Stochastic Gradient Descent in NPIV estimation

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1 Problem setup

1.1 Basic definitions

Fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Given $X \in L^2(\Omega; \mathbb{X} \subseteq \mathbf{R}^p)$, we define

$$L^2(X) \triangleq \left\{ h : \mathbb{X} \to \mathbf{R} \ : \ \mathbb{E}[h(X)^2] < \infty \right\},$$

that is, $L^2(X) = L^2(\mathbb{X}, \mathcal{B}(\mathbb{X}), \mathbb{P}_X)^1$, a Hilbert space equipped with the inner product $\langle h, g \rangle_{L^2(X)} = \mathbb{E}[h(X)g(X)]$. The regression problem we are interested in has the form

$$Y = h^\star(X) + \varepsilon, \tag{1}$$

where $h^\star \in L^2(X)$ and ε is an integrable r.v. such that $\mathbb{E}[\varepsilon \mid X] \neq 0$. We assume there exists $Z \in L^2(\Omega; \mathbb{Z} \subseteq \mathbf{R}^q)$ such that $Z \not\perp\!\!\!\perp X$, Z influences Y only through X and $\mathbb{E}[\varepsilon \mid Z] = 0$. This variable is called the instrumental variable. The problem consists of estimating $h^\star$ based on independent joint samples from X, Z and Y.

Conditioning (1) in Z, we find

$$\mathbb{E}[Y \mid Z] = \mathbb{E}[h^\star(X) \mid Z]. \tag{2}$$

This motivates us to introduce the operator $\mathcal{T} : L^2(X) \to L^2(Z)$ defined by

$$\mathcal{T}[h](z) \triangleq \mathbb{E}[h(X) \mid Z = z].$$

Clearly $\mathcal{T}$ is linear and, using Jensen's inequality, one may prove that it's bounded. It's also interesting to notice that its adjoint $\mathcal{T}^* : L^2(Z) \to L^2(X)$ satisfies

$$\mathcal{T}^*[g](x) = \mathbb{E}[g(Z) \mid X = x]. \tag{3}$$

Define $r_0 : \mathbb{Z} \to \mathbf{R}$ by $r_0(Z) = \mathbb{E}[Y \mid Z]$. Again by Jensen's inequality, we have $r_0 \in L^2(Z)$, and thus we can rewrite (2) as

$$\mathcal{T}[h^\star] = r_0. \tag{4}$$

Hence, (1) can be formulated as an inverse problem, where we wish to invert the operator $\mathcal{T}$.

> Discuss the other implication, that if h satisfies $\mathcal{T}[h] = r_0$, then $h = h^\star$. This is false, but the reason can be connected to the strength of the instrument Z.

1.2 Risk measure

Let $\ell : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$ be a pointwise loss function, which, with respect to its second argument, is convex and differentiable. We use the symbol ∂_2 to denote a derivative with respect to the second argument. The example to keep in mind is the quadratic loss function $\ell(y, y') = (y - y')^2$. Given $h \in L^2(X)$, we define the *populational risk* associated with it to be

$$\mathcal{R}(h) \triangleq \mathbb{E}[\ell(r_0(Z), \mathcal{T}[h](Z))].$$

We would like to solve

$$\inf_{h \in \mathcal{F}} \mathcal{R}(h),$$

where $\mathcal{F} \subseteq L^2(X)$ is a closed, convex set such that $h^\star \in \mathcal{F}$.

> Assumption

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[1] We denote by $\mathbb{P}_X$ the distribution of the r.v. X and by $\mathcal{B}(\mathbb{X})$ the Borel σ-algebra in $\mathbb{X}$.

