Linear Mapping

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Definition

Let's consider two vector spaces: V and W. Function L is a linear mapping if for any $\underline{u}$, $\underline{v}$ in V and any scalar α we have:

1. $L(u+v) = L(u) + L(v)$
2. $L(\alpha u) = \alpha L(u)$

Example

Let's consider matrix $A \in \mathbb{R}^{n,m}$ we can define La($\underline{u}$) so that:
$LA(u) = Au$ ($u \in \mathbb{R}^m$) lets show that LA is a linear mapping.

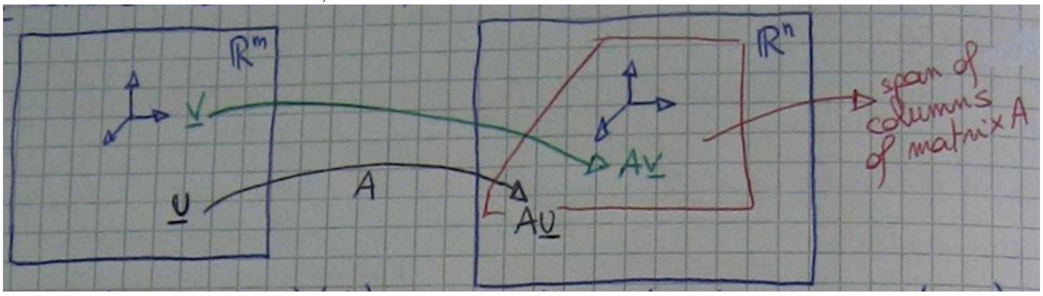
Proof

1. $LA(u+v) = A(u+v) = Au + Av = LA(u) + LA(v)$
2. $LA(\alpha u) = A(\alpha u) = \alpha(A(u)) = \alpha LA(u)$

Therefore LA is a linear mapping.

Example

Lets consider $A \in \mathbb{R}^{n,m}$, $A : \mathbb{R}^m \to \mathbb{R}^n$



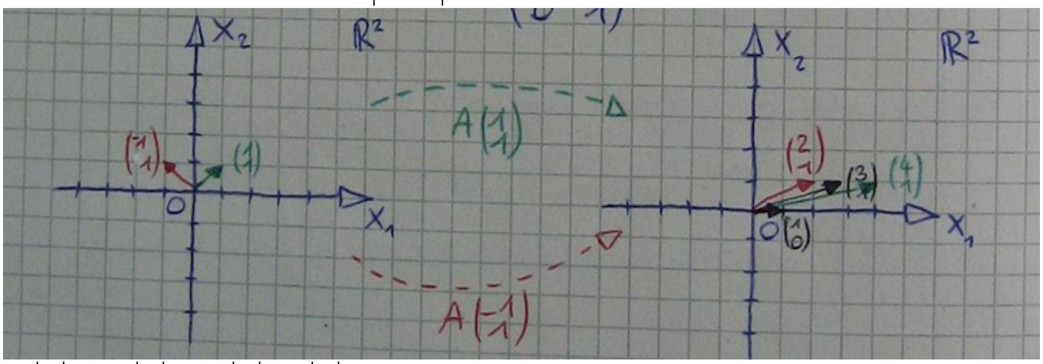
$$Au = \begin{vmatrix} a_{11} & ... & a_{1m} \\ ... & ... & ... \\ a_{n1} & ... & a_{nm} \end{vmatrix} \begin{vmatrix} u_1 \\ ... \\ U_m \end{vmatrix} = \underbrace{u_1 \begin{vmatrix} a_{11} \\ ... \\ a_{n1} \end{vmatrix} + u_2 \begin{vmatrix} a_{12} \\ ... \\ a_{n2} \end{vmatrix} + ... + u_m \begin{vmatrix} a_{1m} \\ ... \\ a_{nm} \end{vmatrix}}_{\text{Linear Combination of columns of A}}$$

Def: Column space

A column space of matrix $A \in \mathbb{R}^{n,m}$ is defined as a span of columns of matrix A Denoted by C(A).

Example

Let's consider matrix $A = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$, $A : \mathbb{R}^2 \to \mathbb{R}^2$



$A \begin{vmatrix} 1 \\ 1 \end{vmatrix} = 1 \begin{vmatrix} 1 \\ 0 \end{vmatrix} + 1 \begin{vmatrix} 3 \\ 1 \end{vmatrix} = \begin{vmatrix} 4 \\ 1 \end{vmatrix}$

