CHAPTER 1

NORMED VECTOR SPACES

1.1 Review of Vector Spaces

In this section, we will review properties of vector spaces with relation to vector spaces.

Definition 1.1.1. A *vector space* $(V, +, \cdot)$ *(over a field* $\mathbb{K}$*)* is a set V and functions $(+)\colon V \times V \to V$ and $(\cdot)\colon \mathbb{K} \times V \to V$ such that:

- $(V, +)$ is an abelian group;
- $\cdot$ is associative over $+$, i.e. for $a, b \in \mathbb{K}$ and $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$;
- $\cdot$ left- and right-distributes over $+$, i.e. for $a \in \mathbb{K}$ and $v, w \in V$, $a \cdot (v+w) = a \cdot v + a \cdot w$.

In this course, we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We are familiar with many vector spaces, e.g. $\mathbb{R}^n$ over $\mathbb{R}$ and $\mathbb{C}^n$ over $\mathbb{C}$ (and $\mathbb{R}$).

We now review the concept of dimensionality.

Definition 1.1.2. Let V be a vector space and let $S \subseteq V$.

- We say that S *spans* V if for all $v \in V$, there exists a collection of scalars $(c_{v_i})_{v_i \in S}$ such that
$$v = \sum_{v_i \in S} c_{v_i} \cdot v_i.$$
- We say that S is *linearly independent* if for all linear combinations
$$\sum_{v_i \in S} c_{v_i} \cdot v_i = 0,$$
we have $c_{v_i} = 0$ for all $v_i \in S$.
- We say that S is a *basis* for V if S spans V and is linearly independent.

For $\mathbb{R}^n$, a basis is given by $\{e_1, e_2, \dots, e_n\}$, with
$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$
for $1 \leq i \leq n$. This basis is not unique, e.g. another basis for $\mathbb{R}^n$ is $\{f_1, f_2, \dots, f_n\}$, with
$$f_i = \sum_{j=1}^{i} e_j.$$
Although the basis is not unique, if it is finite, then any other basis will also be finite and have the same number of elements. This value is defined the

dimension of the vector space. Vector spaces that have a basis with finitely many elements are called *finite-dimensional.* We know that for a field $\mathbb{K}$, if V is an n-dimensional vector space over $\mathbb{K}$, then V is isomorphic to $\mathbb{K}^n$. So, these are all the finite-dimensional vector spaces.

We can represent the vector space $\mathbb{R}^n$ as a function. In particular, for some set $X = \{x_1, x_2, \dots, x_n\}$, let $\text{Fun}(X, \mathbb{R})$ be the set of functions $f\colon X \to \mathbb{R}$. We claim that $\text{Fun}(X, \mathbb{R})$ is isomorphic to $\mathbb{R}^n$, with the isomorphism map $\varphi\colon \text{Fun}(X, \mathbb{R}) \to \mathbb{R}^n$

$$\varphi(f) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

Note however that this format is not limited to finite sets; the space of functions $\text{Fun}(X, \mathbb{R})$ is a vector space even when X is infinite. In particular, we consider the case where X is countable, i.e. $X = \mathbb{Z}_{\geq 1}$. The space $\text{Fun}(X, \mathbb{R})$ in this case is the space of all functions $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{R}$, i.e. sequences in $\mathbb{R}$. We denote this as $\text{Seq}(\mathbb{R})$ as well. The sequences form a vector space with respect to pointwise addition and scalar multiplication. This sequence is infinite-dimensional, i.e. it does not have a finite basis. This is because it has the basis $\{e^{(1)}, e^{(2)}, \dots\}$, with the sequence $(e_n^{(k)})_{n=1}^{\infty}$ given by

$$e_n^{(k)} = \begin{cases} 1 & n = k \\ 0 & \text{otherwise.} \end{cases}$$

We know that every basis of a finite-dimensional space is finite, so $\text{Seq}(\mathbb{R})$ is infinite-dimensional.

Also, the space of continuous functions from the compact subset $[0, 1]$ to $\mathbb{R}$, denoted by $C[0, 1]$, is a vector space- it forms a vector space over pointwise addition and scalar multiplication, i.e. for $c \in \mathbb{R}$ and $f \in C[0, 1]$, we define the function $c \cdot f \in C[0, 1]$ by $(c \cdot f)(x) = c \cdot f(x)$ for $x \in [0, 1]$. This is also an infinite-dimensional space- it has a subspace consisting of polynomial functions, whose basis is given by

$$\{f_n \mid n \in \mathbb{Z}_{\geq 1}\},$$

where $f_n(x) = x^n$ for all $x \in [0, 1]$. Hence, it has an infinite-dimensional subspace, meaning that the entire space must also be infinite-dimensional. We will later see that the space of polynomials is a dense subspace of $C[0, 1]$, i.e. a continuous function can be approximated by a polynomial function arbitrarily well.

1.2 Metrics, Norms and Inner Products

In this section, we will expand the algebraic vector space properties and connect them with analytic ones. In particular, we will look at metrics in vector spaces, and then a stronger concept of norms, and finally inner product spaces.

Definition 1.2.1 (Metric spaces)**.** Let V be a set and let $d\colon V \times V \to \mathbb{R}_{\geq 0}$ be a function. We say that (V, d) is a *metric space* if:

- for all $u, v \in V$, $d(u, v) = 0$ if and only if $u = v$;
- for all $u, v \in V$, $d(u, v) = d(v, u)$;
- for all $u, v, w \in V$, $d(u, w) \leq d(u, v) + d(v, w)$.

If (V, d) is a metric space, we call d a *metric*.

The function d represents a distance function; it allows us to measure distance between two values in V.

There are many examples of metric spaces. In $\mathbb{R}^n$, the following are 3 different norms:

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^{n} |x_i - y_i| \\ d_2(x, y) &= \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \\ d_\infty(x, y) &= \max_{i=1}^{n} |x_i - y_i|. \end{aligned}$$

In general, we can define the d_p-metric for $p \in [1, \infty)$ as follows:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

We can define a lot more metrics on $\mathbb{R}^n$, such as the discrete metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise.} \end{cases}$$

We would like to consider a structure that behaves better with the structure of a vector space, like the d_p-metrics. This gives rise to a norm.

