CHAPTER 1

NORMED VECTOR SPACES

1.1 Review of Vector Spaces

In this section, we will review properties of vector spaces with relation to vector spaces.

Definition 1.1.1. A *vector space* $(V, +, \cdot)$ *(over a field* $\mathbb{K}$*)* is a set V and functions $(+)\colon V \times V \to V$ and $(\cdot)\colon \mathbb{K} \times V \to V$ such that:

- $(V, +)$ is an abelian group;
- $\cdot$ is associative over $+$, i.e. for $a, b \in \mathbb{K}$ and $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$;
- $\cdot$ left- and right-distributes over $+$, i.e. for $a \in \mathbb{K}$ and $v, w \in V$, $a\cdot(v+w) = a \cdot v + a \cdot w$.

In this course, we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We are familiar with many vector spaces, e.g. $\mathbb{R}^n$ over $\mathbb{R}$ and $\mathbb{C}^n$ over $\mathbb{C}$ (and $\mathbb{R}$).

We now review the concept of dimensionality.

Definition 1.1.2. Let V be a vector space and let $S \subseteq V$.

- We say that S *spans* V if for all $v \in V$, there exists a collection of scalars $(c_{v_i})_{v_i \in S}$ such that
$$v = \sum_{v_i \in S} c_{v_i} \cdot v_i.$$
- We say that S is *linearly independent* if for all linear combinations
$$\sum_{v_i \in S} c_{v_i} \cdot v_i = 0,$$
we have $c_{v_i} = 0$ for all $v_i \in S$.
- We say that S is a *basis* for V if S spans V and is linearly independent.

For $\mathbb{R}^n$, a basis is given by $\{e_1, e_2, \dots, e_n\}$, with
$$e_i(j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$
for $1 \leq i \leq n$. This basis is not unique, e.g. another basis for $\mathbb{R}^n$ is $\{f_1, f_2, \dots, f_n\}$, with
$$f_i = \sum_{j=1}^{i} e_j.$$
Although the basis is not unique, if it is finite, then any other basis will also be finite and have the same number of elements. This value is defined the

dimension of the vector space. Vector spaces that have a basis with finitely many elements are called *finite-dimensional*. We know that for a field $\mathbb{K}$, if V is an n-dimensional vector space over $\mathbb{K}$, then V is isomorphic to $\mathbb{K}^n$. So, these are all the finite-dimensional vector spaces.

We can represent the vector space $\mathbb{R}^n$ as a function. In particular, for some set $X = \{x_1, x_2, \dots, x_n\}$, let $\mathrm{Fun}(X, \mathbb{R})$ be the set of functions $f : X \to \mathbb{R}$. We claim that $\mathrm{Fun}(X, \mathbb{R})$ is isomorphic to $\mathbb{R}^n$, with the isomorphism map $\varphi : \mathrm{Fun}(X, \mathbb{R}) \to \mathbb{R}^n$

$$\varphi(f) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

Note however that this format is not limited to finite sets; the space of functions $\mathrm{Fun}(X, \mathbb{R})$ is a vector space even when X is infinite. In particular, we consider the case where X is countable, i.e. $X = \mathbb{Z}_{\geq 1}$. The space $\mathrm{Fun}(X, \mathbb{R})$ in this case is the space of all functions $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$, i.e. sequences in $\mathbb{R}$. We denote this as $\mathrm{Seq}(\mathbb{R})$ as well. The sequences form a vector space with respect to pointwise addition and scalar multiplication. This sequence is infinite-dimensional, i.e. it does not have a finite basis. This is because it has the basis $\{e^{(1)}, e^{(2)}, \dots\}$, with the sequence $(e_n^{(k)})_{n=1}^\infty$ given by

$$e_n^{(k)} = \begin{cases} 1 & n = k \\ 0 & \text{otherwise.} \end{cases}$$

We know that every basis of a finite-dimensional space is finite, so $\mathrm{Seq}(\mathbb{R})$ is infinite-dimensional.

