ST102 Class 11 – Solutions to Additional exercises

1. Each possible sample has an equal probability of occurrence of 1/16. The sampling distribution of $\bar{X}$ is:

Possible samples	(1, 1)	(1, 2) (2, 1)	(1, 3) (3, 1) (2, 2)	(1, 4) (4, 1) (2, 3) (3, 2)	(2, 4) (4, 2) (3, 3)	(3, 4) (4, 3)	(4, 4)
Sample mean, $\bar{X} = \bar{x}$	1	1.5	2	2.5	3	3.5	4
Relative frequency, $P(\bar{X} = \bar{x})$	1/16	1/8	3/16	1/4	3/16	1/8	1/16

2. We have:

$$\mathrm{E}(X) = \int_0^1 x\, f(x)\, \mathrm{d}x = \int_0^1 x\, 3(1-x)^2\, \mathrm{d}x = \int_0^1 (3x-6x^2+3x^3)\, \mathrm{d}x = \left[\frac{3x^2}{2} - 2x^3 + \frac{3x^4}{4}\right]_0^1 = 0.25 = \mu$$

and:

$$\mathrm{E}(X^2) = \int_0^1 x^2 f(x)\, \mathrm{d}x = \int_0^1 x^2 3(1-x)^2\, \mathrm{d}x = \int_0^1 (3x^2-6x^3+3x^4)\, \mathrm{d}x = \left[x^3 - \frac{6x^4}{4} + \frac{3x^5}{5}\right]_0^1 = 0.10$$

hence:

$$\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 0.10 - (0.25)^2 = 0.0375.$$

Therefore, as $n \to \infty$, we have:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

so, for $n = 50$ (treated as 'large') we have:

$$\bar{X} \sim N\left(0.25, \frac{0.0375}{50}\right).$$

Therefore:

$$P(0.225 \leq \bar{X} \leq 0.275) \approx P\left(\frac{0.225 - 0.25}{\sqrt{0.0375/50}} \leq Z \leq \frac{0.275 - 0.25}{\sqrt{0.0375/50}}\right) = P(-0.91 \leq Z \leq 0.91) = 0.6372$$

using Table 3 of Murdoch and Barnes' *Statistical Tables*.

3. We require $P(\bar{X} \geq \bar{Y}) = P(\bar{X} - \bar{Y} \geq 0)$, where:

$$\mathrm{E}(\bar{X} - \bar{Y}) = \mathrm{E}(\bar{X}) - \mathrm{E}(\bar{Y}) = \mu_X - \mu_Y = 2 - 1 = 1$$

and, due to independence of $\bar{X}$ and $\bar{Y}$, we have:

$$\mathrm{Var}(\bar{X} - \bar{Y}) = \mathrm{Var}(\bar{X}) + \mathrm{Var}(\bar{Y}) = \frac{\sigma_X^2}{9} + \frac{\sigma_Y^2}{4} = \frac{2^2}{9} + \frac{1^2}{4} = \frac{25}{36}.$$

This yields:

$$\bar{X} - \bar{Y} \sim N\left(1, \frac{25}{36}\right)$$

therefore:

$$P(\bar{X} - \bar{Y} \geq 0) = P\left(\frac{\bar{X} - \bar{Y} - 1}{\sqrt{25/36}} \geq \frac{0 - 1}{\sqrt{25/36}}\right) = P(Z \geq -1.20) = 0.8849.$$