CHAPTER 1

NORMED VECTOR SPACES

1.1 Introduction to norms

Distance in vector spaces

In $\mathbb{R}^n$, we have seen the following metrics:

- d_1, which is defined by

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} |x_i - y_i|;$$

- d_2, which is defined by

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2};$$

- d_∞, which is defined by

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \sup_{i=1}^{n} |x_i - y_i|;$$

In general, for $p \in [1, \infty)$, we can define the metric d_p, given by

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/p}.$$

We can define metrics on other vector spaces, such as the space of sequences in $\mathbb{R}$ (or any field $\mathbb{K}$). The space of sequences in some field $\mathbb{K}$ is denoted by $\mathrm{Seq}(\mathbb{K})$. For each $p \in [1, \infty)$, we define the set ℓ^p to be a subspace of $\mathrm{Seq}(\mathbb{R})$ for which the distance function d_p is defined. That is, the set ℓ^p contains of all sequences $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that the series

$$\sum_{i=1}^{\infty} |x_i|^p$$

converges- if so, we define the distance between $x^{(n)}$ and $y^{(n)}$ in ℓ^p to be:

$$d_p(x^{(n)}, y^{(n)}) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}.$$

We also define ℓ^∞ to be the set of all bounded sequences in $\mathbb{R}$, with

$$d_\infty(x^{(n)}, y^{(n)}) = \sup_{i=1}^{n} |x_i - y_i|.$$

Further, we can define norms on the set of continuous functions $C[0,1]$. The distance between two functions $f, g \in C[0,1]$ is defined to be:

$$d_p(f,g) = \left(\int_0^1 |f(x) - g(x)|^p \, dx \right)^{1/p}.$$

The value is always finite for f, g continuous, since $|f-g|^p$ is a continuous function.

Norms

We now define a norm for vector spaces.

Definition 1.1.1. Let V be a real vector space and let $\|.\| : V \to \mathbb{R}$ be a function. We say that $\|.\|$ is a *norm* if

N1. for all $\boldsymbol{v} \in V$, $\|\boldsymbol{v}\| \geq 0$, and $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$;

N2. for all $c \in \mathbb{R}$ and $\boldsymbol{v} \in V$, $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$;

N3. for all $\boldsymbol{u}, \boldsymbol{v} \in V$,
$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|.$$

For every norm, there exists a metric; every norm induces a metric.

Proposition 1.1.2. *Let V be a real vector space and let $\|.\|$ be a norm. Define the function $d : V \times V \to \mathbb{R}$ by $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$. Then, (V, d) is a metric space.*

Proof. We check this directly from the axioms of a metric space:

M1. Let $\boldsymbol{x}, \boldsymbol{y} \in V$. We find that

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| \geq 0.$$

Moreover, $d(\boldsymbol{x}, \boldsymbol{y}) = 0$ if and only if $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{0}$, i.e. $\boldsymbol{x} = \boldsymbol{y}$.

M2. Let $\boldsymbol{x}, \boldsymbol{y} \in V$. We find that

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\| = \|-(\boldsymbol{y} - \boldsymbol{x})\| = |-1|\|\boldsymbol{y} - \boldsymbol{x}\| = \|\boldsymbol{y} - \boldsymbol{x}\| = d(\boldsymbol{y}, \boldsymbol{x}).$$

M3. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$. We find that

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{z}) &= \|\boldsymbol{x} - \boldsymbol{z}\| \\ &= \|(\boldsymbol{x} - \boldsymbol{y}) + (\boldsymbol{y} - \boldsymbol{z})\| \\ &\leq \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y} - \boldsymbol{z}\| \\ &= d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}). \end{aligned}$$

So, (V, d) is a metric space. □

However, the converse is not true- not every distance function corresponds to a norm. Before proving this, we characterise the relationship between the distance and the norm in the other direction.

Proposition 1.1.3. *Let V be a real vector space and let d be a metric induced by some norm $\|.\| : V \to \mathbb{R}$. Then, for all $\boldsymbol{v} \in V$, $\|\boldsymbol{v}\| = d(\mathbf{0}, \boldsymbol{v})$.*

Proof. Since the norm $\|.\|$ induces the metric d, we know that for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$. In that case, for $\boldsymbol{v} \in V$,

$$\|\boldsymbol{v}\| = \|\boldsymbol{v} - \mathbf{0}\| = d(\boldsymbol{v}, \mathbf{0}) = d(\mathbf{0}, \boldsymbol{v}).$$

□

Now, we give a counterexample to show that not every metric is induced by some norm. Let d be the discrete metric on $\mathbb{R}$. We claim that $(\mathbb{R}, d)$ is not induced by a norm. Assume, for a contradiction, that there exists a norm $\|.\| : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $d(x, y) = \|x - y\|$. In that case,

$$2 = 2d(1, 0) = 2\|1\| = \|2\| = d(2, 0) = 1.$$

This is a contradiction. So, $(\mathbb{R}, d)$ is not induced by a norm.

Strong equivalence of norms

For a real vector space, we say that two norms on it are strongly equivalent if the induced metrics are strongly equivalent. We know that in $\mathbb{R}^n$, all the ℓ^p norms are strongly equivalent. However, this is not true for sequence and function spaces.

For sequence spaces, the sets ℓ^p are not equal for distinct p. In particular, $\ell^1 \neq \ell^2$. To see this, consider the sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ given by $x_n = 1/n$. Since the harmonic series diverges, we know that (x_n) is not in ℓ^1. However, since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, the sequence is in ℓ^2. So, $\ell^1 \neq \ell^2$.

