

CHAPTER 1

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COMPLEX DIFFERENTIATION

## 1.0 Introduction to Complex Numbers

Complex numbers can be viewed as the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\},$$

where  $i^2 = -1$ . This can be viewed as a 2D plane, e.g.

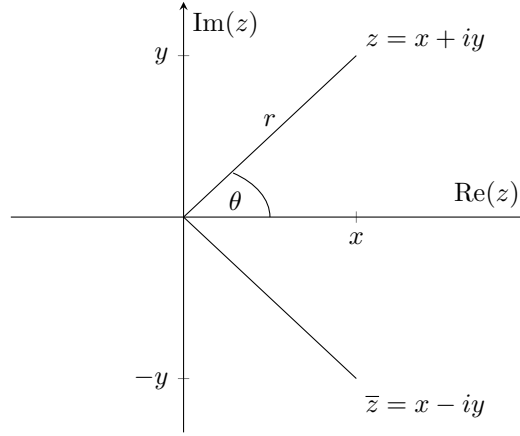


Figure 1.1: The complex plane

The given point is  $z = x + iy$ , for  $x, y \in \mathbb{R}$ . The real part of  $z$ ,  $\text{Re}(z) = x$ , and the imaginary part of  $z$ ,  $\text{Im}(z) = y$ . The complex conjugate of  $z$  is given by  $\bar{z} = x - iy$ . Using the conjugate, we can rewrite the real and the imaginary part of a complex number:

$$\text{Re}(z) = \frac{z + \bar{z}}{2}, \quad \text{Im}(z) = \frac{z - \bar{z}}{2}.$$

The modulus of  $z$  is

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \in \mathbb{R}.$$

Moreover, the argument of  $z$  is

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) + 2k\pi & x > 0 \\ \tan^{-1}(y/x) + (2k+1)\pi & x < 0 \end{cases},$$

for  $k \in \mathbb{Z}$ . It corresponds to the angle  $\theta$ . Moreover, every value of  $\arg(z)$  is  $\theta + 2k\pi$ , for some  $k \in \mathbb{Z}$ . In Euler's notation, we denote  $z = re^{i\theta}$ . By de Moivre's Theorem, we know that

$$z^n = r^n e^{in\theta} = r^n (\cos(n\theta) + i \sin(n\theta)),$$

and if  $z \neq 0$ , then its multiplicative inverse is given by

$$z^{-1} = \frac{1}{z} \frac{\bar{z}}{\bar{z}} = \frac{x - iy}{x^2 + y^2}.$$

Moreover,  $z^{-1} = r^{-1}e^{-i\theta}$  in Euler's notation.

We know that  $\mathbb{C}$  forms a field under addition and multiplication. Adding two complex numbers is equivalent to adding two vectors in  $\mathbb{R}^2$ . Geometrically, this can be thought of as translation. Multiplication is however performed in a way such that the arguments of the two complex numbers gets added. That is, for  $w, z \in \mathbb{C}$  with  $w = re^{i\alpha}$  and  $z = se^{i\beta}$ ,

$$zw = rse^{i(\alpha+\beta)}.$$

Geometrically, it can be thought of as scaling and rotation.

## 1.1 Complex Sequences and Series

### Complex Sequences

We start by considering convergence in  $\mathbb{C}$  before looking at differentiation.

**Definition 1.1.1.** Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$ , and let  $z \in \mathbb{C}$ . We say that  $z_n \rightarrow z$  if for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{Z}_{\geq 1}$  such that for  $n \in \mathbb{Z}_{\geq 1}$ , if  $n \geq N$ , then  $|z_n - z| < \varepsilon$ .

It turns out that a limit exists for a complex sequence if and only if its real part and the complex part converge.

**Proposition 1.1.2.** Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$  and let  $z \in \mathbb{C}$ . Then,  $z_n \rightarrow z$  if and only if  $\operatorname{Re}(z_n) \rightarrow \operatorname{Re}(z)$  and  $\operatorname{Im}(z_n) \rightarrow \operatorname{Im}(z)$ .

We will now look at some proofs of convergence using the result above. We start by showing that  $(i^n)$  does not converge.

**Example 1.1.3.** The sequence  $(z_n)_{n=1}^{\infty}$  given by  $z_n = i^n$  does not converge.

*Proof.* We have

$$\operatorname{Re}(z_n) = \begin{cases} (-1)^{n/2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

Since the subsequence  $\operatorname{Re}(z_{2n}) = (-1)^n$  does not converge, the sequence  $\operatorname{Re}(z_n)$  does not converge. By the result above, we find that  $z_n$  does not converge.  $\square$

Next, we show that  $(\frac{i}{n})$  converges to 0.

**Example 1.1.4.** The sequence  $(z_n)_{n=1}^{\infty}$  given by  $z_n = \frac{i}{n}$  satisfies  $z_n \rightarrow 0$ .

*Proof.* We have

$$\operatorname{Re}(z_n) = 0, \quad \operatorname{Im}(z_n) = \frac{1}{n}.$$

Since  $\operatorname{Re}(z_n) \rightarrow 0$  and  $\operatorname{Im}(z_n) \rightarrow 0$ , we find that  $z_n \rightarrow 0$ .  $\square$

Now, we look at Cauchy sequences.

