NPIV Estimation through Stochastic Gradients and Kernel Methods

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Summary

NPIV estimation

Our approach

Where we are at

Next steps

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- Consider a generic regression problem:

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- We end up estimating $h^\star(X) + \mathbb{E}[\varepsilon \mid X]$. Other problems may occur.

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- How does it help us?

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- Compare:
$$\mathrm{MSE}(h) = \mathbb{E}[(Y - h(X))^2] \quad \text{v.s.} \quad \mathcal{R}(h) = \mathbb{E}\left[(\mathbb{E}[Y - h(X) \mid Z])^2\right].$$

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We have

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- $\mathcal{R}(h) = 0 \iff \mathbb{E}\left[(h^\star - h)\,(X) \mid Z\right] = 0 \iff \mathbb{E}[h^\star(X) \mid Z] = \mathbb{E}[h(X) \mid Z].$

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  1. $Z \not\perp\!\!\!\perp X$,
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- Stands for “Nonparametric Instrumental Variable estimation”.

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$$\mathbb{E}[Y \mid Z] = \mathbb{E}[h^\star(X) \mid Z] \iff r_0 = \mathcal{P}[h^\star].$$
- We wish to "invert" $\mathcal{P}$.
- Risk measure:
$$\mathcal{R}(h) = \mathbb{E}\left[\frac{1}{2}\left(\mathbb{E}\left[Y - h(X) \mid Z\right]\right)^2\right] = \mathbb{E}\left[\frac{1}{2}\left(r_0(Z) - \mathcal{T}[h](Z)\right)^2\right].$$

Problem Formulation

- Our risk measure:

$$\begin{aligned}\mathcal{R}(h) &= \frac{1}{2}\mathbb{E}\left[\left(\mathbb{E}\left[Y - h(X) \mid Z\right]\right)^2\right] \\ &= \frac{1}{2}\mathbb{E}\left[\left(r_0(Z) - \mathcal{P}[h](Z)\right)^2\right] \\ &= \frac{1}{2}\mathbb{E}\left[(r_0 - \mathcal{P}[h])(Z)^2\right].\end{aligned}$$

Stochastic Gradients

- We showed that $\nabla\mathcal{R}(h) = \mathcal{P}^*[\mathcal{P}[h] - r_0]$, where

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- Immediate idea:

$$\begin{cases} h_0 \equiv 0, \\ h_t \leftarrow h_{t-1} - \alpha_t \nabla\mathcal{R}(h_{t-1}) \quad \text{for t } \geq 1 \ . \end{cases}$$

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- There are problems...

Stochastic Gradients

- Another idea: notice that

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\begin{aligned}
\mathcal{P}^*[f](x) &= \mathbb{E}\left[f(Z) \mid X = x\right] \\
&= \int_{\mathcal{Z}} f(z) p(z \mid x) \, \mathrm{d}z \\
&= \int_{\mathcal{Z}} f(z) \frac{p(x,z)}{p(x)p(z)} p(z) \, \mathrm{d}z \\
&= \mathbb{E}\left[f(Z)\Phi(x,Z)\right],
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where $\Phi(x,z) = \frac{p(x,z)}{p(x)p(z)}$.

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  where $\Phi(x,z) = \frac{p(x,z)}{p(x)p(z)}$.
- In the spirit of SGD, $f(Z)\Phi(x,Z)$ is a stochastic estimate for $\mathcal{P}^*[f](x)$.

Stochastic Gradients

- Substitute $\nabla \mathcal{R}(h) = \mathcal{P}^*[\mathcal{P}[h] - r_0]$ for

$$\widehat{\Phi}(\cdot, Z)\left(\widehat{\mathcal{P}}[h](Z) - \widehat{r}_0(Z)\right).$$

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- New algorithm:

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- We chose to use RKHS (kernel) methods for computing $\widehat{\Phi}, \widehat{\mathcal{P}}$ and $\widehat{r_0}$.

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Prototype

Prototype gave reasonable results

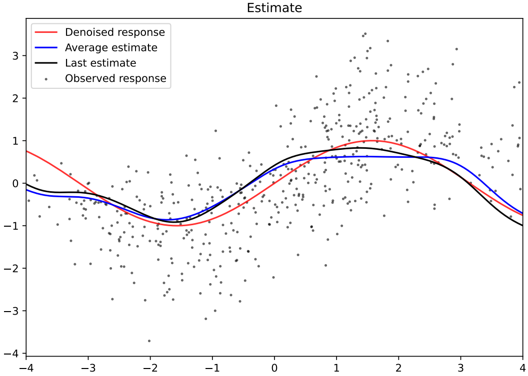

Figure: In red we have $h^\star = \sin$, in black we have h_N and in blue, $\frac{1}{N}\sum_{t=1}^{N} h_N$.

Theoretical properties

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- This is helping us find better ways to estimate Φ and $\mathcal{T}$ (mainly RKHS methods).

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- Finalize convergence guarantees.
- Implement modifications which the theory points to.
- Benchmark against current methods.

References

[1] Yuri R. Fonseca and Yuri F. Saporito. *Statistical Learning and Inverse Problems: A Stochastic Gradient Approach*. 2022. arXiv: 2209.14967 [stat.ML].

[2] Whitney K. Newey and James L. Powell. "Instrumental Variable Estimation of Nonparametric Models". In: *Econometrica* 71.5 (2003), pp. 1565–1578. ISSN: 00129682, 14680262. URL: http://www.jstor.org/stable/1555512 (visited on 07/03/2023).

Thank You!