CHAPTER 1

REVIEW OF 3H

1.1 Complex Numbers

We define

$$\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$$

where $i^2 := -1$. As a vector space (over $\mathbb{R}$), it is isomorphic to $\mathbb{R}^2$. For $z \in \mathbb{C}$ with $z = x + iy$, we defined its conjugate $\overline{z} = x - iy$ and argument

$$\arg z = \tan^{-1} \frac{y}{x}.$$

Its absolute value is given by $|z| = \sqrt{x^2 + y^2}$. We can represent a complex number in polar form, given by $z = |z|e^{i \arg z}$.

For $z, w \in \mathbb{C}$, we define the distance between them by $d(z, w) = |z - w|$. This gives rise to a metric on $\mathbb{C}$. We define an *open disc* of radius $r > 0$ centered at $z \in \mathbb{C}$ by

$$D_r(z) := \{w \in \mathbb{C} \mid |z - w| < r\}.$$

Similarly, the *closed disc* is given by

$$\overline{D}_r(z) := \{w \in \mathbb{C} \mid |z - w| \leq r\}.$$

A set $U \subseteq \mathbb{C}$ is *open* if for any $z \in U$, there exists a radius $r_z > 0$ such that the open disc $D_{r_z}(z) \subseteq U$. We note that U is open if and only if U is a union of open discs. A set $E \subseteq \mathbb{C}$ is *closed* if its complement $E^c = \mathbb{C} \setminus E$ is open. Equivalently, E is closed if and only if for any $\mathbb{C}$-convergent sequence $(z_n)_{n=1}^{\infty}$ in E, the limit lies in E.

For a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$, we say that (z_n) *converges* to $z \in \mathbb{C}$ if $|z_n - z| \to 0$ as $n \to \infty$. The convergence of a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$ can be reduced to convergence of the sequence of its real part $(x_n)_{n=1}^{\infty}$ and the imaginary part $(y_n)_{n=1}^{\infty}$. Hence, it follows that $\mathbb{C}$ is complete. That is, for every Cauchy sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$, (z_n) is convergent.

For a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbb{C}$, the corresponding series $\sum z_n$ *converges* if the sequence of partial sums $(s_n)_{n=1}^{\infty}$ $s_n = \sum_{k=1}^{n} z_n$ converges. The series $\sum z_n$ *converges absolutely* if the series $\sum |z_n|$ converges. We know that if a series is absolutely convergent, then it is convergent. For instance, for $z \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely, to the value e^z. Similarly, the series $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ and $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converge, with

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad e^{iz} = \cos z + i \sin z.$$

A subset $K \subseteq \mathbb{C}$ is *compact* if for every open covering $(K_i)_{i \in I}$ of K has a finite subcovering $(K_{i_n})_{n=1}^{k}$. By Heine-Borel, we know that K is compact in $\mathbb{C}$ if and only if it is closed and bounded. Moreover, by Bolzano-Weierstrass, we know that K is compact if and only if every sequence in K has a convergent subsequence in K. Hence, the open disc $D_1(0)$ is not compact, but the closed disc $D_1(0)$ is.

We will now look at a property of a sequence of compact sets in $\mathbb{C}$.

Proposition 1.1.1. *Let $(\Omega_n)_{n=1}^{\infty}$ be a sequence of non-empty compact sets in $\mathbb{C}$ with $\Omega_n \subseteq \Omega_{n+1}$ such that*

$$\operatorname{diam}(\Omega_n) = \sup_{z,w \in \Omega_n} |z - w| \to 0$$

as $n \to \infty$. Then, there exists a unique $w \in \mathbb{C}$ with $w \in \Omega_n$ for all $n \in \mathbb{Z}_{\geq 1}$.

Proof. Define the sequence $(z_n)_{n=1}^{\infty}$ by $z_n \in \Omega_n$- this is possible since Ω_n is not empty. We show that (z_n) is Cauchy. Let $\varepsilon > 0$. Since $\operatorname{diam}(\Omega_n) \to 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $\operatorname{diam}(\Omega_n) \in [0, \varepsilon)$. In that case, for $m, n \in \mathbb{Z}_{\geq 1}$ with $m, n \geq N$, we have $z_m, z_n \in \Omega_N$, meaning that

$$|z_m - z_n| \leq \operatorname{diam}(\Omega_N) < \varepsilon.$$

Hence, (z_n) is Cauchy. Moreover, for all $n \in \mathbb{Z}_{\geq 1}$, the sequence $(z_m)_{m=n}^{\infty}$ is Cauchy in Ω_n. Since Ω_n is complete, we find that $z_m \to w$, for some $w \in \Omega_n$. That is, there exists a $w \in \mathbb{C}$ with $w \in \Omega_n$ for all $n \in \mathbb{Z}_{\geq 1}$.

We now show that the value w is unique. So, let $z \in \Omega_n$ for all $n \in \mathbb{Z}_{\geq 1}$. In that case, we find that for all $n \in \mathbb{Z}_{\geq 1}$,

$$|z - w| \leq \operatorname{diam}(\Omega_n).$$

Since $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$, we find that $z = w$. So, the value w is unique. $\square$

1.2 Holomorphic Functions

Let $U \subseteq \mathbb{C}$ be open, $f: U \to \mathbb{C}$ be a function and let $c \in U$. We say that f is *holomorphic at* c if the limit

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. If so, we denote the limit by $f'(c)$. It is called *the derivative of f at c*. We say that f is *holomorphic on U* if for all $c \in U$, f is holomorphic at c. For $A \subseteq \mathbb{C}$, we say that f is holomorphic on A if there exists an open set $U \subseteq \mathbb{C}$ with $A \subseteq U$ such that f is holomorphic on U.

We will now look at some examples. For $c \in \mathbb{C}$, the constant function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = c$ is holomorphic, with $f'(z) = 0$. A holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is called *entire*. Also, the identity function $f: \mathbb{C} \to \mathbb{C}$ is entire, with derivative $f'(z) = 1$. On the other hand, the conjugate function $f(z) = \overline{z}$ is not holomorphic.

Given two holomorphic functions f and g on some set $\Omega \subseteq \mathbb{C}$, we know that the following functions are holomorphic:

- $f + g$, with $(f+g)' = f' + g'$;
- fg, with $(fg)' = f'g + fg'$;
- f/g (if $g(z) \neq 0$ for all $z \in \Omega$), with
$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

Moreover, if $f: \Omega \to U$ and $g: U \to \mathbb{C}$ are holomorphic, then their composition is holomorphic with $(g \circ f)'(z) = g'(f(z))f'(z)$. Hence, every rational function p/q is holomorphic on $\mathbb{C} \setminus q^{-1}(0)$.

1.3 Power Series

A *power series* is an expression of the form $\sum_{n=0}^{\infty} a_n z^n$, with $(a_n)_{n=0}^{\infty}$ a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$. Examples of power series include

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

A *geometric series* is the following power series:

$$\sum_{n=0}^{\infty} z^n.$$

We note that if $|z| \geq 1$, then the sequence $(z^n)_{n=0}^{\infty}$ does not converge to 0 as $n \to \infty$, meaning that the power series does not converge. On the other hand, if $|z| < 1$, then

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z} \to \frac{1}{1 - z}.$$

So, the geometric series converges only in the open unit disc $D_1(0)$.

For any power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a unique $R \in [0, \infty]$ such that:

- if $|z| < R$, then the series converges absolutely;
- if $|z| > R$, then the series diverges.

In general, we cannot say what happens for all values $|z| = R$. This value R is called the *radius of convergence*, and the open disc of radius R centered at the origin $D_R(0)$ is the *disc of convergence*. Moreover,

$$\limsup |a_n|^{1/n} = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R},$$

if the limits exist, where we define $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$.

Above, we found that the geometric series has radius of convergence 1. Moreover, we saw that any $z \in \mathbb{C}$ with $|z| = 1$, the geometric series $\sum_{n=0}^{\infty} z^n$ diverges. Next, the power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence 1, but it diverges at $z = 1$ and converges for all $z \neq -1$ with $|z| = 1$.

Now, for a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R, we can consider it as a function $f \colon D_R(0) \to \mathbb{C}$. In this perspective, we find that f is holomorphic on $D_R(0)$, with

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Note that the power series f' also has radius of convergence R. This implies that a power series is infinitely complex-differentiable, since its derivative is also a power series. Using this result, we find that $\exp z$, $\cos z$ and $\sin z$ are infinitely-differentiable on $\mathbb{C}$, with

$$\cos' z = -\sin z, \qquad \sin' z = \cos z, \qquad \exp' z = \exp z.$$

We will now generalise power series. A *power series centered at* $z_0 \in \mathbb{C}$ is an expression of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, with $(a_n)_{n=0}^{\infty}$ a sequence in $\mathbb{C}$. Properties of the generalised power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be inferred from the power series $\sum_{n=0}^{\infty} a_n z^n$. In particular, the radius of convergence $R \in [0,\infty]$ can be defined by

$$\limsup |a_n|^{1/n} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R},$$

with derivative

$$g'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}.$$

This follows from the chain rule using the function $f(z) = g(z+z_0)$. It has disc of convergence $D_R(z_0)$.

Now, let $U \subseteq \mathbb{C}$ be non-empty and open, and let $f\colon U \to \mathbb{C}$ be a function. We say that f is *analytic at* $z_0 \in U$ if there exists a power series $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ with radius of convergence $R > 0$ such that $f(z) = g(z)$ for all $z \in U'$, with U' an open set containing z_0. Alternatively, we say that f has a power series expansion at z. Moreover, we say that f is analytic in U if for every $z \in U$, f is analytic at z_0. By definition, if f is analytic in U, then f is holomorphic on U.

Define the function $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ $f(z) = \frac{1}{z}$. We claim that f is analytic at $\mathbb{C}\setminus\{0\}$. So, let $z_0 \in \mathbb{C}\setminus\{0\}$. We have

$$\begin{aligned} f(z) &= \frac{1}{z} \\ &= \frac{1}{(z-z_0)+z_0} \\ &= \frac{-1}{z_0}\cdot\frac{1}{1-\frac{z+z_0}{z_0}} \\ &= \frac{-1}{z_0}\cdot\sum_{n=0}^{\infty}\left(\frac{z+z_0}{z_0}\right)^n \end{aligned}$$

if

$$\left|\frac{z+z_0}{z_0}\right| < 1 \iff |z+z_0| < |z_0|.$$

So, the function has a local power series expansion on the open set

$$U = \{z \in \mathbb{C} \mid |z+z_0| < |z_0|\}.$$

Hence, f is analytic on $\mathbb{C}\setminus\{0\}$.

1.4 Integration

In this section, we will define integration for complex-valued functions. First, let $f\colon [a,b] \to \mathbb{C}$ be a function. We can define the integral of f over the interval $[a,b]$ by the integral of the real and the imaginary part of f. In particular, assume that the real part $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are (Riemann) integrable. We then define the integral of f over $[a,b]$ by:

$$\int_a^b f(t)\ dt = \int_a^b \mathrm{Re}(f(t))\ dt + i \int_a^b \mathrm{Im}(f(t))\ dt.$$

To extend the concept of integrability in the domain, we define paths. A *path* (in $\mathbb{C}$) is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$. The path γ is *smooth* if it has a continuous derivative. It is *piecewise-smooth* if there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that γ is smooth on $[a_{i-1}, a_i]$ for all $1 \leq i \leq n$. A path γ is *closed* if $\gamma(a) = \gamma(b)$. Finally, it is *simple* if γ is injective on (a,b), i.e. the path doesn't self-intersect except possibly at the endpoints.

