

FUNDAÇÃO GETULIO VARGAS  
SCHOOL OF APPLIED MATHEMATICS

CAIO F. LINS PEIXOTO

**NONPARAMETRIC INSTRUMENTAL VARIABLE REGRESSION  
THROUGH KERNEL METHODS AND STOCHASTIC GRADIENTS**

Rio de Janeiro  
2023

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Bachelor's dissertation presented to  
the School of Applied Mathematics  
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Stochastic Optimization, Machine Learning.

Advisor: Yuri F. Saporito

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Approved on December —, 2023  
By the organizing committee

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Yuri F. Saporito  
School of Applied Mathematics

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Board Member 1  
Institution 1

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Board Member 2  
Institution 2

I dedicate this thesis to ...

# Acknowledgements

Thanks, ...

*“ Biped! boped! bum! ”*

*Albert Einstein*

# Abstract

Keywords:



# Resumo

Palavras-chave:

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# 1 Introduction

Remember to cite every person ([NEWHEY; POWELL, 2003](#)).

## 2 Instrumental Variable Regression

Our goal with this chapter is to provide an introduction to instrumental variables for mathematicians and statisticians unfamiliar with the topic. We assume throughout that all random variables are defined on the same probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . By “random variable” we mean scalar or vector valued measurable functions with  $\Omega$  as their domain.

Rewrite  
chapter  
intro-  
duction.

### 2.1 Endogeneity

We start by introducing the problem of endogenous covariates. The structural equation we consider is the following:

$$Y = h^*(X) + \varepsilon, \quad (2.1)$$

where  $X$  is a vector of explanatory variables,  $Y$  is the scalar response,  $\varepsilon$  is a zero mean noise and the function  $h^*$  is the structural parameter we would like to estimate. The simplest estimation method for this model specification — and, therefore, one we would like to be able to use — is ordinary least squares (OLS), which works by finding, within a given class of functions  $\mathcal{H}$ , the element which minimizes the mean squared error:

$$\hat{h} = \arg \min_{h \in \mathcal{H}} \mathbb{E}[(Y - h(X))^2]. \quad (2.2)$$

A reasonable and ample choice for  $\mathcal{H}$  is the set of all square-integrable functions of  $X$ , that is, such that  $\mathbb{E}[h(X)^2] < \infty$ . Under this choice, we recover the conditional expectation of  $Y$  given  $X$ , i.e.,  $\hat{h}(X) = \mathbb{E}[Y | X]$ . Expanding  $Y$  through (2.1), we find that  $\hat{h}(X) = h^*(X) + \mathbb{E}[\varepsilon | X]$ . Hence, if  $\mathbb{E}[\varepsilon | X]$  is not identically null, we have introduced bias in our estimation.

This is one of the problems which appear when  $\mathbb{E}[\varepsilon | X] \neq 0$ , or, more generally, when  $X$  and  $\varepsilon$  are correlated in some way. When this happens, we say that  $X$  is *endogenous*. There are several causes for endogenous covariates, the most common of which are (WOOLDRIDGE, 2001):

**Omitted Variables** This means  $\varepsilon$  can be decomposed as  $g^*(W) + \eta$ , where  $\mathbb{E}[\eta | X, W] = 0$  a.s. and  $X$  and  $W$  are correlated. Hence, when we don't observe  $W$  and leave it to the error term, we end up estimating  $h^*(X) + \mathbb{E}[g^*(W) | X]$ . For example, if we want to regress a person's wage solely on her number of schooling years, there are other variables unaccounted for which influence both wages and schooling, such as natural ability. Innately skilled people may tend to be successful in school — and, therefore, pursue higher levels of education — as well as show higher performance in their future jobs, resulting in better wages.

**Measurement Error** If we are unable to exactly measure one of the covariates,  $X_k$ , and instead measure  $X'_k$  subject to some stochastic error, by using  $X'_k$  in our regression instead of  $X_k$  we are delegating to  $\varepsilon$  some measure of the difference between  $X_k$  and  $X'_k$ . Depending on how these two variables are related, we may introduce endogeneity. For example,  $X_k$  may be a marginal tax rate, but we may only have access to an average tax rate  $X'_k$ .