2 Gradient computation

We'd like to compute $\nabla\mathcal{R}(h)$ for $h \in L^2(X)$. We start by computing the directional derivative of $\mathcal{R}$ at h in the direction f, denoted by $D\mathcal{R}[h](f)$:

$$
\begin{aligned}
D\mathcal{R}[h](f) &= \lim_{\delta\to 0} \frac{1}{\delta}\left[\mathcal{R}(h+\delta f) - \mathcal{R}(f)\right] \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[\ell(r_0(Z), \mathcal{T}[h+\delta f](Z)) - \ell(r_0(Z), \mathcal{T}[h](Z))\right] \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[\ell(r_0(Z), \mathcal{T}[h](Z) + \delta\mathcal{T}[f](Z)) - \ell(r_0(Z), \mathcal{T}[h](Z))\right] \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[\delta\partial_2\ell(r_0(Z), \mathcal{T}[h](Z))\mathcal{T}[f](Z)\right. \\
&\qquad\qquad\qquad \left. + \frac{\delta^2}{2}\partial_2^2\ell(r_0(Z), \mathcal{T}[h+\theta f](Z))\mathcal{T}[f](Z)^2\right] \\
&= \mathbb{E}\left[\partial_2\ell(r_0(Z), \mathcal{T}[h](Z))\mathcal{T}[f](Z)\right] \\
&\qquad\qquad + \lim_{\delta\to 0}\mathbb{E}\left[\frac{\delta}{2}\partial_2^2\ell(r_0(Z), \mathcal{T}[h+\theta f](Z))\mathcal{T}[f](Z)^2\right] \\
&= \mathbb{E}\left[\partial_2\ell(r_0(Z), \mathcal{T}[h](Z))\mathcal{T}[f](Z)\right],
\end{aligned}
$$

where $\theta \in \mathbf{R}$ is due to Taylor's formula and can be assumed to be inside a fixed interval $(-\theta_0, \theta_0)$, with θ_0 arbitrarily small. We have assumed at the last step that there exists $\theta_0 > 0$ such that

> **Assumption**

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$$
\sup_{|\theta| < \theta_0} \mathbb{E}\left[\partial_2^2\ell(r_0(Z), \mathcal{T}[h+\theta f](Z))\mathcal{T}[f](Z)^2\right] < \infty.
$$

This is a mild integrability condition which can be shown to hold in the quadratic case.

We can in fact expand the calculation a bit more, as follows:

$$
\begin{aligned}
D\mathcal{R}[h](f) &= \mathbb{E}\left[\partial_2\ell(r_0(Z), \mathcal{T}[h](Z))\mathcal{T}[f](Z)\right] \\
&= \langle\partial_2\ell(r_0(\cdot), \mathcal{T}[h](\cdot)), \mathcal{T}[f]\rangle_{L^2(Z)} \\
&= \langle\mathcal{T}^*[\partial_2\ell(r_0(Z), \mathcal{T}[h](\cdot))], f\rangle_{L^2(X)},
\end{aligned}
$$

where we are assuming that $\partial_2\ell(r_0(\cdot), \mathcal{T}[h](\cdot)) \in L^2(Z)$. This shows that $\mathcal{R}$ is Gateux-differentiable, with Gateux derivative at h given by

> **Assumption**

$$
D\mathcal{R}[h] = \mathcal{T}^*[\partial_2\ell(r_0(\cdot), \mathcal{T}[h](\cdot))].
$$

If we assume[2] that $h \mapsto D\mathcal{R}[h]$ is a continuous mapping from $L^2(Z)$ to $L^2(Z)$, then $\mathcal{R}$ is also Fréchet-differentiable, and both derivatives coincide. Therefore, under this assumption, which we henceforth make, $\nabla\mathcal{R}(h) = \mathcal{T}^*[\partial_2\ell(r_0(\cdot), \mathcal{T}[h](\cdot))]$.

> **Assumption**

> Talk about which conditions ℓ can satisfy so that this is continuous.

