CHAPTER 2

DIFFERENTIATION

2.1 Definition of Differentiation

We start by defining derivatives.

Definition 2.1.1 (Derivative)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be a function and let $c \in \mathbb{R}$. We define the *derivative of f at c* to be the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

If the limit exists, we say that f is *differentiable at c*, and denote the value of the limit by $f'(c)$. If f is differentiable at all $c \in \mathbb{R}$, we say that f is *differentiable*.

This limit can also be written in the form

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

where we have made a change of variable in the limit: $h = x - c$. We start by showing that the function x^n is differentiable with derivative nx^{n-1}.

Example 2.1.2. *Let $n \in \mathbb{Z}_{\geqslant 1}$, and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^n$. Then, f is differentiable, with $f'(c) = nc^{n-1}$ for all $c \in \mathbb{R}$.*

Proof. We find that

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{x^n - c^n}{x - c} \\ &= \lim_{x \to c} \frac{(x-c)(x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1})}{x - c} \\ &= \lim_{x \to c} x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1} \\ &= \underbrace{c^{n-1} + c^{n-1} + \cdots + c^{n-1}}_{n \text{ times}} = nc^{n-1}. \end{aligned}$$

□

Next, we show that the absolute value function is differentiable at every non-zero value, but not differentiable at 0.

Example 2.1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$. Then, for $c \neq 0$, f is differentiable at c, with*

$$f'(c) = \begin{cases} 1 & c > 0 \\ -1 & c < 0 \end{cases}.$$

Moreover, f is not differentiable at 0.

Proof.

- First, assume that $c > 0$. Define the function $f_1 : (0, \infty)$ by $f_1(x) = x$. We know that for $x \in \mathbb{R}$, if $|x - c| < \frac{c}{2}$, then $x > 0$ and so $f(x) = f_1(x)$. This implies that
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f_1(x) - f_1(c)}{x - c} = \lim_{x \to c} \frac{x - c}{x - c} = 1.$$
- Next, assume that $c < 0$. Define the function $f_2 : (-\infty, 0)$ by $f_2(x) = -x$. We know that for $x \in \mathbb{R}$, if $|x - c| < \frac{|c|}{2}$, then $x < 0$ and so $f(x) = f_2(x)$. This implies that
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f_2(x) - f_2(c)}{x - c} = \lim_{x \to c} \frac{c - x}{x - c} = -1.$$
- Finally, assume that $c = 0$. We know for $x < 0$, $f(x) = -x$, so the limit
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$
Moreover, for $x > 0$, $f(x) = x$, so the limit
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1.$$
Therefore, the limit
$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$$
cannot exist. So, f is not differentiable at 0.

Therefore, f is differentiable at $x \in \mathbb{R}$ if and only if $x \neq 0$, with
$$f'(c) = \begin{cases} 1 & c > 0 \\ -1 & c < 0 \end{cases}.$$
□

Now, we look at few properties of derivatives. First, differentiability implies continuity.

Proposition 2.1.4. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function, and let $c \in \mathbb{R}$ such that f is differentiable at c. Then, f is continuous at c.*

Proof. Since f is differentiable at c, we know that the limit
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$
exists. In that case,
$$\begin{aligned} \lim_{x \to c} f(x) - f(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}(x - c) \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} x - c \\ &= f'(c) \cdot \lim_{x \to c} x - c \\ &= f'(c) \cdot 0 = 0. \end{aligned}$$
Therefore, $\lim_{x \to c} f(x) = f(c)$. This implies that f is continuous at c. □

Now, we show that a multiple of a differentiable function is differentiable.

Proposition 2.1.5. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function, and let $\lambda \in \mathbb{C}, c \in \mathbb{R}$ such that f is differentiable at c. Then, λf is differentiable at c, with $(\lambda f)'(c) = \lambda f'(c)$.*

Proof. We find that

$$\begin{aligned}(\lambda f)'(c) &= \lim_{x \to c} \frac{(\lambda f)(x) - (\lambda f)(c)}{x - c} \\ &= \lim_{x \to c} \frac{\lambda(f(x) - f(c))}{x - c} \\ &= \lambda \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lambda f'(c).\end{aligned}$$

□

Next, we show that the sum of two differentiable functions is differentiable.

Proposition 2.1.6. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions and let $c \in \mathbb{R}$ such that f and g are differentiable at c. Then, $f+g$ is differentiable at c, with $(f+g)'(c) = f'(c) + g'(c)$.*

Proof. We have

$$\begin{aligned}(f+g)'(c) &= \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= f'(c) + g'(c).\end{aligned}$$

□

Finally, we show that the conjugate of differentiable function is differentiable.

Proposition 2.1.7. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function, and let $c \in \mathbb{R}$ such that f is differentiable at c. Then, $\overline{f}$ is differentiable at c, with $\overline{f}'(c) = \overline{f'(c)}$.*

Proof. We have

$$\overline{f}'(c) = \lim_{x \to c} \frac{\overline{f}(x) - \overline{f}(c)}{x - c} = \lim_{x \to c} \frac{\overline{f(x) - f(c)}}{\overline{x - c}} = \overline{f'(c)}.$$

□

This property allows us to show that the real part and the imaginary part of a function must be differentiable as well.

Corollary 2.1.8. *Let $f : \mathbb{R} \to \mathbb{C}$, and let $c \in \mathbb{R}$ such that f is differentiable at c. Then, the functions $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are differentiable at c, with $\mathrm{Re}(f)'(c) = \mathrm{Re}(f'(c))$ and $\mathrm{Im}(f)'(c) = \mathrm{Im}(f'(c))$.*

Proof. We know that

$$\mathrm{Re}(f)(x) = \frac{f(x) + \overline{f}(x)}{2}, \qquad \mathrm{Im}(f)(x) = \frac{f(x) - \overline{f}(x)}{2}.$$

Therefore,

$$\mathrm{Re}(f)'(c) = \frac{f'(c) + \overline{f}'(c)}{2} = \frac{f'(c) + \overline{f'(c)}}{2} = \mathrm{Re}(f'(c)),$$

and

$$\mathrm{Im}(f)'(c) = \frac{f'(c) - \overline{f}'(c)}{2i} = \frac{f'(c) - \overline{f'(c)}}{2i} = \mathrm{Im}(f'(c)).$$

□

We will now prove the product rule.

Proposition 2.1.9 (Product Rule)**.** *Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions and let $c \in \mathbb{R}$ such that f and g are differentiable at c. Then, fg is differentiable at c, with $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.*

Proof. We have

$$\begin{aligned}
(fg)'(c) &= \lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x \to c} \frac{g(x)[f(x) - f(c)]}{x - c} + \lim_{x \to c} \frac{f(c)[g(x) - g(c)]}{x - c} \\
&= \lim_{x \to c} g(x) \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + f(c) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= g(c)f'(c) + f(c)g'(c).
\end{aligned}$$

□

Next, we look at quotient rule. First, we show that the $\frac{1}{f}$ is differentiable.

Lemma 2.1.10. *Let $f : \mathbb{R} \to \mathbb{C}$ such that f is differentiable at c, with $f(c) \neq 0$. In that case, $\frac{1}{f}$ is differentiable at c, with*

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f(c)^2}.$$

Proof. Since f is continuous at c with $f(c) \neq 0$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{|f(c)|}{2}$, and so $f(x) \neq 0$. Therefore, the expression $\frac{1}{f(x)}$ is well-defined. In that case,

$$\begin{aligned}
\left(\frac{1}{f}\right)'(c) &= \lim_{x\to c} \frac{\frac{1}{f(x)} - \frac{1}{f(c)}}{x-c} \\
&= \lim_{x\to c} \frac{f(c)-f(x)}{f(x)\cdot f(c)\cdot (x-c)} \\
&= -\lim_{x\to c} \frac{1}{f(x)\cdot f(c)} \cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= -\frac{1}{f(c)^2}\cdot f'(c) = -\frac{f'(c)}{f(c)^2}.
\end{aligned}$$

□

Using this result, we prove the quotient rule.

Proposition 2.1.11 (Quotient Rule)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$, and let $c \in \mathbb{R}$ such that both f and g are differentiable at c, with $g'(c) \neq 0$. In that case, $\frac{f}{g}$ is differentiable at c, with*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Proof. We know that

$$\frac{f(c)}{g(c)} = f(c)\cdot\frac{1}{g(c)}.$$

So, the product rule tells us that

$$\begin{aligned}
\left(\frac{f}{g}\right)'(c) &= f'(c)\cdot\frac{1}{g(c)} + f(c)\cdot\left(\frac{1}{g}\right)'(c) \\
&= \frac{f'(c)}{g(c)} - \frac{f(c)g'(c)}{g(c)^2} \\
&= \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.
\end{aligned}$$

□

Next, we look at the chain rule.

