CHAPTER 1

RINGS AND ALGEBRAS

1.0 Recap of Set Theory

Definition 1.0.1. Let X be a set. Then, the *powerset of* X is the set of subsets of X, and is denoted by $\mathcal{P}(X)$.

Definition 1.0.2. Let X be a set, and let $(X_i)_{i\in I}$ be a collection of subsets of X, for some indexing set I. We define the *union* to be:

$$\bigcup_{i\in I} X_i = \{x \in X \mid \exists i \in I \text{ s.t. } x \in X_i\}.$$

Similarly, we define the *intersection* to be:

$$\bigcap_{n=1}^{\infty} A_n = \{x \in X \mid \forall i \in I \text{ s.t. } x \in X_i\}.$$

Definition 1.0.3. Let X be a set, and let $A \subseteq X$. We define the *complement of* A to be:

$$A^c = X \setminus A = \{x \in X \mid x \notin A\}.$$

Proposition 1.0.4 (De Morgan Law)**.** *Let A and B be sets. Then,*

$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c.$$

In general, for a collection of sets $(A_i)_{i\in I}$, where I is an index set,

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c, \qquad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Definition 1.0.5. Let S be a set. We say that S *is countable* if either S is empty, or there exists a surjective function $f : \mathbb{Z}_{\geq 1} \to S$. If so, we can denote

$$S = \{f(1), f(2), f(3), \dots\}.$$

Proposition 1.0.6. *Let S be a countable set, and let $T \subseteq S$. Then, T is countable.*

Proposition 1.0.7. *Let S be a countably infinite set. Then, there exists a bijective function $f : \mathbb{Z}_{\geq 1} \to S$.*

Proposition 1.0.8. *Let S and T be countable sets. Then, their union $S \cup T$ is countable.*

Proposition 1.0.9. *Let $(S_n)_{n=1}^{\infty}$ be a sequence of countable sets. Then, their union*

$$\bigcup_{n=1}^{\infty} S_n$$

is countable.

Proposition 1.0.10. *Let S and T be countable sets. Then, the product $S \times T$ are countable.*

Corollary 1.0.11. *The set $\mathbb{Q}$ is countable.*

Proposition 1.0.12. *The set $[0, 1]$ is not countable.*

Definition 1.0.13. Let A and B be sets. We say that $|A| = |B|$ if there exists a bijection $f : A \to B$. If there exists an injective function $f : A \to B$, then we say that $|A| \leq |B|$.

1.1 Rings and Algebras

Definition 1.1.1. Let X be a set. We say that $\mathcal{R} \subseteq \mathcal{P}(X)$ is a *ring* (of subsets of X) if:

- $\varnothing \in \mathcal{R}$;
- for all $A, B \in \mathcal{R}$, the difference $A \setminus B \in \mathcal{R}$;
- for all $A, B \in \mathcal{R}$, the union $A \cup B \in \mathcal{R}$.

Proposition 1.1.2. *Let X be a set, and let $\mathcal{R} \subseteq \mathcal{P}(X)$ be a ring. Then, for $A, B \in \mathcal{R}$, the intersection $A \cap B \in \mathcal{R}$.*

Proof. □

Definition 1.1.3. Let X be a set. We say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *algebra* (of subsets of X) if $\mathcal{A}$ is a ring with $X \in \mathcal{A}$.

Proposition 1.1.4. *Let X be a set, and $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, $\mathcal{A}$ is an algebra if and only if:*

- *$\varnothing \in \mathcal{A}$;*
- *for all $A \in \mathcal{A}$, the complement $A^c \in \mathcal{A}$; and*
- *for all $A, B \in \mathcal{A}$, the union $A \cup B \in \mathcal{A}$.*

Definition 1.1.5. Let X be a set. We say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *σ-algebra* (of subsets of X) if $\mathcal{A}$ is an algebra such that for all $(A_n)_{n=1}^{\infty}$ in $\mathcal{A}$, the union

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Proposition 1.1.6. *Let X be a set, and $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, $\mathcal{A}$ is a σ-algebra if and only if:*

- *$\varnothing \in \mathcal{A}$;*
- *for all $A \in \mathcal{A}$, the complement $A^c \in \mathcal{A}$; and*
- *for a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{A}$, the union*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Proposition 1.1.7. *Let X be a set, and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. Then, for a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{A}$, the intersection*

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}.$$

1.2 Borel Sets

Definition 1.2.1. We define $\mathcal{E}(\mathbb{R})$ to be the set containing all finite unions of intervals in $\mathbb{R}$.

Proposition 1.2.2. *The set $\mathcal{E}(\mathbb{R})$ is a ring.*

Definition 1.2.3. Let $n \in \mathbb{Z}_{\geq 1}$. We define $\mathcal{E}(\mathbb{R}^n)$ to be the set containing all finite union of intervals in $\mathbb{R}^n$, where an interval in $\mathbb{R}^n$ is a product of n intervals in $\mathbb{R}$.

Proposition 1.2.4. *The set $\mathcal{E}(\mathbb{R}^n)$ is a ring.*

Definition 1.2.5. We define the *Borel set* $\mathcal{B}(\mathbb{R})$ to be the σ-algebra generated by $\mathcal{E}(\mathbb{R})$.