We will now define integration on paths. So, let $X \subseteq \mathbb{C}$ and $\gamma\colon [a,b] \to X$ be a piecewise-smooth path. If $f\colon X \to \mathbb{C}$ is a continuous function on $\Gamma = \gamma[a,b]$, we define the *integral of f along* γ by

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

Now, let $a = a_0 < \cdots < a_n = b$ be such that γ is smooth on $[a_{i-1}, a_i]$ for $1 \leq i \leq n$. Then,

$$\int_\gamma f(z)\ dz = \sum_{i=1}^n \int_{a_{i-1}}^{a_i} f(\gamma(t))\gamma'(t)\ dt.$$

This follows from the properties of Riemann integration. Moreover, the integral $\int_\gamma f(z)\ dz$ only depends on $\Gamma = \gamma[a,b]$ once the orientation is fixed, i.e. two parametrisations of the same curve have the same integral. We can define $\gamma_-\colon [a,b] \to X$ by $\gamma_-(t) = \gamma(b+a-t)$- this curve traverses γ in opposite orientation. It also satisfies

$$\int_{\gamma_-} f(z)\ dz = -\int_\gamma f(z)\ dz.$$

Like with integration in $\mathbb{R}$, the following integration properties hold in $\mathbb{C}$:

- it is linear, i.e. for $f, g\colon X \to \mathbb{C}$,

$$\int_\gamma f(z) + g(z)\ dz = \int_\gamma f(z)\ dz + \int_\gamma g(z)\ dz$$

  and for $\lambda \in \mathbb{C}$ and $f\colon X \to \mathbb{C}$,

$$\int_\gamma \lambda f(z)\ dz = \lambda \int_\gamma f(z)\ dz.$$

- we can estimate the integral, i.e.

$$\left|\int_\gamma f(z)\ dz\right| \leq \sup_{z \in \gamma([a,b])} |f(z)| \cdot \int_a^b |\gamma'(t)|\ dt.$$

  We refer to the $\int_a^b |\gamma'(t)|\ dt$ as the *length of* γ.

Note that since $[a, b]$ is compact, the image $\gamma[a, b]$ is also compact. Hence, γ is bounded and attains its bounds on $[a, b]$ (by the extreme value theorem) and it is uniformly continuous on $[a, b]$.

Now, we will define primitives (i.e. anti-derivatives). Let $U \subseteq \mathbb{C}$ be an open subset and let $f\colon U \to \mathbb{C}$ be a function. A *primitive* of f in U is a holomorphic function $F\colon U \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in U$. The Fundamental Theorem of Calculus connects differentiation with integration via primitives.

Theorem 1.4.1 (Fundamental Theorem of Calculus). *Let $U \subseteq \mathbb{C}$ be open and let $f\colon U \to \mathbb{C}$ be a continuous function with primitive $F\colon U \to \mathbb{C}$. Then, for a piecewise-smooth path $\gamma\colon [a, b] \to U$,*

$$\int_\gamma f(z)\ dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. First, assume that γ is smooth. Then,

$$\begin{aligned}
\int_\gamma f(z)\ dz &= \int_b^a f(\gamma(t))\gamma'(t)\ dt \\
&= \int_\gamma F'(\gamma(t))\gamma'(t)\ dt \\
&= \int_\gamma (F \circ \gamma)'(t)\ dt \\
&= (F \circ \gamma)(b) - (F \circ \gamma)(a) \\
&= F(\gamma(b)) - F(\gamma(a))
\end{aligned}$$

by the Fundamental Theorem of Calculus in $\mathbb{R}$.

Now, if $a = a_0 < a_1 < \cdots < a_n = b$ such that γ is smooth on $[a_{i-1}, a_i]$ for all $1 \leq i \leq n$, then

$$\begin{aligned}
\int_\gamma f(z)\ dz &= \sum_{i=1}^n \int_{a_{i-1}}^{a_i} f(\gamma(t))\gamma'(t)\ dt \\
&= \sum_{i=1}^n F(\gamma(a_i)) - F(\gamma(a_{i-1})) \\
&= F(\gamma(b)) - F(\gamma(b)).
\end{aligned}$$

So, the result holds for a piecewise-smooth path γ. □

Using this result, we find that for $\gamma\colon [a, b] \to U$ a piecewise-smooth, closed path in an open set $U \subseteq \mathbb{C}$ and a continuous function $f\colon U \to \mathbb{C}$ with a primitive F,

$$\int_\gamma f(z)\ dz = F(\gamma(b)) - F(\gamma(a)) = 0$$

since $\gamma(b) = \gamma(a)$. Now, for $n \in \mathbb{Z}$ with $n \neq -1$, we know that the function $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = z^n$ is continuous with primitive $\frac{1}{n+1} z^{n+1}$. Hence, for a closed piecewise-smooth path γ, we find that

$$\int_\gamma f(z)\ dz = 0.$$

Now, if $f(z) = \frac{1}{z}$, then f does not have a primitive in $\mathbb{C} \setminus \{0\}$. To see this, let $C = C_1(0)$ be the unit circle centered at the origin. We can define the smooth path $\gamma\colon [0, 2\pi] \to C$ by $\gamma(t) = e^{it}$. In that case,

$$\int_C f(z)\ dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\ dt = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\ dt = 2\pi i.$$

If f had a primitive, the Fundamental Theorem of Calculus would imply that the integral over the closed smooth curve C is 0, but this is not the case. Hence, f cannot have a primitive.

We will use the Fundamental Theorem of Calculus to show that only constant functions have derivative 0.

Corollary 1.4.2. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function, where $U \subseteq \mathbb{C}$ is an open connected set. If $f'(z) = 0$ for all $z \in U$, then f is a constant on U.*

Proof. Let $w_0 \in U$. We show that $f(w) = f(w_0)$ for all $w \in U$. So, let $w \in U$. Since U is open and connected, there exists a piecewise-smooth path $\gamma\colon [a, b] \to U$ with $\gamma(a) = w_0$ and $\gamma(b) = w$. Since f is a primitive of f', we find that

$$\int_\gamma f'(z)\ dz = f(\gamma(b)) - f(\gamma(a)) = f(w) - f(w_0)$$

by the Fundamental Theorem of Calculus. Moreover, we know that $f'(z) = 0$ for all $z \in U$. So, we also have $\int_\gamma f'(z)\ dz = 0$. This implies that $f(w) = f(w_0)$, meaning that f is constant. □

We can use this result to show that all primitives of a function are just translations of each other.

Corollary 1.4.3. *Let $f\colon U \to \mathbb{C}$ be a continuous function on some open set $U \subseteq \mathbb{C}$, and let $F\colon U \to \mathbb{C}$ be a primitive for f. Then, every primitive of f is of the form $z \mapsto F(z) + z_0$, for some $z_0 \in \mathbb{C}$.*

Proof. Let $G\colon U \to \mathbb{C}$ be a function given by $G(z) = F(z) + z_0$, for some $z_0 \in \mathbb{C}$. By chain rule, we find that G is holomorphic on U, with $G'(z) = f(z)$. Hence, G is a primitive of f.

Now, let $G\colon U \to \mathbb{C}$ be a primitive of f. Then, define the function $g\colon U \to \mathbb{C}$ by $g(z) = F(z) - G(z)$. Then,

$$g'(z) = F'(z) - G'(z) = 0.$$

By the result above, we find that $g'(z) = z_0$, for some $z_0 \in \mathbb{C}$. This implies that $G(z) = F(z) + z_0$ for all $z \in \mathbb{C}$. □

1.5 Integral Formulae

In this section, we will provide different results about integrals, such as Goursat's Theorem and Cauchy's Integral Formulae.

Theorem 1.5.1 (Goursat's Theorem). *Let $f\colon U \to \mathbb{C}$ be a holomorphic function with $U \subseteq \mathbb{C}$ open. Then, for a triangle T contained in U, traced anticlockwise,*

$$\int_T f(z)\ dz = 0.$$

Proof. Let $T_0 = T$. Divide the triangle into 4 sub-triangles by breaking them with respect to the midpoints of the triangles, as shown below:
By the reversing of the order in the triangle T_1^3 as opposed to the shared edges in the other triangles, we find that

$$\int_{T_0} f(z)\ dz = \sum_{i=1}^{4} \int_{T_1^i} f(z)\ dz.$$

Hence, by the Triangle Inequality, we find that

$$\left| \int_{T_0} f(z)\ dz \right| \leq \sum_{i=1}^{4} \left| \int_{T_1^i} f(z)\ dz \right|.$$

Now, if

$$\left| \int_{T_0} f(z)\ dz \right| > 4 \left| \int_{T_1^i} f(z)\ dz \right|$$

for all $1 \leq i \leq 4$, then

$$4 \left| \int_{T_0} f(z)\ dz \right| > 4 \sum_{i=1}^{4} \left| \int_{T_1^i} f(z)\ dz \right|,$$

which contradicts the triangle inequality. Hence, there exists an $1 \leq i \leq 4$ such that

$$\left| \int_{T_0} f(z)\ dz \right| \leq 4 \left| \int_{T_1^i} f(z)\ dz \right|.$$

We denote $T_1 = T_1^i$. By dividing T_1 further into 4 triangles and selecting the right sub-triangle, and so on, we can define a sequence of triangles $(T_n)_{n=0}^{\infty}$ in U. Then, for all $n \in \mathbb{Z}_{\geq 0}$,

$$\left| \int_{T_0} f(z)\ dz \right| \leq 4^n \left| \int_{T_n} f(z)\ dz \right|.$$

Define the sequences $(d_n)_{n=1}^{\infty}$ of diameter of T_n, and $(p_n)_{n=1}^{\infty}$ of perimeter of T_n. Since T_{n+1} is half the size of T_n, we find that

$$d_n = \text{diameter}(T_n) = 2^{-n} d_0, \qquad p_n = \text{perimeter}(T_n) = 2^{-n} p_0$$

for all $n \in \mathbb{Z}_{\geq 1}$. Now, define the sequence $(\hat{T}_n)_{n=1}^{\infty}$ of the solid triangle whose boundary is T_n- each set $\hat{T}_n$ is closed and bounded, meaning that it is compact with $\hat{T}_n \supseteq \hat{T}_{n+1}$ for all $n \in \mathbb{Z}_{\geq 1}$. Since $d_n \to 0$ as $n \to \infty$, we find that

$$\bigcap_{n=0}^{\infty} \hat{T}_n = \{z_0\},$$

for some $z_0 \in U$. Since f is holomorphic at z_0, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

for some function $\psi \colon U \to \mathbb{C}$ with $\psi(z) \to 0$ as $z \to z_0$. Hence,

$$\int_{T_n} f \, dz = \int_{T_n} (f(z_0) + f'(z_0)(z - z_0)) \, dz + \int_{T_n} \psi(z)(z - z_0) \, dz.$$

Since $f(z_0)$ and $f'(z_0)$ are constants, and T_n is a closed curve, we find that

$$\int_{T_n} (f(z_0) + f'(z_0)(z - z_0)) \, dz = 0.$$

This implies that

$$\begin{aligned}
\left| \int_{T_n} f(z) \, dz \right| &= \left| \int_{T_n} \gamma(z)(z - z_0) \, dz \right| \\
&\leq \left(\sup_{z \in T_0} |\psi(z)| \right) \cdot \text{length}(T_n) \\
&= \left(\sup_{z \in T_0} |\psi(z)| \right) \cdot d_n \cdot p_n \\
&= \left(\sup_{z \in T_0} |\psi(z)| \right) \cdot (2^{-n} d_0)(2^{-n} p_0).
\end{aligned}$$

Therefore,

$$\left| \int_{T_0} f(z) \, dz \right| \leq 4^n \left| \int_{T_n} f(z) \, dz \right| \leq \left(\sup_{z \in T_0} |\psi(z)| \right) \cdot d_0 \cdot p_0.$$

We know that as $z \to z_0$, $\phi(z) \to z_0$, with z_0 the unique point in the intersection of (T_n). Thus, $\left(\sup_{z \in T_0} |\psi(z)|\right) \cdot d_0 \cdot p_0 \to 0$ as $n \to \infty$. This implies that

$$\left| \int_{T_0} f(z) \, dz \right| = 0,$$

and so

$$\int_{T_0} f(z) \, dz = 0.$$

□

We will now look at applications of the Goursat's Theorem.

Corollary 1.5.2. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function with $U \subseteq \mathbb{C}$. Then, for a rectangle R contained in U, traced anticlockwise, then*

$$\int_R f(z)\ dz = 0.$$

This follows by breaking the rectangle R into two triangles- T_1 and T_2, as shown below. Since the shared line is going in opposite directions, we find that

$$\int_R f(z)\ dz = \int_{T_1} f(z)\ dz + \int_{T_2} f(z)\ dz.$$

By Goursat's Theorem, this value is 0. We can use Goursat's Theorem for more complex shapes as well, using a similar argument.

