Problem Set 7

COSC 290 Spring 2018

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1 Problem 1: Using induction to prove algorithm correctness, DLN 5.71

Claim: Let $P(n)$ if for a sorted array $A[1...n]$ of length n, $binarySearch(A, x) \iff x \in A$. The claim is that $\forall n \in \mathbb{Z}^{\geq 1} : P(n)$.

Proof. We will prove by strong induction on n.

Base cases: The base case is $n = 1$. $P(1)$ must be true, as an array with length 1 will only have a single element. That element will be x if x is present, in which case the $chooseRandom(1, n)$ call will choose x, thus returning true. If x is not present, the recursive call will be made with an array of length 0, thus returning false. This means that the function will only return true if and only if x is present, thus we have $P(1)$.

Inductive case: Let $n \geq 2$. We will show $P(1) \wedge ... \wedge P(n-1) \iff P(n)$.

- *Given*: Assume $P(1) \wedge ... \wedge P(n-1)$ is true.
- *Want to show*: $P(n)$ is true.

We proceed by cases. There are three ways in which x can exist with relation to $A[1...n]$ and $middle$. These cases are as such:

$$A[1...x = middle...n]$$
$$A[1...x...middle...n]$$
$$A[1...middle...x...n]$$

In Case 1, we have that $x = middle$. In Case 2, we have that $1 \leq x < middle \leq n$. In Case 3, we have that $1 \leq middle < x \leq n$.

- Case 1: In the first case, the algorithm successfully found the item x it was searching for. Thus, the function returns true, acting correctly as x is present.
- Case 2: In the second case, the function will be called recursively on $A[1...middle - 1]$. The length of $A[1...middle - 1]$ *must* be smaller than the length of $A[1...n]$, and thus by the assumption of the inductive case we must have that $P(n')$, where n' is the length of $A[1...middle - 1]$. Therefore, in the second case the function returns correctly.
- Case 3: In the second case, the function will be called recursively on $A[middle + 1...n]$. The length of $A[middle + 1...n]$ *must* be smaller than the length of $A[1...n]$, and thus by the assumption of the inductive case we must have that $P(n')$, where n' is the length of $A[middle - 1...n]$. Therefore, in the third case the function returns correctly.

As in all three cases we have that the function returns correctly, we have that $\forall n \in \mathbb{Z}^{\geq 1} : P(n)$, thus proving the claim.

$\square$

2 Problem 2: proving a relation is a partial order

2.1 DLN 8.131

A relation is a total order if it is a partial order where every pair is comparable ($\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$). To demonstrate that it is not a total order, we must simply identify an (a, b) pair that does not satisfy the total order condition. An example is as follows.
The pair $\langle\langle 1, 2\rangle, \langle 2, 2\rangle\rangle$ would be in R as it meets the condition of the relation. Additionally, the pair $\langle\langle 2, 1\rangle, \langle 2, 2\rangle\rangle$ would also be in R. However, neither $\langle\langle 2, 1\rangle, \langle 1, 2\rangle\rangle$ nor $\langle\langle 1, 2\rangle, \langle 2, 1\rangle\rangle$ would be in R, as neither meet the condition. Thus, this relation R does not meet the definition of a total order, and thus is not a total order.

2.2 DLN 8.132

For R to be a partial order, it must be reflexive, antisymmetric, and transitive. We will show that relation R meets each of these properties.

- **Reflexivity:** $\langle\langle a, b\rangle, \langle x, y\rangle\rangle \in R$ if $a \leq x$ and $b \leq y$. Thus $\langle\langle a, b\rangle, \langle a, b\rangle\rangle \in R$ as $a \leq a$ and $b \leq b$. Therefore this relation is reflexive, as for any (a, b) pair A, $\langle A, A\rangle \in R$.
- **Antisymmetry:** The antisymmetry property is only present if $\langle A, B\rangle \in R \wedge \langle B, A\rangle \in R \implies (A = B)$, where A is the pair (a, b) and B is the pair (x, y). If $\langle A, B\rangle \in R \wedge \langle B, A\rangle \in R$, then by the definition of R, $(a \leq x) \wedge (b \leq y)$ as well as $(x \leq a) \wedge (y \leq b)$. This is only possible if $a = x$ and $b = y$, as it is not possible to have $a < x < a$ or $b < y < b$. Additionally, if $a = x$ and $b = y$, then we must have $A = B$, and thus the relation R is antisymmetric.
- **Transitivity:** The property of transitivity holds that if $\langle A, B\rangle \in R \wedge \langle B, C\rangle \in R$, then we must have that $\langle A, C\rangle \in R$. We have that A is the pair (a, b) and B is the pair (x, y), and C is the pair (c, d). If we have $\langle A, B\rangle \in R \wedge \langle B, C\rangle \in R$, then it must be that $(a \leq x) \wedge (b \leq y)$, as well as $(x \leq c) \wedge (y \leq d)$. It also holds that $(a \leq c) \wedge (b \leq d)$, and thus $\langle A, C\rangle \in R$, fulfilling the property of transitivity.

As the relation R is reflexive, antisymmetric, and transitive, it is a partial order.

3 Problem 3: an equivalence relation and a partial order? DLN 8.155

Claim: There exists a relation $\preceq$ on the set A that is both an equivalence relation and a partial order.

Proof. We will prove by direct proof, by showing an example of a relation that fits both the definition of an equivalence relation and a partial order.

A relation is an equivalence relation if it is reflexive, symmetric, and transitive. A relation is a partial order if it is reflexive, antisymmetric, and transitive. Thus for a relation to be both, it must be reflexive, symmetric, transitive, and antisymmetric simultaneously. We will show that the relation R fits all of these properties. The relationship R is defined as such: with relation to $A[1...n]$, $\langle a, b\rangle \in R$ if $a = b$. We will show that each property holds individually.

- **Reflexivity:** As for any element $a \in A[1...n]$, $a = a$, then $\forall a \in A$, $\langle a, a\rangle \in R$. This is exactly the definition of reflexivity, and thus the relationship R is reflexive.
- **Symmetry:** The property of symmetry exists if $\forall\langle a, b\rangle \in R$, we must have $\langle b, a\rangle \in R$. Since we have $\langle a, b\rangle \in R$, it must be that $a = b$, and that $b = a$. Thus, it holds that $\langle b, a\rangle \in R$. Therefore, the relationship R is symmetric.
- **Antisymmetry:** The property of symmetry exists if for any (a, b) pair where $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$, we must have that $a = b$. By the definition of R, if we have $\langle a, b\rangle \in R$, then we will have $a = b$, fulfilling the property of antisymmetry, as this will always be true, and thus true when $\langle b, a\rangle \in R$.
- **Transitivity:** The transitivity property states that if you have $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$, you must have that $\langle a, c\rangle \in R$. Given the conditions, we would have that $a = b$ and that $b = c$. As the $=$ comparator is transitive, we would have that $a = c$, and by the definition of R, $\langle a, c\rangle \in R$. Thus, the relation R is transitive.

As we have shown that relation R is simultaneously reflexive, symmetric, transitive, and antisymmetric, we can conclude that R is both an equivalence relation and a partial order. Thus, we have directly proven the claim by showing an example. $\square$