Nonetheless, we do have $\ell^1 \subseteq \ell^2$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ with (x_n) in ℓ^1. In that case, the series

$$\sum_{n=1}^{\infty} x_n$$

converges. Therefore, we must have that $x_n \to 0$, i.e. (x_n) is convergent. So, there exists a $K > 0$ such that $|x_n| \leq K$ for all $n \in \mathbb{Z}_{\geq 1}$. In that case, for $n \in \mathbb{Z}_{\geq 1}$,

$$0 \leq |x_n|^2 \leq K|x_n|.$$

So, the comparison test tells us that the series

$$\sum_{n=1}^{\infty} x_n^2$$

converges. This implies that (x_n) is in ℓ^2. Therefore, $\ell^1 \subseteq \ell^2$. [1]

Using the result above, we show that ℓ_1 and ℓ_2 are not equivalent in ℓ^1.[2] To see this, define the sequence $(x^{(k)})_{k=1}^{\infty}$ in ℓ^2 by

$$x^{(k)} = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{k}, 0, 0, \dots).$$

[1] Can we use this approach to show that $\ell^2 \subseteq \ell^3$?

[2] We refer to the set of sequences by ℓ^p, and the corresponding norm by ℓ_p.

For $k \in \mathbb{Z}_{\geq 1}$, we have

$$\|x^{(k)}\|_1 = \sum_{n=1}^{k} \frac{1}{n}.$$

Since the sum is finite, we know that $x^{(k)}$ is in ℓ^1. Now, define also the sequence $(x_n)_{n=1}^{\infty}$ by $x_n = \frac{1}{n}$. We have (x_n) in ℓ^2. Moreover, under the ℓ_2 norm,

$$\|x^{(k)} - x\|_2 = \sum_{n=k+1}^{\infty} \frac{1}{n^2} \to 0$$

as $k \to \infty$. This implies that $(x^{(k)})$ is Cauchy in (ℓ^1, d_1). Now, we show that $(x^{(k)})$ is not Cauchy in (ℓ^1, d_2). Let $N \in \mathbb{Z}_{\geq 1}$. Since the harmonic series diverges, we must have

$$\sum_{n=N}^{\infty} \frac{1}{n} \to \infty.$$

So, there exists an $M \in \mathbb{Z}_{>N}$ such that

$$\sum_{n=N}^{M} \frac{1}{n} > 1.$$

In that case,

$$\|x^{(N)}, x^{(M+1)}\|_1 = \sum_{n=N}^{\infty} \frac{1}{n} - \sum_{n=M+1}^{\infty} \frac{1}{n} = \sum_{n=N}^{M} \frac{1}{n} \geq 1.$$

Therefore, $(x^{(k)})$ is not Cauchy in (ℓ^1, d_2). This implies that ℓ_1 and ℓ_2 are not equivalent in ℓ^1.

Similarly, the norms on the function space $C[0,1]$ are not strongly equivalent. For example, consider the sequence $(f_n(x))_{n=1}^{\infty}$ given by $f_n(x) = x^n$. We know that x^n is increasing on $[0,1]$. So,

$$\|f_n\|_\infty = \sup_{x \in [0,1]} |f_n(x)| = f_n(1) = 1.$$

Moreover,

$$\|f_n\|_1 = \int_0^1 f_n(x)\ dx = \int_0^1 x^n\ dx = \left[\frac{1}{n+1} x^{n+1}\right]_0^1 = \frac{1}{n+1} \to 0$$

as $n \to \infty$. This implies that the two norms are not strongly equivalent.

1.2 Introduction to inner products

Definition 1.2.1. Let V be a real vector space and let $\langle .,. \rangle : V \times V \to \mathbb{R}$ be a function. Then, $\langle .,. \rangle$ is an *inner product* if:

I1. for all $\boldsymbol{v} \in V$, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$, with $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$;

I2. for all $\lambda \in \mathbb{R}$ and $\boldsymbol{v}, \boldsymbol{w} \in V$, $\langle \lambda \boldsymbol{v}, \boldsymbol{w} \rangle = \lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle$;

I3. for all $\boldsymbol{v}, \boldsymbol{w} \in V$, $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$;

I4. for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$,

$$\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle.$$

In $\mathbb{R}^n$, the inner product is the dot product, given by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=1}^{n} x_k y_k$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. For sequences, the inner product is the generalisation of the dot product, given by

$$\langle x^{(n)}, y^{(n)} \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

The sequence space where this series is convergent is precisely ℓ^2. In $C[0,1]$, the inner product is given by

$$\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx.$$

We can define this not just on $C[0,1]$, but also on functions f and g where fg is integrable- this set is called $L^2[0,1]$. In all cases, this follows from the Cauchy-Schwarz inequality.

Proposition 1.2.2 (Cauchy-Schwarz inequality)**.** *Let V be a real vector space and let $\langle .,. \rangle : V \times V \to \mathbb{R}$ be an inner product on V. Then, for all $\boldsymbol{v}, \boldsymbol{w} \in V$,*

$$|\langle \boldsymbol{v}, \boldsymbol{w} \rangle|^2 \leq \langle \boldsymbol{v}, \boldsymbol{v} \rangle \cdot \langle \boldsymbol{w}, \boldsymbol{w} \rangle.$$

Proof. Let $\lambda = \frac{\langle \boldsymbol{v}, \boldsymbol{w} \rangle}{\langle \boldsymbol{w}, \boldsymbol{w} \rangle}$. We know that

$$\langle \boldsymbol{v} - \lambda \boldsymbol{w}, \boldsymbol{v} - \lambda \boldsymbol{w} \rangle \geq 0.$$

