CHAPTER 1

HOLOMORPHIC FUNCTIONS

1.1 Complex Numbers

We define

$$\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$$

where $i^2 := -1$. As a vector space (over $\mathbb{R}$), it is isomorphic to $\mathbb{R}^2$. For $z \in \mathbb{C}$ with $z = x + iy$, we defined its conjugate $\overline{z} = x - iy$ and argument

$$\arg z = \tan^{-1}(y/x).$$

Its absolute value is given by $|z| = \sqrt{x^2 + y^2}$. We can represent a complex number in polar form, given by $z = |z|e^{i \arg z}$. Note that for all $z, w \in \mathbb{C}$, $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$, and $z = \overline{z}$ if and only if $z \in \mathbb{R}$. Using Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can write

$$z = |z|e^{i \arg z} = |z|(\cos(\arg z) + i\sin(\arg z)).$$

This is the polar representation of a complex number. From this formula, if follows that $e^{i\pi} = -1$. Moreover, for $z, w \in \mathbb{C}$ with $z = re^{i\theta}$ and $w = se^{i\phi}$,

$$zw = rse^{i(\theta+\phi)}$$

.

For $z, w \in \mathbb{C}$, we define the distance between them by $d(z, w) = |z - w|$. This gives rise to a metric on $\mathbb{C}$. We define an *open disc* of radius $r > 0$ centered at $z \in \mathbb{C}$ by

$$D_r(z) := \{w \in \mathbb{C} \mid |z - w| < r\}.$$

Similarly, the *closed disc* is given by

$$\overline{D}_r(z) := \{w \in \mathbb{C} \mid |z - w| \leq r\}.$$

A set $U \subseteq \mathbb{C}$ is *open* if for any $z \in U$, there exists a radius $r_z > 0$ such that the open disc $D_{r_z}(z) \subseteq U$. We note that U is open if and only if U is a union of open discs. A set $E \subseteq \mathbb{C}$ is *closed* if its complement $E^c = \mathbb{C} \setminus E$ is open. Equivalently, E is closed if and only if for any $\mathbb{C}$-convergent sequence $(z_n)_{n=1}^{\infty}$ in E, the limit lies in E.

For a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$, we say that (z_n) *converges* to $z \in \mathbb{C}$ if $|z_n - z| \to 0$ as $n \to \infty$. The convergence of a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$ can be reduced to convergence of the sequence of its real part $(x_n)_{n=1}^{\infty}$ and the imaginary part $(y_n)_{n=1}^{\infty}$. Hence, it follows that $\mathbb{C}$ is complete. That is, for every Cauchy sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$, (z_n) is convergent.

For a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$, the corresponding series $\sum z_n$ *converges* if the sequence of partial sums $(s_n)_{n=1}^{\infty}$ $s_n = \sum_{k=1}^{n} z_n$ converges. The series $\sum z_n$ *converges absolutely* if the series $\sum |z_n|$ converges. We claim that a series that is absolutely convergent converges.

Proposition 1.1.1. *Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ such that the series $\sum z_n$ is absolutely convergent. Then, the series is convergent.*

Proof. Let $\varepsilon > 0$. Since the series $\sum z_n$ is absolutely convergent, the series $\sum |z_n|$ is Cauchy. Hence, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq N$, then

$$\left| \sum_{k=1}^{m} |z_k| - \sum_{k=1}^{n} |z_k| \right| = \sum_{k=n}^{m} |z_k| < \varepsilon.$$

Hence,

$$\left| \sum_{k=1}^{m} z_k - \sum_{k=1}^{n} z_k \right| = \left| \sum_{k=n}^{m} z_k \right| \leq \sum_{k=n}^{m} |z_k| < \varepsilon.$$

So, the series is Cauchy, meaning that it is convergent. $\square$

Now, for $z \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely, to the value e^z. Similarly, the series $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ and $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converge, with

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad e^{iz} = \cos z + i \sin z.$$

1.2 Holomorphic Functions

Let $U \subseteq \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ be a function and let $c \in U$. We say that f is *holomorphic at* c if the limit

$$\lim_{z\to c} \frac{f(z)-f(c)}{z-c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

exists. If so, we denote the limit by $f'(c)$, and call it *the derivative of* f *at* c. We say that f is *holomorphic on* U if for all $c \in U$, f is holomorphic at c. For $A \subseteq \mathbb{C}$, we say that f is holomorphic on A if there exists an open set $U \subseteq \mathbb{C}$ with $A \subseteq U$ such that f is holomorphic on U.