Also, the space of continuous functions from the compact subset $[0, 1]$ to $\mathbb{R}$, denoted by $C[0, 1]$, is a vector space- it forms a vector space over pointwise addition and scalar multiplication, i.e. for $c \in \mathbb{R}$ and $f \in C[0, 1]$, we define the function $c \cdot f \in C[0, 1]$ by $(c \cdot f)(x) = c \cdot f(x)$ for $x \in [0, 1]$. This is also an infinite-dimensional space- it has a subspace consisting of polynomial functions, whose basis is given by

$$\{f_n \mid n \in \mathbb{Z}_{\geq 1}\},$$

where $f_n(x) = x^n$ for all $x \in [0, 1]$. Hence, it has an infinite-dimensional subspace, meaning that the entire space must also be infinite-dimensional. We will later see that the space of polynomials is a dense subspace of $C[0, 1]$, i.e. a continuous function can be approximated by a polynomial function arbitrarily well.

1.2 Metrics, Norms and Inner Products

In this section, we will expand the algebraic vector space properties and connect them with analytic ones. In particular, we will look at metrics in vector spaces, and then a stronger concept of norms, and finally inner product spaces.

Definition 1.2.1 (Metric spaces)**.** Let V be a set and let $d\colon V \times V \to \mathbb{R}_{\geq 0}$ be a function. We say that (V, d) is a *metric space* if:

- for all $u, v \in V$, $d(u, v) = 0$ if and only if $u = v$;
- for all $u, v \in V$, $d(u, v) = d(v, u)$;
- for all $u, v, w \in V$, $d(u, w) \leq d(u, v) + d(v, w)$.

If (V, d) is a metric space, we call d a *metric*.

The function d represents a distance function; it allows us to measure distance between two values in V.

There are many examples of metric spaces. In $\mathbb{R}^n$, the following are 3 different norms:

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^{n} |x_i - y_i| \\ d_2(x, y) &= \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \\ d_\infty(x, y) &= \max_{i=1}^{n} |x_i - y_i|. \end{aligned}$$

In general, we can define the d_p-metric for $p \in [1, \infty)$ as follows:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

We can define a lot more metrics on $\mathbb{R}^n$, such as the discrete metric:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise.} \end{cases}$$

We would like to consider a structure that behaves better with the structure of a vector space, like the d_p-metrics. This gives rise to a norm.

Definition 1.2.2 (Normed Vector Space)**.** Let V be a vector space and let $\|\cdot\|\colon V \times V \to \mathbb{R}_{\geq 0}$ be a function. We say that $(V, \|\cdot\|)$ is a *normed vector space* if:

- for all $v \in V$, $\|v\| = 0$ if and only if $v = 0$;
- for all $v \in V$ and $\lambda \in \mathbb{C}$, $\|\lambda v\| = |\lambda| \|v\|$;
- for all $u, v \in V$, $\|u + v\| \leq \|u\| + \|v\|$.

If $(V, \|\cdot\|)$ is a normed vector space, we call $\|\cdot\|$ a *norm*.

The norm function allows us to measure the magnitude of a vector.

In $\mathbb{R}^n$, we have many norms, such as $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ given as follows:

$$\begin{aligned}\|x\|_1 &= \sum_{i=1}^{n} |x_i| \\ \|x\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \\ \|x\|_\infty &= \max_{i=1}^{n} |x_i|.a\end{aligned}$$

These norms are quite closely related to the d_1, d_2 and d_∞-metrics respectively. It turns out that every norm induces a metric, given by

$$d(x, y) = \|x - y\|.$$

However, it is not the case that every metric is induced by a metric, e.g. the discrete metric is not induced by a norm.

We will now look at some norms in infinite-dimensional vector spaces. In particular, if we look at the space of sequences $\mathrm{Seq}(\mathbb{R})$, we can define the norms in a similar manner as above, i.e.

$$\begin{aligned}\|(x_n)\|_1 &= \sum_{n=1}^{\infty} |x_n| \\ \|(x_n)\|_2 &= \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2} \\ \|(x_n)\|_\infty &= \sup_{n=1}^{\infty} |x_n|.\end{aligned}$$

These norms are not defined for all sequences, e.g. the sequence of positive integers has infinite norm with respect to all 3 norms. So, we restrict the norm to those sequences that have a finite value. In particular, we define the following sequence spaces:

- the sequence space ℓ^1, composed of sequences that converge absolutely;
- the sequence space ℓ^2, composed of sequences $(x_n)_{n=1}^\infty$ such that the series
  $$\sum_{n=1}^{\infty} x_n^2$$
  converges;
- the sequence space ℓ^p, composed of sequences $(x_n)_{n=1}^\infty$ such that the series
  $$\sum_{n=1}^{\infty} |x_n|^p$$
  converges;

- the sequence space ℓ^∞, composed of bounded sequences.

