Problem Set 5

COSC 290 Spring 2018

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1 Problem 1: G_n and F_n

Claim: Let $P(n) := G_n = F_{n+3} - 1$. The claim is that $\forall n \in \mathrm{N}^{\geq -1} : P(n)$.

Proof. We will prove by strong induction on n.

Base cases: $n = -1$ and $n = 0$. For the first case, $G_{-1} = F_2 - 1$, and by the definition of F_n and G_n, $F_2 = 1$ and $G_{-1} = 0$, giving us $0 = 1 - 1$, thus the case is true for $n = -1$. For the second case, $G_0 = F_3 - 1$, so $1 = 2 - 1$, thus the second case is also true.

Inductive case: Let $n \geq 1$. We will show $[P(-1) \wedge P(0) \wedge ... \wedge P(n)] \implies P(n+1)$.

- *Given*: Assume $P(-1) \wedge ... \wedge P(n-1) \wedge P(n)$ is true.
- *Want to show*: $P(n+1)$ is true.

Since $P(n-1)$ is true, we have $G_n = F_{n+2} - 1$. Since $P(n)$ is true, we have $G_{n-1} = F_{n+3} - 1$.

We will use this fact to show $P(n+1)$:

$$\begin{aligned}
P(n) := G_n &= F_{n+3} - 1 && \text{definition of P(n)} \\
P(n+1) := G_{n+1} &= F_{n+4} - 1 && \text{inductive hypothesis} \\
G_{n+1} &= G_n + G_{n-1} + 1 && \text{definition of G} \\
F_{n+4} &= F_{n+3} + F_{n+2} && \text{definition of F} \\
P(n+1) := G_n + G_{n-1} &= F_{n+3} + F_{n+2} - 2 && \text{substituting/rearranging terms}
\end{aligned}$$

By the claim, the following are given to be true:

$$\begin{aligned}
G_n &= F_{n+3} - 1 \\
G_{n-1} &= F_{n+2} - 1
\end{aligned}$$

By adding the two, we get:

$$G_n + G_{n-1} = F_{n+3} + F_{n+2} - 2$$

This is exactly equal to the rearranged inductive hypothesis for $P(n+1)$. Thus, the inductive hypothesis must be true. $\square$

2 Problem 2: G_n lower bound

Claim: Let $P_2(h) := G_h \geq 2^{h/2}$. The claim is that $\forall h \in \mathrm{N}^{\geq 0} : P_2(h)$.

Proof. We will prove by strong induction on h.

Base cases: $h = 0$ and $h = 1$. For the first case, $G_0 = 1$, and $2^{0/2} = 1$, giving us $1 \geq 1$ and thus the first case is true. For the second case, $G_1 = 2$, and $2^{1/2} = \sqrt{2}$, giving us $2 \geq \sqrt{2}$, thus the second case is also true.

Inductive case: Let $h \geq 2$. We will show $[P_2(0) \wedge P_2(1) \wedge ... \wedge P_2(h-2) \wedge P_2(h-1)] \implies P_2(h)$.

- *Given*: Assume $P_2(-1) \wedge ... \wedge P_2(h-2) \wedge P_2(h-1)$ is true.
- *Want to show*: $P_2(h)$ is true.

Since $P_2(h-2)$ is true, we have $G_{h-2} \geq 2^{(h-2)/2}$. Since $P_2(h-1)$ is true, we have $G_{h-1} \geq 2^{(h-1)/2}$.

We will use this fact to show $P_2(h)$:

$$\begin{aligned} P_2(h) := G_h &\geq 2^{h/2} && \text{definition of } P_2\text{(h)} \\ G_h &= G_{h-2} + G_{h-1} + 1 && \text{definition of G} \end{aligned}$$

By substituting G_{h-2} and G_{h-1} with the given cases, we get:

$$G_h \geq 2^{(h-2)/2} + 2^{(h-1)/2} + 1$$

It also follows that $2^{(h-2)/2} + 2^{(h-1)/2} + 1 > 2^{h/2}$, for all $h \geq 1$. This is because $2^{(h-2)/2}$ is equal to exactly half of $2^{h/2}$, and because $2^{(h-1)/2} > 2^{(h-2)/2}$. Thus $2^{h/2}$ must be smaller than $2^{(h-2)/2} + 2^{(h-1)/2} + 1$, as it is a number exactly half of $2^{h/2}$ being added to a number bigger than half of $2^{h/2}$, plus one. Thus, if:

$$2^{(h-2)/2} + 2^{(h-1)/2} + 1 > 2^{h/2}$$

And:

$$G_h \geq 2^{(h-2)/2} + 2^{(h-1)/2} + 1$$

It follows that:

$$G_h \geq 2^{h/2}$$

Thus, the claim must be true. $\square$

3 Problem 3: lower bound on height balanced binary trees

Claim: For any height balanced binary tree T, $nodes(T) \geq G_{h(T)}$.