Proposition 2.1.12 (Chain Rule)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{R}$, and let $c \in \mathbb{R}$ such that g is differentiable at c and f is differentiable at $g(c)$. In that case, $f \circ g$ is differentiable at c, with*

$$(f\circ g)'(c) = f'(g(c))\cdot g'(c).$$

Proof. Define the function $h : \mathbb{R} \to \mathbb{C}$ by

$$h(x) = \begin{cases} \frac{f(g(x))-f(g(c))}{g(x)-g(c)} & g(x) \neq g(c) \\ f'(g(c)) & g(x) = g(c). \end{cases}$$

For $x \in \mathbb{R}$ with $x \neq c$, if $g(x) \neq g(c)$, we find that

$$\frac{f(g(x)) - f(g(c))}{x - c} = \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} = h(x) \cdot \frac{g(x) - g(c)}{x - c}.$$

Instead, if $g(x) = g(c)$, then

$$\frac{f(g(x)) - f(g(c))}{x - c} = 0 = f'(g(c)) \cdot \frac{g(x) - g(c)}{x - c} = h(x) \cdot \frac{g(x) - g(c)}{x - c}.$$

Therefore, for $x \in \mathbb{R}$ with $x \neq c$,

$$\frac{f(g(x)) - f(g(c))}{x - c} = h(x) \cdot \frac{g(x) - g(c)}{x - c}.$$

Since f is differentiable at $g(c)$, we know that

$$\lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} = \lim_{x \to g(c)} \frac{f(x) - f(g(c))}{x - g(c)} = f'(g(c)),$$

and so the limit $\lim_{x \to c} h(x) = f'(g(c))$. In that case,

$$\begin{aligned}
(f \circ g)'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} h(x) \cdot \frac{g(x) - g(c)}{x - c} \\
&= \lim_{x \to c} h(x) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= f'(g(c)) \cdot g'(c).
\end{aligned}$$

□

The intuitive idea in the proof of the chain rule is

$$\begin{aligned}
(f \circ g)'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} = f'(g(c)) \cdot g'(c).
\end{aligned}$$

The issue here is the denominator being $g(x) - g(c)$. Since we are taking limits, it is not an issue just when $g(x) = g(c)$. However, if for all $x \in \mathbb{R}$, $g(x) = g(c)$ (i.e. g is a constant), then taking the limit will be an issue. For this reason, we define the function h in the proof that avoids dividing by 0. When $g(x) = g(c)$, we already make it equal to the limit $f'(g(c))$, so h behaves like $\frac{f(g(x)) - f(g(c))}{x - c}$ all the time.

We finish by proving the Inverse Function Theorem.

Theorem 2.1.13 (Inverse Function Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a bijection, and let $c \in \mathbb{R}$ such that f is differentiable at c with $f'(c) \neq 0$. Then, the inverse function f^{-1} is differentiable at $d = f(c)$, with*

$$(f^{-1})'(d) = \frac{1}{f'(c)} = \frac{1}{f'(f^{-1}(d))}.$$

Proof. Define the function $g : \mathbb{R} \to \mathbb{R}$ be the function

$$g(x) = \begin{cases} \frac{x-c}{f(x)-f(c)} & x \neq c \\ \frac{1}{f'(c)} & x = c. \end{cases}$$

We know that the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c),$$

so for $x \in \mathbb{R}$ with $x \neq c$,

$$\lim_{x \to c} \frac{x - c}{f(x) - f(c)} = \frac{1}{f'(c)}.$$

Therefore, the limit

$$\lim_{x \to c} g(x) = \frac{1}{f'(c)}.$$

This implies that the limit

$$\begin{aligned} (f^{-1})'(d) &= \lim_{x \to d} \frac{f^{-1}(x) - f^{-1}(d)}{x - d} \\ &= \lim_{x \to c} \frac{x - c}{f(x) - f(c)} \\ &= \lim_{x \to c} g(x) = \frac{1}{f'(c)}. \end{aligned}$$

□

Like in the chain rule, we define the function g to deal with division by 0. We finish with an example using the Inverse Function Theorem.

Example 2.1.14. *Let $n \in \mathbb{Z}_{\geqslant 1}$ and define the function $f : (0, \infty) \to (0, \infty)$ by $f(x) = x^{1/n}$. Then, f is differentiable, with $f'(x) = \frac{1}{n} x^{1/n-1}$.*

Proof. We know that f is bijective, with $f^{-1}(x) = x^n$ and $(f^{-1})'(x) = nx^{n-1}$. Let $x \in (0, \infty)$. In that case, the Inverse Function Theorem tells us that

$$\begin{aligned} f'(x) &= \frac{1}{(f^{-1})'(f(x))} \\ &= \frac{1}{(f^{-1})'(x^{1/n})} \\ &= \frac{1}{n(x^{1/n})^{n-1}} \\ &= \frac{1}{n(x^{1-1/n})} \\ &= \frac{1}{n} x^{1/n-1}. \end{aligned}$$

□

2.2 The Mean Value Theorem

In this section, we will prove the Mean Value Theorem and look at some results that follow from it. We start by defining local maxima and minima.

Definition 2.2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$. Then, f attains a local minimum at c if there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta$, then $f(c) \leqslant f(x)$. Similarly, f attains a local minimum at c if there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta$, then $f(c) \geqslant f(x)$. If f attains a local minimum or maximum at c, we say that f attains a local extremum at c.

We now show that if a function is differentiable at a local extremum, then the derivative has to be 0.

Lemma 2.2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b), and let $c \in (a, b)$ such that f attains a local extremum at c. Then, $f'(c) = 0$.*

Proof. Without loss of generality, assume that f attains a local maximum at c. In that case, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta$, then $f(x) \leqslant f(c)$. In that case,

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geqslant 0, \qquad \text{and} \qquad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leqslant 0.$$

Since the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists, we find that $f'(c) = 0$. □

Using this result, we prove Rolle's Theorem.

Theorem 2.2.3 (Rolle's Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b), with $f(a) = f(b)$. Then, there exists a $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since f is continuous on $[a, b]$, the Extreme Value Theorem tells us that there exist $u, v \in [a, b]$ such that for all $x \in [a, b]$, $f(u) \leqslant f(x) \leqslant f(v)$.

- If both $u, v \in \{a, b\}$, then $f(u) = f(v)$. In that case, f is a constant. So, take $c = \frac{b-a}{2} \in (a, b)$. Since f is a constant, we know that $f'(c) = 0$.
- Otherwise, $u \notin \{a, b\}$ or $v \notin \{a, b\}$. Take $c \in \{u, v\}$ such that $c \notin \{a, b\}$. In that case, f is differentiable at c, which is a local minimum. Therefore, $c \in (a, b)$ satisfies $f'(c) = 0$.

Therefore, there exists a $c \in (a, b)$ such that $f'(c) = 0$. □

Finally, we prove the Mean Value Theorem.

Theorem 2.2.4 (Mean Value Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define the function $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

We have

$$\begin{aligned} g(a) - g(b) &= f(a) - \frac{f(b) - f(a)}{b - a}(a - a) - f(b) + \frac{f(b) - f(a)}{b - a}(b - a) \\ &= f(a) - f(b) + f(b) - f(a) = 0, \end{aligned}$$

so $g(a) = g(b)$. In that case, Rolle's Theorem tells us that there exists a $c \in (a, b)$ such that $g'(c) = 0$. Therefore,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

We will now look at many of the corollaries of the Mean Value Theorem. We start by showing that the derivative always being zero implies the function is a constant.

Corollary 2.2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) such that for all $x \in (a, b)$, $f'(x) = 0$. Then, f is constant on $[a, b]$.*

Proof. Let $x \in (a, b]$. Since f is continuous on $[a, x]$ and differentiable on (a, x), the Mean Value Theorem tells us that there exists a $c \in (a, x)$ such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

We know that $f'(c) = 0$. Therefore, $f(x) = f(a)$. So, for all $x \in [a, b]$, $f(x) = f(a)$. This implies that f is constant on $[a, b]$. □

We now show that if the derivative is strictly positive, then the function is strictly increasing.

Corollary 2.2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) such that for all $x \in (a, b)$, $f'(x) > 0$. Then, f is strictly increasing on $[a, b]$.*

Proof. Let $x, y \in [a, b]$ such that $x < y$. Since f is continuous on $[x, y]$ and differentiable on (x, y), the Mean Value Theorem tells us that there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

We know that $f'(c) > 0$. Moreover, since $y - x > 0$, we find that $f(y) - f(x) > 0$. Therefore, $f(x) < f(y)$. This implies that f is strictly increasing on $[a, b]$. □

We now show that if the derivative is positive, then the function is increasing.

Corollary 2.2.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) such that for all $x \in (a, b)$, $f'(x) \geqslant 0$. Then, f is increasing on $[a, b]$.*

Proof. Let $x, y \in [a, b]$ such that $x < y$. Since f is continuous on $[x, y]$ and differentiable on (x, y), the Mean Value Theorem tells us that there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

We know that $f'(c) \geqslant 0$. Moreover, since $y - x > 0$, we find that $f(y) - f(x) \geqslant 0$. Therefore, $f(x) \leqslant f(y)$. This implies that f is increasing on $[a, b]$. □

We now show that if the derivative is strictly negative, then the function is strictly decreasing.

