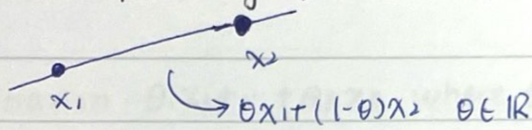
Chapter 2 Convex Set

2.1 Affine and Convex set

2.1.1 Affine

(1) A set is affine: if the line through any 2 distinct points in C lies in C

i.e. if for any $x_1, x_2 \in C$ and $\theta \in \mathbb{R}$, we have $\underbrace{\theta x_1 + (1-\theta) x_2}_{\text{linear combination of } x_1, x_2} \in C$



(2) Affine Combination: of point $x_1, x_2, \cdots, x_k$ is $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$, $\theta_1 + \theta_2 + \cdots + \theta_k = 1$

(3) Affine Set: contains every affine combination of its points

if C is an affine set, $x_1, \cdots, x_k \in C$, and $\sum_i^k \theta_i = 1$ then the point $\theta_1 x_1 + \cdots + \theta_k x_k \in C$

△ The solution set of a system of linear equations, $C = \{x \mid Ax = b\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ is an affine set 解空间的线性变换仍为该线性方程的解

线性变换 → affine combination

仍为该线性方程的解 → still in the set

Converse:

every affine set can be expressed as the solution set of a system of linear equations

(4) Affine Hull of C ⇒ aff C

affine hull of C is the smallest affine set that contains C

$\text{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1\}$

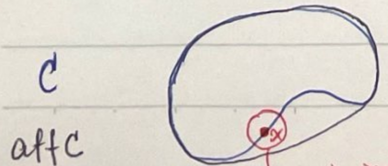
(5) Affine Dimension of set C is the dimension of its affine hull

(6) relative interior 相对内部 of set C (relint C)

$\text{relint}\, C = \{x \in C \mid B(x, r) \cap \text{aff}\, C \subseteq C \text{ for some } r > 0\}$

$B(x, r) = \{y \mid \|y - x\| \le r\}$ → ball with radius r

→ 对于set C中的一点x，如果存在以x为中心的任意大小的一个球，使得该球与affine set of C的交集仍在C中，则x位于C的relative interior



⇒ 这么画不对，但表意即可

→ 以x为中心的球（该例子中，球 $\cap$ aff $C \not\subseteq C$，x并不位于C的relative interior）

2.1.2 Convex

(1) A set is convex: if the line segment between any two points in C lies in C

for any $x_1, x_2 \in C$. and any $\theta \in [0,1]$, $\theta x_1 + (1-\theta)x_2 \in C$

Every affine set is also convex

(2) convex combination: $\theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_1 + \cdots + \theta_k = 1$ and $\theta_i \geq 0, \forall i$

↳ 与 affine combination 相比新增的条件

A set is ~~affine~~ convex if and only if it contains every convex combinations of its points

联系 affine set contains every affine combinations of its points

(3) convex hull of set C, denoted by conv C, is the set of all convex combinations of poins in C

$\text{conv}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \theta_i \geq 0, i = 1, 2, \cdots, k, \theta_1 + \cdots + \theta_k = 1\}$

↳ smallest convex set that contains C

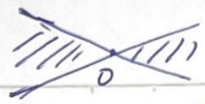
↳ convex combination 可认为是 infinite sums (of pdf, …) $\sum p_i = 1$ 对应 $\sum \theta_i = 1$

↳ if $x_i \in C$, C is convex, $C \subseteq \mathbb{R}^n$ then $\sum_{i=1}^{\infty} \theta_i x_i \in C$ when $\theta_i \geq 0$, $\sum_{i=1}^{\infty} \theta_i = 1$

↳ 进一步抽象为 integral: if $p: \mathbb{R}^n \to \mathbb{R}$, $p(x) \geq 0$, $\int p(x) dx = 1$, $x \in C$, $C \subseteq \mathbb{R}^n$ is convex

then $\int_C p(x) x \, dx \in C$ is also convex

↳ 计样本集合为 convex, 则期望也为 convex



~~[illegible]~~ 是cone但非convex cone

2.1.3 Cone / Negative Homogeneous

(1) convex cone: if a set C is convex and a cone

if ~~a~~ for every $x \in C$ and $\theta \ge 0$, $\theta x \in C$

for any $x_1, x_2 \in C$, $\theta_1, \theta_2 \ge 0$, we have $\theta_1 x_1 + \theta_2 x_2 \in C$