3 Unbiased estimator of the gradient

We have found that

$$
\nabla\mathcal{R}(h)(x) = \mathcal{T}^*[\partial_2\ell(r_0(\cdot), \mathcal{T}[h](\cdot))](x) = \mathbb{E}[\partial_2\ell(r_0(Z), \mathcal{T}[h](Z)) \mid X = x].
$$

This turns out to be hard to estimate in practice, as we have two nested conditional expectation operators. Our objective in this section is to find a random element $u_h \in L^2(X)$ such that $\mathbb{E}[u_h(x)] = \nabla\mathcal{R}(h)(x)$, so we can replace $\nabla\mathcal{R}(h)(x)$ by $u_h(x)$ in a gradient descent algorithm, obtaining a stochastic version which will be easier to compute.

> Should we discuss this further?

Our strategy to obtain u_h will be to write $\nabla\mathcal{R}(h)(x) = \mathbb{E}[\Phi(x, Z)\partial_2\ell(r_0(Z), \mathcal{T}[h](Z))]$, for some suitable kernel Φ. To ease the notation, define $\xi_h(z) \triangleq \partial_2\ell(r_0(z), \mathcal{T}[h](z))$. Assuming that X and

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[2]It is if ℓ is quadratic.

2

44   Z have a joint distribution which is absolutely continuous with respect to Lebesgue measure in $\mathbf{R}^{p+q}$,
45   we can write

$$
\begin{aligned}
\nabla \mathcal{R}(h)(x) &= \mathbb{E}[\xi_h(Z) \mid X = x] \\
&= \int_{\mathbb{Z}} p(z \mid x)\xi_h(z)\, \mathrm{d}z \\
&= \int_{\mathbb{Z}} p(z)\frac{p(z \mid x)}{p(z)}\xi_h(z)\, \mathrm{d}z \\
&= \mathbb{E}\left[\frac{p(Z \mid x)}{p(Z)}\xi_h(Z)\right].
\end{aligned}
$$

46   Thus, we must take

$$
\Phi(x, z) = \frac{p(z \mid x)}{p(z)} = \frac{p(x \mid z)}{p(x)} = \frac{p(x, z)}{p(x)p(z)}.
$$

47   With this choice, setting $u_h(x) = \Phi(x, Z)\xi_h(Z)$ we clearly have $\mathbb{E}[u_h(x)] = \nabla \mathcal{R}(h)(x)$.

> Must discuss why $u_h \in L^2(X)$.

4   Algorithm

49   Having an unbiased estimator of the gradient, we can construct an SGD algorithm for estimating $h^\star$.

> Discuss everything we don't know and must estimate.

> Comment on exactly what is needed to estimate each unknown (samples from which r.v.'s).

> Discuss necessity of discretizing $\mathbb{X}$.

Algorithm 1: SGD-NPIV

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input : Datasets $\mathcal{D}_{r_0} = \{(y_i, \boldsymbol{z}_i)\} \overset{\text{iid}}{\sim} \mathbb{P}_{YZ}$, $\mathcal{D}_\Phi = \{(\boldsymbol{x}_i, \boldsymbol{z}_i)\} \overset{\text{iid}}{\sim} \mathbb{P}_{XZ}$,
         $\mathcal{D}_\mathcal{T} = \{(\boldsymbol{x}_i, \boldsymbol{z}_i)\} \overset{\text{iid}}{\sim} \mathbb{P}_{XZ}$, discretization $\{\boldsymbol{x}_k\}_{k=1}^K$ of $\mathbb{X}$ which contains the observed
         values of X, sequence of learning rates $(\alpha_m)_{m=1}^M$.
output : $\left\{\widehat{h}(\boldsymbol{x}_k)\right\}_{k=1}^K$
Compute $\{\widehat{r_0}(\boldsymbol{z}_m; \mathcal{D}_{r_0})\}_{m=1}^M$;
50   Compute $\widehat{\Phi}(\boldsymbol{x}, \boldsymbol{z}; \mathcal{D}_\Phi)$;
for $1 \le m \le M$ **do**
     Compute $\widehat{\mathcal{T}[\widehat{h}_{m-1}]}(\boldsymbol{z}_m; \mathcal{D}_\mathcal{T})$;
     Set $u_m(\boldsymbol{x}_k) = \widehat{\Phi}(\boldsymbol{x}_k, \boldsymbol{z}_m)\partial_2 \ell\left(\widehat{r_0}(\boldsymbol{z}_m, \mathcal{D}_{r_0}), \widehat{\mathcal{T}[\widehat{h}_{m-1}]}(\boldsymbol{z}_m; \mathcal{D}_\mathcal{T})\right)$ for $1 \le k \le K$;
     Set $\widehat{h}_m(\boldsymbol{x}_k) = \widehat{h}_{m-1}(\boldsymbol{x}_k) - \alpha_m u_m(\boldsymbol{x}_k)$ for $1 \le k \le K$;
end
Set $\widehat{h} = \frac{1}{M}\sum_{m=1}^M \widehat{h}_m$;