Corollary 2.2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) such that for all $x \in (a, b)$, $f'(x) < 0$. Then, f is strictly decreasing on $[a, b]$.*

Proof. Let $x, y \in [a, b]$ such that $x < y$. Since f is continuous on $[x, y]$ and differentiable on (x, y), the Mean Value Theorem tells us that there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

We know that $f'(c) < 0$. Moreover, since $y - x > 0$, we find that $f(y) - f(x) < 0$. Therefore, $f(x) > f(y)$. This implies that f is strictly decreasing on $[a, b]$. □

We now show that if the derivative is negative, then the function is decreasing.

Corollary 2.2.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) such that for all $x \in (a, b)$, $f'(x) \leqslant 0$. Then, f is decreasing on $[a, b]$.*

Proof. Let $x, y \in [a, b]$ such that $x < y$. Since f is continuous on $[x, y]$ and differentiable on (x, y), the Mean Value Theorem tells us that there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

We know that $f'(c) \leqslant 0$. Moreover, since $y - x > 0$, we find that $f(y) - f(x) \leqslant 0$. Therefore, $f(x) \geqslant f(y)$. This implies that f is decreasing on $[a, b]$. □

Finally, we show that if the derivative of a function is bounded, then it is uniformly continuous.

Corollary 2.2.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that is bounded on $\mathbb{R}$. Then, there exists a $K > 0$ such that for all $x, y \in \mathbb{R}$,*

$$|f(x) - f(y)| \leqslant K|x - y|.$$

In particular, f is uniformly continuous.

Proof. Since f is bounded, there exists a $K > 0$ such that for all $x \in \mathbb{R}$, $|f(x)| < K$. Now, let $x, y \in \mathbb{R}$. If $x = y$, we know that

$$|f(x) - f(y)| = K|x - y|.$$

Instead, assume that $x \neq y$. In that case, the Mean Value Theorem tells us that there exists a $c \in (x, y)$ or $c \in (y, x)$ such that

$$|f'(c)| = \frac{|f(x) - f(y)|}{|x - y|} < K.$$

Therefore,

$$|f(x) - f(y)| < K|x - y|.$$

So, for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leqslant K|x - y|$. ☐

We finish by integrating the derivative of the absolute value function.

Example 2.2.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function such that*

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

Then, there exists a $c \in \mathbb{R}$ such that $f(x) = |x| + c$.

Proof. Let $f(0) = c$, for some $c \in \mathbb{R}$. Now, let $x > 0$. The Mean Value Theorem tells us that there exists a $d \in (0, x)$ such that

$$f'(d) = \frac{f(x) - f(0)}{x} = \frac{f(x) - c}{x}.$$

Since $d > 0$, we find that $f'(d) = 1$. In that case, $x = f(x) - c$, and so $f(x) = x + c$. Next, let $x < 0$. Here, the Mean Value Theorem tells us that there exists a $d \in (x, 0)$ such that

$$f'(d) = \frac{f(0) - f(x)}{-x} = \frac{c - f(x)}{-x}.$$

Since $d < 0$, we find that $f'(d) = -1$. In that case, $-x = c - f(x)$, and so $f(x) = -x + c$. This implies that $f(x) = |x| + c$ for all $x \in \mathbb{R}$. ☐

To integrate more complicated functions, we need to define integration rigorously. We will do that later!

2.3 The sine and the cosine function

We will now define the sine and the cosine function. We will state a theorem (without proof) that defines the function e^{ix}, which we will use to define these functions.

Theorem 2.3.1. *Let $z \in \mathbb{C}$ with $|z| = 1$. Then, there exists a unique differentiable function $f : \mathbb{R} \to \mathbb{C}$ such that $f'(t) = if(t)$ for all $t \in \mathbb{R}$ and $f(0) = z$.*

This function is ze^{it}, so we define this notation.

Definition 2.3.2. Define the function $e^{i\bullet} : \mathbb{R} \to \mathbb{C}$ by the unique differentiable function such that $e^{i\bullet}(0) = 1$ and $(e^{i\bullet})'(t) = ie^{i\bullet}(t)$ for all $t \in \mathbb{R}$. We denote $e^{i\bullet}(x) = e^{ix}$.

We now prove important properties of the exponential function. We start with $e^{i(x+y)} = e^{ix}e^{iy}$.

Proposition 2.3.3. *Let $x, y \in \mathbb{R}$. Then,*

$$e^{i(x+y)} = e^{ix}e^{iy}.$$

Proof. Let $x \in \mathbb{R}$ Define the functions $f, g : \mathbb{R} \to \mathbb{C}$ given by $f(y) = e^{i(x+y)}$ and $g(y) = e^{ix}e^{iy}$. We find that

$$f(0) = e^{i(x+0)} = e^{ix} = e^{ix} \cdot e^{i \cdot 0} = g(0).$$

Moreover, for $y \in \mathbb{R}$,

$$f'(y) = i(e^{ix}e^{iy}) = if(y), \qquad g'(y) = ie^{i(x+y)} = ig(y).$$

Since there is a unique function satisfying these conditions, we find that $f(y) = g(y)$ for all $y \in \mathbb{R}$. So, $e^{i(x+y)} = e^{ix}e^{iy}$. □

These properties make use of the uniqueness property of the function. Next, we show that the conjugate of e^{ix} is e^{-ix}, using a similar strategy.

Proposition 2.3.4. *Let $x \in \mathbb{R}$. Then,*

$$\overline{e^{ix}} = e^{-ix}.$$

Proof. Define the functions $f, g : \mathbb{R} \to \mathbb{C}$ given by $f(x) = \overline{e^{ix}}$ and $g(x) = e^{-ix}$. We find that

$$f(0) = \overline{e^{i \cdot 0}} = 1 = 1 = e^{i \cdot (-0)} = g(0).$$

Moreover, for $x \in \mathbb{R}$,

$$f'(x) = \overline{ie^{ix}} = -ie^{ix} = -if(x), \qquad g'(x) = -e^{-ix} = -ig(x).$$

Since there is a unique function satisfying these conditions, we find that $f(x) = g(x)$ for all $x \in \mathbb{R}$. So, $\overline{e^{ix}} = e^{-ix}$. □

Using this result, we can say that the modulus of e^{ix} is always 1.

Proposition 2.3.5. *Let $x \in \mathbb{R}$. Then, $|e^{ix}| = 1$.*

Proof. We find that

$$e^{ix}\overline{e^{ix}} = e^{ix}e^{-ix} = e^{i(x-x)} = e^{i\cdot 0} = 1.$$

Therefore, $|e^{ix}| = 1$. □

We are now ready to define the sine and the cosine function.

Definition 2.3.6. Define the functions $\cos, \sin : \mathbb{R} \to \mathbb{R}$ by

$$\sin(x) = \operatorname{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos(x) = \operatorname{Re}(e^{ix}) = \frac{e^{-ix} + e^{ix}}{2}.$$

By construction, we find that

$$e^{ix} = \cos(x) + i\sin(x).$$

Now, we will prove the familiar properties of the sine and the cosine function.

Proposition 2.3.7. *Let $x \in \mathbb{R}$. Then,*

$$\sin^2(x) + \cos^2(x) = 1.$$

In particular, $-1 \leqslant \sin(x) \leqslant 1$ and $-1 \leqslant \cos(x) \leqslant 1$.

Proof. We find that

$$1 = |e^{ix}| = \operatorname{Re}(e^{ix})^2 + \operatorname{Im}(e^{ix})^2 = \sin^2(x) + \cos^2(x).$$

□

Next, we prove the double angle formula for the sine function.

Proposition 2.3.8. *Let $x, y \in \mathbb{R}$. Then,*

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Proof. We have

$$\begin{aligned}
\sin(x+y) &= \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} \\
&= \frac{2e^{ix}e^{iy} - 2e^{-ix}e^{-iy}}{4i} \\
&= \frac{(e^{ix} - e^{-ix})(e^{iy} + e^{-iy}) + (e^{ix} + e^{-ix})(e^{iy} - e^{-iy})}{4i} \\
&= \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} + e^{-ix}}{2i} \cdot \frac{e^{iy} - e^{-iy}}{2} \\
&= \sin(x)\cos(y) + \cos(x)\sin(y).
\end{aligned}$$

□

Now, we prove the double angle formula for the cosine function.

Proposition 2.3.9. *Let $x, y \in \mathbb{R}$. Then,*

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Proof. We have

$$\begin{aligned}
\cos(x+y) &= \frac{e^{i(x+y)} + e^{-i(x+y)}}{2} \\
&= \frac{2e^{i(x+y)} + 2e^{-i(x+y)}}{4} \\
&= \frac{(e^{ix} + e^{-ix})(e^{iy} + e^{-iy}) + (e^{ix} - e^{-ix})(e^{iy} - e^{-iy})}{4} \\
&= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} - \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{iy} - e^{-iy}}{2i} \\
&= \cos(x)\cos(y) - \sin(x)\sin(y).
\end{aligned}$$

□

Next, we show that the sine function is odd and the cosine function is even.