→ 虽然convex cone中有"convex", 即convex要求 { ① $\sum\theta = 1$ ② $\theta \ge 0$

(2) conic combination: $\theta_1 x_1 + \dots + \theta_k x_k$, with $\theta_1, \dots, \theta_k \ge 0$ 而此处只有 $\theta_i \ge 0$ 的条件

原因在于cone导致 if $x \in C$ then $\theta x \in C$ $\forall \theta \ge 0$

~~so for and any~~ so $\forall \theta_1, \theta_2 \ge 0$

$\theta_1 = \theta_1' \theta_1''$ $\theta_2 = \theta_2' \theta_2''$

with $\theta_1' + \theta_2' = 1$

→ $\theta_1 x_1 + \theta_2 x_2 = \theta_1'(\theta_1'' x_1) + \theta_2'(\theta_2'' x_2)$

$= \theta_1' x_1' + \theta_2' x_2'$

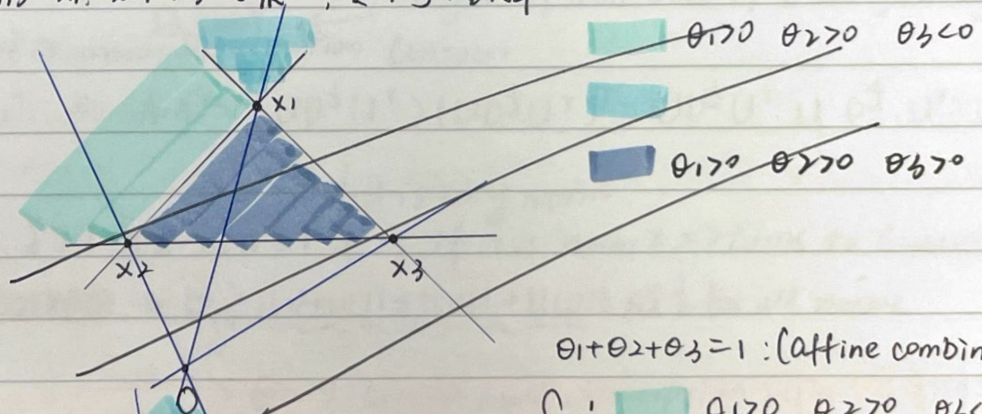
{ $x_1', x_2' \in C$, $\sum\theta_i' = 1$, $\theta_i' \ge 0$

(3) conic hull: is the set of all conic combinations of points in C

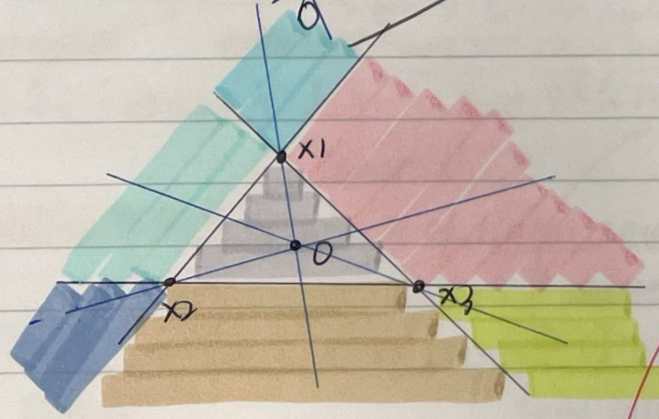
$\{\theta_1 x_1 + \dots + \theta_k x_k \mid x_i \in C, \theta_i \ge 0, i = 1, \dots, k\}$

联系 affine combination ($\sum\theta_i = 1$), convex combination ($\sum\theta_i = 1$, $\theta_i \ge 0$) and conic combination ($\theta_i \ge 0$)

consider 3 points $x_1, x_2, x_3 \in \mathbb{R}^n$, 如何理解 $x = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3$?



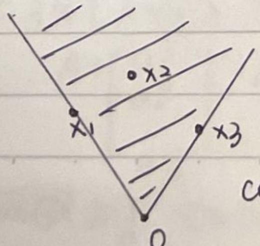
$\theta_1 + \theta_2 + \theta_3 = 1$: (affine combination)



1	$\theta_1 > 0$	$\theta_2 > 0$	$\theta_3 < 0$	+ + −
2	$\theta_1 > 0$	$\theta_2 < 0$	$\theta_3 < 0$	+ − −
3	$\theta_1 < 0$	$\theta_2 > 0$	$\theta_3 < 0$	− + −
4	$\theta_1 < 0$	$\theta_2 > 0$	$\theta_3 > 0$	− + +
5	$\theta_1 > 0$	$\theta_2 > 0$	$\theta_3 > 0$	+ + +
6	$\theta_1 > 0$	$\theta_2 < 0$	$\theta_3 > 0$	+ − +
7	$\theta_1 < 0$	$\theta_2 < 0$	$\theta_3 > 0$	− − +

