CHAPTER 4

FREE GROUPS

4.1 Introduction to Free Groups

Definition 4.1.1. Let S be a set, and fix a set S^- disjoint to S with a bijection $f\colon S \to S^-$, and a singleton set $\{e\}$. Denote $X_S = S \cup S^- \cup \{1\}$. We define the *inverse map* $-1\colon X_S \to X_S$ by

$$s^{-1} = \begin{cases} e & s = e \\ \varphi(s) & s \in S \\ \varphi^{-1}(s) & s \in S^-. \end{cases}$$

Definition 4.1.2. Let S be a set. A *word* on S is an infinite tuple $(s_1, s_2, \dots)$ with values in X_S such that there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $s_n = e$. A *reduced word* on S is a word $(s_1, s_2, \dots)$ such that:

- if $s_N = e$ for some $N \geq 1$, then $s_n = e$ for all $n \geq N$;
- if $s_i \neq e$, then $s_{i+1} \neq s_1^{-1}$ for all $n \in \mathbb{Z}_{\geq 1}$.

We denote a reduced word $(s_1, s_2, \dots, s_n, e, e, \dots)$ by $s_1 s_2 \dots s_n$, where $s_n \neq e$. The set of all reduced words is denoted by $F(S)$. We have the inclusion map $\iota\colon S \to F(S)$ given by $\iota(s) = (s, e, e, \dots)$. We also denote $e = (e, e, e, \dots)$, and call it *identity element*.

Definition 4.1.3. Let S be a set. Define the operation $\cdot\colon F(S) \to F(S)$ by

$$s_1 \dots s_n \cdot t_1 \dots t_k =$$

The operation is called *concatenation*.

Proposition 4.1.4. *Let S be a set. Then, $F(S)$ is a group under concatenation.*

Proof. □

Proposition 4.1.5 (Universal Property of Free Groups)**.** *Let S be a set, G be a group, and $f : S \to G$ be a map. Then, there exists a unique homomorphism $\varphi : F(S) \to G$ such that $\varphi(s) = f(s)$ for all $s \in S$.*

Proof. Define the map $\varphi\colon F(S) \to G$ by

$$\varphi(s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_n^{\varepsilon_n}) = f(s_1)^{\varepsilon_1} f(s_2)^{\varepsilon_2} \dots f(s_n)^{\varepsilon_n}.$$

By construction, this is a group homomorphism. Moreover, it extends f.

Now, let $\psi\colon F(S) \to G$ be such that $\psi(s) = f(s)$ for all $s \in S$. In that case, for all $s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_n^{\varepsilon_n} \in F(S)$, we find that

$$\begin{aligned}\psi(s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_n^{\varepsilon_n}) &= \psi(s_1)^{\varepsilon_1} \psi(s_2)^{\varepsilon_2} \dots \psi(s_n)^{\varepsilon_n} \\ &= f(s_1)^{\varepsilon_1} f(s_2)^{\varepsilon_2} \dots f(s_n)^{\varepsilon_n} = \varphi(s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_n^{\varepsilon_n}).\end{aligned}$$

So, the map is unique. □

Corollary 4.1.6. *Let S be a set, with free groups $F_1(S)$ and $F_2(S)$. Then, there exists a unique isomorphism $\phi\colon F_1(S) \to F_2(S)$ that fixes S.*

Proof. Let $\iota_1\colon S \hookrightarrow F_1(S)$ and $\iota_2\colon S \hookrightarrow F_2(S)$ be the inclusion maps. We can apply the universal property of the free group $F_2(S)$ on the map ι_1 to extend it to a unique homomorphism $\varphi_1\colon F_1(S) \to F_2(S)$. Similarly, we can construct a homomorphism $\varphi_2\colon F_2(S) \to F_1(S)$. Note that, by construction, φ_1 and φ_2 fix S. Now, consider the map $\varphi_2 \circ \varphi_1\colon F_1(S) \to F_1(S)$. This is a group homomorphism that fixes S. We can apply again the universal property of the free group $F_1(S)$ on the map ι_1 to extend it to a unique homomorphism $\psi\colon F_1(S) \to F_1(S)$. Note that the identity map is also a homomorphism $\psi\colon F_1(S) \to F_1(S)$, so by uniqueness we find that $\psi = \varphi_2 \circ \varphi_1$ are the identity map on $F_1(S)$. Similarly, $\varphi_1 \circ \varphi_2$ is the identity map on $F_2(S)$. Hence, φ_1 is an isomorphism with inverse φ_2^{-1}. By construction, the map is unique and fixes S. □