Proposition 2.3.10. *Let $x \in \mathbb{R}$. Then, $\cos(-x) = \cos(x)$ and $\sin(-x) = \sin(x)$.*

Proof. We find that

$$\cos(-x) = \frac{e^{ix} + e^{-ix}}{2} = \cos(x),$$

and

$$\sin(-x) = \frac{e^{-ix} - e^{ix}}{2i} = -\sin(x).$$

□

We finish by proving the derivatives of sine and cosine.

Proposition 2.3.11. *Let $x \in \mathbb{R}$. Then, the sine and the cosine functions are differentiable at x, with $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.*

Proof. We have

$$\sin'(x) = \frac{(e^{ix})' - (e^{-ix})'}{2i} = \frac{ie^{ix} + ie^{-ix}}{2i} = \frac{e^{ix} + e^{-ix}}{2} = \cos(x),$$

and

$$\cos'(x) = \frac{(e^{ix})' + (e^{-ix})'}{2} = \frac{ie^{ix} - ie^{-ix}}{2} = \frac{-e^{ix} + e^{-ix}}{2} = -\sin(x).$$

□

We will now define π- it is double of the first positive root of the cosine function. First, we need to show that the cosine function has a positive root.

Proposition 2.3.12. *There exists an $x \in \mathbb{R}_{>0}$ such that $\cos(x) = 0$.*

Proof. Assume that for all $x \in \mathbb{R}_{>0}$, $\cos(x) \neq 0$. Since $\cos(0) = 1$, the intermediate value theorem tells us that for all $x \in [0, \infty)$, $\cos(x) > 0$. Since $\sin'(x) = \cos(x)$, this implies that the sine function is strictly increasing on $[0, \infty)$. Now, define the function $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = \cos(1) - \cos(x+1) + \sin(1) \cdot (1 - x).$$

We have

$$f'(x) = \sin(x+1) - \sin(1) > 0$$

for all $x \in (0, \infty)$ since the sine function is strictly increasing. This implies that f is strictly increasing on $(0, \infty)$. We have

$$f(0) = \cos(1) - \cos(1) + \sin(1) \cdot 1 = \sin(1) > \sin(0) = 0,$$

so $f(x) > 0$ for all $x \in (0, \infty)$. Now, fix an $x \in (0, \infty)$ such that $x > 1 + \frac{2}{\sin(1)}$. We find that

$$\begin{aligned} 2 &= \sin(1) \cdot \left(1 + \frac{2}{\sin(1)} - 1\right) \\ &< \sin(1) \cdot (x - 1) \\ &= \cos(1) - \cos(x+1) - f(x) \\ &< \cos(1) - \cos(x+1) \leqslant 2 \end{aligned}$$

since $-1 \leqslant \cos(y) \leqslant 1$ for all $y \in \mathbb{R}$. This is a contradiction. Therefore, there must exist an $x \in \mathbb{R}_{>0}$ such that $\cos(x) = 0$. ☐

Now, we show that, for any continuous function, the infimum of the positive roots is the first positive root.

Proposition 2.3.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) > 0$, and that the set*

$$S = \{x \in \mathbb{R}_{>0} \mid f(x) = 0\}$$

is not empty. Then, $\inf(S) \neq 0$ *and* $f(\inf S) = 0$.

Proof. Let $f(0) = \varepsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x| < \delta$, then $|f(x) - f(0)| < \frac{\varepsilon}{2}$. In that case, for all $x \in (0, \frac{\delta}{2}]$, $f(x) \neq 0$. So, $\frac{\delta}{2}$ is a lower bound for S. Therefore, $\inf(S) \geqslant \frac{\delta}{2} > 0$.
Now, we know that there exists a sequence $(s_n)_{n=1}^{\infty}$ in S such that $s_n \to \inf(S)$. Since f is a continuous function, this implies that $f(s_n) \to f(\inf(S))$. By construction, $f(s_n) = 0$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Therefore, $f(\inf S) = 0$. ☐

Using these two results, we define π.

Definition 2.3.14. Define

$$\pi = 2 \inf\{x \in \mathbb{R}_{>0} \mid \cos(x) = 0\}.$$

We will now use π to show that the cosine and the sine functions are periodic. We start by showing that $\sin(\frac{\pi}{2}) = 1$.

Proposition 2.3.15. *Let $x \in \mathbb{R}$. Then,* $\sin(\frac{\pi}{2}) = 1$.

Proof. Since $\cos(\frac{\pi}{2}) = 0$, and

$$\sin(\tfrac{\pi}{2})^2 + \cos(\tfrac{\pi}{2})^2 = 1,$$

either $\sin(\frac{\pi}{2}) = 1$ or $\sin(\frac{\pi}{2}) = -1$. We know that $\cos(0) = 1$ and $\cos(x) \leqslant 1$ for all $x \in \mathbb{R}$. Since the cosine function is continuous, we find that $0 \leqslant \cos(x) \leqslant 1$ for all $x \in [0, \frac{\pi}{2}]$. Therefore, the sine function is increasing on $[0, \frac{\pi}{2}]$. Since $\sin(0) = 0$, we find that $\sin(\frac{\pi}{2}) = 1$. □

Next, we show that $\cos(x + \frac{\pi}{2}) = -\sin(x)$ and $\sin(x + \frac{\pi}{2}) = \cos(x)$.

Proposition 2.3.16. *Let $x \in \mathbb{R}$. Then,*

- $\cos(x + \frac{\pi}{2}) = -\sin(x)$*;*
- $\sin(x + \frac{\pi}{2}) = \cos(x)$*;*

Proof.

- We have

$$\begin{aligned} \cos(x + \tfrac{\pi}{2}) &= \cos(x)\cos(\tfrac{\pi}{2}) - \sin(x)\sin(\tfrac{\pi}{2}) \\ &= \cos(x) \cdot 0 - \sin(x) \cdot 1 \\ &= -\sin(x). \end{aligned}$$

- We have

$$\begin{aligned} \sin(x + \tfrac{\pi}{2}) &= \sin(x)\cos(\tfrac{\pi}{2}) + \cos(x)\sin(\tfrac{\pi}{2}) \\ &= \sin(x) \cdot 0 + \cos(x) \cdot 1 \\ &= \cos(x). \end{aligned}$$

□

Finally, we show the periodicity of the cosine and the sine function.

Proposition 2.3.17. *Let $x \in \mathbb{R}$. Then,*

- $\cos(x + 2\pi) = \cos(x)$*;*
- $\sin(x + 2\pi) = \sin(x)$.

Proof.

- We have

$$\begin{aligned} \cos(x + 2\pi) &= -\sin(x + \tfrac{3\pi}{2}) \\ &= -\cos(x + \pi) \\ &= \sin(x + \tfrac{\pi}{2}) \\ &= \cos(x). \end{aligned}$$

- We have

$$\begin{aligned}\sin(x+2\pi) &= \cos(x+\tfrac{3\pi}{2})\\ &= -\sin(x+\pi)\\ &= -\cos(x+\tfrac{\pi}{2})\\ &= \sin(x).\end{aligned}$$

$\square$

Using this result, we can show that $e^{i\bullet}$ is also periodic:

$$\begin{aligned}e^{i(x+2\pi)} &= \cos(x+2\pi) + i\sin(x+2\pi)\\ &= \cos(x) + i\sin(x) = e^{ix}.\end{aligned}$$

Now, we show that the function $e^{i\bullet} : [0, 2\pi) \to S^1$ is a bijection. First, we show that e^{ix} is only 1 and -1 once between $[0, 2\pi)$.

Lemma 2.3.18. *Let $x \in (0, 2\pi)$ such that $e^{ix} \in \mathbb{R}$. Then, $e^{ix} = -1$.*

Proof. Let $t = \frac{x}{4}$. We know that

$$\begin{aligned}e^{i\cdot 4t} &= (e^{it})^4\\ &= (\cos t + i\sin t)^4\\ &= (\cos^4 t - 6\cos^2 t\sin^2 t + \sin^4 t) + 4\cos t\sin t(\cos^2 t - \sin^2 t)i.\end{aligned}$$

Since $e^{ix} \in \mathbb{R}$, we find that $\cos^2 t = \sin^2 t$. Since $\cos^2 t + \sin^2 t = 1$, we need $\cos^2 t = \sin^2 t = \frac{1}{2}$. In that case,

$$e^{ix} = e^{i\cdot 4t} = \frac{1}{4} - 6\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4} = -1.$$

$\square$

Using this result, we show injectivity of $e^{i\bullet}$ on $[0, 2\pi)$.

Proposition 2.3.19. *For $x, y \in [0, 2\pi)$, $e^{ix} = e^{iy}$ if and only if $x = y$.*

Proof. If $x = y$, then $e^{ix} = e^{iy}$. Instead, assume that $x \neq y$. Without loss of generality, assume that $x < y$. In that case, $0 < x - y < 2\pi$. So, $e^{i(x-y)} \neq 1$. Therefore,

$$e^{ix} = e^{i(y+x-y)} = e^{iy}e^{i(x-y)} \neq e^{iy}.$$

This implies that $e^{ix} = e^{iy}$ if and only if $x = y$. $\square$

Next, we give a stronger bound on the sine function.