**Definition 1.1.5.** Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$ . We say that  $(z_n)$  is *Cauchy* if for all  $\varepsilon > 0$ , there exists an  $N \in \mathbb{Z}_{\geq 1}$  such that for  $m, n \in \mathbb{Z}$ , if  $m, n \geq N$ , then  $|z_n - z_m| < \varepsilon$ .

It turns out that in  $\mathbb{C}$ , a Cauchy sequence is equivalent to a convergent sequence.

**Proposition 1.1.6.** Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$ . Then,  $(z_n)$  is Cauchy if and only if  $z_n \rightarrow z$  for some  $z \in \mathbb{C}$ .

For this reason, we say that  $\mathbb{C}$  is complete.

**Complex series**

Now, we look at series in  $\mathbb{C}$ .

**Definition 1.1.7.** Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$ . Define the *sequence of partial sums*  $(s_n)_{n=1}^{\infty}$

$$s_n = \sum_{i=1}^n z_i.$$

For some  $z \in \mathbb{C}$ , we say that the sum

$$\sum_{n=1}^{\infty} z_n = z$$

if  $s_n \rightarrow z$ .

This is analogous to the real case. Also, by the result above, we know that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re}(z_n) + i \sum_{n=1}^{\infty} \operatorname{Im}(z_n).$$

If a series converges, then the sequence has to converge to 0.

**Proposition 1.1.8.** Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$  such that the series

$$\sum_{n=1}^{\infty} z_n$$

converges. Then,  $z_n \rightarrow 0$ .

*Proof.* Let  $(s_n)_{n=1}^{\infty}$  be the sequence of partial sums of  $(z_n)$ . We know that  $s_n \rightarrow S$ , for some  $S \in \mathbb{C}$ . In that case,

$$z_n = s_{n+1} - s_n \rightarrow S - S = 0.$$

□

We can define absolute convergence of a sequence by looking at the absolute convergence of the sequence of partial sums.

**Definition 1.1.9.** Let  $(z_n)_{n=0}^{\infty}$  be a sequence in  $\mathbb{C}$ . In that case, we say that the series  $\sum_{n=0}^{\infty} z_n$  is absolutely convergent if the series

$$\sum_{n=0}^{\infty} |z_n|$$

converges.

As in  $\mathbb{R}$ , absolute convergence implies convergence in  $\mathbb{C}$ .

**Proposition 1.1.10.** Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$  such that the series  $\sum z_n$  converges absolutely. Then, the series  $\sum_{n=1}^{\infty} z_n$  converges.

*Proof.* Let  $\varepsilon > 0$ . Since  $\sum z_n$  converges absolutely, it is Cauchy. So, there exists an  $N \in \mathbb{Z}_{\geq 1}$  such that for  $m, n \in \mathbb{Z}$ , if  $m \geq n \geq N$ , then

$$\sum_{i=n}^m |z_i| < \varepsilon.$$

In that case, for  $m, n \in \mathbb{Z}$ , if  $m \geq n \geq N$ , then

$$\left| \sum_{i=n}^m z_i \right| \leq \sum_{i=n}^m |z_i| < \varepsilon.$$

Therefore, the series  $\sum z_n$  is Cauchy. This implies that  $\sum z_n$  converges.  $\square$

Now, we will look at the geometric series in  $\mathbb{C}$ . For  $z \in \mathbb{C}$ , define the sequence  $(z_n)_{n=0}^{\infty}$  by  $z_n = z^n$ . If  $z \neq 1$ , we know that the partial sum

$$s_n = \sum_{i=0}^n z_i = \frac{1 - z^{n+1}}{1 - z},$$

using the fact that

$$(1 - z)(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1}.$$

If  $|z| \geq 1$ , then we know that  $|z^n| = |z|^n \geq 1$  for all  $n \in \mathbb{Z}_{\geq 0}$ , so  $z_n \not\rightarrow 0$ . Therefore, the series

$$\sum_{n=0}^{\infty} |z^n|$$

cannot converge. Instead, if  $|z| < 1$ , then

$$|s_n| = \frac{|1 - z^{n+1}|}{|1 - z|} \leq \frac{1 + |z|^{n+1}}{1 - |z|}$$

by the triangle and the reverse triangle inequality. Since

$$\frac{1 + |z|^{n+1}}{1 - |z|} \rightarrow \frac{1}{1 - |z|},$$

we find that the series

$$\sum_{n=0}^{\infty} |z^n|$$

is bounded. The series is monotone increasing, so the monotone convergence theorem tells us that it is absolutely convergent.

Using the geometric series, we can derive the comparison and the ratio test.

