

FUNCTIONAL ANALYSIS PROPER

3.1 More on L^p spaces

In this section, we will consider L^p spaces again and show that they are Banach spaces. Moreover, we show that L^2 is a Hilbert space.

First, we characterise completeness in terms of absolute convergence.

Definition 3.1.1. Let $(V, \|\cdot\|)$ be a normed vector space, and let $(x_n)_{n=1}^\infty$ be a sequence in V . We say that the series $\sum x_n$ is *absolutely convergent* if the series $\sum \|x_n\|$ converges.

Proposition 3.1.2. Let $(V, \|\cdot\|)$ be a normed vector space. Then, V is complete if and only if every absolutely convergent series $\sum x_n$ in V is convergent.

Proof. First, assume that V is complete. Let $(x_n)_{n=1}^\infty$ be a sequence in V such that the series $\sum x_n$ is absolutely convergent. We show that the series $\sum x_n$ is Cauchy. Let $\varepsilon > 0$. Since the series $\sum \|x_n\|$ is convergent, it is Cauchy. Hence, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq N$, then

$$\left| \sum_{k=1}^m \|x_k\| - \sum_{k=1}^n \|x_k\| \right| = \sum_{k=n}^m \|x_k\| < \varepsilon.$$

So, for $k, l \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq N$, then

$$\left\| \sum_{k=1}^m x_k - \sum_{k=1}^n x_k \right\| = \left\| \sum_{k=n}^m x_k \right\| \leq \sum_{k=n}^m \|x_k\| < \varepsilon.$$

Hence, the series $\sum x_n$ is Cauchy. Since V is complete, this implies that $\sum x_n$ is convergent.

Now, assume that every absolutely convergent series is convergent. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in V . We show that (x_n) has a convergent subsequence. Since (x_n) is Cauchy, for each $j \in \mathbb{Z}_{\geq 0}$, we can find an $n_j \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m \geq n \geq n_j$, then

$$\|x_m - x_n\| < 2^{-j}.$$

We can choose $n_{j+1} > n_j$ for all $j \in \mathbb{Z}_{\geq 1}$. Then, $(x_{n_j})_{j=1}^\infty$ is a subsequence of (x_n) . Next, define the sequence $(y_k)_{k=1}^\infty$ in V by $y_1 = x_{n_1}$ and $y_k = x_{n_k} - x_{n_{k-1}}$ for $k \geq 2$. Then, for $l \in \mathbb{Z}_{\geq 1}$, we have

$$\sum_{k=1}^l y_k = x_{n_1} + (x_{n_2} - x_{n_1}) + \cdots + (x_{n_l} - x_{n_{l-1}}) = x_{n_l}.$$

Moreover,

$$\left\| \sum_{k=1}^\infty y_k \right\| = \|x_1\| + \sum_{j=1}^\infty \|x_{n_j} - x_{n_{j-1}}\| \leq \|x_1\| + \sum_{j=1}^\infty 2^{-j} = 1 + \|x_1\|,$$

meaning that $\sum y_k$ is absolutely convergent. By assumption, this implies that $\sum y_k$ is convergent. Hence, the subsequence $(x_{n_j})_{j=1}^\infty$ is convergent. Since the sequence (x_n) is Cauchy, we conclude that (x_n) converges. Hence, V is complete. \square

Proposition 3.1.3. *Let $p \in [1, \infty)$. Then, the L^p space is complete.*

Proof. Let $(f_n)_{n=1}^\infty$ be sequence in L^p such that the series $\sum f_n$ is absolutely convergent, to some $B \in \mathbb{R}$. Define the sequence of functions $(G_n)_{n=1}^\infty$ in L^p by

$$G_n = \sum_{k=1}^n |f_k|,$$

and let $G = \sum_{k=1}^\infty |f_k|$. We have $G_n \geq 0$ and measurable since f_k are measurable, with $G_n \rightarrow G$ pointwise. Moreover,

$$\|G_n\|_p^p = \sum_{k=1}^\infty |f_k|^p \leq \sum_{k=1}^n \|f_k\|_p^p \leq B^p < \infty.$$

This implies that $G_n \in L^p$. Now, Monotone Convergence Theorem tells us that

$$\int_{\mathbb{R}} |G|^p d\mu = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |G_n|^p d\mu \leq B^p < \infty.$$

Hence, $G \in L^p$. So, $G(x)$ is finite for almost all $x \in \mathbb{R}$. That is, the series

$$\sum_{k=1}^\infty f_k(x) = G(x)$$

converges for almost all $x \in \mathbb{R}$. Since \mathbb{R} is complete, we can find a function F such that $\sum_{k=1}^\infty f_k \rightarrow F$ pointwise. Since $|F| \leq G$ and $G \in L^p$, we find that $F \in L^p$.

Now, we note that $|F| \leq G^p$ and $\sum_{k=1}^n f_k \leq G^p$, meaning that

$$\left| F - \sum_{k=1}^n f_k \right|^p \leq (2G)^p$$

for all $n \in \mathbb{Z}_{\geq 1}$. We know that $G \in L^p$, meaning that

$$\int_{\mathbb{R}} G^p d\mu < \infty.$$

So, $G^p \in L^1$. So, we can apply Dominated Convergence Theorem to conclude that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left| F - \sum_{k=1}^n f_k \right|^p d\mu = \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left| F - \sum_{k=1}^n f_k \right|^p d\mu.$$

By construction, we have $\sum_{k=1}^\infty f_k \rightarrow F$ pointwise, so

$$\int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left| F - \sum_{k=1}^n f_k \right|^p d\mu = 0.$$

This means that $\sum_{k=1}^\infty f_k \rightarrow F$ in L^p . Hence, every absolutely convergent sequence is convergent. We conclude that L^p space is complete. \square

We can consider measures on probability spaces as well. We say that a measure space on $[0, 1]$ is a probability, then $\mu[0, 1] = 1$. Then, we have $L^p[0, 1] \subseteq L^1[0, 1]$. More generally, for $p > 1$, $L^p[0, 1] \subseteq L[0, 1]$. This follows from Holder's Inequality:

Proposition 3.1.4. *Let P be a probability space.*

Proof. Let $f \in L^p[0, 1]$ and $g = 1$ on $[0, 1]$. Then, by Holder Inequality, we find that

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_p \|1\|_p = \|f\|_p.$$

Hence, if $f \in L^p[0, 1]$, $f \in L^1[0, 1]$. □

Note that the result does not follow in \mathbb{R} , since $\|1\|_p = \infty$. Also, each inclusion is strict, e.g. $\frac{1}{\sqrt{x}}$ is in $L^2[0, 1]$ but not in $L^1[0, 1]$. We can generalise this to show that for $1 \leq p \leq q$, $L^q[0, 1] \subseteq L^p[0, 1]$, and the inclusion is strict if $p < q$.

3.2 Linear Operators

In this section, we will consider linear operators in more detail and do analysis on them. The concept of a linear operator is the same as a linear function, but being bounded as a linear operator has a different meaning.

Definition 3.2.1. Let V and W be normed spaces, and let $T: V \rightarrow W$ be a function. We say that T is *linear* (or an *operator*) if for all $v_1, v_2 \in V$ and $T(v_1 + v_2) = T(v_1) + T(v_2)$ and for all $v \in V$ and $c \in \mathbb{R}$, $T(cv) = cTv$. We say that T is *bounded* if there exists a $c \geq 0$ such that for all $v \in V$, $\|Tv\|_W \leq c\|v\|_V$. The set of bounded functions $V \rightarrow W$ is denoted by the set $L(V, W)$.

In finite dimensions, every linear function is bounded.

