CHAPTER 1

RINGS AND ALGEBRAS

1.0 Recap of Set Theory

Definition 1.0.1. Let X be a set. Then, the *powerset of* X is the set of subsets of X, and is denoted by $\mathcal{P}(X)$.

Definition 1.0.2. Let X be a set, and let $(X_i)_{i\in I}$ be a collection of subsets of X, for some indexing set I. We define the *union* to be:

$$\bigcup_{i\in I} X_i = \{x \in X \mid \exists i \in I \text{ s.t. } x \in X_i\}.$$

Similarly, we define the *intersection* to be:

$$\bigcap_{n=1}^{\infty} A_n = \{x \in X \mid \forall i \in I \text{ s.t. } x \in X_i\}.$$

Definition 1.0.3. Let X be a set, and let $A \subseteq X$. We define the *complement of* A to be:

$$A^c = X \setminus A = \{x \in X \mid x \notin A\}.$$

Proposition 1.0.4 (De Morgan Law)**.** *Let A and B be sets. Then,*

$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c.$$

In general, for a collection of sets $(A_i)_{i\in I}$, where I is an index set,

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c, \qquad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Definition 1.0.5. Let S be a set. We say that S *is countable* if either S is empty, or there exists a surjective function $f : \mathbb{Z}_{\geq 1} \to S$. If so, we can denote

$$S = \{f(1), f(2), f(3), \dots\}.$$

Proposition 1.0.6. *Let S be a countable set, and let $T \subseteq S$. Then, T is countable.*

Proposition 1.0.7. *Let S be a countably infinite set. Then, there exists a bijective function $f : \mathbb{Z}_{\geq 1} \to S$.*

Proposition 1.0.8. *Let S and T be countable sets. Then, their union $S \cup T$ is countable.*

Proposition 1.0.9. *Let $(S_n)_{n=1}^{\infty}$ be a sequence of countable sets. Then, their union*

$$\bigcup_{n=1}^{\infty} S_n$$

is countable.

Proposition 1.0.10. *Let S and T be countable sets. Then, the product $S \times T$ are countable.*

Corollary 1.0.11. *The set $\mathbb{Q}$ is countable.*

Proposition 1.0.12. *The set $[0, 1]$ is not countable.*

Definition 1.0.13. Let A and B be sets. We say that $|A| = |B|$ if there exists a bijection $f : A \to B$. If there exists an injective function $f : A \to B$, then we say that $|A| \leq |B|$.

1.1 Rings and Algebras

Definition 1.1.1. Let X be a set. We say that $\mathcal{R} \subseteq \mathcal{P}(X)$ is a *ring* (of subsets of X) if:

- $\varnothing \in \mathcal{R}$;
- for all $A, B \in \mathcal{R}$, the difference $A \setminus B \in \mathcal{R}$;
- for all $A, B \in \mathcal{R}$, the union $A \cup B \in \mathcal{R}$.

Proposition 1.1.2. *Let X be a set, and let $\mathcal{R} \subseteq \mathcal{P}(X)$ be a ring. Then, for $A, B \in \mathcal{R}$, the intersection $A \cap B \in \mathcal{R}$.*

Proof. We have
$$A \cap B = (A \cup B) \setminus (A \setminus B \cup B \setminus A).$$
So, $A \cap B \in \mathcal{R}$. □

Definition 1.1.3. Let X be a set. We say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *algebra* (of subsets of X) if $\mathcal{A}$ is a ring with $X \in \mathcal{A}$.

Proposition 1.1.4. *Let X be a set, and $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, $\mathcal{A}$ is an algebra if and only if:*

- *$\varnothing \in \mathcal{A}$;*
- *for all $A \in \mathcal{A}$, the complement $A^c \in \mathcal{A}$; and*
- *for all $A, B \in \mathcal{A}$, the union $A \cup B \in \mathcal{A}$.*

Proof.

$\Longrightarrow$ Since $\mathcal{A}$ is an algebra, we know that $\varnothing \in \mathcal{A}$, and for all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$. Now, let $A \in \mathcal{A}$. Since $X \in \mathcal{A}$, we find that $A^c = X \setminus A \in \mathcal{A}$.

