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COSC 480 - Learning From Data
Fall 2019
Problem Set 5

1. Let X_1 and X_2 be independent random variables. Let $Y = X_1 + X_2$. Show that $V(Y) = V(X_1) + V(X_2)$.

   Before showing $V(Y) = V(X_1) + V(X_2)$, we first start by showing a property that will be helpful in this proof. Let $Z = E[cX_1X_2]$ where X_1, X_2 are independent random variables and c is a constant. We would like to show that the constant can be pulled out such that $Z = cE[X_1X_2]$.

   Let C be a random variable with a single possible outcome: c. The probability that C takes on the value of c is 1, and is thus independent from any other possible random variable. We can rewrite Z as $Z = E[CX_1X_2]$, and given that all three variables are independent, we have $Z = CE[X_1X_2]$ and since C is always c we finally have $Z = cE[X_1X_2]$ for all constants c.

   We now proceed with the target claim.

$$
\begin{aligned}
V(Y) &= E[Y^2] - (E[Y])^2 && \text{definition of V()} \quad (1)\\
&= E[(X_1 + X_2)^2] - (E[X_1 + X_2])^2 && \text{definition of Y} \quad (2)\\
&= E[{X_1}^2 + 2X_1X_2 + {X_2}^2] - (E[X_1] + E[X_2])^2 && \text{expand, } E[X+Y] = E[X] + E[Y] \quad (3)\\
&= E[{X_1}^2] + E[2X_1X_2] + E[{X_2}^2] - E[X_1]^2 - 2E[X_1]E[X_2] - E[X_2]^2 && E[X \cdot Y] = E[X] \cdot E[Y]\text{, expand} \quad (4)\\
&= E[{X_1}^2] + 2E[X_1]E[X_2] + E[{X_2}^2] - E[X_1]^2 - 2E[X_1]E[X_2] - E[X_2]^2 && \text{property shown above} \quad (5)\\
&= E[{X_1}^2] + E[{X_2}^2] - E[X_1]^2 - E[X_2]^2 && \text{algebra} \quad (6)\\
&= V(X_1) + V(X_2) && \text{definition of V()} \quad (7)
\end{aligned}
$$

   Therefore we have that $V(Y) = V(X_1) + V(X_2)$, where X_1, X_2 are random variables and $Y = X_1 + X_2$.

2. Prove the extension of Chebyshev's inequality for sums.

   We proceed by direct proof from the defined variables and Chebyshev's inequality for single random variables.

$$
\begin{aligned}
Pr(|Y - \mu| \geq a) &\leq \frac{V(Y)}{a^2} && \text{definition of Chebyshev's} \quad (8)\\
&\leq \frac{V(\sum_{i=1}^{N} \frac{1}{n} X_i)}{a^2} && \text{definition of Y} \quad (9)\\
&\leq \frac{\sum_{i=1}^{N} V(\frac{1}{n} X_i)}{a^2} && \text{shown in 1} \quad (10)\\
&\leq \frac{\sum_{i=1}^{N} V(X_i)}{N^2 a^2} && \text{pull out constant, squared since } V = \sigma^2 \quad (11)\\
&\leq \frac{V(X_i)}{Na^2} = \frac{\sigma^2}{Na^2} && \text{simplify} \quad (12)
\end{aligned}
$$

   Therefore, proceeding from the definitions given, the proven Chebyshev's inequality for random variables, and the property shown in 1, we have that $Pr(|Y - \mu| \geq a) \leq \frac{\sigma^2}{Na^2}$.