We can also show that any holomorphic function on an open disc has a primitive using Goursat's Theorem.

Theorem 1.5.3. *Let $f\colon D \to \mathbb{C}$ be holomorphic, with $D \subseteq \mathbb{C}$ non-empty open disc. Then, f has a primitive $F\colon D \to \mathbb{C}$.*

Proof. Let D be centered at $a \in \mathbb{C}$. For $z \in D$, define the path γ_z from a to z by moving horizontally first and then vertically, as shown in the diagram below. By definition, γ_z is piecewise-smooth. Now, define $F\colon D \to \mathbb{C}$ by

$$F(z) = \int_{\gamma_z} f(w)\ dw.$$

We show that F is a primitive for f. Let $h \in \mathbb{C}$ such that $z + h \in D$- this is possible since D is open. Define the paths $\gamma_{z,z+h}$, $\delta_{z,z+h}$ and $\eta_{z,z+h}$ as shown in the diagram below: We find that

$$\begin{aligned} F(z+h) - F(z) &= \int_{\gamma_{z+h}} f(w)\ dw - \int_{\gamma_z} f(w)\ dw \\ &= \int_{\gamma_{z,z+h}} f(w)\ dw \\ &= \int_{\delta_{z,z+h}} f(w)\ dw \\ &= \int_{\eta_{z,z+h}} f(w)\ dw. \end{aligned}$$

Since f is continuous, we can write $f(w) = f(z) + \psi(w)$, where $\psi(w) = f(w) - f(z)$, with $\psi(w) \to 0$ as $z \to w$ (by continuity). Hence,

$$\begin{aligned} F(z+h) - F(z) &= \int_{\eta_{z,z+h}} f(z)\ dw + \int_{\eta_{z,z+h}} \psi(w)\ dw \\ &= f(z) \cdot h + \int_{\eta_{z,z+h}} \psi(w)\ dw. \end{aligned}$$

This implies that

$$\frac{F(z+h) - F(z)}{h} = f(z) + \frac{1}{h} \int_{\eta_{z,z+h}} \psi(w)\ dw.$$

We now find that

$$\left| \frac{1}{h} \int_{\eta_{z,z+h}} \psi(w)\ dw \right| \leq \frac{1}{|h|}|h| \cdot \sup_{w \in \eta_{z,z+h}} |\psi(w)| = \sup_{w \in \eta_{z,z+h}} |\psi(w)| \to 0$$

as $h \to 0$, since $\psi(w) \to 0$ as $w \to z$. Therefore,

$$F'(z) = \lim_{h \to \infty} \frac{F(z+h) - F(z)}{h} = f(z).$$

So, F is holomorphic, with $F' = f$. □

Using this result, we can show Cauchy's Theorem for a disc.

Theorem 1.5.4 (Cauchy's Theorem for a disc)**.** *Let $f\colon D \to \mathbb{C}$ be holomorphic on an open disc $D \subseteq \mathbb{C}$. Then, for every closed, piecewise-smooth curve path γ in D,*

$$\int_\gamma f(z)\ dz = 0.$$

This directly follows from applying the Fundamental Theorem of Calculus since we know that f has a primitive.

Corollary 1.5.5. *Let $f\colon U \to \mathbb{C}$ be holomorphic on an open set $U \subseteq \mathbb{C}$ and let C be a circle in U (along with its interior), traced anticlockwise. Then,*

$$\int_C f(z)\ dz = 0.$$

Proof. Let D be a disc with boundary C. Since U is open, there exists a disc D' in U such that $D \subsetneq D'$. Since f is holomorphic in D', Cauchy's Theorem for a disc tells us that

$$\int_C f(z)\ dz = 0.$$

□

Now, we shall consider Cauchy's Theorem in general. Let $U \subseteq \mathbb{C}$ be a path-connected open set, and let $\delta, \gamma\colon [0,1] \to U$ be paths from a to b, i.e. $\delta(0) = \gamma(0) = a$ and $\delta(1) = \gamma(1) = b$. We say that δ and γ are *homotopic in U* if there exists a collection of paths $\gamma_s\colon [0,1] \to U$ for $s \in (0,1)$ such that:

- γ_s is a path from a to b;
- $\gamma_0 = \gamma$ and $\gamma_1 = \delta$
- $\gamma_s(t)$ is a continuous function $[0,1] \times [0,1] \to U$.

A path-connected open set $U \subseteq \mathbb{C}$ is *simply-connected* if for any two paths γ and δ in U with the same endpoints are homotopic.

Theorem 1.5.6 (Cauchy's Theorem for simply connected sets)**.** *Let $U \subseteq \mathbb{C}$ be an open simply connected set, $f\colon U \to \mathbb{C}$ be holomorphic in U and let γ be a piecewise-smooth, closed path in U. Then,*

$$\int_\gamma f(z)\ dz = 0.$$

Sketch Proof. Let $\Gamma = \gamma[a, b]$. Since Γ is compact, we can cover it by finitely many open discs in U. In each disc, we can replace a section of Γ with a line segment by Cauchy's Theorem for open discs, as shown below: We can further assume that Γ is simple, i.e. it doesn't self-intersect. This is because we can break the integral into a sum of integrals, where we have no self-intersections. Since U is simply connected, the interior of the polygon is contained in U. Hence, we can apply Goursat's Theorem on the polygon (by triangulating it) to show that the integral around the polygon is 0. □

Using this result, we can derive Cauchy's Integral Formula.

Theorem 1.5.7 (Cauchy's Integral Formula)**.** *Let $f: U \to \mathbb{C}$ be a holomorphic function on an open set $U \subseteq \mathbb{C}$, let $D \subseteq U$ be an open disc whose interior is contained in U and let C be the boundary of the disc, traced anti-clockwise. Then, for all $z \in D$,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

We will now prove a lemma and then the generalisation of Cauchy's Integral Formula.

Lemma 1.5.8. *Let $\varepsilon > 0$ and $\gamma: [a, b] \to \mathbb{C}$ be a piecewise-smooth path, and denote $\Gamma = \gamma[a, b]$. If $F: D_\varepsilon(0) \times \Gamma \to \mathbb{C}$ is continuous, then*

$$\lim_{n \to 0} \int_\Gamma F(n, z) \, dz = \int_\Gamma F(0, z) \, dz.$$

Proof. □

Theorem 1.5.9 (Cauchy's Integral Formula (for derivatives))**.** *Let $f: U \to \mathbb{C}$ be a holomorphic function on an open set $U \subseteq \mathbb{C}$. Then, f is infinite-differentiable on U, and for C a circle whose interior is contained in U, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

for all z in the interior of C and $n \geq 0$.

Proof. We show this by induction. If $n = 0$, then we get the Cauchy's Integral Formula we saw above. Now, assume that f has $n - 1$ derivatives, with

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^n} \, d\zeta.$$

For small $h \in \mathbb{C}$, we find that

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{h} \left(\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right) d\zeta.$$

Now, let

$$A = \frac{1}{\zeta - z - h} \qquad , B = \frac{1}{\zeta - z}.$$

We know that

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1}).$$

We have

$$A - B = \frac{h}{(\zeta - z - h)(\zeta - z)},$$

and as $h \to 0$, $A \to B$ and so

$$A^{n-1} + A^{n-2}B + \cdots + B^{n-1} \to A^{n-1} + A^{n-1} + \cdots + A^{n-1} = \frac{n}{(\zeta - z)^{n-1}}.$$

Therefore,

$$\lim_{h \to 0} \frac{1}{h} \left(\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right) = \frac{n}{(\zeta - z)^{n+1}}.$$

By the lemma above, this implies that

$$\begin{aligned} \lim_{h \to 0} \int_C \frac{f(\zeta)}{h} \left(\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right) \, d\zeta &= \int_C \lim_{h \to 0} \frac{f(\zeta)}{h} \left(\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right) \, d\zeta \\ &= n \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \end{aligned}$$

Hence, $f^{(n-1)}$ is holomorphic, with

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

So, the result follows by induction. □

When we apply the lemma, we take $\varepsilon = \frac{1}{2} \inf_{\zeta \in C} |\zeta - z|$ and

$$h(h, \zeta) = \frac{f(\zeta)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1} \to A^{n-1} + A^{n-1} + \cdots + A^{n-1})}{(\zeta - z - h)(\zeta - z)}.$$

Using this result, we can derive Cauchy's Inequality.

Corollary 1.5.10 (Cauchy's Inequality). *Let $f \colon U \to \mathbb{C}$ be holomorphic on some open set $U \subseteq \mathbb{C}$ and let $D_R(z_0)$ be an open disc contained in U. Then,*

$$|f^{(n)}(z_0)| \le \frac{n!}{R^n} \|f\|_C,$$

where $C = \partial D_R(z_0)$ is the boundary of the disc, traced anti-clockwise and

$$\|f\|_C = \sup_{z \in C} |f(z)|.$$

Proof. By Cauchy's Integral Formula (for derivatives), we find that

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right| \\ &= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}} \cdot Rie^{i\theta} \, d\theta \right| \\ &= \frac{n!}{2\pi \cdot R^n} \left| \int_0^{2\pi} f(z_0 + Re^{i\theta}) \, d\theta \right| \\ &\le \frac{n!}{2\pi \cdot R^n} \cdot 2\pi \|f\|_C = \frac{n!}{R^n} \|f\|_C. \end{aligned}$$

□

Using this result, we can show that a holomorphic function is analytic.

Theorem 1.5.11 (Taylor's Theorem). *Let $f\colon U \to \mathbb{C}$ be holomorphic on an open set $U \subseteq \mathbb{C}$. If $\overline{D}_R(z_0) \subseteq U$, then f has a power series expansion at z_0 given by*

$$f(z) = \sum_{k=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D_R(z_0)$, where $a_n = \frac{f^{(n)}(z_0)}{n}$. In particular, f is analytic.

1.6 Consequences of Integral Formulae

In this section, we will look at some consequences of Cauchy's Integral Formula.

The first consequence we shall study is the Fundamental Theorem of Algebra. To do so, we require Lioville's Theorem, which follows from Cauchy's Integral Formula.

Theorem 1.6.1 (Lioville's Theorem)**.** *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that f is bounded. Then, f is a constant function.*

Proof. Since $\mathbb{C}$ is connected, we need to show that $f'(z) = 0$ for all $z \in \mathbb{C}$. So, let $z_0 \in \mathbb{C}$ and $R > 0$. By Cauchy's Inequality, we find that

$$|f'(z)| \leq \frac{1}{R}\|f\|_C.$$

We know that f is bounded, so $\|f\|_C$ exists. The inequality holds for all $R > 0$, so we find that $|f'(z)| = 0$. Hence, $f'(z) = 0$ for all $z \in \mathbb{C}$- f is a constant function. □

Theorem 1.6.2 (Fundamental Theorem of Algebra)**.** *Let $p\colon \mathbb{C} \to \mathbb{C}$ be a polynomial in $\mathbb{C}$ of degree $n \geq 1$. Then, p has a root in $\mathbb{C}$, i.e. there exists a $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. Assume that p has no roots. Denote

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

with $a_n \neq 0$. Then, for $z \in \mathbb{C}$ with $z \neq 0$,

$$\frac{p(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right).$$

We know that

$$\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \to 0$$

as $|z| \to \infty$. Hence, there exists an $R > 0$ such that for $z \in \mathbb{C}$, if $|z| > R$, then

$$\left|\frac{p(z)}{z^n}\right| \geq \frac{|a_n|}{2} \iff |p(z)| \geq \frac{|a_n|}{2}|z|^n.$$

Hence, p is bounded below on $\mathbb{C} \setminus \overline{D}_R(0)$. Now, since p is continuous and has no roots, we find that p is bounded below on $\overline{D}_R(0)$ too (since it is compact). Hence, $\frac{1}{p}$ is bounded and entire. By Lioville's Theorem, we find that $\frac{1}{p}$ is constant. This is a contradiction since p is not a constant. □

Now, we can find that a polynomial has precisely n roots.

