CHAPTER 2

INTEGRATION

2.1 Measurable Functions

Definition 2.1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f\colon X \to \mathbb{R} \cup \{\pm\infty\}$ is *measurable* if for all $a \in \mathbb{R}$,

$$f^{-1}(a, \infty] \in \mathcal{A}.$$

Proposition 2.1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Then, the following are equivalent:*

1. *f is measurable;*
2. *for all $a \in \mathbb{R}$, $f^{-1}[a, \infty] \in \mathcal{A}$;*
3. *for all $a \in \mathbb{R}$, $f^{-1}[-\infty, a) \in \mathcal{A}$;*
4. *for all $a \in \mathbb{R}$, $f^{-1}[-\infty, a] \in \mathcal{A}$.*

Proof. For $a \in \mathbb{R}$, we know that

$$f^{-1}[a, \infty] = [f^{-1}[-\infty, a)]^c, \qquad f^{-1}[-\infty, a] = [f^{-1}(a, \infty]]^c.$$

So, $1 \iff 4$ and $2 \iff 3$.

$1 \implies 2$ We know that

$$[a, \infty] = \bigcap_{k=1}^{\infty} (a - \tfrac{1}{k}, \infty].$$

So,

$$f^{-1}[a, \infty] = \bigcup_{k=1}^{\infty} f^{-1}(a - \tfrac{1}{k}, \infty].$$

$2 \implies 1$ We know that

$$(a, \infty] = \bigcup_{k=1}^{\infty} [a + \tfrac{1}{k}, \infty].$$

So,

$$f^{-1}(a, \infty] = \bigcup_{k=1}^{\infty} f^{-1}[a + \tfrac{1}{k}, \infty].$$

□

Proposition 2.1.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Then, the function $|f|\colon X \to \mathbb{R} \cup \{\pm\infty\}$ is measurable, where $|f|(x) = |f(x)|$.*

Proof. Let $a \in \mathbb{R}$. If $a < 0$, then

$$|f|^{-1}(a, \infty] = X \in \mathcal{A}.$$

If $a \geq 0$, then

$$|f|^{-1}(a, \infty) = f^{-1}(a, \infty] \cup f^{-1}[-\infty, -a) \in \mathcal{A}.$$

Hence, $|f|$ is measurable. □

Proposition 2.1.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R}$ be a constant function. Then, f is measurable.*

Proof. Let $x = f(0)$. Then, for all $a \in \mathbb{R}$,

$$f^{-1}(a, \infty] = \begin{cases} \varnothing & a \leq x \\ X & a > x. \end{cases}$$

So, $f^{-1}(a, \infty] \in \mathcal{A}$. This implies that f is measurable. □

Proposition 2.1.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $A \subseteq X$. Define the function $\chi_A \colon X \to \mathbb{R}$ by*

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & \textit{otherwise.} \end{cases}$$

Then, χ_A is measurable if and only if A is measurable.

Proof. Let $a \in \mathbb{R}$. We have

$$\chi_A^{-1}(a, \infty] = \begin{cases} \varnothing & a \geq 1 \\ A & a \in [0, 1) \\ X & a > 0. \end{cases}$$

So, χ_A is measurable if and only if A is measurable. □

Proposition 2.1.6. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, f is measurable.*

Proof. Let $a \in \mathbb{R}$. Since $[-\infty, a)$ is open and f is continuous, we find that

$$f^{-1}[-\infty, a)$$

is open. Hence, the set $f^{-1}[-\infty, a)$ can be written as a union of (open) intervals, meaning that f is measurable. □

Proposition 2.1.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, g\colon \mathcal{A} \to \mathbb{R} \cup \{\pm\infty\}$ be measurable functions. Then, the following functions are measurable:*

1. *$f + g$;*
2. *λf for $\lambda \in \mathbb{R}$;*
3. *$f \cdot g$;*

4. $f \wedge g = \max(f, g)$;

Proof.

1. Let $h = f \wedge g$. Then, for $a \in \mathbb{R}$ and $x \in \mathbb{R}$,
$$\begin{aligned} x \in h^{-1}(a, \infty] &\iff f(x) + g(x) < a \\ &\iff f(x) < a - g(x) \\ &\iff \exists r \in \mathbb{Q} \text{ s.t. } f(x) < r < a - g(x) \\ &\iff \exists r \in \mathbb{Q} \text{ s.t. } f(x) < r \text{ and } g(x) < a - r \\ &\iff x \in f^{-1}(r, \infty] \cap g^{-1}(a - r, \infty]. \end{aligned}$$
Hence,
$$h^{-1}(a, \infty] = \bigcup_{r \in \mathbb{Q}} f^{-1}(r, \infty] \cap g^{-1}(a - r, \infty].$$
Since this is a countable union of sets in $\mathcal{A}$, we find that $h^{-1}(a, \infty] \in \mathcal{A}$. Hence, h is measurable.