Definition 1.2.2 (Normed Vector Space)**.** Let V be a vector space and let $\|\cdot\|\colon V \times V \to \mathbb{R}_{\geq 0}$ be a function. We say that $(V, \|\cdot\|)$ is a *normed vector space* if:

- for all $v \in V$, $\|v\| = 0$ if and only if $v = 0$;
- for all $v \in V$ and $\lambda \in \mathbb{C}$, $\|\lambda v\| = |\lambda| \|v\|$;
- for all $u, v \in V$, $\|u + v\| \leq \|u\| + \|v\|$.

If $(V, \|\cdot\|)$ is a normed vector space, we call $\|\cdot\|$ a *norm*.

The norm function allows us to measure the magnitude of a vector.

In $\mathbb{R}^n$, we have many norms, such as $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ given as follows:

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^n |x_i| \\
\|x\|_2 &= \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \\
\|x\|_\infty &= \max_{i=1}^n |x_i|.
\end{aligned}$$

These norms are quite closely related to the d_1, d_2 and d_∞-metrics respectively. It turns out that every norm induces a metric, given by

$$d(x, y) = \|x - y\|.$$

However, it is not the case that every metric is induced by a metric, e.g. the discrete metric is not induced by a norm.

We will now look at some norms in infinite-dimensional vector spaces. In particular, if we look at the space of sequences $\mathrm{Seq}(\mathbb{R})$, we can define the norms in a similar manner as above, i.e.

$$\begin{aligned}
\|(x_n)\|_1 &= \sum_{n=1}^\infty |x_n| \\
\|(x_n)\|_2 &= \left(\sum_{n=1}^\infty x_n^2\right)^{1/2} \\
\|(x_n)\|_\infty &= \sup_{n=1}^\infty |x_n|.
\end{aligned}$$

These norms are not defined for all sequences, e.g. the sequence of positive integers has infinite norm with respect to all 3 norms. So, we restrict the norm to those sequences that have a finite value. In particular, we define the following sequence spaces:

- the sequence space ℓ^1, composed of sequences that converge absolutely;
- the sequence space ℓ^2, composed of sequences $(x_n)_{n=1}^\infty$ such that the series
  $$\sum_{n=1}^\infty x_n^2$$
  converges;
- the sequence space ℓ^p, composed of sequences $(x_n)_{n=1}^\infty$ such that the series
  $$\sum_{n=1}^\infty |x_n|^p$$
  converges (for $p \in [1, \infty)$);

- the sequence space ℓ^∞, composed of bounded sequences.

We can also define norms in $C[0,1]$, given as follows:

$$\begin{aligned}
\|f\|_1 &= \int_0^1 |f(t)|\ dt \\
\|f\|_2 &= \left(\int_0^1 (f(t))^2\ dt \right)^{1/2} \\
\|f\|_\infty &= \sup_0^1 |f(t)|.
\end{aligned}$$

We will now add even more structure to a vector space, by defining an inner product.

Definition 1.2.3 (Inner Product Space)**.** Let V be a vector space and let $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ be a function. We say that $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space* if:

- for all $v \in V$, $\langle v, v \rangle \in [0, \infty)$ and $\langle v, v \rangle = 0$ if and only if $v = 0$;
- for all $u, v \in V$ and $\lambda \in \mathbb{C}$, $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$;
- for all $v, w \in V$, $\langle v, w \rangle = \overline{\langle w, v \rangle}$;
- for all $u, v, w \in V$, $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$.

If $(V, \langle \cdot, \cdot, \rangle)$ is an inner product space, we call $\langle \cdot, \cdot \rangle$ is an *inner product.*

The inner product allows us to measure angles between two vectors. In particular, the concept of orthogonality gives rise to many powerful results for Hilbert spaces (complete inner product spaces) that do not necessarily hold in Banach spaces (complete normed vector spaces).

In $\mathbb{R}^n$ and $\mathbb{C}^n$, the dot product is an example of an inner product, which is given by

$$\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}.$$

This inner product induces the $\|\cdot\|_2$ norm. In particular, an inner product induces a metric, given by

$$\|x\| = \langle x, x \rangle^{1/2}.$$

To prove this, we require the Cauchy-Schwartz Inequality.

Theorem 1.2.4 (Cauchy-Schwartz Inequality)**.** *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $v, w \in V$, $|\langle v, w \rangle|^2 \leq \langle v, v \rangle \cdot \langle w, w \rangle$.*

Proof. Let $v, w \in V$. If $w = 0$, then the statement is trivial. Otherwise,

$$\begin{aligned}
\langle v, v\rangle - \frac{|\langle v, w\rangle|^2}{\langle w, w\rangle} &= \langle v, v\rangle - \frac{\langle v, w\rangle\overline{\langle v, w\rangle}}{\langle w, w\rangle} \\
&= \langle v, v\rangle - \frac{\langle v, w\rangle^2}{\langle w, w\rangle} - \frac{\langle v, w\rangle\langle w, v\rangle}{\langle w, w\rangle} + \frac{\langle v, w\rangle^2}{\langle w, w\rangle} \\
&= \left\langle v, v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w \right\rangle - \left(\frac{\langle v, w\rangle}{\langle w, w\rangle}\langle w, v\rangle - \frac{\langle v, w\rangle^2}{\langle w, w\rangle^2}\langle w, w\rangle \right) \\
&= \left\langle v, v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w \right\rangle - \left\langle \frac{\langle v, w\rangle}{\langle w, w\rangle} w, v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w \right\rangle \\
&= \left\langle v - \frac{\langle v, v\rangle}{\langle w, w\rangle} w, v - \frac{\langle v, v\rangle}{\langle w, w\rangle} w \right\rangle \geq 0.
\end{aligned}$$

Hence,

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle.$$

□

It is not the case that every inner product is induced by a norm; this is only true for norms that satisfy the Parallelogram identity.

Proposition 1.2.5. *Let $(V, \|\cdot\|)$ be a normed vector space that satisfies the Parallelogram identity, i.e. for all $u, v \in V$,*

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 - \|u - v\|^2.$$

For $\mathrm{Seq}(\mathbb{R})$, an inner product is given by

$$\langle (x_n), (y_n)\rangle = \sum_{n=1}^{\infty} x_i \overline{y_i}.$$

In $C[0, 1]$, we have

$$\langle f, g\rangle = \int_0^1 |f(t)g(t)| \, dt.$$

We will now highlight the difference in metrics/norms for finite- and infinite-dimensional vector spaces. To do so, we will compare equivalence of metrics.