共7种情况. avoid $\theta_1, \theta_2, \theta_3 < 0$

$\theta_1 + \theta_2 + \theta_3 = 1$ 且 $\theta_1 \ge 0$, $\theta_2 \ge 0$, $\theta_3 \ge 0$ (convex combination)



$\theta_1 \ge 0$, $\theta_2 \ge 0$, $\theta_3 \ge 0$

conic combination

2.2 Some Important Examples

(1) Hyperplane convex. affine

(2) Halfspace convex but not affine

(3) (Euclidean) Ball in $\mathbb{R}^n$ convex → 可由以及 $\|v+w\| \le \|v\|+\|w\|$ 证明

$B(x_c, r) = \{x \mid \|x-x_c\|_2 \le r\} = \{x \mid (x-x_c)^T(x-x_c) \le r^2\}$

$= \{x_c + ru \mid \|u\|_2 \le 1\}$

(4) Ellipsoids

$\mathcal{E} = \{x \mid (x-x_c)^T P^{-1}(x-x_c) \le 1\}$, $P \in S^n_{++}$ symmetric positive definite

$= \{x_c + Au \mid \|u\|_2 \le 1\}$

↳ determines how far the ellipsoid extends in every direction from x_c

↳ 可推导得到 $P = AA^T$ 或 $A = P^{\frac{1}{2}}$

∵ $P \in S^n_{++}$ ∴ $P = UDU^T$ where D is 对角线矩阵

↳ lengths of semi-axes of $\mathcal{E}$ are given by $\sqrt{\lambda_i}$ → eigenvalues

联系：ML中 Gaussian Distribution Contour

∴ $P^{\frac{1}{2}} = UD^{\frac{1}{2}}U^T \Rightarrow A \cdot A^T = (UD^{\frac{1}{2}}U^T)(UD^{\frac{1}{2}}U^T)^T = UD^{\frac{1}{2}}U^T \cdot U \cdot D^{\frac{1}{2}} \cdot U^T = UDU^T = P$

(5) norm ball $\{x \mid \|x-x_c\| \le r\}$ $\|\cdot\|$ is any norm. if it is norm 2 ⇒ turns to Euclidean Ball

↳ convex → 由于 $\|v+w\| \le \|v\|+\|w\|$ 成立 for all norms

(6) norm cone convex

$C = \{\binom{x}{t} \mid \|x\| \le t\}$ [illegible]

→ form 4: convex hull description. 任何一个多面体都可以表示成 finite set $\{v_1, \dots, v_k\}$ 的 convex hull 的形式 $P = \mathrm{conv}\{v_1, \dots, v_k\}$

联系后面 intersection of convex set is still convex

(7) Polyhedra 多面体: finite # of linear equalities and inequalities → convex

↳ $a_i^T x \le b_i$ (halfspace)

$c_j^T x = d_j$ (hyperplane)

↳ as intersection of a finite # of hyperplanes and halfspaces

form ①

$P = \{x \mid a_j^T x \le b_j,\ j=1,\dots,m,\ c_j^T x = d_j,\ j=1,\dots,p\}$

form ② 或写成

$P = \{x \mid Ax \preceq b,\ Cx = d\}$

$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ $C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_p^T \end{bmatrix}$ $d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{bmatrix}$

→ vector inequality in $\mathbb{R}^m$: $\vec{v} \preceq \vec{u}$ if $v_i \le u_i$ for $i = 1, \dots, m$

form ③: 由于 $Cx = d$ 可写成 $\begin{cases} Cx \le d \\ -Cx \ge d \end{cases}$ ⇒ $P = \{x \mid Ax \preceq b\}$

if $X \in S_+^n$ then $X = UDU^T$. D is a diagonal matrix, can be written as $D = D^{\frac{1}{2}} \cdot D^{\frac{1}{2}}$

$\Rightarrow X = UD^{\frac{1}{2}} D^{\frac{1}{2}} U^T = (UD^{\frac{1}{2}}U^T)(UD^{\frac{1}{2}}U^T)^T = UD^{\frac{1}{2}}U^T U \cdot D^{\frac{1}{2}T}U^T = UDU^T = X$ ~~$(UD^{\frac{1}{2}})(D^{\frac{1}{2}}U)(D^{\frac{1}{2}}U^T) = z^Tz$ $z = D^{\frac{1}{2}}U$~~ $z = UD^{\frac{1}{2}}U^T$

