ST102 Class 14 – Solutions to Additional exercises

1. The joint probability of the observed sequence of balls is:

$$\pi(1-\pi)(1-\pi)\pi(1-\pi) = \pi^2(1-\pi)^3$$

which also serves as the likelihood function, hence:

$$L(\pi) = \pi^2(1-\pi)^3.$$

For $\pi = 1/3$, we have:

$$L\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{8}{243}$$

which is greater than:

$$L\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Therefore, $\widehat{\pi} = 1/3$.

Note that since there are only two possible values of π, there is no need to derive the log-likelihood function as it is straightforward to enumerate $L(1/3)$ and $L(1/2)$.

2. The joint probability function is:

$$p(x_1, x_2, \ldots, x_8; \pi) = \prod_{i=1}^{8} \pi^{x_i}(1-\pi)^{1-x_i} = \pi^{\sum_{i=1}^{8} x_i}(1-\pi)^{8-\sum_{i=1}^{8} x_i} = \pi^5(1-\pi)^3$$

which also serves as the likelihood function, hence:

$$L(\pi) = \pi^5(1-\pi)^3.$$

The log-likelihood function is:

$$l(\pi) = \log L(\pi) = 5\log(\pi) + 3\log(1-\pi).$$

Differentiating and equating to zero, we have:

$$\frac{\mathrm{d}l}{\mathrm{d}\pi} = \frac{5}{\widehat{\pi}} - \frac{3}{1-\widehat{\pi}} = 0.$$

Hence:

$$\frac{5}{\widehat{\pi}} = \frac{3}{1-\widehat{\pi}}$$

and so $\widehat{\pi} = 5/8$, which is the sample proportion of 1s.

3. (a) Since the X_is are independent and identically distributed, the likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{2\sqrt{X_i}}\, \mathrm{e}^{-\theta\sqrt{X_i}} = \frac{\theta^n}{2^n \prod\limits_{i=1}^{n} \sqrt{X_i}}\, \mathrm{e}^{-\theta \sum\limits_{i=1}^{n} \sqrt{X_i}}.$$

Hence the log-likelihood function is:

$$l(\theta) = \ln L(\theta) = n \ln \theta - \theta \sum_{i=1}^{n} \sqrt{X_i} - \ln\left(2^n \prod_{i=1}^{n} \sqrt{X_i}\right).$$

Differentiating with respect to θ, we obtain:

$$\frac{\mathrm{d}l(\theta)}{\mathrm{d}\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \sqrt{X_i}.$$

Equating to zero, we obtain the maximum likelihood estimator:

$$\widehat{\theta} = \frac{n}{\sum\limits_{i=1}^{n} \sqrt{X_i}}.$$

(b) The maximum likelihood estimator of $\theta/2$ is $\widehat{\theta}/2$. For the given data, the point estimate is:

$$\widehat{\theta} = \frac{4}{2(\sqrt{4.1} + \sqrt{7.3} + \sqrt{6.5} + \sqrt{8.8})} = 0.1953.$$

4. Note that $X \sim \text{Pois}(\lambda^2)$.

(a) The likelihood function is:

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{2X_i} \mathrm{e}^{-\lambda^2}}{X_i!} = \frac{\lambda^{2\sum\limits_{i=1}^{n} X_i} \mathrm{e}^{-n\lambda^2}}{\prod\limits_{i=1}^{n} X_i!}.$$

The log-likelihood function is:

$$l(\lambda) = \ln L(\lambda) = \left(2\sum_{i=1}^{n} X_i\right)(\ln \lambda) - n\lambda^2 - \ln\left(\prod_{i=1}^{n} X_i!\right).$$

Differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} l(\lambda) = \frac{2\sum\limits_{i=1}^{n} X_i}{\lambda} - 2n\lambda = \frac{2\sum\limits_{i=1}^{n} X_i - 2n\lambda^2}{\lambda}.$$

Setting to zero, we re-arrange for the estimator:

$$2\sum_{i=1}^{n} X_i - 2n\widehat{\lambda}^2 = 0 \quad \Rightarrow \quad \widehat{\lambda} = \left(\frac{\sum\limits_{i=1}^{n} X_i}{n}\right)^{1/2} = \bar{X}^{1/2}.$$

(b) By the invariance principle of maximum likelihood estimators:

$$\widehat{\theta} = \widehat{\lambda}^3 = \bar{X}^{3/2}.$$