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On the formation of interaction networks in social coordination games

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Abstract

There are many situations where two interacting individuals can benefit from coordinating their actions. We examine the endogenous choice of partners in such social coordination games and the implications for resulting play. We model the interaction pattern as a network where individuals periodically have the discretion to add or sever links to other players. With such endogenous interaction patterns we see multiple stochastically stable states of play, including some that involve play of equilibria in the coordination game that are neither efficient nor risk-dominant. Thus the endogenous network structure not only has implications for the interaction pattern that emerges, but it also has a significant impact on the play in the coordination game relative to what would arise if the same interaction network were exogenous.

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1. Introduction

There are many situations where two interacting individuals can benefit from coordinating their actions. Examples where coordination is important include

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research and development partnerships, joint production, political and trade alliances, as well as the choice of compatible technologies or conventions such as the choice of a software or language.[1] For instance, a person may select a long distance service or internet provider keeping in mind which is best at facilitating communication with friends and family. Often there is an advantage to having individuals coordinate on selecting the same or compatible systems, with examples including long distance telephone service with MCI's friends and family plan and internet service with AOL's buddy list system. In many such situations, individuals select a strategy (e.g., technology) that they then use in interactions with many other individuals. Such situations are often characterized by multiple equilibria corresponding to coordination on different technologies or strategies, where the equilibria may be Pareto ranked. The welfare implications of such a ranking provide an obvious importance to understanding what behavior might be predicted and which factors determine whether efficient coordination is attained.

In the context of symmetric 2×2 coordination games, Kandori et al. (1993) and Young (1993) have shown that populations of individuals, who are subjected to small random perturbations in their strategy choices, tend in the long run to coordinate on risk-dominant strategies as defined by Harsanyi and Selten (1988). Thus the risk-dominant equilibrium is selected from among the set of strict Nash equilibria, even if the risk-dominant equilibrium is the inefficient equilibrium. This result has a natural and simple intuition: the basin of attraction of the risk-dominant equilibrium is larger than that of the non risk-dominant equilibrium. In particular, more than half of the population must be playing the non-risk-dominant strategy (which may be the efficient strategy) for that strategy to be a best response. So, if everyone initially plays the risk-dominant strategy, then more than half the population must randomly change to the efficient strategy for the dynamics to move towards the efficient equilibrium, while if everyone initially plays the efficient strategy, then less than half the population needs to randomly change to the risk-dominant strategy for the dynamics to move to the risk-dominant solution. In the long run this leads to a higher probability that in any given period players will be playing the risk-dominant equilibrium, and the risk-dominant solution is the stochastically stable convention, in the sense coined by Foster and Young (1990).

In the Kandori et al. (1993) and Young (1993) models each player plays against every other player in the population (or equivalently faces each other player with an equal probability). While this is plausible in some contexts, many situations where social coordination is an issue involve more specific interaction patterns. These results have been shown to hold for some alternative interaction structures, for instance where individuals interact according to certain fixed neighborhood

[1] For overviews and more detailed discussion of other relevant examples and applications, see Young (1998) and the *Journal of Economic Perspectives*, Symposium on Network Externalities (Katz and Shapiro, 1994; Besen and Farrell, 1994; and Liebowitz and Margolis, 1994).

structures as shown by Ellison (1993) (see also Young (1998)). This leads to the somewhat pessimistic result that a society can expect in the long run[2] to coordinate on the risk-dominant equilibrium, even in cases where it is inefficient and not in society's common interest.

In this paper we take a broader look at the role of the interaction pattern, and find that the results above change in this broader analysis. First, we show that the previous analysis depended on the interaction patterns that were considered. In particular, we demonstrate an example of a very simple network structure, yet different from those analyzed previously, that results in multiple stochastically stable states with some involving play of equilibria that are not risk-dominant. This example points out the important role of the particular network structure in determining stochastic stability. This example also motivates our main analysis, which treats the network of interactions as endogenous.

Ours is not the first analysis to endogenize interaction patterns. However, it is quite different in terms of the manner in which individuals choose with whom they interact. Ely (2002) (see also Mailath et al. (1997)) considers models where the interaction structure is endogenized by locational choices. Individuals select a location at which to reside and then interact according to a pattern governed by that location and the location of other individuals. Conditions are given under which the efficient equilibrium is the one that is reached by a society even when it is not risk-dominant. In the model of Ely (2002) if some individual randomly moves to an unoccupied location and plays the efficient strategy, then other individuals would like to move to that location and play the efficient strategy rather than staying at a location where they play the inefficient strategy. This leads to efficient play. While this result is encouraging in showing how endogenizing the interaction pattern can lead to efficiency, this result depends on the locational aspect of the interaction patterns.[3] In particular, in changing locations agents can sever all old ties, form new ties, and switch technologies simultaneously. While there are situations where location is the major factor in determining with whom an individual interacts, in many applications individuals choose with whom they interact in a more discretionary manner, not having to completely uproot to form new relationships.

Our model is not a location model, but rather one where players choose their interaction patterns on an individual-by-individual basis. We bring ideas from the recent literature on the formation of networks to bear on this problem of social coordination. We model the interaction pattern as a network where

[2] Ellison (1993) and Young (1998) also present interesting results on how the time associated with the "long run" varies depending on the network structure.

[3] Mailath, Samuelson, and Shaked discuss the importance of having locations where individuals can isolate themselves from others in order to obtain efficiency. In their model, individuals cannot necessarily escape being matched with undesired opponents, and the ability to isolate is important in determining whether efficiency is attained.

individuals periodically have the discretion to add or sever links to other players. Players choose whether to add or sever links based on their (prospective) partner's past behavior. With such endogenous interaction patterns there exists multiple stochastically stable states of play, including some that involve play of equilibria in the coordination game that are *neither* efficient nor risk-dominant. Thus, it is possible to have inefficient play be stochastically stable *even when* the efficient strategy is risk-dominant, in a model where individuals completely control with whom they interact. This differs significantly from the previous literature (including the locational models mentioned above).

The main insight into how endogenizing the network can affect stochastically stable play is as follows. Consider a situation where there is some cost to at least some players for maintaining a link to another player who is not coordinating on the same action. With an exogenous network structure, a player's choice of action only depends on what actions other players are playing. It may take many changes (trembles or mistakes) by other players in order to get the system to change from one equilibrium to another. With an endogenous network, the process works differently as follows. Consider player i. If some player j that i is linked with changes strategies so that i and j no longer coordinate and if there is a cost to the link between i and j, then player i may wish to sever that link. This can then lead to a new network pattern, and j can form new links to other players who are already playing the strategy j is, or who change to do so. In such a situation, adjustments in the network make it easier for changes in play to build up and persist. This has profound impacts on the way that play changes from one strategy to another, and thus leads to different results from those in the previous models.

Thus the main message is that network endogeneity can have a significant impact on play in the game, even relative to what would happen if the network were fixed at the one that arises endogenously. However, the precise dynamics and the set of networks and play that emerge as stochastically stable depend on the particulars, such as the relative benefits of the play of different actions, the structure of the costs to links, and the number of players in the society. We examine these in some detail in what follows.

2. Network interactions and stochastic stability

2.1. Networks

The set $N = \{1, \dots, n\}$ is a finite set of individual players. These may be people, firms, computers, countries, or other relevant participants. We assume that $n > 2$, as networks among two players are easily handled.

The network relations among these individuals are represented by a graph whose nodes or vertices are the individuals, and whose links or edges are connections between the individuals. The complete network or graph, denoted g^N, is the set of all subsets of N of size 2. The set of all possible networks on N is $\{g \mid g \subset g^N\}$.[4] Let ij denote the subset of N containing i and j and is referred to as the *link* ij. The interpretation of $ij \in g$ is that in the network g, individuals i and j are linked.

Note that i is linked to j if and only if j is linked to i, so that we study non-directed networks where interaction requires mutual consent.

2.2. Costs of links

We will consider cases where links are costly. In particular, player i pays a cost $k(n_i)$ for maintaining each link that i is involved in, leading to a total cost of $n_i k(n_i)$ for i. This cost is a function of n_i, the number of direct links agent i has. We discuss how the structure of this cost function matters in determining behavior.

Before endogenizing the network, let us first discuss how interaction occurs given a fixed network.

2.3. Coordination games played on a fixed network

Consider the following situation as described by Young (1998). A population of n players plays a game repeatedly. Players are located on a fixed network g. In each period t, a player i chooses an action $a_i^t \in \{A, B\}$, and then receives a payoff which is

$$u_i\left(g; a_1^t, \ldots, a_n^t\right) = \sum_{i \neq j} \pi_{ij}(g)\left[v_i\left(a_i^t, a_j^t\right) - k(n_i)\right], \tag{1}$$

where $v_i(a_i^t, a_j^t)$ is a payoff that depends on the actions chosen, and $\pi_{ij}(g) = 1$ if $ij \in g$ and $\pi_{ij}(g) = 0$ if $ij \notin g$. Thus, each player interacts only with the players that he is directly linked to under g.