**Corollary 1.1.11** (Comparison Test). *Let  $(z_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$  and  $\sum_{n=0}^{\infty} r_n$  be a convergent series in  $\mathbb{R}$  with  $r_n \geq 0$  for all  $n \in \mathbb{Z}_{\geq 0}$ . Moreover, assume that for  $k > 0$ ,*

$$|z_n| \leq k r_n$$

*for all  $n \in \mathbb{Z}_{\geq 0}$ . Then, the series  $\sum_{n=0}^{\infty} z_n$  converges absolutely.*

**Corollary 1.1.12** (Ratio Test). *Let  $\sum_{n=0}^{\infty} z_n$  be a sequence in  $\mathbb{C}$  such that  $z_n \neq 0$  for all  $n \in \mathbb{Z}_{\geq 0}$ , and let  $C \in \mathbb{R}$  such that*

$$\lim_{n \rightarrow \infty} \frac{|z_{n+1}|}{|z_n|} = C.$$

- *If  $C < 1$ , then the series  $\sum_{n=0}^{\infty} z_n$  converges absolutely.*
- *If  $C > 1$ , then the series  $\sum_{n=0}^{\infty} z_n$  diverges.*

## 1.2 Introduction to Complex differentiation

### Limits in $\mathbb{C}$

In this section, we will define complex differentiation in open sets. We start by defining open discs.

**Definition 1.2.1.** Let  $z_0 \in \mathbb{C}$  and  $r > 0$ . We define the *open disc around  $z_0$  with radius  $r$*  to be the set

$$D = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

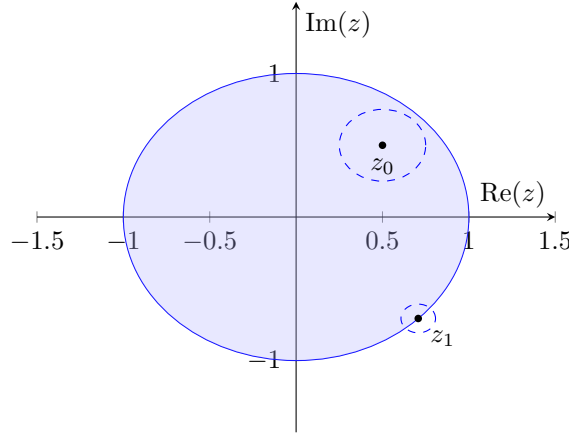
We can generalise this to open sets.

**Definition 1.2.2.** Let  $\Omega \subseteq \mathbb{C}$ . We say that  $\Omega$  is *open* if for every  $z_0 \in \Omega$ , there exists an  $\varepsilon > 0$  such that for all  $z \in \mathbb{C}$ , if  $|z - z_0| < \varepsilon$ , then  $z \in \Omega$ .

In other words, for every  $z_0 \in \Omega$ , the open disc around  $z_0$  of radius  $\varepsilon$  satisfies

$$\{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\} \subseteq \Omega.$$

Intuitively, this means that no point in  $\Omega$  is a boundary point. That is, for any point, there is some  $\varepsilon$ -neighbourhood around it that is still within  $\Omega$ . Consider the closed unit ball in  $\mathbb{C}$  centered at the origin.



In the figure above, the value  $z_0 \in \mathbb{C}$  is not a boundary point of the set since we have an open ball around it that is fully contained in the set. However,  $z_1 \in \mathbb{C}$  is a boundary point since any open ball around it will not be fully contained in the set. So, this set is not open since  $z_1$  is a boundary point.

As we expect, open discs are open.

**Proposition 1.2.3.** Let  $z_0 \in \mathbb{C}$  and let  $r > 0$ . Then, the open disc

$$D = \{z \in \mathbb{C} \mid |z - z_0| < r\}$$

is open.

Also,  $\mathbb{C}$  is open. This is because for all  $z_0, z \in \mathbb{C}$ , if  $|z - z_0| < 1$ , then  $z \in \mathbb{C}$ .<sup>1</sup> We will normally consider functions with domain an open set.

We now define limits.

<sup>1</sup>The choice of 1 here is arbitrary!

**Definition 1.2.4.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a function, and let  $z_0, w \in \mathbb{C}$ . We say that the limit

$$\lim_{z \rightarrow z_0} f(z) = w$$

if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for  $z \in \mathbb{C}$ , if  $0 < |z - z_0| < \delta$ , then  $|f(z) - w| < \varepsilon$ .

As we saw in real analysis, the sequential characterisation of limits still holds in  $\mathbb{C}$ .

**Proposition 1.2.5.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a function, and let  $z_0, w \in \mathbb{C}$ . Then, the limit

$$\lim_{z \rightarrow z_0} f(z) = w$$

if and only if for every sequence  $(z_n)_{n=1}^{\infty}$  in  $\mathbb{C} \setminus \{z_0\}$  with  $z_n \rightarrow z_0$ ,  $f(z_n) \rightarrow w$ .

## Derivatives in $\mathbb{C}$

Now, we will look at differentiation in  $\mathbb{C}$ .

**Definition 1.2.6.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a function, and let  $c \in \mathbb{C}$ . Then, we say that  $f$  is *differentiable at  $c$*  if the limit

$$\lim_{z \rightarrow c} \frac{f(z) - f(c)}{z - c}$$

exists. If the limit exists, we denote the value by  $f'(c)$ . We say that  $f$  is *differentiable* if  $f$  is differentiable at  $c$  for all  $c \in \mathbb{C}$ .

