Problem Set 6

COSC 290 Spring 2018

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1 Problem 1: Equivalence classes

1.1 DLN 8.110

R_1 is an equivalence relationship if and only if it is reflexive, symmetric, and transitive.

- *Reflexivity*: If A is an empty set, then the $A = A = \emptyset$, so $\langle A, A\rangle \in R_1$. If A is not an empty set, then the greatest element in A will equal the greatest element in A, because $A = A$, thus $\langle A, A\rangle \in R_1$. Therefore R_1 is reflexive.
- *Symmetry*: In the case that $A = \emptyset$ and $\langle A, B\rangle \in R_1$, then by the second condition $B = \emptyset$. Therefore, also by the second condition, $\langle B, A\rangle \in R_1$. In the case that $\langle A, B\rangle \in R_1$ and $A \neq \emptyset$, the greatest element in A must equal the greatest element in B. This means that the greatest element of B must equal the largest element in A, and so $\langle B, A\rangle \in R_1$, making R_1 symmetric.
- *Transitivity*: If $\langle A, B\rangle \in R_1$ and $\langle B, C\rangle \in R_1$, and $A = \emptyset$, then $A = B = \emptyset$, and thus $B = C = \emptyset$. Therefore, if A,B, and C are all empty sets, then $A = C = \emptyset$, and so $\langle A, C\rangle \in R_1$. In the case that A is not an empty set, then the greatest element in A must equal the greatest element in B. Since $\langle B, C\rangle \in R_1$, then the greatest element in B must equal the greatest element in C. Thus, the greatest element in A must equal the greatest element in C, therefore $\langle A, C\rangle \in R_1$. This means that the relation must be transitive.

Since the relation R_1 is reflexive, symmetric, and transitive, it is an equivalence relationship. The equivalence clauses are as follows:

- $\{\emptyset\}$
- $\{\{0\}\}$
- $\{\{1\}, \{0, 1\}\}$
- $\{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
- $\{\{3\}, \{0, 3\}, \{1, 3\}, \{2, 3\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}, \{0, 1, 2, 3\}\}$

1.2 DLN 8.111

R_2 is an equivalence relationship if and only if it is reflexive, symmetric, and transitive. $P = \mathcal{P}(\{0, 1, 2, 3\})$.

- *Reflexivity*: The sum of the items in A will always equal the sum of items in A. Thus, for every $A \in P, \langle A, A\rangle \in R_2$. Therefore, the relation is reflexive.
- *Symmetry*: If the sum of items in A equals the sum of items in B, the reverse must also be true. Therefore, for every $A, B \in P$, if $\langle A, B\rangle \in R_2$, then $\langle B, A\rangle \in R_2$. This means that R_2 is symmetric.
- *Transitivity*: If $\langle A, B\rangle \in R_2$, then the sum of elements in A equals the sum of elements in B. If $\langle B, C\rangle \in R_2$, then the sum of elements in B must equal the sum of elements in C. Thus, the sum of elements in all three sets (A, B, and C) must be equal, giving us $\langle A, C\rangle \in R_2$. This means that the relation R_2 is transitive.

Since the relation R_2 is reflexive, symmetric, and transitive, it is an equivalence relationship. The equivalence clauses are as follows:

- $\{\emptyset\}$
- $\{\{0\}\}$
- $\{\{1\}, \{0, 1\}\}$
- $\{\{2\}, \{0, 2\}\}$

- $\{\{3\}, \{0,3\}, \{1,2\}, \{0,1,2\}\}$
- $\{\{1,3\}, \{0,1,3\}\}$
- $\{\{2,3\}, \{0,2,3\}\}$
- $\{\{1,2,3\}, \{0,1,2,3\}\}$

1.3 DLN 8.113

R_4 is an equivalence relationship if and only if it is reflexive, symmetric, and transitive. $P = \mathcal{P}(\{0,1,2,3\})$.

- *Reflexivity*: For any $A \in P$, all items present in A are also present in A. However, if $A = \emptyset$, $A \cap A$ will result in $\emptyset$, and thus $\langle A, A \rangle \notin R_4$ by the definition of R_4. This means that R_4 is not reflexive, as there is an $A \in P$ that does not satisfy $\langle A, A \rangle \in R_4$.