Proposition 3.2.2. *Let V and W be vector spaces such that V is finite-dimensional, and let $T: V \rightarrow W$ be linear. Then, T is bounded.*

Proof. Let the basis of V be: $\{v_1, v_2, \dots, v_n\}$. For each $1 \leq i \leq n$, there exists a $c_i > 0$ such that $\|Tv_i\| \leq c_i\|v_i\|$. Set $c = \max(c_1, c_2, \dots, c_n)$. By construction, we have $\|Tv_i\| \leq c\|v_i\|$ for $1 \leq i \leq n$. Now, let $v \in V$. In that case, there exist $\alpha_i \in \mathbb{R}$ for $1 \leq i \leq n$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n.$$

Hence, □

This result does not hold in infinite dimensions. To see this, define the function $T: \ell^\infty \rightarrow \ell^\infty$ by $T(e_n) = ne_n$ extended linearly. This is linear, but not bounded. In particular, for all $n \in \mathbb{Z}_{\geq 1}$,

$$\|Te_{n+1}\| = (n+1)\|e_{n+1}\| > n+1 = (n+1)\|e_{n+1}\|.$$

We can define isomorphisms and isometries like in metric spaces for linear operators.

Definition 3.2.3. Let V and W be vector spaces, and let $T: V \rightarrow W$ be a bounded function. We say that T is an *isomorphism* if T is a bijection with T^{-1} also bounded. We say that T is an *isometry* if for all $v \in V$, $\|Tv\| = \|v\|$. If T is both an isometry and isomorphism, then it is called an *isometric isomorphism*.

Lemma 3.2.4. *Let V and W be vector spaces, and let $T: V \rightarrow W$ be an isometry. Then, T is injective.*

Proof. Let $v_1, v_2 \in V$ with $Tv_1 = Tv_2$. In that case,

$$\|v_1 - v_2\| = \|T(v_1 - v_2)\| = \|Tv_1 - Tv_2\| = 0.$$

Hence, $v_1 = v_2$. So, T is injective. □

Proposition 3.2.5. *Let V and W be vector spaces, and let $T: V \rightarrow W$ be a surjective isometry. Then, T is an isometric isomorphism.*

Proof. Since T is an isometry, we know that it is injective. So, T is a bijection. Hence, it suffices to show that T^{-1} is bounded. We know that for all $v \in V$,

$$\|Tv\| = \|v\|.$$

So, for all $w \in W$ with $T^{-1}(w) = v$, we find that

$$\|w\| = \|Tv\| = \|v\| = \|T^{-1}(w)\|.$$

Hence, T^{-1} is bounded. So, T is an isometric isomorphism. \square

It turns out that continuity and boundedness are equivalent for linear operators.

Proposition 3.2.6. *Let V and W be normed spaces and let $T: V \rightarrow W$ be a linear operator. Then, the following are equivalent:*

1. T is continuous;
2. T is continuous at $0 \in V$;
3. T is bounded.

Proof.

(1) \implies (2) Trivial.

(2) \implies (3) Assume that T is continuous at 0. In that case, there exists a $\delta > 0$ such that for $v \in V$, if $\|v\|_V < \delta$, then $\|Tv\|_W < \varepsilon$. Now, let $v \in V$. If $v = 0$, then

$$\|Tv\|_W = 0 < \frac{2\varepsilon}{\delta} \|v\|_V.$$

Otherwise, let $v' = \frac{\delta}{2\|v\|_V} v$. Then,

$$\|v'\|_V = \frac{\delta}{2\|v\|_V} \|v\|_V = \frac{\delta}{2} < \delta.$$

Hence,

$$\|Tv'\|_W < \varepsilon \iff \frac{\delta}{2\|v\|_V} \|Tv\|_W < \varepsilon \iff \|Tv\|_W < \frac{2\varepsilon}{\delta} \|v\|_V.$$

So, T is bounded.

(3) \implies (1) Now, assume that T is bounded. In that case, there exists a $c \geq 0$ such that for $v \in V$, $\|Tv\|_W \leq c\|v\|_V$. If $c = 0$, then we find that $\|Tv\|_W = 0$ for all $v \in V$, i.e. $T = 0$. This is a continuous function. Otherwise, let $\varepsilon > 0$. Set $\delta = \varepsilon/c$. In that case, for $v, w \in V$, if $\|v - w\|_V < \delta$, then

$$\|Tv - Tw\|_W = \|T(v - w)\|_W \leq c\|v - w\|_V = \varepsilon.$$

Hence, T is (uniformly) continuous. \square

With this result, we can show that the set of bounded functions is a (normed) vector space.

Proposition 3.2.7. *Let V and W be normed spaces. Define the function $\|\cdot\|: L(V, W) \rightarrow \mathbb{R}_{\geq 0}$ by*

$$\|T\| = \sup_{\|v\|=1} \|Tv\|.$$

Then, $(L(V, W), \|\cdot\|)$ is a normed vector space.

Proof. We know that the sum and the scalar product of bounded functions is still bounded, so $L(V, W)$ is a vector space. Now, let $T \in L(V, W)$ such that $\|T\| = 0$. Then, let $v \in V$. If $v = 0$, then we know that $Tv = 0$. Otherwise, we know that $w = v/\|v\|$ has norm 1, meaning that $Tw = 0$. Hence, $Tv = 0$, so $T = 0$. Moreover,

$$\|0\| = \sup\{0\} = 0.$$

Now, let $c \in \mathbb{R}$ and $T \in L(V, W)$. Then,

$$\|cT\| = \sup_{\|v\|=1} \|cTv\| = \sup_{\|v\|=1} |c| \cdot \|Tv\| = |c| \|T\|.$$

Finally, let $T_1, T_2 \in L(V, W)$. Then,

$$\begin{aligned} \|T_1 + T_2\| &= \sup_{\|v\|=1} \|T_1(v) + T_2(v)\| \\ &\leq \sup_{\|v\|=1} (\|T_1(v)\| + \|T_2(v)\|) \\ &\leq \sup_{\|v\|=1} \|T_1(v)\| + \sup_{\|v\|=1} \|T_2(v)\| = \|T_1\| + \|T_2\|. \end{aligned}$$

So, $L(V, W)$ is a normed vector space. □

Moreover, there are many equivalent definitions for the operator norm.

Proposition 3.2.8. *Let V and W be vector spaces, and let $T: V \rightarrow W$ be a bounded function. Then, all of the following expressions are equal to the operator norm $\|T\|$:*

1. $\sup_{\|v\|=1} \|Tv\|$;
2. $\sup_{\|v\| \leq 1} \|Tv\|$;
3. $\sup_{v \neq 0} \|Tv\|/\|v\|$;
4. $\inf\{c \geq 0 \mid \|Tv\| \leq c\|v\| \ \forall v \in V\}$.

Proof.

(1) = (2) We know that

$$\{\|Tv\| \mid \|v\| = 1\} \subseteq \{\|Tv\| \mid \|v\| \leq 1\},$$

in which case

$$\sup_{\|v\|=1} \|Tv\| \leq \sup_{\|v\| \leq 1} \|Tv\|.$$

Now, let $v \in V$ with $\|v\| \leq 1$. If $v = 0$, then $\|Tv\| = 0 \leq \sup_{\|v\|=1} \|Tv\|$. Otherwise, if $v \neq 0$, then there exists a $c > 0$ such that $\|cv\| = 1$. Since $\|v\| \leq 1$, we can further assume that $c \geq 1$. Hence,

$$\|Tv\| = \frac{1}{c} \|T(cv)\| \leq \|T(cv)\|.$$

Hence,

$$\sup_{\|v\| \leq 1} \|Tv\| \leq \sup_{\|v\|=1} \|Tv\|,$$

meaning that the two values are equal.

(1) = (3) We show that

$$\{\|Tv\| \mid \|v\| = 1\} = \{\|Tv\|/\|v\| \mid v \neq 0\}.$$

Clearly, if $\|v\| = 1$, then $\|Tv\|/\|v\| = \|Tv\|$. Hence,

$$\{\|Tv\| \mid \|v\| = 1\} \subseteq \{\|Tv\|/\|v\| \mid v \neq 0\}.$$

Now, let $v \in V$ be non-zero. In that case, let $c = \frac{1}{\|v\|}$. Then, $\|cv\| = 1$, with

$$\|T(cv)\| = \|Tv\|/\|v\|.$$

Hence,

$$\{\|Tv\|/\|v\| \mid v \neq 0\} \subseteq \{\|Tv\| \mid \|v\| = 1\}.$$

So, the sets are equal, meaning that the supremum values agree too.

(3) = (4) The set

$$\{c \geq 0 \mid \|Tv\| \leq c\|v\| \ \forall v \in V\}$$

is the set of upper bounds of the set

$$\{\|Tv\|/\|v\| \mid v \in V\} \cup \{0\}.$$

So, we find that the infimum of the upper bounds equals the supremum of the set.