$\Longleftarrow$ We know that $\varnothing \in \mathcal{A}$, and for all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$. Now, let $A \in \mathcal{A}$. We know that $A \setminus B = A \cap B^c$.

So, the result holds. □

Definition 1.1.5. Let X be a set. We say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *σ-algebra* (of subsets of X) if $\mathcal{A}$ is an algebra such that for all $(A_n)_{n=1}^{\infty}$ in $\mathcal{A}$, the union
$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Proposition 1.1.6. *Let X be a set, and $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, $\mathcal{A}$ is a σ-algebra if and only if:*

- *$\varnothing \in \mathcal{A}$;*
- *for all $A \in \mathcal{A}$, the complement $A^c \in \mathcal{A}$; and*

- *for a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{A}$, the union*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Proof.

$\implies$

$\impliedby$

$\square$

Proposition 1.1.7. *Let X be a set, and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. Then, for a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{A}$, the intersection*

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Proof. Define the sequence $(B_n)_{n=1}^{\infty}$ by $B_n = A_n^c$. Since $\mathcal{A}$ is an algebra, (B_n) is in $\mathcal{A}$. Moreover,

$$\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}.$$

Hence,

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} B_n\right)^c \in \mathcal{A}.$$

$\square$

1.2 Borel Sets

Definition 1.2.1. We define $\mathcal{E}(\mathbb{R})$ to be the set containing all finite unions of intervals in $\mathbb{R}$.

Proposition 1.2.2. *The set $\mathcal{E}(\mathbb{R})$ is a ring.*

Proof. $\square$

Definition 1.2.3. Let $n \in \mathbb{Z}_{\geq 1}$. We define $\mathcal{E}(\mathbb{R}^n)$ to be the set containing all finite union of intervals in $\mathbb{R}^n$, where an interval in $\mathbb{R}^n$ is a product of n intervals in $\mathbb{R}$.

Proposition 1.2.4. *The set $\mathcal{E}(\mathbb{R}^n)$ is a ring.*

Definition 1.2.5. We define the *Borel set* $\mathcal{B}(\mathbb{R})$ to be the σ-algebra generated by $\mathcal{E}(\mathbb{R})$.

Proposition 1.2.6. *Let $A \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$. Then,*

$$x + A = \{x + a \mid a \in A\} \in \mathcal{B}(\mathbb{R}).$$

Proof. Let $x \in \mathbb{R}$. Define the set

$$\mathcal{A} = \{A \in \mathcal{B}(\mathbb{R}) \mid x + A \in \mathcal{B}(\mathbb{R})\}.$$

We show that $\mathcal{A}$ is a σ-algebra.

- We have $x + \varnothing = \varnothing$, so $\varnothing \in \mathcal{A}$;
- Let $A \in \mathcal{A}$. For $y \in \mathbb{R}$,

$$\begin{aligned} y \in x + A^c &\iff y - x \in A^c \\ &\iff y - x \notin A \\ &\iff y \notin x + A \\ &\iff y \in (x + A)^c. \end{aligned}$$

  So, $(x + A)^c = x + A^c$. Since $x + A \in \mathcal{B}(\mathbb{R})$, we have

$$x + A^c = (x + A)^c \in \mathcal{B}(\mathbb{R}).$$

  Hence, $A^c \in \mathcal{A}$.

