CHAPTER 3

FUNCTIONAL ANALYSIS PROPER

3.1 More on L^p spaces

In this section, we will consider L^p spaces again and show that they are Banach spaces. Moreover, we show that L^2 is a Hilbert space.

First, we characterise completeness in terms of absolute convergence.

Definition 3.1.1. Let $(V, \|\cdot\|)$ be a normed vector space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in V. We say that the series $\sum x_n$ is *absolutely convergent* if the series $\sum \|x_n\|$ converges.

Proposition 3.1.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Then, V is complete if and only if every absolutely convergent series $\sum x_n$ in V is convergent.*

Proof. First, assume that V is complete. Let $(x_n)_{n=1}^{\infty}$ be a sequence in V such that the series $\sum x_n$ is absolutely convergent. We show that the series $\sum x_n$ is Cauchy. Let $\varepsilon > 0$. Since the series $\sum \|x_n\|$ is convergent, it is Cauchy. Hence, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq N$, then

$$\left| \sum_{k=1}^{m} \|x_k\| - \sum_{k=1}^{n} \|x_k\| \right| = \sum_{n=k}^{l} \|x_n\| < \varepsilon.$$

So, for $k, l \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq N$, then

$$\left\| \sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k \right\| = \left\| \sum_{k=m}^{n} x_k \right\| \leq \sum_{k=m}^{n} \|x_n\| < \varepsilon.$$

Hence, the series $\sum x_n$ is Cauchy. Since V is complete, this implies that $\sum x_n$ is convergent.

Now, assume that every absolutely convergent series is convergent. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in V. We show that (x_n) has a convergent subsequence. Since (x_n) is Cauchy, for each $j \in \mathbb{Z}_{\geq 0}$, we can find an $n_j \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq n_j$, then

$$\|x_m - x_n\| < 2^{-j}.$$

We can choose $n_{j+1} > n_j$ for all $j \in \mathbb{Z}_{\geq 1}$. Then, $(x_{n_j})_{j=1}^{\infty}$ is a subsequence of (x_n). Next, define the sequence $(y_k)_{k=1}^{\infty}$ in V by $y_1 = x_{n_1}$ and $y_k = x_{n_k} - x_{n_{k-1}}$ for $k \geq 2$. Then, for $l \in \mathbb{Z}_{\geq 1}$, we have

$$\sum_{k=1}^{l} y_k = x_{n_1} + (x_{n_2} - x_{n_1}) + \cdots + (x_{n_l} - x_{n_l - 1}) = x_{n_l}.$$

Moreover,

$$\left\| \sum_{k=1}^{\infty} y_k \right\| = \|x_1\| + \sum_{j=1}^{\infty} \|x_{n_j} - x_{n_{j-1}}\| \leq \|x_1\| + \sum_{j=1}^{\infty} 2^{-j} = 1 + \|x_1\|,$$

meaning that $\sum y_k$ is absolutely convergent. By assumption, this implies that $\sum y_k$ is convergent. Hence, the subsequence $(x_{n_j})_{j=1}^\infty$ is convergent. Since the sequence (x_n) is Cauchy, we conclude that (x_n) converges. Hence, V is complete. □

Proposition 3.1.3. *Let $p \in [1, \infty)$. Then, the L^p space is complete.*

Proof. Let $(f_n)_{n=1}^\infty$ be sequence in L^p such that the series $\sum f_n$ is absolutely convergent, to some $B \in \mathbb{R}$. Define the sequence of functions $(F_n)_{n=1}^\infty$ in L^p by

$$G_n = \sum_{k=1}^n |f_k|,$$

and let $G = \sum_{k=1}^\infty |f_k|$. We have $G_n \geq 0$ and measurable since f_k are measurable, with $G_n \to G$ pointwise. Moreover,

$$\|G_n\|_p^p = \sum_{k=1}^\infty |f|^p \leq \sum_{k=1}^n \|f_k\|_p^p \leq B^p < \infty.$$

This implies that $G_n \in L^p$. Now, Monotone Convergence Theorem tells us that

$$\int_{\mathbb{R}} |G|^p \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} |G_n|^p \, d\mu \leq B^p < \infty.$$

Hence, $G \in L^p$. So, $G(x)$ is finite for almost all $x \in \mathbb{R}$. That is, the series

$$\sum_{k=1}^\infty f_k(x) = G(x)$$

converges for almost all $x \in \mathbb{R}$. Since $\mathbb{R}$ is complete, we can find a function F such that $\sum_{k=1}^\infty f_k \to F$ pointwise. Since $|F| \leq G$ and $G \in L^p$, we find that $F \in L^p$.

