CHAPTER 1

COMPLEX DIFFERENTIATION

1.0 Introduction to Complex Numbers

Complex numbers can be viewed as the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\},$$

where $i^2 = -1$. This can be viewed as a 2D plane, e.g.

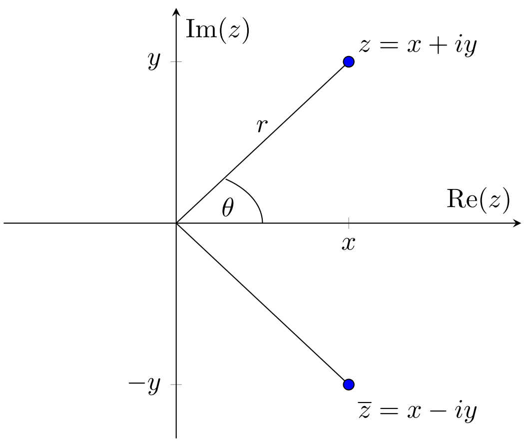


Figure 1.1: The complex plane

The given point is $z = x + iy$, for $x, y \in \mathbb{R}$. The real part of z, $\mathrm{Re}(z) = x$, and the imaginary part of z, $\mathrm{Im}(z) = y$. The complex conjugate of z is given by $\overline{z} = x - iy$. Using the conjugate, we can rewrite the real and the imaginary part of a complex number:

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2}.$$

The modulus of z is

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}} \in \mathbb{R}.$$

Moreover, the argument of z is

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) + 2k\pi & x > 0 \\ \tan^{-1}(y/x) + (2k+1)\pi & x < 0 \end{cases},$$

for $k \in \mathbb{Z}$. It corresponds to the angle θ. Moreover, every value of $\arg(z)$ is $\theta + 2k\pi$, for some $k \in \mathbb{Z}$.

In Euler's notation, we denote $z = re^{i\theta}$. By de Moivre's Theorem, we know that

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta)),$$

and if $z \neq 0$, then its multiplicative inverse is given by

$$z^{-1} = \frac{1}{z}\frac{\overline{z}}{\overline{z}} = \frac{x - iy}{x^2 + y^2}.$$

Moreover, $z^{-1} = r^{-1}e^{-i\theta}$ in Euler's notation.

We know that $\mathbb{C}$ forms a field under addition and multiplication. Adding two complex numbers is equivalent to adding two vectors in $\mathbb{R}^2$. Geometrically, this can be thought of as translation. Multiplication is however performed in a way such that the the arguments of the two complex numbers gets added. That is, for $w, z \in \mathbb{C}$ with $w = re^{i\alpha}$ and $z = se^{i\beta}$,

$$zw = rse^{i(\alpha+\beta)}.$$

Geometrically, it can be thought of as scaling and rotation.

1.1 Complex Sequences and Series

Complex Sequences

We start by considering convergence in $\mathbb{C}$ before looking at differentiation.

Definition 1.1.1. Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$, and let $z \in \mathbb{C}$. We say that $z_n \to z$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geqslant 1}$ such that for $n \in \mathbb{Z}_{\geqslant 1}$, if $n \geqslant N$, then $|z_n - z| < \varepsilon$.

It turns out that a limit exists for a complex sequence if and only if its real part and the complex part converge.

Proposition 1.1.2. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ and let $z \in \mathbb{C}$. Then, $z_n \to z$ if and only if $\mathrm{Re}(z_n) \to \mathrm{Re}(z)$ and $\mathrm{Im}(z_n) \to \mathrm{Im}(z)$.*

We will now look at some proofs of convergence using the result above. We start by showing that (i^n) does not converge.

Example 1.1.3. *The sequence $(z_n)_{n=1}^{\infty}$ given by $z_n = i^n$ does not converge.*

Proof. We have

$$\mathrm{Re}(z_n) = \begin{cases} (-1)^{n/2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

Since the subsequence $\mathrm{Re}(z_{2n}) = (-1)^n$ does not converge, the sequence $\mathrm{Re}(z_n)$ does not converge. By the result above, we find that (z_n) does not converge. □

Next, we show that $(\frac{i}{n})$ converges to 0.

Example 1.1.4. *The sequence $(z_n)_{n=1}^{\infty}$ given by $z_n = \frac{i}{n}$ satisfies $z_n \to 0$.*

Proof. We have

$$\mathrm{Re}(z_n) = 0, \qquad \mathrm{Im}(z_n) = \frac{1}{n}.$$

Since $\mathrm{Re}(z_n) \to 0$ and $\mathrm{Im}(z_n) \to 0$, we find that $z_n \to 0$. □

Now, we look at Cauchy sequences.

