ST102 Class 8 – Solutions to Additional exercises

1. Since $X \sim N(-7, 9)$, we use the transformation $Z = (X - \mu)/\sigma$ with the values $\mu = -7$ and $\sigma = \sqrt{9} = 3$. Note that here $(X + 7)/3 \sim N(0, 1)$, i.e. take care with the negative mean!

   (a) i. We have $P(X > 0.8) = P(Z > (0.8 + 7)/3) = P(Z > 2.60) = 0.00466$.

   ii. We have:

   $$\begin{aligned} P(-10 < X < -4) &= P\left(\frac{-10+7}{3} < Z < \frac{-4+7}{3}\right) \\ &= P(-1 < Z < 1) \\ &= P(Z < 1) - P(Z < -1) \\ &= \Phi(1) - \Phi(-1) \\ &= (1 - 0.1587) - 0.1587 \\ &= 0.6826. \end{aligned}$$

   (b) We want to find the value a such that $P(-7 - a < X < -7 + a) = 0.95$, that is:

   $$\begin{aligned} 0.95 &= P\left(\frac{(-7-a)+7}{3} < Z < \frac{(-7+a)+7}{3}\right) \\ &= P\left(-\frac{a}{3} < Z < \frac{a}{3}\right) \\ &= 1 - P\left(Z > \frac{a}{3}\right) - P\left(Z < -\frac{a}{3}\right) \\ &= 1 - 2 \times P\left(Z > \frac{a}{3}\right). \end{aligned}$$

   This is the same as $2 \times P(Z > a/3) = 0.05$, i.e. $P(Z > a/3) = 0.025$. Hence, from Table 3, $a/3 = 1.96$, and so $a = 5.88$.

   (c) We want to find the value b such that $P(-7 - b < X < -7 + b) = 0.99$. Similar reasoning shows that $P(Z > b/3) = 0.005$. Hence, from Table 3, $b/3 = 2.58$, so that $b = 7.74$.

   (d) We want z such that $P(Z > z) = 0.01$. From Table 3, $z = 2.33$.

   (e) We want z such that $P(Z < z) = 0.05$. This means that $z < 0$ and also $P(Z > |z|) = 0.05$, so, from Table 3, $|z| = 1.65$ and hence $z = -1.65$.

2. – We have:

   $$\begin{aligned} &P(X > s + t \mid X > t) \\ &= \frac{P((X > s+t) \cap (X > t))}{P(X > t)} \qquad \text{(by definition of conditional probability)} \\ &= \frac{P(X > s+t)}{P(X > t)} \qquad \text{(since if } X > s+t, \text{ also } X > t). \end{aligned}$$

Now consider the exponential distribution with parameter $\lambda > 0$. The numerator is $1 - F(s+t) = \mathrm{e}^{-\lambda(s+t)}$, and the denominator is $1 - F(t) = \mathrm{e}^{-\lambda t}$. Therefore:

$$P(X > s + t \,|\, X > t) = \frac{\mathrm{e}^{-\lambda(s+t)}}{\mathrm{e}^{-\lambda t}} = \mathrm{e}^{-\lambda s}.$$

However, also $P(X > s) = 1 - F(s) = \mathrm{e}^{-\lambda s}$, so $P(X > s + t \,|\, X > t) = P(X > s)$, as required.

- For X, the memoryless property states that if a light bulb has worked for t hours, from that point on the probability that it works at least another s hours is the same as the probability that a brand new light bulb works for at least s hours. In other words, a bulb 'forgets' that it has already worked for t hours. For the wine glass random variable Y, the interpretation is similar.

  In essence, the memoryless property states that an object does not wear out from use, so that an object always has the same expected additional lifetime regardless of how long it has already been used. Here this assumption seems more realistic for wine glasses than for light bulbs.

3. Use the random variable $X \sim N(38.74, (15.72)^2)$.

   (a) Using Table 3 of Murdoch and Barnes' *Statistical Tables*, we have:

$$\begin{aligned} P(35 < X < 45) &= P\left(\frac{35 - 38.74}{15.72} < Z < \frac{45 - 38.74}{15.72}\right) \\ &= P(-0.24 < Z < 0.40) \\ &= 0.6554 - 0.4052 \\ &= 0.2502 \quad \text{(i.e. about 25\%).} \end{aligned}$$

   (b) Let $38.74 \pm c$ be the required interval, where $c > 0$, so:

$$P(38.74 - c \leq X \leq 38.74 + c) = 0.50.$$

   It is convenient to write this as $P(38.74 \leq X \leq 38.74 + c) = 0.25$, by symmetry. This also means that $P(X \leq 38.74 + c) = 0.75$, since $P(X < 38.74) = 0.5$, because the normal distribution of X is symmetric around its mean (and median and mode) of 38.74.