□

Using these definitions, we can show the following key lemma.

Lemma 3.2.9. *Let V and W be vector spaces, and let $T: V \rightarrow W$ be a bounded function. Then, for any $v \in V$, $\|Tv\| \leq \|T\|\|v\|$.*

Proof. We know that

$$\|T\| = \sup_{v \neq 0} \|Tv\|/\|v\|.$$

Hence, for all $v \in V$ non-zero,

$$\|T\| \geq \|Tv\|/\|v\| \iff \|T\|\|v\| \geq \|Tv\|.$$

Next, if $v = 0$, then we know that $\|Tv\| = 0 \leq 0 = \|T\|\|v\|$. So, the result follows. □

Proposition 3.2.10. *Let V and W be vector spaces, and let W be complete. Then, $L(V, W)$ is complete.*

Proof. Let $(T_n)_{n=1}^\infty$ be a Cauchy sequence in $L(V, W)$. For $v \in V$ non-zero, consider the sequence $(T_n(v))_{n=1}^\infty$. We show that the sequence is Cauchy. So, let $\varepsilon > 0$. Since (T_n) is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $\|T_m - T_n\| < \frac{\varepsilon}{\|v\|}$. Hence, for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then

$$\|T_m(v) - T_n(v)\| = \|(T_m - T_n)(v)\| \leq \|T_m - T_n\| \|v\| < \varepsilon.$$

Hence, $(T_n(v))$ is Cauchy in W . Since W is complete, there exists a $t_v \in W$ such that $T_n(v) \rightarrow t_v$. Now, define the function $T: V \rightarrow W$ by $T(v) = t_v$. We show that $T_n \rightarrow T$ in $L(V, W)$.

First, we show that $T \in L(V, W)$. Let $v_1, v_2 \in V$. We know that for all $n \in \mathbb{Z}_{\geq 1}$, $T_n(v_1 + v_2) = T_n(v_1) + T_n(v_2)$. Hence, $T(v_1 + v_2) = T(v_1) + T(v_2)$. Now, let $v \in V$ and $c \in \mathbb{R}$. We know that for all $n \in \mathbb{Z}_{\geq 1}$, $T_n(cv) = cT_n(v)$. Hence, $T(cv) = cT(v)$. This implies that $T \in L(V, W)$.

Now, we show that $T_n \rightarrow T$. So, let $\varepsilon > 0$. Since (T_n) is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $\|T_m - T_n\| < \frac{\varepsilon}{3}$. Next, let $v \in V$ with $\|v\| = 1$. Since $T_n(v) \rightarrow T(v)$, we can find a $K \in \mathbb{Z}_{\geq 1}$ such that for $k \in \mathbb{Z}_{\geq 1}$, if $k \geq K$, then $\|T(v) - T_k(v)\| < \frac{\varepsilon}{3}$. Then, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then

$$\begin{aligned} \|(T - T_n)(v)\| &\leq \|T(v) - T_n(v)\| \\ &\leq \|T(v) - T_K(v)\| + \|T_K(v) - T_n(v)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}. \end{aligned}$$

So, $\|T - T_n\| \leq \frac{2\varepsilon}{3} < \varepsilon$. This means that $T_n \rightarrow T$. Hence, $L(V, W)$ is complete. \square

We now define the dual space.

Definition 3.2.11. Let X be a K -normed vector space. We say that a function $T: X \rightarrow K$ is *functional* if it is linear, and the *dual space* X^* is the set of bounded functionals.

It turns out that for $1 < p < \infty$, we have an isometric isomorphism

$$(L^p)^* \cong L^q,$$

where q is the dual number of p , i.e. $1/p + 1/q = 1$. In particular, if $p = 2$, then $(L^2)^* \cong L^2$.

Theorem 3.2.12. *The sequence spaces ℓ^∞ and $(\ell^1)^*$ are isometrically isomorphic.*

Proof. Define the map $T: \ell^\infty \rightarrow (\ell^1)^*$ by $T(x) = f_x$, where $f_x: \ell^1 \rightarrow \mathbb{C}$ is given by

$$f_x(y) = \sum_{n=1}^{\infty} x_n y_n.$$

We first show that f_x is well-defined, i.e. $f_x(y) \in (\ell^1)^*$. By construction, f_x is linear. Now, let $y \in \ell^1$. We know that $|x_n| \leq \|x\|_\infty$ for all $n \in \mathbb{Z}_{\geq 1}$, meaning that

$$|f_x(y)| = \left| \sum_{n=1}^{\infty} x_n y_n \right| \leq \sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_\infty \sum_{n=1}^{\infty} |y_n| = \|x\|_\infty \|y\|_1.$$

Since $y \in \ell^1$, this implies that $f_x(y) \in \ell^1$. So, f_k is well-defined. Moreover, this shows that f_x is bounded, with $\|f_x\| \leq \|x\|_\infty$.

We now show that T is an isometry. So, let $x = (x_n)_{n=1}^\infty \in \ell^\infty$, and let $\varepsilon > 0$. By the supremum property, we can find an $n \in \mathbb{Z}_{\geq 1}$ such that

$$\|x\|_\infty - \varepsilon < |x_n| \leq \|x\|_\infty.$$

Then, define the sequence $e = (e_k)_{k=1}^\infty$ by

$$e_k = \begin{cases} 1 & n = k \\ 0 & \text{otherwise.} \end{cases}$$

In that case, we have $e \in \ell^1$, with

$$\|e\|_1 = \sum_{k=1}^{\infty} |e_k| = 1.$$

Moreover,

$$|f_x(e)| = \left| \sum_{k=1}^{\infty} x_k e_k \right| = |x_n|.$$

This implies that

$$\|f_x\| \geq |x_n| > \|x\|_\infty - \varepsilon.$$

Since the result holds for all $\varepsilon > 0$, it follows that $\|f_x\| \geq \|x\|_\infty$. So, $\|f_x\| = \|x\|_\infty$, meaning that T is an isometry.

Finally, we show that T is surjective. So, let $f \in (\ell^1)^*$. Define the sequence $x = (x_n)_{n=1}^\infty$ in \mathbb{C} by $x_n = f(e_n)$. Then, for all $n \in \mathbb{Z}_{\geq 1}$, we find that

$$|x_n| = |f(e_n)| \leq \|f\| \|e_n\|_1 = \|f\|.$$

Hence, $x \in \ell^\infty$. Moreover, for all $y \in \ell^1$, we find that

$$f(y) = \sum_{n=1}^{\infty} f(y_n e_n) = \sum_{n=1}^{\infty} y_n f(e_n) = \sum_{n=1}^{\infty} y_n x_n = f_x(y).$$

So, $T(x) = f_x = f$, meaning that T is surjective. Hence, T is an isometric isomorphism. \square

3.3 Hahn-Banach

In this section, we define Hahn-Banach theorem, which allows us to extend linear functions. To prove this, we first need to consider partial order and then Zorn's Lemma.

Definition 3.3.1. Let X be a set, and let \leq be a relation on X . We say that (X, \leq) is a *partial order* if:

- for all $x \in X$, $x \leq x$;
- for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$; and
- for all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

Examples of partial order include (\mathbb{R}, \leq) and $(\mathcal{P}(X), \subseteq)$ for some set X . The relation does not require every element to be related- not any 2 subset of X needs to satisfy the condition that one is the subset of the other. However, this is satisfied in (\mathbb{R}, \leq) , which gives rise to a total order.

Definition 3.3.2. Let X be a set, and let \leq be a partial order on X . Then, we say that (X, \leq) is a *total order* if for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Now, in partial order, we can define a maximal element in the set.

Definition 3.3.3. Let X be a set with partial order \leq , and let $x \in X$. We say that x is a *maximal element* if for all $y \in X$ such that $x \leq y$, we have $y = x$.

It is not necessarily the case that a partial order has a maximal element, or that it is unique. We now consider upper bounds.

Definition 3.3.4. Let X be a set with partial order \leq , $E \subseteq X$ and $x \in X$. We say that x is an *upper bound* of E if for all $e \in E$, $e \leq x$.