Definition 1.1.5. Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. We say that (z_n) is *Cauchy* if for all $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geqslant 1}$ such that for $m, n \in \mathbb{Z}$, if $m, n \geqslant N$, then $|z_n - z_m| < \varepsilon$.

It turns out that in $\mathbb{C}$, a Cauchy sequence is equivalent to a convergent sequence.

Proposition 1.1.6. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. Then, (z_n) is Cauchy if and only if $z_n \to z$ for some $z \in \mathbb{C}$.*

For this reason, we say that $\mathbb{C}$ is complete.

Complex series

Now, we look at series in $\mathbb{C}$.

Definition 1.1.7. Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. Define the *sequence of partial sums* $(s_n)_{n=1}^{\infty}$

$$s_n = \sum_{i=1}^{n} z_i.$$

For some $z \in \mathbb{C}$, we say that the sum

$$\sum_{n=1}^{\infty} z_n = z$$

if $s_n \to z$.

This is analogous to the real case. Also, by the result above, we know that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \mathrm{Re}(z_n) + i \sum_{n=1}^{\infty} \mathrm{Im}(z_n).$$

If a series converges, then the sequence has to converge to 0.

Proposition 1.1.8. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ such that the series*

$$\sum_{n=1}^{\infty} z_n$$

converges. Then, $z_n \to 0$.

Proof. Let $(s_n)_{n=1}^{\infty}$ be the sequence of partial sums of (z_n). We know that $s_n \to S$, for some $S \in \mathbb{C}$. In that case,

$$z_n = s_{n+1} - s_n \to S - S = 0.$$

$\square$

We can define absolute convergence of a sequence by looking at the absolute convergence of the sequence of partial sums.

Definition 1.1.9. Let $(z_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$. In that case, we say that the series $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if the series

$$\sum_{n=0}^{\infty} |z_n|$$

converges.

As in $\mathbb{R}$, absolute convergence implies convergence in $\mathbb{C}$.

Proposition 1.1.10. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ such that the series $\sum z_n$ converges absolutely. Then, the series $\sum_{n=1}^{\infty} z_n$ converges.*

Proof. Let $\varepsilon > 0$. Since $\sum z_n$ converges absolutely, it is Cauchy. So, there exists an $N \in \mathbb{Z}_{\geqslant 1}$ such that for $m, n \in \mathbb{Z}$, if $m \geqslant n \geqslant N$, then

$$\sum_{i=n}^{m} |z_i| < \varepsilon.$$

In that case, for $m, n \in \mathbb{Z}$, if $m \geqslant n \geqslant N$, then

$$\left| \sum_{i=n}^{m} z_i \right| \leqslant \sum_{i=n}^{m} |z_i| < \varepsilon.$$

Therefore, the series $\sum z_n$ is Cauchy. This implies that $\sum z_n$ converges. $\square$

Now, we will look at the geometric series in $\mathbb{C}$. For $z \in \mathbb{C}$, define the sequence $(z_n)_{n=0}^{\infty}$ by $z_n = z^n$. If $z \neq 1$, we know that the partial sum

$$s_n = \sum_{i=0}^{n} z_i = \frac{1 - z^{n+1}}{1 - z},$$

using the fact that

$$(1 - z)(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1}.$$

If $|z| \geqslant 1$, then we know that $|z^n| = |z|^n \geqslant 1$ for all $n \in \mathbb{Z}_{\geqslant 0}$, so $z_n \not\to 0$. Therefore, the series

$$\sum_{n=0}^{\infty} |z^n|$$

cannot converge. Instead, if $|z| < 1$, then

$$|s_n| = \frac{|1 - z^{n+1}|}{|1 - z|} \leqslant \frac{1 + |z|^{n+1}}{1 - |z|}$$

by the triangle and the reverse triangle inequality. Since

$$\frac{1 + |z|^{n+1}}{1 - |z|} \to \frac{1}{1 - |z|},$$

we find that the the series

$$\sum_{n=0}^{\infty} |z^n|$$

is bounded. The series is monotone increasing, so the monotone convergence theorem tells us that it is absolutely convergent.

Using the geometric series, we can derive the comparison and the ratio test.

Corollary 1.1.11 (Comparison Test). *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ and $\sum_{n=0}^{\infty} r_n$ be a convergent series in $\mathbb{R}$ with $r_n \geq 0$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Moreover, assume that for $k > 0$,*

$$|z_n| \leqslant k r_n$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. Then, the series $\sum_{n=0}^{\infty} z_n$ converges absolutely.