Now, we note that $|F| \leq G^p$ and $\sum_{k=1}^n f_k \leq G^p$, meaning that

$$\left| F - \sum_{k=1}^n f_k \right|^p \leq (2G)^p$$

for all $n \in \mathbb{Z}_{\geq 1}$. We know that $G \in L^p$, meaning that

$$\int_{\mathbb{R}} G^p \, d\mu < \infty.$$

So, $G^p \in L^1$. So, we can apply Dominated Convergence Theorem to conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \left| F - \sum_{k=1}^n f_k \right|^p \, d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} \left| F - \sum_{k=1}^n f_k \right|^p \, d\mu.$$

By construction, we have $\sum_{k=1}^\infty f_k \to F$ pointwise, so

$$\int_{\mathbb{R}} \lim_{n\to\infty} \left| F - \sum_{k=1}^n f_k \right|^p \, d\mu = 0.$$

This means that $\sum_{k=1}^\infty f_k \to F$ in L^p. Hence, every absolutely convergent sequence is convergent. We conclude that L^p space is complete. □

We can consider measures on probability spaces as well. We say that a measure space on $[0,1]$ is a probability, then $\mu[0,1] = 1$. Then, we have $L^p[0,1] \subseteq L^1[0,1]$. More generally, for $p > 1$, $L^p[0,1] \subseteq L[0,1]$. This follows from Holder's Inequality:

Proposition 3.1.4. *Let P be a probability space.*

Proof. Let $f \in L^p[0,1]$ and $g = 1$ on $[0,1]$. Then, by Holder Inequality, we find that

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_p \|1\|_p = \|f\|_p.$$

Hence, if $f \in L^p[0,1]$, $f \in L^1[0,1]$. □

Note that the result does not follow in $\mathbb{R}$, since $\|1\|_p = \infty$. Also, each inclusion is strict, e.g. $\frac{1}{\sqrt{x}}$ is in $L^2[0,1]$ but not in $L^1[0,1]$. We can generalise this to show that for $1 \leq p \leq q$, $L^q[0,1] \subseteq L^p[0,1]$, and the inclusion is strict if $p < q$.

3.2 Linear Operators

In this section, we will consider linear operators in more detail and do analysis on them. The concept of a linear operator is the same as a linear function, but being bounded as a linear operator has a different meaning.

Definition 3.2.1. Let V and W be normed spaces, and let $T\colon V \to W$ be a function. We say that T is *linear* (or an *operator*) if for all $v_1, v_2 \in V$ and $T(v_1 + v_2) = T(v_1) + T(v_2)$ and for all $v \in V$ and $c \in \mathbb{R}$, $T(cv) = cTv$. We say that T is *bounded* if there exists a $c \geq 0$ such that for all $v \in V$, $\|Tv\|_V \leq c\|v\|_W$. The set of bounded functions $V \to W$ is denoted by the set $L(V, W)$.

In finite dimensions, every linear function is bounded.

Proposition 3.2.2. *Let V and W be vector spaces such that V is finite-dimensional, and let $T\colon V \to W$ be linear. Then, T is bounded.*

Proof. Let the basis of V be: $\{v_1, v_2, \dots, v_n\}$. For each $1 \leq i \leq n$, there exists a $c_i > 0$ such that $\|Tv_i\| \leq c_i\|v_i\|$. Set $c = \min(c_1, c_2, \dots, c_n)$. By construction, we have $\|Tv_i\| \leq c\|v_i\|$ for $1 \leq i \leq n$. Now, let $v \in V$. In that case, there exist $\alpha_i \in \mathbb{R}$ for $1 \leq i \leq n$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Hence, □

This result does not hold in infinite dimensions. To see this, define the function $T\colon \ell^\infty \to \ell^\infty$ by $T(e_n) = ne_n$ extended linearly. This is linear, but not bounded. In particular, for all $n \in \mathbb{Z}_{\geq 1}$,

$$\|Te_{n+1}\| = (n+1)\|e_{n+1}\| > n + 1 = (n+1)\|e_{n+1}\|.$$

We can define isomorphisms and isometries like in metric spaces for linear operators.

Definition 3.2.3. Let V and W be vector spaces, and let $T\colon V \to W$ be a bounded function. We say that T is an *isomorphism* if T is a bijection with T^{-1} also bounded. We say that T is an *isometry* if for all $v \in V$, $\|Tv\| = \|v\|$. If T is both an isometry and isomorphism, then it is called an *isometric isomorphism.*

Lemma 3.2.4. *Let V and W be vector spaces, and let $T\colon V \to W$ be an isometry. Then, T is injective.*

Proof. Let $v_1, v_2 \in V$ with $Tv_1 = Tv_2$. In that case,

$$\|v_1 - v_2\| = \|T(v_1 - v_2)\| = \|Tv_1 - Tv_2\| = 0.$$

Hence, $v_1 = v_2$. So, T is injective. □

Proposition 3.2.5. *Let V and W be vector spaces, and let $T\colon V \to W$ be a surjective isometry. Then, T is an isometric isomorphism.*

Proof. Since T is an isometry, we know that it is injective. So, T is a bijection. Hence, it suffices to show that T^{-1} is bounded. We know that for all $v \in V$,

$$\|Tv\| = \|v\|.$$

So, for all $w \in W$ with $T^{-1}(w) = v$, we find that

$$\|w\| = \|Tv\| = \|v\| = \|T^{-1}(w)\|.$$

Hence, T^{-1} is bounded. So, T is an isometric isomorphism. □

It turns out that continuity and boundedness are equivalent for linear operators.