- Let $(A_n)_{n=1}^\infty$ be a sequence of disjoint sets in $\mathcal{A}$. For $y \in \mathbb{R}$,

$$\begin{aligned} y \in \bigcup_{n=1}^\infty (x + A_n) &\iff \exists n \in \mathbb{Z}_{\geq 1} \text{ s.t. } y \in x + A_n \\ &\iff y - x \in A_n \\ &\iff y - x \in \bigcup_{n=1}^\infty A_n \\ &\iff y \in x + \bigcup_{n=1}^\infty A_n. \end{aligned}$$

Since $x + A_n \in \mathcal{B}(\mathbb{R})$ for all $n \in \mathbb{Z}_{\geq 1}$,

$$x + \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (x + A_n) \in \mathcal{B}(\mathbb{R}).$$

This implies that

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Hence, $\mathcal{A}$ is a σ-algebra. Moreover, for all interval I, $x + I \in \mathcal{B}(\mathbb{R})$. Hence, $\mathcal{A}$ contains all the intervals. Since $\mathcal{A}$ is a σ-algebra, we find that $\mathcal{A} = \mathcal{B}(\mathbb{R})$. This implies that for all $A \in \mathcal{B}(\mathbb{R})$, $x + A \in \mathcal{B}(\mathbb{R})$. $\square$

1.3 Measure on Algebra

Definition 1.3.1. Let X be a set and $\mathcal{R}$ be a ring of subsets of X. We say that $\mu\colon \mathcal{R} \to [0, \infty]$ is an *additive set function* if:

- $\mu(\varnothing) = 0$ and
- for all $A, B \in \mathcal{R}$ with $A \cap B = \varnothing$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Proposition 1.3.2. *Let X be a set and $\mathcal{R}$ be a ring of subsets of X, and $\mu\colon \mathcal{R} \to [0, \infty]$ be an additive set function. Then, for $A, B \in \mathcal{R}$,*

1. *if $A \subseteq B$ then $\mu(A) \leq \mu(B)$;*
2. *$\mu(A \cup B) \leq \mu(A) + \mu(B)$.*

Proof.

1. We have
$$\mu(B) = \mu(A) + \mu(B \setminus A) \leq \mu(A).$$
2. We find that
$$\mu(A \cup B) = \mu(A) + \mu(B \setminus A) \leq \mu(A) + \mu(B)$$
since $B \setminus A \subseteq B$.

□

Definition 1.3.3. Let X be a set and $\mathcal{R}$ be a ring of subsets of X. We say that $\mu\colon \mathcal{R} \to [0, \infty]$ is a *measure* if:

- $\mu(\varnothing) = 0$ and
- for a sequence $(A_n)_{n=1}^\infty$ in $\mathcal{R}$ of pairwise disjoint sets, if $\bigcup_{n=1}^\infty A_n \in \mathcal{R}$, then
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n).$$

Proposition 1.3.4. *Let X be a set, $\mathcal{R}$ be a ring of subsets of X and let $\mu\colon \mathcal{R} \to [0, \infty]$ be a measure. Then, for all sequences $(A_n)_{n=1}^\infty$ in $\mathcal{R}$,*
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n).$$

Proof. Define the sequence $(B_n)_{n=1}^\infty$ inductively by $B_1 = A_1$ and
$$B_n = A_n \setminus \bigcup_{k=1}^{n-1} B_k$$
for $n \geq 2$. Then, (B_n) is a sequence of disjoint sets in $\mathcal{R}$ such that
$$\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n.$$

Hence,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \le \sum_{n=1}^{\infty} \mu(A_n)$$

since $B_n \subseteq A_n$ for all $n \in \mathbb{Z}_{\ge 1}$. □

Definition 1.3.5. Let X be a set, $\mathcal{R}$ a ring of subsets of X, and $\mu\colon \mathcal{R} \to [0, \infty]$ be an additive set function. We say that μ is *σ-finite* if there exists a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{R}$ such that $\mu(A_n) < \infty$ for all $n \in \mathbb{Z}_{\ge 1}$, and

$$X = \bigcup_{n=1}^{\infty} A_n.$$

If we have $X \in \mathcal{R}$ with $\mu(X) < \infty$, then μ is *finite*.