Note the following relations between the sequence spaces:

- the space ℓ^∞ contains all convergent sequences, i.e. a convergent sequence is bounded;
- the space ℓ^∞ contains ℓ^p for all $p \in [1, \infty)$[1];
- the space $\ell^p \subseteq \ell^q$ if $p < q$.

We can also define norms in $C[0, 1]$, given as follows:

$$\begin{aligned} \|f\|_1 &= \int_0^1 |f(t)| \, dt \\ \|f\|_2 &= \left(\int_0^1 (f(t))^2 \, dt \right)^{1/2} \\ \|f\|_\infty &= \sup_0^1 |f(t)|. \end{aligned}$$

We will now add even more structure to a vector space, by defining an inner product.

Definition 1.2.3 (Inner Product Space)**.** Let V be a vector space and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be a function. We say that $(V, \langle \cdot, \cdot \rangle)$ is an *inner product space* if:

- for all $v \in V$, $\langle v, v \rangle \in [0, \infty)$ and $\langle v, v \rangle = 0$ if and only if $v = 0$;
- for all $u, v \in V$ and $\lambda \in \mathbb{C}$, $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$;
- for all $v, w \in V$, $\langle v, w \rangle = \overline{\langle w, v \rangle}$;
- for all $u, v, w \in V$, $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$.

If $(V, \langle \cdot, \cdot, \rangle)$ is an inner product space, we call $\langle \cdot, \cdot \rangle$ is an *inner product.*

The inner product allows us to measure angles between two vectors. In particular, the concept of orthogonality gives rise to many powerful results for Hilbert spaces (complete inner product spaces) that do not necessarily hold in Banach spaces (complete normed vector spaces).

In $\mathbb{R}^n$ and $\mathbb{C}^n$, the dot product is an example of an inner product, which is given by

$$\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}.$$

This inner product induces the $\|\cdot\|_2$ norm. In particular, an inner product induces a metric, given by

$$\|x\| = \langle x, x \rangle^{1/2}.$$

To prove this, we require the Cauchy-Schwartz Inequality.

Theorem 1.2.4 (Cauchy-Schwartz Inequality)**.** *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $v, w \in V$, $|\langle v, w \rangle|^2 \leq \langle v, v \rangle \cdot \langle w, w \rangle$.*

[1] In fact, we know that a sequence in ℓ^p converges to 0.

Proof. Let $v, w \in V$. If $w = 0$, then the statement is trivial. Otherwise,

$$\begin{aligned}
\langle v, v\rangle - \frac{|\langle v, w\rangle|^2}{\langle w, w\rangle} &= \langle v, v\rangle - \frac{\langle v, w\rangle\overline{\langle v, w\rangle}}{\langle w, w\rangle} \\
&= \langle v, v\rangle - \frac{\langle v, w\rangle^2}{\langle w, w\rangle} - \frac{\langle v, w\rangle\langle w, v\rangle}{\langle w, w\rangle} + \frac{\langle v, w\rangle^2}{\langle w, w\rangle} \\
&= \left\langle v, v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w\right\rangle - \left(\frac{\langle v, w\rangle}{\langle w, w\rangle}\langle w, v\rangle - \frac{\langle v, w\rangle^2}{\langle w, w\rangle^2}\langle w, w\rangle\right) \\
&= \left\langle v, v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w\right\rangle - \left\langle \frac{\langle v, w\rangle}{\langle w, w\rangle} w, v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w\right\rangle \\
&= \left\langle v - \frac{\langle v, v\rangle}{\langle w, w\rangle} w, v - \frac{\langle v, v\rangle}{\langle w, w\rangle} w\right\rangle \geq 0.
\end{aligned}$$

Hence,

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle.$$

□

It is not the case that every inner product is induced by a norm; this is only true for norms that satisfy the Parallelogram identity.

Proposition 1.2.5. *Let $(V, \|\cdot\|)$ be a normed vector space that satisfies the Parallelogram identity, i.e. for all $u, v \in V$,*

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 - \|u - v\|^2.$$

For $\mathrm{Seq}(\mathbb{R})$, an inner product is given by

$$\langle (x_n), (y_n)\rangle = \sum_{n=1}^{\infty} x_i \overline{y_i}.$$

In $C[0, 1]$, we have

$$\langle f, g\rangle = \int_0^1 |f(t)g(t)| \; dt.$$