We will now look at some examples. For $c \in \mathbb{C}$, the constant function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = c$ is holomorphic, with $f'(z) = 0$. A holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is called *entire*. Also, the identity function $f\colon \mathbb{C} \to \mathbb{C}$ is entire, with derivative $f'(z) = 1$. On the other hand, the conjugate function $f(z) = \overline{z}$ is not holomorphic. To see this, define the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ by $x_n = \frac{1}{n}$ and $y_n = \frac{i}{n}$. Then, we have

$$\begin{aligned} \frac{f(z+x_n)-f(z)}{x_n} &= \frac{\overline{z}+x_n-\overline{z}}{x_n} = \frac{x_n}{x_n} = 1, \\ \frac{f(z+y_n)-f(z)}{y_n} &= \frac{\overline{z}-y_n-\overline{z}}{y_n} = \frac{-y_n}{y_n} = -1. \end{aligned}$$

Since we have $x_n \to 0$ and $y_n \to 0$, the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

cannot exist for any $z \in \mathbb{C}$.

Given two holomorphic functions f and g on some set $\Omega \subseteq \mathbb{C}$, we know that the following functions are holomorphic:

- $f+g$, with $(f+g)' = f' + g'$;
- fg, with $(fg)' = f'g + fg'$;
- f/g (if $g(z) \neq 0$ for all $z \in \Omega$), with

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

Moreover, if $f\colon \Omega \to U$ and $g\colon U \to \mathbb{C}$ are holomorphic, then their composition is holomorphic with $(g \circ f)'(z) = g'(f(z))f'(z)$. Hence, every rational function p/q is holomorphic on $\mathbb{C} \setminus q^{-1}(0)$.

We now aim to connect differentiability in $\mathbb{R}^2$ with differentiability in $\mathbb{C}$. We know that a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at some $x \in \mathbb{R}^2$ if there exists a linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\|f(x+h)-f(x)-L(h)\|}{\|h\|} \to 0$$

as $h \to 0$ in $\mathbb{R}^2$. This matrix L is unique, if it exists. In particular, it is the Jacobian:

$$L = \begin{bmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix},$$

where $f(x, y) = (f_1(x, y), f_2(x, y))$.

We will now characterise differentiability in $\mathbb{C}$ using this notion of differentiability in $\mathbb{R}^2$.

Proposition 1.2.1. *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function that is complex-differentiable at $x \in U$. Then, it is $\mathbb{R}$-differentiable at x, and if $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$, then the Cauchy-Riemann equations are satisfied:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}.$$

Proof. Since f is differentiable at x, we know that

$$\frac{f(x+h) - f(x)}{h} \to f'(x)$$

as $h \to 0$. In that case,

$$\frac{|f(x+h) - f(x) - f'(x)h|}{|h|} \to 0$$

as $h \to 0$. To show that the function is $\mathbb{R}$-differentiable, it suffices to show that $h \mapsto f'(x)h$ is $\mathbb{R}$-linear.

Let $h = s + it$ and $f'(x) = a + ib$. Then, $h \mapsto f'(x)h$, in $\mathbb{R}^2$, is given by

$$\begin{bmatrix} s \\ t \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{Re}[(a+ib)(s+it)] \\ \mathrm{Im}[(a+ib)(s+it)] \end{bmatrix} = \begin{bmatrix} as - bt \\ at + bs \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}.$$

So, the map $h \mapsto f'(x)h$ is $\mathbb{R}$-linear. Moreover, since the linear matrix represents the Jacobian, we find that

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Hence, the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = a$$
$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x} = b.$$

□

If $f'(x) \neq 0$, then we can write $f'(x) = a + ib$ by $a = r\cos\theta$ and $b = r\sin\theta$. Then,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

So, the matrix just rotates the coordinate by θ and scales it by r. In particular, the angles between two points with non-zero derivatives gets preserved. This is called *conformality.*

Note that the converse of the theorem is not true- we need to add a further assumption to make it true.

