CHAPTER 3

PRODUCT MEASURES

3.1 Product Algebras

Definition 3.1.1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. We define the *product σ-algebra* $\mathcal{A} \otimes \mathcal{B}$ by the σ-algebra generated by sets of the form $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Definition 3.1.2. Let X and Y be sets and let $E \subseteq X \times Y$. For $x \in X$, we define the *x-section of E* by the set

$$E_x = \{y \in Y \mid (x, y) \in E\},$$

and for $y \in Y$, the *y-section of E* by the set

$$E^y = \{x \in X \mid (x, y) \in E\}.$$

Lemma 3.1.3. *Let X and Y be sets and let $E \subseteq X \times Y$. Then,*

1. *for $x \in X$ and $y \in Y$, $(E_x)^c = (E^c)_x$ and $(E^y)^c = (E^c)^y$;*
2. *for a sequence of subsets $(E_n)_{n=1}^{\infty}$ in $X \times Y$,*

$$\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \bigcup_{n=1}^{\infty} (E_n)_x \qquad \left(\bigcap_{n=1}^{\infty} E_n\right)_x = \bigcap_{n=1}^{\infty} (E_n)_x$$
$$\left(\bigcup_{n=1}^{\infty} E_n\right)^y = \bigcup_{n=1}^{\infty} (E_n)^y \qquad \left(\bigcap_{n=1}^{\infty} E_n\right)^y = \bigcap_{n=1}^{\infty} (E_n)^y.$$

Proof.

1. Let $x \in X$. For $y \in Y$, we have

$$\begin{aligned} y \in (E_x)^c &\iff y \notin E_x \\ &\iff (x, y) \notin E \\ &\iff (x, y) \in E^c \\ &\iff y \in (E^c)_x. \end{aligned}$$

   Hence, $(E_x)^c = (E^c)_x$. Similarly, $(E^y)^c = (E^c)^y$.

2. Let $x \in X$. For $y \in Y$, we have

$$\begin{aligned} y \in \left(\bigcup_{n=1}^{\infty} E_n\right)_x &\iff \exists n \in \mathbb{Z}_{\geq 1} \text{ s.t. } (x, y) \in E_n \\ &\iff \exists n \in \mathbb{Z}_{\geq 1} \text{ s.t.} y \in (E_n)_x \\ &\iff y \in \bigcup_{n=1}^{\infty} (E_n)_x. \end{aligned}$$

Also, for $y \in Y$, we have

$$\begin{aligned} y \in \left(\bigcap_{n=1}^{\infty} E_n\right)_x &\iff \forall n \in \mathbb{Z}_{\geq 1}, (x,y) \in E_n \\ &\iff \forall n \in \mathbb{Z}_{\geq 1}, y \in (E_n)_x \\ &\iff y \in \bigcap_{n=1}^{\infty} (E_n)_x. \end{aligned}$$

Hence,

$$\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \bigcup_{n=1}^{\infty} (E_n)_x \qquad \left(\bigcap_{n=1}^{\infty} E_n\right)_x = \bigcap_{n=1}^{\infty} (E_n)_x.$$

Similarly,

$$\left(\bigcup_{n=1}^{\infty} E_n\right)^y = \bigcup_{n=1}^{\infty} (E_n)^y \qquad \left(\bigcap_{n=1}^{\infty} E_n\right)^y = \bigcap_{n=1}^{\infty} (E_n)^y.$$

□

Proposition 3.1.4. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and $E \in \mathcal{A} \otimes \mathcal{B}$. Then, for all $x \in X$ and $y \in Y$, $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$.*

Proof. Let

$$\mathcal{M} = \{S \in \mathcal{A} \otimes \mathcal{B} \mid S_x \in \mathcal{B} \ \forall x \in X, S^y \in \mathcal{A} \ \forall y \in Y\}.$$

We show that $\mathcal{M}$ is a σ-algebra.