Corollary 1.1.12 (Ratio Test). *Let $\sum_{n=0}^{\infty} z_n$ be a sequence in $\mathbb{C}$ such that $z_n \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$, and let $C \in \mathbb{R}$ such that*

$$\lim_{n \to \infty} \frac{|z_{n+1}|}{|z_n|} = C.$$

- *If $C < 1$, then the series $\sum_{n=0}^{\infty} z_n$ converges absolutely.*
- *If $C > 1$, then the series $\sum_{n=0}^{\infty} z_n$ diverges.*

1.2 Introduction to Complex differentiation

Limits in $\mathbb{C}$

In this section, we will define complex differentiation in open sets. We start by defining open discs.

Definition 1.2.1. Let $z_0 \in \mathbb{C}$ and $r > 0$. We define the *open disc around* z_0 *with radius* r to be the set

$$D = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

We can generalise this to open sets.

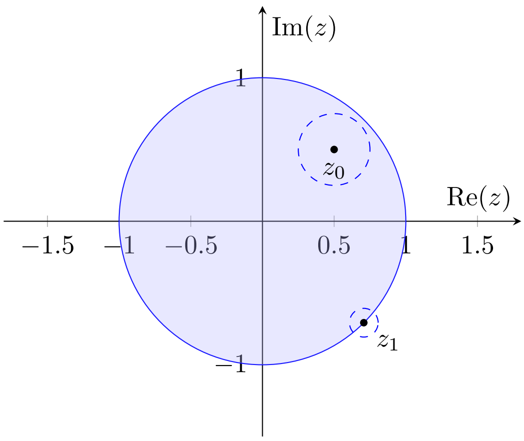
Definition 1.2.2. Let $\Omega \subseteq \mathbb{C}$. We say that Ω is *open* if for every $z_0 \in \Omega$, there exists an $\varepsilon > 0$ such that for all $z \in \mathbb{C}$, if $|z - z_0| < \varepsilon$, then $z \in \Omega$.

In other words, for every $z_0 \in \Omega$, the open disc around z_0 of radius ε satisfies

$$\{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\} \subseteq \Omega.$$

Intuitively, this means that no point in Ω is a boundary point. That is, for any point, there is some ε-neighbourhood around it that is still within Ω.

Consider the closed unit ball in $\mathbb{C}$ centered at the origin.



In the figure above, the value $z_0 \in \mathbb{C}$ is not a boundary point of the set since we have an open ball around it that is fully contained in the set. However, $z_1 \in \mathbb{C}$ is a boundary point since any open ball around it will not be fully contained in the set. So, this set is not open.

As we expect, open discs are open.

Proposition 1.2.3. *Let* $z_0 \in \mathbb{C}$ *and let* $r > 0$. *Then, the open disc*

$$D = \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

is open.

Also, $\mathbb{C}$ is open. This is because for all $z_0, z \in \mathbb{C}$, if $|z - z_0| < 1$, then $z \in \mathbb{C}$.[1] We will normally consider functions with domain an open set.

We now define limits.

[1] The choice of 1 here is arbitrary!

Definition 1.2.4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, and let $z_0, w \in \mathbb{C}$. We say that the limit
$$\lim_{z \to z_0} f(z) = w$$
if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $z \in \mathbb{C}$, if $0 < |z - z_0| < \delta$, then $|f(z) - w| < \varepsilon$.

As we saw in real analysis, the sequential characterisation of limits still holds in $\mathbb{C}$.

Proposition 1.2.5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a function, and let $z_0, w \in \mathbb{C}$. Then, the limit*
$$\lim_{z \to z_0} f(z) = w$$
if and only if for every sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C} \setminus \{z_0\}$ with $z_n \to z_0$, $f(z_n) \to w$.

Derivatives in $\mathbb{C}$

Now, we will look at differentiation in $\mathbb{C}$.

Definition 1.2.6. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, and let $c \in \mathbb{C}$. Then, we say that f is *differentiable at c* if the limit
$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$
exists. If the limit exists, we denote the value by $f'(c)$. We say that f is *differentiable* if f is differentiable at c for all $c \in \mathbb{C}$.

Although the difference quotient is the same in $\mathbb{C}$ as in $\mathbb{R}$, there are a lot of functions in $\mathbb{C}$ that we might 'expect' to be continuous, but aren't. This is because $\mathbb{R}$ can be thought of as a line, so there are only two directions to approach a point. However, $\mathbb{C}$ is a plane, so there are infinitely many directions we can approach a point from.

We will illustrate the difference with an example. Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = |z|^2 = z\overline{z}$. We can consider the function as a map $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = x^2 + y^2$. This is a differentiable function in $\mathbb{R}^2$ because we can derive it partially with respect to both x and y. But, f is only differentiable at $z_0 = 0$.

- If $z_0 = 0$, then
$$f'(0) = \lim_{z \to 0} \frac{z\overline{z} - 0}{z - 0} = \lim_{z \to 0} \overline{z} = \lim_{z \to 0} x - iy = 0.$$
Note that if $z = x + iy \to 0$, then we must have $x \to 0$ and $y \to 0$.

