4.1 operations that preserve convexity.

(1) Non-negative weighted sum

A nonnegative weighted sum of convex functions is still convex.

concave functions is still concave.

expand sum to $\int$ (integral) $g(x)=\int_A w(y)f(x,y)dy$. is convex if $f(x,y)$ is convex and $w(y)\geq 0\ \forall y\in A$

理解：对 f 的非负 scaling and addition 与 epigraph 的线性变化有关

convex set

affine mapping of convex set preserve convexity.

$\mathrm{epi}(wf)=\begin{bmatrix} I & 0 \\ 0 & w\end{bmatrix}\mathrm{epi}(f)$

$\{(x,t)\mid x\in\mathrm{dom}f.\ f(x)\leq t\}$

$\{(x,t)\mid x\in\mathrm{dom}f,\ wf\leq t\}$

$RHS=\{(x,wt)\mid x\in\mathrm{dom}f,\ f(x)\leq t\}$ 由于 $w>0$. 所以可以由 $wf\leq t$ 得到 $f\leq\frac{t}{w}$.

$LHS=\{(x,t)\mid x\in\mathrm{dom}f,\ f(x)\leq\frac{t}{w}\}$ 使 LHS=RHS 即 wf 的 epigraph 是 f epigraph

let $\theta=\frac{t}{w}$. $LHS=\{(x,w\theta)\mid x\in\mathrm{dom}f.\ f(x)\leq\theta\}=RHS.$ 的 affine mapping (preserve convexity)

(2) Composition with affine function

$g(x)=f(Ax+b)$ $\quad \mathrm{epi}\,g(x)=\{(x,t)\mid x\in\mathrm{dom}\,g,\ g(x)\leq t\}$

f is convex, so is g; g is convex. so is f. $\quad \mathrm{epi}\,f(Ax+b)=\{(x,t)\mid Ax+b\in\mathrm{dom}f.\ f(Ax+b)\leq t\}$

又由于 $g(x)=f(Ax+b)$

$\Rightarrow \mathrm{epi}\,g(x)=\mathrm{epi}\,f(Ax+b)$

(3) Pointwise Maximum

$\Rightarrow$ if g is convex $\Rightarrow \mathrm{epi}\,g(x)$ convex

$\Rightarrow \mathrm{epi}\,f(Ax+b)$ convex $\Rightarrow f(Ax+b)$ convex

if f_1 and f_2 are convex functions. their pointwise maximum f. defined by $f(x)=\max\{f_1(x),f_2(x)\}$ with $\mathrm{dom}f=\mathrm{dom}f_1\cap\mathrm{dom}f_2$ is also convex.

can be easily proved by the definition of convex function

(4) Pointwise Supremum

If for each $y\in A$, $f(x,y)$ is convex in x. then the function g. defined as

$g(x)=\sup_{y\in A} f(x,y)$ is convex in x! $\quad \mathrm{epi}\,g=\bigcap_{y\in A}\mathrm{epi}\,f(\cdot,y)$

intersection of epigraph.

pointwise infimum of a set of concave function is a concave function.

No.

Date

☆ Almost every convex function can be expressed as the pointwise supremum of a family of affine functions.

$f(x) = \sup\{g(x) \mid g \text{ affine}, g(z) \le f(z) \text{ for all } z\}$

理解：① $f(x) \ge \sup\{g(x) \mid g \text{ affine}, g(z) \le f(z) \text{ for all } z\}$

由于 $g(x)$ 是 $f(x)$ 的 underestimator

② 由于 $f(x)$ convex, the epigraph of f is a convex set.

⇒ can find a supporting hyperplane at $(x, f(x))$

↳ affine function $g(x_0) = f(x_0)$ at $(x_0, f(x_0))$

(5) Partial Minimization

$\text{epi}\, g = \{(x,t) \mid (x,y,t) \in \text{epi}\, f, \text{ for } \text{some}\ y \in C\}$

if $f(x,y)$ is convex in (x,y) and C is a convex set, then partial minimization f (w.r.t y) ($g(x) = \inf_{y \in C} f(x,y)$) is convex.

↳ with respect to

cause $f(x,y)$ is convex.

