# Problem Set 5

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# 1 Problem 1: $G_n$ and $F_n$

**Claim**: Let $P(n) := G_n = F_{n+3} - 1$. The claim is that $\forall n \in \mathrm{N}^{\geq -1} : P(n)$.

*Proof.* We will prove by strong induction on $n$.

**Base cases**: $n = -1$ and $n = 0$. For the first case, $G_{-1} = F_2 - 1$, and by the definition of $F_n$ and $G_n$, $F_2 = 1$ and $G_{-1} = 0$, giving us $0 = 1 - 1$, thus the case is true for $n = -1$. For the second case, $G_0 = F_3 - 1$, so $1 = 2 - 1$, thus the second case is also true.

**Inductive case**: Let $n \geq 1$. We will show $P(-1) \wedge ... \wedge P(n-1) \wedge P(n) \implies P(n+1)$.

- *Given*: Assume $P(-1) \wedge ... \wedge P(n-1) \wedge P(n)$ is true.
- *Want to show*: $P(n+1)$ is true.

Since $P(n-1)$ is true, we have $G_n = F_{n+2} - 1$. Since $P(n)$ is true, we have $G_{n-1} = F_{n+3} - 1$.

We will use this fact to show $P(n+1)$:

$$\begin{aligned}
P(n) := G_n &= F_{n+3} - 1 && \text{definition of P(n)} \\
P(n+1) := G_{n+1} &= F_{n+4} - 1 && \text{inductive hypothesis} \\
G_{n+1} &= G_n + G_{n-1} + 1 && \text{definition of G} \\
F_{n+4} &= F_{n+3} + F_{n+2} && \text{definition of F} \\
P(n+1) := G_n + G_{n-1} &= F_{n+3} + F_{n+2} - 2 && \text{substituting/rearranging terms}
\end{aligned}$$

By the claim, the following are given to be true:

$$\begin{aligned}
G_n &= F_{n+3} - 1 \\
G_{n-1} &= F_{n+2} - 1
\end{aligned}$$

By adding the two, we get:

$$G_n + G_{n-1} = F_{n+3} + F_{n+2} - 2$$

This is exactly equal to the rearranged inductive hypothesis for $P(n+1)$. Thus, the inductive hypothesis must be true. □

# 2 Problem 2: $G_n$ lower bound

**Claim**: Let $P_2(h) := G_h \geq 2^{h/2}$. The claim is that $\forall h \in \mathrm{N}^{\geq 0} : P_2(h)$.

*Proof.* We will prove by strong induction on $h$.

**Base cases**: $h = 0$ and $h = 1$. For the first case, $G_0 = 1$, and $2^{0/2} = 1$, giving us $1 \geq 1$ and thus the first case is true. For the second case, $G_1 = 2$, and $2^{1/2} = \sqrt{2}$, giving us $2 \geq \sqrt{2}$, thus the second case is also true.

**Inductive case**: Let $h \geq 2$. We will show $[P_2(0) \wedge P_2(1) \wedge ... \wedge P_2(h-2) \wedge P_2(h-1)] \implies P_2(h)$.

- *Given*: Assume $P_2(-1) \wedge ... \wedge P_2(h-2) \wedge P_2(h-1)$ is true.
- *Want to show*: $P_2(h)$ is true.

Since $P_2(h-2)$ is true, we have $G_{h-2} \geq 2^{(h-2)/2}$. Since $P_2(h-1)$ is true, we have $G_{h-1} \geq 2^{(h-1)/2}$.

We will use this fact to show $P_2(h)$:

$$\begin{aligned} P_2(h) := G_h &\geq 2^{h/2} && \text{definition of } P_2\text{(h)} \\ G_h &= G_{h-2} + G_{h-1} + 1 && \text{definition of G} \end{aligned}$$

By substituting $G_{h-2}$ and $G_{h-1}$ with the given cases, we get:

$$G_h \geq 2^{(h-2)/2} + 2^{(h-1)/2} + 1$$

It also follows that $2^{(h-2)/2} + 2^{(h-1)/2} + 1 > 2^{h/2}$, for all $h \geq 1$. This is because $2^{(h-2)/2}$ is equal to exactly half of $2^{h/2}$, and because $2^{(h-1)/2} > 2^{(h-2)/2}$. Thus $2^{h/2}$ must be smaller than $2^{(h-2)/2} + 2^{(h-1)/2} + 1$, as it is a number exactly half of $2^{h/2}$ being added to a number bigger than half of $2^{h/2}$, plus one. Thus, if:

$$2^{(h-2)/2} + 2^{(h-1)/2} + 1 > 2^{h/2}$$

And:

$$G_h \geq 2^{(h-2)/2} + 2^{(h-1)/2} + 1$$

By the transitive property it follows that:

$$G_h \geq 2^{h/2}$$

Thus, the claim must be true. ☐

# 3 Problem 3: lower bound on height balanced binary trees

**Claim**: For any height balanced binary tree $T$, $nodes(T) \geq G_{h(T)}$.

*Proof.* We will prove by structural induction on $h(T)$.

**Base cases**: $n = 1$. There are two base cases. The first base case is an empty tree. For an empty tree, $h(T) = -1$, and $nodes(T) = 0$. Thus, $G_{h(t)} = 0$, and since $0 \geq 0$, the claim is true for this base case. The second base case is a tree with a single node, and two empty subtrees. For this tree, $h(T) = 1 + max\{h(T_l), h(T_r)\} = 1 + -1 = 0$, and so $G_{h(T)} = 1$. In this case, $nodes(T) = 1$, and since $1 \geq 1$, the claim is also true for this base case.