- Now, assume that $z_0 \neq 0$. Define the sequences $x_n = z_0 + \frac{1}{n}$ and $y_n = z_0 + \frac{i}{n}$. We know that $x_n, y_n \to z_0$. Moreover,
$$\begin{aligned}\frac{f(x_n) - f(z_0)}{x_n - z_0} &= \frac{(z_0 + 1/n)(\overline{z_0} + 1/n) - z_0\overline{z}_0}{1/n} \\ &= \frac{1/n \cdot z_0 + 1/n \cdot \overline{z}_0 + 1/n^2}{1/n} \\ &= z_0 + \overline{z}_0 + \frac{1}{n} \to z_0 + \overline{z}_0,\end{aligned}$$

and

$$\begin{aligned}\frac{f(y_n) - f(z_0)}{y_n - z_0} &= \frac{(z_0 + i/n)(\overline{z_0} - i/n) - z_0\overline{z}_0}{i/n} \\ &= \frac{-i/n \cdot z_0 + i/n \cdot \overline{z}_0 + 1/n^2}{i/n} \\ &= -z_0 + \overline{z}_0 - \frac{i}{n} \to -z_0 + \overline{z}_0.\end{aligned}$$

Since $z_0 \neq 0$, we know that $z_0 \neq -z_0$. Therefore, we must find that the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

does not exist by the sequential characterisation of continuity.

So, the function $f'(0) = 0$, and is not differentiable at any non-zero point. In fact, we say that it depends on the value of θ, where $z = re^{i\theta}$. That is, the direction we approach in. In this case, we saw two directions- approaching z_0 from the real line ($\theta = 0$) and the imaginary line ($\theta = \frac{\pi}{2}$).

Similarly, we can show that the complex conjugation function is not differentiable.

Example 1.2.7. *Let $c \in \mathbb{C}$, and define the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \overline{z}$. Then, f is not differentiable at c.*

Proof. Let $z_0 \in \mathbb{C}$. Define the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $\mathbb{C}$ by $x_n = z_0 + \frac{1}{n}$ and $y_n = z_0 + \frac{i}{n}$. We know that $x_n, y_n \to z_0$. Moreover,

$$\frac{f(x_n) - f(z_0)}{x_n - z_0} = \frac{\overline{z}_0 + \frac{1}{n} - \overline{z}_0}{1/n} = \frac{1/n}{1/n} = 1 \to 1,$$

and

$$\frac{f(y_n) - f(z_0)}{y_n - z_0} = \frac{\overline{z}_0 - \frac{i}{n} + \overline{z}_0}{i/n} = \frac{-i/n}{i/n} = -1 \to -1.$$

So, the sequential characterisation of continuity tells us that the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

does not exist. □

Using the properties of limits, we can establish arithmetic properties of derivatives. We start with the sum rule.

Proposition 1.2.8. *Let $f, g : \mathbb{C} \to \mathbb{C}$ be functions and let $c \in \mathbb{C}$ such that f and g are differentiable at c. Then, $f+g$ is differentiable at c, with $(f+g)'(c) = f'(c) + g'(c)$.*

Next, we look at the scalar rule.

Proposition 1.2.9. *Let $f : \mathbb{C} \to \mathbb{C}$ be a function and let $a, c \in \mathbb{C}$ such that f is differentiable at c. Then, af is differentiable at c, with $(af)'(c) = af'(c)$.*

We will now look at the quotient rule.

Proposition 1.2.10. *Let $f, g : \mathbb{C} \to \mathbb{C}$ be functions and let $c \in \mathbb{C}$ such that $g(c) = 0$. In that case, $\frac{f}{g}$ is differentiable at c, with*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Finally, we have the chain rule.

Proposition 1.2.11. *Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be functions such that f is differentiable at $c \in \mathbb{C}$ and g is differentiable at $f(c)$. In that case, the composition $g \circ f$ is differentiable at c, with*

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

We finish by defining holomorphic functions.

Definition 1.2.12. Let Ω be an open subset of $\mathbb{C}$. Then, f is *holomorphic* in Ω if f is differentiable at every $x \in \Omega$.

1.3 Power series

In this section, we will look at power series in $\mathbb{C}$.

Definition 1.3.1. Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$, and let $c, z \in \mathbb{C}$. The power series $\sum a_n(z-c)^n$ is defined by

$$S(z) = \sum_{n=0}^{\infty} a_n(z-c)^n = a_0 + a_1(z-c) + a_2(z-c)^2 + \dots.$$

We refer to the sequence (a_n) as the *coefficient* sequence, z as the *variable*, and c is the *expansion point.* We say that the power series $S(z)$ *converges* at $z_0 \in \mathbb{C}$ if the series $S(z_0)$ converges. Otherwise, we say that it *diverges* at z_0.

