Problem Set 5

COSC 290 Spring 2018

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1 Problem 1: G_n and F_n

Claim: Let $P(n) := G_n = F_{n+3} - 1$. The claim is that $\forall n \in \mathrm{N}^{\geq -1} : P(n)$.

Proof. We will prove by strong induction on n.

Base cases: $n = -1$ and $n = 0$. For the first case, $G_{-1} = F_2 - 1$, and by the definition of F_n and G_n, $F_2 = 1$ and $G_{-1} = 0$, giving us $0 = 1 - 1$, thus the case is true for $n = -1$. For the second case, $G_0 = F_3 - 1$, so $1 = 2 - 1$, thus the second case is also true.

Inductive case: Let $n \geq 1$. We will show $[P(-1) \wedge P(0) \wedge ... \wedge P(n)] \implies P(n+1)$.

- *Given*: Assume $P(-1) \wedge ... \wedge P(n-1) \wedge P(n)$ is true.
- *Want to show*: $P(n+1)$ is true.

Since $P(n-1)$ is true, we have $G_n = F_{n+2} - 1$. Since $P(n)$ is true, we have $G_{n-1} = F_{n+3} - 1$.

We will use this fact to show $P(n+1)$:

$$\begin{aligned}
P(n) := G_n &= F_{n+3} - 1 && \text{definition of P(n)} \\
P(n+1) := G_{n+1} &= F_{n+4} - 1 && \text{inductive hypothesis} \\
G_{n+1} &= G_n + G_{n-1} + 1 && \text{definition of G} \\
F_{n+4} &= F_{n+3} + F_{n+2} && \text{definition of F} \\
P(n+1) := G_n + G_{n-1} &= F_{n+3} + F_{n+2} - 2 && \text{substituting/rearranging terms}
\end{aligned}$$

By the claim, the following are given to be true:

$$\begin{aligned}
P(n) := G_n &= F_{n+3} - 1 \\
P(n-1) := G_{n-1} &= F_{n+2} - 1
\end{aligned}$$

By adding the two, we get:

$$G_n + G_{n-1} = F_{n+3} + F_{n+2} - 2$$

This is exactly equal to the rearranged inductive hypothesis for $P(n+1)$. Thus, the inductive hypothesis must be true. □

2 Problem 2: G_n lower bound

Claim: Let $P_2(h) := G_h \geq 2^{h/2}$. The claim is that $\forall n \in \mathrm{N}^{\geq 0} : P_2(h)$.

Proof. We will prove by strong induction on h.

Base cases: $h = 0$ and $h = 1$. For the first case, $G_0 = 1$, and $2^{0/2} = 1$, giving us $1 \geq 1$ and thus the first case is true. For the second case, $G_1 = 2$, and $2^{1/2} = \sqrt{2}$, giving us $2 \geq \sqrt{2}$, thus the second case is also true.

Inductive case: Let $h \geq 2$. We will show $[P_2(0) \wedge P_2(1) \wedge ... \wedge P_2(h-2) \wedge P_2(h-1)] \implies P_2(h)$.

- *Given*: Assume $P_2(-1) \wedge ... \wedge P_2(h-2) \wedge P_2(h-1)$ is true.
- *Want to show*: $P_2(h)$ is true.

Since $P_2(h-2)$ is true, we have $G_{h-2} \geq 2^{(h-2)/2}$. Since $P_2(h-1)$ is true, we have $G_{h-1} \geq 2^{(h-1)/2}$.

We will use this fact to show $P_2(h)$:

$$\begin{aligned} P_2(h) &:= G_h \geq 2^{h/2} && \text{definition of } P_2\text{(h)} \\ G_h &= G_{h-2} + G_{h-1} + 1 && \text{definition of G} \end{aligned}$$

By substituting G_{h-2} and G_{h-1} with the given cases, we get:

$$G_h \geq 2^{(h-2)/2} + 2^{(h-1)/2} + 1$$

□

3 Problem 3: lower bound on height balanced binary trees

4 Problem 4: false lower bound on binary trees

5 Problem 5: bound on height

6 Example of proof by induction

This is an example proof, provided in LaTeX so that you may copy its basic formatting.

Claim: Let $P(n) := \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. The claim is that $\forall n \in \mathrm{Z}^{\geq 1} : P(n)$.

Proof. We will prove by weak induction on n.

Base case: $n = 1$. In this case $\sum_{i=1}^{n} i = \sum_{i=1}^{1} i = 1$ and $\frac{n(n+1)}{2} = \frac{1\times(1+1)}{2} = 1$. Thus $P(1)$ is true.

Inductive case: Let $n \geq 2$. We will show $P(n-1) \implies P(n)$.

- *Given*: Assume $P(n-1)$ is true.
- *Want to show*: $P(n)$ is true.

Since $P(n-1)$ is true, we have

$$\sum_{i=1}^{n-1} i = \frac{(n-1)((n-1)+1)}{2}$$

We will use this fact to show $P(n)$:

$$\begin{aligned}
\sum_{i=1}^{n} i &= \left(\sum_{i=1}^{n-1} i\right) + n && \text{definition of summation} \\
&= \frac{(n-1)((n-1)+1)}{2} + n && \text{inductive hypothesis} \\
&= \frac{(n-1)n + 2n}{2} && \text{rearanging/simplifying terms} \\
&= \frac{n^2 - n + 2n}{2} = \frac{n^2+n}{2} && \text{algebra} \\
&= \frac{n(n+1)}{2} && \square
\end{aligned}$$

$\square$