Proposition 2.3.20. *Let $x \in \mathbb{R}$. Then, $|\sin(x)| \leqslant |x|$.*

Proof. We know that $\sin(0) = 0$. Moreover, for $x > \frac{\pi}{2} > 1$, we find that $|\sin(x)| \leqslant 1 < x$. Now, assume that $x \in (0, \frac{\pi}{2})$. In that case, the Mean Value Theorem tells us that there exists a $c \in (0, \frac{\pi}{2})$ such that

$$\cos(c) = \frac{\sin(x) - \sin(0)}{x - 0} = \frac{\sin(x)}{x}.$$

Since $c \in (0, \frac{\pi}{2})$, we find that $\cos(c) \geqslant 0$. This implies that $\sin(x) \geqslant x$. Since $|\sin(-x)| = |\sin(x)|$, we find that for all $x \in \mathbb{R}$, $|\sin(x)| \leqslant |x|$. $\square$

Finally, we show surjectivity of $e^{i\bullet}$ on $[0, 2\pi)$ to S^1.

Proposition 2.3.21. *For $z \in \mathbb{C}$ with $|z| = 1$, there exists a $t \in [0, 2\pi)$ such that $e^{it} = z$.*

Proof. Let $z = x + iy$, for $x, y \in \mathbb{R}$. Since $|z| = 1$, we know that $x^2 + y^2 = 1$. We know that $\cos(0) = 1$ and $\cos(\frac{\pi}{2}) = 0$, so the intermediate value theorem tells us that there exists an $s \in [0, \frac{\pi}{2})$ such that $\cos(s) = |x|$. In that case, $\sin(s) = |y|$. We know that

$$\cos(s + \pi) = -|x|, \qquad \sin(s + \pi) = -|y|,$$

so there exists a $t \in [0, 2\pi)$ such that $e^{it} = x + iy = z$. □

We finish this section by defining more trigonometric functions. We start with the tangent function.

Definition 2.3.22. Define the function

$$\tan : \mathbb{R} \setminus \{\tfrac{\pi}{2} + \pi n \mid n \in \mathbb{Z}\} \to \mathbb{R}$$

by $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Now, we show that the tangent function is odd.

Proposition 2.3.23. *Let $x \in \mathbb{R} \setminus \{\frac{\pi}{2} + \pi n \mid n \in \mathbb{Z}\}$. Then, $\tan(-x) = -\tan(x)$.*

Proof. We find that

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x).$$

□

Next, we find the derivative of the tangent function and show it is strictly increasing.

Proposition 2.3.24. *Let $x \in \mathbb{R} \setminus \{\frac{\pi}{2} + \pi n \mid n \in \mathbb{Z}\}$. Then, $\tan'(x) = \frac{1}{\cos^2(x)}$. Moreover, the tan function is strictly increasing at x.*

Proof. We find that

$$\begin{aligned}
\tan'(x) &= \left(\frac{\sin(x)}{\cos(x)}\right)' \\
&= \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} > 0.
\end{aligned}$$

Therefore, the tan function is strictly increasing at x. □

Now, we show that $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$ is bijective.

Proposition 2.3.25. *The function $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$ is bijective.*

Proof.

- We show that for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos(x) > 0$. We already know that if $x \in [0, \frac{\pi}{2})$, then $\cos(x) > 0$. Therefore, $\sin(x) > \sin(0) = 0$ for all $x \in (0, \frac{\pi}{2})$. Moreover, for $x \in (-\frac{\pi}{2}, 0)$,

  $$\cos(x) = \sin(x + \tfrac{\pi}{2}) > 0$$

  since $x + \frac{\pi}{2} \in (0, \frac{\pi}{2})$. This implies that for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\sin'(x) = \cos(x) > 0$. Therefore, the sine function is strictly increasing. This implies that the sine function is injective.

- Next, let $x \in (-1, 1)$. If $x = 0$, we know that $\sin(0) = x$. If $x > 0$, then $\sin(0) < x < \sin(\frac{\pi}{2})$. So, the intermediate value theorem tells us that there exists a $c \in (0, \frac{\pi}{2})$ such that $\sin(c) = x$. Now, assume that $x < 0$. We find that

  $$\sin(-\tfrac{\pi}{2}) = -\sin(\pi - \tfrac{\pi}{2}) = -\sin(\tfrac{\pi}{2}) = -1.$$

  So, we have $\sin(-\frac{\pi}{2}) < x < \sin(0)$. So, the intermediate value theorem tells us that there exists a $c \in (-\frac{\pi}{2}, 0)$ such that $\sin(c) = x$. Therefore, the sine function is surjective onto $(-1, 1)$.

Therefore, the sine function $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$ is bijective. □

Using this result, we define the arcsine function.

Definition 2.3.26. Define the arcsine function $\sin^{-1} : (-1, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ by the inverse function of $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

Next, we find the derivative of this function.

Proposition 2.3.27. *Let $x \in (-1, 1)$. Then,*

$$(\sin^{-1})'(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Proof. By the Inverse Function Theorem, we find that

$$(\sin^{-1})'(x) = \frac{1}{\sin'(\sin^{-1}(x))} = \frac{1}{\cos(\sin^{-1}(x))}.$$

We know that $\cos(x) > 0$ for all $x \in (\frac{\pi}{2}, -\frac{\pi}{2})$, and $\sin^2(x) + \cos^2(x) = 1$. In that case,

$$\begin{aligned}
(\sin^{-1})'(x) &= \frac{1}{\cos(\sin^{-1}(x))} \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1}(x))}} \\
&= \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}$$

□

Now, we show that $\cos : (0, \pi) \to (-1, 1)$ is bijective.

Proposition 2.3.28. *The function* $\cos : (0, \pi) \to (-1, 1)$ *is bijective.*

Proof. We know that $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$ is bijective. Since $\cos(x) = \sin(x + \frac{\pi}{2})$, we find that $\cos : (0, \pi) \to (-1, 1)$ is bijective. $\square$

Using this result, we define the arccosine function.

Definition 2.3.29. Define the arccosine function $\cos^{-1} : (-1, 1) \to (0, \pi)$ by the inverse function of $\cos : (0, \pi) \to (-1, 1)$.

Next, we find the derivative of this function.

Proposition 2.3.30. *Let* $x \in (-1, 1)$*. Then,*

$$(\cos^{-1})'(x) = -\frac{1}{\sqrt{1 - x^2}}.$$

Proof. By the Inverse Function Theorem, we find that

$$(\cos^{-1})'(x) = \frac{1}{\cos'(\cos^{-1}(x))} = \frac{-1}{\sin(\cos^{-1}(x))}.$$

We know that $\sin(x) > 0$ for all $x \in (0, \pi)$ and $\sin^2(x) + \cos(x)^2 = 1$. In that case,

$$\begin{aligned}(\cos^{-1})'(x) &= \frac{-1}{\sin(\cos^{-1}(x))} \\ &= -\frac{1}{\sqrt{1 - \cos^2(\cos^{-1}(x))}} \\ &= -\frac{1}{\sqrt{1 - x^2}}.\end{aligned}$$

$\square$

Now, we show that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is bijective.

Proposition 2.3.31. *The tangent function* $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ *is a bijection.*

Proof. We know that the tangent function is strictly increasing, so it is injective. Now, define the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ by

$$f(x) = \sin(x) - x\cos(x).$$

We have

$$f'(x) = \cos(x) - (\cos(x) - x\sin(x)) = x\sin(x) > 0$$

for $x > 0$. We have $f(0) = 0$, so $f(x) > 0$ for all $x \in (0, \infty)$. In that case, for $x \in (0, \frac{\pi}{2})$, we have

$$\begin{aligned} f(x) &> 0 \\ \sin(x) - x\cos(x) &> 0 \\ \sin(x) &> x\cos(x) \\ \tan(x) = \frac{\sin(x)}{\cos(x)} &> x. \end{aligned}$$

So, for $x > 0$, $0 < x < \tan(x)$. So, the Intermediate Value Theorem tells us that there exists a $c \in (0, x)$ such that $x = \tan(c)$. Moreover, since $0 = \tan(0)$ and $\tan(-x) = -\tan(x)$, we find that the tangent function is surjective onto $\mathbb{R}$. So, the tangent function is a bijection. $\square$

Using this result, we define the arctangent function.

Definition 2.3.32. Define the arctangent function $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ by the inverse function of $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

Next, we find the derivative of the function.

Proposition 2.3.33. *Let $x \in \mathbb{R}$. Then,*

$$(\tan^{-1})'(x) = \frac{1}{1+x^2}.$$

Proof. By the Inverse Function Theorem, we find that

$$\begin{aligned}
(\tan^{-1})'(x) &= \frac{1}{\tan'(\tan^{-1}(x))} \\
&= \frac{1}{\frac{1}{\cos^2(\tan^{-1}(x))}} \\
&= \frac{1}{1 + \tan^2(\tan^{-1}(x))} \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

$\square$

2.4 The exponential function

Now, we look at the exponential function e^x. We start with a similar theorem like in the imaginary case.