总结. if $X \in S_+^n$ 则 X经常转换成 z^Tz 的形式, 其中 ~~$z = D^{\frac{1}{2}}U$~~ $z = UD^{\frac{1}{2}}U^T$ 由于 D is diagonal $D^{\frac{1}{2}T} = D^{\frac{1}{2}} \Rightarrow D^{\frac{1}{2}} \cdot D^{\frac{1}{2}T} = D$

进一步讲. $z^Tz = \langle z, z\rangle = \|z\|^2$ 经常与 ball或其它set联系到一起

△非负象限 nonnegative orthant $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \dots, n\} = \{x \in \mathbb{R}^n \mid x \succeq 0\}$

is polyhedron and a cone (a polyhedron cone)

(8) Simplexes 单纯形 → a special polyhedra ⇒ convex

~~If for~~ k+1 points $v_0, v_1, \dots, v_k \in \mathbb{R}^n$ 并 are affinely independent, which means $v_1 - v_0, \dots, v_k - v_0$ are linearly independent

Simplexes $C = \mathrm{conv}\{v_0, \dots, v_k\} = \{\theta_0 v_0 + \dots + \theta_k v_k \mid \theta_i \geq 0, i = 0, \dots, k, \sum_{i=0}^k \theta_i = 1\}$

$= \underbrace{\mathrm{conv}\{v_0, \dots, v_k\}}_{\text{convex hull}} = \{\theta_0 v_0 + \dots + \theta_k v_k \mid \theta \succeq 0, \underbrace{\mathbf{1}}_{\text{vector with all entries one}}^T \theta = 1\}$

the convex hull of k+1 affinely independent points

→ k is the affine dimension of this simplexes

1-D simplex: a line segment

2-D simplexes: a triangle

3-D simplexes: a tetrahedron 四面体

① $\succeq$ for vector is vector inequality

→ 即vector中每个元素之间比较

② $\succeq$ for symmetric matrix

$A \succeq 0$ 表示 matrix A 半正定

即 $\forall x$, $x^TAx \geq 0$

(9) Symmetric matrix $S^n = \{X \in R^{n \times n} \mid X = X^T\}$

→ S^n: a vector space with dimension $\frac{n(n+1)}{2}$ ①怎么理解

矩阵 $X \succeq 0$ 是什么意思 ② 已解决

Symmetric Positive Semidefinite matrices: $S_+^n = \{X \in S^n \mid X \succeq 0\}$

(看最上面) $z^TXz \geq 0$ for all z

→ 有趣的思路

→ convex cone

→ if $A, B \in S_+^n$, $\theta_1, \theta_2 \geq 0 \Rightarrow \theta_1 A + \theta_2 B \in S_+^n$

→ 角度2: $\{X \in S^n \mid z^TXz \geq 0\}$ z^TXz is linear function of X ⇒ intersection of infinite # of halfspaces → convex

Symmetric positive definite matrices: $S_{++}^n = \{X \in S^n \mid X \succ 0\}$

$\forall x \neq 0$ $x^TXx > 0$

同理正定矩阵 eigenvalues > 0

for n=2 $S_+^2 = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \Leftrightarrow x \geq 0, z \geq 0, xz - y^2 \geq 0$

理解: if $A \succeq 0$. ⇒ A的特征值非负 $A = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} x-\lambda & y \\ y & z-\lambda \end{vmatrix} = (x-\lambda)(z-\lambda) - y^2$

$= \lambda^2 - (x+z)\lambda + (xz - y^2)$ 特征值是 $\lambda^2 - (x+z)\lambda + (xz-y^2) = 0$ 的解

$|A - \lambda I|$: A-λI 的行列式

for $ax^2 + bx + c = 0$. 轴在 $x = \frac{-b}{2a}$ 处

→ 轴在 $\frac{x+z}{2}$ 处 > 0 ⇒ 只要保证 λ=0 处该二次方程非负

λ=0 时 LHS = $xz - y^2$ ⇒ if $xz - y^2 \geq 0$ 且 $x, z \geq 0$, $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$ 特征值非负 ⇒ $A \succeq 0$

2.3 Operations that preserve convexity

(1) Intersection:

Convexity is preserved under intersection: if S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex.

Converse: Every closed convex set S is a (usually infinite) intersection of halfspaces.