We also have

$$\langle \boldsymbol{v} - \lambda \boldsymbol{w}, \boldsymbol{v} - \lambda \boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{v} \rangle - 2\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \lambda^2 \langle \boldsymbol{w}, \boldsymbol{w} \rangle.$$

Treating this as a quadratic on λ, we know that it has at most one real root. So, the discriminant is non-positive. That is,

$$\begin{aligned}
(2\langle \boldsymbol{v}, \boldsymbol{w} \rangle)^2 - 4\langle \boldsymbol{w}, \boldsymbol{w} \rangle \cdot \langle \boldsymbol{v}, \boldsymbol{v} \rangle &\leq 0 \\
4\langle \boldsymbol{v}, \boldsymbol{w} \rangle^2 &\leq 4\langle \boldsymbol{w}, \boldsymbol{w} \rangle \cdot \langle \boldsymbol{v}, \boldsymbol{v} \rangle \\
|\langle \boldsymbol{v}, \boldsymbol{w} \rangle|^2 &\leq \langle \boldsymbol{w}, \boldsymbol{w} \rangle \cdot \langle \boldsymbol{v}, \boldsymbol{v} \rangle.
\end{aligned}$$

□

An inner product induces a norm.

Proposition 1.2.3. *Let V be a real vector space and let $\langle .,. \rangle : V \times V \to \mathbb{R}$ be an inner product on V. Then, define the function $\|.\| : V \to \mathbb{R}$ by*

$$\|\boldsymbol{v}\| = \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{1/2}.$$

Then, $\|.\|$ defines a norm on V.

Proof.

N1. Let $\boldsymbol{v} \in V$. We know that $\|\boldsymbol{v}\| \geq 0$ for all $\boldsymbol{v} \in V$. Moreover, if $\|\boldsymbol{v}\| = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$, then $\boldsymbol{v} = \boldsymbol{0}$.

N2. Let $c \in \mathbb{R}$ and $\boldsymbol{v} \in V$. We find that

$$\|c\boldsymbol{v}\|^2 = \langle c\boldsymbol{v}, c\boldsymbol{v} \rangle = c^2 \langle \boldsymbol{v}, \boldsymbol{v} \rangle = c^2 \|\boldsymbol{v}\|^2.$$

So, $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$.

N3. Let $\boldsymbol{u}, \boldsymbol{v} \in V$. We find that

$$\begin{aligned} \|\boldsymbol{u} + \boldsymbol{v}\|^2 &= \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle \\ &= \langle \boldsymbol{u}, \boldsymbol{u} \rangle + 2\langle \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{v}, \boldsymbol{v} \rangle \\ &\leq \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 \\ &\leq (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2. \end{aligned}$$

This implies that $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$.

So, $\|.\|$ is a norm on V. □

Definition 1.2.4. Let V be a real vector space with inner product $\langle .,. \rangle$. Then, for $\boldsymbol{v}, \boldsymbol{w} \in V$, we say that $\boldsymbol{v}$ and $\boldsymbol{w}$ are *orthogonal* if $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$.

Proposition 1.2.5 (Pythagoras' Theorem)**.** *Let V be a real vector space with inner product $\langle .,. \rangle$, let $\|.\|$ be the norm induced by the inner product, and let $\boldsymbol{v}, \boldsymbol{w} \in V$ be orthogonal. Then,*

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2.$$

Proof. We find that

$$\begin{aligned} \|\boldsymbol{v} + \boldsymbol{w}\|^2 &= \langle \boldsymbol{v} + \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w} \rangle \\ &= \langle \boldsymbol{v}, \boldsymbol{v} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \langle \boldsymbol{w}, \boldsymbol{v} \rangle + \langle \boldsymbol{w}, \boldsymbol{w} \rangle \\ &= \langle \boldsymbol{v}, \boldsymbol{v} \rangle + \langle \boldsymbol{w}, \boldsymbol{w} \rangle \\ &= \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2. \end{aligned}$$

□

Proposition 1.2.6 (Parallelogram Law)**.** *Let V be a real vector space with inner product $\langle .,. \rangle$, and let $\|.\|$ be the norm induced by the inner product. Then, for all $\boldsymbol{v}, \boldsymbol{w} \in V$,*

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 + \|\boldsymbol{v} - \boldsymbol{w}\|^2 = 2\|\boldsymbol{v}\|^2 + 2\|\boldsymbol{w}\|^2.$$

Conversely, if a norm satisfies the equality above for all $\boldsymbol{v}, \boldsymbol{w} \in V$, then it is induced by an inner product.

Proof.

- First, assume that $\|.\|$ is induced by an inner product. Let $\boldsymbol{v}, \boldsymbol{w} \in V$. Then,

$$\begin{aligned}\|\boldsymbol{v}+\boldsymbol{w}\|^2 + \|\boldsymbol{v}-\boldsymbol{w}\|^2 &= \langle \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{v}+\boldsymbol{w}\rangle + \langle \boldsymbol{v}-\boldsymbol{w}, \boldsymbol{v}-\boldsymbol{w}\rangle \\ &= \langle \boldsymbol{v}, \boldsymbol{v}\rangle + \langle \boldsymbol{v}, \boldsymbol{w}\rangle + \langle \boldsymbol{w}, \boldsymbol{v}\rangle + \langle \boldsymbol{w}, \boldsymbol{w}\rangle \\ &+ \langle \boldsymbol{v}, \boldsymbol{v}\rangle - \langle \boldsymbol{v}, \boldsymbol{w}\rangle - \langle \boldsymbol{w}, \boldsymbol{v}\rangle + \langle \boldsymbol{w}, \boldsymbol{w}\rangle \\ &= 2\langle \boldsymbol{v}, \boldsymbol{v}\rangle + 2\langle \boldsymbol{w}, \boldsymbol{w}\rangle \\ &= 2\|\boldsymbol{v}\|^2 + 2\|\boldsymbol{w}\|^2.\end{aligned}$$

