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COSC 302 - Analysis of Algorithms – Spring 2019
Lab 2

5. Find a constant $c < 1$ such that $F_n \leq 2^{cn}$ for all $n \geq 0$.
We proceed by induction. We select the constant $c = 0.9$. First, we show that the base cases are correct. As this problem pertains to the Fibonacci sequence, our base cases are $n = 0$ and $n = 1$.
Base case 1 - $n = 0$
$F_0 = 0 \leq 2^{0.9 \cdot 0} = 1$
As the expression with the constant evaluates to 1, and we know F_0 to be 0, the first base case holds.
Base case 2 - $n - 1$
$F_1 = 1 \leq 2^{0.9 \cdot 1} \approx 1.86$
As the expression with the constant evaluates to approximately 1.86, and we know F_1 to be 1, the second base case holds.

Inductive hypothesis
Through the inductive hypothesis, we assume that the claim is true for all values from 0 to $n + 2$. In other words, we assume the claim is true for n and $n + 1$, and we proceed by showing that it is thus true for $n + 2$.

$$\begin{aligned}
F_{n+2} &= F_n + F_{n+1} && \text{definition of Fibonacci} && (1)\\
&\leq 2^{0.9n} + 2^{0.9n+0.9} && \text{inductive hypothesis} && (2)\\
&\leq 2^{0.9n}(1 + 2^{0.9}) && \text{common factor} && (3)\\
&\leq \frac{2^{1.8}(2^{0.9n}(1 + 2^{0.9}))}{2^{1.8}} && \text{multiply by 1, note } 1.8 = 2\text{c} && (4)\\
&\leq 2^{0.9(n+2)} \frac{(1 + 2^{0.9})}{2^1.8} \approx 0.82(2^{0.9(n+2)}) && \text{exponent rules} && (5)
\end{aligned}$$

Thus, we have shown that:
$F_{n+2} \approx 0.82(2^{0.9(n+2)})$
and therefore that:
$F_{n+2} \leq 2^{c(n+2)}$
where $c = 0.9$. Thus, the claim is true for a constant c of 0.9.