The following matrix describes the payoff function v_i. The matrix lists only the payoff to player i, with the payoff to player j being symmetrically determined.

		Player j	
		A	B
Player i	A	a	c
	B	d	b

Let $a > d$ and $b > c$, so that the game is a coordination game, with two pure strategy equilibria, A, A and B, B. Let $(a - d) > (b - c)$, so that A, A is the risk-dominant equilibrium in the sense of Harsanyi and Selten (1988). This

[4] There are many ways in which graphs may be encoded (with a standard one being in a matrix). This notation proves to be convenient for the study of individual incentives in network formation.

equilibrium has the property that each player is choosing a strategy that is also a best response to the other player mixing $50/50$. Thus, A is the strategy that is a best response to the largest set of beliefs over possible plays of the opponent, and so the risk-dominant equilibrium is the pure strategy equilibrium with a larger basin of attraction than the other pure strategy equilibrium. Specifically, playing A is a player's best response if the fraction of his opponents who play A is greater than or equal to $(b-c)/(a-d+b-c) < 1/2$. Thus if $a=3$, $b=1$, and $c=d=0$, then playing A is a best response if $1/4$ or more of a player's opponents play A.

In the case where n is even, assume that A is the best response to a mixture of $(n-2)/[2(n-1)]$ on A and $n/[2(n-1)]$ on B. So, for instance, if $n=10$ then A is a best response for a player who is playing against 4 other players who play A and 5 who play B. This assumption ensures an asymmetry among strategies, so that our results of multiple stochastically stable states are not artificial.

The dynamic process is described as follows. Each period one player is chosen at random (say with equal probability across players, although that is not important) to update his strategy. A player updates his strategy myopically, best responding to what the other players with whom he interacts did in the *previous* period. There is also a probability $1 > \varepsilon > 0$ that a player trembles, and chooses a strategy that he did not intend to. Thus, with probability $1-\varepsilon$ the strategy chosen is $a_i^t = \operatorname{argmax}_{a_i} u_i(a_i, a_{-i}^{t-1})$ and with probability ε the strategy is $a_i^t \neq \operatorname{argmax}_{a_i} u_i(a_i, a_{-i}^{t-1})$.[5] The probabilities of trembles are identical and independent across players, strategies, and periods.[6] These trembles can be thought of as mistakes made by players or exogenous factors that influence players' choices. Once initial strategies are specified, the above process leads to a well-defined Markov chain where the state is the vector of actions, a^t, that are played in period t. The Markov chain has a unique stationary distribution, denoted $\mu^\varepsilon(a)$. Thus, for any given initial strategies, $\mu^\varepsilon(a)$ describes the probability that a will be the state in some period (arbitrarily) far in the future. Let $\mu = \lim_\varepsilon \mu^\varepsilon$. Following the terminology of Foster and Young (1990), a given state a is *stochastically stable* if it is in the support of μ. Thus, a state is stochastically stable if there is a probability bounded away from zero that the system will be in that state according to the steady state distribution, for arbitrarily small probabilities of trembles.

Let us consider a specific example that illustrates the importance of the network configuration in determining the stochastically stable states. Let $n=4$

[5] Assume that $\operatorname{argmax}_{a_i} u_i(a_i, a_{-i}^{t-1})$ is single valued, which is true generically in choices of the payoff matrix.

[6] The structure of trembles is not innocuous. As detailed by Bergin and Lipman (1996), varying the perturbation structure can have a profound influence on the dynamic process. We return to discuss this shortly.

and for all n_i let $a - k(n_i) = 3$, $b - k(n_i) = 1$, and $c - k(n_i) = d - k(n_i) = 0$. This is a coordination game where A, A is both the efficient and risk-dominant equilibrium and the cost per link is constant. In this situation, for either the complete network or circle networks the unique stochastically stable state is all players playing A; but, for the star network there are two stochastically stable states: all players playing A and all players playing B.

Example (The complete network). First, consider the case where players are located on the complete graph, g^N. Recall that $\pi_{ij} = 1$ for each $ij \in g$. Thus for the graph g^N, each player plays once against every other player in a period (or equivalently faces each other player with an equal probability). The stochastically stable state in this case is for all players to play A, as shown in Kandori et al. (1993) and Young (1993).

Let us briefly go over the intuition behind this example and the Kandori et al. (1993) and Young (1993), as this will be useful later on. If at least one of the other players played A last period, then a player who is updating her strategy will choose to play A. A player who is updating her strategy will choose to play B only if all of the other players played B last period. So, consider a situation where all players are playing A. If due to an ε-error some player switches to B, the other players when called on to update will not choose to switch and will continue to play A. This situation will update back to the situation where all players play A. It takes three trembles on three different players in order to have the remaining player choose to switch to B. So, the process takes at least three trembles (without any intermediate updating by those players) to switch the process from all playing A to all playing B. If all players are playing B, then if one player switches to playing A the other players when called on to update will choose to switch to A. Thus, with only one tremble the process can lead from all playing B to all playing A. While this is not a complete description of the stochastic process it outlines why it is relatively easier (by orders of magnitude in the number of trembles needed) to move from social coordination on B to coordination on A than the other way around.

Example (Circle networks). Next consider the case where players are located on a circle, as for instance in the network $g = \{12, 23, 34, 41\}$. The unique stochastically stable state will be all players playing A as shown by Ellison (1993), and as he shows this result also holds for some more general neighborhood structures on circles.

The intuition behind this example is similar to that of complete networks. Here, each player only cares about what his or her two neighbors are playing. Again, as long as one neighbor is playing A, an updating player will choose to play A. Thus it takes both neighbors switching to B in order to have a player want to switch from playing A to playing B, while it only takes one neighbor switching to A to get a player to want to switch from playing B to playing A.

Example (Star networks). Lastly, consider the situation where players are located on a star such as $g = \{12, 13, 14\}$. The dynamics associated with this network differ in important ways from those described around the previous two examples. With a star network there are two stochastically stable states: one where all players play A and the other where all players play B.

To see the intuition behind this example note that now players 2, 3, and 4 care only about what player 1 is playing, and they will update to play whatever 1 played last period when called on to update. Player 1, in contrast, cares about what all the players are doing. Thus one tremble by player 1 can lead from a network where all play A to one where all play B. Alternatively, any tremble of any player changing from B to A can lead from a situation where all play B to one where all play A. Thus starting from either equilibrium of all play A or all play B, we need only one tremble to have updating lead naturally to the other equilibrium. As the relative number of trembles is the important factor in determining the set of stochastically stable states, both of these states are stochastically stable.

Note that the relative probability in the limiting distribution of the state where all play A is higher than that of the state where all play B. This follows from the fact that any single tremble can lead to a transition from the state where all play B to the state where all play A, while a specific tremble is needed to lead to a transition from the state where all play B to that where all play A. Nevertheless, the state where all play the inefficient and non-risk-dominant equilibrium still receives positive probability in the limit distribution and is stochastically stable.

The star example shows that it is possible for a network of individuals to have multiple stochastically stable states in the coordination game, including some where the inefficient and non risk-dominant strategy is selected. This result is in contrast with the previous literature (e.g., Kandori et al., 1993; Young, 1993; and Ellison, 1993, 2000) where the risk-dominant equilibrium is always selected.[7] This result has seemingly negative implications, as there may be inefficient play. On the other hand, the result can also have positive implications in the case where the risk-dominant equilibrium A, A is inefficient. For instance, if $a - c > b - d$ but $b > a$, then the analysis of the above examples still holds, and under the complete or circle networks only play of the inefficient equilibrium A, A is stochastically

[7] Bergin and Lipman (1996) have shown for any specific equilibrium, there is a set of perturbations varying in their order of magnitude across strategies that will select the desired equilibrium as the stochastically stable state. What we are showing here is different, in that there are identical trembles resulting in multiple stochastically stable states, one of which may be neither risk-dominant nor efficient, and instead it is the network structure that matters.

stable, while on the star network play of the efficient equilibrium B, B is also stochastically stable.

This example seems to contradict Theorem 6.1 (and Corollary 6.1) in Young (1998). Young's result states that for *any* fixed network structure the unique stochastically stable state in a symmetric coordination game is for all players to coordinate on the risk-dominant strategy. However, there is no contradiction between the results as the perturbation structure is different in the two analyses. We assume that the probability of a player trembling (i.e., a perturbation or mistake) from A to B when A is the best response is the same as the probability of a player trembling from B to A when B is the best response. In contrast, Young (1998) assumes that updating takes place according to a distribution that is proportional to a factor that is log–linear in payoff. In particular, as ε goes to 0 in the Young (1998) framework, the probability of any player trembling from A to B when others are playing A becomes *infinitely* less likely than the probability of trembling from B to A when the others are playing B. Such an error structure is very tractable and powerful in its predictions (e.g., see Blume, 1993 and Young, 1998), and reflects the belief that errors should be less relatively likely when they are more costly. However, such an error structure is extreme in that very small differences in relative payoff comparisons across strategies, lead to infinite differences in limiting perturbation probability. If an error reflects exogenous factors or limits in a player's calculation ability, then there is no reason to assume that such an error should become infinitely more likely for some strategies versus others. While we work with the other extreme assumption of equal probabilities of trembles, the results we present here would still hold if the probabilities of single errors vary across strategy choices as long as the ratio of these error probabilities does not become infinite.