⇒ epi f is a convex set.

the projection of a convex set ~~one~~ on some of its components is still convex 联系P7

注意与 pointwise infimum/supremum 的区别：

pointwise infimum: for each $y \in A$, $f(x,y)$ is concave in x, then $g(x) = \inf_{y \in A} f(x,y)$ is a concave function.

pointwise supremum: for each $y \in A$, $f(x,y)$ is convex in x, then $g(x) = \sup_{y \in A} f(x,y)$ is a convex function.

pointwise infimum/supremum 要求固定 y 后, $f(x,y)$ 是 convex/concave.
↳ fix

而 partial minimization 要求 $f(x,y)$ is convex in (x,y)

且 partial minimization 要求 C 是 convex set. ~~七月联系~~

(6) Composition

$f = h(g(x))$ ($f = h \circ g$)

① case I: ~~h(x)=~~ $h: R \to R$, $g: R \to R$.

$f = h(g(x))$

↳ $f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$

⇒ $f(x)$ is convex if → h convex and nondecreasing, g is convex

一致 → h convex and nonincreasing, g is concave

$f(x)$ is concave if → h concave and nondecreasing, g is concave

一致 → h ~~convex~~ concave and nonincreasing, g is convex

~~$h \circ g$~~ $h(g(x)) = h(g_1(x), g_2(x), \cdots, g_k(x))$

② General case $h: R^k \to R$. $g: R^n \to R^k$ → Extended-value extension of h

$f(x)$ is convex if → h is convex. $\tilde{h}$ nondecreasing. g_i is convex.
→ h is convex. $\tilde{h}$ nonincreasing. g_i is concave.

$f(x)$ is concave if → h is concave. $\tilde{h}$ nondecreasing. g_i is concave.
→ h is convex. $\tilde{h}$ nonincreasing. g_i is convex.

△ h is convex and $\tilde{h}$ is nondecreasing 意味着什么?

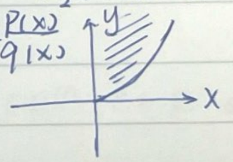
$$\tilde{h} = \begin{cases} +\infty & 当 x \notin \mathrm{dom} f \\ h(x) & \text{otherwise} \end{cases}$$

⇒ the domain of h extends infinitely in the negative direction.

即 if $x, y \in R$. $\tilde{h}(x) \le \tilde{h}(y)$. $y \in \mathrm{dom}\, h$ ⇒ then $x \in \mathrm{dom}\, h$.
(if $x \notin \mathrm{dom}\, h$. $\tilde{h}(x) = +\infty$. $\tilde{h}(x) = +\infty$ 不可能 $\le \tilde{h}(y)$)

同理: h is convex and $\tilde{h}$ is nonincreasing 意味着 dom h extends infinitely in the positive direction.

△怎么运用到实际例题: 怎么判断 $f(x) = \frac{p(x)}{q(x)}$ ($f: R^n \to R$, $x \in R^n$) 的 convexity?



$h(x, y) = \frac{x^2}{y}$. dom $h = \{(x, y) \mid x > 0, y > 0\}$

$h(x, y)$ is a convex function

↳ $g_1(x) = p(x)$. $g_2(x) = q(x)$ $g: R^n \to R^2$ $g_1: R^n \to R$. $g_2: R^n \to R$

~~$h(x)$~~ $h(x, t) = \frac{x^2}{t}$ $h: R^2 \to R$

$f(x) = h(g(x)) = h(g_1(x), g_2(x)) = \frac{p^2(x)}{q(x)}$ (composition)

h is convex. ~~so if~~ and h is nondecreasing in the 1st argument.
h is nonincreasing in the 2nd argument

↳ f is convex if $\underset{g_1 = p(x)}{g(x)}$ is convex and $g_2(x) = q(x)$ is concave

重点: ① 将 $f(x) = \frac{p(x)}{q(x)}$ 分解成 $h(x, t) = \frac{x^2}{t}$. $f(x) = h(p(x), q(x))$

② 将 $h(p(x), q(x))$ 整合成 $h(g(x))$ $g: R^n \to R^2$. $g_1: R^n \to R$. $g_2: R^n \to R$
$h: R^2 \to R$
$g_1(x) = p(x)$. $g_2(x) = q(x)$

③ 得到 $f(x) = h(g(x))$ $f: R^n \to R$. $h: R^2 \to R$. $g: R^n \to R^2$
将 f 的 convexity 与 h 与 g 的 convexity 联系起来
且 g_i 的 convexity 与 h 在第几个变参数处的单调性有关

4.2 Prespective and Conjugate.

(1) Perspective

If $f: R^n \to R$, Then the perspective of f is the function $g: R^{n+1} \to R$ defined by

$g(x,t) = t f(\frac{x}{t})$ with domain g $\operatorname{dom} g = \{(x,t) \mid \frac{x}{t} \in \operatorname{dom} f, t > 0\}$!