2. If $\lambda = 0$, then λf is a constant, meaning that it is measurable. Otherwise, $\lambda \neq 0$. In that case, let $a \in \mathbb{R}$. We find that
$$\begin{aligned} (\lambda f)^{-1}(a, \infty] &= \{x \in X \mid \lambda f(x) > a\} \\ &= \{x \in X \mid f(x) > \tfrac{a}{\lambda}\} \\ &= f^{-1}(\tfrac{a}{\lambda}, \infty]. \end{aligned}$$
Hence, since f is measurable, we find that λf is measurable.

3. We first show that $h = f^2$ is measurable. So, let $a \in \mathbb{R}$. If $a < 0$, then $h^{-1}(a, \infty] = X \in \mathcal{A}$. Then,
$$\begin{aligned} h^{-1}(a, \infty] &= \{x \in X \mid h(x) > a\} \\ &= \{x \in X \mid f(x) > \sqrt{a} \text{ or } f(x) < -\sqrt{a}\} \\ &= f^{-1}(\sqrt{a}, \infty] \cup f^{-1}[-\infty, -\sqrt{a}). \end{aligned}$$
Since f is measurable, we find that h is measurable. Hence,
$$f \cdot g = \frac{1}{4}[(f + g)^2 - (f - g)^2]$$
is measurable.

4. Let $h = f \wedge g$. Then, for $a \in \mathbb{R}$,
$$\begin{aligned} h^{-1}(a, \infty] &= \{x \in X \mid h(x) > a\} \\ &= \{x \in X \mid f(x) > a \text{ or } g(x) > a\} \\ &= f^{-1}(a, \infty] \cup g^{-1}(a, \infty]. \end{aligned}$$
Since f and g are measurable, we find that h is measurable.

□

Definition 2.1.8. Let X be a set and $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n\colon X \to \mathbb{R} \cup \{\pm\infty\}$. We define the functions $\inf f_n, \sup f_n \colon X \to \mathbb{R} \cup \{\pm\infty\}$

$$(\inf f_n)(x) = \inf\{f_n(x) \mid n \in \mathbb{Z}_{\geq 1}\}, \qquad (\sup f_n)(x) = \sup\{f_n(x) \mid n \in \mathbb{Z}_{\geq 1}\}.$$

Definition 2.1.9. Let X be a set and $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n\colon X \to \mathbb{R} \cup \{\pm\infty\}$. Then, we define the functions $\inf f_n, \sup f_n, \lim f_n \colon X \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\begin{aligned}(\inf f_n)(x) &= \inf\{f_n(x) \mid n \in \mathbb{Z}_{\geq 1}\}\\ (\sup f_n)(x) &= \sup\{f_n(x) \mid n \in \mathbb{Z}_{\geq 1}\}\\ (\lim f_n)(x) &= \lim_{n\to\infty} f_n(x).\end{aligned}$$

The function $\lim f_n$ need not exist in general.

Definition 2.1.10. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Define

$$\liminf a_n = \sup_{n=1}^{\infty} \inf\{a_m \mid m \geq n\}, \qquad \limsup_{n\to\infty} a_n = \inf_{n=1}^{\infty} \sup\{a_m \mid m \geq n\}.$$

Proposition 2.1.11. *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then,*

1. *The value $\liminf_{n\to\infty} a_n$ is the smallest accumulation point of (a_n);*
2. *The value $\limsup_{n\to\infty} a_n$ is the largest accumulation point of (a_n).*

Proof.

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□

Proposition 2.1.12. *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then,*

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n,$$

with

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

if and only if

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Definition 2.1.13. Let X be a set and $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n\colon X \to \mathbb{R} \cup \{\pm\infty\}$. We define the functions $\liminf f_n, \limsup f_n \colon X \to \mathbb{R} \cup \{\pm\infty\}$

$$\begin{aligned}(\liminf f_n)(x) &= \liminf f_n(x) = \sup_{n=1}^{\infty} \inf\{f_n(x) \mid n \in \mathbb{Z}_{\geq 1}\},\\ (\limsup f_n)(x) &= \limsup f_n(x) = \inf_{n=1}^{\infty} \sup\{f_n(x) \mid n \in \mathbb{Z}_{\geq 1}\}.\end{aligned}$$

Proposition 2.1.14. *Let X be a set and $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $f_n\colon X \to \mathbb{R} \cup \{\pm\infty\}$. Then, the following functions are measurable:*

1. *$\inf f_n$ and $\sup f_n$;*
2. *$\liminf f_n$ and $\limsup f_n$;*
3. *the limit $\lim f_n$, if it exists.*

Proof.