---

51   An option we have is to project onto the closed, convex, bounded set $\mathcal{F}$ after applying the stochastic
52   gradient, that is, constructing the new estimate as

> Should we do this?

$$
\widehat{h}_m = P_\mathcal{F}\left[\widehat{h}_{m-1} - \alpha_m u_m\right].
$$

53   From what I can see, this would require minor changes to the proof and would justify the assumption
54   that $\widehat{h}_m \in \mathcal{F}$ for all m.

55   A possible choice for the set $\mathcal{F}$ is

$$
\mathcal{F} \triangleq \left\{h \in L^2(X) : \|h\|_\infty \le M\right\},
$$

56   where $M > 0$ is a constant chosen *a priori*. This set is obviously closed, convex and bounded in
57   the $L^2(X)$ norm. Furthermore, the operator $P_\mathcal{F}$ is very easy to compute, as $P_\mathcal{F}[h]$ is obtained by
58   cropping h inside $[-M, M]$. More formally,

$$
P_\mathcal{F}[h] = h^+ \wedge M - h^- \wedge M.
$$

3

5 Proof of convergence

60 The first problem is proving our sequence of estimates is, in fact, contained in $L^2(X)$. This amounts
61 to proving $u_m \in L^2(X)$ for every m. It's not even immediate why $u_h(x) = \Phi(x, Z)\xi_h(Z)$ (the
62 unbiased gradient when we know r_0, Φ and $\mathcal{T}$) belongs to $L^2(X)$ _____

> We'll need to bound the norm of u_m by a constant later in the proof.

63 After doing this, we check that $\mathcal{R}$ is convex in $\mathcal{F}$: if $h, g \in \mathcal{F}$ and $\lambda \in [0, 1]$, then

$$\begin{aligned}
\mathcal{R}(\lambda h + (1 - \lambda)g) &= \mathbb{E}[\ell(r_0(Z), \mathcal{T}[\lambda h + (1 - \lambda)g](Z))] \\
&= \mathbb{E}[\ell(r_0(Z), \lambda \mathcal{T}[h](Z) + (1 - \lambda)\mathcal{T}[g](Z))] \\
&\leq \lambda \mathbb{E}[\ell(r_0(Z), \mathcal{T}[h](Z))] + (1 - \lambda)\mathbb{E}[\ell(r_0(Z), \mathcal{T}[g](Z))] \\
&= \lambda \mathcal{R}(h) + (1 - \lambda)\mathcal{R}(g).
\end{aligned}$$

64 To lighten the notation, we denote the norm and inner product in $L^2(X)$ by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively.
65 By the Algorithm 1 procedure, we have

$$\begin{aligned}
\frac{1}{2}\left\|\widehat{h}_m - h^\star\right\|^2 &= \frac{1}{2}\left\|\widehat{h}_{m-1} - \alpha_m u_m - h^\star\right\|^2 \\
&= \frac{1}{2}\left\|\widehat{h}_{m-1} - h^\star\right\|^2 - \alpha_m \langle u_m, \widehat{h}_{m-1} - h^\star \rangle + \frac{\alpha_m^2}{2}\|u_m\|^2.
\end{aligned}$$