**Inductive case**: The inductive case is a height balanced binary tree $T$ that contains *at least* one non-empty subtree.

- *Given*: Assume the claim is true for the subtrees $T_l$ and $T_r$ of tree $T$.
- *Want to show*: The claim must be true for T.

By the definition of a binary tree, we have

$$nodes(T) = 1 + nodes(T_l) + nodes(T_r)$$

and,

$$h(T) = 1 + max\{h(T_l) + h(T_r)\}$$

By rearranging the second, we have

$$max\{h(T_l) + h(T_r)\} = h(T) - 1$$

Since the tree is height balanced, $h(Tl)$ and $h(Tr)$ differ by at most 1, thus they can both be equal to each other, or the smaller of the two must be $max\{h(T_l) + h(T_r)\} - 1 = h(T) - 2$. We will first focus on the case where the height of the subtrees are *not* equal. It is irrelevant which one of the two the larger one. Using these terms, we can rearrange the definition of $G_h$ to have the following

$$G_{h(T)} = G_{h(T)-2} + G_{h(T)-1} + 1$$

$$G_{h(T)} = G_{h(T_l)} + G_{h(T_r)} + 1$$

We will use this fact to show that the claim is true for T:

$$\begin{aligned}
nodes(T) &\geq G_{h(T)} && \text{claim} \\
nodes(T) &\geq G_{h(T_l)} + G_{h(T_r)} + 1 && \text{substituting terms} \\
1 + nodes(T_l) + nodes(T_r) &\geq G_{h(T_l)} + G_{h(T_r)} + 1 && \text{definition of } nodes(T) \\
nodes(T_l) + nodes(T_r) &\geq G_{h(T_l)} + G_{h(T_r)} && \text{algebra}
\end{aligned}$$

By the assumption, the claim is true for $T_l$ and $T_r$, thus

$$nodes(T_l) \geq G_{h(T_l)}$$

and

$$nodes(T_r) \geq G_{h(T_r)}$$

are both true. By adding both sides these, we have □

$$nodes(T_l) + nodes(T_r) \geq G_{h(T_l)} + G_{h(T_r)}$$

This is exactly equal to the rearranged claim for $T$ above. Thus, if the claim is true for the subtrees of $T$ and $T$ is height balanced, the claim must be true for $T$. There is also the case where $h(T_l) = h(T_r)$. In this scenario, the claim $nodes(T) \geq G_{h(T)}$ would still be true, as this change would only increase the left hand side of the expression. As the right hand side is a function of $h(T)$, which is defined by the maximum of the two subtree heights, it is irrelevant whether they are equal or one is smaller.

# 4 Problem 4: false lower bound on binary trees

In order to disprove the claim that for any binary tree $T$, $nodes(T) \geq G_{h(T)}$ we must simply provide a counterexample where the claim is false. A simple example of this is a scenario where the right subtree has a height of 2, and the left subtree is empty. We have

$$h(T_l) = -1$$

and

$$h(T_r) = 2$$

The height of this tree $T$ would be defined as

$$\begin{aligned} h(T) &= 1 + max\{h(T_l), h(T_r)\} && \text{definition of } h(T) \\ &= 1 + 2 && \text{substituting terms} \\ &= 3 && \text{algebra} \end{aligned}$$

For the nodes, we would have

$$nodes(T_l) = 0$$

and

$$nodes(T_r) = 3$$

Since $T_l$ is empty, it must have 0 nodes. For $T_r$, there are multiple valid amounts of nodes. This specific tree has 3 nodes in the subtree $T_r$. The total nodes for $T$ would be

$$\begin{aligned} nodes(T) &= 1 + nodes(T_l) + nodes(T_r) && \text{definition of } nodes(T) \\ &= 1 + 0 + 3 && \text{substituting terms} \\ &= 4 && \text{algebra} \end{aligned}$$

By the definition of $G_h$, and by substituting in the value above for $h(T)$, we have

$$G_{h(T)} = G_3 = 7$$

We will use these facts to show that the claim is not true for this $T$:

$$\begin{aligned} nodes(T) &\geq G_{h(T)} && \text{claim} \\ 4 &\geq 7 && \text{substituting terms} \end{aligned}$$

This statement is clearly false, as 4 is *not* greater than or equal to 7. The claim is then false for this tree $T$, and thus disproven.

# 5 Problem 5: bound on height

**Claim**: The claim is that any non-empty height balanced tree with $n$ nodes has a height of at most $2log_2n$. In mathematical terms this is represented as $2log2n \geq h(T)$, where $n = nodes(T)$ and $T$ is a non-empty height balanced binary tree.

*Proof.* We will prove by direct proof.

By the previous proofs on $G_h$ and on height balanced binary trees, we have that

$$n \geq G_{h(T)}$$

And

$$G_h \geq 2^{h/2}$$

We can use these facts to prove the claim

$$\begin{aligned}
G_h(T) &\geq 2^{(h(T)/2)} && \text{substituting } h(T) \text{ in for } h \\
n &\geq 2^{(h(T)/2)} && \text{transitive property} \\
log_2(n) &\geq h(T)/2 && \text{algebra} \\
2log_2(n) &\geq h(T) && \text{algebra}
\end{aligned}$$

This is exactly equal to the claim, thus given the previous proofs and the definition of $G_h$, the claim must be true.

$\square$