Proposition 3.2.6. *Let V and W be normed spaces and let $T\colon V \to W$ be a linear operator. Then, the following are equivalent:*

1. *T is continuous;*
2. *T is continuous at $0 \in V$;*
3. *T is bounded.*

Proof.

(1) $\implies$ (2) Trivial.

(2) $\implies$ (3) Assume that T is continuous at 0. In that case, there exists a $\delta > 0$ such that for $v \in V$, if $\|v\|_V < \delta$, then $\|Tv\|_W < \varepsilon$. Now, let $v \in V$. If $v = 0$, then

$$\|Tv\|_W = 0 < \frac{2\varepsilon}{\delta}\|v\|_V.$$

Otherwise, let $v' = \frac{\delta}{2\|v\|_V}v$. Then,

$$\|v'\|_V = \frac{\delta}{2\|v\|_V}\|v\|_v = \frac{\delta}{2} < \delta.$$

Hence,

$$\|Tv'\|_W < \varepsilon \iff \frac{\delta}{2\|v\|_V}\|Tv\|_W < \varepsilon \iff \|Tv\|_V < \frac{2\varepsilon}{\delta}\|v\|_V.$$

So, T is bounded.

(3) $\implies$ (1) Now, assume that T is bounded. In that case, there exists a $c \geq 0$ such that for $v \in V$, $\|Tv\|_W \leq c\|v\|_V$. If $c = 0$, then we find that $\|Tv\|_V = 0$ for all $v \in V$, i.e. $T = 0$. This is a continuous function. Otherwise, let $\varepsilon > 0$. Set $\delta = \varepsilon/c$. In that case, for $v, w \in V$, if $\|v - w\|_V < \delta$, then

$$\|Tv - Tw\|_W = \|T(v - w)\|_W \leq c\|v - w\|_V = \varepsilon.$$

Hence, T is (uniformly) continuous.

□

With this result, we can show that the set of bounded functions is a (normed) vector space.

Proposition 3.2.7. *Let V and W be normed spaces. Define the function $\|\cdot\| \colon L(V, W) \to \mathbb{R}_{\geq 0}$ by*

$$\|T\| = \sup_{\|v\|=1} \|Tv\|.$$

Then, $(L(V, W), \|\cdot\|)$ is a normed vector space.

Proof. We know that the sum and the scalar product of bounded functions is still bounded, so $L(V, W)$ is a vector space. Now, let $T \in L(V, W)$ such that $\|T\| = 0$. Then, let $v \in V$. If $v = 0$, then we know that $Tv = 0$. Otherwise, we know that $w = v/\|v\|$ has norm 1, meaning that $Tw = 0$. Hence, $Tv = 0$, so $T = 0$. Moreover,

$$\|0\| = \sup\{0\} = 0.$$

Now, let $c \in \mathbb{R}$ and $T \in L(V, W)$. Then,

$$\|cT\| = \sup_{\|v\|=1} \|cTv\| = \sup_{\|v\|=1} |c| \cdot \|Tv\| = |c|\|T\|.$$

Finally, let $T_1, T_2 \in L(V, W)$. Then,

$$\begin{aligned}
\|T_1 + T_2\| &= \sup_{\|v\|=1} \|T_1(v) + T_2(v)\| \\
&\leq \sup_{\|v\|=1} (\|T_1(v)\| + \|T_2(v)\|) \\
&\leq \sup_{\|v\|=1} \|T_1(v)\| + \sup_{\|v\|=1} \|T_2(v)\| = \|T_1\| + \|T_2\|.
\end{aligned}$$

So, $L(V, W)$ is a normed vector space. □

Moreover, there are many equivalent definitions for the operator norm.

Proposition 3.2.8. *Let V and W be vector spaces, and let $T \colon V \to W$ be a bounded function. Then, all of the following expressions are equal to the operator norm $\|T\|$:*

1. $\sup_{\|v\|=1} \|Tv\|$;
2. $\sup_{\|v\|\leq 1} \|Tv\|$;
3. $\sup_{v \neq 0} \|Tv\|/\|v\|$;
4. $\inf\{c \geq 0 \mid \|Tv\| \leq c\|v\| \ \forall v \in V\}$.

