1. $\bar{x} \pm \cdot z/t \cdot SE(\bar{x})$.

① Normal $X_i \sim N(\mu, \sigma^2)$

i) σ^2 is known $\qquad \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N$

$(1-2\alpha)\%$ $\quad \bar{x} \pm z_\alpha \cdot \frac{\sigma}{\sqrt{n}}$

ii) σ^2 is unknown $\quad S^2 - \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$(1-2\alpha)\%$ $\quad \bar{x} \pm t_{n-1,\alpha} \cdot \frac{S}{\sqrt{n}}$ $\qquad \frac{\bar{x}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

② Non-normal.

i) CLT. when $\boxed{n \text{ is large}}$ $\frac{\bar{x}-\mu}{S/\sqrt{n}} \sim N$.

$(1-2\alpha)\%$ $\quad \bar{x} \pm z_\alpha \cdot \frac{S}{\sqrt{n}}$.

ii). MLE, $\quad \hat{\theta} \sim N(\theta, (nI(\theta))^{-1})$.

$(1-2\alpha)\%$ $\quad \hat{\theta} \pm . z_\alpha \sqrt{\frac{1}{nI(\hat{\theta})}}$

EX 1. $X_i \sim Bin(1, \pi)$

$\hat{\pi} = \bar{X}$

$I(\pi) = -\int_R \frac{\partial^2 \ln f(x;\pi)}{\partial \pi^2} f(x;\pi) dx$

$= E[-\frac{\partial^2 \ln P(x;\pi)}{\partial \pi^2}]$

$P(x;\pi) = \pi^x (1-\pi)^{1-x} \quad \log P(x;\pi) = x\log\pi + (1-x)\log(1-\pi)$

$\frac{\partial \ln P(x;\pi)}{\partial \pi} = \frac{x}{\pi} + (-1)\frac{1-x}{1-\pi} = \frac{x}{\pi} - \frac{1-x}{1-\pi}$

$\frac{\partial^2 \ln P(x;\pi)}{\partial \pi} = (-1)\frac{x}{\pi^2} - \frac{(1-x)}{(1-\pi)^2} \cancel{(-1)(-1)}$

$= -\frac{x}{\pi^2} - \frac{1-x}{(1-\pi)^2}$

$I(\pi) = E[\frac{x}{\pi^2} + \frac{1-x}{(1-\pi)^2}]$, $\quad E[X] = \pi$

$= \frac{\pi}{\pi^2} + \frac{1-\pi}{(1-\pi)^2} = \frac{1}{\pi} + \frac{1}{1-\pi}$

$= \frac{1}{\pi(1-\pi)}$

$\boxed{\hat{\pi}} \sim N(\pi. (nI(\pi))^{-1}) = N(\pi, \frac{\pi(1-\pi)}{n})$.

$E[\hat{\pi}] = \boxed{\pi}$

2. $n = 23$. $\bar{x} = 18.4$, $S = 3.9$.

$\bar{x} \pm t_{n-1, 0.005} \cdot \frac{S}{\sqrt{n}} = CI(n)$

(a) $t_{22, 0.005} = 2.819$, $CI(23) = (16.11, 20.69)$.

(b). $t_{120, 0.005} = 2.617$, $CI(121) = (17.47, 19.33)$

$z_{0.005} \approx 2.58$

width CI: $2 \cdot t_{n-1, 0.005} \cdot \frac{S}{\sqrt{n}} \downarrow$

3. $\bar{x} = \frac{\sum x_i}{n} = \frac{370}{425} \approx 0.87058$

(a) ① CLT: $S^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2$

$= \frac{1}{n-1}(\sum_{i=1}^n x_i^2 - n\bar{x}^2)$. $\qquad \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i = \sum_{i=1}^n x_i^k$ $\;k>0$

$= 0.11293$ $\qquad = 370$

$\bar{x} \pm z_{0.025} \frac{S}{\sqrt{n}} = (0.8386, 0.9025)$

② MLE $\quad \bar{x} \sim N(\pi, \frac{\pi(1-\pi)}{n})$ From Q1.

