CHAPTER 1

NORMED VECTOR SPACES

1.1 Review of Vector Spaces

In this section, we will review properties of vector spaces with relation to vector spaces.

Definition 1.1.1. A *vector space* $(V, +, \cdot)$ *(over a field* $\mathbb{K}$*)* is a set V and functions $(+)\colon V \times V \to V$ and $(\cdot)\colon \mathbb{K} \times V \to V$ such that:

- $(V, +)$ is an abelian group;
- $\cdot$ is associative over $+$, i.e. for $a, b \in \mathbb{K}$ and $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$;
- $\cdot$ left- and right-distributes over $+$, i.e. for $a \in \mathbb{K}$ and $v, w \in V$, $a\cdot(v+w) = a \cdot v + a \cdot w$.

In this course, we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We are familiar with many vector spaces, e.g. $\mathbb{R}^n$ over $\mathbb{R}$ and $\mathbb{C}^n$ over $\mathbb{C}$ (and $\mathbb{R}$).

We now review the concept of dimensionality.

Definition 1.1.2. Let V be a vector space and let $S \subseteq V$.

- We say that S *spans* V if for all $v \in V$, there exists a collection of scalars $(c_{v_i})_{v_i \in S}$ such that
$$v = \sum_{v_i \in S} c_{v_i} \cdot v_i.$$
- We say that S is *linearly independent* if for all linear combinations
$$\sum_{v_i \in S} c_{v_i} \cdot v_i = 0,$$
we have $c_{v_i} = 0$ for all $v_i \in S$.
- We say that S is a *basis* for V if S spans V and is linearly independent.

For $\mathbb{R}^n$, a basis is given by $\{e_1, e_2, \ldots, e_n\}$, with
$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$
for $1 \leq i \leq n$. This basis is not unique, e.g. another basis for $\mathbb{R}^n$ is $\{f_1, f_2, \ldots, f_n\}$, with
$$f_i = \sum_{j=1}^{i} e_j.$$
Although the basis is not unique, if it is finite, then any other basis will also be finite and have the same number of elements. This value is defined the

dimension of the vector space. Vector spaces that have a basis with finitely many elements are called *finite-dimensional*. We know that for a field $\mathbb{K}$, if V is an n-dimensional vector space over $\mathbb{K}$, then V is isomorphic to $\mathbb{K}^n$. So, these are all the finite-dimensional vector spaces.

We can represent the vector space $\mathbb{R}^n$ as a function. In particular, for some set $X = \{x_1, x_2, \dots, x_n\}$, let $\text{Fun}(X, \mathbb{R})$ be the set of functions $f\colon X \to \mathbb{R}$. We claim that $\text{Fun}(X, \mathbb{R})$ is isomorphic to $\mathbb{R}^n$, with the isomorphism map $\varphi\colon \text{Fun}(X, \mathbb{R}) \to \mathbb{R}^n$

$$\varphi(f) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

Note however that this format is not limited to finite sets; the space of functions $\text{Fun}(X, \mathbb{R})$ is a vector space even when X is infinite. In particular, we consider the case where X is countable, i.e. $X = \mathbb{Z}_{\geq 1}$. The space $\text{Fun}(X, \mathbb{R})$ in this case is the space of all functions $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{R}$, i.e. sequences in $\mathbb{R}$. We denote this as $\text{Seq}(\mathbb{R})$ as well. The sequences form a vector space with respect to pointwise addition and scalar multiplication. This sequence is infinite-dimensional, i.e. it does not have a finite basis. This is because it has the basis $\{e^{(1)}, e^{(2)}, \dots\}$, with the sequence $(e_n^{(k)})_{n=1}^{\infty}$ given by

$$e_n^{(k)} = \begin{cases} 1 & n = k \\ 0 & \text{otherwise.} \end{cases}$$

We know that every basis of a finite-dimensional space is finite, so $\text{Seq}(\mathbb{R})$ is infinite-dimensional. This space has many interesting subspaces, including:

- the space of bounded sequences ℓ^∞;
- the space of convergent sequences c;
- the space of sequences that converge to 0 c_0;
- the space of sequences $(x_n)_{n=1}^{\infty}$ such that the series
$$\sum_{n=1}^{\infty} |x_n|^p$$
converges, denoted ℓ^p, for some $p \in [1, \infty)$;
- the space of sequences that are eventually zero c_{00}.

Also, the space of continuous functions from the compact subset $[0, 1]$ to $\mathbb{R}$, denoted by $C[0, 1]$, is a vector space- it forms a vector space over pointwise addition and scalar multiplication, i.e. for $c \in \mathbb{R}$ and $f \in C[0, 1]$, we define the function $c \cdot f \in C[0, 1]$ by $(c \cdot f)(x) = c \cdot f(x)$ for $x \in [0, 1]$. This is also an infinite-dimensional space- it has a subspace consisting of polynomial functions, whose basis is given by

$$\{f_n \mid n \in \mathbb{Z}_{\geq 1}\},$$

where $f_n(x) = x^n$ for all $x \in [0, 1]$. Hence, it has an infinite-dimensional subspace, meaning that the entire space must also be infinite-dimensional. We will later see that the space of polynomials is a dense subspace of $C[0,1]$, i.e. a continuous function can be approximated by a polynomial function arbitrarily well.