# Problem Set 7

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# 1 Problem 1: Using induction to prove algorithm correctness, DLN 5.71

**Claim**: Let $P(n)$ if for a sorted array $A[1...n]$ of length $n$, $binarySearch(A, x) \iff x \in A$ . The claim is that $\forall n \in \mathrm{Z}^{\geq 0} : P(n)$.

*Proof.* We will prove by strong induction on $n$.

**Base cases**: The base cases are $n = 0$ and $n = 1$. Both $P(0)$ and $P(1)$ are true, as any array of length 0 or 1 is sorted.

**Inductive case**: Let $n \geq 2$. We will show $P(0) \wedge ... \wedge P(n-1) \iff P(n)$.

- *Given*: Assume $P(0) \wedge ... \wedge P(n-1)$ is true.
- *Want to show*: $P(n)$ is true.

We proceed by cases. There are three ways in which $x$ can exist with relation to $A[1...n]$ and $middle$. These cases are as such:

$$A[1...x = middle...n]$$
$$A[1...x...middle...n]$$
$$A[1...middle...x...n]$$

In Case 1, we have that $x = middle$. In Case 2, we have that $1 \leq x < middle \leq n$. In Case 3, we have that $1 \leq middle < x \leq n$.

*Case 1*: In the first case, the algorithm successfully found the item $x$ it was searching for. Thus, the function returns true.
*Case 2*: In the second case, the function will be called recursively on $A[1...middle - 1]$. From there
*Case 3*: In the third case, the function will be called recursively on $A[middle + 1...n]$. From there

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# 2 Problem 2: proving a relation is a partial order

## 2.1 DLN 8.131

A relation is a total order if it is a partial order where every pair is comparable ($\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$). To demonstrate that it is not a total order, we must simply identify an $(a, b)$ pair that does not satisfy the total order condition. An example is as follows.
The pair $\langle\langle 1, 2\rangle, \langle 2, 2\rangle\rangle$ would be in $R$ as it meets the condition of the relation. Additionally, the pair $\langle\langle 2, 1\rangle, \langle 2, 2\rangle\rangle$ would also be in $R$. However, neither $\langle\langle 2, 1\rangle, \langle 1, 2\rangle\rangle$ nor $\langle\langle 1, 2\rangle, \langle 2, 1\rangle\rangle$ would be in $R$, as neither meet the condition. Thus, this relation $R$ does not meet the definition of a total order, and thus is not a total order.

## 2.2 DLN 8.132

For $R$ to be a partial order, it must be reflexive, antisymmetric, and transitive. We will show that relation $R$ meets each of these properties.

- **Reflexivity:** $\langle\langle a, b\rangle, \langle x, y\rangle\rangle \in R$ if $a \leq x$ and $b \leq y$. Thus $\langle\langle a, b\rangle, \langle a, b\rangle\rangle \in R$ as $a \leq a$ and $b \leq b$. Therefore this relation is reflexive, as for any $(a, b)$ pair $A$, $\langle A, A\rangle \in R$.
- **Antisymmetry:** The antisymmetry property is only present if $\langle A, B\rangle \in R \wedge \langle B, A\rangle \in R \implies (A = B)$, where $A$ is the pair $(a, b)$ and $B$ is the pair $(x, y)$. If $\langle A, B\rangle \in R \wedge \langle B, A\rangle \in R$, then by the definition of $R$, $(a \leq x) \wedge (b \leq y)$ as well as $(x \leq a) \wedge (y \leq b)$. This is only possible if $a = x$ and $b = y$, as it is not possible to have $a < x < a$ or $b < y < b$. Additionally, if $a = x$ and $b = y$, then we must have $A = B$, and thus the relation $R$ is antisymmetric.
- **Transitivity:** The property of transitivity holds that if $\langle A, B\rangle \in R \wedge \langle B, C\rangle \in R$, then we must have that $\langle A, C\rangle \in R$. We have that $A$ is the pair $(a, b)$ and $B$ is the pair $(x, y)$, and $C$ is the pair $(c, d)$. If we have $\langle A, B\rangle \in R \wedge \langle B, C\rangle \in R$, then it must be that $(a \leq x) \wedge (b \leq y)$, as well as $(x \leq c) \wedge (y \leq d)$. It also holds that $(a \leq c) \wedge (b \leq d)$, and thus $\langle A, C\rangle \in R$, fulfilling the property of transitivity.

As the relation $R$ is reflexive, antisymmetric, and transitive, it is a partial order.

# 3 Problem 3: an equivalence relation and a partial order? DLN 8.155

**Claim**: There exists a relation $\preceq$ on the set $A$ that is both an equivalence relation and a partial order.

*Proof.* We will prove by direct proof, by showing an example of a relation that fits both the definition of an equivalence relation and a partial order.
A relation is an equivalence relation if it is reflexive, symmetric, and transitive. A relation is a partial order if it is reflexive, antisymmetric, and transitive. Thus for a relation to be both, it must be reflexive, symmetric, transitive, and antisymmetric simultaneously. We will show that the relation $R$ fits all of these properties. The relationship $R$ is defined as such: with relation to $A[1...n]$, $\langle a, b\rangle \in R$ if $a = b$. We will show that each property holds individually.

- **Reflexivity:** As for any element $a \in A[1...n]$ , $a = a$, then $\forall a \in A$, $\langle a, a\rangle \in R$. This is exactly the definition of reflexivity, and thus the relationship $R$ is reflexive.
- **Symmetry:** The property of symmetry exists if $\forall\langle a, b\rangle \in R$, we must have $\langle b, a\rangle \in R$. Since we have $\langle a, b\rangle \in R$, it must be that $a = b$, and that $b = a$. Thus, it holds that $\langle b, a\rangle \in R$. Therefore, the relationship $R$ is symmetric.
- **Antisymmetry:**
- **Transitivity:**

$\square$