Proposition 1.3.6. *Let X be a set, $\mathcal{R}$ a ring of subsets of X, and $\mu\colon \mathcal{R} \to [0, \infty)$ be an additive set function. Then, the following are equivalent:*

1. *μ is countably additive (i.e. a measure);*
2. *If $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{R}$ with $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{Z}_{\ge 1}$ with*
$$A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{R},$$
then
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
3. *If $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{R}$ with $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{Z}_{\ge 1}$ with*
$$\bigcap_{n=1}^{\infty} A_n = A,$$
then
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
4. *If $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{R}$ with $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{Z}_{\ge 1}$ with*
$$\bigcap_{n=1}^{\infty} A_n = \varnothing,$$
then
$$\lim_{n\to\infty} \mu(A_n) = 0 = \mu(\varnothing).$$

Proof.

(1) $\implies$ (2) Define the sequence $(B_n)_{n=1}^{\infty}$ in $\mathcal{R}$ inductively by $B_1 = A_1$ and $B_n = A_n \setminus B_{n-1}$ for $n \ge 2$. Then, (B_n) is a sequence of disjoint sets with

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \qquad \text{and} \qquad \bigcup_{n=1}^{N} A_n = A_N$$

for all $N \geq 1$. Hence,

$$\begin{aligned}
\mu(A) &= \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \\
&= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \\
&= \sum_{n=1}^{\infty} \mu(B_n) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} B_n\right) \\
&= \lim_{N\to\infty} \mu(A_N).
\end{aligned}$$

(2) $\Longrightarrow$ (3) Define the sequence $(B_n)_{n=1}^{\infty}$ in $\mathcal{R}$ by $B_1 = A_1^c$. Then, $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{Z}_{\geq 1}$ with

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n^c = \left(\bigcap_{n=1}^{\infty} A_n\right)^c = A^c.$$

Hence,

$$\mu(A^c) = \lim_{n\to\infty} \mu(B_n).$$

This implies that

$$\begin{aligned}
\mu(A) &= \mu(X) - \mu(A^c) \\
&= \mu(X) - \lim_{n\to\infty} \mu(A_n^c) \\
&= \mu(X) - \lim_{n\to\infty} \mu(X) - \mu(A_n) \\
&= \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

(3) $\Longrightarrow$ (4) This follows if we set $A = \varnothing$.

(4) $\Longrightarrow$ (1) Let $(A_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{R}$ of disjoint sets. Define the sequence $(B_n)_{n=1}^{\infty}$ in $\mathcal{R}$ by

$$B_n = \bigcup_{k=n}^{\infty} A_k.$$

By definition, $B_{n+1} \supset B_{n+1}$ for all $n \in \mathbb{Z}_{\geq 1}$. Moreover, we claim that

$$\bigcap_{n=1}^{\infty} B_n = \varnothing.$$

So, assume for a contradiction, that

$$x \in \bigcap_{n=1}^{\infty} B_n.$$

In that case, $x \in B_1$. In particular, there exists an $k \in \mathbb{Z}_{\geq 1}$ such that $x \in A_k$. Since (A_n) is a sequence of disjoint sets, we know that $x \notin A_n$ for $n \geq k+1$. Hence, $x \notin B_{k+1}$. This is a contradiction. So,

$$\bigcap_{n=1}^{\infty} B_n = \varnothing.$$

This implies that

$$\lim_{n\to\infty} \mu(B_n) = 0.$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu(A_n) &= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} A_n\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{\infty} A_n \setminus B_N\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) - \mu(B_N) \\
&= \mu\left(\bigcup_{n=1}^{\infty} A_n\right).
\end{aligned}$$

□

Definition 1.3.7. We define the *Lebesgue measure* $\lambda \colon \mathcal{E}(\mathbb{R}) \to [0, \infty]$ as the extension of $\lambda(I) = \sup I - \inf I$, for some interval I. In particular, for disjoint intervals $(I_k)_{k=1}^{n}$, we define

$$\lambda\left(\bigcup_{k=1}^{n} I_k\right) = \sum_{k=1}^{n} \lambda(I_k).$$

Lemma 1.3.8. *The Lebesgue measure $\lambda \colon \mathcal{E}(\mathbb{R}) \to [0, \infty]$ is well-defined.*