(2) Affine Mapping

A function is affine: $f: \mathbb{R}^n \to \mathbb{R}^m$ is affine if it is a sum of a linear function and a constant, i.e. if it has the form $f(x) = Ax + b$, where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

△ if $S_1 \xrightarrow{f_1(\cdot)} \{x\} \xrightarrow{f_2(\cdot)} S_2$ $f_1(\cdot)$与$f_2(\cdot)$是affine function

Affine Mapping $\Rightarrow$ ① if $\{x\}$ is convex, then $S_2 = f_2(x)$ is convex

$x \xrightarrow{f_2(\cdot)} S_2$: S_2 is the image of x under $f_2(\cdot)$

② if $\{x\}$ is convex, then $S_1 = f_1^{-1}(x)$ is convex

即要 $S_1 = f_1^{-1}(x)$ 但描述时写成 inverse image of x under $f_1(\cdot)$ 没有 -1

$\begin{cases} S_1 \xrightarrow{f_1(\cdot)} x : S_1 \text{ is the inverse image of } x \text{ under } f_1(\cdot) \\ x \xrightarrow{f_1^{-1}(\cdot)} S_1 \end{cases}$

→思路如果想检测一个集合 ~~是~~ $\{x\}$是否是convex.

once ①. 构造affine function $f_1(\cdot)$ 使得 ~~S...f~~ $f_1(S_1) = x$ 且 S_1是convex

once ②: 构造affine function $f_2(\cdot)$使得 $f_2(x) = S_2$ 且S_2是convex

→注意: affine mapping是指可表示成 $Ax+b$的functions

once ③ 利用其它方式如是否可表示成 convex set 的交集

Example: ~~x_1A_1+~~ $\{x \mid x_1A_1 + x_2A_2 + \cdots + x_nA_n \preceq B\}$ ~~x~~ $B, A_i \in S^m$, $x \in \mathbb{R}^n$

想法1: $x_1A_1 + \cdots + x_nA_n \preceq B$ 与 $a^Tx = a_1x_1 + \cdots + a_nx_n \le b$ 类似

↳ linear matrix inequality of x (LMI of x) ↳ linear inequality of x

for convenience 写成 $A(x) \preceq B$

而 $\{x \mid a^Tx \le b\}$ 可看成 halfspaces 的交集. ∵ halfspace is convex

∴ $\{x \mid a^Tx \le b\}$ is convex

同理: $\{x \mid A_1x_1 + \cdots + A_nx_n \preceq B\}$ is convex

想法2: 构造 ~~S conv~~ convex set S 与 affine function $f(\cdot)$ 使得 $f(x) = S$ 或 $f(S) = x$

▲想法: 有x, 已知 $A(x) \preceq B$. 如何构造$f(\cdot)$与convex set S?

$A(x) \preceq B \Rightarrow B - A(x) \succeq 0 \Rightarrow B - A(x) \in S_+^m$. S_+^m is convex. 且 $f(x) = B - A(x)$ is affine function

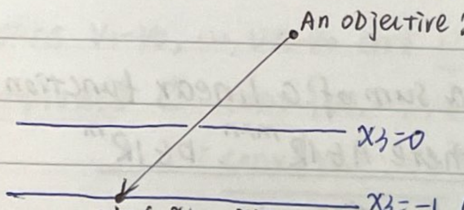
∴ $\{x \mid A(x) \preceq B\}$ 可看成 the inverse image of the positive semidefinite ~~matrix~~ cone under the affine function $f: \mathbb{R}^n \to S^m$ given by $f(x) = B - A(x)$

(3) linear fractional and perspective functions.

perspective function: $P: R^{n+1} \to R^n$ $\quad P(x,t) = \frac{x}{t}$ $\quad \mathrm{dom}\,P = \{ \binom{x}{t} \mid t > 0 \} = R^n \times R_{++}$

① if $C \subseteq \mathrm{dom}\,P$ is convex, then its image is convex $\quad P(C) = \{ P(x) \mid x \in C \}$ → a line segment viewed through a pin-hole camera, yields a line segment image

理解 perspective function：看作 pin-hole camera. (in R^3)



$x' = (-\frac{x_1}{x_3}, -\frac{x_2}{x_3}, -1)$

$= -(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1)$

→ if we drop the last component (cause it is alway -1) then the image of a point at x appears at $x' = -P(x)$

② The inverse image of a convex set under ~~the~~ the perspective function is also convex.

if $C \subseteq R^n$ is convex, then $P^{-1}(C) = \{ (x,t) \in R^{n+1} \mid \frac{x}{t} \in C,\ t > 0 \}$

linear fractional function: composing a perspective function with an affine function

suppose affine function $g: R^{n} \to R^{m+1}$:

$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix} \quad A \in R^{m \times n},\ c \in R^n,\ b \in R^m,\ d \in R$$

function $f: R^n \to R^m$ given $f = P \circ g$, i.e.

$$f(x) = \frac{Ax + b}{c^T x + d} \quad \mathrm{dom}\,f = \{ x \mid c^T x + d > 0 \}$$ is linear-fractional function

$f(x)$ 相当于先对 $x \in R^n$ 做 affine function g, 得到 $g(x) \in R^{m+1}$.