66 After adding and subtracting $\alpha_m \langle \nabla \mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle$, we are left with

$$\frac{1}{2}\left\|\widehat{h}_{m-1} - h^\star\right\|^2 - \alpha_m \langle u_m - \nabla \mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle + \frac{\alpha_m^2}{2}\|u_m\|^2 - \alpha_m \langle \nabla \mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle.$$

67 Applying the basic convexity inequality on the last term give us, in total,

$$\begin{aligned}
\frac{1}{2}\left\|\widehat{h}_m - h^\star\right\|^2 &\leq \frac{1}{2}\left\|\widehat{h}_{m-1} - h^\star\right\|^2 - \alpha_m \langle u_m - \nabla \mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle \\
&\quad + \frac{\alpha_m^2}{2}\|u_m\|^2 - \alpha_m(\mathcal{R}(\widehat{h}_{m-1}) - \mathcal{R}(h^\star)).
\end{aligned}$$

68 Rearranging terms, we get

$$\begin{aligned}
\mathcal{R}(\widehat{h}_{m-1}) - \mathcal{R}(h^\star) &\leq \frac{1}{2\alpha_m}\left(\left\|\widehat{h}_{m-1} - h^\star\right\|^2 - \left\|\widehat{h}_m - h^\star\right\|^2\right) \\
&\quad + \frac{\alpha_m}{2}\|u_m\|^2 - \langle u_m - \nabla \mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle.
\end{aligned}$$

69 Finally, summing over $1 \leq m \leq M$ leads to

$$\begin{aligned}
\sum_{n=1}^{M}\left[\mathcal{R}(\widehat{h}_{m-1}) - \mathcal{R}(h^\star)\right] &\leq \sum_{m=1}^{M}\frac{1}{2\alpha_m}\left(\left\|\widehat{h}_{m-1} - h^\star\right\|^2 - \left\|\widehat{h}_m - h^\star\right\|^2\right) \\
&\quad + \sum_{m=1}^{M}\frac{\alpha_m}{2}\|u_m\|^2 \\
&\quad - \sum_{m=1}^{M}\langle u_m - \nabla \mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle.
\end{aligned}$$

70 We then treat each term (summation) separately:

71 - The first term is bounded using the assumption that $\operatorname{diam}\mathcal{F} = D < \infty$. [Assumption]

72 - The bound on the second term depends on bounding $\mathbb{E}\left[\|u_m\|_{L^2(X)}^2\right]$ by a constant indepen-
73 dent of m or M.

74 - The third term must vanish because of the unbiasedness of u_m, but we don't know that our
75 u_m is unbiased, and it may very well not be.

4

First term By assumption, we have $\operatorname{diam} \mathcal{F} = D < \infty$. Hence

$$
\begin{aligned}
\sum_{m=1}^{M} \frac{1}{2\alpha_m} \left(\left\| \widehat{h}_{m-1} - h^\star \right\|^2 - \left\| \widehat{h}_m - h^\star \right\|^2 \right) &= \sum_{m=2}^{M} \left(\frac{1}{2\alpha_m} - \frac{1}{2\alpha_{m-1}} \right) \left\| \widehat{h}_{m-1} - h^\star \right\|^2 \\
&\quad + \frac{1}{2\alpha_1} \left\| \widehat{h}_0 - h^\star \right\|^2 - \frac{1}{2\alpha_M} \left\| \widehat{h}_M - h^\star \right\|^2 \\
&\leq \sum_{m=2}^{M} \left(\frac{1}{2\alpha_m} - \frac{1}{2\alpha_{m-1}} \right) D^2 + \frac{1}{2\alpha_1} D^2 = \frac{D^2}{2\alpha_M}.
\end{aligned}
$$