Definition 1.2.6. Let (V, d) and (V, d') be metric spaces. We say that the metrics d and d' are *equivalent* if there exist $c, C > 0$ such that for all $x, y \in V$,

$$cd'(x, y) \leq d(x, y) \leq Cd'(x, y).$$

If V is a vector space with two norms $\|\cdot\|$ and $\|\cdot\|'$, we say that the norms are *equivalent* if the induced metrics are equivalent.

It turns out that in finite-dimensional vector spaces, all norms are equivalent. We will show that the p-norms and the ∞-norm are equivalent.

Proposition 1.2.7. *Let $n \in \mathbb{Z}_{\geq 1}$, $p \in [1, \infty)$ and $x \in \mathbb{R}^n$. Then,*

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty.$$

In particular, all these norms are equivalent.

Proof. Let $x = [x_1, \ldots, x_n]$. So, we have $\|x\|_\infty = |x_i|$, for some $i \in \{1, \ldots, n\}$. Then,

$$\|x\|_\infty = |x_i| = (|x_i|^p)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = \|x\|_p.$$

Moreover,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} \|x\|_\infty^p \right)^{1/p} = n^{1/p} \|x\|_\infty.$$

So,

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty.$$

Hence, for $x, y \in \mathbb{R}^n$, we find that

$$d_\infty(x, y) \leq d_p(x, y) \leq n^{1/p} d_\infty(x, y).$$

So, d_p and d_∞ are equivalent norms. Since metric equivalence is an equivalence relation, this implies that d_p and d_q norms are also equivalent. So, all the norms are equivalent. □

However, in infinite-dimensional vector spaces, the norms are not equivalent. In fact, in sequence spaces, we know that ℓ^1, ℓ^2 and ℓ^∞ are different sequence spaces. So, we will focus on L_1 and L_∞ norms in $C[0, 1]$, and show that they are not equivalent. To see this, consider the sequence of functions $f_n(x) = t^n$. Then,

$$\begin{aligned} \|f_n\|_1 &= \int_0^1 |f_n(t)| \ dt = \int_0^1 t^n \ dt = \frac{1}{n+1} \\ \|f_n\|_\infty &= \sup_{t \in [0,1]} |f_n(t)| = 1. \end{aligned}$$

So,

$$\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{1}{n+1} \to 0.$$

This implies that the norms are not equivalent.

1.3 Sequence Spaces

In this section, we will study sequence spaces in more detail. First, we define more sequence spaces:

- The sequence space c contains all convergent sequences in $\mathbb{R}$;
- The sequence space c_0 contains all sequences in $\mathbb{R}^n$ that converge to 0;
- The sequence space c_{00} contains all sequences that are eventually zero, i.e. $(x_n)_{n=1}^{\infty}$ is in c_{00} if and only if there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \geq N$, $x_n = 0$.

We will now prove some containment relations between the sequence spaces.

Proposition 1.3.1. *We have*

$$c \subsetneq \ell^{\infty}.$$

That is, every convergent sequence is bounded but not every bounded sequence is convergent.

Proof. We first show that $c \subseteq \ell^{\infty}$. So, let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, with $x_n \to L$. In that case, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \geq N$, $|x_n - L| < 1$, i.e. $|x_n| < |L| + 1$. So, set

$$K = \max(|x_1|, \dots, |x_{N-1}|, |L| + 1) > 0.$$

By construction, $|x_n| \leq K$ for all $n < N$. Moreover, for $n \geq N$, $|x_n| < |L|+1 \leq K$. So, K is a bound for (x_n). That is, (x_n) is bounded. Hence, $c \subseteq \ell^{\infty}$.

Now, consider the sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = (-1)^n$. Although the sequence (x_n) is bounded, it does not converge. Hence, $c \subsetneq \ell^{\infty}$. □

Proposition 1.3.2. *We have*

$$c_{00} \subsetneq \ell^1.$$

Proof. We first show that $c_{00} \subseteq \ell^1$. So, let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \geq N$, $x_n = 0$. In that case, the series

$$\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{N} |x_n|$$

converges- it is a finite sum. Hence, $c_{00} \subseteq \ell^1$.

Now, consider the sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{1}{n^2}$. Although the sequence (x_n) is in ℓ^1, it is not in c_{00} (i.e. for all $n \in \mathbb{Z}_{\geq 1}$, $x_n > 0$). So, $c_{00} \subsetneq \ell^1$. □

Proposition 1.3.3. *Let $p \in [1, \infty)$. Then, $\ell^p \subsetneq c_0$.*

Proof. We first show that $\ell^p \subseteq c_0$. So, let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

In that case, we find that $|x_n|^p \to 0$ as $n \to \infty$. Hence, $|x_n| \to 0$ as $n \to \infty$, meaning that $x_n \to 0$. So, $\ell^p \subseteq c_0$.

Now, consider the sequence $(x_n)_{n=1}^\infty$ given by $x_n = \frac{1}{n^p}$. Although the sequence $x_n \to 0$, we find that

$$\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty \frac{1}{n}$$

diverges. Hence, $\ell^p \subsetneq c_0$. $\square$

Proposition 1.3.4. *Let $1 \leq q < p$. Then, $\ell^p \subsetneq \ell^q$.*

Proof. We first show that $\ell^p \subseteq \ell^q$. So, let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^\infty |x_n|^p < \infty.$$

In that case, for all $n \in \mathbb{Z}_{\geq 1}$, we find that

$$\begin{aligned}\sum_{n=1}^\infty |x_n|^q &= \sum_{n=1}^\infty |x_n|^{q-p}|x_n|^p \\ &\leq \sum_{n=1}^\infty \|x\|_\infty^{q-p}|x_n|^p \\ &= \|x\|_\infty^{q-p}\sum_{n=1}^\infty |x_n|^p.\end{aligned}$$

Since $\ell^p \subseteq \ell^\infty$, we know that $\|x\|_\infty < \infty$. Hence,

$$\sum_{n=1}^\infty |x_n|^q < \infty.$$

So, $\ell^p \subseteq \ell^q$.