$s^2 = \hat{\pi}(1-\hat{\pi}) = 0.11265$ $\quad$ roughly same.

$\bar{x} \pm z_{0.025}\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} = (\cdot \cdot \cdot \cdot \cdot)$

(b). $2 \cdot z_{0.025} \cdot \frac{S}{\sqrt{n}} \rightarrow 2 \cdot z_{0.005} \frac{S}{\sqrt{n}}$

$z_{0.005} > z_{0.025}$

4. $\boxed{n = 400}$ $\quad \bar{x} = 36,000$. $\quad S = 17,000$.

(a). $\bar{x} \pm \underset{\boxed{1.645}}{z_{0.05}} \cdot \frac{\sigma}{\sqrt{n}} = (35,013, 36.987)$

(b) $\bar{x} \pm z_{0.05}\frac{S}{\sqrt{n}} = (34602, 37398)$.

(c) $\boxed{n = 400}$, sample variance is more reliable.

$\frac{\bar{x}-\mu}{S/\sqrt{n}} \sim N$

AEX 15

Q2. (a) $P(z > 0.67) = 0.2514 = \alpha$.

Coverage probability $1 - 2\alpha = 0.4972 < 0.5$

(b) $(\frac{x}{n}, \infty)$

$P(\frac{x}{n} < \mu < \infty)$ $\qquad \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim N$

$= P(0 < \mu - \frac{x}{n} < \infty)$ $\qquad -z, z$

$= P(0 < \frac{\mu - \frac{x}{n}}{\sqrt{\frac{\frac{x}{n}(1-\frac{x}{n})}{n}}} < \infty)$ $\sim z$

$= P(0 < z < \infty) = 0.5$.

Q4. $\bar{x} = \frac{\sum x_i}{n}$

CI: $\bar{x} \pm z_{0.005} \cdot \sqrt{\frac{\bar{x}(1-\bar{x})}{n}}$

width: $2 \cdot z_{0.005} \sqrt{\frac{\bar{x}(1-\bar{x})}{n}}$, $\quad \bar{x} \in [0, 1]$

$f(\bar{x}) = \bar{x}(1-\bar{x}) = \bar{x} - \bar{x}^2$

$f'(\bar{x}) = 1 - 2\bar{x} = 0 \quad \bar{x}_m = \frac{1}{2}$

$f(\bar{x}) = S^2 \leqslant f(\bar{x}_m) = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$.

$2 \cdot z_{0.005}\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} \leqslant \boxed{2 \cdot z_{0.005}\sqrt{\frac{1}{4n}}} \leqslant \boxed{0.01}$

$n \geqslant \frac{z^2_{0.005}}{4 \cdot 0.005^2} = 66,357.76$.

Q5. $X_i \sim Bin(1, \pi)$. , $\boxed{\sum x_i = R}$

Derive unbiased estimator of $\pi(1-\pi)$

$$\begin{cases} S^2_{MM} = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2 \\ E[S^2_{MM}] = \left(\frac{n-1}{n}\right)\sigma^2. \\ S^2 = \frac{n}{n-1} \cdot S^2_{MM} = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2. \end{cases}$$

$Y = \hat{\pi}(1-\hat{\pi}) = \frac{R}{n}(1-\frac{R}{n}) = \frac{R}{n} - \frac{R^2}{n^2}$

$R \sim Bin(n, \pi)$.

$E[R] = n\pi$. $\quad Var(R) = n\pi(1-\pi)$

$E[R^2] = n\pi(1-\pi) + n^2\pi^2$

$E[Y] = \frac{n\pi}{n} - \frac{1}{n^2}(n\pi(1-\pi) + n^2\pi^2)$.

$= \pi - \frac{\pi(1-\pi)}{n} - \pi^2$

$= \pi(1-\pi) - \frac{1}{n}\pi(1-\pi)$

$= \boxed{\left(\frac{n-1}{n}\right)} \boxed{\pi(1-\pi)}$

$E[\frac{n}{n-1}Y] = \pi(1-\pi)$

$\frac{n}{n-1}Y = \frac{n}{n-1}\left(\frac{R}{n}\left(1-\frac{R}{n}\right)\right)$