- Let $a \in \mathbb{R}$. Then, for $x \in \mathbb{R}$,

$$\begin{aligned} x \in (\inf f_n)^{-1}(a, \infty] &\iff (\inf f_n)(x) > a \\ &\iff f_n(x) > a \ \forall n \in \mathbb{Z}_{\geq 1} \\ &\iff x \in \bigcap_{n=1}^{\infty} f_n^{-1}(a, \infty]. \end{aligned}$$

  Since f_n is measurable for all $n \in \mathbb{Z}_{\geq 1}$, we find that

$$(\inf f_n)^{-1}(a, \infty] = \bigcap_{n=1}^{\infty} f_n^{-1}(a, \infty]$$

  is measurable. Hence, $\inf f_n$ is measurable. We have

$$\sup f_n = -\inf(-f_n),$$

  so $\sup f_n$ is measurable as well.

- Define the sequence of functions $(g_n)_{n=1}^{\infty}$, $g_n \colon X \to \mathbb{R}$ by

$$g_n(x) = \inf\{f_m(x) \mid m \geq n\}.$$

  By the result above, we know that (g_n) is a sequence of measurable functions. We know that

$$\liminf f_n = \sup g_n,$$

  so $\liminf f_n$ is measurable. Similarly, $\limsup f_n$ is measurable.

- If $\lim f_n$ exists, then $\lim f_n = \liminf f_n$. So, $\lim f_n$ is measurable.

□

Definition 2.1.15. Let X be a set and $f \colon X \to \mathbb{R}$ be a function. We say that f is *simple* if the image $f(X)$ is finite.

Definition 2.1.16. Let X be a set and $(f_n)_{n=1}^{\infty}$ a sequence of functions $f_n \colon X \to \mathbb{R}$, and let $f \colon X \to \mathbb{R}$ be a function.

- We say that (f_n) *converges pointwise* to f if for every $\varepsilon > 0$ and $x \in X$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$.
- We say that (f_n) *converges uniformly* to f if for every $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $x \in X$ and $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$.

Theorem 2.1.17. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f\colon X \to \mathbb{R}$ be a function. Then, there exists a sequence $(s_n)_{n=1}^{\infty}$ of simple functions $s_n\colon X \to \mathbb{R}$ such that $s_n \to f$ pointwise. Moreover,*

1. *if f is bounded, then the sequence (s_n) can be chosen such that $s_n \to f$ uniformly.*
2. *if $f \geq 0$, then the sequence (s_n) can be chosen to be positive increasing (i.e. $0 \leq s_1 \leq s_2 \leq \dots$).*
3. *if f is measurable, then the sequence (s_n) can be chosen such that s_n is measurable for all $n \in \mathbb{Z}_{\geq 1}$.*

Proof.

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Definition 2.1.18. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $s\colon X \to \mathbb{R}$ be a simple measurable function denoted by

$$s = \lambda_1 \chi_{E_1} + \lambda_2 \chi_{E_2} + \dots + \lambda_n \chi_{E_n},$$

where $E_k \in \mathcal{A}$ for $1 \leq i \leq n$. We define the *integral of s over X* as

$$\int_X s \, d\mu = \sum_{k=1}^{n} \lambda_k \mu(E_k).$$

In general, for $A \in \mathcal{A}$, we define the *of s over A* to be

$$\int_X s \, d\mu = \sum_{k=1}^{n} \lambda_k \mu(E_k \cap A).$$

Proposition 2.1.19. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $s, t\colon X \to \mathbb{R}$ be simple and measurable and let $A \in \mathcal{A}$. Then,*

1. *$s + t$ is simple and measurable and*
$$\int_A s + t \, d\mu = \int_A s \, d\mu + \int_A t \, d\mu.$$
2. *$s \geq t$ implies*
$$\int_A s \, d\mu \geq \int_A t \, d\mu.$$
3. *for all $\lambda \in \mathbb{R}$, λs is simple and measurable, with*
$$\int_A \lambda s \, d\mu = \lambda \int_A s \, d\mu.$$

Proof.

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Proposition 2.1.20. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $s\colon X \to \mathbb{R}$ be a simple measurable function. Then, the function $\nu\colon \mathcal{A} \to \mathbb{R}$ given by*

$$\nu(A) = \int_A s \; d\mu$$

is a measure.