As an equivalence relationship depends on reflexivity, symmetry, and transitivity, proving that one of these three properties is not met is enough to show the relationship is not an equivalence relationship. Therefore, since R_4 is not reflexive, it is not an equivalence relationship.

2 Problem 2: DLN 8.84

Claim: The claim is that for any relation $R \subseteq A \times A$ that is both irreflexive and transitive, R must also be asymmetric.

Proof. We will prove by assuming the claim is false and showing a contradiction.

- *Given*: Assume a relation $R \subseteq A \times A$ is irreflexive, transitive, and not asymmetric.
- *Want to show*: The assumption leads to a contradiction.

By the definition of asymmetry, since R is not asymmetric there must be one pair $\langle a, b \rangle \in R$ such that $\langle b, a \rangle \in R$. By the definition of transitivity, if $\langle a, b \rangle \in R$, and $\langle b, a \rangle \in R$, then $\langle a, a \rangle \in R$. Thus, if R is not asymmetric and transitive, then we must have $\langle a, a \rangle \in R$.

By the definition of irreflexivity, for every $a \in A, \langle a, a \rangle \notin R$. By the assumption, R is not asymmetric and is transitive, thus $\langle a, a \rangle \in R$, and is also irreflexive, thus for every $a \in A, \langle a, a \rangle \notin R$. These two statements are in direct contradiction, therefore if R is both irreflexive and transitive, it must be asymmetric, proving the claim.

$\square$

3 Problem 3: DLN 8.130

Claim: Let $P(k)$ if there are no non-trivial cycles of length k in any relationship R that is both transitive and antisymmetric. The claim is that $\forall k \in Z^{\geq 2} : P(k)$.

Proof. We will prove by weak induction on k.

Base case: The base case is that $k = 2$, and thus $P(2)$. The definition of a non-trivial cycle is that R contains a sequence of distinct elements $a_0, a_1, ...a_{k-1}$ such that $\langle a_i, a_{(i+1)mod(k)}\rangle \in R$ for each $i \in \{0, 1, ...k-1\}$, and $k \geq 2$. Since $k = 2$, the cycle would consist of pairs $\langle a_0, a_1\rangle$, and $\langle a_1, a_0\rangle$. By antisymmetry, since $\langle a_0, a_1\rangle \in R$ and $\langle a_1, a_0\rangle \in R$, $a_0 = a_b$. However, the definition of non-trivial cycles holds that all elements in the cycle must be distinct, which is contradiction to this, therefore no non-trivial cycles of length $k = 2$ can exist.

Inductive case: Let $k \geq 3$. We will show $P(k-1) \implies P(k)$.

- *Given*: Assume $P(k-1)$ is true, and that relationship R is transitive and antisymmetric.
- *Want to show*: $P(k)$ is true.

By the definition of a cycle, and by using $i = k-1$, we have the following:

$$\langle a_{k-1}, a_{(k)mod(k)}\rangle \in R$$

By the definition of the mod operator, this can be simplified to:

$$\langle a_{k-1}, a_0\rangle \in R$$

This means that the last item in the sequence (a_{k-1}) will always be paired with the first (a_0) if there is a non-trivial cycle present.
Again by the definition of non-trivial cycles, we have:

$$\langle a_i, a_{(i+1)mod(k)}\rangle \in R$$

As the upper bound for i is $k-1$, for all other values of i we have $i < k-1$. This means that the expression $a_{(i+1)mod(k)}$, can be simplified to a_{i+1}, as since $i < k-1$, $i+1$ must always be smaller than k. Using this simplification, we have:

$$\langle a_i, a_{i+1}\rangle \in R$$

This means that for every $i \in \{0, 1, ...k-1\}$, i is paired with $i+1$. This creates a chain from a_0 to a_{k-1} by using the transitive property, meaning that $\langle a_0, a_{k-1}\rangle \in R$. We now have that both:

$$\langle a_0, a_{k-1}\rangle \in R$$

$$\langle a_{k-1}, a_0\rangle \in R$$

Since R is antisymmetric, $a_0 = a_{k-1}$. However, the definition of non-trivial cycles holds that all elements in the cycle are distinct, therefore this *cannot* be a non-trivial cycle. Thus, we have $P(k)$, proving the claim. $\square$