NLP 201: FSAs and Regular Expressions

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Slides adapted from Chris Dyer

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The plan for the next few lectures

For the next few lectures, we'll start with the simplest models of language.

- DFAs and NFAs (today)
- Regular languages (today)
- Finite state transducers (FSTs) and applications

Definition: DFA

A **deterministic finite automaton** is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

Q is a finite **set of states**

Σ is a finite **alphabet**

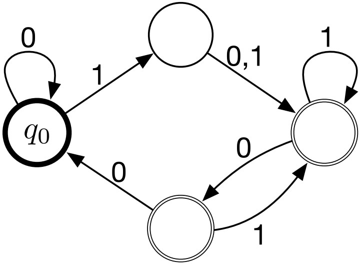
$\delta : Q \times \Sigma \to Q$ is the **transition function**

$q_0 \in Q$ is the **start (initial) state**

$F \subseteq Q$ is the **set of final (accept) states**

$L(M) \subseteq \Sigma^*$ is **the language of** M, i.e. the set of strings M accepts

DFAs



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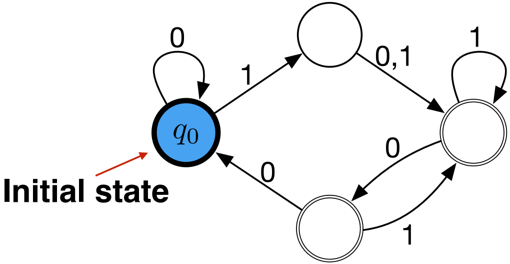

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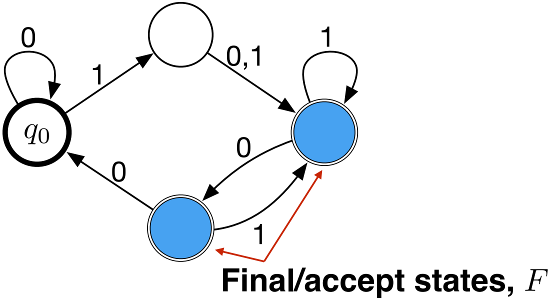

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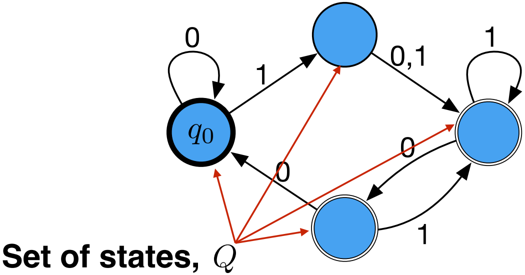

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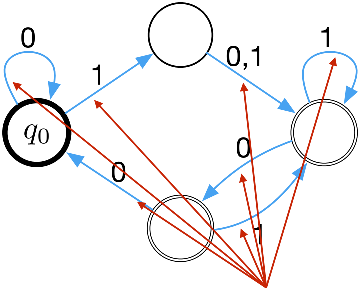


Transition function ("edges"), δ

Acceptance

- M is a DFA with alphabet Σ
- An input sentence (word / string) ***w*** is a sequence $w_1 w_2 w_3 \ldots w_n$ where each $w_i \in \Sigma$
- M accepts ***w*** if after starting in q_0 and reading ***w*** it ends in an accept state, i.e., if there is a sequence of states $s_0, s_1, \ldots, s_n$ such that

$$s_0 = q_0$$
$$s_i = \delta(s_{i-1}, w_i)$$
$$s_n \in F$$

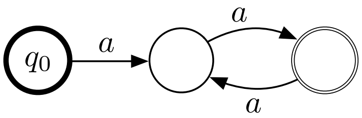
Acceptance and Analysis

- In many problems in natural language processing, we want to analyze a strings in terms of underlying operations.
- With finite state models, this usually corresponds to the question: what was the sequence of states $s_0,s_1,\ldots,s_n$ taken to produce some $\boldsymbol{w}$?

Example

The following DFA accepts the language

$L = \{a^{2n} \mid n \geq 1\}$



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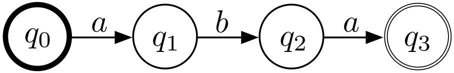
Remarks

- A DFA is required to have a **complete** transition function, whereas an NFA can have an **incomplete** transition function

- This is useful for specifying simple language generating automata, e.g. “linear chain” automata

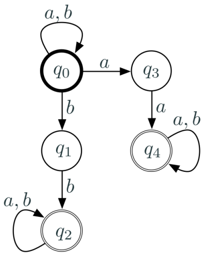


$$L(M) = \{aba\}$$

Board Work: NFA Example

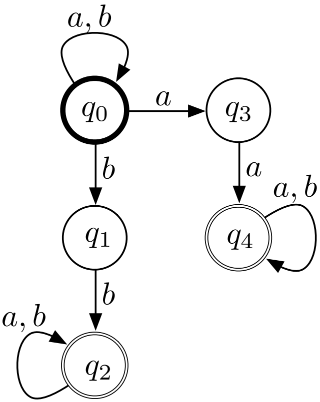
(5 Min) For the following NFA, does it accept the string $\mathbf{w} = abaa$? What sequence of states does it go through?

