# ST102 Class 3 – Solutions to Additional exercises

1. [1] Just 1 possibility: No-one is connected with anyone else.

   [2] 6: As given in the question.

   [3] 3: This looks similar to [2], but is not quite the same. Here it is sufficient to pick just one person, and choose the connection for that person. For example, person A can be paired with each of the 3 other people. Once that is done, the remaining two people must be connected with each other. The full set of patterns is (AB, CD), (AC, BD) and (AD, BC).

   [4] 12: The person with two connections can be chosen in 4 ways, and then the person with no connections in 3 ways. The remaining two are then those with one connection.

   [5] 12: There are 6 ways of choosing the two people with 2 connections each. For each such choice, the remaining two can be in two different orders, depending on who is connected with whom. For example, if AB is the pair in the middle, the remaining connections can be (AC, BD) or (AD, BC).

   [6] 4: Choosing the person with three connections determines the rest.

   [7] 4: Choosing the person with no connections determines the rest.

   [8] 3: This uses the same logic as [3]. Starting with one person (for example, A), there are 3 ways of choosing the person with which the first person is *not* connected. Once these are selected, the remaining two people must also be unconnected with each other. The full set of patterns, identified by the pairs who are unconnected, is (AB, CD), (AC, BD) and (AD, BC).

   [9] 12: First choose the person with one connection (4 choices) and then the person with which the first is connected (3 choices).

   [10] 6: The number of choices of the pair who are not connected with each other.

   [11] Just 1 possibility: The network where everyone is connected with everyone else.

   Therefore, the total is $2 \times 1 + 2 \times 4 + 2 \times 3 + 2 \times 6 + 3 \times 12 = 64$ different networks of connections.

2. There are two ways to approach this.

   Method 1: This involves use of the total probability formula. There are three different types of square on a chessboard, which we could describe as corner squares (C), side squares (S) and middle squares (M):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| C | S | S | S | S | S | S | C |
| S | M | M | M | M | M | M | S |
| S | M | M | M | M | M | M | S |
| S | M | M | M | M | M | M | S |
| S | M | M | M | M | M | M | S |
| S | M | M | M | M | M | M | S |
| S | M | M | M | M | M | M | S |
| C | S | S | S | S | S | S | C |

There are 4 C squares, 24 S squares and 36 M squares. Looking at the diagram, C squares have 2 adjacent squares, S squares have 3 adjacent squares, and M squares have 4 adjacent squares.

So:

$$
\begin{aligned}
&P(\text{two squares selected are adjacent}) \\
&= P(\text{first square C})\, P(\text{second square adjacent} \mid \text{first square C}) \\
&\quad + P(\text{first square S})\, P(\text{second square adjacent} \mid \text{first square S}) \\
&\quad + P(\text{first square M})\, P(\text{second square adjacent} \mid \text{first square M}) \\
&= \frac{4}{64} \times \frac{2}{63} + \frac{24}{64} \times \frac{3}{63} + \frac{36}{64} \times \frac{4}{63} \\
&= \frac{8 + 72 + 144}{64 \times 63} \\
&= \frac{1}{18} \\
&= 0.0556.
\end{aligned}
$$

<u>Method 2</u>: This involves counting the possibilities.

The probability of choosing two adjacent squares is:

$$\frac{\text{number of ways of choosing two adjacent squares}}{\text{total number of ways of choosing two squares}}.$$

The denominator is $\binom{64}{2} = (64 \times 63)/2 = 32 \times 63 = 2{,}016$.

We can choose two adjacent squares in 7 ways from the first row, 7 ways from the second row, ... and 7 ways from the last row, i.e. we can choose a horizontal pair of adjacent squares in $8 \times 7 = 56$ ways. Similarly, we can choose two adjacent squares in 7 ways from the first column, 7 ways from the second column, ... and 7 ways from the last column, i.e. we can choose a vertical pair of adjacent squares in $8 \times 7 = 56$ ways. So the total number of ways of choosing two adjacent squares is $2 \times 56 = 112$ ways. This is the numerator required.

Therefore, the probability is $112/2{,}016 = 1/18 = 0.0556$.

3. (a) Once the initial 50 fish have been tagged and returned to the lake, there are 50 tagged fish and $N - 50$ untagged fish, where $N = 50, 51, \ldots$. Catching 50 fish again from the $N$ fish can be achieved in $\binom{N}{50}$ ways (size of the sample space), and it is possible to catch three tagged fish and 47 untagged fish in $\binom{50}{3} \times \binom{N-50}{47}$ ways (the observed event). Therefore, the required probability is:

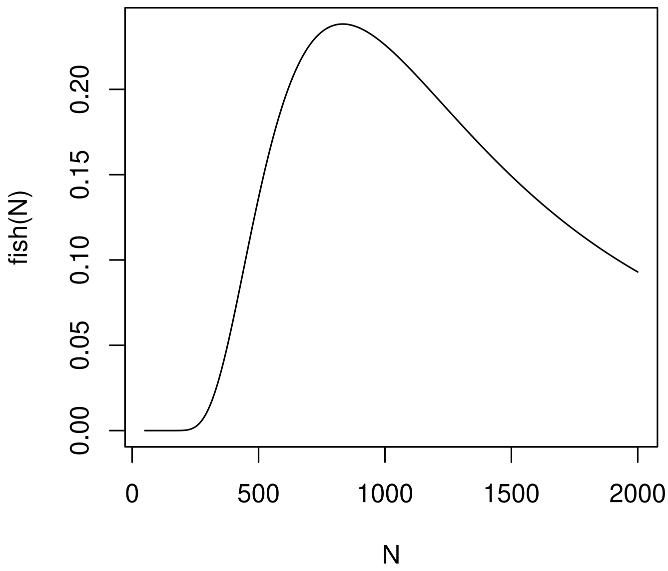
$$\frac{\binom{50}{3} \times \binom{N-50}{47}}{\binom{N}{50}}.$$

In R this can be computed by creating a simple function called 'fish', say:

```
> fish <- function(N) choose(50,3)*choose(N-50,47)/choose(N,50)
```

If we specify a realistic range of values of $N$, such as 500 to 2,000, we can easily plot these as a line graph as follows:

```
> N <- 50:2000
> plot(N, fish(N), type="l")
```




(b) Using R, the maximum can be determined as:

```
> max(fish(N))
[1] 0.2382917
```

although what we really want is the value of $N$ which achieves this maximum. To obtain this, we seek the component(s) of the vector $N$ which result in the maximum. Here, we can see from the above plot that there is only one such component.

```
> N[fish(N)==max(fish(N))]
[1] 833
```

(c) A fish population size of 833 is most likely given the evidence of three tagged fish in the second catch of 50 fish. However, values of $N$ in the vicinity of 833 produce similarly probable outcomes, for example:

```
> fish(835)
[1] 0.238289
```

which, as with the maximum probability, is just under 24%. So none of the values of $N$ are *very* likely. However, based on the data evidence, our best estimate would be around 830 to 835.

We will formally cover *maximum likelihood estimation* in Chapter 7, although this example illustrates the main principle, i.e. we estimate something (here it is $N$) with the most plausible value conditional on the data observed!