再对 $g(x)$ 做 perspective function, 得到 $f(x) \in R^m$

$$\underset{R^n}{x} \xrightarrow{g} \underset{R^{m+1}}{g(x)} \xrightarrow{P} \underset{R^m}{f(x)} \quad \Rightarrow f: P \circ g$$

if x is convex $\underset{\Leftarrow}{\Rightarrow}$ $g(x)$ is convex (affine mapping) $\underset{\Leftarrow}{\Rightarrow}$ $f(x)$ is convex (perspective function ~~per~~ preserves convexity)

2.4 Generalized inequality

(1) proper cone → Example: R^n_+ S^n_+

for $\forall x_1, x_2 \in C$, $\theta_1, \theta_2 \geq 0$. we have $\theta_1 x_1 + \theta_2 x_2 \in C$

A convex cone $K \subseteq R^n$ is called a proper cone if

- K is convex
- K is closed
- K is solid (has ~~no em~~ nonempty interior $int(S)$)
- K is pointed (has no line) → not all cone has no line
  counter example: halfspace.

(2) Generalize Inequality

A proper cone K is used to define a generalize inequality.

$$x \preceq_K y \iff y - x \in K$$

(或 $y \succeq_K x$)

Strict partial ordering: $x \prec_K y \iff y - x \in int\, K$

i. two vectors $x, y \in R^n$ if $x \preceq_K y$, $K = R^n_+$ which means $x_i \leq y_i \Rightarrow$ componentwise inequality

↳ usually drop the subscript R^n_+ when symbol $\preceq$ or $\prec$ appears between vectors

ii. Matrix inequality $X \preceq_K Y$, $K = S^n_+ \iff Y - X \in S^n_+$ is positive semidefinite

↳ X is positive semidefinite 并 is written by $X \succeq 0$

↳ drop the subscribe S^n_+ ~~S^n_+~~ for matrix inequality

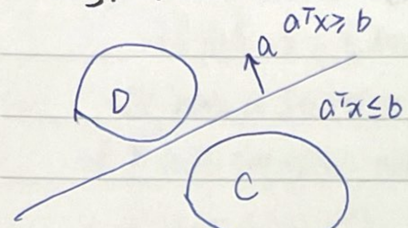
注意:没有closed的条件

条件① 条件② 条件③

2.5 Seperating and supporting hyperplanes

(1) Seperating hyperplane theorem: suppose C and D are nonempty disjoint convex sets, i.e. $C \cap D = \emptyset$, then there exist $a \neq 0$ ~~and~~ and b, such that $a^T x \le b$ for all $x \in C$ and $a^T x \ge b$ for all $x \in D$.

The hyperplane $\{x \mid a^T x = b\}$ is called a seperating hyperplane for the sets C and D.



Assume. C and D are closed.

(if C is not closed ( ), 用 $cl(C)$ 代替 C.

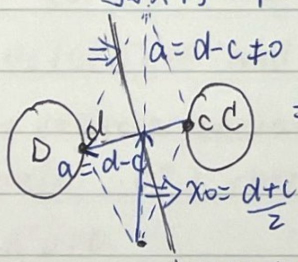
$cl(C)$ is the smallest closet set including C. $cl(C) = C \cup bd(C)$

$dist(cl(C), D) = dist(C, D)$

①假设 $dist(C, D) = \inf\{\|u - v\|_2 \mid u \in C, v \in D\} > 0$

可找到 a point $c \in C$. $d \in D$ such that $\|c - d\|_2$ ~~dist(C,D)~~ $= dist(C, D)$

$\Rightarrow a = d - c \neq 0$. $x_0 = \frac{1}{2}(d + c)$



$\Rightarrow$ we can prove $f(x) \ge 0$ for $\forall x \in D$.