If f is a convex function, then so is its perspective function g.

(If f is concave, so is g)

联系P9 perspective function : $P: R^{n+1} \to R^n$ $P(x,t) = \frac{x}{t}$ $\operatorname{dom} P = \{\cancel{(\frac{x}{t})} (\frac{x}{t}) \mid t > 0\}$

if $C \subseteq \operatorname{dom} P$ is convex. then $P(C)$ is a convex set

if C is convex. $P^{-1}(C)$ is a convex set

$\hookrightarrow (x,t,s) \in \operatorname{epi} g \Leftrightarrow t f(\frac{x}{t}) \le s$

$\Leftrightarrow f(\frac{x}{t}) \le \frac{s}{t}$

$\Leftrightarrow (\frac{x}{t}, \frac{s}{t}) \in \operatorname{epi} f$

$\Rightarrow$ epi g is the inverse image of epi f under the perspective mapping that takes (u, v, w) to $(u, w)/v$.

△ Extension:

Suppose $f: R^m \to R$ is convex, and $A \in R^{m \times n}$, $b \in R^m$, $c \in R^n$ and $d \in R$, then

$g(x) = (c^T x + d) f((Ax+b)/(c^T x + d))$

with $\operatorname{dom} g = \{x \mid c^T x + d > 0, \frac{Ax+b}{c^T x + d} \in \operatorname{dom} f\}$ is convex.

$\hookrightarrow$ f is convex $\Rightarrow$ $g(x,t) = t f(\frac{x}{t})$ is convex. in (x,t) for $t > 0$

$\Rightarrow g(\cancel{c^Tx+d}, Ax+b, c^T x + d)$ is convex (composition with affine function)

$= g(x)$

(2) Conjugate

$f^*(y) = \sup_{x \in \operatorname{dom} f} (\underbrace{y^T x - f(x)}_{\text{affine function of } y})$ → pointwise supremum of affine functions

$f: R^n \to R$. $f^*: R^n \to R$

$f^*(y)$ is convex whether or not f is convex

The domain of the conjugate function ~~is~~ consists of $y \in R^n$ for which supremum is finite

对于较复杂的 $f(x)$，如 $f(x) = \log \Sigma e^{x_i}$，对 $x^T y - f(x)$ 求导 (w.r.t x_i)

通常得到 y_i 的表达式而非 x_i 的表达式

例 $y_i = e^{x_i} / \Sigma e^{x_i}$ 但仍注意应代入 $x_i = ?$

重点 对x求导

△例题 conjugate of $f(x) = -\log x$ with $\operatorname{dom} f = R_{++}$

▷将 $x = \cdots$ 代入 $x^T y - f(x)$ 中得到 f^*

$f^*(y) = \sup_{x \in R_{++}} \{xy + \log x\}$. $xy + \log x$ is unbounded above if $y \ge 0$ and $\frac{d(xy + \log x)}{dx} = y - \frac{1}{x}$

$\Rightarrow xy + \log x$ reaches its maximum at $x = -\frac{1}{y}$ substitude $x = -\frac{1}{y}$ into $f^*(y)$

$\Rightarrow f^*(y) = -1 - \log(-y)$ $\operatorname{dom} f^* = \{y \mid y < 0\}$

if f is convex and $\mathrm{dom} f = R^n$ then $f^{**} = f$

重要前提

<pf> ① for affine function $f(x) = a^T x + b \quad x \in R^n$

重点1. $f^{**}(x) = f(x)$

$f^*(y) = \sup_x \{x^T y - a^T x - b\} = \sup_x \{(y-a)^T x - b\}$ ← affine unbounded if $y \neq a$

$$= \begin{cases} -b & y = a \\ +\infty & \text{otherwise} \end{cases}$$

$$f^{**}(x) = \sup_{y=a} \{x^T y + b\} = a^T x + b = f(x)$$

② for general case. if f is convex. then $f^{**} = f$

重点2. 联系 Pro: if f is convex. then f can be expressed as the pointwise supremum of a family of affine functions