Proof.

(1) = (2) We know that

$$\{\|Tv\| \mid \|v\| = 1\} \subseteq \{\|Tv\| \mid \|v\| \leq 1\},$$

in which case

$$\sup_{\|v\|=1} \|Tv\| \leq \sup_{\|v\|\leq 1} \|Tv\|.$$

Now, let $v \in V$ with $\|v\| \leq 1$. If $v = 0$, then $\|Tv\| = 0 \leq \sup_{\|v\|=1}\|Tv\|$. Otherwise, if $v \neq 0$, then there exists a $c > 0$ such that $\|cv\| = 1$. Since $\|v\| \leq 1$, we can further assume that $c \geq 1$. Hence,

$$\|Tv\| = \frac{1}{c}\|T(cv)\| \leq \|T(cv)\|.$$

Hence,

$$\sup_{\|v\|\leq 1}\|Tv\| \leq \sup_{\|v\|=1}\|Tv\|,$$

meaning that the two values are equal.

(1) = (3) We show that

$$\{\|Tv\| \mid \|v\| = 1\} = \{\|Tv\|/\|v\| \mid v \neq 0\}.$$

Clearly, if $\|v\| = 1$, then $\|Tv\|/\|v\| = \|Tv\|$. Hence,

$$\{\|Tv\| \mid \|v\| = 1\} \subseteq \{\|Tv\|/\|v\| \mid v \neq 0\}.$$

Now, let $v \in V$ be non-zero. In that case, let $c = \frac{1}{\|v\|}$. Then, $\|cv\| = 1$, with

$$\|T(cv)\| = \|Tv\|/\|v\|.$$

Hence,

$$\{\|Tv\|/\|v\| \mid v \neq 0\} \subseteq \{\|Tv\| \mid \|v\| = 1\}.$$

So, the sets are equal, meaning that the supremum values agree too.

(3) = (4) The set

$$\{c \geq 0 \mid \|Tv\| \leq c\|v\| \ \forall v \in V\}$$

is the set of upper bounds of the set

$$\{\|Tv\|/\|v\| \mid v \in V\} \cup \{0\}.$$

So, we find that the infimum of the upper bounds equals the supremum of the set.

□

Using these definitions, we can show the following key lemma.

Lemma 3.2.9. *Let V and W be vector spaces, and let $T\colon V \to W$ be a bounded function. Then, for any $v \in V$, $\|Tv\| \leq \|T\|\|v\|$.*

Proof. We know that

$$\|T\| = \sup_{v\neq 0}\|Tv\|/\|v\|.$$

Hence, for all $v \in V$ non-zero,

$$\|T\| \geq \|Tv\|/\|v\| \iff \|T\|\|v\| \geq \|Tv\|.$$

Next, if $v = 0$, then we know that $\|Tv\| = 0 \leq 0 = \|T\|\|v\|$. So, the result follows. □

Proposition 3.2.10. *Let V and W be vector spaces, and let W be complete. Then, $L(V, W)$ is complete.*

Proof. Let $(T_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L(V, W)$. For $v \in V$ non-zero, consider the sequence $(T_n(v))_{n=1}^{\infty}$. We show that the sequence is Cauchy. So, let $\varepsilon > 0$. Since (T_n) is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $\|T_m - T_n\| < \frac{\varepsilon}{\|v\|}$. Hence, for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then

$$\|T_m(v) - T_n(v)\| = \|(T_m - T_n)(v)\| \leq \|T_m - T_n\| \|v\| < \varepsilon.$$

Hence, $(T_n(v))$ is Cauchy in W. Since W is complete, there exists a $t_v \in W$ such that $T_n(v) \to t_v$. Now, define the function $T \colon V \to W$ by $T(v) = t_v$. We show that $T_n \to T$ in $L(V, W)$.

First, we show that $T \in L(V, W)$. Let $v_1, v_2 \in V$. We know that for all $n \in \mathbb{Z}_{\geq 1}$, $T_n(v_1 + v_2) = T_n(v_1) + T_n(v_2)$. Hence, $T(v_1 + v_2) = T(v_1) + T(v_2)$. Now, let $v \in V$ and $c \in \mathbb{R}$. We know that for all $n \in \mathbb{Z}_{\geq 1}$, $T_n(cv) = cT_n(v)$. Hence, $T(cv) = cT(v)$. This implies that $T \in L(V, W)$.