The upper bound need not lie in E , or be unique. Finally, we define well-orderedness.

Definition 3.3.5. Let X be a set with partial order \leq . We say that it is *well-ordered* if for every $E \subseteq X$, E has a minimal element.

For example, \mathbb{R} is not well-ordered using the normal ordering. We now consider Zorn's Lemma.

Lemma 3.3.6 (Zorn's Lemma). *Let X be a non-empty set with partial order \leq , and let $E \subseteq X$ be a non-empty totally ordered with respect to \leq that has an upper bound. Then, X has a maximal element.*

This lemma is equivalent to the axiom of choice (ZFC).

We will now look at Hahn-Banach Theorem. Before doing so, we define a sublinear functional.

Definition 3.3.7. Let V be a normed vector space and let $p: X \rightarrow \mathbb{R}$ be a function. We say that p is *sublinear* if

- for all $x, y \in X$, $p(x + y) \leq p(x) + p(y)$; and

- for all $x \in X$ and $c \geq 0$, $p(cx) = cp(x)$.

Theorem 3.3.8 (Hahn-Banach). *Let V be a real normed vector space and let $p: V \rightarrow \mathbb{R}$ be a sublinear function, $M \subseteq V$ be a subspace and let $f: M \rightarrow \mathbb{R}$ be a linear functional, and $f \leq p$. Then, there exists an $F: V \rightarrow \mathbb{R}$ such that $F|_M = f$ and $f \leq p$.*

Proof. If $M = X$, then there is nothing to show. So, assume that $M \subsetneq X$. We prove this using Zorn's Lemma. So, define the set

$$X = \{F: Y \rightarrow \mathbb{R} \mid F \subsetneq M, F|_M = f, F \leq p\}.$$

We can define the partial order \leq on X by inclusion of domains, and extensionality, i.e. for F_1 and F_2 in X with domains Y_1 and Y_2 , we say that $F_1 \leq F_2$ if $Y_1 \subseteq Y_2$, and $F_2|_{Y_1} = F_1$.

First, we show that X is non-empty. Let $x \in X \setminus M$. For $y_1, y_2 \in M$, we find that

$$\begin{aligned} f(y_1) + f(y_2) &= f(y_1 + y_2) \\ &\leq p(y_1 + y_2) \\ &= p(y_1 - x + x + y_2) \\ &\leq p(y_1 - x) + p(x + y_2). \end{aligned}$$

So, we find that

$$f(y_1) - p(y_1 - x) \leq p(x + y_2) - f(y_2)$$

for all $y_1, y_2 \in M$. Hence,

$$\sup_{y \in M} f(y) - p(y - x) \leq \inf_{y \in M} p(x + y) - f(y).$$

So, we can find an $\alpha \in \mathbb{R}$ such that

$$\sup_{y \in M} f(y) - p(y - x) \leq \alpha \leq \inf_{y \in M} p(x + y) - f(y).$$

Now, define $F: M + \mathbb{R}x \rightarrow \mathbb{R}$ by $F(y + \lambda x) = f(y) + \lambda\alpha$. We claim that $F \in X$. Since $x \notin M$, we find that $M + \mathbb{R}x \subsetneq M$, and that F extends f . Moreover, for all $y + \lambda x \in M + \mathbb{R}x$, if $\lambda \neq 0$, then

$$\begin{aligned} F(y + \lambda x) &= f(y) + \lambda\alpha \\ &= \lambda(f(y/\lambda) + \alpha) \\ &\leq \lambda(f(y/\lambda) + p(x + y/\lambda) - f(y/\lambda)) \\ &= \lambda(p(x + y/\lambda)) \\ &= p(y + \lambda x). \end{aligned}$$

So, $F \leq p$, meaning that $F \in X$.

We now show that every totally ordered set $E \subseteq X$ has a upper bound, with

$$E = \{F_i: Y_i \rightarrow \mathbb{R} \in X \mid i \in I\},$$

for some indexing set I . Now, define the set

$$Y = \bigcup_{i \in I} Y_i$$

and the map $F: Y \rightarrow \mathbb{R}$ by $F(y) = F_i(y)$, where $y \in Y_i$. Since the order assumes extensionality, it is well-defined. We claim that F is an upper bound for E . By construction, F extends each $F_i \in E$, and $Y \subseteq Y_i$ for all $i \in I$. So, F is an upper bound for E .

Finally, we can apply Zorn's Lemma to find a maximal element $F: V \rightarrow \mathbb{R}$. \square

The Hahn-Banach theorem can be generalised for complex functions as well.

Theorem 3.3.9 (Hahn-Banach for complex functions). *Let V be a complex normed vector space and let $p: V \rightarrow \mathbb{C}$ be a sublinear function, $M \subseteq V$ be a subspace and let $f: M \rightarrow \mathbb{R}$ be a seminorm, and $|f| \leq p$. Then, there exists an $F: V \rightarrow \mathbb{C}$ such that $F|_M = f$ and $|f| \leq p$.*

A seminorm is a norm such that $\|x\| = 0$ does not imply that $x = 0$.

We will now look at some consequences of the Hahn-Banach Theorem.

Corollary 3.3.10. *Let V be a normed vector space, $M \subseteq V$ be a subspace and let $f \in M^*$. Then, there exists a linear functional $F \in V^*$ such that F extends f , with $\|F\| = \|f\|$.*

Proof. Note that for any extension F of f , we have $\|F\| \geq \|f\|$ by the supremum property. Now, define the map $p: X \rightarrow \mathbb{C}$ by $p(x) = \|f\|\|x\|$. By construction, we find that for all $y \in M$,

$$|f(y)| \leq \|f\|\|y\| = p(y),$$

meaning that $|f| \leq p$. We now show that p is a seminorm. For all $\lambda \in \mathbb{C}$ and $y \in M$, we find that

$$p(\lambda y) = \|f\|\|\lambda y\| = |\lambda|\|f\|\|y\| = |\lambda|p(y).$$

Moreover, for all $y_1, y_2 \in M$,

$$\begin{aligned} p(y_1 + y_2) &= \|f\|\|y_1 + y_2\| \\ &\leq \|f\|(\|y_1\| + \|y_2\|) \\ &= \|f\|\|y_1\| + \|f\|\|y_2\| \\ &= p(y_1) + p(y_2). \end{aligned}$$

Hence, p is a seminorm. So, Hahn-Banach allows us to extend f into a function $F \in V^*$. Moreover, since $F \leq p$, we find that for all $v \in V$,

$$|F(v)| \leq p(v) = \|f\|\|v\|.$$

So, $\|F\| \leq \|f\|$, meaning that $\|F\| = \|f\|$. \square

Corollary 3.3.11. *Let V be a normed vector space, $M \subsetneq V$ be closed and let $x \in M \setminus V$, and denote*

$$\delta = \inf_{y \in M} \|x - y\|.$$

Then, there exists an $F \in V^$ such that $\|F\| = 1$, $F(x) = \delta$ and $M \subseteq \ker F$.*

Proof. Define the map $f: M + \mathbb{C}x \rightarrow \mathbb{C}$ by $f(y + \lambda x) = \lambda\delta$. This is a linear functional by definition. Next, define the function $p: V \rightarrow \mathbb{R}$ by $p(x) = \|x\|$. Since this is a norm, it is a seminorm. Moreover, for all $y + \lambda x \in M + \mathbb{C}x$, if $\lambda = 0$, then $f(y + \lambda x) = 0 \leq \|y + \lambda x\|$, and if $\lambda \neq 0$, then

$$|f(y + \lambda x)| = |\lambda|\delta \leq |\lambda| \left\| \frac{1}{\lambda}y + x \right\| = \|y + \lambda x\|.$$

So, $|f| \leq p$. Applying Hahn-Banach, we can find a function $F \in V^*$ that extends f . In particular, we still have $M \subseteq \ker F$ and $F(x) = \delta$. Finally, for all $v \in V$, we have

$$|F(v)| \leq p(v) = \|F\|\|v\|,$$

meaning that $\|F\| \leq 1$. □

Corollary 3.3.12. *Let V be a normed vector space and let $v \in V$ be non-zero. Then, there exists a functional $f \in V^*$ such that $\|f\| = 1$ and $f(v) = \|v\|$.*

Proof. Let $M = \{0\}$. Then, it is a closed proper subset of V , with

$$\delta = \inf_{y \in M} \|x - y\| = \|v\|.$$

Hence, there exists a functional $f \in V^*$ such that $\|f\| = 1$ and $F(v) = \|v\|$. □

Corollary 3.3.13. *Let V be a normed vector space and let $x, y \in V$ be distinct. Then, there exists a functional $f \in V^*$ such that $f(x) \neq f(y)$. In particular, linear functionals separate the vector space.*

Proof. Since x and y are distinct, we find that $x - y \neq 0$. Hence, there exists a functional $f \in V^*$ such that $f(x - y) = \|x - y\| \neq 0$. So, $f(x) \neq f(y)$. □

Now, we show that the double dual of a vector space always has an isometry from the vector space.

