Project Euler - Exercise 25

The Fibonacci Sequence is given by $f_n = f_{n-1} + f_{n-2}$ where $f_0 = f_1 = 1$. If we define the linear transformation above

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

We notice that, if we apply it to the vector $\begin{bmatrix} f_{n-2} \\ f_{n-1} \end{bmatrix}$, we have

$$\Rightarrow T \begin{bmatrix} f_{n-2} \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n-1} \\ f_{n-1} + f_{n-2} \end{bmatrix} = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$$

Starting with the vector $v_0 = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and successively applying the transformation to the new result, we have:

$$\begin{aligned} Tv_0 = T \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = v_1 \\ \Rightarrow Tv_1 = T(Tv_0) = T^2 v_0 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} = v_2 \\ \Rightarrow Tv_2 = T(T^2 v_0) = T^3 v_0 &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} = v_3 \end{aligned}$$

If we continue the iterations, it's easy to notice that the pattern holds. Then, we have $T^n v_0 = v_n = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}$. This relation means that one way to calculate the n'th term of the Fibonacci Sequence is computing T^{n-1} and apply to v_0.

Since multiplication of matrices is a operation with high computational cost, we want to find a way to compute T^n with more efficiency.

Let's calculate the eigenvalues of T, i.e, the scalars $\lambda \in \mathbb{C}$ and $v \in \mathbb{R}^2$ such that:

$$Tv = \lambda v \tag{1}$$

$$\begin{aligned} &\Rightarrow Tx - \lambda v = 0 \\ &\Rightarrow (T - \lambda I)v = 0 \end{aligned}$$

Since we want non trivial solutions, we want that the matrix $T - \lambda I$ be singular (not invertible). Then:

$$\begin{aligned} \det(T - \lambda I) &= 0 \\ \det(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}) \\ = \det(\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}) \\ = \lambda^2 - \lambda - 1 &= 0 \end{aligned}$$

Then, the eigenvalues of T are $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$. Since the eigenvalues are distinct, the eigenvectors v_1, v_2 associated are linearly independent.

Let $V = [v_1, v_2]$ the matrix where the eigenvectors are the columns and $\Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$ the diagonal matrix of eigenvalues.

We can write the equation (1) as:

$$\begin{aligned} TV &= V\Lambda \\ \Rightarrow T &= V\Lambda V^{-1} \end{aligned}$$

since V is invertible (linearly independent eigenvectors).

Notice now, that if we want to calculate powers of T, we have:

$$\begin{aligned} T^2 &= (V\Lambda V^{-1})(V\Lambda V^{-1}) \\ &= V\Lambda(VV^{-1})\Lambda V^{-1} \\ &= V\Lambda I\Lambda V^{-1} \\ &= V\Lambda^2 V^{-1} \end{aligned}$$

This means, that if we want to compute T^n, we just have to compute $\Lambda^n = \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix}$.

This means that the n'th term of the Fibonacci Sequence is given by

$$\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} = V \begin{bmatrix} \lambda_1^{n-1} & \\ & \lambda_2^{n-1} \end{bmatrix} V^{-1}$$

One heuristic to find the first element of the Fibonacci sequence with 1000 digits is using its greatest eigenvalue.

Since the eigenvectors can be normalized, i.e, $||v_1|| = ||v_2|| = 1$, the value of f_n is going to be most influenced by $lambda_1^{n-1}$.

One way to estimate this first value, is calculating the which $n \in \mathbb{N}$ most approximate the relation $\lambda_1^{n-1} = 10^{1000}$. Let's calculate n:

$$
\begin{aligned}
\lambda_1^{n-1} &\approx 10^{1000} \\
\Rightarrow \log_{10}(\lambda_1^{n-1}) &\approx \log_{10}(10^{1000}) \\
\Rightarrow (n-1)\log_{10}(\lambda_1) &\approx 1000 \\
\Rightarrow n &= \left\lfloor \frac{1000}{\log_{10}(\lambda_1)} + 1 \right\rfloor
\end{aligned}
$$