Orthogonal Matrices

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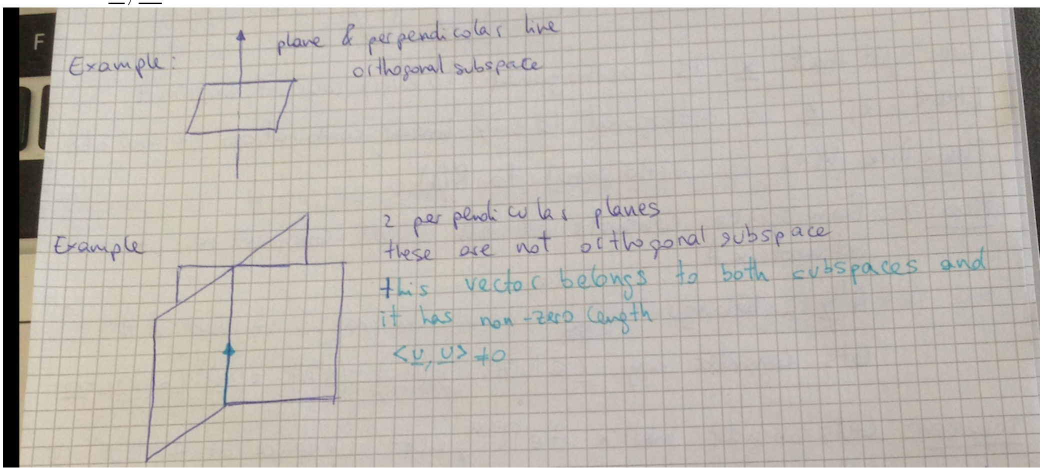
Definition

Two subspaces V & W are orthogonal if for any $\underline{v} \in V$ and any $\underline{w} \in W$ we have $< \underline{v}, \underline{w} >= 0$.



note

if vector $\underline{r}$ belongs to two orthogonal subspaces, then it has to be zero vector.
Lets consider matrix $A \in \mathbb{R}^{n,m}$. **Nullspace**, $N(A) \subset \mathbb{R}^m$ consists of all $x \in \mathbb{R}^m$ such that $A\underline{x} = \underline{0}$.
Columnspace,$C \subset \mathbb{R}$, span of columns of matrix A.
We also define **rowspace** $R(A) \subset \mathbb{R}^m$, span of rows of matrix A.
Left Nullspace, $N(A^T) \subset \mathbb{R}$ consisto of all $y \in \mathbb{R}^n$ such that:
$A^T\underline{y} = \underline{0}, \underline{y}A^T = \underline{0}$

note

$C(A) = R(A^T)$ and viceversa.

Theorem

Lets consider Matrix $A \in \mathbb{R}^{m,n}$.
Nullspace N(A) is orthogonal to rowspace of A, R(A).

proof1

lets consider any arbitrary $\underline{x} \in N(A)$ so that $A\underline{x} = \underline{0}$.

$$A\underline{x} = \begin{vmatrix} -row1ofA \\ -row2ofA \\ . \\ . \\ rownofA \end{vmatrix} \underline{x} = \begin{vmatrix} < row1ofA, \underline{x} > \\ < row2ofA, \underline{x} > \\ . \\ . \\ < rownofA, \underline{x} > \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

we got that x is orthogonal to every row of matrix A therefore, x is orthogonal to any combination of row of A therefore N(A) is orthogonal to R(A).//

Proof2

Rowspace consists of all linear combination of rows of A.
$A^T\underline{y}$ belongs to rowspace. It is the linear combination of rows of A.
Lets consider any $x \in N(A), \underline{x}^T(A^T\underline{y}) = (\underline{x}^T A^T)\underline{y} = (A\underline{x})^T\underline{y} = \underline{0}\underline{y} = \underline{0}$

0.1 Theorem

Lets consider matrix$A \in \mathbb{R}^{nm}$Left nullspace of A, N(A^T) is orthogonal to the columnspace of A, C(A).

proof

Remember that C(A)=R(A^T). Then we can apply the proof above to A^T

Definition

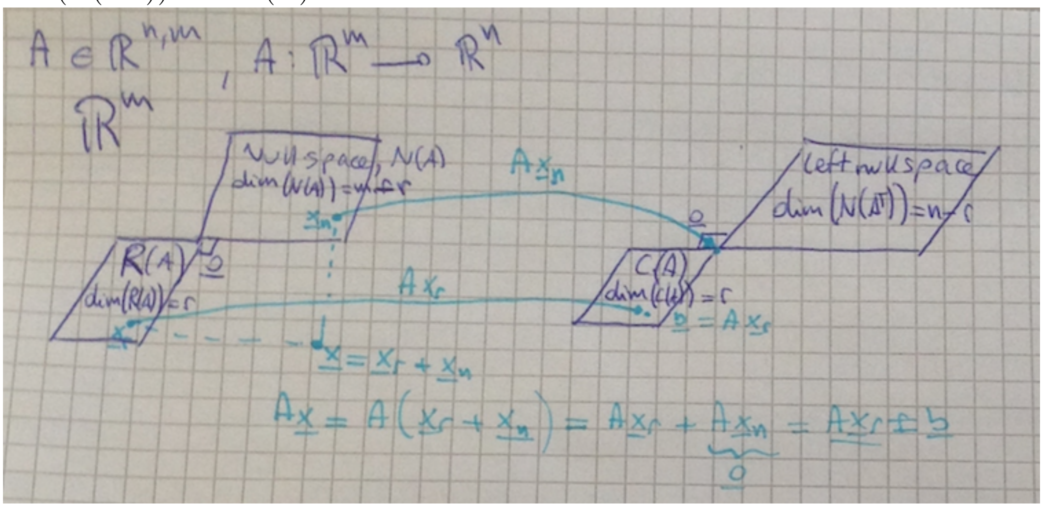
An orthogonal complement of subspace M of vectors $s \in V$ consist of all vectors orthogonal to m. this subspace is denoted by $M^\perp$
Remark: Dim M + dim $M^\perp$ = dim V
based on this definition nullspace of A is an orthogonal complement of rowspace of A

Theorem

N(A) is orthogonal complement of R(A) and dim(N(A))+dim(R(A))= m

Theorem

N($A^{\perp}$) is orthogonal cmplement of C(A).
dim(N($A^{\perp}$))+ di C(A)= n



Theorem

if $\underline{b} \in C(A)$ then there exist one and only one vector $\underline{x}_r \in R(A)$ such that A$\underline{x}_r = \underline{b}$

Proof

Lets assume that $\underline{x}_r$ and $\underline{x}_r^i$ both belong to R(A) and also $A\underline{x}_r = \underline{b}$ and $A\underline{x}_r^i = \underline{b}$
Then $A\underline{x}_r - A\underline{x}_r^i = \underline{0}$
$A(underlinex_r - underlinex_r^i) = \underline{0}$
and so $A\underline{x}_r - A\underline{x}_r^i \in N(A)$
but since xr and xr prime, belongs to R(A).
$A\underline{x}_r - A\underline{x}_r^i \in R(A)$
it is only possible if xr minus xr prime is equal to $\underline{0}$