- We have $\varnothing \in \mathcal{M}$ since $\varnothing_x = \varnothing \in \mathcal{B}$ and $\varnothing^y = \varnothing \in \mathcal{A}$ for all $x \in X$ and $y \in Y$.
- Let $S \in \mathcal{M}$. Then, for all $x \in X$ and $y \in Y$,
$$(S^c)_x = (S_x)^c \in \mathcal{B}, \qquad (S^c)^y = (S^y)^c \in \mathcal{A}$$
since $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras.
- Let $(S_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{M}$. Then, for all $x \in X$ and $y \in Y$,
$$\left(\bigcup_{n=1}^{\infty} S_n\right)_x = \bigcup_{n=1}^{\infty} (S_n)_x \in \mathcal{B}, \qquad \left(\bigcup_{n=1}^{\infty} S_n\right)^y = \bigcup_{n=1}^{\infty} (S_n)^y \in \mathcal{A}$$
since $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras.

Hence, $\mathcal{M}$ is a σ-algebra. Now, let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. For $x \in X$ and $y \in Y$, we have

$$(A \times B)_x = \begin{cases} B & x \in A \\ \varnothing & \text{otherwise,} \end{cases} \qquad (A \times B)^y = \begin{cases} A & y \in B \\ \varnothing & \text{otherwise.} \end{cases}$$

So, $A \times B \in \mathcal{M}$. Since $\mathcal{A} \otimes \mathcal{B}$ is generated by $\mathcal{A} \times \mathcal{B}$, we find that $\mathcal{M} = \mathcal{A} \times \mathcal{B}$. That is, for all $E \in \mathcal{A} \otimes \mathcal{B}$, $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$. □

Definition 3.1.5. Let X and Y be sets and let $f\colon X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function. For $x \in X$, define the function $f_x\colon Y \to \mathbb{R} \cup \{\pm\infty\}$ by $f_x(y) = f(x, y)$, and for $y \in Y$, define the function $f_y\colon X \to \mathbb{R} \cup \{\pm\infty\}$ by $f_y(x) = f(x, y)$.

Lemma 3.1.6. *Let X and Y be sets and let $f\colon X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Then, for $a \in \mathbb{R}$, $x \in X$ and $y \in Y$,*

$$(f_x)^{-1}(a, \infty] = (f^{-1}(a, \infty])_x, \qquad (f_y)^{-1}(a, \infty] = (f^{-1}(a, \infty])_y.$$

Proof. Let $a \in \mathbb{R}$ and $x \in X$. For $y \in Y$, we find that

$$\begin{aligned} y \in (f_x)^{-1}(a, \infty] &\iff f(x, y) = f_x(y) \in (a, \infty] \\ &\iff (x, y) \in f^{-1}(a, \infty] \\ &\iff y \in (f^{-1}(a, \infty])_x. \end{aligned}$$

Hence, $(f_x)^{-1}(a, \infty] = (f^{-1}(a, \infty])_x$ for all $x \in X$. Similarly, $(f_y)^{-1}(a, \infty] = (f^{-1}(a, \infty])^y$ for all $y \in Y$. ☐

Proposition 3.1.7. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and $f\colon X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be measurable with respect to $\mathcal{A} \otimes \mathcal{B}$. Then, for all $x \in X$, f_x is measurable with respect to $\mathcal{B}$ and for all $y \in Y$, f_y is measurable with respect to $\mathcal{A}$.*

Proof. Let $a \in \mathbb{R}$. Since f is measurable, we find that

$$S = f^{-1}(a, \infty] \in \mathcal{A} \otimes \mathcal{B}.$$

Now, let $x \in X$ and $y \in Y$. We have shown above that $S_x \in \mathcal{B}$ and $S^y \in \mathcal{A}$. In that case,

$$(f_x)^{-1}(a, \infty] = S_x \in \mathcal{B}, \qquad (f_y)^{-1}(a, \infty] = S^y \in \mathcal{A}.$$

This implies that f_x is measurable with respect to $\mathcal{B}$, and f_y is measurable with respect to $\mathcal{A}$. ☐

Definition 3.1.8. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Define the set $\mathcal{E}$ by the elements in $X \times B$ that are a finite union of elements in $\mathcal{A} \times \mathcal{B}$.