Note that the star example is different from results found in Morris (2000). Morris (2000) gives conditions on the network structure (the structure must be low growth which roughly means that my neighbor's neighbors are likely to be my neighbors) and payoffs a, b, c, d that allow different players to take different actions in a non-stochastic equilibrium (equilibrium in the absence of trembles). For instance if players are on a line and if $a = b = 1$ and $c = d = 0$ then it is possible in a non-stochastic equilibrium for half the players to play A and the other half to play B. However, our star example is quite different since it shows that in the presence of trembles it is possible for two equilibria to exist: one where all play A and the other where all play B. Note that in the absence of trembles it is not possible in the star to get the Morris result where in a non-stochastic equilibrium some play A and others play B.

Given that the set of stochastically stable states varies with the network structure, we now turn to the question of which networks arise endogenously if the network structure is at the players' discretion. If players choose both with whom they interact as well as what strategies they play, then which networks should we expect to see? This is the question we address next.

3. Endogenous networks and stochastic stability

The following outlines an approach for endogenizing the network. It is a simple variation on the process discussed above.

3.1. A dynamic process

Let g^{t-1} denote the network at the end of period $t-1$ and a^{t-1} denote the action profile at the end of period $t-1$. In an arbitrary period t three things occur:

1. First, one link ij is chosen at random according to the fixed probability distribution $\{p_{ij}\}$ where $p_{ij} > 0$ for each ij. This is the only link that can be formed or severed at time t. Players decide whether to add or sever the link, and make this choice based on the assumption that players (including themselves) will play the same strategy as in the previous period. If the link is not in the network then it is added if at least one player's utility increases and the other player's does not decrease. If the link is already in the network then it is severed if either player would benefit from its removal. After the choice is made, with probability $1 > \gamma > 0$ the choice is reversed by a tremble. This process determines a network g^t according to well-defined probabilities.[8]
2. Second, one player i is randomly selected to adjust their strategy according to the fixed probability distribution $\{q_i\}$ where $q_i > 0$ for each i. This player chooses the strategy that is a best response to the current network g^t and to the previous periods' play configuration a^{t-1}.[9] After the choice is made, with probability $1 > \varepsilon > 0$ it is reversed by a tremble. All trembles and random selections are independent. This determines a strategy profile a^t according to well defined probabilities.[10]

[8] So, conditional on ij being selected the process is as follows: If $ij \in g^t$ and $u_i(g^{t-1} - ij, a^{t-1}) > u_i(g^{t-1}, a^{t-1})$ or $u_j(g^{t-1} - ij, a^{t-1}) > u_j(g^{t-1}, a^{t-1})$ then $g^t = g^{t-1} - ij$ with probability $(1-\gamma)$ and g^{t-1} with probability γ. If $ij \in g^t$ and $u_i(g^{t-1} - ij, a^{t-1}) \leqslant u_i(g^{t-1}, a^{t-1})$ and $u_j(g^{t-1} - ij, a^{t-1}) \leqslant u_j(g^{t-1}, a^{t-1})$ then $g^t = g^{t-1}$ with probability $(1-\gamma)$ and $g^{t-1} - ij$ with probability γ. If $ij \notin g^t$ and $u_i(g^{t-1} - ij, a^{t-1}) \geqslant u_i(g^{t-1}, a^{t-1})$ and $u_j(g^{t-1} - ij, a^{t-1}) \geqslant u_j(g^{t-1}, a^{t-1})$ with one inequality strict, then $g^t = g^{t-1} + ij$ with probability $(1-\gamma)$ and g^{t-1} with probability γ. Otherwise, $g^t = g^{t-1}$ with probability $(1-\gamma)$ and $g^t = g^{t-1} + ij$ with probability γ. Taking probabilities across ij according to p_{ij} leads to a distribution over g^t as a function of g^{t-1}.

[9] Again, assume that the a_i that maximizes $u_i(g^t, a_i, a_{-i}^{t-1})$ is unique for every g^t, a_{-i}^{t-1}.

[10] Conditional on i being selected, $a_{-i}^t = a_{-i}^{t-1}$ and $a_i^t = \operatorname{argmax} u_i(g^t, a_i, a_{-i}^{t-1})$ with probability $1-\varepsilon$ and $a_i^t \neq \operatorname{argmax} u_i(g^t, a_i, a_{-i}^{t-1})$ with probability ε. Taking probabilities across i according to q_i leads to a distribution over a^t as a function of a^{t-1}.

3. Lastly, players play the coordination game with the other players that they are directly connected to in the network and receive the payoff $u_i(g^t, a^t)$, as defined in Eq. (1), where $\pi_{ij}(g^t) = 1$ if $ij \in g^t$, and $\pi_{ij}(g^t) = 0$ if $ij \notin g^t$.

Assume that disconnected players, when identified in step 2, choose a best response to being uniformly randomly matched with any other player and to the previous period's play configuration a^{t-1}. As before, the process determines a finite state, irreducible, aperiodic Markov chain, and thus has a unique stationary distribution $\mu^{\gamma,\varepsilon}$ over states, where states are now network/strategy configurations.

A network/strategy configuration g, a is *stochastically stable* if it is in the support of $\mu = \lim_{\gamma=k,\varepsilon\to 0} \mu^{\gamma,\varepsilon}$, where we take γ and ε to zero at the same rate; so $\gamma = f\varepsilon$ for some $f > 0$.[11]

In the above process, players adjust their links and strategies independently. Players do not consider the possibility of changing their strategy when adding or severing a link and the possible implications that this might have for the future evolution of play. This sort of consideration may be important when there are relatively small numbers of forward-looking players who are well-informed about the network, strategies played, and the motivation of others. However, in larger networks and situations where players' information might be local and limited, or in situations where players significantly discount the future, myopic behavior is a more natural assumption.

3.2. Constant costs of maintaining links

Recall from Eq. (1) that agents must pay a cost for maintaining a link. First we consider the case where every player pays a constant cost k for each direct link that he maintains in graph g, independent of the number of links maintained. Thus, $k(n_i) = k$ for any n_i (and the total cost of links paid by i is $n_i k$).

In the following proposition we only consider cases where at least one of $(a-k)$ and $(b-k)$ is strictly greater than 0. The other case is trivial as then no links will ever form.

Proposition 1. *Let $k(n_i) = k$ for all n_i.*

(i) *If either $(a-k) > 0$ and $(b-k) < 0$ or $(a-k) < 0$ and $(b-k) > 0$, then the unique stochastically stable state is the fully-connected network with all players playing A or B, respectively; except when $(b-c)/(a-d+b-c) \leqslant 2/(n-1)$ and $(a-k) < 0$ and $(b-k) > 0$ in which case having all players*

[11] We do not have to worry about cycles or mixed strategies given this "one at a time" process and the structure of the payoffs.

disconnected and playing A is also stochastically stable and is the unique stochastically stable state when $(b-c)/(a-d+b-c) \leqslant 1/(n-1)$.

(ii) *If $(c-k) > 0$ and $(d-k) > 0$, then the unique stochastically stable state is the fully-connected network with all playing A.*

(iii) *If $(c-k) < 0$ and/or $(d-k) < 0$ and $(a-k) > 0$ and $(b-k) > 0$, then if $(b-c)/(a-d+b-c) > 1/(n-1)$, there are two stochastically stable states, a fully-connected network with all players playing A and a fully-connected network with all players playing B; while if $(b-c)/(a-d+b-c) \leqslant 1/(n-1)$ then the fully-connected network with all playing A is the unique stochastically stable state.*

Note that the cases are exhaustive except for allowing some of the terms to be equal to 0, which allows for hybrids of the cases. The proof of Proposition 1 is in Appendix A. Let us outline the intuition here.

Case (i) of Proposition 1 is relatively straightforward. Here there is only one possible action that can lead to a positive payoff, and so players coordinate on the action associated with the positive payoff. Given that everyone plays this action, the payoff from forming a link is always positive and all links will form. The only potential exception is that if nobody is playing this action to begin with and so all players stay disconnected. In such cases, it takes at most two players to tremble to start the formation of links, and so such states are relatively unstable. The exception is the case were $(b-c)/(a-d+b-c) \leqslant 2/(n-1)$, $(a-k) < 0$, and $(b-k) > 0$. If $1/(n-1) < (b-c)/(a-d+b-c) \leqslant 2/(n-1)$ then it takes only two trembles to leave the fully-connected network with all players playing B and so both this state and the state where all play A and are disconnected are stochastically stable. While if $(b-c)/(a-d+b-c) \leqslant 1/(n-1)$, $(a-k) < 0$, and $(b-k) > 0$ then it takes only one tremble to leave the fully-connected network with all players playing B and so the state where all play A and are disconnected is the unique stochastically stable state.