Can you describe in English the strings that it accepts?

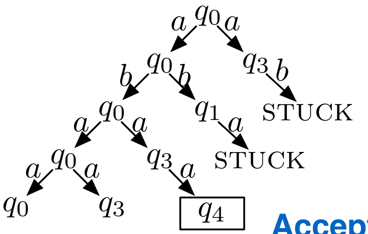


(3 Min) Discuss with a partner.

Example: NFA



Computation on $\boldsymbol{w} = abaa$



Accept!

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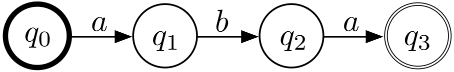

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DFA & NFA Equivalence

Theorem. For every NFA A there exists a DFA A' s.t. $L(A) = L(A')$.

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Example regular languages:

$$L = \Sigma^*$$

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Example regular languages:

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Example non-regular languages (more later):

$$L = \{a^n b^n \mid n \geq 0\}$$
$$L = \{\boldsymbol{w} \mid \boldsymbol{w} \text{ is a grammatical sentence of English}\}$$

Generalization

It is helpful to generalize the transition function of DFAs in terms of **words** (instead of single symbols).

$$\delta(q, a) = q'$$

$$\hat{\delta} : Q \times \Sigma^* \to Q$$

$$\hat{\delta}(q, \varepsilon) = q$$

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Formal definition of $L(M)$

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We also generalize in terms of **sets of states**:

$$P \in 2^Q$$

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Formal definition of $L(M)$

$$L(M) = \{\boldsymbol{w} \in \Sigma^* \mid \delta(q_0, \boldsymbol{w}) \cap F \neq \emptyset\}$$

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DFA & NFA Equivalence

Theorem. For every NFA A there exists a DFA A' s.t. $L(A) = L(A')$.

Proofs

- Proofs in formal language theory often use **set containment** and **induction**

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- **Set containment**
  To prove that $(L_1 \cup L_2).L_3 = L_1.L_3 \cup L_2.L_3$

  one can show that (i)
  $\mathbf{w} \in (L_1 \cup L_2).L_3 \implies \mathbf{w} \in L_1.L_3 \cup L_2.L_3$

  and the other direction that (ii)
  $\mathbf{w} \in L_1.L_3 \cup L_2.L_3 \implies \mathbf{w} \in (L_1 \cup L_2).L_3$

Proofs

- Proofs in formal language theory often use **set containment** and **induction**
- **Induction**
  Induction is usually on the length of the word, i.e. to prove P is true for all $\mathbf{w} \in \mathbf{L}$, we:
  1. show P is true for words length 0 or 1
  2. hypothesize that P is true for all words $\leq$ length $n-1$
  3. prove that if P is true for all words $\leq$ length $n-1$, then P is true for for all words of length n

Board work: Proof by induction

On your whiteboards, take 5 min to do 1. and 3.
Then discuss with a partner for 3 min

- **Induction example**
  Prove that $\forall n \geq 0$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

1. base: verify for $n = 0$
2. hypothesis: assume true for $n \leq k - 1$
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Proofs

- **Induction example**
  Prove that $\forall n \geq 0$
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  1. base: $n = 0$: $2^0 = 1 = 2^{0+1} - 1$
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$$\begin{aligned}\sum_{i=0}^{k} 2^i &= 2^k + \sum_{i=0}^{k-1} 2^i \\ &= 2^k + 2^k - 1 \quad \text{(by inductive hypothesis)} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1 \quad \square\end{aligned}$$

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Proof. *This is a constructive proof.* That is, given an NFA $A = \langle Q, \Sigma, \delta, q_0, F \rangle$ we construct a DFA $A' = \langle Q', \Sigma', \delta', q_0', F' \rangle$ s.t. $L(A) = L(A')$

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Powerset construction.

$\Sigma' = \Sigma$

$Q' = 2^Q$ Constructed DFA states are **sets** of NFA states

$q_0' = \{q_0\}$

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Transition function.

