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 COSC 302 - Analysis of Algorithms – Spring 2019  
 Lab 1

1. Proof

A.

$$a = b^{\log_b a} \quad (1)$$

$$\log_b a = \log_b a \quad \text{definition of log} \quad (2)$$

$$a = a \quad \text{simplify} \quad (3)$$

B.

Let  $a = c^x$  and  $b = c^y$ .

$$\log_c(ab) = \log_c(c^x \cdot c^y) \quad \text{substitution} \quad (4)$$

$$= \log_c(c^{x+y}) \quad \text{exponent property} \quad (5)$$

$$= x + y \quad \text{A} \quad (6)$$

$$= \log_c(a) + \log_c(b) \quad \text{definition of log} \quad (7)$$

2. Proof

C.

Let  $a = b^x$  and thus  $\log_b(a) = x$

$$\log_b(a^n) = \log_b((b^x)^n) \quad \text{substitution} \quad (8)$$

$$= \log_b(b^{xn}) \quad \text{exponent property} \quad (9)$$

$$= xn \quad \text{A} \quad (10)$$

$$= n\log_b(a) \quad \text{substitution} \quad (11)$$

D.

Let  $\log_b(a) = x$

$$b^x = a \quad \text{log definition} \quad (12)$$

$$\log_c(b^x) = \log_c(a) \quad \text{multiply both sides by } \log_c \quad (13)$$

$$x\log_c(b) = \log_c(a) \quad \text{C} \quad (14)$$

$$x = \frac{\log_c(a)}{\log_c(b)} \quad \text{algebra} \quad (15)$$

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)} \quad \text{substitution} \quad (16)$$

3. Proof

E.

$$\log_b(1/a) = \log_b(1) - \log_b(a) \quad \text{quotient rule} \quad (17)$$

$$= -\log_b(a) \quad \log_b(1) = 0 \text{ for any number } b \quad (18)$$

F.

$$\log_b(a) = \frac{\log_a(a)}{\log_a(b)} \quad \text{D} \quad (19)$$

$$= \frac{1}{\log_a(b)} \quad \text{if } a^x = 1, a \text{ must equal } 1 \quad (20)$$

4. Proof  
G.

$$\begin{aligned}
 a^{\log_b c} &= c^{\log_b a} \\
 \log_a(a^{\log_b c}) &= \log_a(c^{\log_b a}) && \text{multiply both sides by } \log_a \\
 \log_b c &= \log_b a \cdot \log_a c && \text{A for LHS and C for RHS} \\
 \frac{\log_b c}{\log_b a} &= \log_a c && \text{algebra, true by D}
 \end{aligned}$$

5. Proof  
H.

**Claim:** if  $f(n)$  and  $g(n)$  are monotonically increasing,  $f(n) + g(n)$  must also be monotonically increasing.

We proceed by direct proof. With what we are given, if  $n_1 < n_2$ , it must be by the definition of monotonically increasing that  $f(n_1) \leq f(n_2)$  and  $g(n_1) \leq g(n_2)$ . We start from the second equality and add  $g(n_1)$  to both sides:

$$f(n_1) + g(n_1) \leq f(n_2) + g(n_1)$$

Since we know that  $g(n_1) \leq g(n_2)$ , we can substitute on the right hand side and result in:

$$f(n_1) + g(n_1) \leq f(n_2) + g(n_2)$$

This precisely meets the definition of monotonically increasing, and matches the function we are trying to prove. Thus, the claim is true.

I.

**Claim:** if  $f(n)$  and  $g(n)$  are monotonically increasing,  $f(g(n))$  must also be monotonically increasing. We proceed by direct proof. With what we are given, if  $n_1 < n_2$ , it must be by the definition of monotonically increasing that  $f(n_1) \leq f(n_2)$  and  $g(n_1) \leq g(n_2)$ . Then if  $f(n_1) \leq f(n_2)$  and  $g(n_1) \leq g(n_2)$ , it must be true that  $g(f(n_1)) \leq g(f(n_2))$ , as can effectively substitute  $n_1$  and  $n_2$  for  $f(n_1)$  and  $f(n_2)$  based on the monotonically increasing property of  $f(n)$ . Thus the claim is true, as  $g(f(n_1)) \leq g(f(n_2))$  both meets the desired function in the claim and is the exact definition of monotonically increasing.