Theorem 2.4.1. *Let $x \in \mathbb{R}_{>0}$. Then, there exists a unique differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$ and $f(0) = x$.*

We use this result to define the exponential function.

Definition 2.4.2. Define the function $\exp : \mathbb{R} \to \mathbb{R}$ by the unique differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$ and $f(0) = 1$. We denote $e^x = \exp(x)$.

Now, we prove some properties of the exponential function. First, we show that $e^{x+y} = e^x e^y$.

Proposition 2.4.3. *Let $x, y \in \mathbb{R}$. Then, $e^{x+y} = e^x e^y$.*

Proof. Let $x \in \mathbb{R}$. Define the function $f, g : \mathbb{R} \to \mathbb{R}$ by $f(y) = e^{x+y}$ and $g(y) = e^x e^y$. We find that

$$f(0) = e^x = e^x e^0 = g(0).$$

Moreover,

$$f'(y) = (e^{x+y})' = e^{x+y} = f(y), \qquad g'(y) = (e^x e^y)' = e^x e^y = g(y).$$

Since there is a unique function $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = e^x$ and $h'(y) = h(y)$, we find that $f(x) = g(x)$ for all $x \in \mathbb{R}$. So, $e^{x+y} = e^x e^y$. □

Next, we show that $e^{-x} = \frac{1}{e^x}$.

Proposition 2.4.4. *Let $x \in \mathbb{R}$. Then,*

$$e^{-x} = \frac{1}{e^x}.$$

In particular, $e^x \neq 0$.

Proof. We find that

$$e^x e^{-x} = e^{x-x} = e^0 = 1.$$

Therefore, $e^{-x} = \frac{1}{e^x}$. □

We now show that the exponential function is bijective onto $(0, \infty)$.

Proposition 2.4.5. *The function $\exp : \mathbb{R} \to (0, \infty)$ is a bijection.*

Proof. We know that $e^0 = 1 > 0$. Moreover, since $e^x \neq 0$ for all $x \in \mathbb{R}$, the intermediate value theorem tells us that $e^x > 0$ for all $x \in \mathbb{R}$. We know that $\exp'(x) = \exp(x)$, so the exponential function is strictly increasing. Therefore, the function is injective.
Now, define the function $f : [0, \infty) \to \mathbb{R}$ by $f(x) = e^x - x$. We find that $f'(x) = e^x - 1 > 0$ for all $x \in (0, \infty)$. Therefore, f is strictly increasing on $(0, \infty)$. We have $f(0) = e^0 - 0 = 1 > 0$, so $f(x) > 0$ for all $x \in [0, \infty)$. This

implies that $e^x > x$ for all $x \in [0, \infty)$. We know that $1 = e^0$. Now, assume that $x > 1$. We find that $e^0 < x < e^x$, so the intermediate value theorem tells us that there exists a $c \in (0, x)$ such that $e^c = x$. Finally, assume that $0 < x < 1$. Since $\frac{1}{x} > 1$, we can find a $c \in (0, \infty)$ such that $e^c = \frac{1}{x}$. In that case, $e^{-c} = x$. Therefore, the exponential function is surjective onto $(0, \infty)$. □

We can now define the logarithmic function.

Definition 2.4.6. Define the function $\log : (0, \infty) \to \mathbb{R}$, where log is the inverse of the exponential function.

Using properties of the exponential function, we show that $\log(xy) = \log(x) + \log(y)$.

Proposition 2.4.7. *Let $x, y \in (0, \infty)$. Then,*

$$\log(xy) = \log(x) + \log(y).$$

Proof. Since the exponential function is surjective on $(0, \infty)$, there exist $a, b \in \mathbb{R}$ such that $e^a = x$ and $e^b = y$. In that case,

$$\log(xy) = \log(e^a e^b) = \log(e^{a+b}) = a + b = \log(x) + \log(y).$$

□

Now, we use the Inverse Function Theorem to derive the logarithmic function.

Proposition 2.4.8. *Let $x \in (0, \infty)$. Then, the logarithmic function is differentiable on x, with*

$$\log'(x) = \frac{1}{x}.$$

Proof. By the Inverse Function Theorem, we find that

$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{\exp(\log(x))} = \frac{1}{x}.$$

□

We finish by generalising the theorem of $f(0) = 1$ and $f'(t) = f(t)$ or $f'(t) = if(t)$ to all the complex numbers.

Proposition 2.4.9. *Let $z \in \mathbb{C}$. Then, there exists a differentiable function $f : \mathbb{R} \to \mathbb{C}$ such that $f(0) = 1$ and $f'(t) = zf(t)$ for all $t \in \mathbb{R}$.*

Proof. Let $z = x + iy$, for $x, y \in \mathbb{R}$. Define the function $f : \mathbb{R} \to \mathbb{C}$ by $f(t) = e^{xt} e^{iyt}$. We have

$$\begin{aligned} f'(t) &= (e^{xt})' \cdot e^{iyt} + e^{xt} \cdot (e^{iyt})' \\ &= xe^{xt} e^{iyt} + e^{xt} \cdot iye^{iyt} \\ &= (x + iy) e^{xt} e^{iyt} \\ &= ze^{xt} e^{iyt} = zf(t). \end{aligned}$$

Moreover,

$$f(0) = e^{x \cdot 0} e^{iy \cdot 0} = 1 \cdot 1 = 1.$$

□

2.5 L'Hopital's Rule

In this section, we prove three versions of L'Hopital's Rule. We start by proving the Cauchy Mean Value Theorem.

Theorem 2.5.1 (Cauchy Mean Value Theorem). *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions that are continuous on $[a, b]$ and differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there exists a $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Since $g'(x) \neq 0$ for all $x \in (a, b)$, Rolle's Theorem tells us that $g(a) \neq g(b)$. So, define the function $h : [a, b] \to \mathbb{R}$ by

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x).$$

$$\begin{aligned} h(b) - h(a) &= f(b) - \frac{f(b) - f(a)}{g(b) - g(a)} g(b) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} g(a) \\ &= [f(b) - f(a)] - (g(b) - g(a)) \cdot \frac{f(b) - f(a)}{g(b) - g(a)} = 0. \end{aligned}$$

This implies that $h(a) = h(b)$. Therefore, Rolle's Theorem tells us that there exists a $c \in (a, b)$ such that $h'(c) = 0$. In that case,

$$f'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(c).$$

Since $g'(c) \neq 0$, we find that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

□

We will now use the Cauchy Mean Value Theorem to prove the first version of L'Hopital's rule.

Proposition 2.5.2 (L'Hopital's Rule, Version I). *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and let $c \in [a, b]$, $L \in \mathbb{R}$ such that f and g are continuous on $[a, b] \setminus \{c\}$ and differentiable on $(a, b) \setminus \{c\}$, with $f(c) = g(c) = 0$, and $g'(x) \neq 0$ for all $x \in (a, b) \setminus \{c\}$. In that case,*

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\varepsilon > 0$. Since the limit

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L,$$

there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta$, then

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Now, for $x \in \mathbb{R}$, if $|x - c| < \delta$, the Cauchy Mean Value Theorem tells us that there exists a $y \in (x, c)$ or $y \in (c, x)$ such that

$$\frac{f'(y)}{g'(y)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f(x)}{g(x)}.$$

We have $|x - y| < |x - c| < \delta$, so we find that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(y)}{g'(y)} - L \right| < \varepsilon.$$

Therefore, the limit

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

$\square$

We illustrate this rule with an example.

Example 2.5.3. *The limit*

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}.$$

Proof. We know that $1 - \cos(0) = 1 - 1 = 0$ and $x^2 = 0$, so L'Hopital's Rule, Version I tells us that

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x}.$$

Now, $\sin(0) = 0$ and $2x = 0$. So, L'Hopital's rule Version I tells us that

$$\lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2}.$$

Therefore, the limit

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$

$\square$

Now, we prove the second version of L'Hopital's rule.

Proposition 2.5.4 (L'Hopital's Rule, Version II)**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and let $L \in \mathbb{R}$ such that f and g are continuous and differentiable on $\mathbb{R}$, with $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$, and $g'(x) \neq 0$ for all $x \in (a, \infty)$. In that case,*

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to \infty} \frac{f(x)}{g(x)} = L.$$

Proof. Define the function $f_0, g_0 : [0,1] \to \mathbb{R}$ given by

$$f_0(x) = \begin{cases} f(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}, \qquad g_0(x) = \begin{cases} g(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Since the limits

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f_0(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = \lim_{x\to 0^+} g_0(x) = 0,$$

the functions f_0 and g_0 are continuous on $[0,1]$ and differentiable on $(0,1)$. Moreover, since $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, we find that the limit

$$\lim_{x\to 0} \frac{f_0'(x)}{g_0'(x)} = L.$$

In that case, L'Hopital's Rule (Version I) tells us that the limit

$$\lim_{x\to 0} \frac{f_0(x)}{g_0(x)} = L.$$

Therefore, the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

□

We illustrate this rule with an example.