Although the difference quotient is the same in  $\mathbb{C}$  as in  $\mathbb{R}$ , there are a lot of functions in  $\mathbb{C}$  that we might ‘expect’ to be continuous, but aren’t. This is because  $\mathbb{R}$  can be thought of as a line, so there are only two directions to approach a point. However,  $\mathbb{C}$  is a plane, so there are infinitely many directions we can approach a point from.

We will illustrate the difference with an example. Consider the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by  $f(z) = |z|^2 = z\bar{z}$ . We can consider the function as a map  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $g(x, y) = x^2 + y^2$ . This is a differentiable function in  $\mathbb{R}^2$  because we can derive it partially with respect to both  $x$  and  $y$ . But,  $f$  is only differentiable at  $z_0 = 0$ .

- If  $z_0 = 0$ , then

$$f'(0) = \lim_{z \rightarrow 0} \frac{z\bar{z} - 0}{z - 0} = \lim_{z \rightarrow 0} \bar{z} = \lim_{z \rightarrow 0} x - iy = 0.$$

Note that if  $z = x + iy \rightarrow 0$ , then we must have  $x \rightarrow 0$  and  $y \rightarrow 0$ .

- Now, assume that  $z_0 \neq 0$ . Define the sequences  $x_n = z_0 + \frac{1}{n}$  and  $y_n = z_0 + \frac{i}{n}$ . We know that  $x_n, y_n \rightarrow z_0$ . Moreover,

$$\begin{aligned} \frac{f(x_n) - f(z_0)}{x_n - z_0} &= \frac{(z_0 + 1/n)(\bar{z}_0 + 1/n) - z_0\bar{z}_0}{1/n} \\ &= \frac{1/n \cdot z_0 + 1/n \cdot \bar{z}_0 + 1/n^2}{1/n} \\ &= z_0 + \bar{z}_0 + \frac{1}{n} \rightarrow z_0 + \bar{z}_0, \end{aligned}$$



and

$$\begin{aligned}\frac{f(y_n) - f(z_0)}{y_n - z_0} &= \frac{(z_0 + i/n)(\bar{z}_0 - i/n) - z_0\bar{z}_0}{i/n} \\ &= \frac{-i/n \cdot z_0 + i/n \cdot \bar{z}_0 + 1/n^2}{i/n} \\ &= -z_0 + \bar{z}_0 - \frac{i}{n} \rightarrow -z_0 + \bar{z}_0.\end{aligned}$$

Since  $z_0 \neq 0$ , we know that  $z_0 \neq -z_0$ . Therefore, we must find that the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

does not exist by the sequential characterisation of continuity.

So, the function  $f'(0) = 0$ , and is not differentiable at any non-zero point. In fact, we say that it depends on the value of  $\theta$ , where  $z = re^{i\theta}$ . That is, the direction we approach in. In this case, we saw two directions- approaching  $z_0$  from the real line ( $\theta = 0$ ) and the imaginary line ( $\theta = \frac{\pi}{2}$ ).

Similarly, we can show that the complex conjugation function is not differentiable.

**Example 1.2.7.** Let  $c \in \mathbb{C}$ , and define the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  by  $f(z) = \bar{z}$ . Then,  $f$  is not differentiable at  $c$ .

*Proof.* Let  $z_0 \in \mathbb{C}$ . Define the sequences  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  in  $\mathbb{C}$  by  $x_n = z_0 + \frac{1}{n}$  and  $y_n = z_0 + \frac{i}{n}$ . We know that  $x_n, y_n \rightarrow z_0$ . Moreover,

$$\frac{f(x_n) - f(z_0)}{x_n - z_0} = \frac{\bar{z}_0 + \frac{1}{n} - \bar{z}_0}{1/n} = \frac{1/n}{1/n} = 1 \rightarrow 1,$$

and

$$\frac{f(y_n) - f(z_0)}{y_n - z_0} = \frac{\bar{z}_0 - \frac{i}{n} - \bar{z}_0}{i/n} = \frac{-i/n}{i/n} = -1 \rightarrow -1.$$

So, the sequential characterisation of continuity tells us that the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

does not exist. □

Using the properties of limits, we can establish arithmetic properties of derivatives. We start with the sum rule.

**Proposition 1.2.8.** Let  $f, g : \mathbb{C} \rightarrow \mathbb{C}$  be functions and let  $c \in \mathbb{C}$  such that  $f$  and  $g$  are differentiable at  $c$ . Then,  $f + g$  is differentiable at  $c$ , with  $(f + g)'(c) = f'(c) + g'(c)$ .

Next, we look at the scalar rule.

**Proposition 1.2.9.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a function and let  $a, c \in \mathbb{C}$  such that  $f$  is differentiable at  $c$ . Then,  $af$  is differentiable at  $c$ , with  $(af)'(c) = af'(c)$ .

We will now look at the quotient rule.

**Proposition 1.2.10.** *Let  $f, g : \mathbb{C} \rightarrow \mathbb{C}$  be functions and let  $c \in \mathbb{C}$  such that  $g(c) \neq 0$ . In that case,  $\frac{f}{g}$  is differentiable at  $c$ , with*

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Finally, we have the chain rule.