77 **Second term** Define $\mathcal{D}$ to be the set of all observed data, that is, all of the variables in $\mathcal{D}_\Phi, \mathcal{D}_{r_0}, \mathcal{D}_\mathcal{T}$
78 and $\{z_i\}_{i=1}^{M}$. Let's evaluate $\mathbb{E}\left[\|u_m\|^2 \right]$:

$$
\mathbb{E}\left[\|u_m\|^2 \right] = \mathbb{E}_\mathcal{D} \left[\mathbb{E}_X \left[\widehat{\Phi}(X, z_m; \mathcal{D}_\Phi)^2 \partial_2 \ell \left(\widehat{r_0}(z_m; \mathcal{D}_{r_0}), \widehat{\mathcal{T}[\widehat{h}_{m-1}]}(z_m; \mathcal{D}_\mathcal{T}) \right)^2 \right] \right],
$$

79 where the second expectation is with respect to a copy of X which is independent of $\mathcal{D}$. Continuing:

$$
\begin{aligned}
&\mathbb{E}_\mathcal{D} \left[\mathbb{E}_X \left[\widehat{\Phi}(X, z_m; \mathcal{D}_\Phi)^2 \partial_2 \ell \left(\widehat{r_0}(z_m; \mathcal{D}_{r_0}), \widehat{\mathcal{T}[\widehat{h}_{m-1}]}(z_m; \mathcal{D}_\mathcal{T}) \right)^2 \right] \right] \\
&\quad = \mathbb{E}_\mathcal{D} \left[\partial_2 \ell \left(\widehat{r_0}(z_m; \mathcal{D}_{r_0}), \widehat{\mathcal{T}[\widehat{h}_{m-1}]}(z_m; \mathcal{D}_\mathcal{T}) \right)^2 \mathbb{E}_X \left[\widehat{\Phi}(X, z_m; \mathcal{D}_\Phi)^2 \right] \right].
\end{aligned}
$$

80 If ℓ is quadratic, we have

$$
\begin{aligned}
&\mathbb{E}_\mathcal{D} \left[\partial_2 \ell \left(\widehat{r_0}(z_m; \mathcal{D}_{r_0}), \widehat{\mathcal{T}[\widehat{h}_{m-1}]}(z_m; \mathcal{D}_\mathcal{T}) \right)^2 \mathbb{E}_X \left[\widehat{\Phi}(X, z_m; \mathcal{D}_\Phi)^2 \right] \right] \\
&\quad = \mathbb{E}_\mathcal{D} \left[\left(\widehat{\mathcal{T}[\widehat{h}_{m-1}]}(z_m; \mathcal{D}_\mathcal{T}) - \widehat{r_0}(z_m; \mathcal{D}_{r_0}) \right)^2 \mathbb{E}_X \left[\widehat{\Phi}(X, z_m; \mathcal{D}_\Phi)^2 \right] \right].
\end{aligned}
$$

81

82 **Third term** We have to work with

$$
\mathbb{E}\left[\langle u_m - \nabla\mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle \right].
$$

> Find a way to finish this bound. Maybe switch the order of expectations? First in X and then in $\mathcal{D}$.

83 The strategy employed on the SIP paper was to condition on the σ-algebra generated by the training
84 samples observed up until iteration iteration $m-1$. In our case, that would be $z_1, \ldots, z_{m-1}$. The
85 problem which arises is that we no longer have measurability of $\widehat{h}_{m-1}$ with respect to this σ-algebra,
86 as it depends on the datasets $\mathcal{D}_\Phi, \mathcal{D}_\mathcal{T}, \mathcal{D}_{r_0}$, used to estimate $\widehat{\Phi}, \widehat{\mathcal{T}}$ and $\widehat{r_0}$ in an offline manner.
87 The other option would be to condition on more things, namely the σ-algebra generated by
88 $z_1, \ldots, z_{m-1}, \mathcal{D}_\Phi, \mathcal{D}_\mathcal{T}, \mathcal{D}_{r_0}$. We gain measurability of $\widehat{h}_{m-1}$, but we are no longer integrating
89 out $\mathcal{D}_\Phi, \mathcal{D}_\mathcal{T}, \mathcal{D}_{r_0}$, which is needed to use some sort of unbiasedness of the estimators $\widehat{\Phi}, \widehat{\mathcal{T}}, \widehat{r_0}$.
90 Let's try the latter and see what we end up with. Define $\mathcal{D}_{m-1} \triangleq \mathcal{D}_\Phi \cup \mathcal{D}_\mathcal{T} \cup \mathcal{D}_{r_0} \cup \{z_1, \ldots, z_{m-1}\}$.
91 Then,

