ST102 Class 7 – Solutions to Additional exercises

1. We have:

$$M_X(t) = \mathrm{E}(\mathrm{e}^{tX}) = \sum_{x=1}^{k} \mathrm{e}^{tx}\,\frac{1}{k} = \frac{\mathrm{e}^t}{k} + \frac{\mathrm{e}^{2t}}{k} + \cdots + \frac{\mathrm{e}^{kt}}{k} = \frac{1}{k}(\mathrm{e}^t + \mathrm{e}^{2t} + \cdots + \mathrm{e}^{kt}).$$

The bracketed part of this expression is a geometric series with a first term of e^t and a common ratio also of e^t. Hence:

$$M_X(t) = \frac{1}{k}\,\frac{\mathrm{e}^t(1-(\mathrm{e}^t)^k)}{1-\mathrm{e}^t} = \frac{\mathrm{e}^t(1-\mathrm{e}^{kt})}{k(1-\mathrm{e}^t)}.$$

2.* This is the proof of the Poisson approximation of the binomial distribution. From the second step onwards, we substitute $\pi = \lambda/n$. We have:

$$\begin{aligned}
\lim_{n\to\infty} P(X=x) &= \lim_{n\to\infty} \binom{n}{x} \pi^x (1-\pi)^{n-x} \\
&= \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \\
&= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(\frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x} \right) \\
&= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) \right) \\
&\quad \times \lim_{n\to\infty} \left(\left(1-\frac{\lambda}{n}\right)^n \right) \times \lim_{n\to\infty} \left(\left(1-\frac{\lambda}{n}\right)^{-x} \right) \\
&= \frac{\lambda^x}{x!} \times 1 \times \mathrm{e}^{-\lambda} \times 1 \\
&= \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!} \\
&= P(Y=x)
\end{aligned}$$

where we have used the result that:

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = \mathrm{e}^{-\lambda}$$

applying the general result stated in the second hint to the question.

3. (a) Let X denote the number of cars which pass in 2 minutes. Since a rate of 150 cars per hour is a rate of 5 per two minutes, $X \sim \text{Poisson}(5)$. Using this distribution:

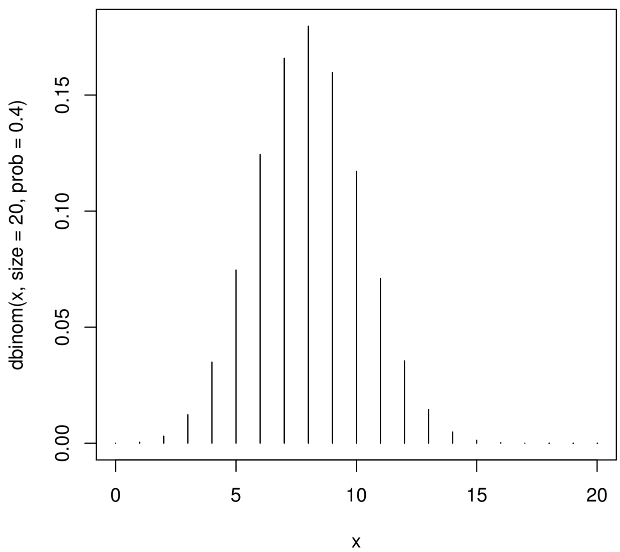
$$P(X=0) = \frac{\mathrm{e}^{-5}(5)^0}{0!} = \mathrm{e}^{-5} = 0.0067.$$

(b) Let Y denote the number of cars passing in one minute. This has a Poisson distribution with mean $\lambda = 150/60 = 2.5$, so the expected number of cars passing in one minute is $\mathrm{E}(Y) = \lambda = 2.5$.

(c) The probability of more than 2 cars passing in one minute is given by $P(Y > 2) = 1 - P(Y \leq 2)$, which is:

$$1 - \sum_{x=0}^{2} \frac{\mathrm{e}^{-2.5}(2.5)^x}{x!} = 1 - (0.0821 + 0.2052 + 0.2565) = 0.4562.$$

4. (a) The function `dbinom` calculates point probabilities for the specified binomial distribution. The use of 'd' is due to *d*ensity, although this makes more sense for continuous distributions! Note `dbinom` takes three arguments: (i) the values of X, (ii) the number of trials, n, and (iii) the probability of success, π. The `type="h"` is to specify a *h*istogram. We have:



(b) The function `pbinom` calculates cumulative *p*robabilities. We have:

```
> pbinom(5,size=20,prob=0.4)
[1] 0.125599
```

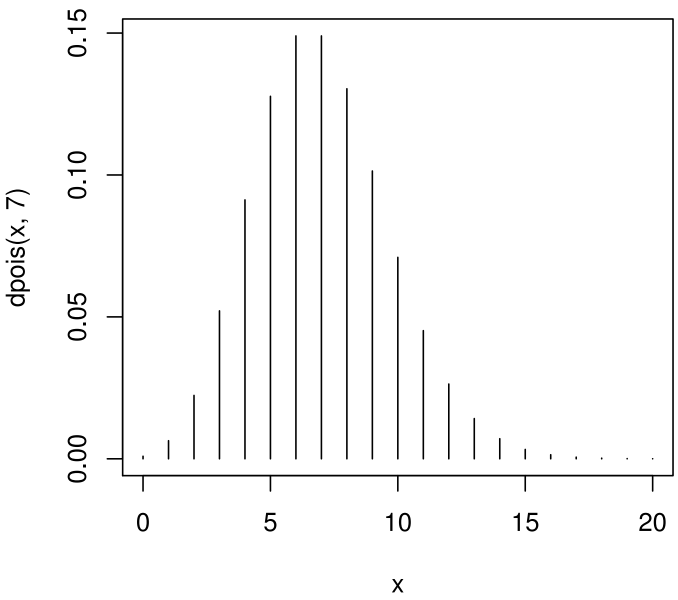
hence $P(X = 5)$ is:

```
> pbinom(5,size=20,prob=0.4)-pbinom(4,size=20,prob=0.4)
[1] 0.07464702
```

(c) The function `rbinom` returns a *r*andom sample of the specified size. For example:

```
> rbinom(10,size=20,prob=0.4)
 [1] 12  6  8  9  5 10  6 12  9  8
```

5. (a) The function dpois calculates point probabilities for the specified Poisson distribution. The use of 'd' is due to *d*ensity, although this makes more sense for continuous distributions! Note dpois takes two arguments: (i) the values of X, and (ii) the rate parameter λ. The type="h" is to specify a *h*istogram. We have:



(b) The function ppois calculates cumulative *p*robabilities. We have:

```
> ppois(5,7)
[1] 0.3007083
```

hence $P(X = 5)$ is:

```
> ppois(5,7)-ppois(4,7)
[1] 0.1277167
```

(c) The function rpois returns a *r*andom sample of the specified size. For example:

```
> rpois(10,7)
 [1] 5  8  8  7  9 12  7  7  8  7
```