For example, consider the power series

$$S(z) = \sum_{n=0}^{\infty} (z-c)^n,$$

for some $c \in \mathbb{C}$. Then, we saw before that the interval of convergence is the open disc

$$D = \{z \in \mathbb{C} \mid |z-c| < 1\}.$$

We can also consider absolute convergence at a point.

Definition 1.3.2. Let $S(z)$ be a power series. We say that $S(z)$ *converges absolutely* at $z_0 \in \mathbb{C}$ if the series $S(z_0)$ converges absolutely.

Similarly, we can define uniform convergence.

Definition 1.3.3. Let $S(z)$ be a power series. We say that $S(z)$ *converges uniformly* on some open disc $D \subseteq \mathbb{C}$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 0}$ such that for $n \in \mathbb{Z}$, if $n \geq N$, then

$$|S_n(z) - S(z)| < \varepsilon$$

for all $z \in D$. We define by $S_n(z)$ the partial sum

$$S_n(z) = \sum_{i=1}^{n} a_i(z-c)^i.$$

Now, we define the radius of convergence and its properties.

Theorem 1.3.4. *Let $S(z)$ be a power series. Then, there exists an $R \in [0, \infty)$ such that*

- *$S(z)$ diverges for all $z \in \mathbb{C}$ with $|z-c| > R$;*
- *$S(z)$ converges absolutely and uniformly for all $z \in \mathbb{C}$ with $|z-c| \leq r$, for some $0 \leq r < R$; and*
- $$R = \sup\{r \geq 0 \mid |a_n|r^n \text{ is a bounded sequence}\}.$$

We call R the radius of convergence*.*

It is possible for $S(z)$ to converge for all $z \in \mathbb{C}$, in which case we say that $R = \infty$.

Now, consider the power series

$$S(z) = \sum_{n=0}^{\infty} z^{2m}, \qquad T(z) = \sum_{n=0}^{\infty} (-1)^n z^{2m}.$$

Since these are geometric series with variables $z_S = z^2$ and $z_T = -z^2$, they both have radius of convergence $R = 1$. Moreover,

$$S(z) = \frac{1}{1-z^2}, \qquad T(z) = \frac{1}{1+z^2}.$$

If we consider $\frac{1}{1-z^2}$ for $z \in \mathbb{R}$, we would have singularities at $z = \pm 1$ and the series would diverge for $|z| \geq 1$. But, $\frac{1}{1+z^2}$ does not have any singularities for $z \in \mathbb{R}$. Instead, if we look at the complex series, there are singularities at $z = \pm i$.

Like in the real case, we can use the ratio test to characterise the radius of convergence.

Theorem 1.3.5. *Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$, and let*

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

where $L \in [0, \infty]$. Then, the radius of convergence $R = \frac{1}{L}$.

Proof. We find that

$$\frac{|z_{n+1}(z-c)^{n+1}|}{|a_n(z-c)^n|} = \frac{|a_{n+1}|}{|a_n|}|z-c| \to L|z-c|.$$

So, if $|z-c| < \frac{1}{L}$, then the ratio test tells us that the power series $S(z)$ converges. Moreover, if $|z-c| > \frac{1}{L}$, then the ratio test tells us that the power series $S(z)$ diverges. So, the radius of convergence $R = \frac{1}{L}$. $\square$

We can use the ratio test to show that the derivative has the same radius of convergence. Let

$$S(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

have radius of convergence R. Then, we have the derivative

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z-c)^{n-1} = \sum_{n=0}^{\infty} (n-1) a_{n-1} (z-c)^n.$$

So, the ratio is

$$\left|\frac{a_{n+1}}{a_n}\right| \left(1 + \frac{1}{n}\right) \to \frac{1}{R} \cdot 1 = \frac{1}{R}.$$

Therefore, it has radius of convergence R. Similarly, for the antiderivative $F(z) = w + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \left(1 - \frac{1}{n+2}\right) \to \frac{1}{R} \cdot 1 = \frac{1}{R}.$$

Therefore, it has radius of convergence R.

So, a power series gives us a holomorphic function.

Proposition 1.3.6. *Let*

$$S(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$$

be a power series with radius of convergence $R > 0$. Then, $S : D \to \mathbb{C}$ is holomorphic, where

$$D = \{z \in \mathbb{C} \mid |z-c| < R\},$$

with

$$S'(z) = \sum_{n=1}^{\infty} a_n n(z-c)^{n+1}.$$

Moreover, every holomorphic function can always be written as a power series. This is not true in the real case. For a power series, it is its own Taylor series, i.e.

$$S(z) = \sum_{n=0}^{\infty} \frac{S^{(n)}(c)}{n!}(z-c)^n.$$

Also, its antiderivative has the same radius of convergence.