Case (ii) of Proposition 1 states that if the payoff to miscoordination is positive, then the unique stochastically stable state involves formation of the full network and all players coordinating on the risk-dominant equilibrium regardless of whether or not it is efficient. In this case, all links are valuable and so the complete network forms, and players' best responses look very similar to that in the previous literature, and so do the results.

Case (ii) also has some interesting implications regarding speed of convergence to the stochastically stable state. As discussed in Ellison (1993, 2000) and Young (1998), network structures with features that are similar to that of a circle, i.e., where players have tight local interactions and are more loosely linked across neighborhoods, have faster speeds of convergence to the stochastically stable state than networks with more interconnections. Young (1998) provides an elegant characterization of such "close knit" networks. However, in case (ii) of Proposition 1, we will not end up with the close knit networks that have the nice speed-of-convergence properties, but instead we end up with fully-connected networks. Case (ii) should be interpreted cautiously, however, as it depends critically on there being no significant costs to having a link. This is captured in $c - k > 0$ and $d - k > 0$, where players still receive a positive payoff when they fail to coordinate.

Case (iii) breaks into two parts. The second part where $(b - c)/(a - d + b - c) \leqslant 1/(n - 1)$, is an extreme case where if any player starts playing A, then all players would like to play A. This makes it very easy to get to the state where all players play A, and thus this state becomes the unique stochastically stable state.

The other part of case (iii), where $(b - c)/(a - d + b - c) > 1/(n - 1)$, is perhaps the most interesting in Proposition 1 in terms of the role of the network and in particular how the endogeneity of the network is important. In this situation, if the network were exogenously fixed, then A would be the risk-dominant strategy and all playing A would be the stochastically stable state. The idea is that it would take more strategy trembles to get from all playing A to all playing B, then vice versa. However, when the network is endogenous, the network changes in responses to changes in strategies. To see this, suppose that we are initially in a network where all players are playing A and are fully-connected. Suppose that two trembles occur and two players start playing B. Then, players playing A would like to sever links to these players, as it is costly to maintain a link with B players. These B players would form a component playing B. This component could then continue to grow as additional trembles occur. This process turns out to be symmetric in the way it moves between A and B. It only takes two trembles to start a new component, and then that component can continue to grow as trembles can accumulate one by one. The other candidates for stochastically stable states are situations where there are two separate, fully-connected components to the network, one playing all A and the other playing all B. However, such states are not stochastically stable, as they can move to other states via a single tremble where it takes two trembles to leave a state where there is just one component where all players play A, or all players play B.

Case (iii) shows that endogenizing the network has implications beyond predictions of the network structure; it also has implications for the strategies that are chosen in the game. Case (iii) shows that there are stochastically stable states where actions are played that are neither risk-dominant nor efficient. That is, all playing B can be part of a stochastically stable state even when $a > b$. The important aspect is that the adjustments and flexibility in the network change the stochastic process so that the frictions between various states of play are fundamentally different from when the interaction pattern is fixed (or even locational).

Let us make one further remark concerning case (iii). Although case (iii) has all players fully-connected in the stochastically stable states, the speed of transition from any state to one of the stochastically stable states can be faster than with an exogenous fully-connected network. That is, the network adjusts as players' strategies change. So, even though the stable states have fully-connected networks, the transitions involve changes in those connections that allow trembles to build up one-at-a-time.[12] In the exogenous fully-connected network case, to go from all playing B to all playing A requires m trembles (where $m = n(b-c)/(a-d+b-c) < n/2$), which must occur simultaneously. In case (iii), to go from all playing B to all playing A requires $(n-1)$ trembles, but these trembles can build up one-at-a-time, or in other combinations. Thus case (iii) will have a faster speed of convergence than the fully-connected network if m is large, but will have a slower speed of convergence if m is small.

The analysis above assumes that in each period, players get a separate payoff for each player with whom they are connected. If instead, there is a limit to how many other players a player can interact with and if a player still bears some cost for all of the links that she maintains, then the results can change and in particular we can find mixed stochastically stable states where there is play of different strategies by different groups. We examine this situation next.

3.3. Constrained costs of maintaining links

The fact that the complete network forms in all stochastically stable states in Proposition 1 (except when $(b-c)/(a-d+b-c) \leqslant 1/(n-1)$ and $(a-k) < 0$ and $(b-k) > 0$) is due to the simple form of the link costs which have no scale effects. We now consider a situation with stark scale effects. This changes the resulting network structure, but the play in the game is similar to that in Proposition 1. This situation is outlined in Proposition 2.

We now consider a case where each player pays a cost k for maintaining each direct link provided that he has no more than $m > 1$ direct links. If a player has more than m links then the cost of maintaining each link is larger than any possible benefit received from maintaining the link. Thus, $k(n_i) = k$ for all $n_i \leqslant m$ and $k(n_i) > \max\{a, b\}$ for all $n_i > m$. Of course, if $m \geqslant n$, then we are back in the case of constant costs considered previously. But if $m < n$, then a player has a capacity constraint on how many interactions he may be involved in. So, we restrict attention to $m < n$, as Proposition 1 already addresses the other case. In Proposition 3 we consider cases where $k(n_i)$ takes more general convex forms.

Again we examine the cases where at least one of $(a-k)$ and $(b-k)$ is strictly greater than 0, as otherwise no links will form.

[12] Speed of convergence and its relationship to one-at-a-time trembles is discussed formally in Ellison (2000).

Proposition 2. *Let $k(n_i) = k$ for all $n_i \leqslant m$ and $k(n_i) > \max\{a, b\}$ if $n_i > m$. Let m be even*[13] *and such that $n > m > 1$. Also, let $(b - c)/(a - d + b - c) > 2/m$.*[14]

(i) *If either $(a - k) > 0$ and $(b - k) < 0$ or $(a - k) < 0$ and $(b - k) > 0$, then the set of stochastically stable states involve everyone playing A or B, respectively. There can be a variety of network configurations in stochastically stable states (as illustrated below).*

(ii) *If $(c - k) > 0$ and $(d - k) > 0$, then in any stochastically stable state each player has m links and plays A.*

(iii) *If $(c - k) < 0$ and/or $(d - k) < 0$ and $(a - k) > 0$ and $(b - k) > 0$, then in any stochastically stable state either all players play A or all players play B. There can be a variety of network configurations in stochastically stable states.*

The proof of Proposition 2 is in Appendix A.

The similarity of play under Propositions 1 and 2 shows that adding a capacity constraint on how many interactions a player may have does not substantially affect the results, except in terms of the number of links that players form. Although the statements of the Propositions are somewhat parallel, the proof of Proposition 2 (especially part (ii)) is significantly more complicated, due to the numerous network configurations that can conceivably be part of the set of stochastically stable states.

The case (ii) result that all agents form the full number of m ties is actually somewhat surprising. Consider the following 10 player example where $m = 6$ and $(a - k) > 0$ and consider the following two states where all agents play A. In the first state 7 agents are in a completely connected component and the remaining 3 agents are in a completely connected component. Although each of the 3 agents would like to form a link with one of the 7 agents, the 7 agents will refuse since they each have 6 links already. In the second state all agents have exactly 6 ties; the second state is stochastically stable while the first state is not. To leave the first state and move toward the second state we only need one tremble to break one tie among the 7 agents. Then two of the 3 agents will be able to form a new tie. However, to leave the second state and move toward the first state takes more than one tremble, since if one link is broken the agents will have no one new to link to so the link will reform. Thus it is easy to leave the first state but hard to leave the

[13] We focus on the case of m even for $m \geqslant 2$ and $n > m$ since then there always exists a graph g where everyone has m ties. Note that if $m = 1$, then each player maintains at most one link. Stochastically stable states are networks of pairs (with one player left out if n is odd) with each pair playing the same strategy, but different pairs possibly playing different strategies (if it is true that both $a - k > 0$ and $b - k > 0$, otherwise they play the only profitable strategy).

[14] The case where $(b - c)/(a - d + b - c) \leqslant 2/m$ adds qualifications to part (i) and (iii) analogous to those in the corresponding parts of Proposition 1.

second state, which is why the first state is not stochastically stable. Accounting for all the various networks that could form accounts for the complications to the proof of Proposition 2.

To see the variety of network configurations that can emerge in cases (i) and (iii) in Proposition 2, it is useful to look at some examples. Let us concentrate on case (i), as case (iii) has similar examples.

