CHAPTER 1

REVIEW OF 3H ALGEBRA

1.1 Isomorphism Theorems

Theorem 1.1.1 (First Isomorphism Theorem)**.** *Let G and H be groups, and let $\varphi : G \to H$ be a homomorphism. Then, $G/\ker\varphi \cong \operatorname{Im}(\varphi)$.*

Proof. Let $H = \ker\varphi$. Define the map $\psi : G/H \to \operatorname{Im}(\varphi)$ by $\psi(gH) = \varphi(g)$. Let $g_1H, g_2H \in G/H$. We know that $g_2^{-1}g_1 \in H$, and so $\varphi(g_1) = \varphi(g_2)$. So, ψ is well-defined. Moreover, since φ is a homomorphism, we find that ψ is a homomorphism. Also, by construction, ψ is surjective.

Now, we claim that ψ is injective. Let $g_1H, g_2H \in G/H$ such that $\psi(g_1H) = \varphi(g_2H)$. In that case, $\varphi(g_1) = \varphi(g_2)$. Hence, $g_2^{-1}g_1 \in H$, meaning that $g_1H = g_2H$. This implies that ψ is injective. So, ψ defines an isomorphism. □

Theorem 1.1.2 (Second Isomorphism Theorem)**.** *Let G be a group, and let $H, N \leq G$ with $N \lhd G$. Then, $HN \leq G$, $H \cap N \lhd H$, and*

$$H/(H \cap N) \cong HN/N.$$

Proof. Define the map $\varphi : H \to H/N$ by $\varphi(h) = hN$. This is a homomorphism, with

$$\ker\varphi = \{g \in H \mid \varphi(g) = N\} = \{g \in H \mid g \in N\} = H \cap N,$$

and

$$\operatorname{Im}\varphi = \{hN \mid h \in H\} = HN/N.$$

Hence,

$$H/(H \cap N) \cong HN/N.$$

□

Theorem 1.1.3 (Correspondence Theorem for Subgroups)**.** *Let G be a group, and let $N \lhd G$. Then, there exists a bijection $f : S \to X$, where S is the set of subgroups of G containing N, and X is the set of subgroups of G/N.*

Proof. Let $q : G \to G/N$ be the quotient map. Define the map $f : S \to X$ by

$$f(H) = q(H) = \{hN \mid h \in H\} =: H/N.$$

We show that f is bijective. Let $L \leq G/N$. Then, set

$$K = q^{-1}(L) = \{g \mid gN \in L\}.$$

We have $N \in L$, so $N \leq K$. This implies that $K \in S$. Moreover,

$$gN \in L \iff g \in K \iff gN \in K/N.$$

So, $L = K/N$. This implies that f is surjective. Also, for $H/N = K/N$, we have

$$g \in H \iff gN \in H/N \iff gN \in K/N \iff g \in K.$$

So, $H = K$. This implies that f is injective as well. Hence, f is a bijection. □

Theorem 1.1.4 (Third Isomorphism Theorem)**.** *Let G be a group, and let $H, K \lhd G$, wth $K \leq H$. Then,*

$$(G/K)/(H/K) \cong G/H.$$

Proof. Define the map $\psi : G/K \to G/H$ by $\psi(gK) = gH$. For $g_1K, g_2K \in G/H$, if $g_1K = g_2K$, then $g_2^{-1}g_1 \in K \subseteq H$. So, $g_1H = g_2H$, meaning that ψ is well-defined. Moreover, the map ψ is surjective by construction. The map ψ is also a homomorphism by definition of quotients. Now,

$$\ker\psi = \{gK \in G/K \mid gK = H\} = \{gK \in G/K \mid g \in H\} = H/K.$$

So, the First Isomorphism Theorem tells us that

$$(G/K)/(H/K) \cong G/H.$$

□

1.2 Intersection, Product and Join

Proposition 1.2.1. *Let G be a group and $H, K \leq G$ with $H \lhd G$. Then, $HK \leq G$.*

Definition 1.2.2. Let G be a group and $H, K \leq G$. Then, the *join* of H and K is given by

$$H \wedge K := \bigcap_{\substack{N \leq G \\ H, K \leq N}} N.$$

Proposition 1.2.3. *Let G be a group and $H, K \leq G$. Then, $HK = H \wedge K$ if and only if $HK \leq G$.*

Proof. If $HK = H \wedge K$, then $HK \leq G$. So, assume that $HK \leq G$. We have $H, K \leq HK$, so $H \wedge K \leq HK$ by definition. Now, let $hk \in HK$ and $N \leq G$ such that $H, K \leq N$. Then, $h, k \in N$, meaning that $hk \in N$. Hence, $hk \in H \wedge K$. So, $HK = H \wedge K$. □

Proposition 1.2.4. *Let G be a group and $H, K \leq G$ be finite. Then,*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Proof. □

1.3 Composition Series

Definition 1.3.1. Let G be a group, and let $H_i \leq G$ for all $i \in \{1, \ldots, n-1\}$. We say that

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$$

is a *group series* if $H_i \leq H_{i+1}$ for all $i \in \{0, \ldots, n-1\}$. The group series

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$$

is a *normal* series if $H_i \lhd G$ for all $i \in \{0, \ldots, n-1\}$. Also, the group series

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$$

is *subnormal* if $H_i \lhd H_{i+1}$ for all $i \in \{0, \ldots, n-1\}$.

Definition 1.3.2. Let G be a group, and let

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$$

be a subnormal series. We say that the group series is a *composition series* if for all $n \in \{0, \ldots, n-1\}$, H_i/H_{i+1} is simple. If

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$$

is a normal series such that for all $n \in \{0, \ldots, n-1\}$, H_i/H_{i+1} is simple, then the group series is a *principal series*.

Proposition 1.3.3. *$\mathbb{Z}$ has no composition series.*

Proof. Let the following be a subnormal series for $\mathbb{Z}$:

$$\{0\} = G_0 \lhd G_1 \lhd \ldots G_n = \mathbb{Z}.$$

We know that the subgroup $G_1 = m\mathbb{Z}$, for some $m \in \mathbb{Z}$. Then, the quotient $G_1/G_0 \cong m\mathbb{Z}$ is not simple. So, the subnormal series is not a composition series. □

Theorem 1.3.4 (Jordan-Holder Theorem)**.**

Proof. □

Definition 1.3.5. Let G be a group. We say that G is *solvable* if there exists a normal series

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$$

such that for all $i \in \{1, 2, \ldots, n-1\}$, H_i/H_{i+1} is abelian.

Example 1.3.6. *The group S_5 is not solvable.*

Proof. □