ST102 Class 12 – Solutions to Additional exercises

1. (a) Since $X_1 + 3X_2 \sim N(0, 30)$, then:

$$P(X_1 + 3X_2 > 4) = P\left(\frac{X_1 + 3X_2}{\sqrt{30}} > \frac{4}{\sqrt{30}}\right) = P(Z > 0.73) = 0.2327$$

   where $Z \sim N(0, 1)$.

   (b) $X_i^2/3 \sim \chi_1^2$ for $i = 1, 2, 3$, hence $(X_1^2 + X_2^2 + X_3^2)/3 \sim \chi_3^2$, so:

$$P(X_1^2 + X_2^2 + X_3^2 < k) = P\left(X < \frac{k}{3}\right) = 0.90$$

   where $X \sim \chi_3^2$. Hence $k/3 = 6.251$, so $k = 18.753$.

   (c) $X_1/\sqrt{3} \sim N(0, 1)$ and $(X_2^2 + X_3^2)/3 \sim \chi_2^2$, hence:

$$\frac{X_1/\sqrt{3}}{\sqrt{\frac{(X_2^2+X_3^2)/3}{2}}} = \frac{\sqrt{2}X_1}{\sqrt{X_2^2 + X_3^2}} \sim t_2.$$

   Therefore:

$$P(T < \sqrt{2}k) = 0.05$$

   where $T \sim t_2$. Hence $\sqrt{2}k = -2.920$, so $k = -2.065$.

2. If $X_i \sim N(50, 100)$, for $i = 1, 2, 3$, then:

$$\frac{X_i - 50}{10} \sim N(0, 1) \quad \Rightarrow \quad \sum_{i=1}^{3} \left(\frac{X_i - 50}{10}\right)^2 \sim \chi_3^2.$$

   The central 95% of the χ_3^2 distribution lies in the interval $(0.216, 9.348)$ since:

$$\chi^2_{0.975,\,3} = 0.216 \quad \text{and} \quad \chi^2_{0.025,\,3} = 9.348.$$

   Given the observed random sample, we have:

$$\sum_{i=1}^{3} \left(\frac{x_i - 50}{10}\right)^2 = \left(\frac{65 - 50}{10}\right)^2 + \left(\frac{30 - 50}{10}\right)^2 + \left(\frac{55 - 50}{10}\right)^2 = 6.50.$$

   Since $0.216 < 6.50 < 9.348$, then the data are *consistent* with the original sample being drawn from $N(50, 100)$.

   Note this approach of checking whether sample data are ‘consistent’ with being drawn from a particular probability distribution is central to ‘classical’ hypothesis testing, which will be covered in Chapter 9.

3. Both differences represent intervals associated with 5% probability under a t_n distribution. However, because the pdf is closer to the x-axis the further t is away from 0, we must have that:

$$t_{0.05,\,n} - t_{0.10,\,n} > t_{0.15,\,n} - t_{0.10,\,n}.$$

4. Let $Y = (n-1)S^2/\sigma^2$, then (using the hint):

$$\mathrm{Var}(Y) = \frac{(n-1)^2\mathrm{Var}(S^2)}{\sigma^4} = 2(n-1).$$

   It follows that $\mathrm{Var}(S^2) = 2\sigma^4/(n-1)$.

5. Let $Y \sim \chi^2_{200}$, then:

$$\frac{Y-200}{\sqrt{400}} \doteq Z \sim N(0,1).$$

   Since:

$$P(Z \leq -0.25) \approx 0.40$$

   then:

$$P\left(\frac{Y-200}{\sqrt{400}} \leq -0.25\right) \approx 0.40.$$

   Equivalently, $Y \leq 200 - 0.25 \times \sqrt{400} = 195$, implying that the 40th percentile of a χ^2_{200} random variable is approximately 195. (The actual value is 194.3193.)