Example 2.5.5. *Let $n \in \mathbb{Z}_{\geqslant 1}$. Then, the limit*

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Proof. We know that $e^x > x$, so $0 < \frac{1}{e^x} < \frac{1}{x}$. Therefore, the Sandwich Theorem tells us that the limit

$$\lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Now, if the limit $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$, then by L'Hopital's rule, we find that

$$\lim_{x\to\infty} \frac{x^{n+1}}{e^x} = \lim_{x\to\infty} \frac{(n+1)x^n}{e^x} = (n+1) \lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Therefore, for all $n \in \mathbb{Z}_{\geqslant 1}$,

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

by induction. □

We look at another example.

Example 2.5.6. *The limit*

$$\lim_{x\to 0^+} x \log x = 0.$$

Proof. We find that

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

□

We finish by stating (without proof) the third version of L'Hopital's Rule.

Proposition 2.5.7 (L'Hopital's Rule, Version III)**.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and let $L \in \mathbb{R}$ such that f and g are continuous on $(a, b]$ and differentiable on (a, b), with $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty$, and $g'(x) \neq 0$ for all $x \in (a, b)$. In that case,*

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \implies \lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\varepsilon > 0$. Since the limit $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, there exists a $\delta_1 > 0$ such that for $x \in (a, a+\delta_1)$,

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\varepsilon}{3}.$$

Since the limit $\lim_{x\to a^+} g(x) = \infty$, the limit

$$\lim_{x\to\infty} \frac{f(x+\delta_1)}{g(x) - g(x+\delta_1)} = 0.$$

So, we can find a $\delta_2 \in (0, \delta_1)$ such that for $x \in (a, a+\delta_2)$,

$$\left|\frac{f(x+\delta_1)}{g(x) - g(x+\delta_1)}\right| < \frac{\varepsilon}{3}.$$

Furthermore, since

$$\lim_{x\to a^+} \frac{1}{|1 - f(x+\delta_2)/f(x)|} = 1,$$

we can find a $\delta_3 \in (0, \delta_2)$ such that for $x \in (a, a+\delta_2)$,

$$\left|\frac{1}{1 - f(x+\delta_1)/f(x)} - 1\right| < \frac{1}{|g(x+\delta_2)| \cdot (|L| + \frac{\varepsilon}{3})}.$$

Moreover, since

$$\lim_{x\to a^+} \frac{1}{g(x)} = 0,$$

we can find a $\delta_4 \in (0, \delta_3)$ such that for $x \in (a, a+\delta_3)$,

$$\frac{1}{|g(x)|} < \frac{\varepsilon}{3}.$$

In that case, for $x \in (0, \delta_4)$

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &= \left|\frac{f(x)}{g(x)} - \frac{f(x)-f(x+\delta_1)}{g(x)-g(x+\delta_1)} + \frac{f(x)-f(x+\delta_1)}{g(x)-g(x+\delta_1)} - L\right| \\
&\leqslant \left|\frac{f(x)}{g(x)} - \frac{f(x)-f(x+\delta_1)}{g(x)-g(x+\delta_1)}\right| + \left|\frac{f(x)-f(x+\delta_1)}{g(x)-g(x+\delta_1)} - L\right| \\
&= \left|\frac{f(x)}{g(x)} - \frac{f(x)-f(x+\delta_1)}{g(x)-g(x+\delta_1)}\right| + \left|\frac{f'(c)}{g'(c)} - L\right| \\
&< \left|\frac{g(x)f(x+\delta_1) - f(x+\delta_1)}{g(x)[g(x)-g(x+\delta_1)]}\right| + \frac{\varepsilon}{3} \\
&\leqslant \left|\frac{f(x+\delta_1)}{g(x)-g(x+\delta_1)}\right| + \left|\frac{f(x)g(x+\delta_1)}{g(x)(g(x)-g(x+\delta_1))}\right| + \frac{\varepsilon}{3} \\
&< \frac{2\varepsilon}{3} + \left|\frac{f(x)g(x+\delta_1)}{g(x)(g(x)-g(x+\delta_1))}\right| \\
&= \frac{2\varepsilon}{3} + \left|\frac{f(x)-f(x+\delta_1)}{g(x)-g(x+\delta_1)}\right| \cdot \left|\frac{f(x)}{f(x)-f(x+\delta_1)}\right| \cdot \left|\frac{g(x+\delta_1)}{g(x)}\right| \\
&= \frac{2\varepsilon}{3} + \left|\frac{f'(c)}{g'(c)}\right| \cdot \left|\frac{1}{1-f(x+\delta_1)/f(x)}\right| \cdot \left|\frac{g(x+\delta_1)}{g(x)}\right| \\
&< \frac{2\varepsilon}{3} + \left(|L| + \frac{\varepsilon}{3}\right) \cdot \left(\frac{1}{|g(x+\delta_1) \cdot (|L| + \frac{\varepsilon}{3})|}\right) \cdot \frac{|g(x+\delta_1)|}{|g(x)|} \\
&= \frac{2\varepsilon}{3} + \frac{1}{|g(x)|} \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Therefore, the limit

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

□

Example 2.5.8. *The limit*

$$\lim_{x \to \infty} \frac{\log x}{x} = 0.$$

Proof. We find that $\lim_{x\to\infty} \log x = \infty$ and $\lim_{x\to\infty} x = \infty$. So, L'Hopital's Rule tells us that the limit

$$\lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

□

2.6 Taylor's Theorem

In this section, we generalise the Mean Value Theorem. This is called Taylor's Theorem. We first prove it in $\mathbb{R}$.

Theorem 2.6.1 (Taylor's Theorem in $\mathbb{R}$)**.** *Let $n \in \mathbb{Z}_{\geqslant 1}$ and let $f : [a, b] \to \mathbb{R}$ be such that the k-th derivative $f^{(k)}$ exists and is continuous for all $1 \leqslant k < n$ and $f^{(n)}$ exists on (a, b). Then, there exists a $c \in (a, b)$ such that*

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(c)}{n!}(b-a)^n.$$

Proof. Let $g : [a, b] \to \mathbb{R}$ be given by $g(x) = (x-a)^{n-1}$. Now, define the functions $F, G : [a, b] \to \mathbb{R}$ by

$$F(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k, \qquad G(x) = \sum_{k=0}^{n} \frac{g^{(k)}(x)}{k!}(b-x)^k.$$

We find that

$$\begin{aligned} F'(x) &= \sum_{k=0}^{n} \frac{f^{(k+1)}(x)}{k!}(b-x)^k + \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \cdot -k(b-x)^{k-1} \\ &= \sum_{k=0}^{n} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1} \\ &= \sum_{k=0}^{n} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(b-x)^k \\ &= \frac{f^{(n+1)}(x)}{n!}(b-x)^n. \end{aligned}$$

Similarly, $G'(x) = \frac{g^{(n+1)}(x)}{n!}(b-x)^n$.
Now, we show that

$$G^{(k)}(x) = \frac{(n+1)!}{(n+1-k)!}(x-a)^{n+1-k}$$

for $k \in \{0, \dots, n+1\}$ by induction. If $k = 0$, then

$$G(x) = (x-a)^{n+1} = \frac{(n+1)!}{(n+1)!}(x-a)^{n+1}$$

by construction. Next, if

$$G^{(k)}(x) = \frac{(n+1)!}{(n+1-k)!}(x-a)^{n+1-k},$$

for $k \in \{0, 1, \dots, n\}$, then

$$\begin{aligned} g^{k+1}(x) &= \left(\frac{(n+1)!}{(n+1-k)!}(x-a)^{n+1-k} \right)' \\ &= \frac{(n+1)!}{(n+1-k)!} \cdot (n+1-k)(x-a)^{n-k} \\ &= \frac{(n+1)!}{(n+1-(k+1))!}(x-a)^{(n+1)-(k+1)}. \end{aligned}$$

So, the result follows by induction. In that case,

$$\begin{aligned} G'(x) &= \frac{g^{(n+1)}(x)}{n!}(b-x)^n \\ &= \frac{1}{n!}(b-x)^n \cdot \frac{(n+1)!}{0!}(x-a)^0 \\ &= (n+1)(b-x)^n. \end{aligned}$$

We have

$$\begin{aligned} G(a) &= \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(b-a)^k \\ &= \sum_{k=0}^{n} \frac{1}{k!}(b-a)^k \cdot \frac{(n+1)!}{(n+1-k)!}(a-a)^{n+1-k} = 0, \end{aligned}$$

and

$$\begin{aligned} G(a) &= \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(b-a)^k \\ &= \sum_{k=0}^{n} \frac{1}{k!}(b-b)^k \cdot \frac{(n+1)!}{(n+1-k)!}(b-a)^{n+1-k} \\ &= \frac{1}{0!} \cdot 1 \cdot \frac{(n+1)!}{(n+1)!}(b-a)^{n+1} = (b-a)^{n+1}. \end{aligned}$$

We know that F and G are continuous on $[a,b]$ and differentiable on (a,b). So, we can apply Cauchy's Mean Value Theorem to find that there exists a $c \in (a,b)$ such that

$$\frac{F'(c)}{G'(c)} = \frac{F(b)-F(a)}{G(b)-G(a)}.$$

In that case,

$$\begin{aligned} G'(c)(F(b)-F(a)) &= F'(c)(G(b)-G(a)) \\ (n+1)(b-c)^n(f(b)-F(a)) &= \frac{f^{(n+1)}(c)}{n!}(b-c)^n \cdot (b-a)^{n+1} \\ f(b)-F(a) &= \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \\ f(b) &= F(a) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-a)^k + \frac{f^{(n+1)(c)}}{(n+1)!}(b-a)^{n+1}. \end{aligned}$$

□

If we take $n = 1$, we find that

$$f(b) = f(a) + f'(c)(b-a).$$

This is the Mean Value Theorem.

