ST102 Class 6 – Solutions to Additional exercises

1. Since $k > 0$, then $2x/(k(k+1)) \geq 0$ for $x = 1, 2, \ldots, k$. Therefore, $p(x) \geq 0$ for all real x. Also, noting that:
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
we have:
$$\begin{aligned}\sum_{x=1}^{k} \frac{2x}{k(k+1)} &= \frac{2}{k(k+1)} + \frac{4}{k(k+1)} + \cdots + \frac{2k}{k(k+1)} \\ &= \frac{2}{k(k+1)}(1 + 2 + \cdots + k) \\ &= \frac{2}{k(k+1)} \frac{k(k+1)}{2} \\ &= 1.\end{aligned}$$
Hence $p(x)$ is a valid probability function.

2. Clearly, $f(x) \geq 0$ since $1/x^2 \geq 0$ for $x \geq 1$. Also:
$$\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_1^{\infty} = 1.$$
Hence $f(x)$ is a valid pdf. However, the expected value would be:
$$\mathrm{E}(X) = \int_1^{\infty} x\,\frac{1}{x^2}\,\mathrm{d}x = \int_1^{\infty} \frac{1}{x}\,\mathrm{d}x = \Big[\ln x\Big]_1^{\infty}$$
but this quantity is infinite.

3. From the lectures, we know that the cdf of this distribution is:
$$F(x) = 1 - \mathrm{e}^{-\lambda x} = 1 - \mathrm{e}^{-0.25x}.$$
The expected value is $\mathrm{E}(X) = 1/\lambda = 1/0.25 = 4$ and the standard deviation is $\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} = 1/\lambda = 4$. Using these, we get the following.
   (a) $P(X = 5) = 0$. The probability that a continuous distribution has any particular value *exactly* is always 0.
   (b) $P(X \leq 4) = F(4) = 1 - \mathrm{e}^{-0.25\times 4} = 1 - \mathrm{e}^{-1} = 0.6321$.
   (c) $P(X \geq 8) = P(X > 8) = 1 - P(X \leq 8) = 1 - F(8) = \mathrm{e}^{-0.25\times 8} = \mathrm{e}^{-2} = 0.1353$.

(d) We have:

$$\begin{aligned}
P(\mathrm{E}(X) - 0.5 \times \mathrm{sd}(X) \leq X \leq \mathrm{E}(X) + 0.5 \times \mathrm{sd}(X)) &= P(4 - 2 \leq X \leq 4 + 2) \\
&= P(2 \leq X \leq 6) \\
&= F(6) - F(2) \\
&= (1 - \mathrm{e}^{-1.5}) - (1 - \mathrm{e}^{-0.5}) \\
&= \mathrm{e}^{-0.5} - \mathrm{e}^{-1.5} \\
&= 0.3834.
\end{aligned}$$

4. (a) The area of the circle of points which are at most x units from the centre is $x^2\pi$ and the area of the whole board where the dart is certain to land (according to the statement in the question) is $(10)^2\pi = 100\pi$. Therefore:

$$F(x) = P(X \leq x) = \frac{x^2\pi}{100\pi} = \frac{x^2}{100}$$

for $0 \leq x \leq 10$. Furthermore, $F(x) = 0$ for all $x < 0$ and $F(x) = 1$ for all $x > 10$. In full:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2/100 & \text{for } 0 \leq x \leq 10 \\ 1 & \text{for } x > 10. \end{cases}$$

(b) Differentiating, $f(x) = F'(x) = x/50$ for $0 \leq x \leq 10$, and $f(x) = 0$ otherwise. In full:

$$f(x) = \begin{cases} x/50 & \text{for } 0 \leq x \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this satisfies $f(x) \geq 0$ for all x. Also:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^{10} \frac{x}{50}\,\mathrm{d}x = \frac{1}{100}\Big[x^2\Big]_0^{10} = \frac{1}{100}\Big[100 - 0\Big] = 1$$

so $f(x)$ satisfies the necessary conditions for being a probability density function.

(c) Since the radius of the circle which includes the innermost (i.e. 10) circle and the 9-ring is 2, the probability is:

$$P(X \leq 2) = F(2) = \frac{4}{100} = 0.04.$$