Motivation

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Definition 1. Let A be a ring, and $\mathbb{P}$ be a preorder - we will refer to this preorder as a *map.* Consider the *space of connections* $\mathbb{L} = \mathbb{P} \cup \mathbb{P}^{-1}$. If $(a, b) \in \mathbb{L}$, we denote a—b. Now, consider $\mathbb{V}$ to be the free A-module generated by $\mathbb{L}$. We calll it *solution* of $\mathbb{P}$.
Now define the sets $S = \{a—b - b—a\}_{a \leq b}$ and $T = \{a—b + b—c - a—c\}_{a \leq b \leq c}$. These sets are called the *coagulant* and *solvent*, and its elements are called *symmetries* and *transitivities*, respectively. We define the *congeal* (name and notation for the congeal are not yet stablished) as $\mathbb{V}_S = \mathbb{V}/\langle S \rangle$, where $\langle X \rangle$ is the submodule generated by X. The *hydrosphere* is $V_{\mathbb{P}} = \mathbb{V}_S/\langle \pi(T') \rangle$, where π is the projection $\mathbb{V} \to \mathbb{V}_S$. If a—b, we write its image in $V_{\mathbb{P}}$ as $[a—b]$, and call it a *polarity.*

When it is not ambiguous, I will call the hydrosphere V.

Proposition 1. Given a hydrosphere V, we have:

- (*linear symmetry*)If a—b, then $[a—b] = [b—a]$;
- (*additive transitivity*) If $a \leq b \leq c$, then $[a—b] + [b—c] = [a—c]$;
- (*nilreflexivity*) When a is an element of dom$\mathbb{P}$, $[a—a] = 0$.

Proof. I will only prove that last one: since $a \leq a \leq a$, $[a—a] + [a—a] = [a—a]$. Thus, $[a—a] = 0$. □

A *map* will be illustrated by its Hasse diagram, and we will see that this diagram will be an important part of our work. For clarity:

Definition 2. Given a finite preorder $\mathbb{P}$, its Hasse diagram (which we will call a *hydrography*) is the graph when the vertices are the elements of dom$\mathbb{P}$, and the edges are the pairs $a \leq b$, when there is no $c \notin \{a, b\}$ such that $a \leq c \leq b$.

If $a \leq b$ is an edge of the Hasse diagram of $\mathbb{P}$, then $[a—b]$ is called an *edge* of V.

Proposition 2. Let $\mathbb{P}$ be a finite preorder. Then the edges generate the hydrosphere.

Proof. It is clear that the polarities generate V. Let $[a—b]$ be a polarity that is not an edge. Then there exists a chain $a := c_1 \leq \cdots \leq c_n =: b$, $n > 2$, where each $[c_i—c_{i+1}]$, $i = 1, \ldots, n-1$, is an edge. Thus $[a—b] = \sum_i [c_i—c_{i+1}]$. □