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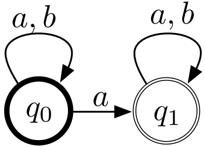
$\delta'(\emptyset, \sigma) = \emptyset \; \forall \sigma \in \Sigma'$ "Failure state"

$$\delta'(\{q_1, q_2, \ldots, q_i\}, \sigma) = \bigcup_{q \in \{q_1, q_2, \ldots, q_i\}} \delta(q, \sigma)$$

Example



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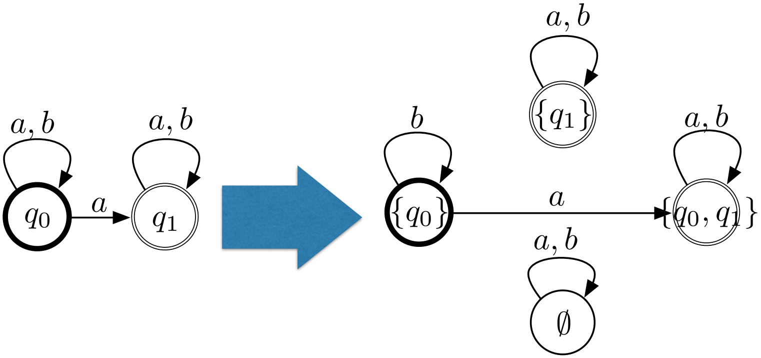

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$$\delta'(\{p_1, p_2, \dots, p_k\}, \sigma) = \{r_1, r_2, \dots, r_j\} \iff \delta(\{p_1, p_2, \dots, p_k\}, \sigma) = \{r_1, r_2, \dots, r_j\}$$

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Proof of theorem.

By definition, A' is a complete function, so it is a DFA

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Proof of theorem.

By definition, A' is a complete function, so it is a DFA

From the lemma,

$$\delta'(q_0', \boldsymbol{w}) = \{p_1, p_2, \ldots, p_j\} \iff \delta(q_0, \boldsymbol{w}) = \{p_1, p_2, \ldots, p_j\}$$

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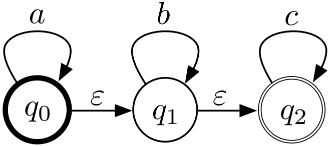
Remarks

- NFAs therefore only accept regular languages
- NFAs and DFAs can be used interchangeably

NFA with Epsilons

A **NFA with ε-transitions** is an NFA that may change states without reading an input symbol.



NFA with Epsilons

A **nondeterministic finite automaton** is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

Q is a finite **set of states**

Σ is a finite **alphabet**

~~$\delta : Q \times \Sigma \to 2^Q$ is the **transition relation**~~

$\delta : Q \times \Sigma \cup \{\varepsilon\} \to 2^Q$ is the **transition relation**

$q_0 \in Q$ is the **start (initial) state**

$F \subseteq Q$ is the **set of final (accept) states**

$L(M) \subseteq \Sigma^*$ is **the language of** M, i.e. the set of strings M accepts

NFA& ε-NFA Equivalence

Theorem. For every NFA A with epsilon moves there is an equivalent NFA A' without, s.t. $L(A) = L(A')$

Regular Expression

A regular expression is a way of describing the languages accepted by FSAs.

Defined recursively:

1. $\emptyset$ is an RE denoting the empty set
2. ε is an RE denoting the set $\{\varepsilon\}$
3. for each $a \in \Sigma$, a is a RE denoting $\{a\}$
4. If `r` and `s` are REs denoting the languages R and S

   `(r|s)` denotes $R \cup S$

   `(rs)` denotes $R.S$

   `r*` denotes R^*

Precedence means parentheses can sometimes be omitted: `*.|`

Examples

`(0|1)*` denotes all finite words over $\Sigma = \{0, 1\}$

`0*|1*` denotes all finite words containing only 0's and 1's

Board Work: Regex Examples

(5 Min) For the following languages, write it's regular expression

- Strings over $\{a, b\}^*$ that start with an a
- Strings over $\{a, b\}^*$ that contain ab or ba

(3 Min) Discuss with a partner.

Regex in practice

Most programming languages, grep, etc use a slightly different syntax. By default, the regex only needs to match a substring.

Character	Meaning	Example
*	Match **zero, one or more** of the previous	`Ah*` matches "`Ahhhhh`" or "`A`"
?	Match **zero or one** of the previous	`Ah?` matches "`Al`" or "`Ah`"
+	Match **one or more** of the previous	`Ah+` matches "`Ah`" or "`Ahhh`" but not "`A`"
\	Used to **escape** a special character	`Hungry\?` matches "`Hungry?`"
.	Wildcard character, matches **any** character	`do.*` matches "`dog`", "`door`", "`dot`", etc.
( )	**Group** characters	See example for |
[ ]	Matches a **range** of characters	`[cbf]ar` matches "car", "bar", or "far" `[0-9]+` matches any positive integer `[a-zA-Z]` matches ascii letters a-z (uppercase and lower case) `[^0-9]` matches any character not 0-9.
|	Matche previous **OR** next character/group	`(Mon|Tues)day` matches "Monday" or "Tuesday"
{ }	Matches a specified **number of occurrences** of the previous	`[0-9]{3}` matches "315" but not "31" `[0-9]{2,4}` matches "12", "123", and "1234" `[0-9]{2,}` matches "1234567..."
^	**Beginning** of a string. Or within a character range `[]` negation.	`^http` matches strings that begin with http, such as a url. `[^0-9]` matches any character not 0-9.
$	**End** of a string.	`ing$` matches "exciting" but not "ingenious"

