CHAPTER 3

FUNCTIONAL ANALYSIS PROPER

3.1 More on L^p spaces

In this section, we will consider L^p spaces again and show that they are Banach spaces. Moreover, we show that L^2 is a Hilbert space.

First, we characterise completeness in terms of absolute convergence.

Definition 3.1.1. Let $(V, \|\cdot\|)$ be a normed vector space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in V. We say that the series $\sum x_n$ is *absolutely convergent* if the series $\sum \|x_n\|$ converges.

Proposition 3.1.2. *Let $(V, \|\cdot\|)$ be a normed vector space. Then, V is complete if and only if every absolutely convergent series $\sum x_n$ in V is convergent.*

Proof. First, assume that V is complete. Let $(x_n)_{n=1}^{\infty}$ be a sequence in V such that the series $\sum x_n$ is absolutely convergent. We show that the series $\sum x_n$ is Cauchy. Let $\varepsilon > 0$. Since the series $\sum \|x_n\|$ is convergent, it is Cauchy. Hence, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq N$, then

$$\left|\sum_{k=1}^{m} \|x_k\| - \sum_{k=1}^{n} \|x_k\|\right| = \sum_{n=k}^{l} \|x_n\| < \varepsilon.$$

So, for $k, l \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq N$, then

$$\left\|\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k\right\| = \left\|\sum_{k=m}^{n} x_k\right\| \leq \sum_{k=m}^{n} \|x_n\| < \varepsilon.$$

Hence, the series $\sum x_n$ is Cauchy. Since V is complete, this implies that $\sum x_n$ is convergent.

Now, assume that every absolutely convergent series is convergent. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in V. We show that (x_n) has a convergent subsequence. Since (x_n) is Cauchy, for each $j \in \mathbb{Z}_{\geq 0}$, we can find an $n_j \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq n_j$, then

$$\|x_m - x_n\| < 2^{-j}.$$

We can choose $n_{j+1} > n_j$ for all $j \in \mathbb{Z}_{\geq 1}$. Then, $(x_{n_j})_{j=1}^{\infty}$ is a subsequence of (x_n). Next, define the sequence $(y_k)_{k=1}^{\infty}$ in V by $y_1 = x_{n_1}$ and $y_k = x_{n_k} - x_{n_{k-1}}$ for $k \geq 2$. Then, for $l \in \mathbb{Z}_{\geq 1}$, we have

$$\sum_{k=1}^{l} y_k = x_{n_1} + (x_{n_2} - x_{n_1}) + \cdots + (x_{n_l} - x_{n_l - 1}) = x_{n_l}.$$

Moreover,

$$\left\|\sum_{k=1}^{\infty} y_k\right\| = \|x_1\| + \sum_{j=1}^{\infty} \|x_{n_j} - x_{n_{j-1}}\| \leq \|x_1\| + \sum_{j=1}^{\infty} 2^{-j} = 1 + \|x_1\|,$$

meaning that $\sum y_k$ is absolutely convergent. By assumption, this implies that $\sum y_k$ is convergent. Hence, the subsequence $(x_{n_j})_{j=1}^{\infty}$ is convergent. Since the sequence (x_n) is Cauchy, we conclude that (x_n) converges. Hence, V is complete. □

Proposition 3.1.3. *Let $p \in [1, \infty)$. Then, the L^p space is complete.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be sequence in L^p such that the series $\sum f_n$ is absolutely convergent, to some $B \in \mathbb{R}$. Define the sequence of functions $(F_n)_{n=1}^{\infty}$ in L^p by

$$G_n = \sum_{k=1}^{n} |f_k|,$$

and let $G = \sum_{k=1}^{\infty} |f_k|$. We have $G_n \geq 0$ and measurable since f_k are measurable, with $G_n \to G$ pointwise. Moreover,

$$\|G_n\|_p^p = \sum_{k=1}^{\infty} |f|^p \leq \sum_{k=1}^{n} \|f_k\|_p^p \leq B^p < \infty.$$

This implies that $G_n \in L^p$. Now, Monotone Convergence Theorem tells us that

$$\int_{\mathbb{R}} |G|^p \, d\mu = \lim_{n \to \infty} \int_{\mathbb{R}} |G_n|^p \, d\mu \leq B^p < \infty.$$

Hence, $G \in L^p$. So, $G(x)$ is finite for almost all $x \in \mathbb{R}$. That is, the series

$$\sum_{k=1}^{\infty} f_k(x) = G(x)$$

converges for almost all $x \in \mathbb{R}$. Since $\mathbb{R}$ is complete, we can find a function F such that $\sum_{k=1}^{\infty} f_k \to F$ pointwise. Since $|F| \leq G$ and $G \in L^p$, we find that $F \in L^p$.

Now, we note that $|F| \leq G^p$ and $\sum_{k=1}^{n} f_k \leq G^p$, meaning that

$$\left| F - \sum_{k=1}^{n} f_k \right|^p \leq (2G)^p$$

for all $n \in \mathbb{Z}_{\geq 1}$. We know that $G \in L^p$, meaning that

$$\int_{\mathbb{R}} G^p \, d\mu < \infty.$$

So, $G^p \in L^1$. So, we can apply Dominated Convergence Theorem to conclude that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \left| F - \sum_{k=1}^{n} f_k \right|^p \, d\mu = \int_{\mathbb{R}} \lim_{n \to \infty} \left| F - \sum_{k=1}^{n} f_k \right|^p \, d\mu.$$

By construction, we have $\sum_{k=1}^{\infty} f_k \to F$ pointwise, so

$$\int_{\mathbb{R}} \lim_{n \to \infty} \left| F - \sum_{k=1}^{n} f_k \right|^p \, d\mu = 0.$$

This means that $\sum_{k=1}^{\infty} f_k \to F$ in L^p. Hence, every absolutely convergent sequence is convergent. We conclude that L^p space is complete. □