Proposition 1.3.7. *Let*

$$S(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$$

be a power series with radius of convergence $R > 0$. Then, $S : D \to \mathbb{C}$ has an antiderivative, where

$$D = \{z \in \mathbb{C} \mid |z-c| < R\},$$

given by

$$F(z) = w + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1},$$

for some $w \in \mathbb{C}$

1.4 Elementary functions

Trigonometric functions

The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is given by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \dots,$$

for some $z \in \mathbb{C}$. If we let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ defined by $a_n = \frac{1}{n!}$, then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

So, the ratio test tells us that the radius of convergence of the power series e^x $R = \infty$.

The properties of the real exponential function extend to the complex exponential function. First, the derivative of the exponential function is itself.

Proposition 1.4.1. *Let $z \in \mathbb{C}$. Then,*

$$e'(x) = e(x).$$

Proof. We find that

$$\begin{aligned} e'(x) &= \sum_{n=1}^{\infty} \frac{1}{n!} n z^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e(x). \end{aligned}$$

□

Moreover, the following properties hold.

Proposition 1.4.2. *Let $z, w \in \mathbb{C}$. Then, $e^{z+w} = e^z e^w$ and $e^{-z} = \frac{1}{e^z}$.*

Proof. Define the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^{z+w} e^{-z}$. Then,

$$f'(z) = e^{z+w} e^{-z} + e^{z+w} \cdot -e^{-z} = 0.$$

In that case, f is a constant. Then,

$$f(z) = e^{z+w} e^{-z} = f(0) = e^{0+w} e^0 = e^w.$$

So, $e^{z+w} e^{-z} = e^w$ for all $z, w \in \mathbb{C}$. This means that $e^{z+w} = e^z e^w$, and

$$e^{z+0} e^{-z} = e^0 = 1,$$

and so $e^{-z} = \frac{1}{e^z}$. □

We can further define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

Just using the power series, we will still find that the radii of convergence $R = \infty$. Also, we find that

$$\begin{aligned} \cos'(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 2nz^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} z^{2n-1} \\ &= \sum_{n=0}^{\infty} \frac{-(-1)^n}{(2n+1)!} z^{2n+1} = -\sin(z), \end{aligned}$$

and

$$\begin{aligned} \sin'(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) \cdot z^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos(z). \end{aligned}$$

Moreover,

$$e^{iz} = \frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2} = \cos(z) + i\sin(z).$$

We can also define the following trigonometric functions:

- $$\tan(z) = \frac{\sin(z)}{\cos(z)},$$
- $$\cosh(z) = \frac{e^z + e^{-z}}{2},$$
- $$\sinh(z) = \frac{e^z - e^{-z}}{2}.$$

Logarithmic function and powers

In $\mathbb{C}$, the exponential function is not injective, since

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1 = e^z.$$

So, we cannot define the logarithm function as the inverse of $\exp : \mathbb{C} \to \mathbb{C}$. We need to restrict both the domain and the codomain. Similarly, for any $n \in \mathbb{Z}$, the function $z \mapsto z^n$ is not injective, since

$$(ze^{\frac{2\pi i}{n}})^n = z^n.$$

So, we would need to restrict the complex power function to find the inverse.

To compute the inverse, we can use the inverse function theorem.

Theorem 1.4.3. *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function, and let $z \in \Omega$ such that $f'(z) \neq 0$. Then, there exist open sets $U \subseteq \Omega$ and $V \subseteq \mathbb{C}$ with $z \in U$ and $w = f(z) \in V$, such that $f : U \to V$ is bijective, and $f^{-1} : V \to U$ is holomorphic with*

$$\frac{d}{dz} f^{-1}(w) = \frac{1}{f'(z)} = \frac{1}{f'(f^{-1}(w))}.$$

Now, let $z \in \mathbb{C}$. We know that

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

For all $x \in \mathbb{R}$, $e^x \neq 0$. Moreover, for all $y \in \mathbb{R}$, we cannot have both $\cos y = 0$ and $\sin y = 0$. Therefore, $e^z \neq 0$. So, the inverse function theorem tells us that there are open sets U, V such that $\exp : U \to V$ is bijective and holomorphic. A possible open set is

$$U = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi)\}.$$

This choice ensures that for all $u_1, u_2 \in U$, $u_1 - u_2 \neq 2k\pi$, for some $k \in \mathbb{Z}$, and so the function is injective. Moreover,

$$V = \exp(\mathbb{C}) = \mathbb{C} \setminus (-\infty, 0].$$

This gives us a bijection $\exp : U \to V$. The inverse function is $\log : V \to U$, and is called the principal branch of the (complex) logarithm.