Corollary 1.6.3. *Let $p\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $n \geq 1$. Then, p has n roots.*

We shall now consider the Identity Theorem. We need the concept of a region to do so- a *region* is a connected open subset of $\mathbb{C}$.

Theorem 1.6.4 (Identity Theorem for zero functions)**.** *Let $U \subseteq \mathbb{C}$ be a region and $f : U \to \mathbb{C}$ a holomorphic function. If there exists a convergent sequence $(z_n)_{n=1}^{\infty}$ of distinct points in U (with limit in U) such that $f(z_n) = 0$ for all $n \in \mathbb{Z}_{\geq 1}$, then $f(z) = 0$ for all $z \in U$.*

Proof. Let $z_n \to w$, for some $w \in U$, and let $D \subseteq U$ be an open disc centered at w. Since f is holomorphic, we know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$. Now, assume that f is not the zero function on D. Let a_m be the first non-zero coefficient in the power series expansion. Then,

$$f(z) = a_m (z - w)^m (1 + g(z)),$$

where

$$g(z) = \sum_{j=1}^{\infty} \frac{a_{m+j}}{a_m} (z - w)^j.$$

The function g is continuous with $g(w) = 0$. So, we can find an open disc $D' \subseteq D$ centered at w such that $1 + g(z) \neq 0$ for all $z \in D'$. We know that $z_n \to w$, so we can find an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \geq N$, $z_n \in D'$. In that case, we can find an $n \geq N$ such that

$$f(z_n) = a_m (z_n - w)^m (1 + g(z)) \neq 0$$

This is a contradiction. Hence, f must be identically 0, i.e. $f(z) = 0$ for all $z \in D$.

Now, let

$$V = \{z \in U \mid f \equiv 0 \text{ on an open disc } D \text{ centered at } z\}.$$

We know that $z_0 \in V$ by above. Moreover, V is open[1]. Also, V is closed- let $(z_n)_{n=1}^{\infty}$ be a sequence in V that converges to $z \in V$, we can use the argument above to show that $f \equiv 0$ on an open disc centered at z. Since U is connected, this implies that $V = U$. Hence, $f \equiv 0$ in U. □

Intuitively, this result means that a non-zero holomorphic function that has isolated zeros. We can generalise the result to arbitrary functions.

Theorem 1.6.5 (Identity Theorem)**.** *Let $U \subseteq \mathbb{C}$ be a region and $f, g : U \to \mathbb{C}$ holomorphic functions. If there exists a convergent sequence $(z_n)_{n=1}^{\infty}$ of distinct points in U (with limit in U) such that $f(z_n) = g(z_n)$ for all $n \in \mathbb{Z}_{\geq 1}$, then $f \equiv g$ in U.*

This follows by using the previous result on the function $g - f$.

The identity theorem can be used to show the trigonometric identity

$$\sin^2 z + \cos^2 z = 1$$

for all $z \in \mathbb{C}$. To do so, define the sequence $(z_n)_{n=1}^{\infty}$ by $z_n = \frac{1}{n}$. We assume the identity holds on $\mathbb{R}$ (which can be shown using real analysis). With that, we

[1] for all $z \in V$, there exists an open disc D centered at z such that $f \equiv 0$, meaning that $D \subseteq V$ with $z \in D$.

can apply the identity theorem using the sequence (z_n) and functions $f(z) = \sin^2 z + \cos^2 z, g(z) = 1$ to show that $f(z) = g(z)$ for all $z \in \mathbb{C}$.

Note that we cannot drop the assumption that the limit lies in U in the identity theorem. To see this, define the functions

$$f(z) = \sin(1/z), \qquad g(z) = 0$$

and the sequence $(z_n)_{n=1}^{\infty}$ by $z_n = \frac{1}{n\pi}$. We know that $z_n \to 0$, but f is not defined on 0. Moreover, $f(z_n) = g(z_n)$ for all $n \in \mathbb{Z}_{\geq 1}$, but $f \not\equiv g$ in $\mathbb{C} \setminus \{0\}$. So, the assumption that the limit lies in the set is a necessary one.

We will now prove the converse of Goursat's Theorem- Morera's Theorem.

Theorem 1.6.6 (Morera's Theorem)**.** *Let $f \colon D \to \mathbb{C}$ be a continuous function in an open disc $D \subseteq \mathbb{C}$. If*

$$\int_T f(z)\ dz = 0$$

for all triangles T whose interior is contained in D, then f is holomorphic.

Proof. By a previous result, we know that f has a primitive F in D. Since F is infinitely differentiable, we find that $f = F'$ is differentiable. □

We can use this to show that a convergent sequence of holomorphic functions is holomorphic.

Theorem 1.6.7. *Let $U \subseteq \mathbb{C}$ be an open subset, and $(f_n)_{n=1}^{\infty}$ a sequence of holomorphic functions $f_n \colon U \to \mathbb{C}$ that converges uniformly to some function $f \colon U \to \mathbb{C}$ in every compact subset of U. Then, f is holomorphic in U.*

Proof. Let $D \subseteq \mathbb{C}$ be an open disc such that its closure is contained in U and let T be a triangle in D. Since f_n is holomorphic for all $n \in \mathbb{Z}_{\geq 1}$, we find that

$$\int_T f_n(z)\ dz = 0$$

by Goursat's Theorem. Since $f_n \to f$ uniformly in the closure of D, f is continuous with

$$\int_T f_n(z)\ dz \to \int_T f(z)\ dz$$

as $n \to \infty$. Hence,

$$\int_T f(z)\ dz = 0.$$

By Morera's Theorem, we find that f is holomorphic in D. Since D was an arbitrary disc whose closure is contained in U, we find that f is holomorphic in U. □

1.7 Isolated Singularities

A function $f\colon U \to \mathbb{C}$ on an open set $U \subseteq \mathbb{C}$ has an *isolated singularity* at $z_0 \in U$ if f is holomorphic at $U \setminus \{z_0\}$. There are 3 types of isolated singularities:

- it is *removable* if f can be extended to a holomorphic function in U;
- it is a *pole* if the limit $|f(z)| \to \infty$ as $z \to z_0$;
- otherwise, it is *essential*.

We will now look at some examples:

- The functions $\frac{\sin z}{z}$ and $\frac{\exp z - 1}{z}$ have removable singularities at $z = 0$- this follows from their power series expansions.
- The functions $\frac{1}{z}$ and $\frac{1}{\sin z}$ have poles at $z = 0$.
- The functions $\sin(1/z)$ and $\exp(1/z)$ have essential singularities at $z = 0$.

We will now look at a theorem that gives a sufficient condition for removable singularities.

Theorem 1.7.1 (Riemann's Theorem on Removable Singularities)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on $U \setminus \{z_0\}$, for some $z_0 \in U$. If f is bounded on $U \setminus \{z_0\}$, then f has a removable singularity at z_0.*

Proof. Define the function $h\colon U \to \mathbb{C}$ by

$$h(z) = \begin{cases} 0 & z = z_0 \\ (z - z_0)^2 f(z) & \text{otherwise.} \end{cases}$$

We claim that h is holomorphic in U. By product rule, h is holomorphic at all $z \in U \setminus \{z_0\}$. By assumption, f is bounded, so there exists an $M > 0$ such that $|f(z)| \leq M$ for all $z \in U \setminus \{z_0\}$. In that case, for $z \in U \setminus \{z_0\}$,

$$\left| \frac{h(z) - h(z_0)}{z - z_0} \right| = |z - z_0| \cdot |f(z)| \leq M|z - z_0|.$$

As $z \to z_0$, we find that $M|z - z_0| \to 0$. Hence, h is holomorphic at z_0, with $h'(z) = 0$. So, h is holomorphic in U.

Since h is holomorphic in U, it has a power series representation

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(z - z_0)^n$$

around some open disc D centered at z_0. We have $a_0 = h(z_0)$ and $a_1 = h'(z_0)$, so

$$h(z) = \sum_{n=2}^{\infty} \frac{a_n}{n!}(z - z_0)^n = (z - z_0)^2 \sum_{n=2}^{\infty} a_n (z - z_0)^{n-2}.$$

Hence, for all $z \in D \setminus \{z_0\}$,

$$f(z) = (z - z_0)^{-2} f(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^{n-2}.$$

We know that the power series for f is holomorphic in D, we can use it to extend f to U. So, z_0 is a removable singularity. $\square$

We will now look at different kinds of isolated singularities. For instance, consider the function

$$f(z) = \frac{1}{\sin(1/z)}.$$

We claim that 0 is not an isolated singularity for f. Note that for every $n \in \mathbb{Z}_{\geq 1}$, the value $z_n = \frac{1}{\pi n}$ satisfies $\sin(z_n) = 0$. This implies that there cannot be an open disc D such that f is holomorphic at $D \setminus \{0\}$- we will have a value of form $\frac{1}{\pi n}$ for sufficiently large $n \in \mathbb{Z}_{\geq 1}$.

Next, we show that $f(z) = \sin(1/z)$ has an essential singularity at $z = 0$. To prove this, we first show that f is not removable, i.e. the limit $\lim_{z \to 0} f(z)$ does not exist. Consider the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ by $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$. Then,

$$f(x_n) = \sin(2\pi n) = 0, \qquad f(y_n) = \sin(2\pi n + \tfrac{\pi}{2}) = 1$$

for all $n \in \mathbb{Z}_{\geq 1}$. Hence, $f(x_n) \to 0$ and $f(y_n) \to 1$. We have $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$, so we must have that the limit $\lim_{z \to 0} f(z)$ does not exist, i.e. the isolated singularity is not removable. Moreover, we also cannot have $\lim_{z \to 0} f(z) = \infty$, meaning that the isolated singularity is not a pole. Hence, the isolated singularity is essential.

We will now show that the image of a function with an essential singularity is dense.

Theorem 1.7.2 (Casarati-Weierstrass Theorem). *Let $f\colon D_R(z_0) \to \mathbb{C}$ be a holomorphic function on the punctured open disc $D_R(z_0) \setminus \{z_0\}$ for some $z_0 \in U$, and let f have an essential singularity at z_0. Then, the image of the set $D_R(z_0) \setminus \{z_0\}$ is dense in $\mathbb{C}$.*

Proof. Assume that the image of the set $D_R(z_0) \setminus \{z_0\}$ is dense in $\mathbb{C}$. In that case, there exists a $w \in \mathbb{C}$ and a $\delta > 0$ such that the open disc $D_\delta(w)$ intersects the image trivially, i.e. $|f(z) - w| \geq \delta$ for all $z \in D_R(z_0) \setminus \{z_0\}$. Hence, the closed ball $\overline{D}_\delta(w) \subseteq (D_R(z_0) \setminus \{z_0\})^c$. Now, define the function $g\colon D_R(z_0) \setminus \{z_0\} \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z) - w}.$$

Then, g is holomorphic in $U \setminus \{z_0\}$ since f is. Moreover,

$$|g(z)| = \frac{1}{|f(z) - w|} \leq \frac{1}{\delta}$$

for all $z \in D_R(z_0) \setminus \{z_0\}$. That is, g is bounded on $D_R(z_0) \setminus \{z_0\}$. So, by Riemann's Theorem on Removable Singularities, we find that g has a removable singularity at z_0. So, we can find an extension of g, $\overline{g}\colon D_R(z_0) \to \mathbb{C}$, that is holomorphic on $D_R(z_0)$. We now consider two cases:

- If $\overline{g}(z_0) = 0$, then

$$\lim_{z \to z_0} f(z_0) = \lim_{z \to z_0} \frac{1}{\overline{g}(z_0)} = \infty,$$

so f has a pole at z_0.

- Otherwise, $\overline{g}(z_0) \neq 0$. Hence, $f(z) - w$ is holomorphic at z_0, meaning that f has a removable singularity at z_0.

Hence, f does not have an essential singularity at z_0. Taking the contrapositive, we find that if f has an essential singularity at z_0, then the image is dense in $\mathbb{C}$. □

Now, we will study poles in more detail. They are closely related to zeros of a function. Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open set $U \subseteq \mathbb{C}$. Then, $z_0 \in U$ is a *zero* of f if $f(z_0) = 0$. By the identity theorem, we know that a non-zero holomorphic function cannot have a neighbourhood around z_0 that is identically zero, i.e. any zeros of a holomorphic function are isolated.