Consider a situation where $n = 5$ and $m = 2$, where $(a - k) > 0$ and other payoffs are negative. In any stochastically stable state, all players play A. There are three types of network configurations that are candidates for stochastic stability: all players connected in a circle of 5 players (denoted as a 5,0); 4 players in a circle with 1 isolated (denoted 4,1); and 3 players in a circle with 2 players in a separate pair (denoted 3,2). In this situation, all of these network configurations are in the set of stochastically stable states. To see this, we note that any of these networks moves to another with just a single tremble and so they all have the same stochastic properties (the formal proof is a bit more complicated, but not much more so). A 5,0 goes to any of five different 4,1's by having just one player tremble. This happens when some player trembles from A to play B. The links to this player are then severed and a new link forms between the two players who severed the links, leading to a 4,1. Similarly, a 4,1 can lead to any of four different 3,2's and to any of four different 5's in one tremble. A 3,2 goes to six different 5's in one tremble and to 6 different 4,1's in one tremble. In this case, all of these network configurations are stochastically stable.

Next, consider the same setting except with $n = 7$ and $m = 2$. Using a similar notation for network configurations we find the following. A 7,0 goes to seven different 6,1's via one tremble. Any 6,1 can go to six different 5,1's and also to six different 7's via one tremble. Any 5,2 can go to a number of 4,3's and also to several 7's via one tremble. Any 4,3 can go to several 3,3,1's and any 3,3,1 can go to several 4,3's via one tremble. However, a 4,3 or a 3,3,1 cannot go to any 5,2 or 6,1 or 7, without at least two trembles (one in each of the components where all players are at their capacity). So here the 4,3's and the 3,3,1's are the only networks that are part of the set of stochastically stable states.

The reasoning in the second example resembles that behind part (ii) of Proposition 2. But we see from the combination of the two examples that the set of stochastically stable network configurations does not have an easy pattern. One thing that is clear, is that the set of stable networks includes only those such that if there are two (or more) players with fewer than m links, then these players must be connected to each other as otherwise they would gain by linking.

While the network configurations that arise in cases (i) and (iii) do not have an easy pattern, there are some properties that we can point out that are interesting, at least in some special cases. For instance, from the second example we see that the set of stochastically stable networks tends to consist of smaller components, where the players in each component are as completely connected as possible.

To see this in more detail, consider the case where $m = 2$, so that each player can link with at most two other players. If n is divisible by 3 $(= m + 1)$, then the only stochastically stable states will consist of networks where players are grouped in fully-connected components of three players. This is proven in Appendix A, and we conjecture that it is also true for n divisible by $m + 1$, when $m > 2$. The intuition behind this is as follows. Consider such a network where all components are fully-connected subnetworks and each player has m links. If one player trembles and changes strategies the other players might sever their links to that player. However, they then have no one to link to, other than the player they just severed the link with. That player will switch strategies back to the best response, and the links will reform. Such states need at least two trembles to lead to some other state. If instead, the network does not consist of fully-connected subnetworks, then when a player trembles and links to that player are severed, there will be opportunities for the remaining players to form new links (such as to each other). Thus, the networks with fully-connected components are more robust, and hence become the only stochastically stable networks.[15]

3.4. Convex costs of maintaining links

Lastly, we examine the case where player j's total cost of maintaining n_j links, $n_j k(n_j)$, is increasing and convex in n_j.

Define m_α such that $\alpha < (m_\alpha + 1)k(m_\alpha + 1) - (m_\alpha)k(m_\alpha)$ and $\alpha \geqslant (m_\alpha)k(m_\alpha) - (m_\alpha - 1)k(m_\alpha - 1)$, for each $\alpha \in \{a, b, c, d\}$. Thus if an A player has m_a links he will refuse additional links to A players, and similarly for B and m_b.

Proposition 3. *Let $n_j k(n_j)$ be increasing and convex in n_j and let $(b - c)/(a - d + b - c) > 2/m_b$.*[16]

(i) *Assume m_a and m_b are even. If either $a - k(1) > 0$ and $b - k(1) < 0$ or $a - k(1) < 0$ and $b - k(1) > 0$, then the set of stochastically stable states involve all players playing A (respectively B). There can be a variety of network configurations in stochastically stable states.*

(ii) *If $c - k(1) > 0$ and $d - k(1) > 0$ then the set of stochastically stable states depends on the specifics of the cost function. There are examples where each*

[15] The difficulty in establishing a formal proof for $m > 2$ comes from the fact that one needs to show that all networks other than those with fully-connected components can move from one to another via single trembles in such a way to form a path leading to one of those with fully-connected components. This can be shown for $m = 2$, but we have not found an argument to establish this generally for $m > 2$, as the potential configurations grow in number exponentially in m.

[16] Again, the case where $(b - c)/(a - d + b - c) \leqslant 2/m_b$ adds qualifications to part (i) and (iii) analogous to those in the corresponding parts of Proposition 1.

player has m_a links and plays A, and there are other examples where each player has m_b links and plays B.

(iii) *Assume m_a and m_b are even. If $c - k(1) < 0$ and/or $d - k(1) < 0$, and $a - k(1) > 0$, $b - k(1) > 0$, then in any stochastically stable state all players play A or all players play B. There can be a variety of network configurations in stochastically stable states.*

The proof of Proposition 3 is in Appendix A.

Cases (i) and (iii) are similar to those of Propositions 1 and 2. However, case (ii) is quite different. Contrary to Propositions 1 and 2, for some cost functions it is now the case that all play B is part of a stochastically stable state, while all play A is not.

We prove case (ii) with a 5 player example where A, A is the risk-dominant equilibrium and B, B is the efficient equilibrium and where $(b - c)/(a - d + b - c) > \frac{1}{4}$. Let us outline that example here. Consider $k(\cdot)$ such that $m_b = 4$, $m_a = 3$, $m_c = 2$, and $m_d = 1$. The following two states are both stable in the absence of trembles. In the first state everyone plays B and has 4 ties. In the second state everyone plays A and players are in an augmented circle where 4 players have 3 ties each and the fifth player has only has two ties. Since $m_a = 3$ this second state is stable in the absence of trembles. To leave the first state and move towards the second state takes at least two trembles. If one tie is severed it is reformed. If one person changes from B to A then since $m_d = 1$ everyone will sever ties to this person. This person will then change back to A and will relink with everyone. So at least two trembles are needed to leave this first state. However, to leave the second state and move towards the first state only one tremble is needed. If in the second state we change the player with two ties from A to B, then the remaining A players will sever ties to this B player (since $m_c = 2$) and the A players will then connect with each other so that all 4 have 3 ties each. We can keep changing the A players to B one-at-a-time; each instance that we do this we end up at another state which is stable in the absence of trembles. Thus we can reach the state where all play B by a series of one-at-a-time trembles. This reasoning can be shown to imply that each player playing B and having 4 ties is the unique stochastically stable state.

4. Concluding remarks

From the previous literature one might take away two predictions: (1) With fixed networks of interaction society will coordinate on the risk-dominant equilibrium, and (2) with endogenous interaction patterns as determined by location (with homogeneous players[17]) society will coordinate on the efficient equilibrium. Here we have shown via an example that (1) may fail and one can see multiple stochastically stable states on a fixed network; and that (2) may fail with endogenous networks, as stochastically stable states can include coordination on equilibria that are neither efficient nor risk-dominant. Moreover, the characterization of the set of stochastically stable states depends in non-trivial ways on payoffs and costs of links.

There are several aspects of the analysis here that deserve further attention.

First, the analysis here assumes similar rates of recognition and trembles for link changes and strategy changes. If instead, for instance, link patterns are much more rigid than strategies (by an order of $1/\varepsilon$), then some of the reasoning above may not apply as players' strategies may readjust before the network adjusts. The relative ease of change, of course, will depend on the application, but is an important consideration.

Second, the analysis depends on myopic choices on the part of players. Players are not forecasting the responses of other players, in terms of the strategies played or the links chosen when they decide on their own actions and links. This myopia is clearly important in the reasoning behind the results, and thus the analysis is best suited for large settings where history is the best benchmark for predicting behavior. Note that this is a caution that applies fairly broadly to the literature on stochastic stability and it has to be kept in mind when interpreting the results.

There are two new papers which examine some variations on what is considered here: Goyal and Vega-Redondo (1999) and Droste et al. (2000).[18] Goyal and Vega-Redondo consider links that are formed unilaterally (one player does not need another player's permission to form a link with him). This addresses a very different set of applications than those captured here. Moreover, the ability of players to unilaterally connect to other players leads to very different strategic dynamics and conclusions, and they find that either risk-dominant play or efficient play is always reached in such models.

Droste et al. (2000) is closer to the analysis here in that links are formed bilaterally (the consent of both players necessary to form a link) and there exists a significant cost to forming a link. However, their model differs in having a geographic cost to forming links, which again is very different in terms of the applications. That geographical cost leads to the formation of networks with specific neighborhood structures and results in the risk-dominant equilibrium as the unique stochastically stable state. Thus, the results are again quite different.

[17] Mailath et al. (1997) give an example of a locational model where there exists a stable state with heterogeneous play among the population. Their example builds off of heterogeneity of players in the population, where players' types affect their matching probabilities and their ability to choose with whom to interact. We are referring to situations where players are initially homogeneous.