Proof. We will prove by structural induction on $h(T)$.

Base cases: $n = 1$. There are two base cases. The first base case is an empty tree. For an empty tree, $h(T) = -1$, and $nodes(T) = 0$. Thus, $G_{h(t)} = 0$, and since $0 \geq 0$, the claim is true for this base case. The second base case is a tree with a single node, and two empty subtrees. For this tree, $h(T) = 1 + max\{h(T_l), h(T_r)\} = 1 + -1 = 0$, and so $G_{h(T)} = 1$. In this case, $nodes(T) = 1$, and since $1 \geq 1$, the claim is also true for this base case.

Inductive case: The inductive case is a height balanced binary tree T that contains *at least* one non-empty subtree.

- *Given*: Assume the claim is true for the subtrees T_l and T_r of tree T.
- *Want to show*: The claim must be true for T.

By the definition of a binary tree, we have

$$nodes(T) = 1 + nodes(T_l) + nodes(T_r)$$

and,

$$h(T) = 1 + max\{h(T_l) + h(T_r)\}$$

By rearranging the second, we have

$$max\{h(T_l) + h(T_r)\} = h(T) - 1$$

Since the tree is height balanced, $h(Tl)$ and $h(Tr)$ differ by at most 1, thus the smaller of the two must be equal to $h(T) - 2$. It is irrelevant which one of the two the larger one. Thus, we can rearrange the definition of G_h to have the following

$$G_{h(T)} = G_{h(T)-2} + G_{h(T)-1} + 1$$
$$G_{h(T)} = G_{h(T_l)} + G_{h(T_r)} + 1$$

We will use this fact to show that the claim is true for T:

$$\begin{aligned}
nodes(T) &\geq G_{h(T)} && \text{claim} \\
nodes(T) &\geq G_{h(T_l)} + G_{h(T_r)} + 1 && \text{substituting terms} \\
1 + nodes(T_l) + nodes(T_r) &\geq G_{h(T_l)} + G_{h(T_r)} + 1 && \text{definition of nodes(T)} \\
nodes(T_l) + nodes(T_r) &\geq G_{h(T_l)} + G_{h(T_r)} && \text{algebra}
\end{aligned}$$

□

4 Problem 4: false lower bound on binary trees

5 Problem 5: bound on height

6 Example of proof by induction

This is an example proof, provided in LaTeX so that you may copy its basic formatting.

Claim: Let $P(n) := \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. The claim is that $\forall n \in \mathrm{Z}^{\geq 1} : P(n)$.

Proof. We will prove by weak induction on n.

Base case: $n = 1$. In this case $\sum_{i=1}^{n} i = \sum_{i=1}^{1} i = 1$ and $\frac{n(n+1)}{2} = \frac{1\times(1+1)}{2} = 1$. Thus $P(1)$ is true.

Inductive case: Let $n \geq 2$. We will show $P(n-1) \implies P(n)$.

- *Given*: Assume $P(n-1)$ is true.
- *Want to show*: $P(n)$ is true.

Since $P(n-1)$ is true, we have

$$\sum_{i=1}^{n-1} i = \frac{(n-1)((n-1)+1)}{2}$$

We will use this fact to show $P(n)$:

$$\begin{aligned}
\sum_{i=1}^{n} i &= \left(\sum_{i=1}^{n-1} i\right) + n && \text{definition of summation} \\
&= \frac{(n-1)((n-1)+1)}{2} + n && \text{inductive hypothesis} \\
&= \frac{(n-1)n + 2n}{2} && \text{rearanging/simplifying terms} \\
&= \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} && \text{algebra} \\
&= \frac{n(n+1)}{2} && \square
\end{aligned}$$

$\square$