Lemma 1.2.2. *Let $U \subseteq \mathbb{C}$ be open, and let $f\colon U \to \mathbb{R}$ have continuous partial derivatives. Then, f is $\mathbb{R}$-differentiable.*

Proposition 1.2.3. *Let $U \subseteq \mathbb{C}$ be an oepn set and let $f\colon U \to \mathbb{C}$ be a function. Denote by u and v the real and the imaginary parts of f, as functions $\mathbb{R}^2 \to \mathbb{R}^2$. If u and v have continuous first partial derivatives on U and satisfy the Cauchy-Riemann equations, then f is holomorphic on U, with*

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Proof. We know that $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are $\mathbb{R}$-differentiable. Hence, $f\colon U \to \mathbb{C}$ is $\mathbb{R}$-differentiable. We know that the total derivative of f is given by

$$L = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let $x \in U$ and let $h = [s, t] \in \mathbb{R}^2$ small enough such that By the Mean Value Theorem, we can find $\alpha, \beta \in \mathbb{R}^2$ such that

$$f(x+s) - f(x) = \frac{\partial f}{\partial x}(\alpha) \cdot s, \qquad f(x+h) - f(x+s) = \frac{\partial f}{\partial y}(\beta) \cdot t.$$

Hence,

$$f(x+h) - f(x) = \frac{\partial f}{\partial x}(\alpha) \cdot s + \frac{\partial f}{\partial y}(\beta) \cdot t.$$

We also have

$$\begin{aligned} \frac{\partial f}{\partial x}(\alpha) \cdot s &= \frac{\partial f}{\partial x}(x) \cdot s + \left(\frac{\partial f}{\partial x}(\alpha) - \frac{\partial f}{\partial x}(x)\right) \cdot s \\ \frac{\partial f}{\partial y}(\beta) \cdot t &= \frac{\partial f}{\partial y}(x) \cdot t + \left(\frac{\partial f}{\partial y}(\beta) - \frac{\partial f}{\partial y}(x)\right) \cdot t. \end{aligned}$$

Since the partial derivatives are continuous, we can bound

$$\frac{\partial f}{\partial y}(\beta) - \frac{\partial f}{\partial y}(x), \qquad \text{and} \qquad \frac{\partial f}{\partial x}(\alpha) - \frac{\partial f}{\partial x}(x).$$

Hence, we find that

$$\frac{f(x+h) - f(x)}{s + it} \to \frac{\partial f}{\partial x}(x) + i\frac{\partial f}{\partial y}(x)$$

as $h \to 0$. So, the result follows. □

1.3 Power Series

A *power series* is an expression of the form $\sum_{n=0}^{\infty} a_n z^n$, with $(a_n)_{n=0}^{\infty}$ a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$. Examples of power series include

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

A *geometric series* is the following power series:

$$\sum_{n=0}^{\infty} z^n.$$

We note that if $|z| \geq 1$, then the sequence $(z^n)_{n=0}^{\infty}$ does not converge to 0 as $n \to \infty$, meaning that the power series does not converge. On the other hand, if $|z| < 1$, then

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z} \to \frac{1}{1 - z}.$$

So, the geometric series converges only in the open unit disc $D_1(0)$.

For any power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a unique $R \in [0, \infty]$ such that:

- if $|z| < R$, then the series converges absolutely;
- if $|z| > R$, then the series diverges.

In general, we cannot say what happens for all values $|z| = R$. This value R is called the *radius of convergence*, and the open disc of radius R centered at the origin $D_R(0)$ is the *disc of convergence*. Moreover,

$$\limsup |a_n|^{1/n} = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R},$$

if the limits exist, where we define $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$.

Above, we found that the geometric series has radius of convergence 1. Moreover, we saw that any $z \in \mathbb{C}$ with $|z| = 1$, the geometric series $\sum_{n=0}^{\infty} z^n$ diverges. Next, the power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence 1, but it diverges at $z = 1$ and converges for all $z \neq -1$ with $|z| = 1$.

Now, for a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R, we can consider it as a function $f \colon D_R(0) \to \mathbb{C}$. In this perspective, we find that f is holomorphic on $D_R(0)$, with

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Note that the power series f' also has radius of convergence R. This implies that a power series is infinitely complex-differentiable, since its derivative is also a power series. Using this result, we find that $\exp z$, $\cos z$ and $\sin z$ are infinitely-differentiable on $\mathbb{C}$, with

$$\cos' z = -\sin z, \qquad \sin' z = \cos z, \qquad \exp' z = \exp z.$$