NLP 201: Finite State Automata

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Fall 2023

Learning Objectives of NLP 201-203

- Ability to take a description of a model or system and implement it
- Ability to read current papers, understand them, and implement them
- Knowledge of lots of methods for solving problems (including older or alternative methods)
- Intuition into why the methods work and why they were developed
- Theoretical knowledge of why the methods work
- Debugging skills (difficult skill, *very important*)
- Ability to develop new solutions to problems

The plan for the next few lectures

For the next few lectures, we'll start with the simplest models of language.

- Finite state automata (FSAs, **today**)
- Regular languages
- Finite state transducers (FSTs)
- Later: Weighted FSAs and FSTs (WFSAs and WFSTs)

Why learn FSAs?

- Many models (HMMs, n-gram LMs, etc) are special cases of WFSAs
- FSAs and FSTs are used to model morphology
- First step in the formal language hierarchy
- Useful for understanding the modeling capacity of deep NN (for example, A Formal Hierarchy of RNN Architectures `https://arxiv.org/pdf/2004.08500.pdf`)
- Another tool for your toolbox
  - FSAs are incredibly fast

Goals of Today's Lecture

- Understand concepts of **formal languages**
- Understand **operations** on formal languages
- Refresher on proof techniques of **set containment** and **induction**
- Understand two FSAs: **deterministic finite automata** (DFAs), **nondeterministic finite automata** (NFAs) and their equivalence using the **powerset construction**.

Natural languages

- **Natural languages** are languages learned without explicit instruction by children from speakers in their environments
    - e.g., English, Mandarin, Hindi, Cantonese, German
- Natural languages link sounds or visual information (**linguistic forms**) with conceptual/intentional representations (**linguistic functions**)

Linguistic Concept: Grammar

Native speakers have a **grammar**, a (infinite) set of sentences that they accept as being "well-formed" **a sentence can either be in the speaker's grammar, or not in the speaker's grammar**

Example:

- They says "hello."
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What about:

- Colorless green ideas sleep furiously. **Another example: "Colorless green idea sleep furiously." doesn't work, but "A colorless green idea sleeps furiously" does.**

Formal languages

- **Formal languages** are (usually) infinite sets of strings described mathematically
- Formal languages may be used to describe things that are not natural languages, e.g. computer languages

Formal languages

- In this lecture, I will use the terms
  - **alphabet**, the symbols in the language
  - **word**, a string of symbols

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  - **alphabet**, the symbols in the language
  - **word**, a string of symbols
- Could also have used the terms
  - **vocabulary**, the symbols in the language
  - **sentence**, a string of symbols

Fundamental definitions

Σ is a finite, nonempty **alphabet**

Each symbol is σ in the set is a "letter"

Usually denoted by lowercase letters $a, b, c, \ldots$

A **word** (sentence, string) $\boldsymbol{w} = w_1 w_2 \ldots w_n$

Usually denoted by bold, italic lowercase $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$

$|\boldsymbol{x}|$ is the **length** of word $\boldsymbol{x}$

ε is the **empty word**, and $|\varepsilon| = 0$

A **language** L is a *set* (finite or infinite) of words from a given alphabet Σ

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Careful!

$L_1 = \{\varepsilon\}$ is **not the same** as $L_2 = \emptyset$

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Formal language operations

Set theoretic **operations** apply to languages

$$L_1 \cup L_2 = \{\boldsymbol{w} \in \Sigma^* \mid \boldsymbol{w} \in L_1 \vee \boldsymbol{w} \in L_2\}$$ **union**

$$L_1 \cap L_2 = \{\boldsymbol{w} \in \Sigma^* \mid \boldsymbol{w} \in L_1 \wedge \boldsymbol{w} \in L_2\}$$ **intersection**

$$\overline{L}\ (= L^C) = \{\boldsymbol{w} \in \Sigma^* \mid \boldsymbol{w} \notin L\}$$ **complementation**

$$\left.\begin{aligned} \overline{(L_1 \cup L_2)} &= \overline{L_1} \cap \overline{L_2} \\ \overline{(L_1 \cap L_2)} &= \overline{L_1} \cup \overline{L_2} \end{aligned}\right\}$$ **de Morgan's laws**