**Proposition 1.2.11.** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  and  $g : \mathbb{C} \rightarrow \mathbb{C}$  be functions such that  $f$  is differentiable at  $c \in \mathbb{C}$  and  $g$  is differentiable at  $f(c)$ . In that case, the composition  $g \circ f$  is differentiable at  $c$ , with*

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

We finish by defining holomorphic functions.

**Definition 1.2.12.** Let  $\Omega$  be an open subset of  $\mathbb{C}$ . Then,  $f$  is *holomorphic* in  $\Omega$  if  $f$  is differentiable at every  $x \in \Omega$ .

### 1.3 Power series

In this section, we will look at power series in  $\mathbb{C}$ .

**Definition 1.3.1.** Let  $(a_n)_{n=0}^\infty$  be a sequence in  $\mathbb{C}$ , and let  $c, z \in \mathbb{C}$ . The power series  $\sum a_n(z - c)^n$  is defined by

$$S(z) = \sum_{n=0}^{\infty} a_n(z - c)^n = a_0 + a_1(z - c) + a_2(z - c)^2 + \dots$$

We refer to the sequence  $(a_n)$  as the *coefficient* sequence,  $z$  as the *variable*, and  $c$  is the *expansion point*. We say that the power series  $S(z)$  *converges* at  $z_0 \in \mathbb{C}$  if the series  $S(z_0)$  converges. Otherwise, we say that it *diverges* at  $z_0$ .

For example, consider the power series

$$S(z) = \sum_{n=0}^{\infty} (z - c)^n,$$

for some  $c \in \mathbb{C}$ . Then, we saw before that the interval of convergence is the open disc

$$D = \{z \in \mathbb{C} \mid |z - c| < 1\}.$$

We can also consider absolute convergence at a point.

**Definition 1.3.2.** Let  $S(z)$  be a power series. We say that  $S(z)$  *converges absolutely* at  $z_0 \in \mathbb{C}$  if the series  $S(z_0)$  converges absolutely.

Similarly, we can define uniform convergence.

**Definition 1.3.3.** Let  $S(z)$  be a power series. We say that  $S(z)$  *converges uniformly* on some open disc  $D \subseteq \mathbb{C}$  if for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{Z}_{\geq 0}$  such that for  $n \in \mathbb{Z}$ , if  $n \geq N$ , then

$$|S_n(z) - S(z)| < \varepsilon$$

for all  $z \in D$ . We define by  $S_n(z)$  the partial sum

$$S_n(z) = \sum_{i=0}^n a_i(z - c)^i.$$

Now, we define the radius of convergence and its properties.

**Theorem 1.3.4.** Let  $S(z)$  be a power series. Then, there exists an  $R \in [0, \infty)$  such that

- $S(z)$  diverges for all  $z \in \mathbb{C}$  with  $|z - c| > R$ ;
- $S(z)$  converges absolutely and uniformly for all  $z \in \mathbb{C}$  with  $|z - c| \leq r$ , for some  $0 \leq r < R$ ; and
- 

$$R = \sup\{r \geq 0 \mid |a_n|r^n \text{ is a bounded sequence}\}.$$

We call  $R$  the radius of convergence.

It is possible for  $S(z)$  to converge for all  $z \in \mathbb{C}$ , in which case we say that  $R = \infty$ .

Now, consider the power series

$$S(z) = \sum_{n=0}^{\infty} z^{2n}, \quad T(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

Since these are geometric series with variables  $z_S = z^2$  and  $z_T = -z^2$ , they both have radius of convergence  $R = 1$ . Moreover,

$$S(z) = \frac{1}{1 - z^2}, \quad T(z) = \frac{1}{1 + z^2}.$$

If we consider  $\frac{1}{1-z^2}$  for  $z \in \mathbb{R}$ , we would have singularities at  $z = \pm 1$  and the series would diverge for  $|z| \geq 1$ . But,  $\frac{1}{1+z^2}$  does not have any singularities for  $z \in \mathbb{R}$ . Instead, if we look at the complex series, there are singularities at  $z = \pm i$ .

Like in the real case, we can use the ratio test to characterise the radius of convergence.

**Theorem 1.3.5.** *Let  $(a_n)_{n=0}^{\infty}$  be a sequence in  $\mathbb{C}$ , and let*

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

*where  $L \in [0, \infty]$ . Then, the radius of convergence  $R = \frac{1}{L}$ .*

*Proof.* We find that

$$\frac{|z_{n+1}(z - c)^{n+1}|}{|a_n(z - c)^n|} = \frac{|a_{n+1}|}{|a_n|} |z - c| \rightarrow L|z - c|.$$

So, if  $|z - c| < \frac{1}{L}$ , then the ratio test tells us that the power series  $S(z)$  converges. Moreover, if  $|z - c| > \frac{1}{L}$ , then the ratio test tells us that the power series  $S(z)$  diverges. So, the radius of convergence  $R = \frac{1}{L}$ .  $\square$

We can use the ratio test to show that the derivative has the same radius of convergence. Let

$$S(z) = \sum_{n=0}^{\infty} a_n(z - c)^n$$

have radius of convergence  $R$ . Then, we have the derivative

$$S'(z) = \sum_{n=1}^{\infty} n a_n(z - c)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1}(z - c)^n.$$

So, the ratio is

$$\left| \frac{a_{n+1}}{a_n} \right| \left( 1 + \frac{1}{n} \right) \rightarrow \frac{1}{R} \cdot 1 = \frac{1}{R}.$$

Therefore, it has radius of convergence  $R$ . Similarly, for the antiderivative  $F(z) = w + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-c)^{n+1}$ , we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \left( 1 - \frac{1}{n+2} \right) \rightarrow \frac{1}{R} \cdot 1 = \frac{1}{R}.$$

Therefore, it has radius of convergence  $R$ .