Proposition 3.3.14. *Let V be a vector space, $v \in V$ and consider the evaluation map $\hat{v}: V^* \rightarrow \mathbb{C}$ given by $\bar{v}(f) = f(v)$. Then, the map $T: V \rightarrow V^{**}$ given by $T(v) = \hat{v}$ is an isometry.*

Proof. Let $v \in V$. We show that $\|v\| = \|\hat{v}\|$ for all $v \in V$. So, let $f \in V^*$. Then,

$$\|\hat{v}(f)\| = \|f(v)\| \leq \|f\|\|v\|.$$

This implies that $\|\hat{v}\| \leq \|v\|$. Now, consider the identity map $f \in V^*$. We know that $\|f\| = 1$, with

$$\|\hat{v}(f)\| = \|f(v)\| = \|v\|.$$

Hence, we find that $\|\hat{v}\| = \|v\|$. □

This is a key result- assuming that the field is complete (which is true for \mathbb{R} and \mathbb{C}), we have found an embedding of V into a complete space V^{**} . Hence, we can identify its completion as a concrete subspace of V^{**} . There are cases when V^{**} is also isometrically isometric, e.g. L^p for $p > 1$, in which case we say that the vector space is *reflexive*.

3.4 Baire-Category Theorem

Definition 3.4.1. Let X be a topological space and let $E \subseteq X$.

- We say that E is *open dense* if E is open with closure X .
- We say that E is *nowhere dense* if the complement of its closure \overline{E}^c is open dense.
- E is a *meagre* (or *first category*) if it is a countable union of nowhere dense sets.
- E is of *second category* if it is not of first category.

Theorem 3.4.2 (Baire-Category Theorem). *Let X be a complete metric space, and let $(U_n)_{n=1}^\infty$ be a sequence of open dense sets in X . Then, the intersection*

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X .

Proof. Let $W \subseteq X$ be open. We show that for all $n \in \mathbb{Z}_{\geq 1}$, $U_n \cap W$ is non-empty. Since U_1 is open dense, we know that $U_1 \cap W$ is a non-empty open set. Hence, there exists an open ball $B_{r_0}(x_0) \subseteq U_1 \cap W$. Without loss of generality, assume that $r_0 \leq 1$. Now, since U_2 is open dense, we can find an open ball $B_{r_1}(x_1)$ such that $\overline{B_{r_1}(x_1)} \subseteq U_1 \cap B_{r_0}(x_0)$, with $r_1 < 2^{-1}$. We can continue on finding open balls $(B_{r_n}(x_n))_{n=0}^\infty$ such that $\overline{B_{r_n}(x_n)} \subseteq U_n \cap B_{r_{n-1}}(x_{n-1})$ for $n \geq 1$.

We now claim that the sequence $(x_n)_{n=0}^\infty$ is a Cauchy sequence. Let $\varepsilon > 0$. Select an $N \in \mathbb{Z}_{\geq 1}$ such that $2^{N+1} > \frac{1}{\varepsilon}$. In that case, for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then we know that $x_m, x_n \in U_{N+1} \cap B_{r_N}(x_N)$, in which case

$$|x_m - x_n| \leq |x_m - x_N| + |x_N - x_n| < r_N + r_N = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

So, (x_n) is Cauchy.

Since X is complete, $x_n \rightarrow x$ for some $x \in X$. By construction, we have

$$x \in \bigcap_{n=1}^{\infty} \overline{B_{r_n}(x_n)} \subseteq \bigcap_{n=0}^{\infty} U_n \cap B_{r_{n-1}}(x_{n-1}) \subseteq \bigcap_{n=0}^{\infty} U_n \cap W.$$

So, the intersection is non-empty, meaning that it is dense in X . □

Corollary 3.4.3. *Let X be a complete metric space. Then, it is of second category.*

Proof. Let $(E_1)_{n=1}^\infty$ be a sequence of nowhere dense sets. We show that the union

$$\bigcup_{n=1}^{\infty} E_n \subsetneq X.$$

We know that $(\overline{E_n}^c)$ is a sequence of open dense sets. By Baire-Category Theorem, we know that the intersection

$$\bigcap_{n=1}^{\infty} \overline{E_n}^c$$

is dense in X . In particular, the intersection is non-empty, meaning that its complement

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E_n} \neq X.$$

So, X cannot be of first category. □

3.5 Open Mapping Theorem

Definition 3.5.1. Let X and Y be topological spaces and let $f: X \rightarrow Y$ be a map. We say that f is *open* if for all $U \subseteq X$ open, $f(U) \subseteq Y$ is open.

Proposition 3.5.2. Let X and Y be metric spaces and let $f: X \rightarrow Y$ be a function. Then, f is an open map if and only if for every $x \in X$ and $\delta > 0$, there exists an $\varepsilon > 0$ such that for $y \in X$, if $d_Y(f(x), f(y)) < \varepsilon$, then there exists a $z \in X$ such that $d_X(x, z) < \delta$ with $f(y) = f(z)$.

Proof.

- Assume that f is an open map. Let $x \in X$ and $\delta > 0$. Let

$$U = B_X(x, \delta).$$

Since f is an open map, we know that $f(U)$ is open. We have $f(x) \in f(U)$. In that case, there exists an $\varepsilon > 0$ such that for $y \in X$, if $d_Y(f(x), f(y)) < \varepsilon$, then $f(y) \in f(B_X(x, \delta))$. So, there exists a $z \in X$ with $d_X(x, z) < \delta$ with $f(y) = f(z)$. That is, for all $x \in X$ and $\delta > 0$, there exists an $\varepsilon > 0$ such that for $y \in X$, if $d_Y(f(x), f(y)) < \varepsilon$, then there exists a $z \in X$ such that $d_X(x, z) < \delta$ with $f(y) = f(z)$.

- Assume that for every $x \in X$ and $\delta > 0$, there exists an $\varepsilon > 0$ such that for $y \in X$, if $d_Y(f(x), f(y)) < \varepsilon$, then there exists a $z \in X$ such that $d_X(x, z) < \delta$ with $f(y) = f(z)$. Let $U \subseteq X$ be open. We show that $f(U)$ is open. So, let $x \in U$. Since U is open, there exists a $\delta_x > 0$ such that for all $y \in X$, if $d_X(x, y) < \delta_x$, then $y \in U$. Now, we can find an $\varepsilon_x > 0$ such that for $y \in X$, if $d_Y(f(x), f(y)) < \varepsilon_x$, then there exists a $z \in X$ such that $d_X(x, z) < \delta_x$ with $f(y) = f(z)$. This implies that $z \in U$. So, $f(y) = f(z) \in f(U)$. In that case, for all $x \in X$, there exists an ε_x such that for all $y \in X$, if $d_Y(f(x), f(y)) < \varepsilon_x$, then $f(y) \in f(U)$. This implies that $f(U)$ is open. In other words, f is an open map.

□

Another way of writing this is the following- $f: X \rightarrow Y$ is an open map if and only if for every $x \in X$ and $\delta > 0$, there exists an $\varepsilon > 0$ such that

$$B_Y(f(x), \varepsilon) \subseteq f(B_X(x, \delta)).$$

Proposition 3.5.3. Let V and W be normed vector spaces, and let $T: V \rightarrow W$ be a linear operator. Then, T is an open map if and only if there exists an $\varepsilon > 0$ such that for $\mathbf{v} \in V$, if $\|T(\mathbf{v})\|_W < \varepsilon$, then there exists a $\mathbf{u} \in V$ such that $\|\mathbf{u}\|_V < 1$ with $T(\mathbf{u}) = T(\mathbf{v})$.