It turns out that we can factorise a holomorphic function around its zero locally.

Theorem 1.7.3. *Let $f\colon U \to \mathbb{C}$ be a non-zero holomorphic function on a region $U \subseteq \mathbb{C}$ and has a zero $z_0 \in U$. Then, there exists a neighbourhood $V \subseteq U$ of z_0 and a holomorphic function $g\colon V \to \mathbb{C}$ and a unique $n \in \mathbb{Z}_{\geq 1}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^n g(z)$ for all $z \in V$.*

Proof. We can write f by its power series expansion, i.e.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $z \in D_R(z_0)$, where $R > 0$. Since f is not identically 0 on any neighbourhood around z_0, so so the identity theorem tells us that there exists an $n \in \mathbb{Z}_{\geq 1}$ such that $a_n \neq 0$, and n is the smallest integer satisfying the condition. Note that we have $a_0 = f(z_0) = 0$, so $n \geq 1$. Hence,

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}.$$

So, let

$$g(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}.$$

Then, g is holomorphic in some open disc $D \subseteq U$ centered at z_0, with $g(z_0) = a_n \neq 0$.

Now, we show that the value n is unique. So, let $f(z) = (z - z_0)^n g(z)$ and $f(z) = (z - z_0)^m h(z)$. Without loss of generality, assume that $m \geq n$. In that case, we have $g(z) = (z - z_0)^{m-n} h(z)$. Since $g(z_0) \neq 0$, we must find that $m = n$. □

We define the value n as the *order* (or *multiplicity*) of the zero z_0. We will use this to define the order of a pole.

Theorem 1.7.4. *Let $f\colon U \to \mathbb{C}$ be a function that is holomorphic at $U \setminus \{z_0\}$, for some $z_0 \in U$, and let f have a pole at z_0. Then, there exists a holomorphic function $h\colon V \to \mathbb{C}$ in some open neighbourhood $V \subseteq U$ containing z_0 and a unique $n \in \mathbb{Z}_{\geq 1}$ such that $f(z) = (z - z_0)^{-n} h(z)$ for all $z \in V$.*

Proof. □

We now define the *order of a pole* of the function f at z_0 as the order of the zero of the function g, as defined above, at z_0. We say that a pole (or a zero) is *simple* if it has order 1.

Theorem 1.7.5. *Let $U \subseteq \mathbb{C}$ be an open set and let $f\colon U \to \mathbb{C}$ be a holomorphic function on $U \setminus \{z_0\}$, for some $z_0 \in U$. If f has a pole of order n at z_0, then*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + g(z),$$

where $g\colon V \to \mathbb{C}$ is holomorphic and $V \subseteq U$ is a neighbourhood around z_0, for all $z \in V$.

Proof. We know that

$$f(z) = (z - z_0)^{-n} h(z),$$

where $h\colon V \to \mathbb{C}$ is holomorphic in some neighbourhood around z_0. Hence, we can write z_0 by its power series expansion, i.e.

$$h(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

for all $z \in V$. This implies that

$$\begin{aligned} f(z) &= (z - z_0)^{-n} \sum_{k=0}^{\infty} b_k (z - z_0)^k \\ &= \sum_{j=0}^{\infty} a_j (z - z_0)^{j-k} \\ &= \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + g(z), \end{aligned}$$

where

$$g(z) = \sum_{j=n}^{\infty} a_j (z - z_0)^{j-n}$$

is holomorphic on V. □

We call the expression

$$\frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0}$$

the *principal part of f*, and the value a_{-1} the *residue of f*. We denote this by $\mathrm{res}_{z_0}(f)$.

We will now look at a way to compute the residue for a pole.

Theorem 1.7.6. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on $U \setminus \{z_0\}$, where U is open and $z_0 \in U$, with a pole of order n at z_0. Then,*

$$\mathrm{res}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

Proof. Let the principal part of f be

$$\frac{a_{-n}}{(z-z_0)^n}+\cdots+\frac{a_{-1}}{z-z_0}.$$

□

We will now compute some residues. Define the function

$$f(z)=\frac{z}{(z-1)^3(z+3)}.$$

It is holomorphic on $\mathbb{C}\setminus\{1,-3\}$. Clearly, it has a simple pole at $z=-3$ and a pole of order 3 at $z=1$. Moreover,

$$\begin{aligned}\operatorname{res}_1(f)&=\frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\frac{z}{z+3}=\lim_{z\to 1}\frac{-3}{(z+3)^3}=\frac{-3}{64},\\ \operatorname{res}_{-3}(f)&=\lim_{z\to -3}\frac{z}{(z-1)^3}=\frac{3}{64}.\end{aligned}$$

Next, consider the function

$$f(z)=\frac{e^z}{z^5}.$$

The function has a pole of order 5 at $z=0$. Moreover, by the power series expansion of $\exp z$ around $z=0$, we find that the principal part of f is:

$$\frac{1}{z^5}+\frac{1}{z^4}+\frac{1}{2z^3}+\frac{1}{6z^2}+\frac{1}{24z}.$$

Hence, the residue is $\frac{1}{24}$. Alternatively,

$$\operatorname{res}_0(f)=\frac{1}{4!}\lim_{z\to 0}\frac{d^5}{dz^5}e^z=\frac{1}{4!}\lim_{z\to 0}e^z=\frac{1}{24}.$$

Now, we shall prove the Residue Formula.

Theorem 1.7.7 (Residue Formula for a single pole)**.** *Let $f\colon U\to\mathbb{C}$ be holomorphic on an open set $U\subseteq\mathbb{C}$ and let $C\subseteq U$ be a circle whose iterior is contained in U, except for a pole at z_0 inside C. Then,*

$$\int_C f(z)\ dz=2\pi i\cdot\operatorname{res}_{z_0}(f).$$

Proof. Let p be the principal part of f at z_0, given by

$$p(z)=\frac{a_{-n}}{(z-z_0)^n}+\cdots+\frac{a_{-1}}{z-z_0}.$$

Then, p is holomorphic in $\mathbb{C}\setminus\{z_0\}$, and the function $f-p$ has a removable singularity at z_0. Hence, $f-p$ extends to a holomorphic function $g\colon U\to\mathbb{C}$. By Cauchy's Theorem, we find that

$$\int_C f(z)-p(z)\ dz=0.$$

This implies that

$$\int_C f(z)\ dz = \int_C p(z)\ dz = \sum_{i=1}^{n} \int_C \frac{a_{-i}}{(z-z_0)^i}\ dz.$$

The function $\frac{1}{(z-z_0)^n}$ has a primitive if and only if $n \neq -1$. Therefore,

$$\int_C f(z)\ dz = \int_C \frac{a_{-1}}{z-z_0}\ dz = 2\pi i \cdot a_{-1} = 2\pi i \cdot \text{res}_f(z_0).$$

□

Corollary 1.7.8 (Residue Formula). *Let $f\colon U \to \mathbb{C}$ be holomorphic on an open set $U \subseteq \mathbb{C}$ and let $C \subseteq U$ be a circle whose interior is contained in U, except for finitely many poles $z_1, \dots, z_n$ inside C. Then,*

$$\int_C f(z)\ dz = 2\pi i \cdot \sum_{k=1}^{n} \text{res}_{z_n}(f).$$

This follows by breaking the circle down into smaller circles, each of which contains a single pole.

Typically, it is easier to compute the residues than the integral directly. We will now illustrate this. For instance, let

$$f(z) = \frac{z}{(z-1)^3(z-5)}.$$

Then, f is holomorphic in $\mathbb{C} \setminus \{1, 5\}$ and has a simple pole at $z = 5$ and a pole of order 3 at $z = 1$. If we want to compute the integral over the circle C of radius 2 centered at 0, then we just need to consider the pole at $z = 1$. We have

$$\text{res}_1(f) = \frac{1}{2!} \cdot \lim_{z\to 1} \frac{d}{dz^2} \frac{z}{z-5} = \lim_{z\to 1} \frac{-5}{(z-5)^3} = \frac{-5}{64}.$$

Hence,

$$\int_C f(z)\ dz = 2\pi i \cdot \text{res}_1(f) = \frac{-5}{32}\pi i.$$

A function $f\colon U \to \mathbb{C}$ is *meromorphic* on an open set $U \subseteq \mathbb{C}$ if there exists a set of poles $P \subseteq U$ such that f is holomorphic in $U \setminus P$. By the definition of isolated singularities, we find that P is closed (and discrete). Examples of meromorphic functions include holomorphic functions, a rational function $\frac{p(z)}{q(z)}$, where p and q are polynomials, $\frac{1}{\sin z}$ (which has poles at πk for $k \in \mathbb{Z}$), and in general, functions of the form $\frac{f(z)}{g(z)}$, where f and g are holomorphic in $U \subseteq \mathbb{C}$ open, and $g \not\equiv 0$ on any open subset of U. We can now prove the argument principle.

Theorem 1.7.9 (The Argument Principle). *Let $f\colon U \to \mathbb{C}$ be meromorphic on an open set $U \subseteq \mathbb{C}$, and let $C \subseteq U$ be a circle whose interior is in C. If f has no poles and no zeros on C, then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\ dz = z - p,$$

where z is the number of zeros of f inside C (with respect to their order), and p the number of poles inside C (with respect to their order).

Proof. By the quotient rule, we know that $\frac{f'}{f}$ is holomorphic on U except for the zeros and poles of f. We will show the result using residue formula.

First, assume that f has a zero of order n at z_0 inside C. Then, we can find a function $g\colon D \to \mathbb{C}$ holomorphic on an open disc D centered at z_0 with $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^n g(z)$ for all $z \in D$. Hence,

$$\frac{f'(z)}{f(z)} = \frac{n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)}{(z - z_0)^n g(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

We know that $g(z_0) \neq 0$, so $\frac{g'}{g}$ is holomorphic around some neighbourhood of z_0. Hence, $\operatorname{res}_{f'/f}(z_0) = n$.

Next, assume that f has a pole of order n at z_0 inside C. Then, we can find a function $g\colon D \to \mathbb{C}$ holomorphic on an open disc D centered at z_0 with $g(z) \neq 0$ such that $f(z) = (z - z_0)^{-n} g(z)$ for all $z \in D$. Hence,

$$\frac{f'(z)}{f(z)} = \frac{-n(z - z_0)^{-n-1} g(z) + (z - z_0)^{-n} g'(z)}{(z - z_0)^{-n} g(z)} = \frac{-n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

So, we find that $\operatorname{res}_{f/f}(z_0) = -n$.

Now, by residue formula, we find that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = \sum_{z_0 \text{ pole of } f'/f \text{ in } C} \operatorname{res}_{f'/f}(z_0) = z - p.$$

$\square$

We can use the argument principle to compute integrals. For instance, consider the integral

$$\int_C \frac{z^4}{z^5 - 1}\, dz,$$

where C is a circle with radius 2 centered at 0. Here, we have $f(z) = z^5 - 1$. The function f has 5 zeros that are roots of unity, meaning that they lie inside C, and it has no poles. Hence,

$$\int_C \frac{f'(z)}{f(z)}\, dz = 2\pi i \cdot 5 = 10\pi i.$$

This implies that

$$\int_C \frac{z^4}{z^5 - 1}\, dz = \frac{1}{5} \int_C \frac{f'(z)}{f(z)}\, dz = 2\pi i.$$

1.8 Rouché's Theorem

In this section, we will prove Rouché's Theorem. First, we prove a lemma about continuous functions.

Lemma 1.8.1. *Let $h\colon [0,1] \to \mathbb{C}$ be a continuous function. If $h(t) \in \mathbb{Z}$ for all $t \in [0,1]$, then h is constant.*

Proof. □

We make use of the lemma in Rouché's Theorem below.

Theorem 1.8.2 (Rouché's Theorem)**.** *Let $f, g\colon U \to \mathbb{C}$ be non-zero holomorphic functions in an open set $U \subseteq \mathbb{C}$ and let $C \subseteq U$ be a circle whose interior is in U. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside C.*

Proof. For every $0 \leq t \leq 1$, define the function $f_t\colon U \to \mathbb{C}$ by $f_t(z) = f(z) + tg(z)$ and let n_t be the number of zeros the function f_t has inside C. Then, $f_0 = f$ and $f_1 = f + g$. Since f_t is holomorphic for all $t \in [0,1]$ and not identically equal to 0, we note that $n_t \in \mathbb{Z}_{\geq 0}$. We will show that $t \mapsto n_t$ is a continuous function, and hence a constant.