Proof. □

Lemma 1.3.9. *Let $A \in \mathcal{E}(\mathbb{R})$ with $\lambda(A) > 0$. Then, for all $\delta \in (0, 1)$, there exists a closed $A' \in \mathcal{E}(\mathbb{R})$ such that $A' \subseteq A$ and $\lambda(A') = (1-\delta)\lambda(A)$. In particular, for every $\varepsilon > 0$, there exists a closed $A' \in \mathcal{E}(\mathbb{R})$ such that $\lambda(A \setminus A') < \varepsilon$.*

Proof. □

Theorem 1.3.10. *The Lebesgue measure $\lambda \colon \mathcal{E}(\mathbb{R}) \to [0, \infty]$ is a measure.*

Proof. □

1.4 Outer Measure

Definition 1.4.1. Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0, \infty]$. Then, we define $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$ by:

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(E_j) \mid (E_j)_{j=1}^{\infty} \text{ in } \mathcal{R}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$$

and $\mu^*(A) = \infty$ if there is no $(E_j)_{j=1}^{\infty}$ in $\mathcal{R}$ containing A.

Lemma 1.4.2. *Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0, \infty]$. Then,*

1. $\mu^*(\varnothing) = 0$;
2. *for* $A \subseteq B \subseteq X$, $\mu^*(A) \leq \mu^*(B)$;
3. *for all* $A \in \mathcal{R}$, $\mu^*(A) = \mu(A)$;
4. *for a sequence* $(A_n)_{n=1}^{\infty}$ *in* X,
$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Proof.

1. Define the sequence $(E_j)_{j=1}^{\infty}$ in $\mathcal{R}$ by $E_j = \varnothing$. We know that
$$\varnothing \subseteq \bigcup_{j=1}^{\infty} E_j,$$
meaning that
$$\mu^*(\varnothing) \leq \sum_{j=1}^{\infty} \mu(E_j) = 0.$$
We also know that $\mu(E) \geq 0$ for all $E \in \mathcal{R}$, meaning that $\mu^*(A) \geq 0$ for all $A \subseteq X$. Hence, $\mu^*(\varnothing) = 0$.

2. Let $(E_j)_{j=1}^{\infty}$ be a sequence in $\mathcal{R}$ such that
$$B \subseteq \bigcup_{j=1}^{\infty} E_j.$$
In that case,
$$A \subseteq \bigcup_{j=1}^{\infty} E_j$$
as well. Hence, by the infimum property, we find that $\mu^*(A) \leq \mu^*(B)$.

3. Define the sequence $(E_j)_{j=1}^{\infty}$ in $\mathcal{R}$ by $E_1 = A$ and $E_n = \varnothing$ for $n \geq 2$. Then,
$$A \subseteq \bigcup_{j=1}^{\infty} E_j,$$
meaning that $\mu^*(A) \leq \mu(A)$.
Now, let $(E_j)_{j=1}^{\infty}$ in $\mathcal{R}$ such that
$$A \subseteq \bigcup_{j=1}^{\infty} E_j.$$
In that case,
$$\mu(A) \leq \mu\left(\bigcup_{j=1}^{\infty} E_j\right).$$
Hence, $\mu(A) \leq \mu^*(A)$. This implies that $\mu(A) = \mu^*(A)$.

4.

□

Definition 1.4.3 (Caratheodory's Condition)**.** Let X be a set, $\mathcal{R}$ a ring, a measure $\mu\colon \mathcal{R} \to [0, \infty]$, and $A \subseteq X$. We say that A is μ^**-measurable* if for all $S \subseteq X$,
$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c).$$
We denote by $\mathcal{M}_{\mu^*}$ the set of μ^*-measurable sets of X.

