Y, it would hold that $|X| + |Y| = n$, and subsequently $n > n$, an obviously contradictory statement. Thus, the claim that for any array A only a single strictly majority element can exist must be true.

3 Problem 3: Proof analysis

3.1 DLN 4.101

The error in the proof lies at the very end of the second part of the proof. The proof states that because $r < 12$, adding r^2 to a number divisible by 12 will always result in a number *not* divisible by 12. While that would be true for simply adding r, as a number smaller than 12 cannot be divisible by 12, but raising r to the 2nd power creates the possibility of resulting in a number divisible by 12. For instance, if $r = 6$, then $r^2 = 36$. Adding 36 to a multiple of 12 would not make it no longer a multiple of 12, and thus the statement that adding r^2 always results in a number not divisible by 12 is false.

3.2 DLN 4.102

The clearest way to disprove the claim that $12|n \iff 12|n^2$ is to provide a counterexample. The proof represents a number n not divisible by 12 as $n = 12k + r$, where k is an integer and r is an integer belonging to $1, 2, ..., 11$. n^2 would thus be equal to $12(12k^2 + 2kr) + r^2$. If there is a k, r pair such that $\neg 12|n$ and $12|n^2$, the claim would be disproved.
One such example is $k = 1$ and $r = 6$. In this case, $n = 12 + 6$, thus being not divisible, as a number not divisible by 12 is being added to a multiple of 12. On the other hand, $n^2 = 12(...) + r^2 = 12(...) + 36$. The contents of the parentheses have been omitted for simplicity as they are irrelevant. For n^2, a number divisible by 12 is being added to another multiple of 12, resulting in $12|n^2$. This means that $12|n^2 \implies 12|n$ is false, as the left hand side is true while the right hand side is false, thus making the overall claim $12|n \iff 12|n^2$ false.