ST102 Class 4 – Solutions to Additional exercises

1. Six equally-likely orderings are possible for any set of three distinct random numbers, say x_1, x_2 and x_3. These are $x_1 < x_2 < x_3$, $x_1 < x_3 < x_2$, $x_2 < x_1 < x_3$, $x_2 < x_3 < x_1$, $x_3 < x_1 < x_2$ and $x_3 < x_2 < x_1$. By inspection, $P(A) = 2/6 = 1/3$ and $P(B) = 1/6$, so:

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18} = 0.0556.$$

2. Let A be the event that James A is in class, and let B be the event that James B is in class. Let C be the event that at least one of the Jameses is in class. Hence $C = A \cup B$.

   (a) We require $P(C)$. Since A and B are independent, we have $P(A \cap B) = P(A)\,P(B)$. Hence:

$$P(C) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.8 \times 0.6 = 0.92.$$

   (b) We require $P(A \,|\, C)$. Since $A \subset C$, we have $P(A \cap C) = P(A) = 0.8$. Therefore:

$$P(A \,|\, C) = \frac{P(A \cap C)}{P(C)} = \frac{0.8}{0.92} = 0.8696.$$

3. (a) The last toss must be a '1'. Therefore, there are three '1's among the first $x-1$ tosses. Hence the probability of x tosses to obtain four '1's is (noting the independence of the tosses):

$$\binom{x-1}{3} \left(\frac{5}{6}\right)^{x-4} \left(\frac{1}{6}\right)^{4} = \frac{(x-1)!}{3!\,(x-4)!} \left(\frac{5}{6}\right)^{x-4} \left(\frac{1}{6}\right)^{4}$$
$$= (x-1)(x-2)(x-3) \left(\frac{5}{6}\right)^{x-4} /6^5.$$

   Therefore:

$$p(x) = \begin{cases} 5^{x-4}(x-1)(x-2)(x-3)/6^{x+1} & \text{for } x = 4, 5, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

   (b) We are given that $p(x)$ is a valid probability function, hence:

$$\sum_{x=4}^{\infty} p(x) = \sum_{x=4}^{\infty} 5^{x-4}(x-1)(x-2)(x-3)/6^{x+1}$$
$$= \sum_{x=4}^{\infty} (x-1)(x-2)(x-3) \left(\frac{5}{6}\right)^{x-4} /6^5$$
$$= 1.$$

   Therefore:

$$\sum_{x=4}^{\infty} (x-1)(x-2)(x-3) \left(\frac{5}{6}\right)^{x-4} = 6^5 = 7{,}776.$$

4. One has to consider whether such glimpses are really possible. It seems doubtful whether the spectator failed to distinguish the distinctive ace of spades from the other aces – especially the red aces. As such, knowing in (b) that the card was a black ace would suggest the ace of clubs. In practice, your (subjective) opinion about the spectator's eyesight, memory, honesty etc. would affect your probability calculations.

   Regardless, let us define the events:

   - $A_i = \{\text{the hand has at least } i \text{ aces}\}$
   - $B = \{\text{the hand has a black ace}\}$
   - $S = \{\text{the hand has the ace of spades}\}$.

   The required conditional probabilities can be obtained with R by calculating the relevant classical probabilities.

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> A2 <- choose(4,2)*choose(48,3) + choose(4,3)*choose(48,2) +
+ choose(4,4)*choose(48,1)
> A1 <- A2 + choose(4,1)*choose(48,4)
> B <- choose(2,1)*choose(50,4) + choose(2,2)*choose(50,3)
> S <- choose(51,4)
> BA2 <- choose(2,1)*choose(2,1)*choose(48,3) +
+ choose(2,1)*choose(2,2)*choose(48,2) +
+ choose(2,2)*choose(50,3)
> SA2 <- choose(3,1)*choose(48,3) +
+ choose(3,2)*choose(48,2) + choose(48,1)
> A2/A1
[1] 0.1221849
> BA2/B
[1] 0.1895877
> SA2/S
[1] 0.2213685
```

   Hence we have the following.

   (a) $P(\text{at least two aces} \mid \text{an ace seen}) = 0.1221849$.
   (b) $P(\text{at least two aces} \mid \text{a black ace seen}) = 0.1895877$.
   (c) $P(\text{at least two aces} \mid \text{ace of spades seen}) = 0.2213685$.
   (d) The first conditional probability is $P(A_2 \mid A_1)$, while the second is $P(A_2 \mid B)$, and the third is $P(A_2 \mid S)$. On first thought, you might think that knowledge of an ace being in the hand does not have much/any impact on the *other* cards in the hand. However, the conditional probabilities increase non-negligibly as we learn more information about the observed ace!