5. Convex Optimization Problems

(1) Optimization Problem in standard form

minimize $f_0(x)$ objective function

subject to $f_i(x) \le 0$ $i=1,\dots,m$ inequality constraint function

$h_i(x) = 0$ $i=1,\dots,p$ equality constraint function

x is feasible if $x \in \mathrm{dom} f_0$ and it satisfies the constraints.

$f_i(x)\le 0$ $i=1,\dots,m$
$h_i(x)=0$ $i=1,\dots,p$
are explicit constraints

domain of the optimization problem: $D = \bigcap_{i=0,\dots,m} \mathrm{dom} f_i \bigcap_{i=1,\dots,p} \mathrm{dom}\, h_i$

x is feasible if the domain is not empty.

→ Implicit constraint $x \in D$

A problem is unconstraint if no explicit constraints

optimal value $p^* = \begin{cases} +\infty & \text{problem is infeasible} \\ -\infty & f_0(x) \text{ is unbounded below} \\ \text{finite value} & \text{otherwise} \end{cases}$

optimal point x^* if x^* is feasible and $f_0(x^*) = p^*$

optimal set $X_{opt} = \{x \mid f_i \le 0, i=1,\dots,m,\ h_i = 0, i=1,\dots,p,\ f_0(x) = p^*\}$

problem is solvable: there is an optimal point (optimal set is not empty)

active constraint: if x is feasible and $f_i(x)=0$, we say the ith inequality constraint $f_i(x)\le 0$ is active at x

if $f_i(x)<0 \Rightarrow f_i(x)\le 0$ is inactive at x

→ feasible是形容x与constraint的，而非optimization problem

Feasibility Problem: (objective function is 0. $\Rightarrow p^* = \begin{cases} 0 \\ +\infty \end{cases}$ constraints are feasible → Any feasible x is optimal; constraints are infeasible

写法: minimize 0

subject to $f_i(x)\le 0$ $i=1,\dots,m$

$h_i(x)=0$ $i=1,\dots,p$

(2) Standard form convex optimization problem

minimize $f_0(x)$

subject to $f_i(x)\le 0$ $i=1,\dots,m$

~~$h_i(x)=0$ $i=1,\dots,p$~~

$a_i^T x = b$ $i=1,\dots,p$

$f_i(x), i=0,\dots,m$ are convex, equality constraints are affine

feasible set of a convex optimization problem is convex $\Rightarrow$ Minimize a convex function over a convex set.

In particular, the optimal set is also convex

$S_{i0} = \{x \mid f_i(x)\le 0\}$ is the sublevel set of $f_i(x)$

if $f_i(x)$ is convex $\Rightarrow S_{i0}$ is convex

$\Rightarrow$ feasible set $= \bigcap_i S_{i0}$ is convex

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(3) Local and Global Optima

Any locally optimal point of a convex problem is globally optimal.

For convex problem, x is optimal iff ① x is feasible
and ② $\nabla f_0(x)^T(y-x) \geq 0$ for all feasible y

<pf> if f_0 is convex $\Rightarrow f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x)$ for all x and all y in $\operatorname{dom} f_0$

if x^* is optimal

$f_0(y) \geq f_0(x^*) + \nabla f_0(x^*)^T(y-x^*)$ for all feasible y,

$\Rightarrow \nabla f_0(x^*)^T(y-x^*) \geq 0$

$\nabla f_0(x^*)^T(y-x^*)$ is a hyperplane. for $\forall y \in$ feasible set, $\nabla f_0(x^*)^T(y-x) \geq 0$

$\Rightarrow$ supporting plane of feasible set (convex problem 中, feasible set is a convex set)

For general optimization problem:

x^* is optimal $\Rightarrow \nabla f_0(x^*)^T(y-x) \geq 0$ for all feasible y
$\not\Leftarrow$

(4) Linear Optimization Problem (convex problem)

(General form)

minimize $c^T x + d$
subject to $Gx \preceq h$
$Ax = b$

} feasible set is a polyhedron

Special Case ① standard form LP

minimize $c^T x + d$
subject to $\boxed{x \succeq 0}$
$Ax = b$

→ All variables are non-negative

Special Case ② An Inequality form LP

minimize $c^T x + d$
subject to $Ax \preceq b$

理解 all variables

Adding a slack variable s, convert general form LP into standard form:

$Gx \preceq h \Rightarrow \begin{cases} Gx + s = h \\ \boxed{s \succeq 0} \end{cases} \Rightarrow$ let $x = x^+ - x^-$, $x^+ \succeq 0$, $x^- \succeq 0$ $\Rightarrow$ minimize $c^T x^+ - c^T x^- + d$

subject to $Gx^+ - Gx^- + s = h$
$Ax^+ - Ax^- = b$
$\boxed{\begin{matrix} x^+ \succeq 0 \\ x^- \succeq 0 \\ s \succeq 0 \end{matrix}}$

(解 不足够
仅令 $s \succeq 0$.)

(General) LP:

It is common to refer to a problem that can be formulated as a LP as an LP, even if it ~~is not~~ does not have the general form. ~~or standard form or~~

↳ Example: pointwise Linear minimization: minimize $f_0(x) = \max_i (a_i^T x + b_i)$ is an LP

↳ cause it ~~equati~~ equavalent to: minimize t

subject to $a_i^T x + b \le t$ $i=1,\cdots,m$

(5) Quadratic optimization problems (QP) (convex problem) if $P=0$. $QP \Rightarrow LP$

minimize $\frac{1}{2} x^T P x + q^T x + r$ (objective function is quadratic) $P \in S_+^n$

subject to $Gx \preceq h$ (constraints are affine)

$Ax = b$

Minimize a convex quadratic function over a polyhedron.

(6) 问 QCQP

minimize $\frac{1}{2} x^T P_0 x + q_0^T x + r_0$ if $P_i = 0$, 则 $QCQP \Rightarrow QP$ $P_i \in S_+^n$

subject to $\frac{1}{2} x^T P_i x + q_i^T x + r_i \le 0$ $i=1,\cdots,m$

$Ax = b$

↳ quadratically constrained quadratic problem

feasible region is the intersection of ellipsoids (when $P_i \succ 0$)

(7) Transform Problems

① change variables: $x = \phi(z)$

例: $f_0(x) = \frac{x_1}{x_2} + \frac{x_3}{x_1} \xrightarrow{x=\phi(z)=\exp(z)} \tilde{f}_0(z) = \frac{\exp(z_1)}{\exp(z_2)} + \frac{\exp(z_3)}{\exp(z_1)}$
$= \exp(z_1 - z_2) + \exp(z_3 + z_1)$

minimize $f_0(x)$

subject to $f_i(x) \le 0$ $i=1,\cdots,m$

$h_i(x) = 0$, $i=1,\cdots,p$

$\phi: R^n \to R^n$ is one2one with $\phi(\mathrm{dom}\,\phi) \supseteq D$

minimize (over z) $\tilde{f}_0(z)$

subject to $\tilde{f}_i(z) \le 0$ $i=1,\cdots,m$

$\tilde{h}_i(z) = 0$ $i=1,\cdots,p$

$\tilde{f}_i(z) = f_i(\phi(z))$

$\tilde{h}_i(z) = h_i(\phi(z))$

② $f_0(x) \Rightarrow \phi_0(f_0(x))$ $f_i(x) \Rightarrow \psi_i(f_i(x))$, $h_i(x) \Rightarrow \varphi_i(h_i(x))$ 例:

if ϕ_0 单增, $\psi_i(x) \le 0$ iff $x \le 0$ $i=1,\cdots,m$, $\varphi_i(x) = 0$ iff $x=0$ $i=1,\cdots,p$ $\|Ax+b\|_2 \Rightarrow \|Ax+b\|_2^2$

③ Eliminate Equality Constraint

minimize $f_0(x)$
subject to $f_i(x) \le 0$
$Ax = b$
} convex problem

find x_0 such that $Ax_0 = b$
find F such that $R(F) \subseteq Null(A)$
if $x \in Null A$, then $Ax = 0$

minimize $f_0(Fz + x_0)$
subject to $f_i(Fz + x_0) \le 0$

④ Introduce Equality constraint

minimize $f_0(A_0 x + b_0)$ subject to $f_i(A_i x + b_0) \le 0 \Longrightarrow$ minimize over x and y_i $f_0(y_0)$ subject to $f_i(y_i) \le 0$, ~~$A_i x + b_i$~~ $y_i = A_i x + b_i$