**Simultaneity** Simultaneity arises when one covariate  $X_k$  is determined simultaneously with  $Y$ . For example, if we are regressing neighborhood murder rates using the size of the local task force as a covariate, there is a simultaneity problem, since larger murder rates in a place cause a larger task force to be allocated there.

Bias in the estimation procedure is only one of the problems which arise when there are endogenous covariates. It's well known that the OLS estimate for linear regression fails to be consistent if any one of the covariates is endogenous (WOOLDRIDGE, 2001). To overcome endogeneity a few approaches exist, but by far the one most used by empirical economic research is instrumental variable estimation (WOOLDRIDGE, 2001).

## 2.2 Instrumental Variables

**2.1 Definition** An *instrumental variable* for regression problem (2.1) is a random variable  $Z$  such that

- (i) There is some influence of  $Z$  upon  $X$ , that is, the marginal distribution of  $X$  is not the same as the distribution of  $X$  conditioned on  $Z$ ;
- (ii) The conditional mean of  $\varepsilon$  given  $Z$  is almost surely null, i.e.,  $\mathbb{E}[\varepsilon \mid Z] = 0$ .

The idea behind an instrumental variable is that it is exogenous (ii) while still influencing  $Y$  through  $X$  (i). An exogenous covariate, in contrast to an endogenous one, is a variable that is determined outside of the system described by (2.1).

Condition (ii) is only one of the possible meanings for the statement that  $Z$  is exogenous. Two possible alternatives are requiring that  $Z$  be (1) independent from, or (2) uncorrelated with  $\varepsilon$ . Of course, (1) is a much more strict requirement which implies (ii), while (2) is a softer condition, implied by (ii). Independence is almost always impossible to verify in real scenarios, so (1) is not a good option. In contrast, there are situations where condition (2) is enough for ensuring good properties of IV estimators, including one we will present shortly, the linear model (WOOLDRIDGE, 2001). However, in order to prepare grounds for the nonparametric methods that will come later, we chose to use the definition which serves both.

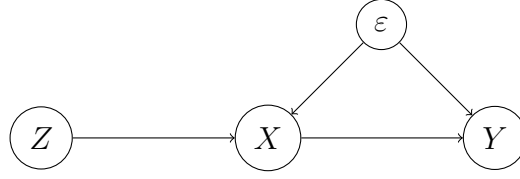


Figure 1 – Causal diagram for equation (2.1), where  $X$  is endogenous and  $Z$  is an IV. Source: prepared by the author.

Instrumental variables are also studied in the context of causal inference, where the conditions above are presented differently, in terms of causal diagrams. In this field, instrumental variables are also required that to satisfy a third condition, phrased in terms of the causal diagram describing the relations between variables of interest (HERNÁN; ROBINS, 2020):

- (iii) All paths from  $Z$  to  $Y$  must pass through  $X$ , that is,  $Z$  *only* influences  $Y$  through  $X$ .

In this sense, a typical causal diagram for an IV problem is the one in Figure 1.

## 2.3 Two Stages Least Squares (2SLS)

In this section, we restrict the structural function  $h^*$  in (2.1) to be affine:

$$h^*(x) = \beta_0 + \beta_1 x_1 + \cdots + \beta_{d_X} x_{d_X}, \quad (2.3)$$

and assume to have access to a random variable  $Z$ , taking values in  $\mathbf{R}^{d_Z}$ , satisfying conditions 2.1 (i) and (ii), so that  $Z$  is a valid instrumental variable. Hence, our data is composed of  $n$  independent joint samples  $\{(X_i, Z_i, Y_i)\}_{i=1}^n$ . Let  $\mathbf{X} \in \mathbf{R}^{n \times (d_X+1)}$  and  $\mathbf{Z} \in \mathbf{R}^{n \times (d_Z+1)}$  be the experiment design matrices with 1's in the first column.