- Now, assume that for all $\boldsymbol{v}, \boldsymbol{w} \in V$,

$$\|\boldsymbol{v}+\boldsymbol{w}\|^2 + \|\boldsymbol{v}-\boldsymbol{w}\|^2 = 2\|\boldsymbol{v}\|^2 + 2\|\boldsymbol{w}\|^2.$$

  Define the function $\langle ., . \rangle : V \times V \to \mathbb{R}$ by

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \frac{1}{4}(\|\boldsymbol{v}+\boldsymbol{w}\|^2 - \|\boldsymbol{v}-\boldsymbol{w}\|^2).$$

  We claim that $\langle ., . \rangle$ is an inner product on V and that the inner product induces the norm $\|.\|$.

  I1. Let $\boldsymbol{v} \in V$. We have

$$\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \frac{1}{4} \cdot \|2\boldsymbol{v}\|^2 = \|\boldsymbol{v}\|^2 \geq 0.$$

  Moreover, if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$, then $\boldsymbol{v} = \boldsymbol{0}$.

  I2. Let $\lambda \in \mathbb{R}$ and $\boldsymbol{v}, \boldsymbol{w} \in V$. Then,

$$\begin{aligned}\langle \lambda\boldsymbol{v}, \boldsymbol{w}\rangle &= \frac{1}{4}(\|\lambda\boldsymbol{v}-\boldsymbol{w}\|^2 - \|\lambda\boldsymbol{v}-\boldsymbol{w}\|^2) \\ &= \frac{1}{4}(2\|\lambda\boldsymbol{v}\|^2 + 2\|\boldsymbol{w}\|^2 - 2\|\lambda\boldsymbol{v}-\boldsymbol{w}\|^2)\end{aligned}$$

  I3. Let $\boldsymbol{v}, \boldsymbol{w} \in V$. Then,

$$\begin{aligned}\langle \boldsymbol{v}, \boldsymbol{w}\rangle &= \frac{1}{4}(\|\boldsymbol{v}+\boldsymbol{w}\|^2 - \|\boldsymbol{v}-\boldsymbol{w}\|^2) \\ &= \frac{1}{4}(\|\boldsymbol{w}+\boldsymbol{v}\|^2 - \|\boldsymbol{w}-\boldsymbol{v}\|^2) \\ &= \langle \boldsymbol{w}, \boldsymbol{v}\rangle.\end{aligned}$$

  I4. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$. Then,

$$\langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{w}\rangle =$$

□

Using this result, we can show that the norm $\|.\|_1$ on $\mathbb{R}^2$ is not induced by a norm. Let $\boldsymbol{v} = [1, 0]$ and $\boldsymbol{w} = [0, 1]$. Then, we have

$$\|\boldsymbol{v}+\boldsymbol{w}\|_1^2 + \|\boldsymbol{v}-\boldsymbol{w}\|_1^2 = \|[1,1]\|_1^2 + \|[1,-1]\|_1^2 = 4 + 4 = 8,$$

but

$$2\|\boldsymbol{v}\|_1 + 2\|\boldsymbol{w}\|_1 = 2\|[1,0]\|_1^2 + 2\|[0,1]\|_1^2 = 2 + 2 = 4.$$

We can use the same values to show that $\|.\|_\infty$ on $\mathbb{R}^2$ is not induced by a norm.

1.3 Sequence spaces

In this section, we will have a look at sequence spaces in more detail- we will show that they are vector spaces and characterise the containment of sequence spaces. First, we will consider the sequence spaces:

- For $p \in [1, \infty)$, the sequence space ℓ^p contains sequences $(x_n)_{n=1}^\infty$ in $\mathbb{R}$ such that the series
$$\sum_{k=1}^\infty |x_k|^p$$
converges. We show that ℓ^p is a vector space.
  - Let $(x_n)_{n=1}^\infty$ be in ℓ^p and $\lambda \in \mathbb{R}$. Since (x_n) is in ℓ^p, the series
  $$\sum_{n=1}^\infty |x_n|^p$$
  converges. In that case,
  $$\sum_{n=1}^\infty |\lambda x_n|^p = |\lambda|^p \sum_{n=1}^\infty |x_n|^p.$$
  So, the series (λx_n) is in ℓ^p.
  - Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be in ℓ^p. Since (x_n) and (y_n) are in ℓ^p, the series
  $$\sum_{n=1}^\infty |x_n|^p, \qquad \sum_{n=1}^\infty |y_n|^p$$
  converge. In that case, for all $n \in \mathbb{Z}_{\geq 1}$,
  $$(|x_n| + |y_n|)^p = |x_n|^p + \cdots + |y_n|^p$$
  So, the comparison test tells us that
  $$\sum_{n=1}^\infty |x_n + y_n|^p$$
  converges. So, the series $(x_n + y_n)$ is in ℓ^p.
- The sequence space ℓ^∞ is the set where the infinity norm is defined, i.e. for a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}$, the supremum
$$\sup_{k=1}^\infty |x_n|$$
exists. Therefore, the space is precisely the set of all bounded sequences. We show that ℓ^∞ is a vector space.
  - Let $(x_n)_{n=1}^\infty$ be in ℓ^∞ and $\lambda \in \mathbb{R}$. Since (x_n) is in ℓ^∞, there exists a $K > 0$ such that for all $n \in \mathbb{Z}_{\geq 1}$, $|x_n| \leq K$. In that case, for all $n \in \mathbb{Z}_{\geq 1}$, $|\lambda x_n| \leq |\lambda| K$. So, the sequence (λx_n) is in ℓ^∞.

  - Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be in ℓ^∞. In that case, there exist $K_1, K_2 > 0$ such that for all $n \in \mathbb{Z}_{\geq 1}$, $|x_n| \leq K_1$ and $|y_n| \leq K_2$. In that case, for all $n \in \mathbb{Z}_{\geq 1}$,
$$|x_n + y_n| \leq |x_n| + |y_n| \leq K_1 + K_2.$$
    So, the sequence $(x_n + y_n)$ is in ℓ^∞.

- The sequence space c is the set of convergent sequences. We show that c is a vector space.
  - Let $(x_n)_{n=1}^\infty$ be in c and $\lambda \in \mathbb{R}$. In that case, the sequence $x_n \to L$, for some $L \in \mathbb{R}$. Therefore, $\lambda x_n \to \lambda L$. This implies that (λx_n) is in c.
  - Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be in c. In that case, we have $x_n \to L_1$ and $y_n \to L_2$, for some $L_1, L_2 \in \mathbb{R}$. Therefore, $x_n + y_n \to L_1 + L_2$. This implies that $(x_n + y_n)$ is in c.
- The sequence space c_0 is the set of convergent sequences that converge to 0. We show that c_0 is a vector space.
  - Let $(x_n)_{n=1}^\infty$ be in c_0 and $\lambda \in \mathbb{R}$. In that case, the sequence $x_n \to 0$. Therefore, $\lambda x_n \to 0$. This implies that (λx_n) is in c_0.
  - Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be in c. In that case, we have $x_n \to 0$ and $y_n \to 0$. Therefore, $x_n + y_n \to 0$. This implies that $(x_n + y_n)$ is in c_0.
- Finally, the sequence space c_{00} is the set of convergent sequences $(x_n)_{n=1}^\infty$ such that there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $x_n = 0$.[3] We show that c_{00} is a vector space.
  - Let $(x_n)_{n=1}^\infty$ be in c_{00} and $\lambda \in \mathbb{R}$. In that case, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $x_n = 0$. So, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $\lambda x_n = 0$. So, (λx_n) is in c_{00}.
  - Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be in c_{00}. In that case, there exist $N_1, N_2 \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N_1$, then $x_n = 0$, and if $n \geq N_2$, then $y_n = 0$. So, set $N = \max(N_1, N_2)$. Then, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $x_n + y_n = 0$. This implies that $(x_n + y_n)$ is in c_{00}.

The following is the containment relationship between the sequence spaces:
$$c_{00} \subseteq c_0 \subseteq c \subseteq \ell^\infty.$$
Moreover, the quotient vector space c/c_0 is isomorphic to $\mathbb{R}$- this is because the relation $\sim$ on c given by
$$(x_n) \sim (y_n) \iff (x_n) \text{ and } (y_n) \text{ converge to the same limit}$$
corresponds to $x_n - y_n \to 0$, i.e. $(x_n - y_n)$ in c_0. In other words, it is the kernel of the surjective vector space homomorphism $v : c \to \mathbb{R}$ where $v(x) = L$, with $x_n \to L$. Then, the result follows from the first isomorphism theorem.

[3] That is, it is the set of sequences that eventually become 0.

Now, let $p \in [1, \infty)$. We will place ℓ^p in the containment relationship. Let $(x_n)_{n=1}^\infty$ be in ℓ^p. We know that the series

$$\sum_{n=1}^\infty |x_n|^p$$

converges. In that case, we have $|x_n|^p \to 0$. Therefore, $x_n \to 0$. So, $\ell^p \subseteq c_0$. However, $\ell^p \not\subseteq c_{00}$ since there are infinite sums that converge. So, we have $c_{00} \subseteq \ell^p \subseteq c_0$.

Next, let $p, q \in [1, \infty)$ with $p < q$. We show that $\ell^p \subseteq \ell^q$. We start with a lemma.

Lemma 1.3.1. *Let $p \in [1, \infty)$, and let $(x_n)_{n=1}^\infty$ be in ℓ^p. Then, $\|x\|_\infty \leq \|x\|_p$.*

Proof. Let $\varepsilon > 0$. Since

$$\|x\|_\infty = \sup_{n=1}^\infty |x_n|,$$

we can find a $k \in \mathbb{Z}_{\geq 1}$ such that $|x_k| \geq \|x\|_\infty - \varepsilon$. In that case,

$$\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} \geq (|x_k|^p)^{1/p} = |x_k| \geq \|x\|_\infty - \varepsilon.$$

So, for all $\varepsilon > 0$, $\|x\|_p \geq \|x\|_\infty - \varepsilon$. In that case, we must have $\|x\|_\infty \leq \|x\|_p$. ☐

Now, we prove that $\ell^p \subseteq \ell^q$.

Proposition 1.3.2. *Let $p, q \in [1, \infty)$ with $p < q$. Then, $\ell^p \subseteq \ell^q$.*

Proof. Let $(x_n)_{n=1}^\infty$ be in ℓ^p. We find that

$$\begin{aligned}
\|x\|_q^q &= \sum_{n=1}^\infty |x_n|^q \\
&= \sum_{n=1}^\infty |x_k|^p \cdot |x_n|^{q-p} \\
&\leq \sum_{n=1}^\infty |x_n|^p \cdot \|x\|_\infty^{q-p} \\
&= \|x\|_\infty^{q-p} \sum_{n=1}^\infty |x_n|^p \\
&= \|x\|_\infty^{q-p} \|x\|_p^p \\
&= \|x\|_p^{q-p} \|x\|_p^p \\
&= \|x\|_p^q.
\end{aligned}$$

This implies that $\|x\|_q \leq \|x\|_p$. So, (x_n) is in ℓ^q. Therefore, $\ell^p \subseteq \ell^q$. ☐

Finally, we have the complete characterisation of the containment:

$$c_{00} \subseteq \ell^1 \subseteq \ell^2 \subseteq \cdots \subseteq c_0 \subseteq c \subseteq \ell^\infty.$$

At each level, the containment is strict.