Proposition 3.1.9. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Then, $\mathcal{E}$ is an algebra.*

Proof.

- We have $\varnothing \in \mathcal{E}$ since $\varnothing \in \mathcal{A} \times \mathcal{B}$.
- Let $E, F \in \mathcal{E}$. Then, E and F are both finite union of elements in $\mathcal{A} \times \mathcal{B}$. Hence, their union $E \times F$ is a finite union of elements in $\mathcal{A}$.
- Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. For $(x, y) \in X \times Y$, we have

$$\begin{aligned} (x, y) \in (A \times B)^c &\iff (x, y) \notin A \times B \\ &\iff x \notin A \text{ or } y \notin B \\ &\iff (x \in A^c \text{ and } y \in B) \text{ or } (x \in A^c \text{ and } y \in B^c) \\ &\quad \text{or } (x \in A \text{ and } y \in B^c) \\ &\iff (x, y) \in (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c). \end{aligned}$$

Hence,

$$(A \times B)^c = (A^c \times B) \cup (A \times B^c) \times (A^c \times B^c),$$

where each of the 3 subsets is disjoint. This implies that $(A \times B)^c \in \mathcal{E}$. We know that $\mathcal{E}$ is closed under the union of finite intervals from above, so in general, for all $E \in \mathcal{E}$, $E^c \in \mathcal{E}$.

So, $\mathcal{E}$ is an algebra. □

Proposition 3.1.10. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Define the map $\mu \otimes \nu \colon \mathcal{A} \times \mathcal{B} \to [0, \infty]$ by*

$$(\mu \otimes \nu)\left(\bigcup_{k=1}^{n} A_k \times B_k\right) = \sum_{k=1}^{n} \mu(A_k) \cdot \mu(B_k)$$

whenever $(A_k \times B_k)_{k=1}^n$ is pairwise disjoint. Then, $\mu \otimes \nu$ is a measure.

Proof.

- We first show that $\mu \otimes \nu$ is well-defined. So, let $(A_i \times B_i)_{i=1}^n$ and $(C_j \times D_j)_{j=1}^m$ be disjoint sequences of sets in $\mathcal{A} \times \mathcal{B}$ such that

$$\bigcup_{i=1}^{n} A_i \times B_i = \bigcup_{j=1}^{m} C_j \times D_j.$$

□

Proposition 3.1.11. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Then, there exists a measurable functions $\mu \otimes \nu$ on $\mathcal{A} \otimes \mathcal{B}$ such that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,*

$$(\mu \otimes \nu)(A \otimes B) = \mu(A) \cdot \nu(B).$$

Proof. By Caratheodory Extension Theorem, we know that the measure $\mu \otimes \nu$ extends to the σ-algebra generated by $\mathcal{A} \times \mathcal{B}$, i.e. $\mathcal{A} \otimes \mathcal{B}$. □

Proposition 3.1.12. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and $A \in \mathcal{A}$, $B \in \mathcal{B}$. If there exists a sequence of disjoint subsets $(A_n \times B_n)_{n=1}^\infty$ such that*

$$A \times B = \bigcup_{n=1}^{\infty} A_n \times B_n,$$

then

$$(\mu \otimes \nu)(A \otimes B) = \mu(A) \cdot \nu(B) = \sum_{n=1}^{\infty} (\mu \otimes \nu)(A_n \otimes B_n).$$

Proof. □

Lemma 3.1.13. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $E \in \mathcal{A} \otimes \mathcal{B}$. Then, the functions $f \colon X \to [0, \infty]$ and $g \colon Y \to [0, \infty]$ defined by $f(x) = \nu(E_x)$ and $g(x) = \mu(E_y)$ are measurable with*

$$(\mu \otimes \nu)(E) = \int_X f \; d\mu = \int_Y g \; d\nu,$$

meaning that

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\ d\mu(x) = \int_Y \mu(E^y)\ d\nu(y).$$