Now, we will find all the complex logarithms for some $z \in \mathbb{C}$.

Example 1.4.4. *The complex logarithms of $z = -1 + i\sqrt{3}$ is given by*

$$\log z = \log 2 + \frac{2\pi}{3} i + 2k\pi i,$$

for some $k \in \mathbb{Z}$.

Proof. We find that

$$|z| = \sqrt{1+3} = 2.$$

We have $\arg z = \theta + 2k\pi$ for $k \in \mathbb{Z}$, where $\cos\theta = -1$ and $\sin\theta = \sqrt{3}$. So,

$$\theta = \pi - \tan^{-1}\sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Therefore,

$$\log z = \log(|z| e^{i \arg z}) + \log|z| + i \arg z = \log 2 + \frac{2\pi}{3} i + 2k\pi i,$$

for some $k \in \mathbb{Z}$. □

Now, we will define complex powers for a complex number:

Definition 1.4.5. Let $z, w \in \mathbb{C}$. We define

$$z^w = e^{w \log z},$$

where $\log z$ are the complex logarithms of z.

A complex number has many complex logarithms, so we can limit ourselves with the principal branch

$$\log : \mathbb{C} \setminus (-\infty, 0] \to \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi)\}.$$

Then, the function $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ given by $f(z) = z^w$ is a well-defined holomorphic function for all $w \in \mathbb{C}$, and $z^u z^v = z^{u+v}$ for all $u, v \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$.

We will now compute all the values of z^i, for some $z \in \mathbb{C}$. By definition, $z^i = e^{i \log z}$. Moreover, we know that

$$\log z = \log |z| + i \arg z.$$

Therefore,

$$z^i = e^{i \log z} = e^{i \log |z|} e^{-\arg z}.$$

1.5 Cauchy-Riemann equations

We saw that a vector function $f : \mathbb{R}^2 \to \mathbb{R}$ being differentiable is not equivalent to saying the corresponding function $f : \mathbb{C} \to \mathbb{R}$ is differentiable. In this section, we characterise complex differentiation in terms of vector differentiation.

Consider a holomorphic function $f : \Omega \to \mathbb{C}$ by

$$f(x + iy) = u(x, y) + iv(x, y).$$

We can see this as a vector function $f : \overline{\Omega} \to \mathbb{R}^2$, where

$$\overline{\Omega} = \{(x, y) \in \mathbb{R}^2 \mid x + iy \in \Omega\} \subseteq \mathbb{R}^2,$$

and $f(x, y) = (u(x, y), v(x, y))$. Since f is differentiable for all $z \in \Omega$, the limit

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists, no matter how h approaches 0. In particular, if we restrict $h = \varepsilon$, for $\varepsilon \in \mathbb{R}$, we find that

$$\begin{aligned}\lim_{\varepsilon \to 0} \frac{f(z + \varepsilon) - f(z)}{\varepsilon} &= \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon, y) - u(x, y)}{\varepsilon} + i \lim_{\varepsilon \to 0} \frac{v(x + \varepsilon, y) - v(x, y)}{\varepsilon} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}.\end{aligned}$$

Instead, if we restrict $h = i\varepsilon$, for $\varepsilon \in \mathbb{R}$, we find that

$$\begin{aligned}\lim_{\varepsilon \to 0} \frac{f(z + i\varepsilon) - f(z)}{i\varepsilon} &= \lim_{\varepsilon \to 0} \frac{u(x, y + \varepsilon) - u(x, y)}{i\varepsilon} + i \lim_{\varepsilon \to 0} \frac{v(x, y + \varepsilon) - v(x, y)}{i\varepsilon} \\ &= \frac{1}{i}\frac{\partial u}{\partial x} + \frac{i}{i}\frac{\partial v}{\partial y} = -i\frac{\partial f}{\partial y}.\end{aligned}$$

Therefore,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}.$$

Equating the real and the imaginary parts, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the Cauchy-Riemann equations. Geometrically, this means that a differentiable function in $\mathbb{C}$ preserves $90°$ angles any two lines in the complex plane[2].

We have shown that if a complex function $f : \Omega \to \mathbb{C}$ is holomorphic, then the vector function $f : \overline{\Omega} \to \mathbb{C}$ must satisfy the Cauchy-Riemann equations. It also turns out that if a vector function $f : \overline{\Omega} \to \mathbb{C}$ satisfies the Cauchy-Riemann equations (and the partial derivatives are continuous), then the complex function $f : \Omega \to \mathbb{C}$ is holomorphic.