Now, let $t \in [0,1]$. We first show that f_t has no zeros in C. So, let $z_0 \in C$ be a zero of f_t. In that case, $|f(z_0)| = t|g(z_0)|$. However, since $0 \leq t \leq 1$ and $|f(z_0)| > |g(z_0)|$, this is a contradiction. Hence, f_t has no zeros in C.

Next, since f_t is holomorphic in U and has no poles, the argument principle tells us that

$$n_t = \int_C \frac{f_t'(z)}{f_t(z)}\, dz.$$

Hence, the map $t \mapsto n_t$ is continuous. This implies that the value n_t is constant by the lemma above, meaning that $n_0 = n_1$. Therefore, f and $f+g$ have the same number of zeros. □

We will now use Rouché's Theorem to prove different results. For instance, we can prove the Fundamental Theorem of Algebra using it. So, let p be a polynomial of degree $n \geq 1$, given by

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

with $a_n \neq 0$. Then,

$$\frac{p(z) - a_n z^n}{a_n z^n} = \frac{a_{n-1} z^{-1} + \cdots + a_0 z^{-n}}{a_n} \to 0$$

as $|z| \to \infty$. Hence, there exists a circle C of radius $R > 0$ centered at 0 such that for all $z \in C$, then

$$\left| \frac{p(z) - a_n z^n}{a_n z^n} \right| < 1 \iff |p(z) - a_n z^n| < |a_n z^n|.$$

So, Rouché's Theorem tells us that $z \mapsto a_n z^n$ and $z \mapsto p(z)$ have the same number of zeros inside C. We know that $a_n z^n$ has n zeros inside C (since it contains 0), meaning that p has n zeros inside C. The circle C can be made arbitrarily big (since the radius R can be arbitrarily increased), meaning that p has n zeros in $\mathbb{C}$.

Next, we will use Rouché's Theorem to show that all solutions of $z^6+z+1=0$ lie in the annulus $\frac{1}{2} < |z| < 2$. First, let $C_1 = C_2(0)$. Then, we know that for $z \in C_1$,

$$|z+1| \leq |z|+1 = 3, \qquad |z^6| = 2^6.$$

Hence, $|z+1| < |z^6|$. We know that z^6 has 6 zeros inside C_1, so Rouché's Theorem tells us that z^6+z+1 has 6 zeros inside C_1 and none on C_1. Next, let $C_2 = C_{1/2}(0)$. Then, we know that for $z \in C_2$,

$$|z^6+z| \leq |z^6| + |z| = \frac{1}{2^6} + \frac{1}{2} < 1.$$

We know that the function $z \mapsto 1$ in C_2 has no zeros in C_2, so Rouché's Theorem tells us that z^6+z+1 has no zeros inside C_2 and on C_2. So, the zeros of the polynomial z^6+z+1 satisfy $\frac{1}{2} < |z| < 2$.

Next, we will prove the open mapping theorem. A function $f\colon X \to \mathbb{C}$ is an *open map* in $X \subseteq C$ if for all $U \subseteq X$ open, its image $f(U)$ is open.

Theorem 1.8.3 (Open Mapping Theorem). *Let $f\colon U \to \mathbb{C}$ be a non-constant holomorphic function in some region $U \subseteq \mathbb{C}$. Then, f is an open map.*

Proof. Let $V \subseteq U$ be open. We show that $f(V)$ is open. Let $w_0 \in f(V)$. So, there exists a $z_0 \in V$ such that $f(z_0) = w_0$. Define the function $F\colon U \to \mathbb{Z}$ by $F(z) = f(z) - w_0$. Then, F has a zero at z_0. Moreover, since F is not a constant function, it has isolated zeros. Hence, there exists a $\delta > 0$ such that z_0 is the only isolated zero of F in $\overline{D}_\delta(z_0)$. Set

$$\varepsilon = \inf_{z \in C_\delta(z_0)} |F(z)| > 0.$$

We claim that $D_\varepsilon(w_0) \subseteq f(V)$. So, let $w \in D_\varepsilon(w_0)$. Define the function $G\colon U \to \mathbb{C}$ by $G(z) = f(z) - w$. Then,

$$|G(z) - F(z)| = |w - w_0| < \varepsilon \leq |F(z)|$$

for all $z \in C_\delta(z_0)$. Now, by Rouché's Theorem, we find that F and $F+G$ have the same number of zeros inside $C_\delta(z_0)$. By construction, F has precisely one zero inside $C_\delta(z_0)$, at z_0. Hence, there exists a $z \in D_\delta(z_0) \subseteq V$ such that $G(z) = 0$, meaning that $f(z) = w$. So, $w \in f(V)$. So, $D_\varepsilon(w_0) \subseteq f(V)$, meaning that $f(V)$ is open. □

Note that the Open Mapping Theorem does not hold in $\mathbb{R}$. For instance, consider the function $f(x) = x^2$. Then, $f(-1,1) = [0,1)$ is not open, even though f is a non-constant holomorphic function that is infinitely differentiable and has Maclaurin series with radius of convergence ∞!

Next, we consider the maximum modulus principle.

Theorem 1.8.4 (Maximum Modulus Principle). *Let $f\colon U \to \mathbb{C}$ be a non-constant holomorphic function in a region $U \subseteq \mathbb{C}$. Then, the complex modulus $|f|$ cannot attain a local maximum in U.*

Proof. Note that since f is not a constant, it satisfies the Open Mapping Theorem. Now, assume, for a contradiction, that $|f|$ attains a local maximum at some $z_0 \in U$. Then, there exists an open set $V \subseteq U$ containing z_0 such that $|f(z)| \leq |f(z_0)|$ for all $z \in V$. By the open mapping theorem, we know that $f(V)$ is open. Hence, there exists a $\delta > 0$ such that $D_\delta(f(z_0)) \subseteq f(V)$. In particular, we have $z_0 + \frac{\delta}{2} \in f(V)$ that satisfies $|z_0 + \frac{\delta}{2}| > |z_0|$. This is a contradiction since z_0 is a local maximum. □

1.9 Conformal Mappings

A *conformal map* is a holomorphic bijection $f: U \to V$, where $U, V \subseteq \mathbb{C}$ are open sets. Examples of conformal mappings include:

- $f(z) = az + b$, for $a, b \in \mathbb{C}$ with $a \neq 0$, with inverse $f^{-1}(z) = \frac{1}{a}(z - b)$. These maps are entire bijections.
- $f(z) = e^z$ from the open set
$$\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0, \mathrm{Im}(z) \in (0, \pi)\}$$
to the open set
$$\{z \in \mathbb{C} \mid |z| < 1, \mathrm{Im}(z) > 0\}.$$
- Let
$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}, \qquad \mathbb{D} = D_1(0).$$
Define the functions $F: \mathbb{H} \to \mathbb{D}$ and $G: \mathbb{D} \to \mathbb{H}$ by
$$F(z) = \frac{i - z}{i + z}, \qquad G(z) = i\frac{1 - w}{1 + w}.$$
We claim that F is conformal with inverse G. By definition, they are holomorphic on their respective domains. We have $|i - z| < |i + z|$ for all $z \in \mathbb{H}$, so $|F(z)| < 1$ and hence lies in $\mathbb{D}$. So, the map F is well-defined. Moreover, for all $w \in \mathbb{D}$ with $w = u + iv$ for $u, v \in \mathbb{R}$. Then,
$$\begin{aligned}
\mathrm{Im}(G(w)) &= \mathrm{Im}\left(i\frac{1 - u - iv}{1 + u + iv}\right) \\
&= \mathrm{Re}\left(\frac{1 - u - iv}{1 + u + iv}\right) \\
&= \mathrm{Re}\left(\frac{(1 - u - iv)(1 + u - iv)}{(1 + u)^2 + v^2}\right) \\
&= \frac{1 - u^2 - v^2}{(1 + u)^2 + v^2} > 0
\end{aligned}$$
since $|w|^2 = u^2 + v^2 < 1$. So, G too is well-defined. Finally, for $w \in \mathbb{D}$ and $z \in \mathbb{H}$,
$$\begin{aligned}
F(G(w)) &= \frac{i - i\frac{1-w}{1+w}}{i + i\frac{1-w}{1+w}} \\
&= \frac{1 + w - 1 + w}{1 + w + 1 - w} \\
&= w
\end{aligned} \qquad\qquad G(F(z)) =$$

We will now show that the inverse is holomorphic. First, we prove the lemma below.

Lemma 1.9.1. *Let $f: U \to \mathbb{C}$ be a holomorphic function on an open set $U \subseteq \mathbb{C}$ and $w_0 \in \mathbb{C}$ such that the function $f(z) - w_0$ has an isolated zero of multiplicity $n \in \mathbb{Z}_{\geq 1}$ at some $z_0 \in U$. Then, for all sufficiently small $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |w - w_1| < \delta$, then $f(z) = w$ has precisely n distinct solutions z with $|z - z_0| < \varepsilon$.*

Proof. □

Proposition 1.9.2. *Let $U, V \subseteq \mathbb{C}$ be open and let $f\colon U \to V$ be holomorphic and bijective. Then, its inverse $f^{-1}\colon V \to U$ is also holomorphic, with*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$$

for all $w \in V$. In particular, f^{-1} is conformal.

Proof. We first show that $f'(z) \neq 0$ for all $z \in U$. Assume, for a contradiction, that $f'(z_0) = 0$ for some $z_0 \in U$. Then, the following is a power series expansion of f around z_0:

$$f(z) = a_0 + \sum_{n=2}^{\infty} a_n (z - z_0)^n$$

for all z in some open disc D in U centered at z_0. Since the zeros of f' are isolated, we can assume that $f'(z) \neq 0$ for all $z \in D$ with $z \neq z_0$. By the lemma above, we find that $f(z) = w$ has more than 2 solutions in D. Since f is a bijection, this is a contradiction. Hence, $f'(z) \neq 0$ for all $z \in U$.

We know that the inverse map $f^{-1}\colon V \to U$ is continuous by the open mapping theorem. Let $w_0 \in V$. We show that f is differentiable at w_0. Then, for $w \in V$, with $z = f(w)$ and $z_0 = f(w_0)$, we find that

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \to \frac{1}{f'(z_0)}$$

as $z \to z_0$ by continuity of f^{-1}, i.e. $w \to w_0$ as $z \to z_0$. Hence,

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}.$$

□

We will now consider automorphisms. A *(conformal) automorphism* of an open set $U \subseteq \mathbb{C}$ is a conformal map $f\colon U \to U$. We denote by $\mathrm{Aut}(U)$ the set of automorphisms. We first characterise automorphisms of $\mathbb{C}$.

Theorem 1.9.3. *The automorphisms of the plane are precisely the polynomials of degree 1, i.e.*

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}.$$

Proof. We saw above that polynomials of degree 1 give rise to a conformal map $\mathbb{C} \to \mathbb{C}$. Now, let $f\colon \mathbb{C} \to \mathbb{C}$ be a conformal map. Define the map $g\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g(z) = f(1/z)$. Then, g is holomorphic in $\mathbb{C} \setminus \{0\}$ and injective.

Now, we claim that 0 is not an essential singularity of g. Assume, for a contradiction, that 0 is an essential singularity of g. Let D_1, D_2 be open discs such that D_1 is centered at 0 and $D_1 \cap D_2 = \varnothing$. By Casarati-Weierstrass Theorem, we know that $g(D_1 \setminus \{0\})$ is dense in $\mathbb{C}$. Since g is an open map, $g(D_2)$ is open. So, $g(D_1 \setminus \{0\}) \cap g(D_2) \neq \varnothing$. So, there exists a $z_0 \in D_1 \setminus \{0\}$ and $w_0 \in D_2$ such that $g(z_0) = g(w_0)$. However, since g is bijective, this implies that $z_0 = w_0$. This is a contradiction since $D_1 \cap D_2 = \varnothing$. Hence, 0 is not an essential singularity of g.