## 2.4 Nonparametric Instrumental Variable Regression

In nonparametric regression, we do not specify *a priori* a finite dimensional parametric form for the structural function (such as restricting it to be affine), and so we allow our search space to potentially be infinite dimensional. However, in doing this, we must still precisely define the infinite dimensional space where the solution will be searched for. Hence, we start by precisely defining the nonparametric regression problem given by (2.1)

### 2.4.1 Problem specification

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be the underlying probability space. Assume that  $X : (\Omega, \mathcal{A}) \rightarrow (\mathbf{R}^{d_X}, \mathcal{B}(\mathbf{R}^{d_X}))$  and  $\varepsilon : (\Omega, \mathcal{A}) \rightarrow (\mathbf{R}, \mathcal{B}(\mathbf{R}))$  are measurable<sup>1</sup> and, furthermore, that  $\varepsilon \in$

Finish this section. Explain 2SLS and give examples of good IVs, as well as numerical examples.



$L^1(\Omega, \mathcal{A}, \mathbb{P})$  with  $\mathbb{E}[\varepsilon] = 0$ . We also assume that  $\mathbb{E}[\varepsilon \mid X]$  is *not* almost surely null and, hence,  $X$  is endogenous. Denote by  $\mathbb{P}_X$  the distribution of the random variable  $X$ , that is, the pushforward measure  $\mathbb{P} \circ X^{-1}$  defined on  $\mathcal{B}(\mathbf{R}^{d_X})$ . We write  $L^2(X)$  as a shorthand for the space  $L^2(\mathbf{R}^{d_X}, \mathcal{B}(\mathbf{R}^{d_X}), \mathbb{P}_X)$  of real and square integrable (equivalence classes of) measurable functions defined on the measure space  $(\mathbf{R}^{d_X}, \mathcal{B}(\mathbf{R}^{d_X}), \mathbb{P}_X)$ . It's important to recall that the inner product and norm in  $L^2(X)$  are given by  $\langle h, g \rangle_{L^2(X)} = \mathbb{E}[h(X)g(X)]$  and  $\|h\|_{L^2(X)}^2 = \langle h, h \rangle_{L^2(X)} = \mathbb{E}[h(X)^2]$ .

We assume there exists  $h^* \in L^2(X)$  such that (2.1) holds, that is,  $Y = h^*(X) + \varepsilon$ . Finally, we assume there exists a random variable  $Z : (\Omega, \mathcal{A}) \rightarrow (\mathbf{R}^{d_Z}, \mathcal{B}(\mathbf{R}^{d_Z}))$  such that  $Z$  qualifies as an instrumental variable, i.e.,  $Z$  satisfies conditions 2.1 (i) and (ii). We define  $\mathbb{P}_Z$  and  $L^2(Z)$  in a manner analogous to  $\mathbb{P}_X$  and  $L^2(X)$ . Our goal is to estimate  $h^*$  based on i.i.d. samples from the joint distribution of  $X, Z$  and  $Y$ .

## 2.4.2 Identification

An important question to ask after specifying the problem is whether the function  $h^*$  is identified. The answer is negative without further assumptions, which will be presented in this subsection. This discussion was inspired on the introduction of (NEWHEY; POWELL, 2003).

Suppose there exists  $\delta \in L^2(X)$  such that  $\delta \neq 0$ , but  $\mathbb{E}[\delta(X) \mid Z] = 0$ . Without loss of generality, we can assume<sup>2</sup>  $\delta(X) \neq \mathbb{E}[\varepsilon \mid X]$ . Defining  $g \triangleq h^* + \delta$  and  $\eta \triangleq \varepsilon - \delta(X)$ , we have

$$Y = g(X) + \eta,$$

where  $\mathbb{E}[\eta \mid Z] = 0$  and  $\mathbb{E}[\eta \mid X] \neq 0$ . Hence,  $g \neq h^*$  but they are indistinguishable from the data generating process perspective. Reciprocally, suppose that the only member of  $L^2(X)$  which has null mean conditioned on  $Z$  is the null function. Then, given  $g \in L^2(X)$  such that

$$Y = g(X) + \eta$$

with  $\mathbb{E}[\eta \mid Z] = 0$ , we have

$$0 = (g - h^*)(X) + \eta - \varepsilon.$$

Conditioning on  $Z$ , we get

$$\mathbb{E}[(g - h^*)(X) \mid Z] = 0,$$

which, by assumption, implies  $h^* = g$ .