We will now use the Taylor's Theorem to find different values. First, we start by defining the value e as a series.

Example 2.6.2. *The value*

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Proof. Let $n \in \mathbb{Z}_{\geqslant 1}$. Taylor's Theorem tells us that there exists a $c_n \in (0,1)$ such that

$$\begin{aligned} e = \exp(1) &= \sum_{k=0}^{n-1} \frac{\exp^{(k)}(0)}{k!} + \frac{\exp^{(n)}(c_n)}{n!} \\ &= \sum_{k=0}^{n-1} \frac{\exp(0)}{k!} + \frac{\exp(c_n)}{n!} \\ &= \sum_{k=0}^{n-1} \frac{1}{k!} + \frac{\exp(c_n)}{n!}. \end{aligned}$$

In that case,

$$e - \sum_{k=0}^{n-1} \frac{1}{k!} = \frac{\exp(c_n)}{n!}.$$

Moreover, for all $n \in \mathbb{Z}_{\geqslant 1}$,

$$0 < \frac{\exp(c_n)}{n!} < \frac{e}{n!}.$$

So, the Sandwich Theorem tells us that

$$\lim_{n \to \infty} \frac{\exp(c_n)}{n!} = 0.$$

In that case, we find that the series

$$\sum_{n=1}^{\infty} \frac{1}{n!} = e.$$

□

Next, we approximate $\sqrt{5}$ using Taylor's Theorem.

Example 2.6.3. *The value*

$$\sqrt{5} \approx \frac{36635}{16384},$$

with error at most $\frac{1}{10000}$.

Proof. Define the function $f : [4,5] \to \mathbb{R}$ by $f(x) = \sqrt{x}$. We find that for $x \in (4,5)$,

$$\begin{aligned} f(x) &= \sqrt{x} & f'(x) &= \frac{1}{2\sqrt{x}} & f''(x) &= -\frac{1}{4x^{3/2}} \\ f'''(x) &= \frac{3}{8x^{5/2}} & f^{(iv)}(x) &= -\frac{15}{16x^{7/2}} & f^{(v)}(x) &= \frac{105}{32x^{9/2}} \end{aligned}$$

By Taylor's Theorem, we can find a $c \in (4,5)$ such that

$$\begin{aligned}
f(5) &= f(4) + f'(4) + \frac{f''(4)}{2} + \frac{f'''(4)}{6} + \frac{f^{(iv)}(4)}{24} + \frac{f^{(v)}(c)}{120} \\
\sqrt{5} &= \sqrt{4} + \frac{1}{2\sqrt{4}} - \frac{1}{2 \cdot 4\sqrt{64}} + \frac{3}{6 \cdot 8\sqrt{1024}} - \frac{15}{24 \cdot 16\sqrt{16384}} + \frac{f^{(v)}(c)}{120} \\
&= 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} - \frac{5}{16384} + \frac{f^{(v)}(c)}{120} \\
&= \frac{36635}{16384} + \frac{f^{(v)}(c)}{120}.
\end{aligned}$$

In that case, $\sqrt{5} \approx \frac{36635}{16384}$, with error at most

$$\left| \frac{f^{(v)}(c)}{120} \right| = \frac{105}{120 \cdot 32 \cdot c^{9/2}} < \frac{105}{120 \cdot 32 \cdot 4^{9/2}} = \frac{7}{131072} < \frac{1}{10000}.$$

□

Now, we approximate $\sqrt{10}$ using Taylor's Theorem.

Example 2.6.4. *The value*

$$\sqrt{10} \approx \frac{98371}{31104},$$

with error at most $\frac{1}{10000}$.

Proof. Define the function $f : [9,10] \to \mathbb{R}$ by $f(x) = \sqrt{x}$. By Taylor's Theorem, we can find a $c \in (9,10)$ such that

$$\begin{aligned}
f(10) &= f(9) + f'(9) + \frac{f''(9)}{2} + \frac{f'''(9)}{6} + \frac{f^{(iv)}(9)}{24} + \frac{f^{(v)}(c)}{120} \\
\sqrt{10} &= \sqrt{9} + \frac{1}{2\sqrt{9}} - \frac{1}{2 \cdot 4\sqrt{729}} + \frac{3}{6 \cdot 8\sqrt{6561}} - \frac{15}{24 \cdot 16\sqrt{59049}} + \frac{f^{(v)}(c)}{120} \\
&= 3 + \frac{1}{6} - \frac{1}{216} + \frac{1}{1296} - \frac{5}{31104} + \frac{f^{(v)}(c)}{120} \\
&= \frac{98371}{31104} + \frac{f^{(v)}(c)}{120}.
\end{aligned}$$

In that case, $\sqrt{5} \approx \frac{98371}{31104}$, with error at most

$$\left| \frac{f^{(v)}(c)}{120} \right| = \frac{105}{120 \cdot 32 \cdot c^{9/2}} < \frac{105}{120 \cdot 32 \cdot 9^{9/2}} = \frac{7}{5038848} < \frac{1}{10000}.$$

□

We shall now look at Taylor's Theorem in $\mathbb{C}$.

Theorem 2.6.5 (Taylor's Theorem in $\mathbb{C}$)**.** *Let $n \in \mathbb{Z}_{\geqslant 1}$ and let $f : [a,b] \to \mathbb{C}$ be such that the k-th derivative $f^{(k)}$ exists and is continuous for all $1 \leqslant k < n$ and $f^{(n)}$ exists on (a,b). Then, there exists a $c \in (a,b)$ such that*

$$\left| f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k \right| \leqslant \frac{2|f^{(n)}(c)|}{n!}|b-a|^n.$$

Proof. Define the functions $g, h : [a, b] \to \mathbb{R}$ by $g(x) = \mathrm{Re}(f(x))$ and $h(x) = \mathrm{Im}(f(x))$. By Taylor's Theorem in $\mathbb{R}$, we can find $c_1, c_2 \in (a, b)$ such that

$$g(b) = \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(b-a)^k + \frac{g^{(n)(c_1)}}{n!}(b-a)^n,$$
$$h(b) = \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{k!}(b-a)^k + \frac{h^{(n)(c_2)}}{n!}(b-a)^n.$$

We know that

$$\begin{aligned} f(b) &= g(b) + ih(b) \\ &= \sum_{k=0}^{n-1} \frac{g^{(k)}(a) + ih^{(k)}(a)}{k!}(b-a)^k + \frac{(b-a)^n}{n!}[g^{(n)}(c_1) + ih^{(n)}(c_2)] \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{(b-a)^n}{n!}[g^{(n)}(c_1) + ih^{(n)}(c_2)]. \end{aligned}$$

Moreover,

$$\begin{aligned} |g^{(n)}(c_1) + ih^{(n)}(c_2)| &\leqslant |g^{(n)}(c_1)| + |h^{(n)}(c_2)| \\ &\leqslant |f^{(n)}(c_1)| + |f^{(n)}(c_2)| \\ &\leqslant 2|f^{(n)}(c)|, \end{aligned}$$

where $c \in \{c_1, c_2\}$ satisfies $|f^n(c)| = \max(|f^{(n)}(c_1)|, |f^{(n)}(c_2)|)$. In that case,

$$\begin{aligned} \left| f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k \right| &= \frac{|g^{(n)}(c_1) + ih^{(n)}(c_2)|}{n!}|b-a|^n \\ &\leqslant \frac{2|f^{(n)}(c)|}{n!}|b-a|^n. \end{aligned}$$

□

We illustrate this theorem with an example.

Example 2.6.6. *Let $x > 0$. Then,*

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^k}{k!}x^k.$$

Proof. Let $f : [0, x] \to \mathbb{C}$ be given by $f(t) = e^{it}$. We find that $f^{(n)}(t) = i^n e^{it}$ for all $n \in \mathbb{Z}_{\geqslant 1}$. In that case, Taylor's Theorem tells us that there exists a $c \in (0, x)$ such that

$$\begin{aligned} \left| e^{ix} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}x^k \right| &\leqslant 2\frac{|f^{(n)}(c)|}{n!}|x|^n \\ \left| e^{ix} - \sum_{k=0}^{n-1} \frac{i^k}{k!}x^k \right| &\leqslant 2\frac{|i^n e^{it}|}{n!}|x|^n \\ &= \frac{2}{n!}|x|^n. \end{aligned}$$

We find that $\frac{2}{n!}|x|^n \to 0$, and so

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^k}{n!} x^n.$$

$\square$