So, a power series gives us a holomorphic function.

**Proposition 1.3.6.** *Let*

$$S(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$$

*be a power series with radius of convergence  $R > 0$ . Then,  $S : D \rightarrow \mathbb{C}$  is holomorphic, where*

$$D = \{z \in \mathbb{C} \mid |z-c| < R\},$$

*with*

$$S'(z) = \sum_{n=1}^{\infty} a_n n(z-c)^{n+1}.$$

Moreover, every holomorphic function can always be written as a power series. This is not true in the real case. For a power series, it is its own Taylor series, i.e.

$$S(z) = \sum_{n=0}^{\infty} \frac{S^{(n)}(c)}{n!} (z-c)^n.$$

Also, its antiderivative has the same radius of convergence.

**Proposition 1.3.7.** *Let*

$$S(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$$

*be a power series with radius of convergence  $R > 0$ . Then,  $S : D \rightarrow \mathbb{C}$  has an antiderivative, where*

$$D = \{z \in \mathbb{C} \mid |z-c| < R\},$$

*given by*

$$F(z) = w + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-c)^{n+1},$$

*for some  $w \in \mathbb{C}$*

## 1.4 Elementary functions

### Trigonometric functions

The exponential function  $\exp : \mathbb{C} \rightarrow \mathbb{C}$  is given by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \dots,$$

for some  $z \in \mathbb{C}$ . If we let  $(a_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{C}$  defined by  $a_n = \frac{1}{n!}$ , then

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

So, the ratio test tells us that the radius of convergence of the power series  $e^x$  is  $R = \infty$ .

The properties of the real exponential function extend to the complex exponential function. First, the derivative of the exponential function is itself.

**Proposition 1.4.1.** *Let  $z \in \mathbb{C}$ . Then,*

$$e'(x) = e(x).$$

*Proof.* We find that

$$\begin{aligned} e'(x) &= \sum_{n=1}^{\infty} \frac{1}{n!} n z^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e(x). \end{aligned}$$

□

Moreover, the following properties hold.

**Proposition 1.4.2.** *Let  $z, w \in \mathbb{C}$ . Then,  $e^{z+w} = e^z e^w$  and  $e^{-z} = \frac{1}{e^z}$ .*

*Proof.* Define the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  by  $f(z) = e^{z+w} e^{-z}$ . Then,

$$f'(z) = e^{z+w} e^{-z} + e^{z+w} \cdot -e^{-z} = 0.$$

In that case,  $f$  is a constant. Then,

$$f(z) = e^{z+w} e^{-z} = f(0) = e^{0+w} e^0 = e^w.$$

So,  $e^{z+w} e^{-z} = e^w$  for all  $z, w \in \mathbb{C}$ . This means that  $e^{z+w} = e^z e^w$ , and

$$e^{z+0} e^{-z} = e^0 = 1,$$

and so  $e^{-z} = \frac{1}{e^z}$ .

□

We can further define

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.\end{aligned}$$

Just using the power series, we will still find that the radii of convergence  $R = \infty$ . Also, we find that

$$\begin{aligned}\cos'(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 2nz^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} z^{2n-1} \\ &= \sum_{n=0}^{\infty} \frac{-(-1)^n}{(2n+1)!} z^{2n+1} = -\sin(z),\end{aligned}$$

and

$$\begin{aligned}\sin'(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) \cdot z^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos(z).\end{aligned}$$

Moreover,

$$e^{iz} = \frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2} = \cos(z) + i \sin(z).$$

We can also define the following trigonometric functions:

•

$$\tan(z) = \frac{\sin(z)}{\cos(z)},$$

•

$$\cosh(z) = \frac{e^z + e^{-z}}{2},$$

•

$$\sinh(z) = \frac{e^z - e^{-z}}{2}.$$

### Logarithmic function and powers

In  $\mathbb{C}$ , the exponential function is not injective, since

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z \cdot 1 = e^z.$$

So, we cannot define the logarithm function as the inverse of  $\exp : \mathbb{C} \rightarrow \mathbb{C}$ . We need to restrict both the domain and the codomain. Similarly, for any  $n \in \mathbb{Z}$ , the function  $z \mapsto z^n$  is not injective, since

$$(ze^{\frac{2\pi i}{n}})^n = z^n.$$

So, we would need to restrict the complex power function to find the inverse. To compute the inverse, we can use the inverse function theorem.

