ST102 Class 12 – Additional exercises

1. Suppose $X_i \sim N(0, 3)$, for $i = 1, 2, 3, 4$. Assume all these random variables are independent. Derive the value of k in each of the following.

   (a) $P(X_1 + 3X_2 > 4) = k$.

   (b) $P(X_1^2 + X_2^2 + X_3^2 < k) = 0.90$.

   (c) $P(X_1 < k\sqrt{X_2^2 + X_3^2}) = 0.05$.

$Y_1 = \sum_{i=1}^{3} \left(\frac{X_i - 50}{10}\right)^2 = 6.5 \sim \chi^2_3$

2. A random sample of size $n = 3$ is observed such that $x_1 = 65$, $x_2 = 30$ and $x_3 = 55$. Using a chi-squared distribution, are the data consistent with being drawn from $N(50, 100)$?

$P(Z_1 < Y_1 < Z_3) = 0.95$

$[Z_1, Z_3]$

3. Of the following two differences:

$$t_{0.05,\,n} - t_{0.10,\,n} \quad \text{and} \quad t_{0.10,\,n} - t_{0.15,\,n}$$

   where $t_{\alpha,\,n}$ is such that $P(T > t_{\alpha,\,n}) = \alpha$ for $T \sim t_n$, which is larger?

4. Use the fact that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ to prove that:

$$\text{Var}(S^2) = \frac{2\sigma^4}{n-1}.$$

   Hint: Use the fact that the variance of a chi-squared random variable with k degrees of freedom is $2k$.

5. If $Y \sim \chi^2_n$, it can be shown that:

$$\frac{Y - n}{\sqrt{2n}} \to N(0, 1)$$

   as $n \to \infty$. Use the asymptotic normality of $(Y - n)/\sqrt{2n}$ to approximate the 40th percentile of a chi-squared random variable with 200 degrees of freedom.