ST102 Class 13 – Solutions to Additional exercises

1. We have:

$$\mathrm{E}(X) = \sum_{x=0}^{1} x\, p(x;\pi) = 0 \times (1-\pi) + 1 \times \pi = \pi.$$

   Hence:

$$\widehat{\pi} = \bar{X}$$

   is the method of moments estimator of π.

2. We have:

$$\mathrm{E}(X) = \int_0^\theta x\, \frac{2x}{\theta^2}\, \mathrm{d}x = \left[\frac{2x^3}{3\theta^2}\right]_0^\theta = \frac{2\theta}{3}.$$

   Setting:

$$\frac{2\widehat{\theta}}{3} = \bar{X}$$

   we obtain:

$$\widehat{\theta} = \frac{3\bar{X}}{2}$$

   as the method of moments estimator of θ.

3. Using the results derived in lectures, we have:

$$\widehat{\mu} = \widehat{k}\widehat{\theta} = \bar{X} \quad \text{and} \quad \widehat{\sigma}^2 = \widehat{k}\widehat{\theta}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

   Rearranging, we have:

$$\widehat{k} = \frac{\bar{X}}{\widehat{\theta}}$$

   which upon substituting gives:

$$\widehat{k}\widehat{\theta}^2 = \left(\frac{\bar{X}}{\widehat{\theta}}\right)\widehat{\theta}^2 = \bar{X}\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

   Therefore:

$$\widehat{\theta} = \frac{1}{n\bar{X}}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

   Finally:

$$\widehat{k} = \frac{\bar{X}}{\widehat{\theta}} = \frac{\bar{X}}{\sum_{i=1}^{n}(X_i - \bar{X})^2/(n\bar{X})} = \frac{n\bar{X}^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}.$$