**Theorem 1.4.3.** *Let  $\Omega \subseteq \mathbb{C}$  be an open set, and let  $f : \Omega \rightarrow \mathbb{C}$  be a holomorphic function, and let  $z \in \Omega$  such that  $f'(z) \neq 0$ . Then, there exist open sets  $U \subseteq \Omega$  and  $V \subseteq \mathbb{C}$  with  $z \in U$  and  $w = f(z) \in V$ , such that  $f : U \rightarrow V$  is bijective, and  $f^{-1} : V \rightarrow U$  is holomorphic with*

$$\frac{d}{dz}f^{-1}(w) = \frac{1}{f'(z)} = \frac{1}{f'(f^{-1}(w))}.$$

Now, let  $z \in \mathbb{C}$ . We know that

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y).$$

For all  $x \in \mathbb{R}$ ,  $e^x \neq 0$ . Moreover, for all  $y \in \mathbb{R}$ , we cannot have both  $\cos y = 0$  and  $\sin y = 0$ . Therefore,  $e^z \neq 0$ . So, the inverse function theorem tells us that there are open sets  $U, V$  such that  $\exp : U \rightarrow V$  is bijective and holomorphic. A possible open set is

$$U = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi)\}.$$

This choice ensures that for all  $u_1, u_2 \in U$ ,  $u_1 - u_2 \neq 2k\pi$ , for some  $k \in \mathbb{Z}$ , and so the function is injective. Moreover,

$$V = \exp(\mathbb{C}) = \mathbb{C} \setminus (-\infty, 0].$$

This gives us a bijection  $\exp : U \rightarrow V$ . The inverse function is  $\log : V \rightarrow U$ , and is called the principal branch of the (complex) logarithm.

Now, we will find all the complex logarithms for some  $z \in \mathbb{C}$ .

**Example 1.4.4.** *The complex logarithms of  $z = -1 + \sqrt{3}i$  is given by*

$$\log z = \log 2 + \frac{2\pi}{3}i + 2k\pi i,$$

for some  $k \in \mathbb{Z}$ .

*Proof.* We find that

$$|z| = \sqrt{1+3} = 2.$$

We have  $\arg z = \theta + 2k\pi$  for  $k \in \mathbb{Z}$ , where  $\cos \theta = -1$  and  $\sin \theta = \sqrt{3}$ . So,

$$\theta = \pi - \tan^{-1} \sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Therefore,

$$\log z = \log(|z|e^{i \arg z}) = \log |z| + i \arg z = \log 2 + \frac{2\pi}{3}i + 2k\pi i,$$

for some  $k \in \mathbb{Z}$ . □

Now, we will define complex powers for a complex number:



**Definition 1.4.5.** Let  $z, w \in \mathbb{C}$ . We define

$$z^w = e^{w \log z},$$

where  $\log z$  are the complex logarithms of  $z$ .

A complex number has many complex logarithms, so we can limit ourselves with the principal branch

$$\log : \mathbb{C} \setminus (-\infty, 0] \rightarrow \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi)\}.$$

Then, the function  $f : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$  given by  $f(z) = z^w$  is a well-defined holomorphic function for all  $w \in \mathbb{C}$ , and  $z^u z^v = z^{u+v}$  for all  $u, v \in \mathbb{C}$  and  $z \in \mathbb{C} \setminus (-\infty, 0]$ .

We will now compute all the values of  $z^i$ , for some  $z \in \mathbb{C}$ . By definition,  $z^i = e^{i \log z}$ . Moreover, we know that

$$\log z = \log |z| + i \arg z.$$

Therefore,

$$z^i = e^{i \log z} = e^{i \log |z|} e^{-\arg z}.$$

### 1.5 Cauchy-Riemann equations

We saw that a vector function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  being differentiable is not equivalent to saying the corresponding function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is differentiable. In this section, we characterise complex differentiation in terms of vector differentiation.

Consider a holomorphic function  $f : \Omega \rightarrow \mathbb{C}$  by

$$f(x + iy) = u(x, y) + iv(x, y).$$

We can see this as a vector function  $f : \bar{\Omega} \rightarrow \mathbb{R}^2$ , where

$$\bar{\Omega} = \{(x, y) \in \mathbb{R}^2 \mid x + iy \in \Omega\} \subseteq \mathbb{R}^2,$$

and  $f(x, y) = (u(x, y), v(x, y))$ . Since  $f$  is differentiable for all  $z \in \Omega$ , the limit

$$\lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h}$$

exists, no matter how  $h$  approaches 0. In particular, if we restrict  $h = \varepsilon$ , for  $\varepsilon \in \mathbb{R}$ , we find that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{f(z + \varepsilon) - f(z)}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \frac{u(x + \varepsilon, y) - u(x, y)}{\varepsilon} + i \lim_{\varepsilon \rightarrow 0} \frac{v(x + \varepsilon, y) - v(x, y)}{\varepsilon} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}. \end{aligned}$$

Instead, if we restrict  $h = i\varepsilon$ , for  $\varepsilon \in \mathbb{R}$ , we find that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{f(z + i\varepsilon) - f(z)}{i\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \frac{u(x, y + \varepsilon) - u(x, y)}{i\varepsilon} + i \lim_{\varepsilon \rightarrow 0} \frac{v(x, y + \varepsilon) - v(x, y)}{i\varepsilon} \\ &= \frac{1}{i} \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = -i \frac{\partial f}{\partial y}. \end{aligned}$$

Therefore,

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating the real and the imaginary parts, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the Cauchy-Riemann equations. Geometrically, this means that a differentiable function in  $\mathbb{C}$  preserves angles any two lines in the complex plane.