Proof. Let

$$\mathcal{M} = \left\{ E \in \mathcal{A} \otimes \mathcal{B} \mid (\mu \otimes \nu)(E) = \int_X \nu(E_x)\ d\nu(x) = \int_Y \mu(E^y)\ d\mu(y) \right\}.$$

We claim that $\mathcal{M}$ is a σ-algebra containing $\mathcal{A} \times \mathcal{B}$.

- We have

  $$\begin{aligned} \mu(\varnothing) &= 0 \\ \int_X \nu(\varnothing_x)\ d\mu(x) = \int_X 0\ d\mu(x) &= 0 \\ \int_Y \mu(\varnothing_y)\ d\nu(y) = \int_Y 0\ d\nu(y) &= 0. \end{aligned}$$

  So, $\varnothing \in \mathcal{M}$.

- Let $(E_n)_{n=1}^{\infty}$ be a sequence of disjoint sets in $\mathcal{M}$. We have

  $$\int_X \nu\left(\bigcup_{n=1}^{\infty} (E_n)_x\right)\ d\mu(x) = \int_X \sum_{n=1}^{\infty} \nu((E_n)_x)\ d\mu(x)$$

  since ν is a measure. Since $\nu((E_n)_x) \geq 0$, Monotone Convergence Theorem tells us that

  $$\int_X \sum_{n=1}^{\infty} \nu((E_n)_x)\ d\mu(x) = \sum_{n=1}^{\infty} \int_X \nu((E_n)_x)\ d\mu(x).$$

  Since $E_n \in \mathcal{M}$ for all $n \in \mathbb{Z}_{\geq 1}$,

  $$\sum_{n=1}^{\infty} \int_X \nu((E_n)_x)\ d\mu(x) = \sum_{n=1}^{\infty} (\mu \otimes \nu)((E_n)_x).$$

  Now, since $\mu \otimes \nu$ is a measure, we find that

  $$\sum_{n=1}^{\infty} (\mu \otimes \nu)((E_n)_x) = (\mu \otimes \nu)(\bigcup_{n=1}^{\infty} E_n)_x.$$

  Hence,

  $$\int_X \nu\left(\bigcup_{n=1}^{\infty} (E_n)_x\right)\ d\mu(x) = (\mu \otimes \nu)(\bigcup_{n=1}^{\infty} E_n)_x.$$

  Similarly,

  $$\int_Y \mu\left(\bigcup_{n=1}^{\infty} (E_n)^y\right)\ d\nu(y) = (\mu \otimes \nu)(\bigcup_{n=1}^{\infty} E_n)^y.$$

  This implies that

  $$\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}.$$

- By Dynkin's Lemma, we find that $\mathcal{M}$ is closed under complements.
- Now, let $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and denote $E = A \times B$. We know that for all $x \in X$,
$$\nu(E_x) = \begin{cases} \nu(B) & x \in A \\ 0 & x \in B. \end{cases}$$
Hence,
$$\int_X \nu(E_x)\ d\mu(x) = \int_X \chi_A \nu(B)\ d\mu(x) = \nu(B) \cdot \mu(X) = (\mu \otimes \nu)(A \times B).$$
Similarly,
$$\int_Y \mu(E^y)\ d\nu(y) = \int_Y \chi_B \mu(A)\ d\nu(y) = (\mu \otimes \nu)(A \times B).$$
This implies that $E \in \mathcal{M}$.