Now, consider the sequence $(x_n)_{n=1}^\infty$ given by $x_n = \frac{1}{n^{2/q}}$. Then, we have

$$\sum_{n=1}^\infty |x_n|^q = \sum_{n=1}^\infty \frac{1}{n^2} < \infty,$$

but

$$\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty \frac{1}{n^{2p/q}}$$

diverges since $2p/q < 2$. So, $\ell^p \subsetneq \ell^q$. $\square$

In summary, we have the following containment:

$$c_{00} \subsetneq \ell^1 \subsetneq \ell^2 \subsetneq \cdots \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty.$$

Note that each containment is strict. Also,

$$\bigcup_{p\in[1,\infty)} \ell^p \subsetneq c_0$$

since the series $\sum \frac{1}{(\log n)^p}$ diverges for all $p \in [1, \infty)$.

1.4 Topology

In this section, we will review some topological properties and related them to vector spaces. As we know, the concept of a topological space generalises metric spaces, while preserving open and closed sets.

Definition 1.4.1 (Open sets)**.** Let (X, d) be a metric space and let $U \subseteq X$. We say that U is *open* if for all $x \in U$, there exists an $\varepsilon > 0$ such that for $y \in X$, if $d(x, y) < \varepsilon$, then $y \in U$, i.e. $B_\varepsilon(y) \subseteq U$. We say that U is *closed* if its complement is open.

Definition 1.4.2 (Topological space)**.** Let X be a set and let $\mathcal{T} \subseteq \mathbb{P}(X)$. We say that $(X, \mathcal{T})$ is a *topological space* if:

- $\varnothing, X \in \mathcal{T}$;
- if $(U_i)_{i \in I}$ is a collection of subsets in $\mathcal{T}$, then its union
$$\bigcup_{i \in I} U_i \in \mathcal{T}.$$
- if $(U_i)_{i=1}^n$ is a finite collection of subsets in $\mathcal{T}$, then its intersection
$$\bigcap_{i=1}^n U_i \in \mathcal{T}.$$

If $(X, \mathcal{T})$ is a topological space, then we denote $U \in \mathcal{T}$ by an *open set*.

These topological axioms are satisfied by open sets in a metric space. We can define convergence in a topological space.

Definition 1.4.3. Let X be a topological space, $(x_n)_{n=1}^\infty$ be a sequence in X and $x \in X$. We say that $x_n \to x$ if for all $U \subseteq X$ open, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $x_n \in U$.

Next, we define the closure of a set.

Definition 1.4.4. Let X be a topological space and let $W \subseteq X$. The *closure of* W, denoted $\overline{W}$, is the set
$$\{x \in X \mid \exists (x_n)_{n=1}^\infty \text{ in } W \text{ s.t. } x_n \to x\}.$$

That is, the closure contains all the limit points of W in X.

By construction, the closure of a set is closed. We will now define compactness.

Definition 1.4.5 (Compact Sets)**.** Let X be a topological space and let $C \subseteq X$. We say that C is *compact* if for any open cover of C admits a finite subcover.

There is another type of compactness- sequential compactness.

Definition 1.4.6 (Sequentially Compact Sets)**.** Let X be a topological space and let $C \subseteq X$. We say that C is *sequentially compact* if any sequence $(x_n)_{n=1}^\infty$ in C has a convergent subsequence.

For metric spaces, compact and sequentially compact sets are equivalent. Moreover, in $\mathbb{R}^n$, the Heine-Borel theorem characterises compactness.

Theorem 1.4.7 (Heine-Borel Theorem)**.** *Let $K \subseteq \mathbb{R}^n$. Then, J is compact if and only if it is closed and bounded.*

The Heine-Borel theorem does not hold in infinite-dimensional vector spaces. We know that in a metric space, compactness implies closed and bounded. We will illustrate the converse is not true in infinite-dimensional vector spaces. In particular, we will show that the unit ball in ℓ^∞ is not compact. This set is by definition closed, and bounded since it has sequences with norm at most 1. To do so, we will construct a sequence which has no convergent subsequence. So, define the sequence $(e^{(i)})_{i=1}^n$ in the unit ball given by

$$e_n^{(i)} = \begin{cases} 1 & n = i \\ 0 & \text{otherwise.} \end{cases}$$

We know that for any $i, j \in \mathbb{Z}_{\geq 1}$, with $i \neq j$, $\|e^{(i)} - e^{(j)}\|_\infty = 1$. Hence, the sequence cannot have a convergent subsequence- the distance is always bounded above by 1. So, the unit ball cannot be (sequentially) compact.

We aim to characterise compact sets in $C[0,1]$. To do so, we first consider pointwise and uniform convergence.

Definition 1.4.8. Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $f\colon [0,1] \to \mathbb{R}$, and let $f\colon [0,1] \to \mathbb{R}$.

- We say that $f_n \to f$ pointwise if for any $\varepsilon > 0$ and $t \in [0,1]$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|f_n(t) - f(t)| < \varepsilon$;
- We say that $f_n \to f$ uniformly if for any $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $t \in [0,1]$ and $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|f_n(t) - f(t)| < \varepsilon$.

The difference between pointwise and uniform convergence is the choice of N- in pointwise convergence, the value N can depend on $t \in [0,1]$, but in uniform convergence, the value N cannot. Hence, if $(f_n)_{n=1}^\infty$ is a sequence of functions that converge uniformly to some function f, then it also converges pointwise to the same function f.

Uniform convergence in $C[0,1]$ implies that the limit also lies in $C[0,1]$.

Proposition 1.4.9. *Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $C[0,1]$ that converge uniformly to the function $f\colon [0,1] \to \mathbb{R}$. Then, $f \in C[0,1]$.*

Proof. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $t \in [0,1]$ and $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|f_n(t) - f(t)| < \frac{\varepsilon}{3}$. Set $N = n$. Since f_n is continuous, there exists a $\delta > 0$ such that for $s, t \in [0,1]$ if $|s-t| < \delta$, then $|f_n(s) - f_n(t)| < \frac{\varepsilon}{3}$. Then, for $s, t \in [0,1]$, if $|s - t| < \delta$, then

$$\begin{aligned} |f(s) - f(t)| &\leq |f(s) - f_n(s)| + |f_n(s) - f_n(t)| + |f_n(t) - f(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Moreover, we can define the concept of uniform Cauchy, which we shall use in the proof.

