CHAPTER 4

FREE GROUPS

4.1 Introduction to Free Groups

Definition 4.1.1. Let S be a set, and fix a set S^- disjoint to S with a bijection $f\colon S \to S^-$, and a singleton set $\{e\}$. Denote $X_S = S \cup S^- \cup \{1\}$. We define the *inverse map* $-1\colon X_S \to X_S$ by

$$s^{-1} = \begin{cases} e & s = e \\ \varphi(s) & s \in S \\ \varphi^{-1}(s) & s \in S^-. \end{cases}$$

Definition 4.1.2. Let S be a set. A *word* on S is an infinite tuple $(s_1, s_2, \dots)$ with values in X_S such that there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $s_n = e$. A *reduced word* on S is a word $(s_1, s_2, \dots)$ such that:

- if $s_N = e$ for some $N \geq 1$, then $s_n = e$ for all $n \geq N$;
- if $s_i \neq e$, then $s_{i+1} \neq s_1^{-1}$ for all $n \in \mathbb{Z}_{\geq 1}$.

We denote a reduced word $(s_1, s_2, \dots, s_n, e, e, \dots)$ by $s_1 s_2 \dots s_n$, where $s_n \neq e$. The set of all reduced words is denoted by $F(S)$. We have the inclusion map $\iota\colon S \to F(S)$ given by $\iota(s) = (s, e, e, \dots)$. We also denote $e = (e, e, e, \dots)$, and call it *identity element.*

Definition 4.1.3. Let S be a set. Define the operation $\cdot\colon F(S) \to F(S)$ by

$$s_1 \dots s_n \cdot t_1 \dots t_k =$$

The operation is called *concatenation.*

Proposition 4.1.4. *Let S be a set. Then, $F(S)$ is a group under concatenation.*

Proof. □

Proposition 4.1.5. *Let S be a set, G be a group, and $f : S \to G$ be a map. Then, there exists a unique homomorphism $\varphi : F(S) \to G$ such that $\varphi(s) = f(s)$ for all $s \in S$.*

Proof. □

Corollary 4.1.6. *Let S be a set. Then, $F(S)$ is unique.*

Proof. □

4.2 Group Relation and Presentation