Now, we show that $T_n \to T$. So, let $\varepsilon > 0$. Since (T_n) is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $\|T_m - T_n\| < \frac{\varepsilon}{3}$. Next, let $v \in V$ with $\|v\| = 1$. Since $T_n(v) \to T(v)$, we can find a $K \in \mathbb{Z}_{\geq 1}$ such that for $k \in \mathbb{Z}_{\geq 1}$, if $k \geq K$, then $\|T(v) - T_m(v)\| < \frac{\varepsilon}{3}$. Then, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then

$$\begin{aligned}
\|(T - T_n)(v)\| &\leq \|T(v) - T_n(v)\| \\
&\leq \|T(v) - T_K(v)\| + \|T_K(v) - T_n(v)\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\end{aligned}$$

So, $\|T - T_n\| \leq \frac{2\varepsilon}{3} < \varepsilon$. This means that $T_n \to T$. Hence, $L(V, W)$ is complete. $\square$

We now define the dual space.

Definition 3.2.11. Let X be a K-normed vector space. We say that a function $T \colon X \to K$ is *functional* if it is linear, and the *dual space* X^* is the set of bounded functionals.

It turns out that for $1 < p < \infty$, we have an isometric isomorphism

$$(L^p)^* \cong L^q,$$

where q is the dual number of p, i.e. $1/p + 1/q = 1$. In particular, if $p = 2$, then $(L^2)^* \cong L^2$.

Theorem 3.2.12. *The sequence spaces ℓ^∞ and $(\ell^1)^*$ are isometrically isomorphic.*

Proof. Define the map $T \colon \ell^\infty \to (\ell^1)^*$ by $T(x) = f_x$, where $f_x \colon \ell^1 \to \mathbb{C}$ is given by

$$f_x(y) = \sum_{n=1}^{\infty} x_n y_n.$$

We first show that f_x is well-defined, i.e. $f_x(y) \in (\ell^1)^*$. By construction, f_x is linear. Now, let $y \in \ell^1$. We know that $|x_n| \leq \|x\|_\infty$ for all $n \in \mathbb{Z}_{\geq 1}$, meaning that

$$|f_x(y)| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \leq \sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_\infty \sum_{n=1}^{\infty} |y_n|.$$

Since $y \in \ell^1$, this implies that $f_x(y) \in \ell^1$. So, f_k is well-defined. Now, we show that T is bounded. So, let $y \in \ell^1$. Then,

$$\|f_x(y)\|_1 = \sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_\infty \sum_{n=1}^{\infty} |y_n| = \|x\|_\infty \|y\|_1.$$

Hence, f_x is bounded with $\|f_x\| \leq \|x\|_\infty$

We now show that T is an isometry. So, let $x = (x_n)_{n=1}^\infty \in \ell^\infty$. Define the sequences $(r_n)_{n=1}^\infty, (\theta_n)_{n=1}^\infty$ by the polar decomposition of x_n, i.e. $x_n = r_n e^{i\theta_n}$. For each $k \in \mathbb{Z}_{\geq 1}$, define the sequence $(y_n^{(k)})_{n=1}^\infty$

$$y_n^{(k)} = \begin{cases} e^{-i\theta_n} & n = k \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\|y^{(k)}\|_\infty = 1$, with

$$\|f_x(y^{(k)})\|_1 = \sum_{n=1}^{\infty} |x_n y_n| = |x_n e^{-i\theta_n}| = r_n = |x_n|.$$

So, we find that

$$\{|x_n| \mid n \in \mathbb{Z}_{\geq 1}\} \subseteq \{\|f_x(y)\|_1 \mid \|y\|_1 = 1\}.$$

So, the supremum property tells us that

$$\|x\|_\infty = \sup_{n=1}^{\infty} |x_n| \leq \sup_{\|y\|_1 = 1} \|f_x(y)\|_1 = \|f_x\|.$$

Hence, it follows that

$$\|f_x\| = \|x\|_\infty.$$

This implies that T is isometric.

Finally, we show that T is surjective. So, let $f \in (\ell^1)^*$. Define the sequence $x = (x_n)_{n=1}^\infty$ in $\mathbb{C}$ by $x_n = f(e_n)$. Then, for all $n \in \mathbb{Z}_{\geq 1}$, we find that

$$|x_n| = |f(e_n)| \leq \|f\| \|e_n\|_1 = \|f\|.$$

Hence, $x \in \ell^\infty$. Moreover, for all $y \in \ell^1$, we find that

$$f(y) = \sum_{n=1}^{\infty} f(y_n e_n) = \sum_{n=1}^{\infty} y_n f(e_n) = \sum_{n=1}^{\infty} y_n x_n = f_x(y).$$

So, $T(x) = f_x = f$, meaning that T is surjective. Hence, T is an isometric isomorphism. $\square$

3.3 Hahn-Banach

In this section, we define Hahn-Banach theorem, which allows us to extend linear functions. To prove this, we first need to consider partial order and then Zorn's Lemma.