Therefore, a necessary and sufficient condition for identification of our regression problem is the following:

<sup>1</sup> We denote by  $\mathcal{B}(\mathbf{R}^k)$  the Borel  $\sigma$ -algebra in  $\mathbf{R}^k$ .

<sup>2</sup> If it happens to be the case that  $\mathbb{E}[\mathbb{E}[\varepsilon \mid X] \mid Z] = 0$  (which is *not* implied by our assumptions so far), we can simply take  $\delta(X) = \lambda \mathbb{E}[\varepsilon \mid X]$ , for some  $\lambda \in \mathbf{R} \setminus \{0, 1\}$ . Since, by hypothesis,  $\mathbb{E}[\varepsilon \mid X] \neq 0$ , this satisfies our requirements and is different from  $\mathbb{E}[\varepsilon \mid X]$ .

**Assumption (Identification)** If  $\delta \in L^2(X)$  satisfies  $\mathbb{E}[\delta(X) \mid Z] = 0$ , then  $\delta = 0$ .

This condition has an interpretation in terms of the conditional expectation operator, which will be a key object in the construction of our estimator for  $h^*$ . Let  $h \in L^2(X)$ . Notice that, by Jensen's inequality,

$$\mathbb{E}[(\mathbb{E}[h(X) \mid Z])^2] \leq \mathbb{E}[\mathbb{E}[h(X)^2 \mid Z]] = \mathbb{E}[h(X)^2] < +\infty. \quad (2.4)$$

Furthermore, since  $\mathbb{E}[h(X) \mid Z]$  is a  $\sigma(Z)$ -measurable<sup>3</sup> random variable, by the Doob-Dynkin Lemma there exists a measurable function  $f_h : (\mathbf{R}^{dz}, \mathcal{B}(\mathbf{R}^{dz})) \rightarrow (\mathbf{R}, \mathcal{B}(\mathbf{R}))$  such that

$$\mathbb{E}[h(X) \mid Z] = f_h(Z).$$

Under these conditions, we write  $\mathbb{E}[h(X) \mid Z = z]$  for  $f_h(z)$ . The computation in (2.4) shows that  $f_h \in L^2(Z)$  for every  $h \in L^2(X)$ . Therefore, we can define the operator  $\mathcal{P} : L^2(X) \rightarrow L^2(Z)$  given by  $\mathcal{P}[h] = f_h = \mathbb{E}[h(X) \mid Z = \cdot]$ . This operator, called the *conditional expectation operator*, is clearly linear and, again by (2.4), also bounded, satisfying  $\|\mathcal{P}\|_{\text{op}} \leq 1$ . The identification assumption thus amounts to saying that the kernel of  $\mathcal{P}$  is trivial, i.e.,  $\mathcal{P}$  is injective.

It is hard to quantify how restrictive this condition is for arbitrary  $X$  and  $Z$ , so we analyze it in a more familiar setting, the exponential family. We will use a classic completeness result for statistics in this family of distributions to reformulate the identification assumption in terms of more familiar objects. We first define completeness:

**2.2 Definition (LEHMANN, 1959)** We say that a family  $\mathcal{P}$  of probability distributions on a measurable space  $(E, \mathcal{E})$  is *complete* if

$$\int_E f(x) P(dx) = 0 \quad \text{for all } P \in \mathcal{P}$$

implies  $f(x) = 0$   $\mathcal{P}$ -a.e.<sup>4</sup>.

Then, a remarkable fact about the exponential family is the completeness of the natural statistics under a mild condition on the set of parameters:

**2.3 Theorem (LEHMANN, 1959)** Let  $\Xi$  be a subset of an Euclidian space with nonempty interior. Let  $X$  be a random vector with distribution  $P^\theta$  parametrized by  $\theta \in \Xi$  in the following manner:

$$P^\theta(dx) = C(\theta) \exp \left\{ \sum_{i=1}^s \theta_i T_i(x) \right\} \mu(dx),$$

where  $\mu$  is the underlying measure. Then, the family  $\mathcal{P}_T$ , formed by the distributions of the random vector  $T(X) = (T_1(X), \dots, T_s(X))$  as  $\theta$  ranges through  $\Xi$ , is complete.