Definition 1.4.10. Let $(g_n)_{n=1}^{\infty}$ be a sequence in $C[0,1]$. We say that (g_n) is *uniformly Cauchy* if for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$ and $t \in [0,1]$, if $m, n \geq N$, then $|g_m(t) - g_n(t)| < \varepsilon$.

Lemma 1.4.11. *Let $(g_n)_{n=1}^{\infty}$ be a sequence in $C[0,1]$ that is uniformly Cauchy. Then, $(g_n)_{n=1}^{\infty}$ is uniformly convergent.*

Proof. Let $t \in [0,1]$ and $\varepsilon > 0$. Since (g_n) is uniformly Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $|g_m(t) - g_n(t)| < \varepsilon$. Hence, the sequence $(g_n(t))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists a $g_t \in \mathbb{R}$ such that $g_n(t) \to g_t$. So, define the function $g\colon [0,1] \to \mathbb{R}$ by $g(t) = g_t$. We show that $g_n \to g$ uniformly.

Let $\varepsilon > 0$. Since (g_n) is uniformly Cauchy, there exists a $K \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$ and $t \in [0,1]$, if $m, n \geq K$, then $|g_m(t) - g_n(t)| < \frac{\varepsilon}{2}$. Now let $t \in [0,1]$. Since $g_k(t) \to g(t)$, there exists a $K' \in \mathbb{Z}_{\geq 1}$ such that for $k \in \mathbb{Z}_{\geq 1}$, if $k \geq K'$, then $|g_k(t) - g(t)| < \frac{\varepsilon}{2}$. Now, fix $N = \max(K, K')$. Hence, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq K$, then

$$|g_n(t) - g(t)| \leq |g_n(t) - g_N(t)| + |g_N(t) - g(t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $g_n \to g$ uniformly. □

Now, we consider the concept of equi-continuity, which will allow us to characterise compactness in $C[0,1]$.

Definition 1.4.12. Let $K \subseteq C[0,1]$. We say that K is *equi-continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $f \in K$ and $s, t \in [0,1]$, if $|s - t| < \delta$, then $|f(s) - f(t)| < \varepsilon$.

The concept of equi-continuous functions is a generalisation of continuity- it establishes that functions in K are continuous in a 'similar' manner. We will now characterise compactness in $C[0,1]$.

Theorem 1.4.13 (Arzela-Ascoli Theorem)**.** *Let $K \subseteq C[0,1]$. Then, K is compact if and only if K is closed, bounded and equi-continuous.*

Proof. Assume that K is compact. We know that K is closed and bounded. So, we show that K is equi-continuous. Let $\varepsilon > 0$. We know that $(B_{\varepsilon/3}(f))_{f \in K}$ is an open cover of K, so it has a finite subcover $(B_{\varepsilon/3}(f_i))_{i=1}^{n}$. Now, let $\delta > 0$ such that for $s, t \in [0,1]$, if $|s - t| < \delta$, then $|f_i(s) - f_i(t)| < \frac{\varepsilon}{3}$ for $i \in \{1, \dots, n\}$. Now, let $f \in K$. Let $i \in \{1, \dots, n\}$ such that $\|f - f_i\|_\infty < \frac{\varepsilon}{3}$. Then, for $s, t \in [0,1]$ with $|s - t| < \delta$, then

$$\begin{aligned}
|f(s) - f(t)| &\leq |f(s) - f_i(s)| + |f_i(s) - f_i(t)| + |f_i(t) - f(t)| \\
&\leq \|f - f_i\|_\infty + |f_i(s) - f_i(t)| + \|f_i - f\|_\infty \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence, K is equi-continuous.

Now, assume that K is closed, bounded and equi-continuous. We show that K is (sequentially) compact. So, let $(f_n)_{n=1}^\infty$ be a sequence in K. Enumerate

$$\mathbb{Q} = \{r_1, r_2, \dots\}.$$

Consider the sequence $(f_n(r_1))_{n=1}^\infty$ in $\mathbb{R}$. Since K is bounded, the sequence $(f_n(r_1))_{n=1}^\infty$ is bounded. So, by Bolzano-Weierstrass, we know that the sequence has a convergent subsequence $(f_{1,n}(r_1))_{n=1}^\infty$ in $\mathbb{R}$. Now, consider the sequence $(f_{1,n}(r_2))_{n=1}^\infty$ in $\mathbb{R}$. By Bolzano-Weierstrass again, we know that the sequence has a convergent subsequence $(f_{2,n}(r_2))_{n=1}^\infty$. Since $(f_{2,n})$ is a subsequence of $(f_{1,n})$, we know that $(f_{2,n}(r_1))_{n=1}^\infty$ is still convergent. We can continue this for every $k \in \mathbb{Z}_{\geq 1}$, to define the sequences $(f_{k,n})_{n=1}^\infty$. Now, define the sequence $(g_n)_{n=1}^\infty$ in K by $g_n = f_{n,n}$. This is a subsequence of $(f_n)_{n=1}^\infty$ by construction. Moreover, for $r_j \in \mathbb{Q}$, the sequence $(f_{j,n}(r_j))_{n=1}^\infty$ is convergent, and since $(g_n)_{n=j}^\infty$ is a subsequence of $(f_{j,n})_{n=1}^\infty$, we find that $(g_n(r_j))_{n=1}^\infty$ is convergent.

