CHAPTER 2

MEASURE THEORY

2.1 Rings and Algebras

In this section, we will define measures and see why they are necessary. In relation to functional analysis, measure theory is needed to formally define function spaces that complement the sequence spaces ℓ^p. We will be able to extend $C[0,1]$ to a bigger set of functions by defining integration for more general spaces.

Intuitively, a measure on $\mathbb{R}$ is a function $\mu\colon \mathbb{P}(\mathbb{R}) \to [0,\infty]$ such that

- for disjoint collection of countable sets $(A_i)_{i\in I}$,
$$\mu\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} \mu(A_i);$$
- for any $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, $\mu(x+A) = \mu(A)$ (translation-invariant); and
- the measure of the unit interval $\mu([0,1]) = 1$.

It turns out that it is not possible to find such a function. We will prove this by showing that satisfying the first 2 axioms implies that the third axiom cannot be satisfied.

So, assume that we have a measure function μ on $\mathbb{R}$ satisfying the first two axioms. Then, define the equivalence relation $\sim$ on $[0,1]$ by

$$x \sim y \iff x - y \in \mathbb{Q}.$$

From each equivalence class, choose a specific element x, and let N be the set containing all such elements. Now, for $r \in \mathbb{Q} \cap [0,1)$, define

$$N_r = \{x + r \mid x \in N, x \leq 1 - r\} \cup \{x + r - 1 \mid x \in N, x > 1 - r\}.$$

We know that $N_r \cap N_q \neq \varnothing$ if and only if $r = q$, with

$$\bigcup_{r\in\mathbb{Q}\cap[0,1)} N_r = [0,1).$$

Since the measure function μ is translation-invariant, we know that $\mu(N_r) = N$ for all $r \in \mathbb{Q} \cap [0,1]$. Moreover, since $\mathbb{Q} \cap [0,1]$ is countable with N_r disjoint for all $r \in \mathbb{Q} \cap [0,1]$, we find that

$$\mu([0,1)) = \mu\left(\bigcup_{r\in\mathbb{Q}\cap[0,1)}\right) = \sum_{r\in\mathbb{Q}\cap[0,1)} \mu(N_r) = \sum_{r\in\mathbb{Q}\cap[0,1)} \mu(N).$$

The value $\mu(N)$ is a constant, so either $\mu(N) = 0$, in which case $\mu([0,1)) = 0$, or $\mu(N) > 0$, in which case $\mu([0,1)) = \infty$. We would like to give the interval

$[0,1)$ a non-zero finite value (in particular 1), and to do so, we cannot allow every subset of $\mathbb{R}$ to be measurable.

We will now define the theory of rings and algebras, and later use this to define measurable functions, including the one we want in $\mathbb{R}^n$.

Definition 2.1.1. Let X be a set and let $\mathcal{A} \subseteq \mathbb{P}(X)$ be a non-empty collection of subsets of X.

- We say that $(X, \mathcal{A})$ is a *ring* if for all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$.
- We say that $(X, \mathcal{A})$ is an *algebra* if for all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$ and $A^c \in \mathcal{A}$.
- We say that $(X, \mathcal{A})$ is a *σ-algebra* if for all $A \in \mathcal{A}$, $A^c \in \mathcal{A}$, and for a collection $(A_i)_{i=1}^{\infty}$ of sets in $\mathcal{A}$, the union

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}.$$

We will only be focusing on σ-algebras, but many of the results will hold for rings or algebras. A σ-algebra is an algebra, and an algebra is a ring. An example of a σ-algebra is $\mathcal{P}(X)$, for some set X. For any set $A \subseteq X$, the smallest σ-algebra containing A is denoted $\sigma(A)$. It is given by the intersection of σ-algebras containing A- this is always a σ-algebra. For a topological space X, the set $\mathcal{B}(X)$ denotes the Borel sets, which is the σ-algebra generated by open sets in X.

We want to define measurable functions on σ-algebras. This can be defined on a ring.

Definition 2.1.2. Let $(X, \mathcal{A})$ be a ring, and let $\mu \colon \mathcal{A} \to [0, \infty]$ be a function.

- We say that μ is *additive* if for $A, B \in \mathcal{A}$ disjoint, $\mu(A \cup B) = \mu(A) + \mu(B)$.
- We say that μ is *σ-subadditive* if for a collection of disjoint sets $(A_i)_{i=1}^{\infty}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

  if the set $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.
- We say that μ is *σ-additive* if for a collection of disjoint sets $(A_i)_{i=1}^{\infty}$,

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  if the set $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

An example of a measurable function on the natural numbers $\mathbb{Z}_{\geq 1}$ is the counting measure i.e. $m(A) = |A|$. In this case, it is possible to define the measure on the entire set. We will later see that integration defined on this set will precisely give us the ℓ^p sequence spaces.