[18] See also Skyrms and Pemantle (2000) for a look at a stochastic dynamic network model with interactions that depend on a learning dynamic.

This sensitivity to a wide variety of assumptions suggests that the conclusions in this literature be interpreted cautiously.[19] Given this sensitivity to a number of aspects of behavior and interaction technology, there may not be broad-sweeping predictions that one can make concerning a society's ability to reach efficient coordination. The main message here is that such predictions are dependent on the details of the costs and benefits of interaction, and most importantly that endogenizing the network has nontrivial and sometimes subtle consequences.

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Appendix A

Let $x = (g, a)$ represent a network and strategy combination.

Two network-strategy combinations x and x' are *adjacent* if they differ by at most one link and/or one player's strategy.

Define a *path* $(x_1, \ldots, x_k)$ as a sequence of network-strategy combinations such that x_i and x_{i+1} are adjacent but $x_i \neq x_{i+1}$ for each $i = 1, \ldots, k-1$. Let $p(x_1, x_k)$ represent the set of all possible paths starting at x_1 and ending at x_k.

Given a two adjacent networks, (x, x'), let the *resistance* $r(x, x')$ be the number of trembles needed to go from x to x'. A tremble is needed if the players involved are not willing to make the change of the link and/or the strategy that is needed to move from x to x'. Thus to go from x to x' takes at most 2 trembles (to change both the link and the strategy) and at least 0 trembles.

To be precise, let $x = (g, a)$ and $x' = (g', a')$. No tremble is needed to move from g to g' (given a) if $g' = g$; $g' = g - ij$, and either $u_i(g, a) < u_i(g - ij, a)$ or $u_j(g, a) < u_j(g - ij, a)$; or if $g' = g + ij$ and $u_i(g + ij, a) \geqslant u_i(g, a)$ and $u_j(g + ij, a) \geqslant u_j(g, a)$ with one inequality holding strictly. Otherwise, one tremble is needed to move from g to g' (given a). No tremble is needed to move from a to a' (given g') if $a = a'$ or if $a_i \neq a'_i$ and $a'_i = \operatorname{argmax} u_i(g', a_i, a'_{-i})$. Otherwise, one tremble is needed to move from a to a' (given g'). Then, $r(x, x') = 2$ if one tremble is needed to go from g to g' (given a) and one tremble is needed to go from a to a' (given g'). $r(x, x') = 1$ if one tremble is needed to go from g to g' (given a) and no trembles are needed to go from a to a' (given g'), or if no trembles are needed to go from g to g' (given a) and one trembles is needed to go from a to a' (given g'). $r(x, x') = 0$ if no trembles are needed to go from g to g' (given a) and no trembles are needed to go from a to a' (given g').

Define *the resistance of a path* $r(x_1, \ldots, x_k)$ as $\sum_{i=1}^{k-1} r(x_i, x_{i+1})$.

[19] This is complementary to a caution raised by Bergin and Lipman (1996) detailing the sensitivity of stochastic stability to the perturbation technology.

The path $(x_1, \ldots, x_k)$ is an *improving path* if $r(x_1, \ldots, x_k) = 0$. For further discussion of improving paths, see Jackson and Watts (1998). A state x is *statically stable* if there are no improving paths leaving it.

Let $r(x_1, x_k) = \min_{(X_1,\ldots,X_k)\in p(X_1,X_k)} r(x_1, \ldots, x_k)$.

A theorem from Young (1993) is instrumental in the proofs of Propositions 1 and 2. Before stating Young's (1993) theorem, the following definitions from Young (1993) are needed.

Consider a stationary Markov process on a finite state space X with transition matrix P.

A set of *mutations of* P is a range $(0, a]$ and a stationary Markov process on X with transition matrix $P(\varepsilon)$ for each ε in $(0, a]$, such that (i) $P(\varepsilon)$ is aperiodic and irreducible for each ε in $(0, a]$, (ii) $P(\varepsilon) \to P$, and (iii) $P(\varepsilon)_{xy} > 0$ implies that there exists $r \geqslant 0$ such that $0 < \lim \varepsilon^{-r} P(\varepsilon)_{xy} < \infty$.

The number r in (iii) above is the *resistance* of the transition from state x to y. There is a path from x to z of *zero resistance* if there is a sequence of states starting with x and ending with z such that the transition from each state to the next state in the sequence is of zero resistance. Note that from (ii) and (iii), this implies that if there is a path from x to z of zero resistance, then the nth order transition probability associated with P of x to z is positive for some n.

The *recurrent communication classes* of P, denoted $X_1, \ldots, X_J$, are disjoint subsets of states such that (i) from each state there exists a path of zero resistance leading to a state in at least one recurrent communication class, (ii) any two states in the same recurrent communication class are connected by a path of zero resistance (in both directions), and (iii) for any recurrent communication class X_j and states x in X_j and y not in X_j such that $P(\varepsilon)_{xy} > 0$, the resistance of the transition from x to y is positive.

For two communication classes X_i and X_j, since each $P(\varepsilon)$ is irreducible, it follows that there is a sequence of states $x_1, \ldots, x_k$ with x_1 in X_i and x_k in X_j such that the resistance of transition from x_k to x_{k+1} is defined by (iii) and finite. Denote this by $r(x_k, x_{k+1})$. Let the resistance of transition from X_i to X_j be the minimum over all such sequences of $\sum_1^{K-1} r(x_k, x_{k+1})$, and denote it by $r(X_i, X_j)$.

Given a recurrent communication class X_i, an i*-tree* is a directed graph with a vertex for each communication class and a unique directed path leading from each class j $(\neq i)$ to i. The *stochastic potential* of a recurrent communication class X_j is then defined by finding an i-tree that minimizes the summed resistance over directed edges, and setting the stochastic potential equal to that summed resistance.

Given any state x, an x*-tree* is a directed graph with a vertex for each state and a unique directed path leading from each state y $(\neq x)$ to x. The resistance of x is then defined by finding an x-tree that minimizes the summed resistance over directed edges.

The following theorem is a combination of Theorem 4 and Lemmas 1 and 2 in Young (1993).

Theorem (Young (1993)). *Let P be the transition matrix associated with a stationary Markov process on a finite state space with a set of mutations $\{P(\varepsilon)\}$ and with corresponding (unique) stationary distributions $\{m(\varepsilon)\}$. Then $m(\varepsilon)$ converges to a stationary distribution m of P, and a state x has $m_x > 0$ if and only if x is in a recurrent communication class of P which has a minimal stochastic potential. This is equivalent to x having minimum resistance.*

Proof of Proposition 1. The stochastic process of Proposition 1 determines a finite state, irreducible, aperiodic Markov chain, and thus has a unique stationary distribution $\mu^{\gamma,\varepsilon}$ over states, where states are network/strategy combinations. Given that $\gamma = f\varepsilon$, we can write $\mu^{\gamma,\varepsilon}$ as μ^{ε}.

Case (i). Consider the case where $(a - k) > 0$ and $(b - k) < 0$. The statically stable states are those where there is a fully-connected component of players playing A, the remaining players play B and are not connected to anyone, and if there are any such B players then there must be enough of them so that their best response to the average play of all other players is B (which requires that there are more than $n/2$ such players). Let (g^N, A) represent the state where players are in the fully-connected network g^N and all players play strategy A. From the fully disconnected network with all players playing B it takes at most two trembles (two players changing to A) to get a link to form. From that state with one link, any single tremble of a player from B to A leads to a state where that player becomes fully linked with the other players who play A. This continues, and so by single trembles we move from state to state until it is a best response for all disconnected players to play A, in which case the process moves directly to the complete network with all players playing A. From (g^N, A), it takes more than $(n+1)/2$ trembles to get to any other statically stable state. So, whenever $(n+1)/2 > 2$ it follows that (g^N, A) is the unique stochastically stable state. The only remaining case is when $n = 3$. However, in that case it takes only one tremble to change from the fully disconnected network with all players playing B to get a link to form, as one player trembling to A can lead another player to switch to A as a best response to the average play. The case of $(a-k) < 0$ and $(b-k) > 0$ is similar where now (g^N, B) is the unique stochastically stable state; except for the situation where $(b-c)/(a-d+b-c) \leqslant 2/(n-1)$ and $(a-k) < 0$ and $(b-k) > 0$. If $1/(n-1) < (b-c)/(a-d+b-c) \leqslant 2/(n-1)$ then two trembles (two players changing to A) can lead from (g^N, B) to the fully connected network where all play A except for one player who plays B, and from here players will sever all ties. In this case, the fully disconnected network with all players playing A is also stochastically stable (it takes two trembles to leave either of these states, and only one tremble to transition between other states and from neighboring states into these states). Similarly, if $(b-c)/(a-d+b-c) \leqslant 1/(n-1)$ then one tremble can lead from (g^N, B) to the disconnected network where all play A and thus the disconnected network where all play A is the unique stochastically stable state.