   Standardise by way of $Z = (X - 38.74)/15.72$ to give:

$$P\left(Z \leq \frac{c}{15.72}\right) = 0.75$$

   i.e. $P(Z > c/15.72) = 0.25$. From Table 3, $P(Z > 0.67) \approx 0.25$.

   Therefore, we have $c/15.72 = 0.67$ and $c = 10.53$. Hence the required interval is 38.74 ± 10.53 or $(28.21, 49.27)$.

   (c) Here we need k such that $P(X > k) = 0.10$. Standardising:

$$P\left(Z > \frac{k - 38.74}{15.72}\right) = 0.10.$$

Table 3 tells us that $P(Z > 1.28) \approx 0.10$, so here:

$$\frac{k - 38.74}{15.72} = 1.28.$$

Hence $k = 1.28 \times 15.72 + 38.74 = 58.87$ hours.

4.* (a) Using the total probability formula:

$$\begin{aligned} P(T < 4) &= P(T < 4 \mid \text{no notice})p + P(T < 4 \mid \text{processing problems})q \\ &= p \int_0^4 \lambda \mathrm{e}^{-\lambda t}\, \mathrm{d}t + q \int_0^4 \frac{\lambda}{4}\, \mathrm{e}^{-\lambda t/4}\, \mathrm{d}t \\ &= p(1 - \mathrm{e}^{-4\lambda}) + q(1 - \mathrm{e}^{-\lambda}) \\ &= 1 - p\mathrm{e}^{-4\lambda} - q\mathrm{e}^{-\lambda}. \end{aligned}$$

(b) Using Bayes' theorem, we have:

$$P(\text{problems} \mid T > 4) = \frac{P(T > 4 \mid \text{problems})\, P(\text{problems})}{P(T > 4)} = \frac{q\mathrm{e}^{-\lambda}}{p\mathrm{e}^{-4\lambda} + q\mathrm{e}^{-\lambda}}.$$

(c) To find the value of x which makes the conditional probability $1/2$, consider:

$$\begin{aligned} P(\text{problems} \mid T > x) &= \frac{q \int_x^\infty (\lambda/4)\mathrm{e}^{-\lambda t/4}\, \mathrm{d}t}{p \int_x^\infty \lambda \mathrm{e}^{-\lambda t}\, \mathrm{d}t + q \int_x^\infty (\lambda/4)\mathrm{e}^{-\lambda t/4}\, \mathrm{d}t} \\ &= \frac{q\mathrm{e}^{-\lambda x/4}}{p\mathrm{e}^{-\lambda x} + q\mathrm{e}^{-\lambda x/4}} \\ &= \frac{q}{p\mathrm{e}^{-3\lambda x/4} + q}. \end{aligned}$$

Setting the above to $1/2$, we have:

$$x = \frac{4}{3\lambda} \ln\left(\frac{p}{q}\right).$$

(d) Since $q = 1 - p$, we have:

$$\mathrm{E}(T) = \mathrm{E}(T \mid \text{no notice})\, p + \mathrm{E}(T \mid \text{problems})\, q = \frac{p}{\lambda} + \frac{4q}{\lambda} = \frac{4 - 3p}{\lambda}.$$

and:

$$\mathrm{E}(T^2) = \mathrm{E}(T^2 \mid \text{no notice})\, p + \mathrm{E}(T^2 \mid \text{problems})\, q = \frac{2p}{\lambda^2} + \frac{2q}{(\lambda/4)^2} = \frac{2p + 32q}{\lambda^2} = \frac{32 - 30p}{\lambda^2}.$$

Hence:

$$\mathrm{Var}(T) = \mathrm{E}(T^2) - (\mathrm{E}(T))^2 = \frac{32 - 30p}{\lambda^2} - \frac{16 - 24p + 9p^2}{\lambda^2} = \frac{16 - 6p - 9p^2}{\lambda^2}.$$