Next, we show that (g_n) is uniformly Cauchy. Let $\varepsilon > 0$. Since K is equi-continuous, there exists a $\delta > 0$ such that for all $k \in \{1, \dots, n\}$ and $s, t \in [0,1]$, if $|s-t| < \delta$, then $|g_k(s) - g_k(t)| < \frac{\varepsilon}{3}$. Now, partition $[0,1]$ into intervals $(I_k)_{k=1}^n$ of length $< \delta$, and fix $r_k \in I_k$ with $r_k \in \mathbb{Q}$. For each $k \in \{1, \dots, n\}$, there exists an $N_k \in \mathbb{Z}_{\geq 1}$ such that for all $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N_k$, then $|g_m(r_k) - g_n(r_k)| < \frac{\varepsilon}{3}$. Now, let $N = \max(N_1, \dots, N_n)$, and let $t \in [0,1], m, n \in \mathbb{Z}_{\geq 1}$. We have $t \in I_k$ for some $k \in \{1, \dots, n\}$. By construction, we have $|r_k - t| < \delta$. Hence, if $m, n \geq N$, then

$$\begin{aligned} |g_m(t) - g_n(t)| &\leq |g_m(t) - g_m(r_k)| + |g_m(r_k) - g_n(r_k)| + |g_n(r_k) - g_n(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So, (g_n) is uniformly Cauchy. This implies that (g_n) is uniformly convergent. This means that $g_n \to g$, for some $g \in C[0,1]$. Since K is closed, we have $g \in K$. Therefore, $(f_n)_{n=1}^\infty$ has a subsequence $(g_n)_{n=1}^\infty$ such that $g_n \to g$. Hence, K is compact. □

1.5 Separability

In this section, we will look at the concept of separablility.

Definition 1.5.1. Let X be a topological space and let $V \subseteq X$.

- We say that V is *dense* in X if for any $x \in X$, there exists a sequence $(v_n)_{n=1}^{\infty}$ in V such that $v_n \to x$.
- We say that X is *separable* if there exists a countable dense set $V \subseteq X$.

We know that $\mathbb{R}^n$ and $\mathbb{C}^n$ are measurable, since $\mathbb{Q}^n \subseteq \mathbb{R}^n$ and $\mathbb{Q}[i]^n \subseteq \mathbb{R}^n$ are dense and countable. It is a measurable of the size of the space- even though the space might be uncountable, it can still have a countable dense set.

We can characterise denseness in metric spaces in terms of non-empty intersection of open balls.

Lemma 1.5.2. *Let X be a metric space and let $V \subseteq X$. Then, V is dense in X if and only if for any non-empty open set $U \subseteq X$, $U \cap V$ is not empty.*

Proof. Assume that V is dense in X, and let $U \subseteq X$ be non-empty. So, there exists a $x \in U$. Since V is dense, there exists a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \to x$. In that case, we can find an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $x_n \in U$. So, we have $x_N \in U \cap V$, meaning that $U \cap V$ is not empty.

Now, assume that for any non-empty open set $U \subseteq X$, $U \cap V$ is not empty. Let $x \in X$. For $n \in \mathbb{Z}_{\geq 1}$, we know that $B_{1/n}(x)$ is a non-empty open set, so $B_{1/n}(x) \cap V$ is not empty. So, define the sequence $(x_n)_{n=1}^{\infty}$ by $x_n \in B_{1/n}(x) \cap V$. By construction, we have $d(x_n, x) < \frac{1}{n}$, so $x_n \to x$. Moreover, (x_n) is a sequence in V. Hence, V is dense in X. □

We will now show some properties about separable spaces.

Proposition 1.5.3. *Let X, W be separable topological spaces. Then, $X \times W$ is a separable topological space. In particular, a finite product of separable topological spaces is still separable.*

Proof. Since X and Y are separable, there exist subsets $A \subseteq X$ and $B \subseteq Y$ that are countable and dense. Then, $A \times B$ is countable. Moreover, let $(x, y) \in X \times Y$. We can find sequences $(x_n)_{n=1}^{\infty}$ in A and $(y_n)_{n=1}^{\infty}$ in B such that $x_n \to x$ and $y_n \to y$. Hence, $(x_n, y_n) \to (x, y)$ in the product topology. This implies that $A \times B$ is dense. So, $X \times W$ is separable. □

Proposition 1.5.4. *Let X be a topological space and let $(X_n)_{n=1}^{\infty}$ be a sequence of separable subsets of X. Then, the union*

$$Y = \bigcup_{n=1}^{\infty} X_n$$

is separable.

Proof. For each $n \in \mathbb{Z}_{\geq 1}$, since X_n is separable, we can find a countable dense set $V_n \subseteq X_n$. Then, the union

$$V = \bigcup_{n=1}^{\infty} V_n$$

is countable. We claim that V is also dense in Y. So, let $y \in Y$. By definition, we can find an $i \in \mathbb{Z}_{\geq 1}$ such that $y \in X_i$. Since $V_i \subseteq X_i$ is dense, there exists a sequence $(v_n)_{n=1}^{\infty}$ in $V_i \subseteq V$ such that $v_n \to y$. Hence, V is dense in X. So, X is separable. □

Proposition 1.5.5. *Let X be a separable metric space, and let $V \subseteq X$. Then, V is separable.*

Proof. Since X is separable, it has a countable dense set $D \subseteq V$. So, for all $U \subseteq X$ non-empty, $D \cap X$ is not empty. We show that $D \cap V$ is dense in V. So, let $U \cap V \subseteq V$ be open and non-empty, for $U \subseteq X$ open. Then, we know that U is not empty, meaning that $U \cap D$ is not empty. □

Proposition 1.5.6. *Let X be a metric space and let $V \subseteq X$ be separable. Then, its closure $\overline{V} \subseteq X$ is separable.*

Proof. Since V is separable, it has a countable dense set $D \subseteq V$. So, for all $U \subseteq V$ non-empty, $D \cap U$ is not empty. Now, let $U \subseteq \overline{V}$ be open and non-empty. In that case, there exists $x \in U$. Since $x \in \overline{V}$, we know that $U \cap V$ is not empty. Now, since $U \cap V$ is open in V and D dense in V, we know that $(U \cap V) \cap D$ is not empty. Hence,

$$U \cap D = U \cap (V \cap D) = (U \cap V) \cap D = \varnothing.$$

So D is dense in $\overline{V}$. Since D is countable, $\overline{V}$ is separable. □

Using these properties, we will show that ℓ^p is separable for $p \in [1, \infty)$.

