Physical Interpretation of the KKT Conditions

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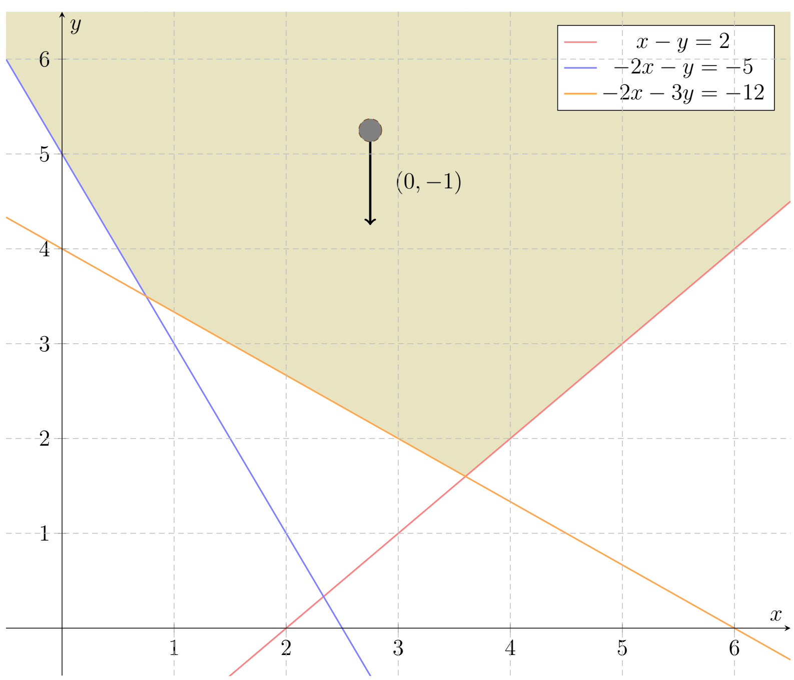
1 Linear Programming case

Let us consider the following optimization problem:

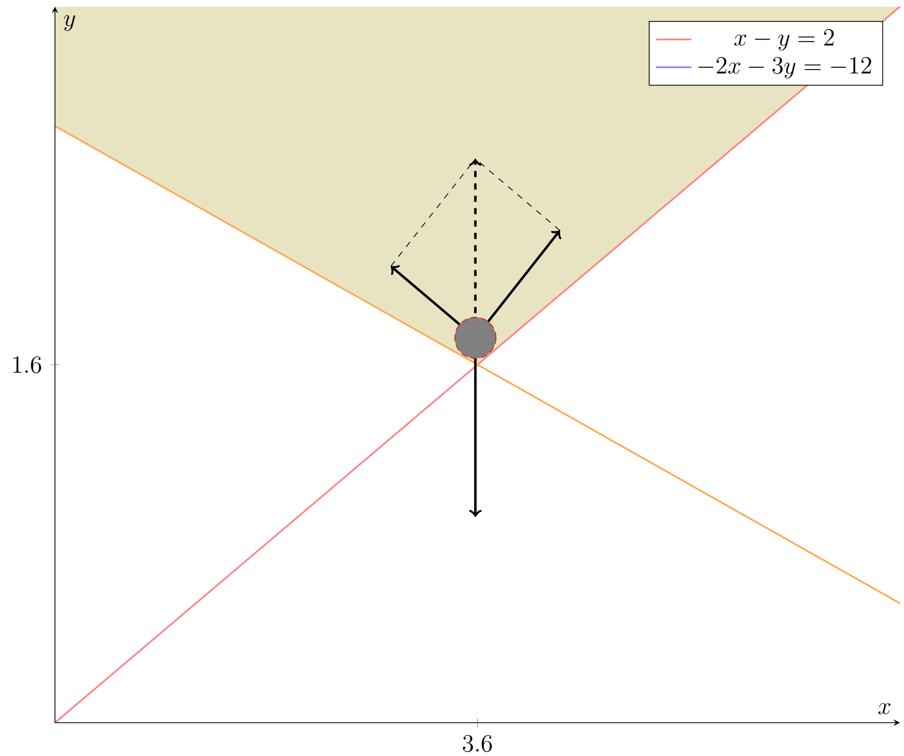
$$\begin{aligned} \text{minimize} \quad & y \\ \text{subject to} \quad & x - y \leq 2, \\ & -2x - y \leq -5, \\ & -2x - 3y \leq -12. \end{aligned}$$

We can easily solve it by drawing the feasible region and then looking for the point with minimum "height":



The solution is the intersection of the lines $x - y = 2$ and $-2x - 3y = -12$, i.e., the point $(\frac{18}{5}, \frac{8}{5}) = (3.6, 1.6)$.

