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COSC 302 - Analysis of Algorithms – Spring 2019
Lab 2

5. Claim: For all $n \geq 6$, $F_n \geq 2^{0.5n}$ where F_n is the Fibonacci sequence for n.
We proceed by induction. First, we show that the base cases are correct. As our lower bound is 6, we use the base cases $n = 6$ and $n = 7$.
Base case 1 - $n = 6$
$F_6 = 8 \geq 2^{0.5 \cdot 6} = 8$
By the definition of the Fibonacci sequence, we have that $F_6 = 8$, and since the right hand side of the inequality evaluates to 8, the claim holds for the first base case.
Base case 2 - $n = 7$
$F_7 = 13 \geq 2^{0.5 \cdot 7} \approx 11.3$ By the definition of the Fibonacci sequence, we have that $F_7 = 13$, and since the right side of the inequality evaluates to approximately 11.3, the claim holds for the second base case.

Inductive Hypothesis
Through the inductive hypothesis, we assume that the claim is true for all values from 6 to $n + 2$. In other words, we assume that the claim is true for n and $+1$, and we proceed by showing that it is thus true for $n + 2$.

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n && \text{definition of Fibonacci sequence} && (1)\\
&\geq 2^{0.5n} + 2^{0.5n+0.5} && \text{substitution of inductive hypothesis} && (2)\\
&\geq 2^{0.5n}(1 + 2^{0.5}) && \text{common factor} && (3)\\
&\geq 2^{0.5n+1}\frac{1 + 2^{0.5}}{2} && \text{multiply by } \frac{2}{2} && (4)\\
&\geq 2^{0.5(n+2)}\frac{1 + 2^{0.5}}{2} \approx 1.2(2^{0.5(n+2)}) && \text{simplify and evaluate, note } 0.5n + 1 = 0.5(n + 2) && (5)\\
F_{n+2} &\geq 2^{0.5(n+2)} && && (6)
\end{aligned}$$

As we can arrange the left hand side of the inequality to result approximately $1.2(2^{0.5(n+2)})$, we can conclude that the left hand side, F_{n+2} is certainly greater than or equal to $2^{0.5(n+2)}$. We conclude this through the approximate factor of 1.2, showing that the left hand side is greater by at least that factor. Thus, the claim is true.

6. I accidentally did this problem instead of the one I was assigned initially so I figured I'd just include it too. The one I'm assigned is the one above, problem 5.
Find a constant $c < 1$ such that $F_n \leq 2^{cn}$ for all $n \geq 0$.
We proceed by induction. We select the constant $c = 0.9$. First, we show that the base cases are correct. As this problem pertains to the Fibonacci sequence, our base cases are $n = 0$ and $n = 1$.
Base case 1 - $n = 0$
$F_0 = 0 \leq 2^{0.9 \cdot 0} = 1$
As the expression with the constant evaluates to 1, and we know F_0 to be 0, the first base case holds.
Base case 2 - $n - 1$
$F_1 = 1 \leq 2^{0.9 \cdot 1} \approx 1.86$
As the expression with the constant evaluates to approximately 1.86, and we know F_1 to be 1, the second base case holds.

Inductive hypothesis
Through the inductive hypothesis, we assume that the claim is true for all values from 0 to $n + 2$. In other words, we assume the claim is true for n and $n + 1$, and we proceed by showing that it is thus

true for $n+2$.

$$
\begin{aligned}
F_{n+2} &= F_n + F_{n+1} && \text{definition of Fibonacci} && (7)\\
&\leq 2^{0.9n} + 2^{0.9n+0.9} && \text{inductive hypothesis} && (8)\\
&\leq 2^{0.9n}(1 + 2^{0.9}) && \text{common factor} && (9)\\
&\leq \frac{2^{1.8}(2^{0.9n}(1 + 2^{0.9}))}{2^{1.8}} && \text{multiply by 1, note 1.8 = 2c} && (10)\\
&\leq 2^{0.9(n+2)} \frac{(1 + 2^{0.9})}{2^1.8} \approx 0.82(2^{0.9(n+2)}) && \text{exponent rules} && (11)
\end{aligned}
$$

Thus, we have shown that:
$F_{n+2} \approx 0.82(2^{0.9(n+2)})$
and therefore that:
$F_{n+2} \leq 2^{c(n+2)}$
where $c = 0.9$. Thus, the claim is true for a constant c of 0.9.