Now, let $\mathcal{E}(\mathbb{R})$ be the set of finite union of intervals in $\mathbb{R}$. This is a ring- the union of two unions is still a union, and the set difference of two intervals is a union of intervals. However, it is not an algebra since it does not contain countable union of intervals. We can define the following measure μ on $\mathcal{E}(\mathbb{R})$. First, for an interval I_k, we define

$$\mu(I_k) = \begin{cases} \infty & I_k \text{ not bounded} \\ \sup I_k - \inf I_k & \text{otherwise.} \end{cases}$$

Now, for a union of intervals

$$I = \bigcup_{k=1}^{n} I_k,$$

we can make it a disjoint union of intervals

$$I = \bigcup_{k=1}^{n} J_k$$

(a possible choice is $J_1 = I_1$, $J_2 = I_2 \setminus J_1$, $J_3 = I_3 \setminus (J_1 \cup J_2)$ and so on). Then, we define

$$\mu(I) = \sum_{k=1}^{n} \mu(J_k).$$

By definition, μ is additive. It is also σ-additive. We will extend μ to the σ-algebra generated by $\mathcal{E}(\mathbb{R})$ in the next section.

2.2 Outer Measure

In this section, we will define the concept of outer measure that allows us to extend a measure on a ring to the σ-algebra generated by the ring. First, we define the outer measure on every subset:

Definition 2.2.1. Let X be a set and let $\mathbb{R}$ be a ring on X and μ a measure on $\mathcal{R}$. Define the map $\mu^* \colon \mathbb{P}(X) \to [0, \infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{k=1}^{\infty} m(I_k) \mid (I_k) \text{ collection in } \mathcal{R} \text{ s.t. } A \subseteq \bigcup_{i=1}^{\infty} I_k \right\}$$

It turns out that $\mu*$ is not a measure on $\mathbb{P}(X)$, but we can restrict it to a measure on the σ-algebra generated by the ring.

We first show some properties about μ^*:

Proposition 2.2.2. *Let X be a set, $\mathcal{R}$ a ring on X and μ a measure on $\mathcal{R}$. Then,*

- *for $A \subseteq B \subseteq X$, $\mu^*(A) \leq \mu^*(B)$;*
- *μ^* is σ-subadditive, i.e. for a collection $(A_k)_{k=1}^{\infty}$ of subsets of X,*
$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu^*(A_k).$$
- *μ^* extends μ, i.e. for $A \in \mathcal{R}$, $\mu^*(A) = \mu(A)$.*

Proof.

- Let $A \subseteq B \subseteq X$. Then, if $(I_k)_{k=1}^{\infty}$ is a collection in $\mathcal{R}$ such that
$$B \subseteq \bigcup_{k=1}^{\infty} I_k,$$
then
$$A \subseteq \bigcup_{k=1}^{\infty} I_k.$$
So, by the definition of the infimum, we find that $\mu^*(A) \leq \mu^*(B)$.
- Let $\varepsilon > 0$ and set
$$A = \bigcup_{k=1}^{\infty} A_k.$$
For $k \in \mathbb{Z}_{\geq 1}$, by the definition of $\mu^*(A_k)$, we can find a collection $(I_n^{(k)})_{n=1}^{\infty}$ such that
$$\mu^*(A_k) + \frac{\varepsilon}{2^{k+1}} \geq \sum_{n=1}^{\infty} \mu(I_n^{(k)}).$$
Then, we know that
$$A \subseteq \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} I_n^{(k)},$$

with

$$\begin{aligned}\mu^*(A) &\leq \sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\mu(I_n^{(k)}) \\ &\leq \sum_{k=1}^{\infty}\mu^*(A_k) + \frac{\varepsilon}{2^{k+1}} \\ &= \varepsilon + \sum_{k=1}^{\infty}\mu^*(A_k).\end{aligned}$$

So,

$$\mu^*(A) \leq \sum_{k=1}^{\infty}\mu^*(A_k).$$

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□

As we mentioned before, to make μ^* a σ-algebra, we need to restrict it to the σ-algebra generated by the ring. For example, for the set $\mathbb{R}$ with the norm measure on the ring $\mathcal{E}(\mathbb{R})$, we can generate the Lebesgue measure λ^* on the σ-algebra generated by the ring, which coincides with the Borel σ-algebra, $\mathcal{B}(\mathbb{R})$, which is the σ-algebra generated by the open sets in $\mathbb{R}$. Note that the Lebesgue-measurable sets $\mathcal{L}(\mathbb{R})$ is the Borel sets $\mathcal{B}(\mathbb{R})$ with union those that have measure 0 in $\mathbb{R}$, i.e.

$$\mathcal{L}(\mathbb{R}) = \{B \cup N \mid B \in \mathcal{B}(\mathbb{R}), \mu^*(N) = 0\}.$$

2.3 Measurable Functions

In this section, we will consider measurable functions- these will be equivalent to integrable functions, but will also allow the integral to be infinite.

Definition 2.3.1. Let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{B}, \nu)$ be measure spaces, and let $f \colon X \to Y$ be a function. Then, f is *measurable* if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

If $X = Y = \mathbb{R}$, then this definition is equivalent to asking that $f^{-1}(-\infty, a)$ is measurable for all $a \in \mathbb{R}$. By definition, all continuous functions are measurable- the preimage of an open set is open, which is measurable.