Since f is conformal in $\mathbb{C}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \mathbb{C}$. Hence,

$$g(z) = \sum_{n=0}^{\infty} a_n z^{-n}.$$

Since 0 is not an essential singularity of g, there exists an $m \geq 0$ such that $a_n = 0$ for all $n > m$. So,

$$f(z) = \sum_{n=0}^{m} a_n z^m.$$

By the Fundamental Theorem of Algebra, f has n zeros. Since f is injective, we require f to have precisely 1 zero. Hence, we must have $m = 1$. That is, $f(z) = a_0 + a_1 z$, with $a_1 \neq 0$. $\square$

Next, we characterise automorphisms of an open disc. We will first prove Schwarz Lemma.

Lemma 1.9.4 (Schwarz Lemma). *Let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function from the unit disc $\mathbb{D} = D_1(0)$ such that $f(0) = 0$. Then, the function f is a contraction, i.e. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, and $|f'(0)| \leq 1$. Moreover, the following are equivalent:*

1. *There exists a $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$.*
2. *The value $|f'(0)| = 1$.*
3. *The function $f = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. By Taylor's Theorem, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \mathbb{D}$. Since $f(0) = 0$, we find that $a_0 = 0$. Hence, the function $\frac{f(z)}{z}$ is holomorphic in $\mathbb{D} \setminus \{0\}$ and has a removable singularity at $z = 0$ (as seen by the power series expansion). We note that

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0).$$

So, we can define the function $g \colon \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z)/z & z \in \mathbb{D} \setminus \{0\} \\ f'(0) & \text{otherwise} \end{cases}$$

so that g is holomorphic in $\mathbb{D}$. Let $r \in (0, 1)$. If $z \in \mathbb{D}$ with $|z| = r$, then

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

By the maximum modulus principle, $|g(z)| \leq \frac{1}{r}$ for all $z \in \overline{D}_1(r)$ as well. If we fix z and let $r \to 1$, we find that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Hence, $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. In particular, $|f'(0)| = |g(0)| \leq 1$.

Now, we show that (1) $\iff$ (2) $\iff$ (3). Clearly, (3) $\implies$ (1) and (3) $\implies$ (2). We will now show (1) $\implies$ (3) and (2) $\implies$ (3).

- Assume that there exists a $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$. Then, $|g(z_0)| = 1$, so g attains the maximum in the interior of $\mathbb{D}$. Hence, the maximum modulus principle tells us that g is a constant. That is, $g(z) = c$ for all $z \in \mathbb{D}$. Hence, $f = cz$, with $|c| = |g(0)| = 1$.

- Now, assume that the value $|f'(0)| = 1$. Then, $|g(0)| = 1$, so using the same argument as above, we find that $f = cz$, with $|c| = 1$.

□

We refer to maps $z \mapsto cz$ with $|c| = 1$ as *rotation* maps.

Now, we will characterise automorphisms of the disc $\mathbb{D}$. For $a \in \mathbb{D}$, let $\varphi_a : \partial\mathbb{D} \to \mathbb{C} \setminus \{1 - \overline{a}\}$ given by $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$. By definition, φ_a is holomorphic. Moreover, $\frac{1}{\overline{a}} \notin \overline{D_1}$, so φ_a is holomorphic in $\mathbb{D}$. We claim that $\varphi_a(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$. Let $e^{i\theta} \in \partial\mathbb{D}$. Then,

$$\varphi_a(e^{i\theta}) = \frac{e^{i\theta} - a}{1 - \overline{a}e^{i\theta}} = \frac{e^{i\theta} - a}{e^{i\theta}(e^{-i\theta} - \overline{a})} = e^{-i\theta} \cdot \frac{e^{i\theta} - a}{\overline{e^{i\theta} - a}}.$$

Hence, $|\varphi_a(e^{i\theta})| = 1$. We now claim that $\varphi_a : \mathbb{D} \to \mathbb{D}$ defines an automorphism. So, we now require $\varphi_a(\mathbb{D}) \subseteq \mathbb{D}$. By the Maximum Modulus Principle, $|\varphi_a|$ attains its maximum on the boundary $\overline{\mathbb{D}}$. Hence, $|\varphi_a(z)| \leq 1$ for all $z \in \mathbb{D}$ by the result above. Since φ_a is not a constant function, we further know that $|\varphi_a(z)| < 1$ for all $z \in \mathbb{D}$, so $\varphi_a(\mathbb{D}) \subseteq \mathbb{D}$. Now, we find that for all $z \in \mathbb{D}$,

$$\begin{aligned}
\varphi_a(\varphi_{-a}(z)) &= \varphi_a\left(\frac{z+a}{1+\overline{a}z}\right) \\
&= \frac{\frac{z+a}{1+\overline{a}z} - a}{1 - \overline{a}\frac{z+a}{1+\overline{a}z}} \\
&= \frac{z + a - a(1+\overline{a}z)}{1+\overline{a}z} \cdot \frac{1+\overline{a}z}{(1+\overline{a}z) - \overline{a}(z+a)} \\
&= \frac{z - |a|^2 z}{1 - |a|^2} = z.
\end{aligned}$$

So, φ_a is bijective, with $(\varphi_a)^{-1} = \varphi_{-a}$. Hence, $\varphi_a \in \mathrm{Aut}(\mathbb{D})$ for all $a \in \mathbb{D}$. We can now characterise the automorphisms of $\mathbb{D}$.

Theorem 1.9.5. *We have*

$$\mathrm{Aut}(\mathbb{D}) = \{c\varphi_a \mid c \in \partial\mathbb{D}, a \in \mathbb{D}\}.$$

Proof. Since φ_a and $z \mapsto cz$ are bijections, their composition is also bijective. Moreover, it is a conformal map $\mathbb{D} \to \mathbb{D}$, so an automorphism of $\mathbb{D}$.

Now, let $f \in \mathrm{Aut}(\mathbb{D})$. Then, there exists an $a \in \mathbb{D}$ such that $f(a) = 0$. Define the function $g = f \circ \varphi_{-a}$. Then, $g \colon \mathbb{D} \to \mathbb{D}$ is a holomorphic bijection, with

$$g(0) = f(\varphi_{-a}(0)) = f\left(\frac{0+a}{1+\overline{a}\cdot 0}\right) = f(a) = 0.$$

Now, by Schwarz Lemma, we find that $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. We also have $g^{-1}(0) = 0$, so $|g^{-1}(w)| \leq |w|$ for all $w \in \mathbb{D}$. Hence, for all $z \in \mathbb{D}$, we have

$$|z| = |g^{-1}(g(z))| \leq |g(z)| \leq |z|,$$

meaning that $|g(z)| = |z|$ for all $z \in \mathbb{D}$. Hence, by Schwarz Lemma again, we find that $g = cz$, for some $c \in \partial\mathbb{D}$. We know that

$$c\varphi_a(z) = g(\varphi_a(z)) = f(\varphi_{-a}(\varphi_a(z))) = f(z)$$

for all $z \in \mathbb{D}$. □

We can now characterise automorphisms in the following way.

Theorem 1.9.6. *Let $z_0, w_0 \in \mathbb{D}$ and $\theta_0 \in (-\pi, \pi)$. Then, there exists a unique automorphism f of $\mathbb{D}$ such that $f(z) = w_0$ and $\arg(f'(z_0)) = \theta_0$.*

Proof. Let $c \in \delta\mathbb{D}$, let $f_c \colon \mathbb{D} \to \mathbb{D}$ be given by $f_c(z) = \varphi_{-w_0}(c\varphi_{z_0}(z))$. Then, $f_c(z_0) = w_0$ and

$$\begin{aligned} f_c'(z_0) &= \varphi_{-w_0}'(c\varphi_{z_0}(z_0)) \cdot c\varphi_{z_0}'(z_0) \\ &= \varphi_{-w_0}'(0) \cdot c\varphi_{z_0}'(z_0) \\ &= (1-|w_0|^2) \cdot \frac{c}{1-|z_0|^2}. \end{aligned}$$

So,

$$|f_c'(z_0)| = \frac{1-|w_0|^2}{1-|z_0|^2}.$$

Hence,

$$c \cdot \frac{1-|w_0|^2}{1-|z_0|^2} = f_c'(z_0) = |f_c'(z_0)|e^{i\arg f_c'(z_0)} = \frac{1-|w_0|^2}{1-|z_0|^2}e^{i\arg f_c'(z_0)}.$$

So, $c = e^{i\arg f_c'(z_0)}$. Therefore, we can fix $c = e^{i\theta_0}$ so that $\arg(f'(z_0)) = \theta_0$.

We now show uniqueness. Let f_1, f_2 be two automorphisms of $\mathbb{D}$ such that $f_1(z) = f_2(z) = w_0$ and $\arg(f_1'(z)) = \arg(f_2'(z)) = \theta_0$. Define the function $g = \varphi_{z_0} \circ f_2^{-1} \circ f_1 \circ \varphi_{-z_0}$. Then, $g(0) = 0$, and g is automorphism of $\mathbb{D}$. So, there exists a $c \in \delta\mathbb{D}$ and $a \in \mathbb{D}$ by $g = c\varphi_a$. Hence,

$$-ac = c\varphi_a(0) = g(0) = 0,$$

so $a = 0$. This implies that $g(z) = cz$ and $g'(0) = c$. So, we find that

$$\begin{aligned} c &= g'(0) \\ &= \varphi_{z_0}'(z_0) \cdot (f_2^{-1})'(w_0) \cdot f_1'(z_0) \cdot \varphi_{-z_0}'(0) \\ &= \frac{1}{1-|z_0|^2}\frac{1}{f_2'(z_0)}f_1'(z_0)(1-|z_0|^2) \\ &= \frac{f_1'(z_0)}{f_2'(z_0)}. \end{aligned}$$

Hence, $f_1'(z_0) = cf_2'(z_0)$. Since $\arg(f_1'(z_0)) = \arg(f_2'(z_0))$, we find that $c \in \mathbb{R}$ with $c > 0$. Since $|c| = 1$, we must have $c = 1$. So, $g = cz$ is the identity map. Hence,

$$\begin{aligned} \varphi_{z_0} \circ f_2^{-1} \circ f_1 \circ \varphi_{-z_0} = id &\iff f_2^{-1} \circ f_1 = \varphi_{z_0}^{-1} \circ \varphi_{-z_0}^{-1} = id \\ &\iff f_2 = f_1. \end{aligned}$$

□

We can use the computation to find particular automorphisms. For instance, we will find the automorphism $f \colon \mathbb{D} \to \mathbb{D}$ with $f(0) = \frac{1}{2}$ and $f'(0) = \frac{\pi}{2}$. We know that $f = c\varphi_a$, for $c \in \partial\mathbb{D}$ and $a \in \mathbb{D}$. Hence,

$$\frac{1}{2} = f(0) = c\varphi_a(0) = -ac, \qquad \frac{\pi}{2} = f'(0) = c(1 - |a|^2).$$

Since $1 - |a|^2 > 0$, we find that $\arg(f'(0)) = c$, meaning that $\arg(c) = \frac{\pi}{2}$. So, we can set $c = e^{\pi i/2} = i$. In that case,

$$\frac{1}{2} = -ac \iff a = \frac{-1}{2c} = \frac{1}{2}i.$$

Therefore, $f(z) = \varphi_{i/2}(z)$.

1.10 Riemann Mapping Theorem

In this section, we will look at the Riemann Mapping Theorem and its relation to conformal equivalence.

Let $U_1, U_2 \subseteq \mathbb{C}$ be open. We say that U_1 is *conformally equivalent* to U_2 if there exists a conformal mapping $f \colon U_1 \to U_2$. By properties of bijections and holomorphisms, we find that conformal equivalence forms an equivalence relation. We will now try to understand the equivalence classes of conformal equivalence. In particular, given $U \subseteq \mathbb{C}$ open, which sets $V \subseteq \mathbb{C}$ are conformally equivalent to U? We will consider this question where $U = \mathbb{D}$. This is given by the Riemann mapping theorem.