Concatenation

Concatenation: denoted by a decimal point (but often omitted when meaning is clear from context)

String concatenation

$\boldsymbol{x} = aa \quad \boldsymbol{y} = bb \quad \boldsymbol{x}.\boldsymbol{y} = \boldsymbol{x}\boldsymbol{y} = aabb$

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$$L.\{\varepsilon\} = L$$

Power operation

Power on letters/strings

$$\boldsymbol{w}^n = \boldsymbol{w}.\boldsymbol{w}^{n-1} \quad \boldsymbol{w}^0 = \varepsilon$$

$$\boldsymbol{w} = ab$$

$$\boldsymbol{w}^2 = abab$$

$$\dots$$

Power on languages

$$L^n = L.L^{n-1} \quad L^0 = \{\varepsilon\}$$

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$$L^n = L.L^{n-1} \quad L^0 = \{\varepsilon\}$$

The Kleene *

Kleene * on letters

$$a^* = \bigcup_{i=0}^{\infty} a^i = \{\varepsilon, a, aa, aaa, \ldots\}$$

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Kleene * on languages

$$\begin{aligned} L^* &= \bigcup_{i=0}^{\infty} L^i = \{\varepsilon\} \cup L \cup L^2 \cup L^3 \ldots \\ &= \{\boldsymbol{w} \mid \exists i \text{ s.t. } \boldsymbol{w} \in L^i\} \end{aligned}$$

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Important notation

$\Sigma^* =$ set of all finite words over Σ

$\Sigma^i =$ set of all words of length i over Σ

The Kleene *

Kleene * on letters

$$a^* = \bigcup^{\infty} a^i = \{\varepsilon, a, aa, aaa, \ldots\}$$

Kleen

Notational variant, the **Kleene** +

$$L^+ = \bigcup_{i=1}^{\infty} L^i = L \cup L^2 \cup L^3 \cup \ldots$$

Does not *add* the empty string ε

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Finite-State Automata

Definition: DFA

A **deterministic finite automaton** is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

Q is a finite **set of states**

Σ is a finite **alphabet**

$\delta : Q \times \Sigma \to Q$ is the **transition function**

$q_0 \in Q$ is the **start (initial) state**

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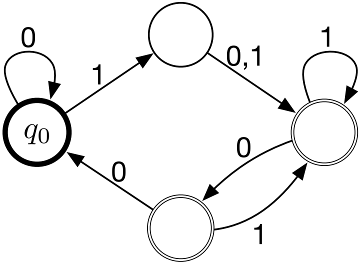
$L(M) \subseteq \Sigma^*$ is **the language of** M, i.e. the set of strings M accepts

DFAs



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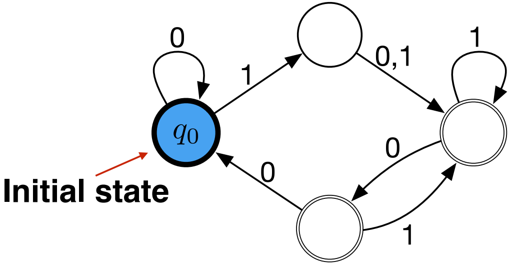

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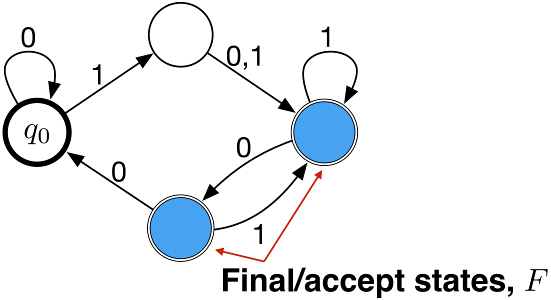

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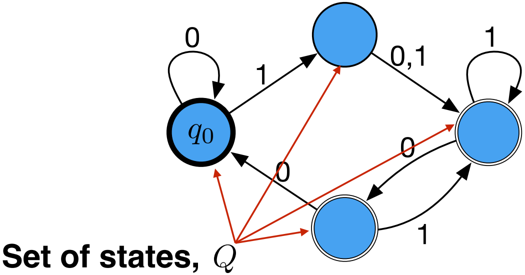