Theorem 1.5.1 (Cauchy-Riemann equations). *Let $\Omega \subseteq \mathbb{C}$ be open, and let $w : \Omega \to \mathbb{C}$ be a function, denoted by $w(x + iy) = u(x, y) + iv(x, y)$. Then,*

[2]Actually, all angles are preserved, and we will see that later!

- *if w is holomorphic, then w satisfies the Cauchy-Riemann equations, i.e.*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

- *if w satisfies the Cauchy-Riemann equations and the partial derivatives are continuous, then w is holomorphic.*

We will use the Cauchy-Riemann equations to show that the exponential function is holomorphic in $\mathbb{C}$.

Example 1.5.2. *The function $f : \mathbb{C} \to \mathbb{C}$ given by*

$$f(z) = e^z$$

is holomorphic.

Proof. For $x + iy \in \mathbb{C}$, we have

$$f(x + iy) = e^{x+iy} = e^x \cos(y) + ie^x \sin(y).$$

Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be given by $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$. Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos(y) & \qquad \frac{\partial u}{\partial y} &= -e^x \sin(y) \\ \frac{\partial v}{\partial x} &= e^x \sin(y) & \qquad \frac{\partial v}{\partial y} &= e^x \cos(y). \end{aligned}$$

So, the Cauchy-Riemann equations are satisfied for all $x + iy \in \mathbb{C}$, and the partial derivatives are continuous. This implies that the function f is holomorphic. $\square$

1.6 Harmonic functions

A function $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic if it satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

It turns out that each complex-differentiable function defines two harmonic functions.

Proposition 1.6.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Denote*

$$f(x+iy) = u(x,y) + iv(x,y),$$

where u and v have continuous 2nd order partial derivatives. Then, u and v are harmonic.

Proof. We find that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial u}{\partial x} \\ &= \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0.\end{aligned}$$

So, u is harmonic. Similarly, v is harmonic. □

For a complex-differentiable function, we have 2 harmonic functions- one for the real part and one for the complex part. We say that the 2 vector functions are harmonic conjugates.

Definition 1.6.2. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Denote

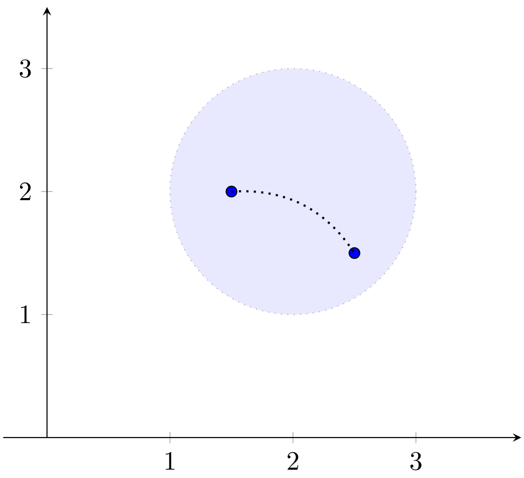
$$f(x+iy) = u(x,y) + iv(x,y).$$

We say that u and v are *harmonic conjugates* of each other.

We can find the Harmonic conjugate for any harmonic function if the domain is path connected.

Proposition 1.6.3. *Let $\Omega \subseteq \mathbb{C}$ be open and path connected, and let $u : \Omega \to \mathbb{R}$ be harmonic such that u has continuous second order derivatives. Then, u has a harmonic conjugate.*

A set V is path connected if for any two values $x_1, x_2 \in V$, there is a path from x_1 to x_2. For example, the following is path-connected.



As we can see, there is a path between the two points. Clearly, any two points in the set can be connected by a path. So, it is path-connected.

We will now find the harmonic conjugate for a vector function.

Example 1.6.4. *The function* $u : \mathbb{R}^2 \to \mathbb{R}$ *given by* $u(x, y) = x^3 - 3xy^2$ *is harmonic, with a harmonic conjugate*

$$v(x, y) = 3x^2y - y^3.$$

Proof. We find that

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

Therefore, the function u is harmonic. Now, let $v : \mathbb{R}^2 \to \mathbb{R}$ be the harmonic conjugate of u. We know that

$$3x^2 - 3y^2 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

So,

$$v = \int 3x^2 - 3y^2 \; dy = 3x^2y - y^3 + f(x),$$

for some $f : \mathbb{R} \to \mathbb{R}$. Moreover, we know that

$$-6xy = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy - f'(x).$$

So, f is a constant. Setting $f(x) = 0$, we get the harmonic conjugate

$$v(x, y) = 3x^2y - y^3.$$

□

Using this result, we find that there is an analytic function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(x + iy) = (x^3 - 3xy^2) + i(3x^2y - y^3) = x^3 + 3ix^2y - 3xy^2 - iy^3.$$

In fact, this function is $z \mapsto z^3$.