Definition 3.3.1. Let X be a set, and let $\leq$ be a relation on X. We say that $(X, \leq)$ is a *partial order* if:

- for all $x \in X$, $x \leq x$;
- for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$; and
- for all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

Examples of partial order include $(\mathbb{R}, \leq)$ and $(\mathcal{P}(X), \subseteq)$ for some set X. The relation does not require every element to be related- not any 2 subset of X needs to satisfy the condition that one is the subset of the other. However, this is satisfied in $(\mathbb{R}, \leq)$, which gives rise to a total order.

Definition 3.3.2. Let X be a set, and let $\leq$ be a partial order on X. Then, we say that $(X, \leq)$ is a *total order* if for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Now, in partial order, we can define a maximal element in the set.

Definition 3.3.3. Let X be a set with partial order $\leq$, and let $x \in X$. We say that x is a *maximal element* if for all $y \in X$ such that $x \leq y$, we have $y = x$.

It is not necessarily the case that a partial order has a maximal element, or that it is unique. We now consider upper bounds.

Definition 3.3.4. Let X be a set with partial order $\leq$, $E \subseteq X$ and $x \in X$. We say that x is an *upper bound* of E if for all $e \in E$, $e \leq x$.

The upper bound need not lie in E, or be unique. Finally, we define well-orderedness.

Definition 3.3.5. Let X be a set with partial order $\leq$. We say that it is *well-ordered* if for every $E \subseteq X$, E has a minimal element.

For example, $\mathbb{R}$ is not well-ordered using the normal ordering. We now consider Zorn's Lemma.

Lemma 3.3.6 (Zorn's Lemma)**.** *Let X be a non-empty set with partial order $\leq$, and let $E \subseteq X$ be a non-empty totally ordered with respect to $\leq$ that has an upper bound. Then, X has a maximal element.*

This lemma is equivalent to the axiom of choice (ZFC).

We will now look at Hahn-Banach Theorem. Before doing so, we define a sublinear functional.

Definition 3.3.7. Let V be a normed vector space and let $p \colon X \to \mathbb{R}$ be a function. We say that p is *sublinear* if

- for all $x, y \in X$, $p(x + y) \leq p(x) + p(y)$; and

- for all $x \in X$ and $c \geq 0$, $p(cx) = cp(x)$.

Theorem 3.3.8 (Hahn-Banach). *Let V be a real normed vector space and let $p\colon V \to \mathbb{R}$ be a sublinear function, $M \subseteq V$ be a subspace and let $f\colon M \to \mathbb{R}$ be a linear functional, and $f \leq p$. Then, there exists an $F\colon V \to \mathbb{R}$ such that $F|_M = f$ and $f \leq p$.*

Proof. If $M = X$, then there is nothing to show. So, assume that $M \subsetneq X$. We prove this using Zorn's Lemma. So, define the set

$$X = \{F\colon Y \to \mathbb{R} \mid F \subsetneq M, F|_M = f, F \leq p\}.$$

We can define the partial order $\leq$ on X by inclusion of domains, and extensionality, i.e. for F_1 and F_2 in X with domains Y_1 and Y_2, we say that $F_1 \leq F_2$ if $Y_1 \subseteq Y_2$, and $F_2|_{Y_1} = F_1$.

First, we show that X is non-empty. Let $x \in X \setminus M$. For $y_1, y_2 \in M$, we find that

$$\begin{aligned} f(y_1) + f(y_2) &= f(y_1 + y_2) \\ &\leq p(y_1 + y_2) \\ &= p(y_1 - x + x + y_2) \\ &\leq p(y_1 - x) + p(x + y_2). \end{aligned}$$

So, we find that

$$f(y_1) - p(y_1 - x) \leq p(x + y_2) - f(y_2)$$

for all $y_1, y_2 \in M$. Hence,

$$\sup_{y \in M} f(y) - p(y - x) \leq \inf_{y \in M} p(x + y) - f(y).$$

So, we can find an $\alpha \in \mathbb{R}$ such that

$$\sup_{y \in M} f(y) - p(y - x) \leq \alpha \leq \inf_{y \in M} p(x + y) - f(y).$$

Now, define $F\colon M + \mathbb{R}x \to \mathbb{R}$ by $F(y + \lambda x) = f(y) + \lambda\alpha$. We claim that $F \in X$. Since $x \notin M$, we find that $M + \mathbb{R}x \subsetneq M$, and that F extends f. Moreover, for all $y + \lambda x \in M + x\mathbb{R}$, if $\lambda \neq 0$, then

$$\begin{aligned} F(y + \lambda x) &= f(y) + \lambda\alpha \\ &= \lambda(f(y/\lambda) + \alpha) \\ &\leq \lambda(f(y/\lambda) + p(x + y/\lambda) - f(y/\lambda)) \\ &= \lambda(p(x + y/\lambda)) \\ &= p(y + \lambda x). \end{aligned}$$

So, $F \leq p$, meaning that $F \in X$.