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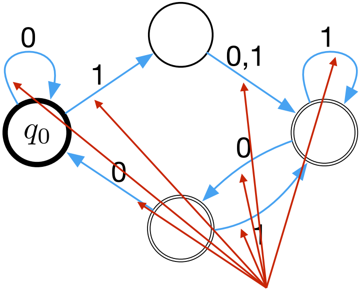


Transition function (“edges”), δ

Acceptance

- M is a DFA with alphabet Σ
- An input sentence (word / string) ***w*** is a sequence $w_1 w_2 w_3 \ldots w_n$ where each $w_i \in \Sigma$
- M accepts ***w*** if after starting in q_0 and reading ***w*** it ends in an accept state, i.e., if there is a sequence of states $s_0, s_1, \ldots, s_n$ such that

$$s_0 = q_0$$
$$s_i = \delta(s_{i-1}, w_i)$$
$$s_n \in F$$

Acceptance and Analysis

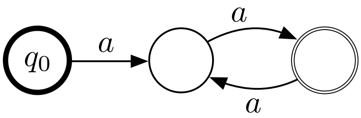
- In many problems in natural language processing, we want to analyze a strings in terms of underlying operations.

- With finite state models, this usually corresponds to the question: what was the sequence of states $s_0,s_1,\ldots,s_n$ taken to produce some $\boldsymbol{w}$?

Example

The following DFA accepts the language

$L = \{a^{2n} \mid n \geq 1\}$



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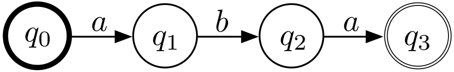
Remarks

- A DFA is required to have a **complete** transition function, whereas an NFA can have an **incomplete** transition function
- This is useful for specifying simple language generating automata, e.g. “linear chain” automata

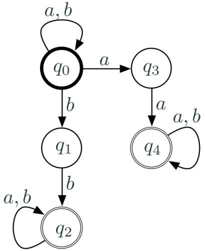


$$L(M) = \{aba\}$$

Board Work: NFA Example

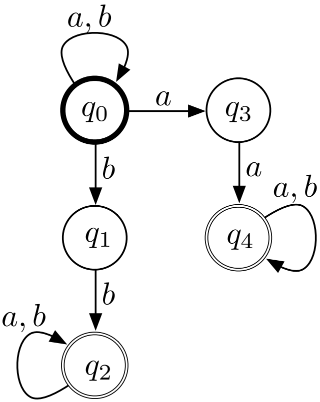
(5 Min) For the following NFA, does it accept the string $\mathbf{w} = abaa$? What sequence of states does it go through?

Can you describe in English the strings that it accepts?

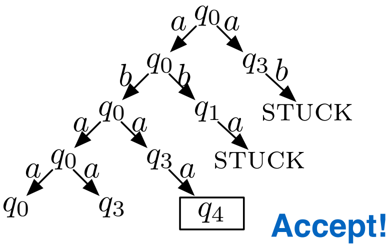


(3 Min) Discuss with a partner.

Example: NFA



Computation on $\boldsymbol{w} = abaa$



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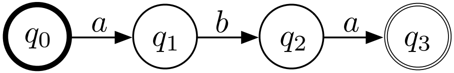

$$L(M) = \{aba\}$$

DFAs vs NFAs

- For each state, a DFA has exactly one outgoing edge for each symbol in Σ.
- Otherwise, it's a NFA

DFA & NFA Equivalence

Theorem. For every NFA A there exists a DFA A' s.t. $L(A) = L(A')$.