Proposition 1.5.7. *Let $p \in [1, \infty)$. Then, the sequence space ℓ^p is separable.*

Proof. We show that $c_{00} \subseteq \ell^p$ is dense in ℓ^p. So, let $x \in \ell^p$. Define the sequence $(x^{(k)})_{k=1}^{\infty}$ in c_{00} by

$$x_n^{(k)} = \begin{cases} x_n & n \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we know that $x^{(k)} \to x$ in ℓ^{∞}, so $x^{(k)} \to x$ in ℓ^p as well. Hence, $c_{00} \subseteq \ell^p$ is dense in ℓ^p. Now, for $n \in \mathbb{Z}_{\geq 1}$, define the sequence space c_{0n} by

$$c_{0n} = \{x \in c_{00} \mid x_k = 0 \ \forall k \geq n\}.$$

There exists a canonical isomorphism $c_{0n} \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is separable, c_{0n} must also be separable. Hence, the countable union

$$c_{00} = \bigcup_{n=1}^{\infty} c_{0n}$$

is also separable. So, $c_{00} \subseteq \ell^p$ is countable and dense in ℓ^p- ℓ^p is separable. □

We shall now show that ℓ^∞ is not separable. To do so, we require another characterisation of dense sets in metric spaces.

Lemma 1.5.8. *Let X be a metric space and let $V \subseteq X$. Then, V is dense in X if and only if for any $x \in X$ and $\varepsilon > 0$, the intersection $V \cap B_\varepsilon(x)$ is not empty.*

Proof. First, assume that V is dense in X, and let $x \in X$ and $\varepsilon > 0$. Since V is dense, there exists a sequence $(v_n)_{n=1}^\infty$ in V such that $v_n \to x$. In that case, we can find an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $d(v_n, x) < \varepsilon$. Hence, we have $x_N \in V \cap B_\varepsilon(x)$- it is not empty.

Now, assume that for any $x \in X$ and $\varepsilon > 0$, the intersection $V \cap B_\varepsilon(x)$ is not empty, and let $x \in X$. Define the sequence $(v_n)_{n=1}^\infty$ by $v_n \in B_{\frac{1}{n}}(x) \cap V$. By definition, this is a sequence in V. Moreover, since $d(v_n, x) \to 0$ as $n \to \infty$, we find that $v_n \to x$. So, there exists a sequence $(v_n)_{n=1}^\infty$ in V such that $v_n \to x$- V is dense in X. □

Proposition 1.5.9. *The space ℓ^∞ is not separable.*

Proof. For every $S \subseteq \mathbb{Z}_{\geq 1}$, define the sequence $x^{(S)}$ in ℓ^∞ by

$$x_n^{(S)} = \chi_S := \begin{cases} 1 & n \in S \\ 0 & n \notin S \end{cases}.$$

For $S, T \subseteq \mathbb{Z}_{\geq 1}$ distinct, we find that

$$\|x^{(S)} - x^{(T)}\|_\infty = 1.$$

So, if we have $x \in B_{1/2}(x^{(S)}) \cap B_{1/2}(x^{(T)})$,

$$\|x^{(S)} - x^{(T)}\|_\infty \leq \|x^{(S)} - x\|_\infty + \|x - x^{(T)}\|_\infty < \frac{1}{2} + \frac{1}{2} = 1,$$

which is not possible. Hence, for $S, T \subseteq \mathbb{Z}_{\geq 1}$ distinct, we have $B_{1/2}(x^{(S)}) \cap B_{1/2}(x^{(T)}) = \varnothing$.

Now, assume that $V \subseteq \ell^\infty$ is dense. We show that V is not countable. Since V is dense, we know that for any $x \in X$, the intersection $V \cap B_{1/2}(x)$ is not empty. Hence, for every $S \subseteq \mathbb{Z}_{\geq 1}$, there exists a $y^{(S)} \in B_{1/2}(x^{(S)})$. For $S, T \subseteq \mathbb{Z}_{\geq 1}$ distinct, we know that $B_{1/2}(x^{(S)}) \cap B_{1/2}(x^{(T)}) = \varnothing$, so $y^{(S)} \neq y^{(T)}$. Since there are uncountably many subsets of $\mathbb{Z}_{\geq 1}$, we know that the collection of sequences $(x^{(S)})_{S \subseteq \mathbb{Z}_{\geq 1}}$ in V is uncountable. Hence, V is uncountable. This implies that ℓ^∞ is not separable. □

Although ℓ^∞ is not separable, it turns out that $C[0,1]$ under the L^∞-norm is separable.

Proposition 1.5.10. *The space $C[0,1]$ under the L^∞-norm is separable.*

Proof. The Weierstrass Theorem tells us that polynomials in $\mathbb{R}$ are dense in L^∞, so their closure is L^∞. Hence, we shall show that the (single-variable) polynomials in $\mathbb{R}$, $\mathbb{R}[x]$, are separable.

We claim that the polynomials in $\mathbb{Q}$, $\mathbb{Q}[x]$, is a countable dense subset of $\mathbb{R}[x]$. Since $\mathbb{Q}$ is countable, we find that $\mathbb{Q}[x]$ is countable- it is the countable union of polynomials in $\mathbb{Q}$ of degree $n \in \mathbb{Z}_{\geq 1}$. Now, let $f \in \mathbb{R}[x]$ be given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

Since $\mathbb{Q} \subseteq \mathbb{R}$ is dense, we can find a sequence $(a_n^{(k)})_{n=1}^{\infty}$ for every $k \in \{0, 1, \ldots, n\}$ such that $a^{(k)} \to a_k$. So, define the sequence $(f_s)_{s=1}^{\infty}$ in $\mathbb{Q}[x]$ by

$$f_s(x) = a_s^{(n)} x^n + a_s^{(n-1)} x^{n-1} + \cdots + a_s^{(0)}.$$

By construction, we have $f_s \to f$, so $\mathbb{Q}[x] \subseteq \mathbb{R}[x]$ is countable and dense, so $\mathbb{R}[x]$ is separable. Hence, its closure, $C[0,1]$, under the L^{∞}-norm is separable. $\square$

1.6 Completeness

In this section, we consider the concept of completeness.

Definition 1.6.1. Let V be a metric space.

- We say that V is *complete* if every Cauchy sequence in V is convergent.
- We say that V is a *Banach space* if it is a complete normed vector space.
- We say that V is a *Hilbert space* if it is a complete inner product space.

We show that $(C[0,1], \|.\|_\infty)$ is complete.