Proof.

- Assume that T is an open map. Set $\mathbf{x} = \mathbf{0}$ and $\delta = 1$. In that case, there exists an $\varepsilon > 0$ such that for $\mathbf{v} \in V$, if $\|T(\mathbf{v}) - T(\mathbf{x})\|_W < \varepsilon$, then there exists a $\mathbf{u} \in V$ such that $\|\mathbf{u} - \mathbf{x}\|_V < \delta$ with $T(\mathbf{u}) = T(\mathbf{v})$. That is, there exists an $\varepsilon > 0$ such that for $\mathbf{v} \in V$, if $\|T(\mathbf{v})\|_W < \varepsilon$, then there exists a $\mathbf{u} \in V$ such that $\|\mathbf{u}\|_V < 1$ with $T(\mathbf{u}) = T(\mathbf{v})$.

- Assume that there exists an $\varepsilon > 0$ such that for $\mathbf{v} \in V$, if $\|T(\mathbf{v})\|_W < \varepsilon$, then there exists a $\mathbf{u} \in V$ such that $\|\mathbf{u}\|_V < 1$ with $T(\mathbf{u}) = T(\mathbf{v})$. We show that T is an open map. Let $\mathbf{x} \in V$ and $\delta > 0$. We can find an $\varepsilon > 0$ such that for $\mathbf{v} \in V$, if $\|T(\mathbf{v} - \mathbf{x})\|_W < \frac{\varepsilon}{\delta}$, then there exists a $\mathbf{u} \in V$ such that $\|\frac{1}{\delta}\mathbf{u}\|_V < 1$ with $T(\mathbf{u}) = T(\mathbf{v} - \mathbf{x})$. Using linearity and replacing \mathbf{u} with $\mathbf{u} + \delta\mathbf{x}$; \mathbf{v} with $\delta\mathbf{v}$; \mathbf{x} with $\delta\mathbf{x}$, we find that for $\mathbf{v} \in V$, if $\|T(\mathbf{v}) - T(\mathbf{x})\|_W < \varepsilon$, then there exists a $\mathbf{u} \in V$ such that $\|\mathbf{u} - \mathbf{x}\|_V < \delta$ with $T(\mathbf{u}) = T(\mathbf{v})$. So, T is an open map.

□

Theorem 3.5.4 (Open Mapping Theorem). *Let V and W be Banach spaces and let $T: V \rightarrow W$ be a surjective linear operator. Then, T is open.*

Proof.

□

Corollary 3.5.5. *Let V and W be Banach spaces and let $T: V \rightarrow W$ be a bijective linear operator. Then, T is an isomorphism.*

Proof. Since T is surjective, we know that T is an open map. We know that $T^{-1}: W \rightarrow V$ is a linear operator, so we show that T^{-1} is continuous. So, let $U \subseteq V$ be open. Since T is an open map, we know that $T(U) \subseteq W$ is open. Now, we know that for $w \in W$,

$$w \in (T^{-1})^{-1}(U) \iff T^{-1}(w) \in U \iff w \in T(U).$$

Hence, we find that $(T^{-1})^{-1}(U) = T(U)$ is open. So, T^{-1} is bounded operator, meaning that T is an isomorphism. □

3.6 Closed Graph Theorem

In this section, we will be proving the closed graph theorem. We begin by defining graphs.

Definition 3.6.1. Let V and W be sets and $T: V \rightarrow W$ be a function. Then, the *graph* of T is the set

$$\text{Graph}(T) = \{(v, Tv) \mid v \in V\} \subseteq V \times W.$$

If V and W have structure, then this structure can be easily extended to $V \times W$. In particular,

- if V and W are vector spaces, then $\text{Graph}(T)$ is a subspace of $V \times W$;
- if V and W are normed vector spaces, then the function $\|\cdot\|: V \times W \rightarrow [0, \infty)$ given by $\|(v, w)\| = \max(\|v\|_V, \|w\|_W)$ is a norm on $V \times W$; and
- if V and W are complete (metric) spaces, then $V \times W$ is a complete (metric) space.

We can show that in metric spaces, $\text{Graph}(T)$ is closed in $V \times W$.

Proposition 3.6.2. *Let V and W be metric spaces and let $T: V \rightarrow W$ be continuous. Then, $\text{Graph}(T)$ is closed in $V \times W$.*

Proof. Let $(v_n, Tv_n)_{n=1}^\infty$ be a sequence in $\text{Graph}(T)$ that converges to $(v, w) \in V \times W$. We show that $(v, w) \in \text{Graph}(T)$, i.e. $w = Tv$. By the definition of the product metric, we know that $v_n \rightarrow v$ and $Tv_n \rightarrow w$. Since T is continuous, it follows that $Tv_n \rightarrow Tv$. Since limits are unique, we find that $w = Tv$, meaning that $(v, w) \in \text{Graph}(T)$. So, $\text{Graph}(T)$ is closed. \square

We can now ask the converse- if $\text{Graph}(T)$ is closed, then does this imply that T is bounded. This is what the closed graph theorem answers.

Theorem 3.6.3 (Closed Graph Theorem). *Let V and W be Banach spaces and let $T: V \rightarrow W$ be a linear operator such that $\text{Graph}(T)$ is closed. Then, T is bounded.*

Proof. Since V and W are complete, we find that $V \times W$ is complete. Hence, $\text{Graph}(T)$ is a Banach space. Now, consider the projection maps $\pi_V: V \times W \rightarrow V$ and $\pi_W: V \times W \rightarrow W$ given by $\pi_V(v, w) = v$ and $\pi_W(v, w) = w$. By definition, π_V and π_W are linear operators. Moreover, for all $(v, w) \in V$,

$$\|\pi_V(v, w)\|_V = \|v\|_V \leq \|(v, w)\|,$$

meaning that π_V is bounded. Similarly, π_W is bounded. Now, consider the restriction $\pi_V: \text{Graph}(T) \rightarrow V$. This is a bijection, with inverse $\pi_V^{-1}(v) = (v, Tv)$. Since $\text{Graph}(T)$ is a Banach space, the Open Mapping Theorem tells us that π_V^{-1} is bounded. Now, for all $v \in V$, we find that

$$\pi_W(\pi_V^{-1}(v)) = \pi_W(v, Tv) = Tv.$$

So, $\pi_W \circ \pi_V^{-1} = T$. Since both π_W and π_V^{-1} are bounded, it follows that T is bounded. \square

3.7 Hilbert Spaces

In this section, we look at Hilbert spaces. Hilbert spaces are complete inner product spaces. Examples include the dot product in \mathbb{C}^n , i.e.

$$\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i.$$

Moreover, in $L^2[0, 1]$, we have the inner product

$$\langle f, g \rangle = \int_{[0,1]} f g \, d\lambda.$$

Note that this integral is finite since

$$\int |fg| \, d\lambda \leq \frac{1}{2} \left(\int_{[0,1]} |f|^2 + |g|^2 \, d\lambda \right) < \infty.$$

A Hilbert space \mathcal{H} satisfies the Cauchy-Schwarz Inequality, i.e.

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

for all $x, y \in \mathcal{H}$. Hence, the inner product map $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is continuous.

A characterisation of a Hilbert space is the *Parallelogram law*. It states that a norm on a vector space V is induced by an inner product if and only if for all $x, y \in X$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The main property that a Hilbert space adds to a Banach space is *orthogonality*. We say that two vectors x, y in a Hilbert space \mathcal{H} are *orthogonal* if $\langle x, y \rangle = 0$. We can define the orthogonal complement of a subset $E \subseteq \mathcal{H}$ by

$$E^\perp = \{y \in \mathcal{H} \mid \langle x, y \rangle = 0 \, \forall x \in E\}.$$

This is a closed subspace of \mathcal{H} .

By the property of orthogonality, we can generalise Pythagoras' Theorem.