Regular Languages

A language $L \subseteq \Sigma^*$ is **regular** if there exists an FSA M such that $L(M) = L$

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Example regular languages:

$$L = \Sigma^*$$

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Example non-regular languages (more later):

$$L = \{a^n b^n \mid n \geq 0\}$$
$$L = \{\boldsymbol{w} \mid \boldsymbol{w} \text{ is a grammatical sentence of English}\}$$

Generalization

It is helpful to generalize the transition function of DFAs in terms of **words** (instead of single symbols).

$$\delta(q, a) = q'$$

$$\hat{\delta} : Q \times \Sigma^* \to Q$$

$$\hat{\delta}(q, \varepsilon) = q$$

$$\hat{\delta}(q, \boldsymbol{x}\sigma) = \delta(\hat{\delta}(q, \boldsymbol{x}), \sigma)$$

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Formal definition of $L(M)$

$$L(M) = \left\{ \boldsymbol{w} \in \Sigma^* \mid \hat{\delta}(q_0, \boldsymbol{w}) \in F \right\}$$

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We also generalize in terms of **sets of states**:

$$P \in 2^Q$$

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> Formal definition of $L(M)$
>
> $$L(M) = \{\boldsymbol{w} \in \Sigma^* \mid \delta(q_0, \boldsymbol{w}) \cap F \neq \emptyset\}$$

We

$$\delta(P, \boldsymbol{x}) = \bigcup_{p \in P} \delta(p, \boldsymbol{x})$$

Again, we will not distinguish between the generalized form of the function and the basic form.

DFA & NFA Equivalence

Theorem. For every NFA A there exists a DFA A' s.t. $L(A) = L(A')$.

Proofs

- Proofs in formal language theory often use **set containment** and **induction**

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- **Set containment**
  To prove that $(L_1 \cup L_2).L_3 = L_1.L_3 \cup L_2.L_3$

  one can show that (i)
  $\mathbf{w} \in (L_1 \cup L_2).L_3 \implies \mathbf{w} \in L_1.L_3 \cup L_2.L_3$

  and the other direction that (ii)
  $\mathbf{w} \in L_1.L_3 \cup L_2.L_3 \implies \mathbf{w} \in (L_1 \cup L_2).L_3$

Proofs

- Proofs in formal language theory often use **set containment** and **induction**

- **Induction**
  Induction is usually on the length of the word, i.e. to prove P is true for all $\mathbf{w} \in \mathbf{L}$, we:
  1. show P is true for words length 0 or 1
  2. hypothesize that P is true for all words $\leq$ length $n - 1$
  3. prove that if P is true for all words $\leq$ length $n - 1$, then P is true for for all words of length n

Board work: Proof by induction

On your whiteboards, take 5 min to do 1. and 3.
Then discuss with a partner for 3 min

- **Induction example**
  Prove that $\forall n \geq 0$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

1. base: verify for $n = 0$
2. hypothesis: assume true for $n \leq k - 1$
3. inductive step: prove for $n = k$

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

Proofs

- **Induction example**
  Prove that $\forall n \geq 0$
  $$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

  1. base: $n = 0$: $2^0 = 1 = 2^{0+1} - 1$
  2. hypothesis: assume true for $n \leq k - 1$
  3. inductive step: prove for $n = k$

$$\begin{aligned}\sum_{i=0}^{k} 2^i &= 2^k + \sum_{i=0}^{k-1} 2^i \\ &= 2^k + 2^k - 1 \quad \text{(by inductive hypothesis)} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1 \quad \square\end{aligned}$$

DFA & NFA Equivalence

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Powerset construction.

$\Sigma' = \Sigma$

$Q' = 2^Q$ Constructed DFA states are **sets** of NFA states

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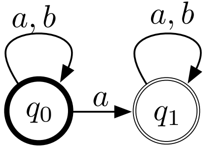
Example



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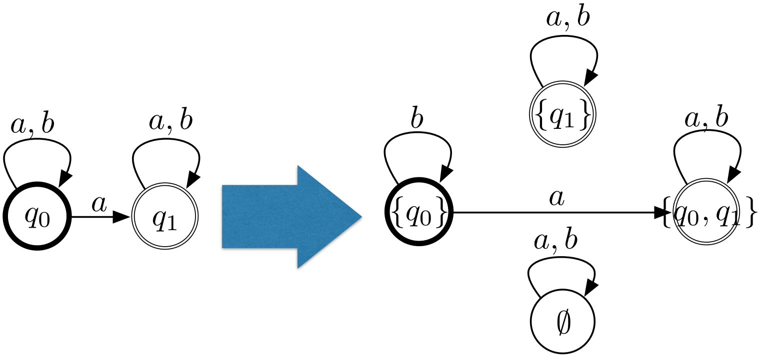

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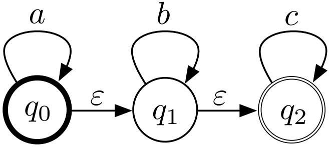
Remarks

- NFAs therefore only accept regular languages
- NFAs and DFAs can be used interchangeably

NFA with Epsilons

A **NFA with ε-transitions** is an NFA that may change states without reading an input symbol.