1.4 Topology of vector spaces

Compactness

We know that a subspace $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded. This does not hold in sequence spaces, under any norm. First, we consider the ℓ_∞ norm. Let

$$A = \{(x_n) \in \ell^\infty \mid \|x\|_\infty \leq 1\}.$$

This is a closed space since the norm map is continuous. Moreover, it is bounded by the triangle inequality. However, we claim that the space is not compact. To see this, let $(e_k)_{k=1}^\infty$ be a sequence in ℓ^∞ given by

$$e_k = \begin{cases} 0 & n \neq k \\ 1 & n = k \end{cases}.$$

By definition, for all $m, n \in \mathbb{Z}_{\geq 1}$, if $m \neq n$, then $\|e_m - e_n\|_\infty = 1$. This implies that any subsequence cannot be Cauchy, and so is not convergent. Therefore, A is not sequentially compact. Since the topology on A is induced by a metric, this further implies that A is not compact. This is true for other norms as well.

Definition 1.4.1. Let X be a real vector space, and let $(f_n)_{n=1}^\infty$ be a sequence of functions $f_n : X \to \mathbb{R}$, and let $f : X \to \mathbb{R}$. We say that (f_n) *converges to f pointwise* if for all $x \in X$, $f_n(x) \to f(x)$. In other words, for every $x \in X$ and $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$.

Definition 1.4.2. Let X be a real vector space, and let $(f_n)_{n=1}^\infty$ be a sequence of functions $f_n : X \to \mathbb{R}$, and let $f : X \to \mathbb{R}$. We say that (f_n) *converges to f uniformly* if for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $x \in X$ and $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$.

Proposition 1.4.3. *Let X be a normed vector space, and let $(f_n)_{n=1}^\infty$ be a sequence of continuous functions $f_n : X \to \mathbb{R}$, and let $f : X \to \mathbb{R}$ such that $f_n \to f$ uniformly. Then, f is continuous.*

Proof. Let $x \in X$ and $\varepsilon > 0$. Since $f_n \to f$ uniformly, we can find an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$ and $a \in X$, if $n \geq N$, then $|f_n(a) - f(a)| < \frac{\varepsilon}{3}$. Moreover, since f_N is continuous, we can find a $\delta > 0$ such that for $y \in X$, if $\|x - y\| < \delta$, then $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$. In that case, for $y \in X$, if $\|x - y\| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This implies that f is continuous. □

Definition 1.4.4. Let $K \subseteq C[0,1]$. We say that K is *equi-continuous* if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $f \in K$ and $s, t \in [0,1]$, if $|s - t| < \delta$, then $|f(s) - f(t)| < \varepsilon$.

Theorem 1.4.5 (Arzela-Ascoli)**.** *Let $K \subseteq C[0,1]$. Then, K is compact if and only if K is closed, bounded and equi-continuous, under the L_∞ norm.*

Proof.

- First, assume that K is compact. We know that K is closed and bounded. We show that K is equi-continuous. So, let $\varepsilon > 0$. We know that $(B_K(f, \frac{\varepsilon}{3}))_{f \in K}$ is an open cover of K. So, it has a finite subcover $(B_K(f_i, \frac{\varepsilon}{3}))_{i=1}^n$. For $i \in \{1, 2, \dots, n\}$, we can find a $\delta_i > 0$ such that for $s, t \in [0, 1]$, if $|s - t| < \delta$, then $|f_i(s) - f_i(t)| < \frac{\varepsilon}{3}$. Set $\delta = \min_{i=1}^n \delta_i$. Now, let $g \in K$. We can find a $j \in \{1, 2, \dots, n\}$ such that
$$g \in B_K(f_j, \tfrac{\varepsilon}{3}).$$
In that case, $\|f_j - g\|_\infty < \frac{\varepsilon}{3}$. So, for all $x \in [0, 1]$, $|f_j(x) - g(x)| < \frac{\varepsilon}{3}$. Now, for $s, t \in [0, 1]$, if $|s - t| < \delta$, then
$$\begin{aligned} |g(s) - g(t)| &\leq |g(s) - f_j(s)| + |f_j(s) - f_j(t)| + |f_j(t) - g(s)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$
So, K is equi-continuous.
- Now, assume that K is closed, bounded and equi-continuous. Let $(f_n)_{n=1}^\infty$ be a sequence in K. We show that (f_n) has a convergent subsequence.

□

Separability

Definition 1.4.6. Let X be a metric space. We say that X is *separable* if it contains a countable dense subset.

We know that $\mathbb{R}$ is separable- $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. In general, $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$.