Theorem 3.7.1 (Pythagoras' Theorem). *Let \mathcal{H} be a Hilbert space and let $(x_i)_{i=1}^n$ be a sequence of mutually orthogonal vectors in \mathcal{H} , i.e. $\langle x_i, x_j \rangle = 0$ for all $1 \leq i, j \leq n$ if $i \neq j$. Then,*

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2.$$

Proof. We have

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \left\| \sum_{i=1}^n x_i, \sum_{j=1}^n x_j \right\| = \sum_{i=1}^n \sum_{j=1}^n \langle x_i, x_j \rangle.$$

By orthogonality, we have $\langle x_i, x_j \rangle = 0$ if $i \neq j$. Hence,

$$\sum_{i=1}^n \sum_{j=1}^n \langle x_i, x_j \rangle = \sum_{i=1}^n \langle x_i, x_i \rangle = \sum_{i=1}^n \|x_i\|^2.$$

So, the result follows. □

Theorem 3.7.2 (Decomposition Theorem). *Let \mathcal{H} be a Hilbert space and let $M \subseteq \mathcal{H}$ be a closed subspace. Then, $\mathcal{H} = M \oplus M^\perp$. That is, for all $x \in \mathcal{H}$, there exist unique $y \in M$ and $z \in M^\perp$ such that $x = y + z$.*

Proof. Let $x \in \mathcal{H}$ and let

$$\delta = \inf_{y \in M} \|x - y\|.$$

By the infimum property, we can find a sequence $(y_n)_{n=1}^\infty$ in M such that $\|x - y_n\| \rightarrow \delta$. Then, the Parallelogram Law tells us that

$$\begin{aligned} 2\|y_n - x\| + 2\|y_m - x\| &= \|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2 \\ &= \|y_n - y_m\|^2 + 4\|\tfrac{1}{2}(y_n + y_m) - x\|^2 \end{aligned}$$

for all $m, n \in \mathbb{Z}_{\geq 1}$. We have $\tfrac{1}{2}(y_n + y_m) \in M$. So,

$$\|\tfrac{1}{2}(y_n + y_m) - x\|^2 \geq \delta.$$

Hence,

$$\begin{aligned} \|y_n - y_m\|^2 &= 2\|y_n - x\| + 2\|y_m - x\| - 4\|\tfrac{1}{2}(y_n + y_m) - x\|^2 \\ &\leq 2\|y_n - x\| + 2\|y_m - x\| - 4\delta^2. \end{aligned}$$

We have $\|y_n - x\| \rightarrow \delta$, so $2\|y_n - x\| + 2\|y_m - x\| \rightarrow 4\delta^2$. Hence, $\|y_n - y_m\|^2 \rightarrow 0$, meaning that (y_n) is Cauchy. Since M is complete, there exists a $y \in M$ such that $y_n \rightarrow y$. Now, define $z = y - x$.

We claim that $z \in M^\perp$. So, let $y' \in M$. □

Lemma 3.7.3. *Let \mathcal{H} be a Hilbert space and let $y \in \mathcal{H}$. Define the map $f_y: \mathcal{H} \rightarrow \mathbb{C}$ by $f_y(x) = \langle x, y \rangle$. Then, $f_y \in \mathcal{H}^*$ with $\|f_y\| = \|y\|$.*

Proof. For $x_1, x_2 \in \mathcal{H}$, we have

$$f_y(x_1 + x_2) = \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle = f_y(x_1) + f_y(x_2),$$

and for $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, we have

$$f_y(\lambda x) = \langle \lambda x, y \rangle = \lambda \langle x, y \rangle = \lambda f_y(x)$$

by Cauchy-Schwarz inequality. So, f_y is linear. Moreover, for all $x \in \mathcal{H}$,

$$|f_y(x)| = |\langle x, y \rangle| \leq \|x\| \|y\|.$$

Hence, f_y is bounded, with $\|f_y\| \leq \|y\|$. We also have

$$|f_y(y)| = |\langle y, y \rangle| = \|y\| \|y\|,$$

meaning that $\|f_y\| \geq \|y\|$. This implies that $\|f_y\| = \|y\|$. □

Theorem 3.7.4 (Riesz Representation Theorem). *Let \mathcal{H} be a Hilbert space. Then, \mathcal{H} is isometrically isomorphic to its dual \mathcal{H}^* .*

Proof. Define the map $T: \mathcal{H} \rightarrow \mathcal{H}^*$ by $T(y) = f_y$, with $f_y(x) = \langle x, y \rangle$. We have shown that T is an isometry, so it suffices to show that T is surjective.

So, let $f \in \mathcal{H}^*$. If $f = 0$, then we have $f = f_0 = T(0)$. Otherwise, let $K = \ker f$. This is a closed proper subspace of \mathcal{H} . In that case, we can find a $z \in K^\perp$ with $\|z\| = 1$. Next, let $x \in \mathcal{H}$. Define $u_x = f(x)z - f(z)x$. Then,

$$f(u_x) = f(x)f(z) - f(z)f(x) = 0,$$

meaning that $u_x \in K$. Hence,

$$\begin{aligned} 0 &= \langle u_x, z \rangle \\ &= \langle f(x)z - f(z)x, z \rangle \\ &= f(x)\langle z, z \rangle - f(z)\langle x, z \rangle \\ &= f(x)\|z\|^2 - f(z)\langle x, z \rangle \\ &= f(x) - f(z)\langle x, z \rangle. \end{aligned}$$

This implies that $f(x) = f(z)\langle x, z \rangle = \langle x, \overline{f(z)}z \rangle$. So, we find that $f = f_{\overline{f(z)}z}$. Hence, T is surjective. So, T is an isometric isomorphism. \square

We say that a sequence $(u_n)_{n=1}^\infty$ in a Hilbert space is *orthonormal* if for all $n \in \mathbb{Z}_{\geq 1}$, $\|u_n\| = 1$, and $\langle u_n, u_m \rangle = 0$ for all $m, n \in \mathbb{Z}_{\geq 1}$ if $m \neq n$.

Theorem 3.7.5 (Gram-Schmidt Decomposition). *Let $(x_n)_{n=1}^\infty$ be a sequence of linearly independent sequence in a Hilbert space \mathcal{H} . Then, there exists a sequence $(y_n)_{n=1}^\infty$ of orthonormal vectors in \mathcal{H} such that*

$$\text{span}(u_1, u_2, \dots) = \text{span}(e_1, e_2, \dots).$$

Proof. Define the sequence $(z_n)_{n=1}^\infty$ in \mathcal{H} inductively, as follows: $z_1 = x_1$ and

$$z_n = x_n - \sum_{i=1}^{n-1} \langle x_n, z_i \rangle z_i.$$

Then, (z_n) is an orthogonal sequence with equal span. So, we can now define the normalised sequence $(y_n)_{n=1}^\infty$ of (z_n) - this is an orthonormal sequence with equal span. \square

Proposition 3.7.6 (Bessel's Inequality). *Let $(u_n)_{n=1}^\infty$ be a sequence of orthonormal vectors in a Hilbert space \mathcal{H} . Then, for any $x \in \mathcal{H}$,*

$$\sum_{n=1}^{\infty} |\langle u_n, x \rangle|^2 \leq \|x\|^2.$$

Proof. Let $N \in \mathbb{Z}_{\geq 1}$. Then, we find that

$$\begin{aligned}
0 &\leq \left\| x - \sum_{n=1}^N \langle u_n, x \rangle u_n \right\|^2 \\
&= \left\langle x - \sum_{n=1}^N \langle u_n, x \rangle u_n, x - \sum_{n=1}^N \langle u_n, x \rangle u_n \right\rangle \\
&= \langle x, x \rangle - 2 \operatorname{Re} \left\langle x, \sum_{n=1}^N \langle u_n, x \rangle u_n \right\rangle + \left\langle \sum_{n=1}^N \langle u_n, x \rangle u_n, \sum_{n=1}^N \langle u_n, x \rangle u_n \right\rangle \\
&= \|x\|^2 - 2 \operatorname{Re} \sum_{n=1}^N \langle x, \langle u_n, x \rangle u_n \rangle + \left\| \sum_{n=1}^N \langle u_n, x \rangle u_n \right\|^2.
\end{aligned}$$