Definition 4.1.7. Let S be a set. We say that $F(S)$ is the *free group* on S. We say that S is the set of *free generators* (or *free basis*) of $F(S)$. The *rank* of the free group $F(S)$ is the cardinality of S.

Proposition 4.1.8. *A free group of rank 0 is isomorphic to the trivial group.*

Proof. □

Proposition 4.1.9. *A free group of rank 1 is isomorphic to $\mathbb{Z}$.*

Proof. □

Proposition 4.1.10. *A free group of rank $n \geq 2$ is not abelian.*

Proof. □

Proposition 4.1.11. *A free group has no torsion elements.*

Proof. □

Theorem 4.1.12 (Neilson-Schrier Theorem)**.** *Let F be a free group and let $G \subseteq F$. Then, G is free.*

4.2 Group Relations and Presentation

Lemma 4.2.1. *Let G be a group. Then, G is the image of some free group. In particular, there exists a free group F and a surjective group homomorphism $\varphi\colon F \to G$.*

Proof. Consider the free group $F(G)$. By the universal propery of free groups on the identity map $id\colon G \to G$, we can extend it to a group homomorphism $\varphi\colon F(G) \to G$. By construction, we know that $\varphi(g) = g$ for all $g \in G$, meaning that φ is surjective. □

Definition 4.2.2. Let G be a group and let $R \subseteq G$. Then, the *normal closure* of R is the intersection of all normal subgroups of G containing R. It is denoted by $\langle\langle R\rangle\rangle$.

Proposition 4.2.3. *Let G be a group and let $R \subseteq G$. Then, $\langle\langle R\rangle\rangle$ is the subgroup generated by the conjugates of R.*

Proof. Since the normal closure $\langle\langle R\rangle\rangle$ is normal, we know that the conjugates of R are in the subgroup. Moreover, a subgroup generated by the conjugates of R is closed under conjugation by construction, meaning that it is normal, and contains R. Hence, it is contained in $\langle\langle R\rangle\rangle$. So, the normal closure is the subgroup generated by the conjugates of R. □

Proposition 4.2.4. *Let G, H be groups, $R \subseteq G$ and let $\varphi\colon G \to H$ be a homomorphism with $R \subseteq \ker\varphi$. Then, $\langle\langle R\rangle\rangle \leq \ker\varphi$. In particular, $\langle\langle R\rangle\rangle$ is the smallest unique kernel of a group homomorphism that sends R to the identity.*

Proof. Since $\ker\varphi$ is a normal subgroup, and $R \subseteq \ker\varphi$, it follows that $\langle\langle R\rangle\rangle \leq \ker\varphi$. □

Definition 4.2.5. Let G be a group and S a generating set of G. A *presentation* is a pair (S, R), where R is a set of words in $F(S)$ such that the normal closure $\langle\langle R\rangle\rangle$ is the kernel of the homomorphism $\varphi\colon F(S) \to G$ that fixes S. The set R is called the *relators*. We denote $G = \langle S \mid R\rangle$.

We say that G is *finitely presented* if there exists a presentation of G, (S, R), such that both S and R are finite. We say that G is *finitely generated* if there exists a presentation of G, (S, R), such that S is finite.

Proposition 4.2.6. *Let G be a finite group. Then, G is finitely presented.*

Proof. □

Proposition 4.2.7. *Let G and H be groups with bijective presentations. Then, there exists a group isomorphism $G \to H$.*

Proof. □

Proposition 4.2.8. *There is one non-abelian group of order 10 up to isomorphism.*

Proof. □