J.

**Claim:** if  $f(n)$  and  $g(n)$  are monotonically increasing, and  $f(n)$  and  $g(n)$  are nonnegative,  $f(g(n))$  must also be monotonically increasing.

We proceed by direct proof. With what we are given, if  $n_1 < n_2$ , it must be by the definition of monotonically increasing that  $f(n_1) \leq f(n_2)$  and  $g(n_1) \leq g(n_2)$ . We start from the second equality and multiply  $g(n_1)$  to both sides:

$$f(n_1)g(n_1) \leq f(n_2)g(n_1)$$

It should be noted that the above only holds for nonnegative values of  $f(n)$  and  $g(n)$ , but this is covered in what we are given. Next, as we know  $g(n_1) \leq g(n_2)$  for nonnegative values, we can substitute one for another:

$$f(n_1)g(n_1) \leq f(n_2)g(n_2)$$

As in the first case, this both matches the definition of monotonically increasing and the designated function in the claim, thus the claim is true.

6. Proof  
K.

By the definition of the golden ratio, we have that  $\phi^i = \frac{1+\sqrt{5}}{2}$ , and consequently  $\hat{\phi}^i = \frac{1-\sqrt{5}}{2}$ . As this proof relates to the Fibonacci sequence, we have two base cases,  $F = 0$  and  $F = 1$ .

First base case:

$$\frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = \frac{0}{\sqrt{5}} = 0 = F_0$$

Second base case:

$$\frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = F_1$$

Through the inductive hypothesis, we can assume that:

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}}$$

For all nonnegative  $n$  less than or equal to  $i$ . We must then show that the following holds:

$$F_{i+1} = \frac{\phi^{i+1} - \hat{\phi}^{i+1}}{\sqrt{5}}$$

By the definition of the Fibonacci sequence, we can rearrange:

$$F_{i+1} = F_i + F_{i-1} = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} + \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

I have been unable to finish the problem from this point. I believe I should either substitute the definition for phi and rearrange, or use exponent rules to rearrange and obtain the equality in the claim, but I have not been able to get this to work.

## 7. Proof

The base case for this problem is  $i = 1$ . The result is as follows:

$$\sum_{i=1}^1 (-1)^1 1^2 = \frac{(-1)^1 1(1+1)}{2} \quad (21)$$

$$-1 = -1 \quad (22)$$

Therefore the base case holds. By the inductive hypothesis, we assume that the claim is true for  $n = k$ , and thus want to show that it holds for  $k + 1$ . By the definition of summation, we have that:

$$\sum_{i=1}^{k+1} (-1)^i i^2 = \sum_{i=1}^k (-1)^i i^2 + (-1)^{k+1} (k+1)^2 \quad (23)$$

Using the inductive hypothesis, we can then substitute and simplify:

$$= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1} (k+1)^2 \quad \text{inductive hypothesis} \quad (24)$$

$$(25)$$

I became stuck with this problem as well. I am certain that I am close, and that to finish this proof I would have to rearrange the above expression with algebra to result in:

$$= \frac{(-1)^k + 1(k+1)^2}{2} \quad (26)$$

## 8. Proof

We start with the base case  $n = 1$ . The base case is as follows:

$$\sum_{i=1}^1 \frac{1}{1(1+1)} = \frac{1}{1+1} \quad (27)$$

$$\frac{1}{2} = \frac{1}{2} \quad (28)$$

Through the inductive hypothesis, we assume that the claim holds for  $n = k$ , and thus want to show that it holds for  $k + 1$ . By the definition of summation, we have that:

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^k \frac{1}{i(i+1)} + \frac{1}{k+1(k+2)} \quad (29)$$

We use the inductive hypothesis to substitute and simplify:

$$= \frac{k}{k+1} + \frac{1}{k+1(k+2)} \quad \text{induction hypothesis} \quad (30)$$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)} \quad \text{algebra} \quad (31)$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} \quad \text{algebra} \quad (32)$$

$$= \frac{k+1}{k+2} \quad \text{algebra} \quad (33)$$