CHAPTER 4

FREE GROUPS

4.1 Introduction to Free Groups

Definition 4.1.1. Let S be a set, and fix a set S^- disjoint to S with a bijection $f\colon S \to S^-$, and a singleton set $\{e\}$ disjoint to $S \cup S^-$. Denote $X_S = S \cup S^- \cup \{e\}$. We define the *inverse map* $-1\colon X_S \to X_S$ by

$$s^{-1} = \begin{cases} e & s = e \\ f(s) & s \in S \\ f^{-1}(s) & s \in S^-. \end{cases}$$

Definition 4.1.2. Let S be a set. A *word* on S is an infinite tuple $(s_1, s_2, \dots)$ with values in X_S such that there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $s_n = e$. A *reduced word* on S is a word $(s_1, s_2, \dots)$ such that:

- if $s_N = e$ for some $N \geq 1$, then $s_n = e$ for all $n \geq N$;
- if $s_n \neq e$, then $s_{n+1} \neq s_n^{-1}$ for all $n \in \mathbb{Z}_{\geq 1}$.

We denote a reduced word $(s_1, s_2, \dots, s_n, e, e, \dots)$ by $s_1 s_2 \dots s_n$, where $s_n \neq e$. The set of all reduced words is denoted by $F(S)$. We also denote $e = (e, e, e, \dots)$, and call it *empty word*.

Definition 4.1.3. Let S be a set. Define the operation $\cdot\colon F(S) \to F(S)$ as follows: let $s_1 \dots s_m, t_1 \dots t_n \in F(S)$, with $m \leq n$. Fix $k \in \mathbb{Z}$ such that $s_{m-k+1} \neq t_k^{-1}$. Then,

$$(s_1 \dots s_m) \cdot (t_1 \dots t_n) = \begin{cases} s_1 \dots s_{m-k+1} t_k \dots t_n & k \leq m \\ t_{m+1} \dots t_n & k = m+1 \leq n \\ 1 & k = m+1, m = n \end{cases}$$

We can similarly define the operation if $m \geq n$. The operation is called *concatenation.*

Proposition 4.1.4. *Let S be a set. Then, $F(S)$ is a group under concatenation.*

Proof. By definition, we have an identity element, and every element in $F(S)$ has an inverse:

$$(s_1 \dots s_m)^{-1} = s_m^{-1} \dots s_1^{-1}.$$

We now show that the operation is associative. First, for $s \in X_S$, define the map $\sigma_s\colon F(S) \to F(S)$ by:

$$\sigma_s(s_1 \dots s_n) = \begin{cases} s s_1 \dots s_n & s^{-1} \neq s_1 \\ s_2 \dots s_n & \text{otherwise.} \end{cases}$$

We note that $\sigma_s \circ \sigma_{s^{-1}} = id$, meaning that σ_s is a bijection. Hence, $\sigma_s \in \mathrm{Perm}(F(S))$. So, define

$$B(S) = \langle \sigma_s \mid s \in X_S \rangle \leq \mathrm{Perm}(F(S)),$$

and the map $f\colon F(S) \to B(S)$ by

$$f(s_1 \dots s_n) = \sigma_{s_1} \dots \sigma_{s_n}.$$

Since f obeys concatenation in $B(S)$, and $B(S)$ is associative, it follows that concatenation is associative. Hence, $F(S)$ is a group. □

Proposition 4.1.5 (Universal Property of Free Groups)**.** *Let S be a set, G be a group, and $f : S \to G$ be a map. Then, there exists a unique homomorphism $\varphi : F(S) \to G$ such that $\varphi(s) = f(s)$ for all $s \in S$.*

Proof. Define the map $\varphi\colon F(S) \to G$ by

$$\varphi(s_1 s_2 \dots s_n) = f(s_1) f(s_2) \dots f(s_n).$$

By construction, this is a group homomorphism. Moreover, it extends f by definition.

Now, let $\psi\colon F(S) \to G$ be such that $\psi(s) = f(s)$ for all $s \in S$. In that case, for all $s_1 s_2 \dots s_n \in F(S)$, we find that

$$\begin{aligned} \psi(s_1 s_2 \dots s_n) &= \psi(s_1)\psi(s_2)\dots\psi(s_n) \\ &= f(s_1) f(s_2) \dots f(s_n) \\ &= \varphi(s_1 s_2 \dots s_n). \end{aligned}$$

So, the map is unique. □

Corollary 4.1.6. *Let S be a set, with free groups $F_1(S)$ and $F_2(S)$. Then, there exists a unique isomorphism $\phi\colon F_1(S) \to F_2(S)$ that fixes S.*