Proposition 1.4.4. *Let X be a set, $\mathcal{R}$ a ring, measure $\mu\colon \mathcal{R} \to [0, \infty]$, and $A \subseteq X$. Then, for all $S \subseteq X$,*
$$\mu^*(S \cap A) + \mu^*(S \cap A^c) \geq \mu^*(S).$$

Proof. Define the sequence $(E_j)_{j=1}^{\infty}$ in $\mathcal{P}(X)$ by $E_1 = S \cap A$, $E_2 = S \cap A^c$, $E_n = \varnothing$ for $n \geq 3$. Then,
$$\mu^*(S) = \mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n) = \mu^*(S \cap A) + \mu^*(S \cap A^c).$$
□

Proposition 1.4.5. *Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0, \infty]$. Then,*

1. *$\mathcal{R} \subseteq \mathcal{M}_{\mu^*}$;*
2. *$\mathcal{M}_{\mu^*}$ is an algebra;*
3. *$\mathcal{M}_{\mu^*}$ is a σ-algebra;*
4. *μ^* is a measure on $\mathcal{M}_{\mu^*}$.*

Proof.

1. Let $A \in \mathcal{R}$, $S \subseteq X$ and let $\varepsilon > 0$. We show that
$$\mu^*(S) > \mu^*(S \cap A) + \mu^*(S \cap A^c) - \varepsilon.$$
By definition of infimum, we can find a sequence $(E_j)_{j=1}^\infty$ in $\mathcal{R}$ with
$$S \subseteq \bigcup_{j=1}^\infty E_j \qquad \text{s.t.} \qquad \mu^*(S) \leq \sum_{j=1}^\infty \mu(E_j) < \mu^*(S) + \varepsilon.$$
We know that
$$S \cap A \subseteq \bigcup_{j=1}^\infty (E_j \cap A), \qquad S \cap A^c \subseteq \bigcup_{j=1}^\infty (E_j \cap A^c).$$
Since rings are closed under intersection, we find that
$$\begin{aligned} \mu^*(S \cap A) + \mu^*(S \cap A^c) &\leq \sum_{j=1}^\infty \mu(E_j \cap A) + \sum_{j=1}^\infty \mu(E_j \cap A^c) \\ &= \sum_{j=1}^\infty \mu(E_j) \\ &< \mu^*(S) + \varepsilon. \end{aligned}$$
Hence,
$$\mu^*(S) > \mu^*(S \cap A) + \mu^*(S \cap A^c) - \varepsilon$$
for all $\varepsilon > 0$. So,
$$\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \cap A^c).$$
This implies that $A \in \mathcal{M}_{\mu^*}$.

2.
   - For all $S \subseteq X$, we find that
   $$\mu^*(S \cap \varnothing) + \mu^*(S \cap \varnothing^c) = \mu^*(\varnothing) + \mu^*(S) = \mu^*(S).$$
   So, $\varnothing \in S$.
   - Let $A \in \mathcal{M}_{\mu^*}$. In that case,
   $$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c).$$
   Hence, $A^c \in \mathcal{M}_{\mu^*}$.
   - Let $A, B \in \mathcal{M}_{\mu^*}$. For $S \subseteq X$, we find that
   $$\begin{aligned} \mu^*(S) &= \mu^*(S \cap A) + \mu^*(S \cap A^c) \\ &= \mu^*(S \cap A \cap B) + \mu^*(S \cap A \cap B^c) + \mu^*(S \cap A^c \cap B) \\ &\quad + \mu^*(S \cap A^c \cap B^c) \\ &= \mu^*(S \cap (A \cup B)) + \mu^*(S \cap (A \cup B)^c). \end{aligned}$$
   So, $A \cup B \in \mathcal{M}_{\mu^*}$.

3. Let $(A_n)_{n=1}^\infty$ be a sequence of disjoint sets in $\mathcal{M}_{\mu^*}$.

4.

$\square$

Proposition 1.4.6 (Caratheodory Extension Theorem)**.** *Let X be a set, $\mathcal{R}$ a ring, and a measure $\mu\colon \mathcal{R} \to [0, \infty]$. Then, μ extends to a measure on the σ-algebra $\mathcal{A}(\mathcal{R})$ generated by $\mathcal{R}$.*

Proof. $\square$

Proposition 1.4.7. *The Lebesgue measure $\lambda\colon \mathcal{E}(\mathbb{R}) \to [0, \infty]$ extends to a unique measure $\lambda^*\colon \mathcal{B}(\mathbb{R}) \to [0, \infty]$.*

Proof. $\square$

Proposition 1.4.8. *Let $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$. Then,*

$$\lambda(x + A) = \lambda(A).$$

Proof. $\square$

1.5 Probability and Independence

Definition 1.5.1. Let $(\Omega, \mathcal{A}, P)$ be a measure space. We say that it is a *probability space* if $P(\Omega) = 1$.