Case (ii). Note that from any $x = (g, a)$ there is a path of zero resistance leading to either (g^N, A) or (g^N, B), where (g^N, A) and (g^N, B) represent the states where players are in the fully-connected network g^N and all players play strategy A (or B, respectively). This follows since all payoffs are strictly positive and so players always prefer to add a link when given the opportunity, and the fact that all players must coordinate on strategy A or B given the symmetry of payoffs and the assumed uniqueness of the best response in the coordination game. Note also that any path from (g^N, A) or (g^N, B) to any adjacent network-strategy pair has positive resistance.

Thus, the recurrent communication classes as defined above must be $\{(g^N, A)\}$ and $\{(g^N, B)\}$. The set of stochastically stable states depends on which of these states has the smallest resistance. Given the theorem above, we need only find the minimum resistance over paths from $\{(g^N, A)\}$ to $\{(g^N, B)\}$ and compare that to the minimum resistance over paths from $\{(g^N, A)\}$ to $\{(g^N, B)\}$.

If agents are in the g^N network and all playing B, then given the payoff structure if we consider a path from (g^N, A) to (g^N, B) where only strategies are changed it takes less than $n/2$ trembles to get all agents to switch to all playing A. So, $r((g^N, B), (g^N, A)) < n/2$. Next we bound $r((g^N, A), (g^N, B))$. If agents are in a g^N network and all agents play A, then given the payoff structure if no links are changed it will take at least $n/2$ trembles to get players to want to switch to play B. So, if the resistance is less than $n/2$ it must involve some trembles on links. Let us consider what it would take to get some single player to switch from playing A to B. For a player to switch from A to B more than half of the players that this player is linked to must play B. Thus, if k links are severed, then $(n-k)/2$ players must be playing B if $n-k-1$ is odd and $(n-k+1)/2$ if $n-k-1$ is even. So starting at (g^N, A) to get a single player to want to switch from A to B involves at least $k + (n-k)/2$ trembles and this is at least $n/2$. Thus $r((g^N, A), (g^N, B)) \geqslant n/2$. Since $r(g^N, A) < r(g^N, B)$, (g^N, A) is the unique stochastically stable state.

Case (iii). If $a - k > 0 > d - k$ and $b - k > 0 > c - k$ then players will prefer to sever any link to a player who plays a strategy that is different than their own, and add a link to any player who plays a strategy that is the same as their own. Thus, the set of recurrent communication classes are $\{(g^N, A)\}, \{(g^N, B)\}$, and each $\{(g^m, A/g^{n-m}, B)\}$ for $2 \leqslant m \leqslant n-2$. Here $(g^m, A/g^{n-m}, B)$ represents any state where there are two separate fully connected components, one of size m and the other of size $(n-m)$ with all players in the m-size component playing A and all players in the $(n-m)$ size component playing B. Note that for any m, there are many different states $(g^m, A/g^{n-m}, B)$ since here are many ways to select m individuals. Each is a distinct recurrent communication class.

Given Young's theorem, we look for the set of recurrent communication classes with minimum resistance, found by constructing restricted (g, a)-trees. First assume that $(b-c)/(a-d+b-c) > 1/(n-1)$. We next construct a restricted (g^N, A)-tree. Direct (g^N, B) to any of the $(g^2, A/g^{n-2}, B)$ vertices; (g^N, B) has resistance 2 to an improving path leading to any $(g^2, A/g^{n-2}, B)$ vertex. (Starting at (g^N, B) allow two trembles to change two player's strategies to A. Then there is an improving path leading to a $(g^2, A/g^{n-2}, B)$ vertex. This improving path exists since all of the B players will receive a negative payoff from their link with the A players, and will sever it when given the opportunity. Note that (g^N, B) has a distance of more than 2 from an improving path to any other communication class, since more than 2 players strategies must be changed.) Any of the $(g^m, A/g^{n-m}, B)$, $2 \leqslant m \leqslant n-3$, vertices has a distance of 1 from an improving path leading to one of the $(g^{m+1}, A/g^{n-m-1}, B)$ vertices; simply allow a tremble to change one of the B players to strategy A. Any of the $(g^{n-2}, A/g^2, B)$ vertices has a distance of 1 from an improving path leading to (g^N, A). Simply allow a tremble to change one of the B players to strategy A, then the remaining B player will sever the tie to this player. The remaining B player will now be unlinked; since we assume that disconnected players, choose a best response to the current average play of all other players, this B player will change strategies to A and will then form links with all the other A players. Thus $r(g^N, A)$ is equal to the number of recurrent communication classes. Similar reasoning shows that $r(g^N, B)$ is the same.

Next we compute $r(g^m, A/g^{n-m}, B)$ for $2 \leqslant m \leqslant n-2$. From the above, we know that g^N/A and g^N/B are both a distance of 2 or more from an improving path leading to any $(g^m, A/g^{n-m}, B)$. Thus $r(g^m, A/g^{n-m}, B)$ is strictly greater than the number of recurrent communication classes, and so larger than $r(g^N, A) = r(g^N, B)$. It follows from Young's theorem that the stochastically stable states are (g^N, A) and (g^N, B).

However if $(b-c)/(a-d+b-c) \leqslant 1/(n-1)$ then to go from (g^N, B) to any of the $(g^2, A/g^{n-2}, B)$ vertices takes only one tremble. (Starting at (g^N, B) allow one tremble to change one player's strategy from B to A. Since $(b-c)/(a-d+b-c) \leqslant 1/(n-1)$, another B player will switch to A if given the chance to update. Additionally the remaining B players will sever ties to these A players when given the opportunity.) So $r(g^N, A)$ is now equal to the number of recurrent communication classes minus one. Thus (g^N, A) now has the smallest resistance and is the unique stochastically stable state. □

Proof of Proposition 2.

Case (i). This is a straightforward extension of the Proof of (i) in Proposition 1.

We now prove the following claim mentioned after Proposition 2.

Claim 1. If $m = 2$ and n is divisible by 3 then the stochastically stable states are networks of $n/(m+1)$ fully-connected components each of size $m+1$.

Proof of Claim 1. Let s be the state where all play A and players are in $n/(m+1)$ fully-connected components each of size $(m+1)$. State s is statically stable since no one wants to form a new tie (everyone already has m links) and no one wants to sever a tie (since all play A, all ties up to m are beneficial). □

Next we show that s is the unique stochastically stable state. To leave state s and go to any other statically stable state takes at least two trembles. To see this note that if there is only one tremble, say one link is severed, then the link will be reformed since agents have no one else to link to (all other agents already have m ties). If one agent trembles from A to B then links with this agent may be severed. The agent will change back to A when given the opportunity to update his strategy and all links will be reformed.

However, we can go from any other statically stable state s' to s by a series of one-at-a-time trembles. State s' must contain at least one component bigger than 3. In this component let one agent, say i, tremble from A to B. Ties to this agent will be severed. The two agents who were next to i on the circle will then link with each other (they are not already linked since the component was originally bigger than 3). Agent i is now unlinked. Repeat this process with any component of size greater than 3. Eventually a state s will be reached by one-at-a-time trembles.

Next we show that state s is stochastically stable. Consider the minimal resistance g-tree for state s'. Change the g-tree as follows: direct state s' to s by a series of one-at-a-time trembles. Sever the arrow leaving state s (this arrow must involve at least two trembles). Sever any other arrows necessary to make this a tree. We have constructed an s-tree. We have added only arrows with one tremble. We have deleted the same number of arrows; at least one of the deleted arrows had two trembles. So the resistance of state s is less than state s'. Thus state s' is not stochastically stable.

Case (ii). First we show that only states where everyone has m links are stochastically stable. To do this we use Claim 2 and the corresponding proof. Note that any player with less than m links is willing to link with any other free player, due to $c - k > 0$ and $d - k > 0$.

Claim 2. Statically stable states where some players have fewer than m links are not stochastically stable.