Proof. Let $\iota_1\colon S \hookrightarrow F_1(S)$ and $\iota_2\colon S \hookrightarrow F_2(S)$ be the inclusion maps. We can apply the universal property of the free group $F_2(S)$ on the map ι_1 to extend it to a unique homomorphism $\varphi_1\colon F_1(S) \to F_2(S)$. Similarly, we can construct a homomorphism $\varphi_2\colon F_2(S) \to F_1(S)$. Note that, by construction, φ_1 and φ_2 fix S. Now, consider the map $\varphi_2 \circ \varphi_1\colon F_1(S) \to F_1(S)$. This is a group homomorphism that fixes S. We can apply again the universal property of the free group $F_1(S)$ on the map ι_1 to extend it to a unique homomorphism $\psi\colon F_1(S) \to F_1(S)$. Note that the identity map is also a homomorphism $\psi\colon F_1(S) \to F_1(S)$, so by uniqueness we find that $\psi = \varphi_2 \circ \varphi_1$ are the identity map on $F_1(S)$. Similarly, $\varphi_1 \circ \varphi_2$ is the identity map on $F_2(S)$. Hence, φ_1 is an isomorphism with inverse φ_2^{-1}. By construction, the map is unique and fixes S. □

Definition 4.1.7. Let S be a set. We say that $F(S)$ is the *free group* on S. We say that S is the set of *free generators* (or *free basis*) of $F(S)$. The *rank* of the free group $F(S)$ is the cardinality of S.

Proposition 4.1.8. *A free group of rank 0 is isomorphic to the trivial group.*

Proof. Let G be a free group of rank 0. In that case, G is the free group on $\varnothing$. Hence, it has precisely one element- the identity. $\square$

Proposition 4.1.9. *A free group of rank 1 is isomorphic to* $\mathbb{Z}$.

Proof. Let F be a free group of rank 1, and let F be generated by some $x \in F$. In that case, we know that every $y \in F$ is of the form $y = x^n$, and so $y \in \langle x \rangle$. Hence, $F = \langle x \rangle$. Since x has infinite order, it follows that F is isomorphic to $\mathbb{Z}$. $\square$

Proposition 4.1.10. *A free group of rank* $n \geq 2$ *is not abelian.*

Proof. Let F be a free group of rank n. Consider distinct elements $a, b \in F$ that generate F. In that case, we know that in $F(S)$, $ab \neq ba$. Hence, $F(S)$ is not abelian. $\square$

Proposition 4.1.11. *A free group has no torsion elements.*

Proof. Let F be a free group, and let $x \in F$ be non-trivial. $\square$

Theorem 4.1.12 (Neilson-Schrier Theorem)**.** *Let* F *be a free group and let* $G \leq F$*. Then,* G *is free.*

4.2 Group Relations and Presentation

Lemma 4.2.1. *Let G be a group. Then, G is the image of some free group. In particular, there exists a free group F and a surjective group homomorphism $\varphi\colon F \to G$.*

Proof. Consider the free group $F(G)$. By the universal propery of free groups on the identity map $id\colon G \to G$, we can extend it to a group homomorphism $\varphi\colon F(G) \to G$. By construction, we know that $\varphi(g) = g$ for all $g \in G$, meaning that φ is surjective. □

Definition 4.2.2. Let G be a group and let $R \subseteq G$. Then, the *normal closure* of R is the intersection of all normal subgroups of G containing R. It is denoted by $\langle\langle R\rangle\rangle$.

Proposition 4.2.3. *Let G be a group and let $R \subseteq G$. Then, $\langle\langle R\rangle\rangle$ is the subgroup generated by the conjugates of R.*

Proof. Since the normal closure $\langle\langle R\rangle\rangle$ is normal, we know that the conjugates of R are in the subgroup. Moreover, a subgroup generated by the conjugates of R is closed under conjugation by construction, meaning that it is normal, and contains R. Hence, it is contained in $\langle\langle R\rangle\rangle$. So, the normal closure is the subgroup generated by the conjugates of R. □

Proposition 4.2.4. *Let G, H be groups, $R \subseteq G$ and let $\varphi\colon G \to H$ be a homomorphism with $R \subseteq \ker\varphi$. Then, $\langle\langle R\rangle\rangle \leq \ker\varphi$. In particular, $\langle\langle R\rangle\rangle$ is the smallest unique kernel of a group homomorphism that sends R to the identity.*

Proof. Since $\ker\varphi$ is a normal subgroup, and $R \subseteq \ker\varphi$, it follows that $\langle\langle R\rangle\rangle \leq \ker\varphi$. □

Definition 4.2.5. Let G be a group and S a generating set of G. A *presentation* is a pair (S, R), where R is a set of words in $F(S)$ such that the normal closure $\langle\langle R\rangle\rangle$ is the kernel of the homomorphism $\varphi\colon F(S) \to G$ that fixes S. The set R is called the *relators*. We denote $G = \langle S \mid R\rangle$.