$$
\begin{aligned}
&\mathbb{E}_\mathcal{D} \left[\langle u_m - \nabla\mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle \right] \\
&\quad = \mathbb{E}_{\mathcal{D}_{m-1}} \left[\mathbb{E}\left[\langle u_m - \nabla\mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle \mid \mathcal{D}_{m-1} \right] \right] \\
&\quad = \mathbb{E}_{\mathcal{D}_{m-1}} \left[\langle \mathbb{E}\left[u_m \mid \mathcal{D}_{m-1} \right] - \nabla\mathcal{R}(\widehat{h}_{m-1}), \widehat{h}_{m-1} - h^\star \rangle \right] \quad \text{(Justify this properly.).}
\end{aligned}
$$

92 Now we must evaluate the expression inside the inner product. We restrict ourselves to the quadratic
93 case. Define $\psi(z) = \mathcal{T}[\widehat{h}_{m-1}](z) - r_0(z)$ and $\widehat{\psi}(z; \mathcal{D}_{\text{proj}}) = \widehat{\mathcal{T}[\widehat{h}_{m-1}]}(z; \mathcal{D}_\mathcal{T}) - \widehat{r_0}(z; \mathcal{D}_{r_0})$, where

94    $\mathcal{D}_{\mathrm{proj}} = \mathcal{D}_{\mathcal{T}} \cup \mathcal{D}_{r_0}$. Then:

$$
\begin{aligned}
\mathbb{E}\left[u_m(x) \mid \mathcal{D}_{m-1}\right] &- \nabla \mathcal{R}(\widehat{h}_{m-1})(x) \\
&= \mathbb{E}\left[\widehat{\Phi}(x, Z; \mathcal{D}_{\Phi})\widehat{\psi}(Z; \mathcal{D}_{\mathrm{proj}}) \mid \mathcal{D}_{m-1}\right] - \mathbb{E}_Z\left[\Phi(x, Z)\psi(Z)\right] \\
&= \mathbb{E}_Z\left[\widehat{\Phi}(x, Z; \mathcal{D}_{\Phi})\widehat{\psi}(Z; \mathcal{D}_{\mathrm{proj}})\right] - \mathbb{E}_Z\left[\Phi(x, Z)\psi(Z)\right] \\
&= \mathbb{E}_Z\left[\left(\widehat{\Phi}(x, Z; \mathcal{D}_{\Phi}) - \Phi(x, Z)\right)\widehat{\psi}(Z; \mathcal{D}_{\mathrm{proj}}) + \left(\widehat{\psi}(Z; \mathcal{D}_{\mathrm{proj}}) - \psi(Z)\right)\Phi(x, Z)\right] \\
&= \left\langle \widehat{\Phi}(x, \cdot; \mathcal{D}_{\Phi}) - \Phi(x, \cdot), \widehat{\psi}(\cdot; \mathcal{D}_{\mathrm{proj}})\right\rangle_{L^2(Z)} + \left\langle \widehat{\psi}(\cdot; \mathcal{D}_{\mathrm{proj}}) - \psi(\cdot), \Phi(x, \cdot)\right\rangle_{L^2(Z)} \\
&\leq \left\|\widehat{\Phi}(x, \cdot; \mathcal{D}_{\Phi}) - \Phi(x, \cdot)\right\|_{L^2(Z)}\left\|\widehat{\psi}(\cdot; \mathcal{D}_{\mathrm{proj}})\right\|_{L^2(Z)} \\
&\qquad + \left\|\widehat{\psi}(\cdot; \mathcal{D}_{\mathrm{proj}}) - \psi(\cdot)\right\|_{L^2(Z)}\|\Phi(x, \cdot)\|_{L^2(Z)} .
\end{aligned}
$$