ST102 Class 5 – Solutions to Additional exercises

1. For completeness, we give here the expected values of these variables for all 11 patterns. For convenience, define $Y = 0$ when $X = 0$. In the following table, each value of X and Y is given, reading clockwise from the bottom-left corner of each square.

	X				E(X)	Y				E(Y)
[1]	0	0	0	0	0	0	0	0	0	0
[2]	0	1	1	0	0.5	0	1	1	0	0.5
[3]	1	1	1	1	1	1	1	1	1	1
[4]	1	2	1	0	1	2	1	2	0	1.25
[5]	1	2	2	1	1.5	2	1.5	1.5	2	1.75
[6]	1	3	1	1	1.5	3	1	3	3	2.5
[7]	2	2	2	0	1.5	2	2	2	0	1.5
[8]	2	2	2	2	2	2	2	2	2	2
[9]	2	2	3	1	2	2.5	2.5	1.67	3	2.42
[10]	2	3	3	2	2.5	3	2.33	2.33	3	2.67
[11]	3	3	3	3	3	3	3	3	3	3

For every pattern, either $\mathrm{E}(X) = \mathrm{E}(Y)$ or $\mathrm{E}(X) < \mathrm{E}(Y)$.

2. (a) The distribution of X is binomial with $n = 4$ and $\pi = 1/3$.

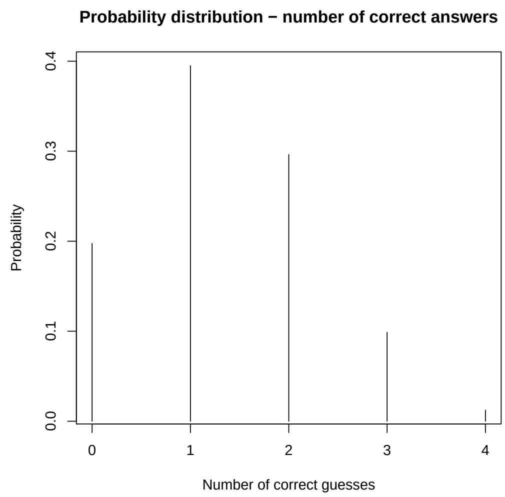
$X = x$	0	1	2	3	4	Total
$P(X = x)$	0.1975	0.3951	0.2963	0.0988	0.0123	1
$x\,P(X = x)$	0	0.3951	0.5926	0.2964	0.0492	1.33
$x^2 P(X = x)$	0	0.3951	1.1852	0.8892	0.1968	2.67

We have:

$$\mathrm{E}(X) = 1.33$$

and:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 2.67 - (1.33)^2 = 0.89.$$



(b) We have:

$$E(Y) = 10 + 22.5 \times E(X) = 10 + 22.5 \times 1.33 = 39.93$$

and:

$$\text{Var}(Y) = (22.5)^2 \times \text{Var}(X) = (22.5)^2 \times 0.89 = 450.6.$$

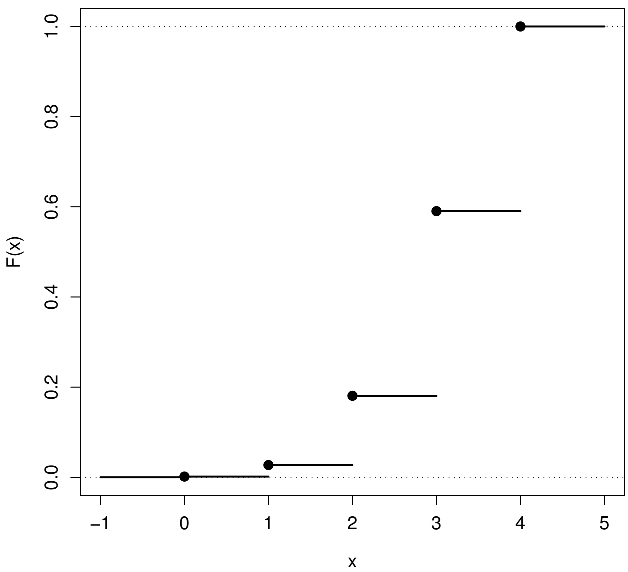
3. (a) The necessary binomial coefficients are:

$$\binom{4}{0} = \binom{4}{4} = 1, \quad \binom{4}{1} = \binom{4}{3} = \frac{4!}{3!\,1!} = 4 \quad \text{and} \quad \binom{4}{2} = 6.$$

The probabilities are:

$$p(0) = 1 \times (0.8)^0 \times (0.2)^4 = 0.0016$$
$$p(1) = 4 \times (0.8)^1 \times (0.2)^3 = 0.0256$$
$$p(2) = 6 \times (0.8)^2 \times (0.2)^2 = 0.1536$$
$$p(3) = 4 \times (0.8)^3 \times (0.2)^1 = 0.4096$$
$$p(4) = 1 \times (0.8)^4 \times (0.2)^0 = 0.4096.$$

(b) The cumulative distribution function looks like:



4. To wait for r heads to show up, suppose x flips are required. The last flip must be a head, with $r-1$ heads randomly appearing in the first $x-1$ flips. In each particular combination of heads and tails, there must be r heads by definition of the experiment, as well as $x-r$ tails (so adding together, x flips in total), with probability due to independence of:

$$\pi^r (1-\pi)^{x-r}.$$

There are $\binom{x-1}{r-1}$ combinations of outcomes with this probability. Hence we have:

$$p(x) = \begin{cases} \binom{x-1}{r-1}\pi^r (1-\pi)^{x-r} & \text{for } x = r, r+1, \ldots \\ 0 & \text{otherwise.} \end{cases}$$