We have shown that if a complex function  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic, then the vector function  $f : \bar{\Omega} \rightarrow \mathbb{C}$  must satisfy the Cauchy-Riemann equations. It also turns out that if a vector function  $f : \bar{\Omega} \rightarrow \mathbb{C}$  satisfies the Cauchy-Riemann equations (and the partial derivatives are continuous), then the complex function  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic.

**Theorem 1.5.1** (Cauchy-Riemann equations). *Let  $\Omega \subseteq \mathbb{C}$  be open, and let  $w : \Omega \rightarrow \mathbb{C}$  be a function, denoted by  $w(x + iy) = u(x, y) + iv(x, y)$ . Then,*

- if  $w$  is holomorphic, then  $w$  satisfies the Cauchy-Riemann equations, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

- if  $w$  satisfies the Cauchy-Riemann equations and the partial derivatives are continuous, then  $w$  is holomorphic.

We will use the Cauchy-Riemann equations to show that the exponential function is holomorphic in  $\mathbb{C}$ .

**Example 1.5.2.** The function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by

$$f(z) = e^z$$

is holomorphic.

*Proof.* For  $x + iy \in \mathbb{C}$ , we have

$$f(x + iy) = e^{x+iy} = e^x \cos(y) + ie^x \sin(y).$$

Let  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given by  $u(x, y) = e^x \cos(y)$  and  $v(x, y) = e^x \sin(y)$ . Then,

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos(y) & \frac{\partial u}{\partial y} &= -e^x \sin(y) \\ \frac{\partial v}{\partial x} &= e^x \sin(y) & \frac{\partial v}{\partial y} &= e^x \cos(y). \end{aligned}$$

So, the Cauchy-Riemann equations are satisfied for all  $x + iy \in \mathbb{C}$ , and the partial derivatives are continuous. This implies that the function  $f$  is holomorphic.  $\square$

## 1.6 Harmonic functions

A function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  is harmonic if it satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

It turns out that each complex-differentiable function defines two harmonic functions.

**Proposition 1.6.1.** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function. Denote*

$$f(x + iy) = u(x, y) + iv(x, y),$$

*where  $u$  and  $v$  have continuous 2-nd order partial derivatives. Then,  $u$  and  $v$  are harmonic.*

*Proof.* We find that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \frac{\partial u}{\partial y} \\ &= \frac{\partial}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial}{\partial y} \frac{\partial u}{\partial x} \\ &= \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0. \end{aligned}$$

So,  $u$  is harmonic. Similarly,  $v$  is harmonic.  $\square$

For a complex-differentiable function, we have 2 harmonic functions- one for the real part and one for the complex part. We say that the 2 vector functions are harmonic conjugates.

**Definition 1.6.2.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function. Denote

$$f(x + iy) = u(x, y) + iv(x, y).$$

We say that  $u$  and  $v$  are *harmonic conjugates* of each other.

We can find the Harmonic conjugate for any harmonic function if the domain is path connected.

**Proposition 1.6.3.** *Let  $\Omega \subseteq \mathbb{C}$  be open and path connected, and let  $u : \Omega \rightarrow \mathbb{R}$  be harmonic such that  $u$  has continuous second order derivatives. Then,  $u$  has a harmonic conjugate.*

A set  $V$  is path connected if for any two values  $x_1, x_2 \in V$ , there is a path from  $x_1$  to  $x_2$ .

We will now find the harmonic conjugate for a vector function.

**Example 1.6.4.** *The function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $u(x, y) = x^3 - 3xy^2$  is harmonic, with a harmonic conjugate*

$$v(x, y) = 3x^2y - y^3.$$

*Proof.* We find that

$$\begin{aligned}\frac{\partial u}{\partial x} &= 3x^2 - 3y^2 & \frac{\partial u}{\partial y} &= -6xy \\ \frac{\partial^2 u}{\partial x^2} &= 6x & \frac{\partial^2 u}{\partial y^2} &= -6x\end{aligned}$$

Therefore, the function  $u$  is harmonic. Now, let  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the harmonic conjugate of  $u$ . We know that

$$3x^2 - 3y^2 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

So,

$$v = \int (3x^2 - 3y^2) dy = 3x^2y - y^3 + f(x),$$

for some  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Moreover, we know that

$$-6xy = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy - f'(x).$$

So,  $f$  is a constant. Setting  $f(x) = 0$ , we get the harmonic conjugate

$$v(x, y) = 3x^2y - y^3.$$

□

Using this result, we find that there is an analytic function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by

$$f(x + iy) = (x^3 - 3xy^2) + i(3x^2y - y^3) = x^3 + 3ix^2y - 3xy^2 - iy^3.$$

In fact, this function is  $z \mapsto z^3$ .