We say that G is *finitely presented* if there exists a presentation of G, (S, R), such that both S and R are finite. We say that G is *finitely generated* if there exists a presentation of G, (S, R), such that S is finite.

Proposition 4.2.6. *Let G be a finite group. Then, G is finitely presented.*

Proof. Let $S = G$, and define the set

$$R = \{ghk^{-1} \in F(S) \mid g, h \in G, gh = h\}.$$

Since G is finite, it follows that R is finite. We claim that $G = \langle S \mid R\rangle$.

Let N be the normal closure of R in $F(S)$. Consider the group $H = F(S)/N$. We know that there is an extension of the inclusion map $\psi\colon F(S) \to G$- this follows from the universal property of free groups. Moreover, $N \leq \ker\psi$, so the universal property of the quotient gives us a map $\varphi\colon H \to G$. By construction, we find that $\{gN \mid g \in G\}$ generates H. Since G is a group, it is also closed under the binary operation. Hence,

$$H = \{gN \mid g \in G\}.$$

In particular, $|H| = |G|$. Hence, we find that $N = \ker \psi$. This implies that $\langle S \mid R\rangle$ is a presentation for G. So, G is finitely presented. □

Proposition 4.2.7. *Let G be a group with the presentation*

$$G = \langle a, b \mid r_1, \ldots, r_k\rangle.$$

Then, for a group H generated by two elements $x, y \in H$ that satisfy the relations $r_1, \ldots, r_k$, there exists a group homomorphism $\Phi\colon G \to H$.

Proof. Consider the presentation homomorphism $\varphi\colon F(a, b) \to G$. This is surjective since it extends the identity map on $\{a, b\}$. Now, consider the map $f\colon \{a, b\} \to H$ given by $f(a) = x$ and $f(b) = y$. By the universal property of free groups, we can extend f to a group homomorphism $\psi\colon F(a, b) \to H$. We know that H satisfies the relations $r_1, \ldots, r_k$, so we have $\ker \varphi \leq \ker \psi$. So, the universal property of quotients tells us that there exists a group homomorphism $\Phi\colon G \to H$. □

Proposition 4.2.8. *There is one non-abelian group of order 10 up to isomorphism.*

Proof. Let G be a non-abelian group of order 10. By Cauchy's Theorem, we can find a $g \in G$ of order 5. Set $H = \langle g\rangle$. We know that H has index 2 in G, so it is a normal subgroup of G. Now, let $k \in G$ such that $k \notin H$. We know that $kH \neq H$, so kH must have order 2. Hence, $k^2 \in H$. If $|k^2| = 5$, then $|k| = 10$, meaning that G is cyclic. This cannot be the case, so we must have $k^2 = e$. Since H is normal in G, we find that $kgk^{-1} \in H$. Since G is not abelian, k and g cannot commute. Hence, $kgk^{-1} \in \{g^2, g^3, g^4\}$. We consider each case separately:

- First, assume that $kgk^{-1} = g^2$. In that case,

$$\begin{aligned} g &= eg \\ &= k^2 g \\ &= k \cdot kg \\ &= k \cdot g^2 k \\ &= kg \cdot gk \\ &= g^2 k \cdot gk \\ &= g^2 \cdot kg \cdot k \\ &= g^2 \cdot g^2 k \cdot k \\ &= g^4 k^2 = g^4. \end{aligned}$$

  Hence, $g^3 = e$. Since g has order 5, this is a contradiction.

- Now, assume that $kgk^{-1} = g^3$. In that case,
$$\begin{aligned} g &= k \cdot kg \\ &= k \cdot g^3 k \\ &= kg \cdot g^2 k \\ &= g^3 k \cdot g^2 k \\ &= g^3 \cdot kg \cdot gk \\ &= g^3 \cdot g^3 k \cdot gk \\ &= g \cdot kg \cdot k \\ &= g \cdot g^3 k \cdot k \\ &= g^4 k^2 = g^4. \end{aligned}$$
  Like above, this is a contradiction.
- Finally, assume that $kgk^{-1} = g^4 = g^{-1}$. In that case, the presentation for G is:
$$G = \langle g, k \mid g^5 = e = k^2, kgk^{-1} = g^{-1} \rangle.$$
  We know that
$$D_5 = \langle r, s \mid r^5 = e = s^2, rs = sr^{-1} \rangle.$$
  So, D_4 and G have isomophic presentations, meaning that $G \cong D_5$.

Hence, there is only one non-abelian group of order 10. □