$f(x) \le 0$ for $\forall x \in C$

hyperplane: $f(x) = \frac{a}{\|a\|}(x - x_0)$

△ Seperating ~~of~~ an affine and a convex set.

if $D = \{Fu + g \mid u \in R^m\} \Rightarrow$ there exist $a \neq 0$. and b such that $a^T x \ge b$ for all $x \in D$

$\hookrightarrow a^T F u \ge b - a^T g$ for all $u \in R^m$

linear function is bounded $\Rightarrow a^T F = 0$ & $b \le a^T g$

(2) strict seperation

if ~~a^Tx~~ $a^T x < b$ for all $x \in C$. $a^T x > b$ for all $x \in D \Rightarrow$ strict seperation of set C and D

$\hookrightarrow$ a point $x_0 \notin C$ and a closed convex set C can be strictly seperated.

可构建 a ball $B(x_0, \epsilon) = \{x_0 + u \mid \|u\|_2 \le \epsilon\}$, 寻找 C 与 B 之间的 seperating hyperplane

$\hookrightarrow a^T(x_0 + u) \ge b$ for all $\|u\|_2 \le \epsilon \Rightarrow$ The u that minimize the LHS is $u = \frac{-\epsilon a}{\|a\|_2}$

$\hookrightarrow$ affine function $f(x) = a^T x - b - \epsilon \frac{\|a\|_2}{2}$ is positive on x_0 and negative on C

$a^T(x + u) = a^T x - \epsilon\|a\|_2 \ge b$

$\hookrightarrow$ A closet convex set is the intersection of all halfspaces that contain it.

let C is a cloest convex set. S is the intersection of all halfspaces that contain $C \Rightarrow$ <pf> $C = S$

① obviously if $x \in C \Rightarrow x \in S$

② show if $x \in S$ then $x \in C$. suppose there exist $x \in S$, $x \notin C$.

by the strict seperation result. there exist a hyperplane that strictly seperates x from C $\Rightarrow x \notin S$.

No.

Date

(3) Converse seperating hyperplane theorems.

converse of seperating hyperplane theorem: if there exist a seperating hyperplane then C, D are disjoint) is <u>not true</u>!

↳ Add additional condition: Any 2 convex sets C and D, at least one of which is open, are disjoint if and only if there exist a seperating plane.

(additional ①条件: convex; additional ②条件: is open)

$\left\{\begin{array}{l}\text{exist seperating plane}\\ \text{convex sets}\\ \text{one is open}\end{array}\right. \Rightarrow$ disjoint

$\left\{\begin{array}{l}\text{disjoint}\\ \text{convex}\\ \text{one is open}\end{array}\right. \Rightarrow$ exist a seperating plane

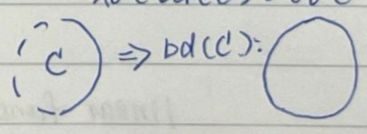
$\left\{\begin{array}{l}\text{disjoint}\\ \text{convex}\\ \text{nonempty}\end{array}\right. \Rightarrow$ exist a seperating plane

exist a seperating plane $\nRightarrow$ disjoint

(4) Supporting Hyperplane

for any nonempty (条件①) convex (条件②) set C and any $x_0 \in bd(C)$, there exists a supporting hyperplane to C at x_0

$x_0 \in bd(C) = cl(C) \setminus int(C)$

if  $\Rightarrow bd(C)$: . bd(C)$\subseteq$C 不一定成立

↳ Partial converse of supporting plane theorem:

if a set is cloest ~~and~~, has nonempty interior, and has a supporting hyperplane at every point in its boundary, then it is convex

$\left\{\begin{array}{l}\text{nonempty}\\ \text{convex}\\ x_0 \in bd(C)\end{array}\right. \Rightarrow$ supporting plane from x_0 and C

$\left\{\begin{array}{l}\text{closed } C\\ \text{nonempty } int(C)\\ \text{supporting plane exist } \forall x_0 \in bd(C)\end{array}\right. \Rightarrow$ convex

↳ 证明: intersection of supporting halfspaces.

2.6 Gordan Theorem & Farkas Theorem

(1) Gordan Theorem

For any $a^1, a^2, \cdots, a^m \in R^n$, exactly one of the following systems has a solution

① $\sum_{i=1}^{m} \lambda_i a_i = 0$, $\sum_{i=1}^{m} \lambda_i = 1$, $0 \le \lambda_1, \cdots, \lambda_m \in R$

② $a^{i^T} x < 0$ for $i = 1, \cdots, m$

if ① has solution $\xLeftrightarrow{}$ ~~then~~ $0 \Leftrightarrow conv(a^1, \cdots, a^m)$

很简单 $x = \sum \lambda_i a_i$, $\sum \lambda_i = 1$, $\lambda_i \ge 0$, then $x \in conv(a^1, \cdots, a^m)$ ~~if 0~~