By Pythagoras' Theorem, we find that

$$\left\| \sum_{n=1}^N \langle u_n, x \rangle u_n \right\|^2 = \sum_{n=1}^N |\langle u_n, x \rangle|^2 \|u_n\|^2 = \sum_{n=1}^N |\langle u_n, x \rangle|^2.$$

Moreover,

$$\sum_{n=1}^N \langle x, \langle u_n, x \rangle u_n \rangle = \sum_{n=1}^N \langle x, u_n \rangle \langle u_n, x \rangle = \sum_{n=1}^N \overline{\langle u_n, x \rangle} \langle u_n, x \rangle = \sum_{n=1}^N |\langle u_n, x \rangle|^2.$$

This implies that

$$\left\| \sum_{n=1}^N \langle u_n, x \rangle u_n \right\|^2 - 2 \operatorname{Re} \sum_{n=1}^N \langle x, \langle u_n, x \rangle u_n \rangle = - \sum_{n=1}^N |\langle u_n, x \rangle|^2.$$

Hence, for all $N \in \mathbb{Z}_{\geq 1}$,

$$\sum_{n=1}^N |\langle u_n, x \rangle| \leq \|x\|^2.$$

Taking limits, we find that

$$\sum_{n=1}^{\infty} |\langle u_n, x \rangle| \leq \|x\|^2.$$

□

Proposition 3.7.7. *Let $(u_n)_{n=1}^{\infty}$ be a sequence of orthonormal vectors in a Hilbert space \mathcal{H} . Then, the following are equivalent:*

1. completeness: for all $x \in \mathcal{H}$, if $\langle x, u_n \rangle = 0$ for all $n \in \mathbb{Z}_{\geq 1}$, then $x = 0$;
2. Parseval's Identity: for all $x \in \mathcal{H}$,

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2;$$

3. for all $x \in \mathcal{H}$,

$$\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n = x.$$

Proof.

(1) \implies (3) We know that

$$\sum_{n=1}^{\infty} \|\langle u_n, x \rangle u_n\|^2 = \sum_{n=1}^{\infty} |\langle u_n, x \rangle|^2.$$

By Bessel's Inequality, the series $\sum_{n=1}^{\infty} |\langle u_n, x \rangle|^2$ is convergent. Hence, the sequence of partial sums is Cauchy. That is, for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$ with $m \geq n$,

$$\sum_{n=N}^M |\langle x, u_n \rangle|^2 < \varepsilon.$$

Since $(u_n)_{n=1}^{\infty}$ is a sequence of orthogonal vectors, we can apply Pythagoras' Theorem to conclude that

$$\left\| \sum_{n=N}^M \langle x, u_n \rangle u_n \right\|^2 < \varepsilon.$$

Since \mathcal{H} is complete, we find that the series $\sum \langle x, u_n \rangle u_n$ converges (absolutely). Now, let

$$y = x - \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n.$$

Then, for all $m \in \mathbb{Z}_{\geq 1}$,

$$\langle y, u_m \rangle = \langle x, u_m \rangle - \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle u_n, u_m \rangle = \langle x, u_m \rangle - \langle x, u_m \rangle = 0.$$

Hence, by assumption, it follows that $y = 0$.

(3) \implies (2) By Bessel's Inequality, we know that for all $N \in \mathbb{Z}_{\geq 1}$,

$$\|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2 = \|x - \sum_{n=1}^N \langle x, u_n \rangle u_n\|^2 = 0.$$

Taking limits, it follows that

$$\|x\|^2 - \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = 0.$$

(2) \implies (1) Let $x \in \mathcal{H}$. Then,

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \sum_{n=1}^{\infty} 0 = 0.$$

So, $x = 0$.

□

Definition 3.7.8. Let \mathcal{H} be a Hilbert space and let $(u_n)_{n=1}^\infty$ be a sequence of orthonormal vectors in \mathcal{H} . We say that (u_n) is an *orthonormal basis* if for all $x \in \mathcal{H}$,

$$\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n = x.$$

Theorem 3.7.9. Let \mathcal{H} be a Hilbert space. Then, \mathcal{H} is separable if and only if \mathcal{H} has a countable orthonormal basis.

Proof. First, assume that \mathcal{H} is separable. In that case, there exists a countable dense subset $(x_n)_{n=1}^\infty$. So, we can remove entries from the sequence so that $(v_n)_{n=1}^\infty$ is linearly independent- we remove the fewest entries to achieve this. By applying Gram-Schmidt, we can construct an orthonormal sequence $(u_n)_{n=1}^\infty$. This is an orthonormal basis for \mathcal{H} .

Now, assume that \mathcal{H} has a countable orthonormal basis $(u_n)_{n=1}^\infty$. In that case, the span of the basis is dense- this is not equal since the span only considers finite sums. Since \mathbb{C} is separable, we can find a countable dense subset $A \subseteq \mathbb{C}$. Then, we can consider the A -span of (u_n) . This is a countable dense subset of \mathcal{H} . Hence, \mathcal{H} is separable. □

Definition 3.7.10. Let $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$ and $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)$ be Hilbert spaces, and let $T: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a linear map. We say that T is *unitary* if for all $x, y \in \mathcal{H}_1$, $\langle Tx, Ty \rangle_2 = \langle x, y \rangle_1$.

Corollary 3.7.11. Let \mathcal{H} is an infinite-dimensional separable Hilbert space. Then, \mathcal{H} is unitarily isomorphic to ℓ^2 .

Proof. Let $(u_n)_{n=1}^\infty$ be a countable orthonormal basis for \mathcal{H} . Define the map $\phi: \mathcal{H} \rightarrow \ell^2$ by $\phi(x) = \hat{x}$, where $\hat{x}_n = \langle x, u_n \rangle$. Since the inner product map is linear in the first variable, we find that ϕ is linear. Moreover, since (u_n) is an orthonormal basis, we find that

$$\|\hat{x}\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$$

by Parseval's Identity. Hence, $\hat{x} \in \ell^1$, meaning that T is well-defined. Moreover, T is an isometry. Furthermore, for $x, y \in \mathcal{H}$, we find that

$$\begin{aligned} \langle x, y \rangle &= \left\langle \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n, \sum_{m=1}^{\infty} \langle y, u_m \rangle u_m \right\rangle \\ &= \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle y, u_n \rangle \langle u_n, u_n \rangle \\ &= \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle y, u_n \rangle = \langle \hat{x}, \hat{y} \rangle. \end{aligned}$$

Hence, T is unitary.

Now, we show that T is surjective. So, let $y \in \ell^2$. Consider the series

$$\sum_{n=1}^{\infty} y_n u_n.$$

For all $M, N \in \mathbb{Z}_{\geq 1}$ with $M \geq N$, we find that

$$\left\| \sum_{n=N}^M y_n u_n \right\| = \sum_{n=N}^M \|y_n u_n\| = \sum_{n=N}^M |y_n|$$

by Pythagoras' Theorem. Since $y \in \ell^2$, this implies that the series is Cauchy, meaning that it converges (absolutely). So, let

$$x_n = \sum_{n=1}^{\infty} y_n u_n.$$

Then, we find that $\phi(x) = \hat{x}$, with

$$\hat{x}_n = \langle x, u_n \rangle = \sum_{m=1}^{\infty} \langle y_m u_m, u_n \rangle = y_n \sum_{m=1}^{\infty} \langle u_m, u_n \rangle = y_n.$$

So, T is surjective, meaning that \mathcal{H} is unitarily isomorphic to ℓ^2 . □

3.8 Adjoint Operators

Proposition 3.8.1. *Let \mathcal{H} be a Hilbert space and let $T: \mathcal{H} \rightarrow \mathcal{H}$ be bounded. Then, there exists a unique bounded operator $T^*: \mathcal{H} \rightarrow \mathcal{H}$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{H}$.*

Definition 3.8.2. Let \mathcal{H} be a Hilbert space and let $T: \mathcal{H} \rightarrow \mathcal{H}$ be bounded. Then, the map T^* is the *adjoint* of T .

Proposition 3.8.3. *Let \mathcal{H} be a Hilbert space and let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a map. Then, T is unitary if and only if T is bijective, with $T^{-1} = T^*$.*

Proof.

□