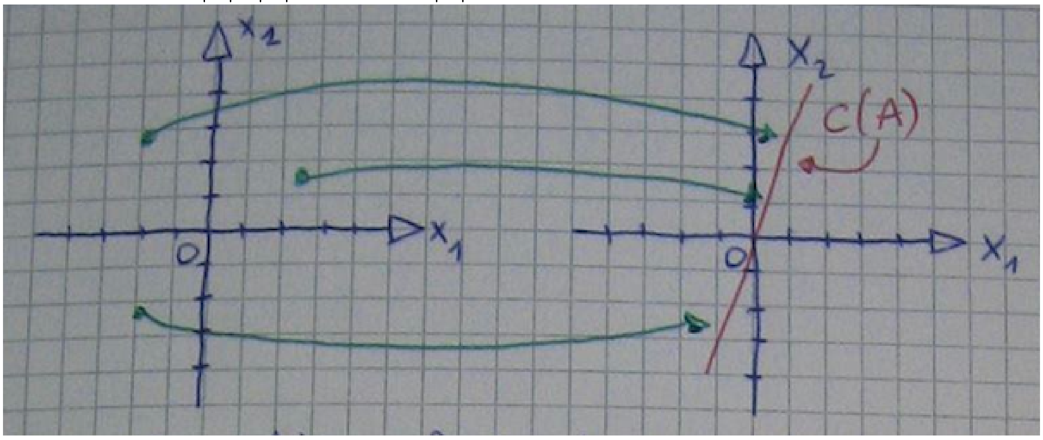
$A \begin{vmatrix} -1 \\ 1 \end{vmatrix} = -1 \begin{vmatrix} 1 \\ 0 \end{vmatrix} + 1 \begin{vmatrix} 3 \\ 1 \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$

Column space, span of $\begin{vmatrix} 1 \\ 0 \end{vmatrix}$ and $\begin{vmatrix} 3 \\ 1 \end{vmatrix}$

Example

Let's consider $A = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}$

$C(A) = span\{\begin{vmatrix} 1 \\ 3 \end{vmatrix}, \begin{vmatrix} 1 \\ 3 \end{vmatrix}\} = span\{\begin{vmatrix} 1 \\ 3 \end{vmatrix}\}$



Note

For the solution of A$\underline{x}$=$\underline{b}$ to exist, $\underline{b}$ should belong to column space C(A).

Def: Null Space

the nullspace of $A \in \mathbb{R}^{n,m}$, N(A), is defined as $N(A) = \{x \in \mathbb{R}^m \mid A\underline{x} = \underline{0}\}$ in other words, N(A) consist of all solution of A$\underline{x}$=$\underline{0}$

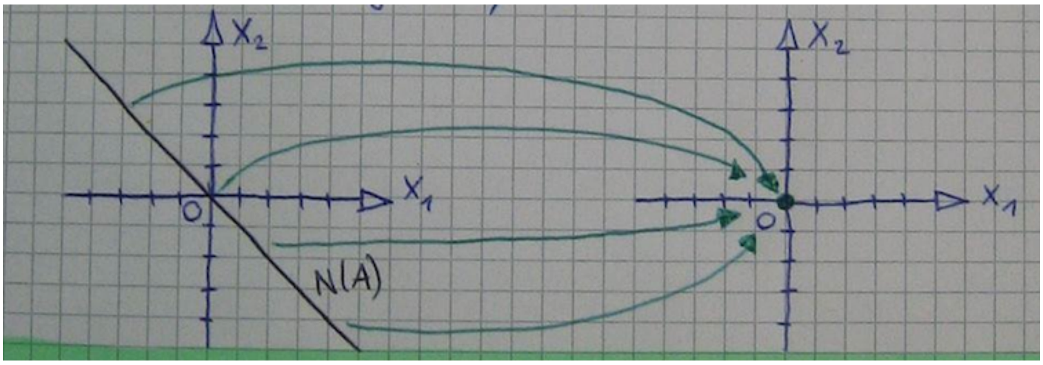
Ex

Let's consider $A = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}$, N(A)=?

A$\underline{x}$=$\underline{0}$ - we look for all solutions.

$\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \underline{0} \rightarrow$

$x_1 + x_2 = 0$
$3x_1 + 3x_2 = 0 \rightarrow$
$x_1 + x_2 = 0$
$0 = 0$ thus all the solutions are given by $x_2 = -x_1$



Theorem

Let's consider $A \in \mathbb{R}^{n,m}$. N(A) is a subspace in $\mathbb{R}^m$.

Proof: 1. $\underline{x}, \underline{x}' \in N(A) \overset{?}{\Rightarrow} \underline{x} + \underline{x}' \in N(A)$

$A(\underline{x} + \underline{x}') = A(\underline{x}) + A(\underline{x}') = \underline{0} + \underline{0} = \underline{0}$

Therefore $\underline{x} + \underline{x}' \in N(A)$

2. $\underline{x} \in N(A), \alpha \in \mathbb{R} \overset{?}{\Rightarrow} \alpha \underline{x} \in N(A)$

$A(\alpha \underline{x}) = \alpha A(\underline{x}) = \alpha \cdot \underline{0} = \underline{0}$

Therefore $\alpha \underline{x} \in N(A)$.

Theorem: Let's consider $A \in \mathbb{R}^{n,m}$. $C(A)$ is a subspace in $\mathbb{R}^n$.