Definition 1.5.2. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $(\mathcal{A}_i)_{i \in I}$ be a sequence of σ-algebras in $\mathcal{A}$, for some indexing set I. We say that $(\mathcal{A}_i)$ is independent if for any $J \subseteq I$ finite,

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j).$$

Definition 1.5.3. Let $(A_n)_{n=1}^{\infty}$ be a sequence of events in some probability space. Define

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m, \qquad \liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

Proposition 1.5.4. *Let $(A_n)_{n=1}^{\infty}$ be a sequence of events in some probability space. Then,*

- $(\limsup A_n)^c = \liminf A_n^c$*;*
- $(\liminf A_n)^c = \limsup A_n^c$*;*
- $\liminf A_n \subseteq \limsup A_n$.

Proof. □

Lemma 1.5.5 (First Borel-Cantelli Lemma)**.** *Let X be a set, $\mathcal{R}$ a ring of subsets of X, P a probability measure on X. Then, for a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{R}$ with*

$$\sum_{n=1}^{\infty} P(A_n)$$

finite, $P(\liminf A_n) = 0$.

Proof. Define the sequence $(B_n)_{n=1}^{\infty}$ in $\mathcal{R}$ by

$$B_n = \bigcup_{m=n}^{\infty} A_m.$$

We have $B_n \supseteq B_{n+1}$ for all $n \in \mathbb{Z}_{\geq 1}$ with

$$\bigcap_{n=1}^{\infty} B_n = \varnothing.$$

Since P is a measure, this implies that

$$P(\liminf A_n) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} P(B_n).$$

We also know that

$$P(B_n) = P\left(\bigcup_{m=n}^{\infty} A_m\right) \leq \sum_{m=n}^{\infty} A_m \to 0$$

by assumption. Hence, $P(\liminf A_n) = 0$. □

Lemma 1.5.6 (Second Borel-Cantelli Lemma)**.** *Let X be a set, $\mathcal{R}$ a ring of subsets of X, P a probability measure on X. Then, for a sequence of independent events $(A_n)_{n=1}^{\infty}$ in $\mathcal{R}$ with*

$$\sum_{n=1}^{\infty} P(A_n) = \infty,$$

$P(\limsup A_n) = 1$.

Proof. We know that for all $a \geq 0$, $1 - a \leq e^{-a}$. Now, let $a_n = P(A_n)$. For $N, n \in \mathbb{Z}_{\geq 1}$ with $N \geq n$, we have

$$P\left(\bigcap_{m=n}^{N} A_m^c\right) = \prod_{m=n}^{N} P(A_m^c)$$

since (A_n) is a sequence of independent events. Moreover,

$$P(A_m^c) = (1 - a_m) \leq e^{-a_m},$$

meaning that

$$P\left(\bigcap_{m=n}^{N} A_m^c\right) \leq \exp\left(-\sum_{m=n}^{N} a_m\right) \to 0$$

as $\sum_{m=n}^{N} a_m \to \infty$ as $N \to \infty$, by assumption. Hence,

$$\lim_{N\to\infty} P\left(\bigcap_{m=n}^{N} P(A_n^c)\right) = P\left(\bigcap_{m=n}^{\infty} P(A_n^c)\right) = 0$$

for all $n \in \mathbb{Z}_{\geq 1}$. So,

$$P(\limsup A_n^c) = P\left(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c\right) \leq \sum_{n=1}^{\infty} P\left(\bigcap_{m=n}^{\infty} A_m^c\right) = 0.$$

Hence,

$$P(\liminf A_n) = 1 - P(\limsup A_n^c) = 1 - 0 = 1.$$

□