Proposition 1.6.2. *The space* $(C[0,1], \|.\|_\infty)$ *is complete.*

Proof. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $C[0,1]$. In that case, for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then

$$\|f_m - f_n\|_\infty < \varepsilon.$$

Hence, for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$ and $t \in [0,1]$, if $m, n \geq N$, then $|f_m(t) - f_n(t)| < \varepsilon$. So, (f_n) is uniformly Cauchy. This implies that (f_n) is uniformly convergent, which implies that $f_n \to f$, for some $f \in C[0,1]$. Hence, $C[0,1]$ is complete. □

Completeness can be inherited by subsets, but only if they are closed.

Lemma 1.6.3. *Let X be a complete metric space and let $V \subseteq X$. Then, V is complete if and only if V is closed.*

Proof. First, assume that V is complete, and let $(x_n)_{n=1}^\infty$ be a sequence in V that converges in X. In that case, (x_n) must be a Cauchy sequence in V. Since V is complete, this implies that (x_n) has a limit in V. Hence, V is closed.

Now, assume that V is closed, and let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in V. Since (x_n) is a Cauchy sequence in X, and X is complete, it must be convergent in X. Furthermore, since V is closed, we find that the limit lies in V. Hence, V is complete. □

Now, we will look at some examples of complete and non-complete spaces.

Proposition 1.6.4. *The sequence space* $(\ell^\infty, \|\cdot\|_\infty)$ *is complete.*

Proof. Let $(x^{(k)})_{k=1}^\infty$ be a Cauchy sequence in ℓ^∞. That is, for all $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $k, l \in \mathbb{Z}_{\geq 1}$, if $k, l \geq N$, then $\|x^{(k)} - x^{(l)}\|_\infty < \varepsilon$. Hence, for all $n \in \mathbb{Z}_{\geq 1}$, $|x_n^{(k)} - x_n^{(l)}| < \varepsilon$, meaning that the sequence $(x_n^{(k)})_{k=1}^\infty$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we can define a sequence $x = (x_n)_{n=1}^\infty$ in $\mathbb{R}$ with $x_n^{(k)} \to x_n$. We show that $x^{(k)} \to x$ in ℓ^∞.

First, we show that (x_n) is in ℓ^∞. Since $(x^{(k)})$ is a Cauchy sequence, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $k, l \in \mathbb{Z}_{\geq 1}$, if $k, l \geq N$, then $\|x^{(k)} - x^{(l)}\|_\infty < 1$. We know that the sequence $x^{(N)}$ is in ℓ^∞, so there exists an $L > 0$ such that $\|x^{(N)}\|_\infty < L$. Now, let $n \in \mathbb{Z}_{\geq 1}$. Since $x_n^{(k)} \to x_n$, we can find a $K \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $k \geq K$, then $|x_n^{(k)} - x_n| < 1$. Now, set $k = \max(N, K)$. Then,

$$|x_n| \leq |x_n - x_n^{(k)}| + |x_n^{(k)} - x_n^{(N)}| + |x_n^{(N)}| < L + 2.$$

Hence, (x_n) is in ℓ^∞.

Finally, we show that $x^{(k)} \to x$. Let $\varepsilon > 0$. Since $(x^{(k)})$ is Cauchy, there exists an $K \in \mathbb{Z}_{\geq 1}$ such that for $k, l \in \mathbb{Z}_{\geq 1}$, if $k, l \geq K$, then $\|x^{(k)} - x^{(l)}\|_\infty < \frac{\varepsilon}{2}$. Now, let $n \in \mathbb{Z}_{\geq 1}$. Since $x_n^{(k)} \to x_n$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $k \in \mathbb{Z}_{\geq 1}$, if $k \geq N$, then $|x_n^{(k)} - x_n| < \frac{\varepsilon}{2}$. Set $M = \max(K, N)$. In that case, for $k \in \mathbb{Z}_{\geq 1}$, if $k \geq K$, then

$$|x_n^{(k)} - x_n| \leq |x_n^{(k)} - x_n^{(M)}| + |x_n^{(M)} - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $x^{(k)} \to x$. □

Proposition 1.6.5. *The sequence space $(c_{00}, \|\cdot\|_\infty)$ is not complete.*

Proof. Consider the sequence $(x^{(k)})_{k=1}^\infty$ in c_{00} by

$$x_n^{(k)} = \begin{cases} 1/n & n \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, define the sequence $x = (x_n)_{n=1}^\infty$ by $x_n = \frac{1}{n}$. We have

$$\|x^{(k)} - x\|_\infty = \frac{1}{n+1} \to 0$$

as $n \to \infty$, meaning that $x^{(k)} \to x$. However, $x \notin c_{00}$, meaning that c_{00} is not a closed subspace of ℓ^∞. This implies that c_{00} is not complete. □

Proposition 1.6.6. *The sequence space $(c_0, \|\cdot\|_\infty)$ is complete.*

Proof. We show that $c_0 \subseteq \ell^\infty$ is closed. Let $(x^{(k)})_{k=1}^\infty$ be a sequence in c_0 that converges to $x = (x_n)_{n=1}^\infty$ in ℓ^∞. We show that $x \in c_0$. Let $\varepsilon > 0$. Since $(x^{(k)})$ is Cauchy, there exists a $K \in \mathbb{Z}_{\geq 1}$ such that for $k, l \in \mathbb{Z}_{\geq 1}$, if $k, l \geq K$, then $\|x^{(k)} - x^{(l)}\|_\infty < \frac{\varepsilon}{3}$. Moreover, since $x^{(K)} \in c_0$, there exists a $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|x_n^{(K)}| < \frac{\varepsilon}{3}$. Finally, since $x^{(k)} \to x$, there exists an $L \in \mathbb{Z}_{\geq 1}$ such that for $k \in \mathbb{Z}_{\geq 1}$, if $k \geq L$, then $\|x^{(k)} - x\|_\infty < \frac{\varepsilon}{3}$. Now, set $K = \max(K, L)$. Then, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then

$$\begin{aligned} |x_n| &\leq |x_n - x_n^{(k)}| + |x_n^{(k)} - x_n^{(K)}| + |x_n^{(K)}| \\ &\leq \|x - x^{(k)}\|_\infty + \|x^{(k)} - x^{(K)}\|_\infty + |x_n^{(K)}| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This implies that $x \in c_0$. So, c_0 is closed in ℓ^∞. Since ℓ^∞ is complete, this implies that c_0 is complete. □