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COSC 480 - Learning From Data
Fall 2019
Problem Set 9

1. Show that the expected squared error can be decomposed into bias, variance, and irreducible error.

   Before showing that the expected error can be decomposed as such, we clarify a few formulas and properties that will be used.

   (a) **Average prediction:** $\mathbb{E}[g(x)] = \overline{g}(x)$
   (b) **Variance:** $\mathbb{E}[(g(x) - \overline{g}(x))^2] = \mathbb{E}[g(x)^2] - \overline{g}(x)^2$
   (c) **Bias:** $(f(x) - \overline{g}(x))^2$
   (d) **Variance of random variable:** $\mathbb{E}[(X - \mathbb{E}[X])^2] = \sigma^2$
   (e) **Expected value of a constant:** $\mathbb{E}[c] = c, \forall$ constants c

   We now proceed with the derivation.

$$
\begin{aligned}
\mathbb{E}[\text{MSE}] &= \mathbb{E}[(g(x) - y)^2] && \text{definition of MSE} && (1) \\
&= \mathbb{E}[g(x)^2 - 2g(x)y + y^2] && \text{expand} && (2) \\
&= \mathbb{E}[g(x)^2 - 2g(x)(f(x) + Z) + (f(x) + Z)^2] && \text{y = f(x) + Z} && (3) \\
&= \mathbb{E}[g(x)^2] - \mathbb{E}[2g(x)f(x) + Z] + \mathbb{E}[f(x)^2 + 2f(x)Z + Z^2] && \text{expand} && (4) \\
&= \mathbb{E}[g(x)^2] - 2\overline{g}(x)\,\mathbb{E}[f(x) + Z] + \mathbb{E}[f(x)^2 + 2f(x)Z + Z^2] && \text{property (a)} && (5) \\
&= \mathbb{E}[g(x)^2] - 2\overline{g}(x)(f(x) + \mathbb{E}[Z]) + f(x)^2 + 2f(x)\,\mathbb{E}[z] + \mathbb{E}[Z^2] && \text{property (e)} && (6) \\
&= \mathbb{E}[g(x)^2] - 2\overline{g}(x)f(x) - 2\overline{g}(x)\,\mathbb{E}[Z] + f(x)^2 + 2f(x)\,\mathbb{E}[Z] + \mathbb{E}[Z^2] && \text{expand} && (7) \\
&= \mathbb{E}[g(x)^2] - 2\overline{g}(x)f(x) + f(x)^2 + \mathbb{E}[Z^2] && \mathbb{E}[Z] = 0 && (8) \\
&= \mathbb{E}[g(x)^2] - \overline{g}(x)^2 + \overline{g}(x)^2 - 2\overline{g}(x)f(x) + f(x)^2 + \mathbb{E}[Z^2] && \text{algebra} && (9) \\
&= \mathbb{E}[g(x)^2] - \overline{g}(x)^2 + (f(x) - \overline{g}(x))^2 + \mathbb{E}[Z^2] && \text{simplify} && (10) \\
&= \text{Var}(x) + \text{Bias}(x) + \mathbb{E}[Z^2] && \text{(b),(c)} && (11) \\
&= \text{Var}(x) + \text{Bias}(x) + \sigma^2 && \text{(d) since } E[Z] = 0 && (12)
\end{aligned}
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