① has a solution $\Leftrightarrow x = 0$ is feasible $\Rightarrow 0 \in conv(a^1, \cdots a^m)$?

if ② has solution ~~then~~ $\Leftrightarrow$ ~~there~~ there is a seperating plane between 0 and $conv(a^1, \cdots a^m)$

~~$\begin{cases} a^{i^T} x < 0 \\ a^{i^T} 0 = 0 \end{cases} \Rightarrow$ there exists a seperating plane between~~

if ② has a solution x, $\Rightarrow$ can find a x such that $a_i^T x < 0$

~~让 x...~~ $a \in conv(a^1, \cdots, a^m)$ then $a = \sum \theta_i a_i$. ~~we have~~ ~~$a^T x < 0$~~

$x^T a = a^T x = \sum \theta_i (a_i^T x) < 0$ and $x^T 0 = 0$ 不对

↳ means there exists a seperating plane between 0 and $conv(a^1, \cdots, a^m)$

同样 if $a \in conv(a^1, \cdots a^m)$, there is a seperating plane between 0 and conv(·)

↳ we have ~~x~~ $\lambda^T 0 = 0$

$\lambda^T a \le 0 \rightarrow \lambda^T(\sum \theta_i a_i) \le 0 \Rightarrow \sum \theta_i (\lambda^T a_i) \le 0 \quad \forall \theta_i \ge 0, \sum \theta_i = 1$

总列结面地方在于 system② 没有对 θ 的约束 $\Rightarrow \lambda^T a_i \le 0$

来, 怎么得到与 conv(·) 有关的解? $\Rightarrow$ ~~can~~ $a_i^T x < 0$ has a solution

if ② has a solution $\Rightarrow$ can find a x such that $a_i^T x < 0$

$\Rightarrow \sum \theta_i a_i^T x < 0 \quad \forall \theta_i \ge 0$ 且 θ_i 不同时为0

$\Rightarrow x^T(\sum \theta_i a_i^T) < 0 \quad \forall \theta_i \ge 0$ 且 θ_i 不同时为0 与 cone 不同. cone 要求 $\theta_i \ge 0$ 即 cone 一定包含 0

let $a = \sum \theta_i a_i^T$ $\theta_i \ge 0$. we have $x^T a < 0$, $x^T 0 = 0$

$\Rightarrow$ 由②有解的条件, 可推出 0 与 $\{\sum \theta_i a_i \mid \theta_i \ge 0$ 且 θ_i 不同时为0$\}$ 之间存在 seperating plane

↳ 即既非 $conv(a_i)$ 也非 conic hull of a_i

由于 ~~$conv(a_i) \subseteq \{\theta_i a_i \mid \theta_i \ge 0\}$~~ $conv(a_i) \subseteq \{\sum \theta_i a_i \mid \theta_i \ge 0$ 且 θ_i 不同时为0$\}$

所以由②有解(a) 也可以推出 there exists a seperating plane between 0 and conv(·) (b)

同样由(b)也可以推出(a).

所以 (a) $\Leftrightarrow$ (b)

No.

Date

(2) Farkas Lemma

for any $a^1, \cdots, a^m$ and $c \in R^n$ exactly one of the following systems has a solution:

① $\sum_{i=1}^{m} \lambda_i a_i = c$, $0 \le \lambda_i$, [redacted]

② $(a^i)^T x \le 0$ for $i = 1, \cdots, m$, $c^T x > 0$, $x \in R^n$

注意. $\sum_{i=1}^{m} \lambda_i a_i = c$ $c \in R^n$ 通常认为 $A = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix}$ $A^T = (a_1, a_2, \cdots, a_m)$ $a_i \in R^n$

$A^T \lambda = (a_1, \cdots, a_m) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \sum^m \lambda_i a_i = c$

即c看成A行向量的线性组合 => 同样Ax是A列向量的线性组合

计① has solution <=> c是ai的非负线性组合

<=> $c \in$ conichull$(a_1, \cdots, a_m)$

计② has solution => $x^T (\sum \theta_i a_i) \le 0$ for all $\theta_i \ge 0$

$\forall$ ~~let~~ $\exists$ $a = \sum \theta_i a_i$ <=> ~~at~~ $\forall a \in$ conichull$(a_1, \cdots, a_m)$

$\begin{cases} x^T a \le 0 \\ \cancel{x \ne 0} \; x^T c > 0 \end{cases}$

=> there is a seperating plane between c and conichull$(a_1 \cdots, a_m)$