NFA with Epsilons

A **nondeterministic finite automaton** is a 5-tuple $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ where

Q is a finite **set of states**

Σ is a finite **alphabet**

~~$\delta : Q \times \Sigma \to 2^Q$ is the **transition relation**~~

$\delta : Q \times \Sigma \cup \{\varepsilon\} \to 2^Q$ is the **transition relation**

$q_0 \in Q$ is the **start (initial) state**

$F \subseteq Q$ is the **set of final (accept) states**

$L(M) \subseteq \Sigma^*$ is **the language of** M, i.e. the set of strings M accepts

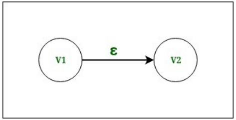
NFA& ε-NFA Equivalence

Theorem. For every NFA A with epsilon moves there is an equivalent NFA A' without, s.t. $L(A) = L(A')$

Conversion of ϵ-NFA to NFA

First do ϵ closure (add all implicit ϵ moves)

Then, for every epsilon move:



1. For all moves to any state that start from V2, duplicate the moves but with the start state V1.
2. If V1 is a start state, make V2 a start state as well
3. If V2 is a final state, make V1 a final state as well
4. Remove the ϵ-move

Regular Expression

A regular expression is a way of describing the languages accepted by FSAs.

Defined recursively:

1. $\emptyset$ is an RE denoting the empty set
2. ε is an RE denoting the set $\{\varepsilon\}$
3. for each $a \in \Sigma$, a is a RE denoting $\{a\}$
4. If `r` and `s` are REs denoting the languages R and S

   `(r|s)` denotes $R \cup S$

   `(rs)` denotes $R.S$

   `r*` denotes R^*

Precedence means parentheses can sometimes be omitted: `*.|`

Examples

`(0|1)*` denotes all finite words over $\Sigma = \{0, 1\}$

`0*|1*` denotes all finite words containing only 0's and 1's

Board Work: Regex Examples

(5 Min) For the following languages, write it's regular expression

- Strings over $\{a, b\}^*$ that start with an a
- Strings over $\{a, b\}^*$ that contain ab or ba

(3 Min) Discuss with a partner.

Regex in practice

Most programming languages, grep, etc use a slightly different syntax. By default, the regex only needs to match a substring.

Character	Meaning	Example
*	Match **zero, one or more** of the previous	`Ah*` matches "`Ahhhhh`" or "`A`"
?	Match **zero or one** of the previous	`Ah?` matches "`Al`" or "`Ah`"
+	Match **one or more** of the previous	`Ah+` matches "`Ah`" or "`Ahhh`" but not "`A`"
\	Used to **escape** a special character	`Hungry\?` matches "`Hungry?`"
.	Wildcard character, matches **any** character	`do.*` matches "`dog`", "`door`", "`dot`", etc.
( )	**Group** characters	See example for |
[ ]	Matches a **range** of characters	`[cbf]ar` matches "car", "bar", or "far" `[0-9]+` matches any positive integer `[a-zA-Z]` matches ascii letters a-z (uppercase and lower case) `[^0-9]` matches any character not 0-9.
|	Matche previous **OR** next character/group	`(Mon|Tues)day` matches "Monday" or "Tuesday"
{ }	Matches a specified **number of occurrences** of the previous	`[0-9]{3}` matches "315" but not "31" `[0-9]{2,4}` matches "12", "123", and "1234" `[0-9]{2,}` matches "1234567..."
^	**Beginning** of a string. Or within a character range `[]` negation.	`^http` matches strings that begin with http, such as a url. `[^0-9]` matches any character not 0-9.
$	**End** of a string.	`ing$` matches "exciting" but not "ingenious"

REs and ε-NFAs

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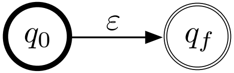

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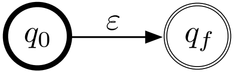

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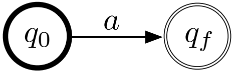

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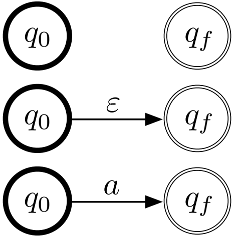

Note: we assume there is **exactly one** final state.