Proof of Claim 2. Let G^0 denote networks where all players have m links. Let G^1 be the networks one tremble from some network in G^0. Define G^2 to be the networks not in G^0 or G^1 that are one tremble from some network in G^1. For $t > 2$, let G^t denote networks not in G^j for any $j < t$, that are one tremble from some network in G^{t-1}. Let us show that these exhaust all networks that could be part of a statically stable states. Consider any statically stable state s where players are in network $g \in G^t$, all play A, and $t > 0$. We show that we can move from g to a network $g' \in G^j$ with $j < t$ by a single tremble. Since $g \notin G^0$ and n is even, there exists either two players i and l who each have less than m ties in g or there exists one only player i who has less than m ties and then given that n is even that player has less than $m - 1$ ties. First consider the case where i has less than $(m - 1)$ ties. Since s is statically stable we know there exists at least one player j who has m ties in g (since all play A, any player with less than m ties wants to link with any free player, thus a state cannot be statically stable unless some players have m ties) and $ij \notin g$. (Since $n > m$ we know in any statically stable state there exists $m + 1$ players with m ties. Agent i cannot be linked to all of these players or i also would have m ties.) Since player j has m ties he must be linked to a player k who is also not linked to i in g. (Such a k exists because player j has m ties and player i cannot be linked to everyone j is linked to otherwise i would also have m ties.) Sever the link jk. Now player i can link to both players j and k. The new graph has more total links than g. The case where players i and l each have less than m ties is similar, except there we find player k who is linked to j but is not linked to l. □

The above argument implies that for any state s with a g in G^t where $t > 0$, there exists some state s' with a $g' \in G^j$ for some $j < t$ such that the minimum resistance of s' is no more than s. (Starting with an s tree, simply redirect an arrow from s to some s' that is just one tremble away, as from the argument above. This s' tree has no more trembles than the s tree.) So to complete the proof, we show that the resistance of states with a network in G^0 is lower than those with networks in G^1. Start with any s with a network in G^1 and find a minimum s tree. Find a statically stable state s' with g' in G^0 such that s is 1 tremble away from s'. Now, alter the s-tree as follows: draw a new arrow from s to s'. This has one tremble. Next, follow the old path away from s' and identify the first s'' on that path such that the arrow away from s'' had more than one tremble. (This could be s'.) There must exist such an s'' since that path leads all the way to s, and g has fewer links than g', so at some point on that path from s' to s there exists an arrow pointing from a network with where all players have m links to one where some players have fewer links. At least two links must be broken by trembles to make such a change. Now delete the arrow away from s''. We have constructed an s''-tree. The added arrow from s to s' had one tremble. The deleted arrow away from s'' had at least two trembles. So the resistance of s is more than that of s'', which concludes the proof of Claim 2.

Next we show that all stochastically stable states have all players playing A. We already know that stochastic stability implies that everyone has m links. First, think of the case where if just one neighbor plays A then I want to play A. Then we can start with a state s where some players play B and g is in G^0 (where G^0 is defined as in the proof of Claim 2). Find an s-tree. Next, find a state s' where g is the same as is s but some component in g has now switched from B to A. By definition it only takes one tremble to get from s to s'. Now follow the path on the s-tree from s' back to s. There must be some s'' on that path where either links changed which needs at least two trembles (since g is in G^0) or where some players switch from A to B which also needs at least two trembles. We construct an s'' tree by deleting the arrow leaving s'' and adding an arrow from s to s'. Since the added arrow has one tremble and the deleted arrow has two trembles we know that s could not have been stochastically stable.

Next, we examine situations where two or more neighbors must play A to get me to switch from B to A. Let us use the following induction argument. Pick a state s which has g in G^0, and has as few players playing B as possible, but still some. Now follow a reasoning similar to that above. Point s to s' where s' has all playing A and the same g as in s. It takes some number of trembles (the minimum threshold to go from B to A) to go from s to s'. Now follow the path on the s-tree back from s' to s. There has to be an s'' where the arrow leaving s'' has some players switch from A to B. This switch must take strictly more than the minimum number of trembles to go from B to A. Next construct an s''-tree as we did above. This new s''-tree has less resistance than the s-tree.

The following concludes the induction argument. Define state s' as any statically stable state which has g in G^0 and some players playing B. Define state s as a statically stable state which has the same g as s' but has more players playing B than does state s' and where the difference between the number of players who play B in s' and who play B in s is as small as possible. Following the same reasoning as above it is possible to find an s''-tree with less resistance than the s-tree.

Case (iii). If $(a-k) > 0 > (d-k)$ and $(b-k) > 0 > (c-k)$ then players will prefer to sever any link to a player who plays a strategy different from their own. Thus the set of recurrent communication classes are $\{(g^m, A)\}$, $\{(g^m, B)\}$, and the set $\{(g^l, A/g^{n-l}, B)\}$ for $2 \leqslant l \leqslant n-2$. Here g^m represents any network where no agent has more than m ties and any agent j with less than m ties has no one to link with (thus any agent unlinked to j already had m ties). And $(g^l, A/g^{n-l}, B)$ represents any state where there are two distinct components, one of size l with everyone playing A and the other of size $(n-l)$ with everyone playing B. Again in g^l (respectively g^{n-l}) no agent has more than m ties and any agent with less than m ties has no other player who plays the same strategy who also has less than m ties and is not already connected to the given agent.

Consider any state in the set $\{(g^m, A)\}$. To move from any such state to a statically stable state where some players are playing B requires at least two trembles, as a lone player who trembles to B will switch back to A when allowed to update his strategy, and so at least one other player must also tremble to B for it to be a best response for those players to stay playing B (and two trembles is sufficient, as then these two players can link and continue to play B). Similarly, it takes at least two trembles to move from any state in $\{(g^m, B)\}$ to a state outside of it. However, for any state in $\{(g^l, A/g^{n-l}, B)\}$, it takes only one tremble to move to a state with a different number $l' \neq l$. In particular, let l^* be such that A is a best response to l^*-1 players playing A and $n-l^*-1$ playing B, but B is a best response to l^*-2 players players playing A and $n-l^*$ players playing B. Then any state in $\{(g^l, A/g^{n-l}, B)\}$ where $l < l^*$ will be one tremble away from a state in $\{(g^{l-1}, A/g^{n-l+1}, B)\}$, and any state where $l > l^*$ will be one tremble away from a state in a state in $\{(g^{l+1}, A/g^{n-l-1}, B)\}$. So consider a least resistance tree leading to some state s in $\{(g^l, A/g^{n-l}, B)\}$. Say that $l > l^*$ and find a state s' in $\{(g^m, A)\}$ that can be reached from state s by successive trembles leading to a sequence of states in $\{(g^{l+1}, A/g^{n-l-1}, B)\}$, $\{(g^{l+2}, A/g^{n-l-2}, B)\}$, etc. Form an s' tree by deleting the arrow leaving s' and forming a path following the sequence just described. This tree will have less resistance than the s tree, and so s is not stochastically stable. A similar argument handles the case where $l < l^*$ working with a state s' in $\{(g^m, B)\}$. □

Proof of Proposition 3.

Case (i). Assume $(a - k(1)) > 0$ and $(b - k(1)) < 0$. Thus any player playing A will sever any ties to agents who are playing B. Also by assumption, no A player is willing to have more than m_a links to other A players. This case is now identical to case (i) of Proposition 2 where no A player is willing to have more than m ties.

Case (ii). First note that we can find $k(\cdot)$ to arbitrarily approximate a constrained cost function of Proposition 2, and so it is possible to find examples where all players have m_a ties and play A. So we show an example where the only stochastically stable states have all players having m_b ties and playing B.

Consider the following 5 player example. Assume that A, A is the risk-dominant equilibrium but that B, B is the efficient equilibrium and assume that $(b - c)/(a - d + b - c) > 1/4$. Then there exists a $k(\cdot)$ such that $m_b = 4$, $m_a = 3$, $m_c = 2$, and $m_d = 1$.

Claim 3. A state where all play A is not stochastically stable.

Proof of Claim 3. Consider a state where all play B. Since $N = 5$ and $m_b = 4$, if all play B then the only stable network is g^N. To leave this state and go to any other state takes at least 2 trembles. (If one tie is severed it is reformed. If one person changes from B to A, then since $m_d = 1$ everyone will sever ties with this person. Next this person will change back to B and reform all his old ties.) □

Next consider a state where all play A. Since $m_a = 3$, each player has at most 3 ties. Thus one possible stable state, say s', is an augmented circle where 4 players have 3 ties each and the fifth player has only 2 ties. Another possible stable state, say s'', is for 4 players to be completely connected and the fifth player to be connected to no one. To leave any of these stable states takes only 1 tremble. To leave s' and go to s'' we just need to change the fifth player from A to B. All ties will be severed to this player (since $m_c = 2$). The remaining agents will then become completely connected. The fifth unconnected player will then change from B to A. Now consider leaving state s'' to move towards state s (where all play B). From state s'' change one of the fully-connected players from A to B. Since $m_c = 2$ all ties to this player will be severed. This player will then connect to the unconnected player who may change from A to B. Now we have a triangle playing A and a two link line playing B. This state is stable. From here we can keep switching the A agents to B one at a time and thus move to other stable states. We will end up at state s where all play B. Note that we got from state s' (or s'') to state s by a series of one-at-a-time trembles.

Thus if we construct an s tree, it must be that states s' and s'' are lead to state s by one-at-a-time trembles (since this is the minimum number of trembles possible). However, to leave state s always takes at least 2 trembles. Therefore states s' and s'' must have a higher resistance than state s. Thus all play A is not stochastically stable.

Case (iii). This case is similar to that of case (iii) from Proposition 2, except now A (respectively B) players are willing to have m_a (respectively m_b) ties to other A (B) players. Thus the only difference with this and Proposition 2 is that now m_a may not equal m_b. But the Proposition 2 proof is easily extended to handle this case, since the trembles considered involved the severing of all links to given players. □

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