Proposition 1.4.7. *The sequence space ℓ^∞ is not separable.*

Proof. Let $C \subseteq \mathbb{Z}_{\geq 1}$. Define the sequence $(x_n^C)_{n=1}^\infty$ in $\mathbb{R}$ by
$$x_n^C = \begin{cases} 1 & n \in C \\ 0 & n \notin C \end{cases}.$$
We have (x_n) in ℓ^∞. Moreover, for $P, Q \subseteq \mathbb{N}$ with $P \neq Q$, $\|x^P - x^Q\|_\infty = 1$. In that case,
$$B_{\ell^\infty}(x^P, \tfrac{1}{2}) \cap B_{\ell^\infty}(x^Q, \tfrac{1}{2}) = \varnothing.$$
So, the set
$$S = \{x^C \mid C \subseteq \mathbb{Z}_{\geq 1}\}$$
is not dense in ℓ^∞- □

Proposition 1.4.8. *The function space $C[0, 1]$ is separable.*

Proof. We show that polynomials is a countable subset in $C[0, 1]$. The space of polynomials is the union of polynomials of degree n, for all $n \in \mathbb{Z}_{\geq 1}$. The polynomials of degree n are isomorphic to $\mathbb{R}^{n+1}$ as vector spaces. So, the space of polynomials is countable. Moreover, the space of polynomials is dense- we will not prove this. For this reason, $C[0, 1]$ is separable. □

Completeness

Definition 1.4.9. Let V be a normed vector space such that the metric induced by the norm is complete. Then, we say that V is a *Banach space.*

Definition 1.4.10. Let V be a inner product space such that the metric induced by the norm is complete. Then, we say that V is a *Hilbert space.*

The sequence space c_{00} is not complete. Consider the sequence $(x^{(n)})_{n=1}^{\infty}$ in c_{00}, given by

$$x^{(n)} = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, 0, 0, \dots).$$

Under the ℓ_∞ norm, this sequence converges to

$$x = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)$$

since

$$\|x^{(n)} - x\|_\infty = \|(0, 0, \dots, 0, -\tfrac{1}{n+1}, -\tfrac{1}{n+2}, \dots)\|_\infty = \tfrac{1}{n+1} \to 0.$$

But, the sequence x is not in c_{00} since it is not eventually 0. So, $(x^{(n)})$ is Cauchy in c_{00} but not convergent- it is not complete.

However, this is not the case for ℓ^∞.

Proposition 1.4.11. *The sequence space ℓ^∞ is complete.*

Proof. Let $(x^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in ℓ^∞. For $k \in \mathbb{Z}_{\geq 1}$, we have the sequence $(x_k^{(n)})_{n=1}^{\infty}$ in $\mathbb{R}$. We claim that $(x_k^{(n)})$ is Cauchy in $\mathbb{R}$. Let $\varepsilon > 0$. Since $(x^{(n)})$ is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then

$$\|x^{(n)} - x^{(m)}\|_\infty < \varepsilon.$$

In that case,

$$|x_k^{(n)} - x_k^{(m)}| \leq \sup_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}| = \|x^{(n)} - x^{(m)}\|_\infty < \varepsilon.$$

So, $(x_k^{(n)})$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists an $x_k \in \mathbb{R}$ such that $x_k^{(n)} \to x_k$. So, we can construct the sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}$, where $x_k^{(n)} \to x_k$ for $k \in \mathbb{Z}_{\geq 1}$.

First, we claim that (x_k) is in ℓ^∞. Since $(x^{(n)})$ is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $\|x^{(m)} - x^{(n)}\|_\infty < 1$. In particular, for all $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $|x_k^{(m)} - x_k^{(n)}| < 1$ for all $k \in \mathbb{Z}_{\geq 1}$. Since $x^{(N)}$ is in ℓ^∞, there exists a $K > 0$ such that for all $k \in \mathbb{Z}_{\geq 1}$, $|x_k^{(N)}| \leq K$. Now, let $k \in \mathbb{Z}_{\geq 1}$. Since $x_k^{(n)} \to x_k$, there exists an $N' \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N'$, then $|x_k - x_k^{(n)}| < 1$. Now, fix $n = \max(N, N')$. In that case,

$$|x_k| \leq |x_k - x_k^{(n)}| + |x_k^{(n)} - x_k^{(N)}| + |x_k^{(N)}| \leq 2 + K.$$

So, (x_k) is in ℓ^∞.

Now, we claim that $x^{(n)} \to x$ under the ℓ_∞ metric. Let $\varepsilon > 0$. Since $(x^{(n)})$ is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $\|x^{(m)} - x^{(n)}\|_\infty < \frac{\varepsilon}{3}$. In particular, for all $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $|x_k^{(m)} - x_k^{(n)}| < \frac{\varepsilon}{3}$ for all $k \in \mathbb{Z}_{\geq 1}$. Now, let $n \in \mathbb{Z}_{\geq 1}$ with $n \geq N$. Moreover, let $k \in \mathbb{Z}_{\geq 1}$. Since $x_k^{(n)} \to x_k$, there exists an $N' \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N'$, then $|x_k^{(n)} - x_k| < \frac{\varepsilon}{3}$. Now, fix $m = \max(N, N')$. Then,

$$|x_k^{(n)} - x_k| \leq |x_k^{(n)} - x_k^{(m)}| + |x_k^{(m)} - x_k| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2}{3}\varepsilon.$$

Therefore, for all $k \in \mathbb{Z}_{\geq 1}$, $|x_k^{(n)} - x_k| < \frac{2}{3}\varepsilon$. By the supremum property, this implies that

$$\|x^{(n)} - x\|_\infty \leq \frac{2}{3}\varepsilon < \varepsilon.$$

So, for all $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then

$$\|x^{(n)} - x\|_\infty < \varepsilon.$$

This implies that $x^{(n)} \to x$. Therefore, ℓ^∞ is complete. □

Using this, we can show that c is complete.