We now show that every totally ordered set $E \subseteq X$ has a upper bound, with

$$E = \{F_i\colon Y_i \to \mathbb{R} \in X \mid i \in I\},$$

for some indexing set I. Now, define the set

$$Y = \bigcup_{i \in I} Y_i$$

and the map $F \colon Y \to \mathbb{R}$ by $F(y) = F_i(y)$, where $y \in Y_i$. Since the order assumes extensionality, it is well-defined. We claim that F is an upper bound for E. By construction, F extends each $F_i \in E$, and $Y \subseteq Y_i$ for all $i \in I$. So, F is an upper bound for E.

Finally, we can apply Zorn's Lemma to find a maximal element $F \colon V \to \mathbb{R}$. □

The Hahn-Banach theorem can be generalised for complex functions as well.

Theorem 3.3.9 (Hahn-Banach for complex functions)**.** *Let V be a complex normed vector space and let $p \colon V \to \mathbb{C}$ be a sublinear function, $M \subseteq V$ be a subspace and let $f \colon M \to \mathbb{R}$ be a seminorm, and $|f| \leq p$. Then, there exists an $F \colon V \to \mathbb{C}$ such that $F|_M = f$ and $|f| \leq p$.*

A seminorm is a norm such that $\|x\| = 0$ does not imply that $x = 0$.

We will now look at some consequences of the Hahn-Banach Theorem.

Corollary 3.3.10. *Let V be a normed vector space, $M \subseteq V$ be a subspace and let $f \in M^*$. Then, there exists a linear functional $F \in V^*$ such that F extends f, with $\|F\| = \|f\|$.*

Proof. Note that for any extension F of f, we have $\|F\| \geq \|f\|$ by the supremum property. Now, define the map $p \colon X \to \mathbb{C}$ by $p(x) = \|f\|\|x\|$. By construction, we find that for all $y \in M$,

$$|f(y)| \leq \|f\|\|y\| = p(y),$$

meaning that $|f| \leq p$. We now show that p is a seminorm. For all $\lambda \in \mathbb{C}$ and $y \in M$, we find that

$$p(\lambda y) = \|f\|\|\lambda y\| = |\lambda|\|f\|\|y\| = |\lambda| p(y).$$

Moreover, for all $y_1, y_2 \in M$,

$$\begin{aligned} p(y_1 + y_2) &= \|f\|\|y_1 + y_2\| \\ &\leq \|f\|(\|y_1\| + \|y_2\|) \\ &= \|f\|\|y_1\| + \|f\|\|y_2\| \\ &= p(y_1) + p(y_2). \end{aligned}$$

Hence, p is a seminorm. So, Hahn-Banach allows us to extend f into a function $F \in V^*$. Moreover, since $F \leq p$, we find that for all $v \in V$,

$$|F(v)| \leq p(v) = \|f\|\|v\|.$$

So, $\|F\| \leq \|f\|$, meaning that $\|F\| = \|f\|$. □

Corollary 3.3.11. *Let V be a normed vector space, $M \subsetneq V$ be closed and let $x \in M \setminus V$, and denote*

$$\delta = \inf_{y \in M} \|x - y\|.$$

Then, there exists an $F \in V^*$ such that $\|F\| = 1$, $F(x) = \delta$ and $M \subseteq \ker F$.

Proof. Define the map $f \colon M + \mathbb{C}x \to \mathbb{C}$ by $f(y + \lambda x) = \lambda\delta$. This is a linear functional by definition. Next, define the function $p \colon V \to \mathbb{R}$ by $p(x) = \|x\|$. Since this is a norm, it is a seminorm. Moreover, for all $y + \lambda x \in M + \mathbb{C}x$, if $\lambda = 0$, then $f(y + \lambda x) = 0 \leq \|y + \lambda x\|$, and if $\lambda \neq 0$, then

$$|f(y + \lambda x)| = |\lambda|\delta \leq |\lambda| \left\| \frac{1}{\lambda} y + x \right\| = \|y + \lambda x\|.$$

So, $|f| \leq p$. Applying Hahn-Banach, we can find a function $F \in V^*$ that extends f. In particular, we still have $M \subseteq \ker F$ and $F(x) = \delta$. Finally, for all $v \in V$, we have

$$|F(v)| \leq p(v) = \|F\|\|v\|,$$

meaning that $\|F\| \leq 1$. □

Corollary 3.3.12. *Let V be a normed vector space and let $v \in V$ be non-zero. Then, there exists a functional $f \in V^*$ such that $\|f\| = 1$ and $f(v) = \|v\|$.*

Proof. Let $M = \{0\}$. Then, it is a closed proper subset of V, with

$$\delta = \inf_{y \in M} \|x - y\| = \|v\|.$$

Hence, there exists a functional $f \in V^*$ such that $\|f\| = 1$ and $F(v) = \|v\|$. □

Corollary 3.3.13. *Let V be a normed vector space and let $x, y \in V$ be distinct. Then, there exists a functional $f \in V^*$ such that $f(x) \neq f(y)$. In particular, linear functionals separate the vector space.*

Proof. Since x and y are distinct, we find that $x - y \neq 0$. Hence, there exists a functional $f \in V^*$ such that $f(x - y) = \|x - y\| \neq 0$. So, $f(x) \neq f(y)$. □

Now, we show that the double dual of a vector space always has an isometry from the vector space.