<sup>3</sup> We denote by  $\sigma(Z)$  the smallest  $\sigma$ -algebra in  $\Omega$  with respect to which  $Z$  is measurable.

<sup>4</sup> We say that a statement  $Q(x)$  is true  $\mathcal{P}$ -a.e. if there exists a set  $N \in \mathcal{E}$  such that  $Q(x)$  is true for  $x \in E \setminus N$  and  $P(N) = 0$  for all  $P \in \mathcal{P}$ .

Using this result and under suitable hypotheses, we can reformulate the identification condition.

**2.4 Theorem** For  $z \in \mathbf{R}^{d_z}$ , let  $\mathbb{Q}_z : \mathcal{B}(\mathbf{R}^{d_x}) \rightarrow [0, 1]$  denote the conditional distribution of  $X$  given  $Z = z$ . Assume there exists  $U \in \mathcal{B}(\mathbf{R}^{d_z})$  such that  $\mathbb{P}_Z(U) = 1$  and for all  $z \in U$  we have

$$\mathbb{Q}_z(dx) = C(z) \exp(\alpha(z)^\top T(x)) \mu(dx)$$

for an underlying measure  $\mu$  on  $\mathcal{B}(\mathbf{R}^{d_x})$  and some functions  $\alpha : \mathbf{R}^{d_z} \rightarrow \mathbf{R}^s$  and  $T : \mathbf{R}^{d_x} \rightarrow \mathbf{R}^s$ . Assume that  $T$  is injective and that the image of  $\alpha$  restricted to  $U$  contains an open set. Then,  $h^*$  is identified.

*Proof.* Taking  $\Xi = \alpha(U)$  and  $\theta = \alpha(z)$ , we see that the hypotheses of Theorem 2.3 are satisfied and, hence,

$$\mathcal{P} \triangleq \{\mathbb{Q}_z \circ T^{-1} : z \in U\}$$

is a complete family of probability distributions. Let  $h \in L^2(X)$  be such that  $\mathbb{E}[h(X) | Z] = 0$ . This means that the function

$$z \mapsto \int_{\mathbf{R}^{d_x}} h(x) \mathbb{Q}_z(dx)$$

is null  $\mathbb{P}_Z$ -a.s. Without loss of generality, we may assume that its null on  $U$ . But notice that, since  $T$  is injective, we can rewrite this integral as

$$\begin{aligned} 0 &= \int_{\mathbf{R}^{d_x}} h(x) \mathbb{Q}_z(dx) = \int_{\mathbf{R}^{d_x}} (h \circ T^{-1})(T(x)) \mathbb{Q}_z(dx) \\ &= \int_{\mathbf{R}^s} (h \circ T^{-1})(t) (\mathbb{Q}_z \circ T^{-1})(dt) \end{aligned}$$

for all  $z \in U$ . By completeness of  $\mathcal{P}$ , this implies  $h \circ T^{-1}(t) = 0$   $\mathcal{P}$ -a.s. which, in turn, means that for all  $z \in U$  we have

$$\begin{aligned} 1 &= (\mathbb{Q}_z \circ T^{-1})[(h \circ T^{-1})(t) = 0] = \mathbb{Q}_z[(h \circ T^{-1})(T(x)) = 0] \\ &= \mathbb{Q}_z[h(x) = 0]. \end{aligned}$$

Now, by the definition of conditional probability we have

$$\begin{aligned} \mathbb{P}_X[h(x) = 0] &= \int_{\mathbf{R}^{d_z}} \mathbb{Q}_z[h(x) = 0] \mathbb{P}_Z(dz) \\ &= \int_U \mathbb{Q}_z[h(x) = 0] \mathbb{P}_Z(dz) \\ &= 0. \end{aligned}$$

Therefore,  $h$  is the null function, which means  $h^*$  is identified. □

See what these conditions imply in the multivariate

### 3 Conclusion

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## Appendix