So, since $\mathcal{M}$ is a σ-algebra containing $\mathcal{A} \times \mathcal{B}$, we find that $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$. Hence,
$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\ d\mu(x) = \int_Y \mu(E^y)\ d\nu(y)$$
for all $E \in \mathcal{A} \otimes \mathcal{B}$. $\square$

Theorem 3.1.14 (Tonelli's Theorem). *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $f \colon X \times Y \to [0, \infty]$ be integrable with respect to $\mu \otimes \nu$. Then, for $F \colon X \to [0, \infty]$ and $G \colon Y \to [0, \infty]$ given by*
$$F(x) = \int_Y f(x, y)\ d\nu(y), \qquad G(y) = \int_X f(x, y)\ d\mu(x)$$
are ν- and μ-measurable respectively. Then, f is integrable,
$$\int_{X \times Y} f\ d(\mu \otimes \nu) = \int_X F\ d\mu = \int_Y G\ d\nu.$$

Proof. Let $E \in \mathcal{A} \otimes \mathcal{B}$. We first show that $f = \chi_E$ satisfies the result. For $x \in X$ and $y \in Y$, we have
$$(\chi_E)_x(y) = \chi_E(x, y) = \begin{cases} 1 & (x, y) \in E \\ 0 & \text{otherwise} \end{cases} = \chi_{E_x}(y),$$
so $(\chi_E)_x = \chi_{E_x}$. Similarly, $(\chi_E)^y = \chi_{E^y}$. This implies that
$$F(x) = \int_Y f_x\ d\nu(y) = \int_Y \chi_{E_x}\ d\nu = \nu(E_x),$$
and
$$G(y) = \int_X f^y\ d\mu(y) = \int_X \chi_{E^y}\ d\mu = \mu(E^y).$$

By the result above, we know that

$$\int_X F \; d\mu = \int_X \nu(E_x) \; d\mu(x) = (\mu \otimes \nu)(E)$$
$$\int_Y G \; d\nu = \int_Y \mu(E^y) \; d\nu(y) = (\mu \otimes \nu)(E).$$

Now, the result follows for a simple measurable function- it is a linear combination of characteristic functions on measurable sets.

Next, let $f \geq 0$ be measurable. We know that there exists a sequence of (monotone) simple functions $(s_n)_{n=1}^{\infty}$ with $s_n \to f$. We find that

$$\int_Y F_n(x) \; d\nu(y) \to \int_Y F \; d\nu(y)$$
$$\int_X G_n(y) \; d\mu(x) \to \int_X G \; d\mu(x)$$

by Monotone Convergence Theorem. Again, by MCT,

$$\int_X F_n \; d\mu(x) = \int_X \int_Y s_n \; d\mu \otimes \nu \to \int_X F(x) \; d\mu(x)$$
$$\int_X G_n \; d\mu(x) = \int_Y \int_X s_n \; d\mu \otimes \nu \to \int_Y G(y) \; d\nu(y).$$

Since

$$\int_X \int_Y s_n \; d\mu \otimes \nu = \int_Y \int_X s_n \; d\mu \otimes d\nu,$$

we find that

$$\int_X F \; d\mu(x) = \int_Y G \; d\nu(y) = \int_{X \times Y} f \; d\mu \otimes d\nu.$$

□

Theorem 3.1.15 (Fubini's Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $f \colon X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be integrable with respect to $\mu \otimes \nu$. Then,*

1. *for μ-almost all x, the function $f_x \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ given by $f_x(y) = f(x, y)$ is ν integrable; for ν-almost all y, the function $f_y \colon X \to \mathbb{R} \cup \{\pm\infty\}$ given by $f_y(x) = f(x, y)$ is μ integrable.*
2. *for $F \colon X \to \mathbb{R} \cup \{\pm\infty\}$ and $G \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by*
$$F(x) = \int_Y f(x, y) \; d\nu(y), \qquad G(y) = \int_X f(x, y) \; d\mu(x),$$
F is μ-integrable and G is ν-integrable.
3. $$\int_{X \times Y} f \; d(\mu \otimes \nu) = \int_X F \; d\mu = \int_Y G \; d\nu.$$

Proof. This follows from Tonelli's Theorem, taking $f = f_+ + f_-$. □