Proposition 1.4.12. *The sequence space $c \subseteq \ell^\infty$ is closed.*

Proof. Let $(x^{(n)})_{n=1}^\infty$ be a convergent sequence in c with $x^{(n)} \to x$, for some x in ℓ^∞. We show that x is Cauchy. Let $\varepsilon > 0$. Since $x^{(n)} \to x$, we can find an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $\|x^{(n)} - x\|_\infty < \frac{\varepsilon}{3}$. In particular, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|x_k^{(n)} - x_k| < \frac{\varepsilon}{3}$ for all $k \in \mathbb{Z}_{\geq 1}$. Since $x^{(N)}$ is in c, we know that $x^{(N)}$ is Cauchy. In that case, there exists an $N' \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N'$, then $|x_m^{(N)} - x_n^{(N)}| < \frac{\varepsilon}{3}$. Therefore, for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N'$, then

$$|x_m - x_n| \leq |x_m - x_m^{(N)}| + |x_m^{(N)} - x_n^{(N)}| + |x_n^{(N)} - x_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This implies that x is Cauchy. Since x is a sequence in $\mathbb{R}$, we find that x is in c. So, c is closed. □

This shows that c is complete- it is a closed subset of a complete space.

We use a similar approach to show that c_0 is complete.

Proposition 1.4.13. *The sequence space $c_0 \subseteq c$ is closed.*

Proof. Let $(x^{(n)})_{n=1}^\infty$ be a convergent sequence in c_0 with $x^{(n)} \to x$, for some x in c. We show that $x_k \to 0$. Let $\varepsilon > 0$. Since $x^{(n)} \to x$, we can find an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $\|x^{(n)} - x\|_\infty < \frac{\varepsilon}{2}$. In particular, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|x_k^{(n)} - x_k| < \frac{\varepsilon}{2}$ for all $k \in \mathbb{Z}_{\geq 1}$. Since $x^{(N)}$ is in c_0, we find that $x^{(N)} \to 0$. In that case, there exists an $N' \in \mathbb{Z}_{\geq 1}$ such that for $k \in \mathbb{Z}_{\geq 1}$, if $k \geq N'$, then $|x_k^{(N)}| < \frac{\varepsilon}{2}$. In that case, for all $k \in \mathbb{Z}_{\geq 1}$, if $k \geq N'$, then

$$|x_k| \leq |x_k - x_k^{(N)}| + |x_k^{(N)}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that $x_k \to 0$. So, x is in c_0, i.e. c_0 is closed. □

Finally, we show that $C[0,1]$ is complete.

Proposition 1.4.14. *The function space $C[0,1]$ is complete.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $C[0,1]$. For $x \in [0,1]$, the sequence $(f_n(x))_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$. We show that $(f_n(x))$ is Cauchy. Let $\varepsilon > 0$. Since (f_n) is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $\|f_m - f_n\|_\infty < \varepsilon$. In that case, for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then

$$|f_m(x) - f_n(x)| \leq \sup_{y \in [0,1]} |f_m(y) - f_n(y)| = \|f_m - f_n\|_\infty < \varepsilon.$$

This implies that $(f_n(x))$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists an $f_x \in \mathbb{R}$ such that $f_n(x) \to f_x$. So, we can construct the function $f : [0,1] \to \mathbb{R}$ by $f(x) = f_x$.

First, we claim that f is in $C[0,1]$. Let $x \in [0,1]$, and $\varepsilon > 0$. Since (f_n) is Cauchy, we can find an $K \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq K$, then $\|f_m - f_n\|_\infty < \frac{\varepsilon}{5}$. In particular, for $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq K$, then $|f_m(y) - f_n(y)| < \frac{\varepsilon}{5}$ for all $y \in [0,1]$. Since f_K is in $C[0,1]$, we can find a $\delta > 0$ such that for $y \in [0,1]$, if $|x - y| < \delta$, then $|f_K(x) - f_K(y)| < \frac{\varepsilon}{5}$. Since $f_n(x) \to f(x)$, we can find an $N_2 \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N_2$, then $|f_n(x) - f(x)| < \frac{\varepsilon}{5}$. Set $N = \max(N_1, N_2)$. Next, let $y \in [0,1]$ with $|x - y| < \delta$. Since $f_n(y) \to f(y)$, we can find an $N_3 \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N_3$, then $|f_n(y) - f(y)| < \frac{\varepsilon}{5}$. Set $M = \max(N_1, N_3)$. Therefore,

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_K(x)| + |f_K(x) - f_K(y)| \\ &\quad + |f_K(y) - f_M(y)| + |f_M(y) - f(y)| \\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.\end{aligned}$$

That is, for all $y \in [0,1]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. This implies that f is in $C[0,1]$.

Now, we claim that $f_n \to f$ under the ℓ_∞ metric. Let $\varepsilon > 0$. Since (f_n) is Cauchy, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $m, n \in \mathbb{Z}_{\geq 1}$, then $\|f_m - f_n\|_\infty < \frac{\varepsilon}{3}$. In particular, for all $m, n \in \mathbb{Z}_{\geq 1}$, if $m, n \geq N$, then $|f_m(x) - f_n(x)| < \frac{\varepsilon}{3}$ for all $x \in [0,1]$. Now, let $n \in \mathbb{Z}_{\geq 1}$ with $n \geq N$. Moreover, let $x \in [0,1]$. Since $f_n(x) \to f(x)$, there exists an $N' \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N'$, then $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. Now, fix $m = \max(N, N')$. Then,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2}{3}\varepsilon.$$

Therefore, for all $x \in [0,1]$, $|f_n(x) - f(x)| < \frac{2}{3}\varepsilon$. By the supremum property, this implies that

$$\|f_n - f\|_\infty \leq \frac{2}{3}\varepsilon < \varepsilon.$$

So, for all $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then

$$\|f_n - f\|_\infty < \varepsilon.$$

This implies that $f_n \to f$. Therefore, $C[0,1]$ is complete. □