But let us see this problem as a physical one. We put a ball inside the feasible region and let only gravity act on it. Let us say the weight of the ball is the vector $(0, -1)$. Where will the ball rest? *It will only rest if the resultant force is zero.*



At the point $(3.6, 1.6)$, we have a normal forces orthogonal to the "walls" $x - y = 2$ and $-2x - 3y = -12$. In other words, the normal forces are parallel to the gradients of $x - y$ and $-2x - 3y$. Therefore, for the resultant force to be zero, we need to find $\lambda, \mu \in \mathbb{R}$ such that

$$(0, -1) + \lambda(1, -1) + \mu(-2, -3) = 0. \tag{1}$$

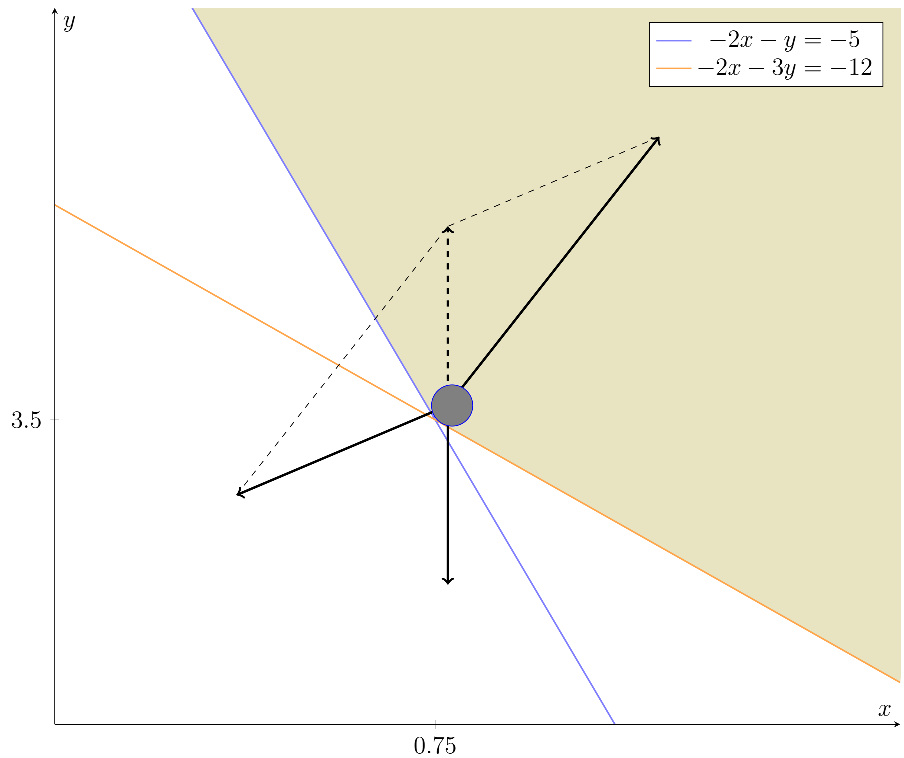
We get $\lambda = -0.4$ and $\mu = -0.2$.

Ok, at this point the normal forces cancel out the weight. Does it happen at any other point? At any interior point, we do not have normal forces to stop the ball. If the ball touches just one "wall," in our example, the normal force is not in the same direction as the weight, so they cannot cancel out each other. It remains to investigate the intersection of $-2x - y = -5$ and $-2x - 3y = -12$.

These lines intersect at $(\frac{3}{4}, \frac{7}{2}) = (0.75, 3.5)$ and their corresponding normal vectors are $(-2, -1)$ and $(-2, -3)$. If we look for $\lambda, \mu \in \mathbb{R}$ such that

$$(0, -1) + \lambda(-2, -1) + \mu(-2, -3) = 0, \tag{2}$$

we get $\lambda = 0.5$ and $\mu = -0.5$.



We have a problem here. Even though we can find vectors orthogonal to the lines that cancel out the weight, one of them is in the wrong direction. Since the ball pushes from inside the feasible region, the normal vector has to point toward the feasible region. The λ has the *wrong sign*.

Now note that the weight $(0, -1)$ is the opposite of the gradient of the objective function $f(x, y) = y$. Changing the signs in Equations (1) and (2), they are saying that we need to find scalars λ_i, $1 \leq I \leq n$, such that

$$\nabla f(x, y) + \sum_{i=1}^{n} \lambda_i \nabla g_i(x, y) = 0,$$

where the restrictions of the programming problem are given by $g_i(x, y) \leq 0$. All λ_i must be non-negative so the "normal forces" point in the right direction and, for each i, if the ball is not touching "wall" $g_i(x, y) = 0$, then $\lambda_i = 0$ (complementary slackness).

Even though in this problem the objective function and the weight vector seemed to be special, any problem can be seen like that: we can always rotate the space so $-\nabla f(x, y)$ points in the direction we want. So every linear programming problem can be seen as finding the point of minimum height.

It only remains to see the case of equality restrictions. But this we let for the reader to think about. Why the sign of the scalar does not matter in this case?

2 Differentiable case

To generalize the linear case to the differentiable one, we need to change the weight to something else. If we want to interpret $-\nabla f(x, y)$ as a force, then it is clear what $f(x, y)$ should be: the potential energy of a conservative field!

The problem is then to minimize the potential energy. If an object starts with velocity zero at the point of minimum potential energy, it cannot move at all since moving would mean increasing the kinetic energy and, hence, diminishing the potential energy. But then, we need to consider the normal forces given by the restrictions so the resultant force equals to zero. This is precisely what the KKT conditions say.