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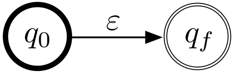

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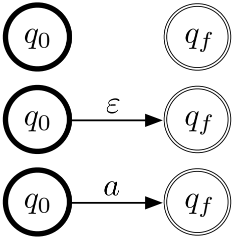

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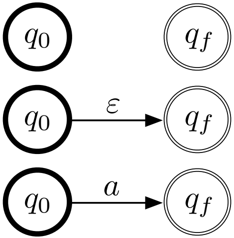

Note: we assume there is **exactly one** final state.

REs and ε-NFAs

Inductive step. We assume hypothesis is true for all REs with $\leq n$ operations, and then prove is true for $n+1$ operations.

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There are three cases to be dealt with:

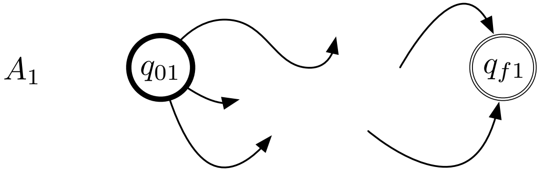
(1) $\mathtt{r} = \mathtt{r}_1 \mid \mathtt{r}_2$

(2) $\mathtt{r} = \mathtt{r}_1\mathtt{r}_2$

(3) $\mathtt{r} = \mathtt{r}_1\mathtt{*}$

Case 1: $\mathtt{r} = \mathtt{r}_1 \mid \mathtt{r}_2$

By the inductive hypothesis, there are two epsilon NFAs A_1 and A_2.



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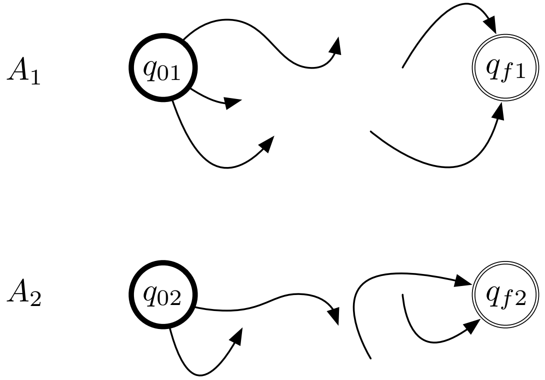

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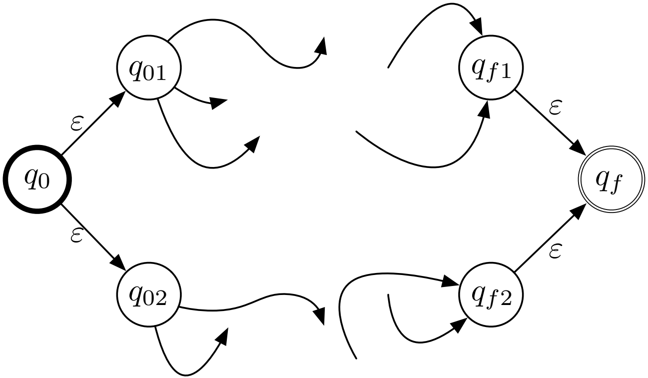

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Formally, if $A_1 = \langle Q_1, \Sigma, \delta_1, q_{01}, \{q_{f1}\} \rangle$

$$A_2 = \langle Q_2, \Sigma, \delta_2, q_{02}, \{q_{f2}\} \rangle$$

then,

$A = \langle Q_1 \cup Q_2 \cup \{q_0\} \cup \{q_f\}, \Sigma, \delta, q_0, \{q_f\} \rangle$

$$\begin{aligned}
\delta(q_0, \varepsilon) &= \{q_{01}, q_{02}\} \\
\delta(q_{f1}, \varepsilon) &= \{q_f\} \\
\delta(q_{f2}, \varepsilon) &= \{q_f\} \\
\delta(q, \sigma) &= \delta_1 q, \sigma \quad \forall q \in Q_1 - \{q_{f1}\}, \sigma \in \Sigma \cup \{\varepsilon\} \\
\delta(q, \sigma) &= \delta_2 q, \sigma \quad \forall q \in Q_2 - \{q_{f2}\}, \sigma \in \Sigma \cup \{\varepsilon\}
\end{aligned}$$

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It remains to show that $L(A) = L(A_1) \cup L(A_2)$

How to do this? Set containment.

Cases 2 & 3

- Strategy for showing this proceeds as with Case 1
- Refer to textbook for details.