

Motivation

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Definition 1. Let A be a ring, and \mathbb{P} be a preorder - we will refer to this preorder as a *map*. Consider the *space of connections* $\mathbb{L} = \mathbb{P} \cup \mathbb{P}^{-1}$. If $(a, b) \in \mathbb{L}$, we denote $a \overset{\circ}{\dashv} b$. Now, consider \mathbb{V} to be the free A -module generated by \mathbb{L} . We call it *solution* of \mathbb{P} .

Now define the sets $S = \{a \overset{\circ}{\dashv} b - b \overset{\circ}{\dashv} a\}_{a \leq b}$ and $T = \{a \overset{\circ}{\dashv} b + b \overset{\circ}{\dashv} c - a \overset{\circ}{\dashv} c\}_{a \leq b \leq c}$. These sets are called the *coagulant* and *solvent*, and its elements are called *symmetries* and *transitivities*, respectively. We define the *congeal* (name and notation for the congeal are not yet established) as $\mathbb{V}_S = \mathbb{V} / \langle S \rangle$, where $\langle X \rangle$ is the submodule generated by X . The *hydrosphere* is $V_{\mathbb{P}} = \mathbb{V}_S / \langle \pi(T') \rangle$, where π is the projection $\mathbb{V} \rightarrow \mathbb{V}_S$. If $a \overset{\circ}{\dashv} b$, we write its image in $V_{\mathbb{P}}$ as $[a \overset{\circ}{\dashv} b]$, and call it a *polarity*.

When it is not ambiguous, I will call the hydrosphere V .

Proposition 1. Given a hydrosphere V , we have:

- (*linear symmetry*) If $a \overset{\circ}{\dashv} b$, then $[a \overset{\circ}{\dashv} b] = [b \overset{\circ}{\dashv} a]$;
- (*additive transitivity*) If $a \leq b \leq c$, then $[a \overset{\circ}{\dashv} b] + [b \overset{\circ}{\dashv} c] = [a \overset{\circ}{\dashv} c]$;
- (*nilreflexivity*) When a is an element of $\text{dom} \mathbb{P}$, $[a \overset{\circ}{\dashv} a] = 0$.

Proof. I will only prove that last one: since $a \leq a \leq a$, $[a \overset{\circ}{\dashv} a] + [a \overset{\circ}{\dashv} a] = [a \overset{\circ}{\dashv} a]$. Thus, $[a \overset{\circ}{\dashv} a] = 0$. \square

A *map* will be illustrated by its Hasse diagram, and we will see that this diagram will be an important part of our work. For clarity:

Definition 2. Given a finite preorder \mathbb{P} , its Hasse diagram (which we will call a *hydrography*) is the graph when the vertices are the elements of $\text{dom} \mathbb{P}$, and the edges are the pairs $a \leq b$, when there is no $c \notin \{a, b\}$ such that $a \leq c \leq b$.

If $a \leq b$ is an edge of the Hasse diagram of \mathbb{P} , then $[a \overset{\circ}{\dashv} b]$ is called an *edge* of V .

Proposition 2. Let \mathbb{P} be a finite preorder. Then the edges generate the hydrosphere.

Proof. It is clear that the polarities generate V . Let $[a \overset{\circ}{\dashv} b]$ be a polarity that is not an edge. Then there exists a chain $a := c_1 \leq \dots \leq c_n := b$, $n > 2$, where each $[c_i \overset{\circ}{\dashv} c_{i+1}]$, $i = 1, \dots, n-1$, is an edge. Thus $[a \overset{\circ}{\dashv} b] = \sum_i [c_i \overset{\circ}{\dashv} c_{i+1}]$. \square