Proposition 3.3.14. *Let V be a vector space, $v \in V$ and consider the evaluation map $\hat{v} \colon V^* \to \mathbb{C}$ given by $\overline{v}(f) = f(v)$. Then, the map $T \colon V \to V^{**}$ given by $T(v) = \hat{v}$ is an isometry.*

Proof. Let $v \in V$. We show that $\|v\| = \|\hat{v}\|$ for all $v \in V$. So, let $f \in V^*$. Then,

$$\|\hat{v}(f)\| = \|f(v)\| \leq \|f\|\|v\|.$$

This implies that $\|\hat{v}\| \leq \|v\|$. Now, consider the identity map $f \in V^*$. We know that $\|f\| = 1$, with

$$\|\hat{v}(f)\| = \|f(v)\| = \|v\|.$$

Hence, we find that $\|\hat{v}\| = \|v\|$. □

This is a key result- assuming that the field is complete (which is true for $\mathbb{R}$ and $\mathbb{C}$), we have found an embedding of V into a complete space V^{**}. Hence, we can identify its completion as a concrete subspace of V^{**}. There are cases when V^{**} is also isometrically isometric, e.g. L^p for $p > 1$, in which case we say that the vector space is *reflexive*.

3.4 Baire-Category Theorem

Definition 3.4.1. Let X be a topological space and let $E \subseteq X$.

- We say that E is *open dense* if E is open with closure X.
- We say that E is *nowhere dense* if the complement of its closure $\overline{E}^c$ is open dense.
- E is a *meagre* (or *first category*) if it is a countable union of nowhere dense sets.
- E is of *second category* if it is not of first category.

Theorem 3.4.2 (Baire-Category Theorem). *Let X be a complete metric space, and let $(U_n)_{n=1}^{\infty}$ be a sequence of open dense sets in X. Then, the intersection*

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X.

Proof. Let $W \subseteq X$ be open. We show that for all $n \in \mathbb{Z}_{\geq 1}$, $U_n \cap W$ is non-empty. Since U_1 is open dense, we know that $U_1 \cap W$ is a non-empty open set. Hence, there exists an open ball $B_{r_0}(x_0) \subseteq U_1 \cap W$. Without loss of generality, assume that $r_0 \leq 1$. Now, since U_2 is open dense, we can find an open ball $B_{r_1}(x_1)$ such that $\overline{B_{r_1}(x_1)} \subseteq U_1 \cap B_{r_0}(x_0)$, with $r_1 < 2^{-1}$. We can continue on finding open balls $(B_{r_n}(x_n))_{n=0}^{\infty}$ such that $\overline{B_{r_n}(x_n)} \subseteq U_n \cap B_{r_{n-1}}(x_{n-1})$ for $n \geq 1$.

We now claim that the sequence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. Let $\varepsilon > 0$. Select an $N \in \mathbb{Z}_{\geq 1}$ such that $2^{N+1} > \frac{1}{\varepsilon}$. In that case, for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then we know that $x_m, x_n \in U_{N+1} \cap B_{r_N}(x_N)$, in which case

$$|x_m - x_n| \leq |x_m - x_N| + |x_N - x_n| < r_N + r_N = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

So, (x_n) is Cauchy.

Since X is complete, $x_n \to x$ for some $x \in X$. By construction, we have

$$x \in \bigcap_{n=1}^{\infty} \overline{B}_{r_n}(x_n) \subseteq \bigcap_{n=0}^{\infty} U_n \cap B_{r_{n-1}}(x_{n-1}) \subseteq \bigcap_{n=0}^{\infty} U_n \cap W.$$

So, the intersection is non-empty, meaning that it is dense in X. □

Corollary 3.4.3. *Let X be a complete metric space. Then, it is of second category.*

Proof. Let $(E_1)_{n=1}^{\infty}$ be a sequence of nowhere dense sets. We show that the union

$$\bigcup_{n=1}^{\infty} E_n \subsetneq X.$$

We know that $(\overline{E}_n^c)$ is a sequence of open dense sets. By Baire-Category Theorem, we know that the intersection

$$\bigcap_{n=1}^{\infty} \overline{E}_n^c$$

is dense in X. In particular, the intersection is non-empty, meaning that its complement

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E}_n \neq X.$$

So, X cannot be of first category. $\square$