REs and ε-NFAs

Inductive step. We assume hypothesis is true for all REs with $\leq n$ operations, and then prove is true for $n+1$ operations.

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There are three cases to be dealt with:

(1) $\mathtt{r} = \mathtt{r}_1 \mid \mathtt{r}_2$

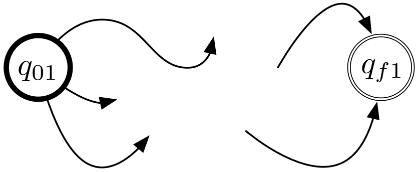
(2) $\mathtt{r} = \mathtt{r}_1\mathtt{r}_2$

(3) $\mathtt{r} = \mathtt{r}_1\texttt{*}$

Case 1: $\mathtt{r} = \mathtt{r}_1 \mid \mathtt{r}_2$

By the inductive hypothesis, there are two epsilon NFAs A_1 and A_2.

A_1



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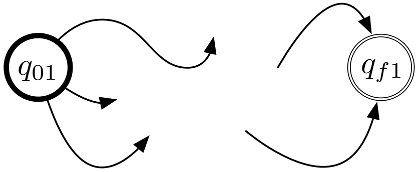

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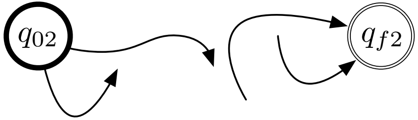

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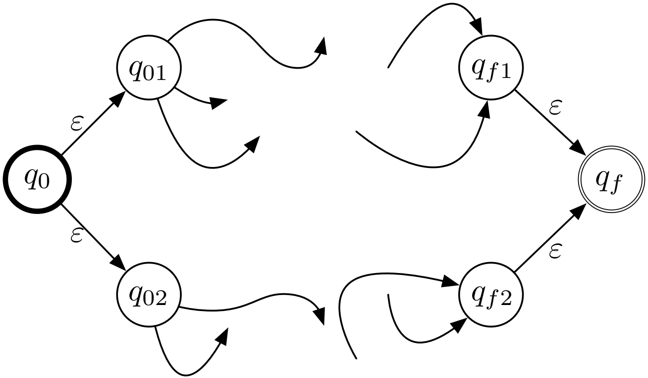

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Formally, if $A_1 = \langle Q_1, \Sigma, \delta_1, q_{01}, \{q_{f1}\} \rangle$

$$A_2 = \langle Q_2, \Sigma, \delta_2, q_{02}, \{q_{f2}\} \rangle$$

then,

$A = \langle Q_1 \cup Q_2 \cup \{q_0\} \cup \{q_f\}, \Sigma, \delta, q_0, \{q_f\} \rangle$

$$\begin{aligned}
\delta(q_0, \varepsilon) &= \{q_{01}, q_{02}\} \\
\delta(q_{f1}, \varepsilon) &= \{q_f\} \\
\delta(q_{f2}, \varepsilon) &= \{q_f\} \\
\delta(q, \sigma) &= \delta_1 q, \sigma \quad \forall q \in Q_1 - \{q_{f1}\}, \sigma \in \Sigma \cup \{\varepsilon\} \\
\delta(q, \sigma) &= \delta_2 q, \sigma \quad \forall q \in Q_2 - \{q_{f2}\}, \sigma \in \Sigma \cup \{\varepsilon\}
\end{aligned}$$

Case 1: $\mathtt{r} = \mathtt{r}_1 \mid \mathtt{r}_2$

It remains to show that $L(A) = L(A_1) \cup L(A_2)$

How to do this? Set containment.

Cases 2 & 3

- Strategy for showing this proceeds as with Case 1
- Refer to textbook for details.