Proposition 1.2.6. *Let $A \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$. Then,*

$$x + A = \{x + a \mid a \in A\} \in \mathcal{B}(\mathbb{R}).$$

1.3 Measure on Algebra

Definition 1.3.1. Let X be a set and $\mathcal{R}$ be a ring of subsets of X. We say that $\mu\colon \mathcal{R} \to [0, \infty]$ is an *additive set function* if:

- $\mu(\varnothing) = 0$ and
- for all $A, B \in \mathcal{R}$ with $A \cap B = \varnothing$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Definition 1.3.2. Let X be a set and $\mathcal{R}$ be a ring of subsets of X. We say that $\mu\colon \mathcal{R} \to [0, \infty]$ is a *measure* if:

- $\mu(\varnothing) = 0$ and
- for a sequence $(A_n)_{n=1}^\infty$ in $\mathcal{R}$ of pairwise disjoint sets, if $\bigcup_{n=1}^\infty A_n \in \mathcal{R}$, then
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n).$$

Definition 1.3.3. Let X be a set, $\mathcal{R}$ a ring of subsets of X, and $\mu\colon \mathcal{R} \to [0, \infty]$ be an additive set function. We say that μ is *σ-finite* if there exists a sequence $(A_n)_{n=1}^\infty$ in $\mathcal{R}$ such that $\mu(A_n) < \infty$ for all $n \in \mathbb{Z}_{\geq 1}$, and
$$X = \bigcup_{n=1}^\infty A_n.$$
If we have $X \in \mathcal{R}$ with $\mu(X) < \infty$, then μ is *finite*.

Proposition 1.3.4. *Let X be a set, $\mathcal{R}$ a ring of subsets of X, and $\mu\colon \mathcal{R} \to [0, \infty)$ be a measure. Then, the following are equivalent:*

- *μ is countably additive (i.e. a measure);*
- *If $(A_n)_{n=1}^\infty$ is a sequence in $\mathcal{R}$ with $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{Z}_{\geq 1}$ with*
$$A = \bigcup_{n=1}^\infty A_n \in \mathcal{R},$$
then
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
- *If $(A_n)_{n=1}^\infty$ is a sequence in $\mathcal{R}$ with $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{Z}_{\geq 1}$ with*
$$\bigcap_{n=1}^\infty A_n = A,$$
then
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
- *If $(A_n)_{n=1}^\infty$ is a sequence in $\mathcal{R}$ with $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{Z}_{\geq 1}$ with*
$$\bigcap_{n=1}^\infty A_n = \varnothing,$$
then
$$\lim_{n\to\infty} \mu(A_n) = 0 = \mu(\varnothing).$$

Definition 1.3.5. We define the *Lebesgue measure* $\lambda \colon \mathcal{E}(\mathbb{R}) \to [0, \infty]$ as the extension of $\lambda(I) = \sup I - \inf I$, for some interval I.

Lemma 1.3.6. *Let $A \in \mathcal{E}(\mathbb{R})$ with $\lambda(A) > 0$. Then, for all $\delta \in (0,1)$, there exists a closed $A' \in \mathcal{E}(\mathbb{R})$ such that $A' \subseteq A$ and $\lambda(A') = (1-\delta)\lambda(A)$. In particular, for every $\varepsilon > 0$, there exists a closed $A' \in \mathcal{E}(\mathbb{R})$ such that $\lambda(A \backslash A') < \varepsilon$.*

Theorem 1.3.7. *The Lebesgue measure $\lambda \colon \mathcal{E}(\mathbb{R}) \to [0, \infty]$ is a measure.*

1.4 Outer Measure

Definition 1.4.1. Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0,\infty]$. Then, we define $\mu^*\colon \mathcal{P}(X) \to [0,\infty]$ by:

$$\mu^*(A) = \inf\left\{\sum_{j=1}^{\infty} \mu(E_j) \mid (E_j)_{j=1}^{\infty} \text{ in } \mathcal{R}, A \subseteq \bigcup_{j=1}^{\infty} E_j\right\}$$

and $\mu^*(A) = \infty$ if there is no $(E_j)_{j=1}^{\infty}$ in $\mathcal{R}$ containing A.

Lemma 1.4.2. *Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0,\infty]$. Then,*

- $\mu^*(\varnothing) = 0$;
- *for* $A \subseteq B \subseteq X$, $\mu^*(A) \le \mu^*(B)$;
- *for all* $A \in \mathcal{R}$, $\mu^*(A) = \mu(A)$;
- *for a sequence* $(A_n)_{n=1}^{\infty}$ *in* X,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 1.4.3 (Caratheodory's Condition)**.** Let X be a set, $\mathcal{R}$ a ring, a measure $\mu\colon \mathcal{R} \to [0,\infty]$, and $A \subseteq X$. We say that A is *μ^*-measurable* if for all $S \subseteq X$,

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c).$$

We denote by $\mathcal{M}_{\mu^*}$ the set of μ^*-measurable sets of X.

Proposition 1.4.4. *Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0,\infty]$. Then,*

- $\mathcal{R} \subseteq \mathcal{M}_{\mu^*}$;
- $\mathcal{M}_{\mu^*}$ *is an algebra;*
- $\mathcal{M}_{\mu^*}$ *is a σ-algebra;*
- μ^* *is a measure on* $\mathcal{M}_{\mu^*}$.

Proposition 1.4.5 (Caratheodory Extension Theorem)**.** *Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0,\infty]$. Then, μ extends to a measure on the σ-algebra $\mathcal{A}(\mathcal{R})$ generated by $\mathcal{R}$.*

Proposition 1.4.6. *The Lebesgue measure $\lambda\colon \mathcal{E}(\mathbb{R}) \to [0,\infty]$ extends to a unique measure $\lambda^*\colon \mathcal{B}(\mathbb{R}) \to [0,\infty]$.*

Proposition 1.4.7. *Let $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$. Then,*

$$\lambda(x + A) = \lambda(A).$$