Theorem 1.10.1 (Riemann Mapping Theorem)**.** *Let $U \subsetneq \mathbb{C}$ be a simply-connected non-empty open set. Then, U is conformally equivalent to the unit disc $\mathbb{D}$.*

Note that all the assumptions- proper, non-empty and simply-connected- are necessary. In particular, $\mathbb{C}$ is not conformally equivalent to $\mathbb{D}$. This follows from Lioville's Theorem (a bounded entire function is constant).

Using the Riemann Mapping Theorem, we can generalise the property about automorphisms of $\mathbb{D}$.

Theorem 1.10.2. *Let $U \subsetneq \mathbb{C}$ be a simply-connected non-empty open set and let $z_0 \in U, w_0 \in \mathbb{D}$ and $\theta_0 \in (-\pi, \pi)$. Then, there exists a unique conformal map $f \colon U \to \mathbb{D}$ such that $f(z_0) = w_0$ and $\arg(f'(z_0)) = \theta_0$.*

Proof. By the Riemann Mapping Theorem, we know that there exists a conformal map $g \colon U \to \mathbb{D}$. Let $\tilde{z}_0 = g(z_0)$ and $\varphi_0 = \arg(g'(z_0))$. Let $h \colon \mathbb{D} \to \mathbb{D}$ be the conformal mapping such that $h(\tilde{z}_0) = w_0$ and $\arg(h'(\tilde{z}_0)) = \theta_0 - \varphi_0$. This is possible by the result above about automorphisms on $\mathbb{D}$. Then, define $f = h \circ g \colon U \to \mathbb{D}$. By construction, we find that $f(z_0) = h(\tilde{z}_0) = w_0$ and

$$\arg(f'(z_0)) = \arg(h'(g(z_0)) \cdot g'(z_0)) = (\theta_0 - \varphi_0) + \varphi_0 = \theta_0.$$

Now, we show that f is unique. So, let $\tilde{f} \colon U \to \mathbb{D}$ be a conformal map such that $\tilde{f}(z_0) = w_0$ and $\arg(\tilde{f}'(z_0)) = \theta_0$. Then, $g := \tilde{f} \circ f^{-1} \colon \mathbb{D} \to \mathbb{D}$ is a conformal mapping such that $g(w_0) = w_0$ to w_0 and

$$\arg(g'(w_0)) = \arg(\tilde{f}'(z) \cdot f'(z_0)) = \theta_0 - \theta_0 = 0.$$

By the uniqueness of automorphisms of $\mathbb{D}$, we find that $g = id$, meaning that $f = \tilde{f}$. $\square$

We will now aim to prove the Riemann Mapping Theorem. To do so, we require Montel's Theorem. Let $U \subseteq \mathbb{C}$ be open and let $\mathcal{F}$ be a collection of continuous functions $f \colon U \to \mathbb{C}$. We say that $\mathcal{F}$ is *normal* if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U. Note that we do not require the limit to be in $\mathcal{F}$. Moreover, $\mathcal{F}$ is *bounded* if there exists an $M > 0$ such that $|f(z)| < M$ for all $f \in \mathcal{F}$ and $z \in U$. Finally, $\mathcal{F}$ is *equi-continuous* at $z_0 \in U$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$ and $z \in U$, if $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$.

Now, consider the family $\mathcal{F}$ of functions $f_n \colon \mathbb{D} \to \mathbb{C}$ given by $f_n(z) = \sum_{k=n}^{\infty} \frac{z^k}{2^k}$. We claim that this family is equi-continuous at $z = 0$.

Next, we will consider Arzela-Ascoli Theorem, a generalisation of the Heine-Borel Theorem.

Theorem 1.10.3 (Arzela-Ascoli Theorem)**.** *Let $U \subseteq \mathbb{C}$ be open and $\mathcal{F}$ be a collection of continuous functions $f\colon U \to \mathbb{C}$. If $\mathcal{F}$ is bounded and equi-continuous at all $z_0 \in U$, then $\mathcal{F}$ is normal.*

Now, we can prove Montel's Theorem.

Theorem 1.10.4 (Montel's Theorem)**.** *Let $U \subseteq \mathbb{C}$ be open and $\mathcal{F}$ be a collection of holomorphic functions $f\colon U \to \mathbb{C}$. If $\mathcal{F}$ is bounded, then $\mathcal{F}$ is normal.*

Proof. By Arzela-Ascoli Theorem, it suffices to show that $\mathcal{F}$ is equi-continuous at all $a \in U$. So, let $a \in U$, $r > 0$ such that $\overline{D}_r(a) \subseteq U$, and let $M > 0$ such that $|f(z)| < M$ for all $f \in \mathcal{F}$ and $z \in U$. Now, let $R > r$ such that $\overline{D}_R(a) \subseteq U$. By Cauchy's Integral Formula, we find that for all $z \in D_r(a)$,

$$\begin{aligned} f(z) - f(a) &= \frac{1}{2\pi i} \int_{\partial D_R(a)} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{\partial D_R(a)} \frac{f(w)}{w-a}\, dw \\ &= \frac{1}{2\pi i} \int_{\partial D_R(a)} \frac{f(w)(z-a)}{(w-z)(w-a)}\, dw. \end{aligned}$$

Hence,

$$\begin{aligned} |f(z) - f(a)| &\leq \frac{M|z-a|}{2\pi} \sup_{w \in \partial D_R(a)} \frac{1}{(w-z)(w-a)} \cdot 2\pi R \\ &= M|z-a| \sup_{w \in \partial D_R(a)} \frac{1}{w-z} \\ &\leq \frac{M}{R-r}|z-a|. \end{aligned}$$

So, now, let $\varepsilon > 0$. Set $\delta = \min(\frac{R-r}{M}, r)$. Then, for all $z \in U$ and $f \in \mathcal{F}$, if $|z-a| < \delta<$ then $z \in D_r(a)$, and so

$$|f(z) - f(a)| \leq \frac{M}{R-r}|z-a| < \varepsilon.$$

Hence, $\mathcal{F}$ is equi-continuous at a. □

We will now use Montel's Theorem to show that a collection $\mathcal{F}$ is normal. So, define the collection $\mathcal{F}$ by $f_n(z) = z^n + z$ for $z \in \mathbb{Z}_{\geq 1}$ on $\mathbb{D}$. We know that all the functions in $\mathcal{F}$ are holomorphic, so we just need to show that they are bounded. So, for $z \in \mathbb{D}$ and $f_n \in \mathcal{F}$, we find that

$$|f(z)| = |z^n + z| \leq |z|^n + |z| < 2.$$

Hence, the collection is bounded, so Montel's Theorem tells us that it is normal.
We will now prove the Riemann Mapping Theorem using Montel's Theorem.

Proof of the Riemann Mapping Theorem. Let

$$\mathcal{F} = \{\psi\colon U \to \mathbb{D} \mid \psi \text{ holomorphic and injective}\}.$$

We first show that $\mathcal{F}$ is not empty. Let $w_0 \in \mathbb{C} \setminus U$. We know that the map $z \mapsto z - w_0$ is holomorphic in U with no zeros. Thus, there exists a holomorphic function $f\colon U \to \mathbb{C}$ such that $f(z)^2 = z - w_0$ for all $z \in U$. We claim that $f \in \mathcal{F}$. Note that for $z_1, z_2 \in U$, if $f(z_1) = f(z_2)$, then

$$z_1 - w_0 = f(z_1)^2 = f(z_2)^2 = z_2 - w_0,$$

meaning that $z_1 = z_2$. So, f is injective. By the Open Mapping Theorem, we know that $f(U)$ is open and hence contains a disc $D_r(a)$ such that $0 < r < |a|$. We know that there are no $z_1, z_2 \in U$ with $z_1 \neq z_2$ such that $f(z_1) = -f(z_2)$. So, $D_r(-a) \cap D_r(a) = \varnothing$. Hence, define the function $\psi\colon U \to \mathbb{C}$ by $\psi(z) = \frac{r}{f(z)+a}$. Then, ψ is holomorphic with

$$|\psi(z)| \leq \frac{r}{|f(z)+a|} < 1$$

for all $z \in U$. So, $\psi \in \mathcal{F}$.

Now, we show that $\mathcal{F}$ is normal. For all $\psi \in \mathcal{F}$ and $z \in U$, we find that $|\psi(z)| < 1$, meaning that $\mathcal{F}$ is bounded. So, Montel's Theorem tells us that $\mathcal{F}$ is normal.

Next, we claim that if $\psi \in \mathcal{F}$ not surjective, then for all $z_0 \in U$, there exists a $\psi_1 \in \mathcal{F}$ such that $|\psi_1'(z_0)| > |\psi'(z_0)|$. So, let $\psi \in \mathcal{F}$ not surjective, and $\alpha \in \mathbb{D}$ with $\alpha \notin \psi(U)$. Then, we know that $\varphi_\alpha \circ \psi \in \mathcal{F}$ and $\varphi_\alpha \circ \psi$ has no zeros in U (since α is the unique zero of φ_α). Hence, so there exists a holomorphic function $g\colon U \to \mathbb{C}$ such that $g^2 = \varphi_\alpha \circ \psi$. Since ψ is injective, we find that g is injective. Therefore, $g \in \mathcal{F}$. Now, define $\psi_1 = \varphi_\beta \circ g$, where $g(z_0) = \beta$. Let $s\colon U \to \mathbb{C}$ be the function $s(w) = w^2$. Then,

$$\psi = \varphi_{-\alpha} \circ s \circ g = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \circ \psi_1.$$

Since $\psi_1(z_0) = 0$, we can apply the chain rule to find that $\psi'(z_0) = F'(0)\psi_1'(z_0)$, where $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}$. We know that F maps $\mathbb{D}$ to $\mathbb{D}$, and F is not injective. Thus, the Schwarz Lemma tells us that $|F'(0)| < 1$. Hence, $|\psi_1'(z_0)| > |\psi'(z_0)|$. Note that $\psi'(z_0) \neq 0$ since ψ is holomorphic and injective in U.

Now, let $z_0 \in U$ and let

$$\eta = \sup_{\psi \in \mathcal{F}} |\psi'(z_0)|.$$

We know that for any $h \in \mathcal{F}$ such that $|h'(z_0)| = \eta$, h is surjective. We will now construct such an h. By the supremum property, there exists a sequence $(\psi_n)_{n=1}^\infty$ in $\mathcal{F}$ such that $|\psi_n'(z_0)| \to \eta$ as $n \to \infty$. Since $\mathcal{F}$ is normal, we can find a subsequence $(\psi_{n_k})_{k=1}^\infty$ in $\mathcal{F}$ that converges uniformly (to some function $h\colon U \to \mathbb{C}$) on every compact subset of U. We know that (ψ_{n_k}) is a sequence of holomorphic functions, so h is holomorphic. We claim that h is injective. Since $\mathcal{F}$ is not empty, $\eta > 0$, meaning that h is not a constant function. We know that $\psi_{n_k}(U) \subseteq \mathbb{D}$ for all $k \in \mathbb{Z}_{\geq 1}$, so the limit function $h(U) \subseteq \overline{\mathbb{D}}$. However, the open mapping theorem tells us that $h(U)$ is open, meaning that $h(U) \subseteq \mathbb{D}$. Since $|h'(z_0)| = \eta$, we find that h is surjective. Now, we show that h is injective. So, let $z_1, z_2 \in U$ with $z_1 \neq z_2$, and set $\alpha = h(z_1)$ and $\alpha_k = \psi_{n_k}(z_1)$ for all $k \in \mathbb{Z}_{\geq 1}$. Let $\overline{D}$ be a closed disc in U centered at z_0 with $z_1 \notin \overline{D}$ and $h - \alpha$ has no zeros on $\partial\overline{D}$. Then, $\psi_n - a_n \to h - \alpha$ uniformly on $\overline{D}$. Moreover,

$\psi_n - \alpha_n$ has no zero in D for all $n \in \mathbb{Z}_{\geq 1}$. Since $\psi_n - \alpha_n$ is injective and has a zero at z_1, Rouché's Theorem tells us that $h - \alpha$ has no zero in D. Hence, $h(z_2) \neq h(z_1)$, meaning that h is injective. So, $h \in \mathcal{F}$ is surjective, meaning that it is a conformal mapping from U to $\mathbb{D}$. $\square$