$f(x) = \sup\{g(x) \mid g \text{ affine}, g(z) \le f(z) \text{ for all } z\}$

(2.1) $f(x) \le f^{**}(x)$

由Definition 可知: $f(x) \ge g(x) \Rightarrow f^*(y) \le g^*(y) \Rightarrow f^{**}(x) \ge g^{**}(x)$ for all x

由于 g is affine. $\Rightarrow g^{**} = g$

$\Rightarrow f^{**}(x) \ge \sup\{g^{**}(x) \mid g \text{ affine}, g(z) \le f(z) \text{ for all } z\}$

$= \sup\{g(x) \mid g \text{ affine}, g(z) \le f(z) \text{ for all } z\}$

$= f(x)$

(2.2) $f(x) \ge f^{**}(x)$

$f^*(y) = \sup_x \{x^T y - f(x)\}$

$\Rightarrow f^*(y) \ge x^T y - f(x)$ for all y, for all x

重点3 $\Rightarrow f(x) \ge x^T y - f^*(y)$ for all y. for all x

$f^{**}(x) = \sup_y \{x^T y - f^*(y)\}$

$\Rightarrow f(x) \ge f^{**}(y)$

combine (2.1) & (2.2) $\Rightarrow f(x) = f^{**}(x)$ if f is a convex function.

Suppose f is differentiable and convex

for any y^*. if we can find a x^* such that $y^* = \nabla f(x^*)$ (即 x^* is a maximizer of $y^T x - f(x)$)

we have $f^*(y^*) = x^{*T} y^* - f(x^*)$

$= x^{*T} \nabla f(x^*) - f(x^*)$

? 为什么要求f is convex?
我的想法是: if $f(x)$ is convex in x
then $y^T x - f(x)$ is concave in x
so by ~~using~~ setting $\frac{\partial (y^T x - f(x))}{\partial x} = 0$
we can find the global maximum point of $y^T x - f(x)$ with fixed y
so $f^*(y) = \sup_x \{x^T y - f(x)\}$
$= x^T y - f(x) \big|_{x^*}$
$\frac{\partial (y^T x - f(x))}{\partial x} \Big|_{x = x^*} = 0$

例. 若想计算 $f^*(2)$. 找 $f(x)$ 是否存在一点 x^* 使得 $\nabla f(x^*) = 2$. 如果存在则不需计算 $f^*(y)$ 表达式. 可用 $f^*(2) = x^{*T} \nabla f(x^*) - f(x^*)$ 直接计算

(3) Quasiconvex

$f: R^n \to R$ is quasiconvex if dom f is convex and the sublevel sets $S_\alpha = \{x \in \text{dom} f \mid f(x) \le \alpha\}$ is convex for all α

联系 P17. sublevel sets of a convex function are convex for all α

$\Rightarrow$ if f is convex $\Rightarrow$ f is also quasiconvex.

f is quasiconcave if $-f$ is quasiconvex

(4) Transformation about $f^*(y)$

① $g(x) = af(x) + b$ $(a>0)$ $\Rightarrow g^*(y) = af^*(\frac{y}{a}) - b$

理解重点在于如何表示 $f^*(\frac{y}{a})$?

$f^*(\frac{y}{a}) = \sup_x \{ x^T y/a - f(x) \}$

其余不变

② if $f(u,v) = f_1(u) + f_2(v)$ and f_1, f_2 are convex

$\Rightarrow f^*(w,z) = f_1^*(w) + f_2^*(z)$

The conjugate of the sum of independent convex functions is the sum of the conjugates

of different variables.

$f^*(w,z) = \sup_{u,v} \{ w^T u + z^T v - f_1(u) - f_2(v) \}$

independent $= \sup_u \{ w^T u - f_1(u) \} + \sup_v \{ z^T v - f_2(v) \}$ ⓘ为什么要有这条件?

$= f_1^*(w) + f_2^*(z)$