Proof. Let

$$s = \lambda_1 \chi_{E_1} + \lambda_2 \chi_{E_2} + \cdots + \lambda_n \chi_{E_n}.$$

We find that

$$\begin{aligned} \nu(\varnothing) &= \int_\varnothing s \; d\mu \\ &= \sum_{k=1}^n \lambda_k \mu(E_k \cap \varnothing) \\ &= \sum_{k=1}^n \lambda_k \mu(\varnothing) = 0. \end{aligned}$$

Now, let $(A_n)_{n=1}^\infty$ be a sequence of disjoint sets in $\mathcal{A}$. Denote

$$A = \bigcup_{j=1}^\infty A_j.$$

We find that

$$\begin{aligned} \nu(A) &= \int_A s \; d\mu \\ &= \sum_{k=1}^n \lambda_k \mu(E_k \cap A) \\ &= \sum_{k=1}^n \lambda_k \mu\left(\bigcup_{j=1}^\infty (E_k \cap A_j)\right) \\ &= \sum_{k=1}^n \lambda_k \cdot \sum_{j=1}^\infty \mu(E_k \cap A_j) \\ &= \sum_{j=1}^\infty \sum_{k=1}^n \lambda_k \mu(E_k \cap A_j) \\ &= \sum_{j=1}^\infty \int_{A_j} s \; d\mu = \sum_{j=1}^\infty \nu(A_j). \end{aligned}$$

Hence, ν is a measure. □

Definition 2.1.21. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \colon X \to \mathbb{R}$ be measurable with $f \geq 0$. The *integral of f over A*, for some $A \in \mathcal{A}$, is given by

$$\int_A f \, d\mu := \sup \left\{ \int_A s \, d\mu \mid 0 \leq s \leq f \text{ simple measurable} \right\}.$$

For $f \colon X \to \mathbb{R}$, define

$$f_+ = \max(f(x), 0), \qquad f_- = \max(-f(x), 0).$$

We say that *f is integrable over A*, for some $A \in \mathcal{A}$, if both

$$\int_A f_+ \, d\mu < \infty, \qquad \int_A f_- \, d\mu < \infty.$$

In that case, we define the *integral of f over A* by

$$\int_A f \, d\mu = \int_A f_+ \, d\mu - \int_A f_- \, d\mu.$$

If f is integrable, we denote $f \in \mathcal{L}_1(X, \mu) = \mathcal{L}(X, \mu)$.

Proposition 2.1.22. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \colon X \to \mathbb{R}$ be a simple measurable function. Then,*

$$\int_A f \, d\mu = \sup \left\{ \int_A s \, d\mu \mid 0 \leq s \leq f \text{ simple measurable} \right\}.$$

Proof. □

Proposition 2.1.23. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $f, g \in \mathcal{L}(X, \mu)$ and $A, B \in \mathcal{A}$. Then,*

1. *a) $\int_A \lambda f \, d\mu = \lambda \int_A f \, d\mu$ for $\lambda \in \mathbb{R}$;*
   b) $\int_A (f + g) \, d\mu \geq \int_A f \, d\mu + \int_A g \, d\mu$;
2. *if $f \leq g$, then $\int_A f \, d\mu \leq \int_A g \, d\mu$;*
3. *$f \in \mathcal{L}(X, \mu)$ if and only if $|f| \in \mathcal{L}(X, \mu)$, with*
$$\left| \int_A f \, d\mu \right| \leq \int_A |f| \, d\mu;$$
4. *a) if $\mu(A) = 0$, then*
$$\int_A f \, d\mu = 0;$$
   b) if $A \subseteq B$ and $\mu(B \setminus A) = 0$,
$$\int_A f \, d\mu = \int_B f \, d\mu;$$
   c) If
$$\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0,$$
   then
$$\int_A f \, d\mu = \int_A g \, d\mu.$$

Proof.

1. a) First, assume that $f \geq 0$. If $\lambda = 0$, then

$$\int_A \lambda f \; d\mu = 0 = \lambda \int_A f \; d\mu.$$

Otherwise, if $\lambda > 0$, then

$$\begin{aligned}
\int_A \lambda f \; d\mu &= \sup \left\{ \int_A \lambda s \; d\mu \mid 0 \leq s \leq \lambda f \text{ sm} \right\} \\
&= \sup \left\{ \lambda \int_A \frac{1}{\lambda} s \; d\mu \mid 0 \leq \frac{1}{\lambda} s \leq f \text{ sm} \right\} \\
&= \lambda \sup \left\{ \int_A t \; d\mu \mid 0 \leq t \leq f \text{ sm} \right\} \\
&= \lambda \int_A f \; d\mu.
\end{aligned}$$

Now, for a general f, we have

$$\begin{aligned}
\lambda \int_A f \; d\mu &= \lambda \left(\int_A f_+ \; d\mu - \int_A f_- \; d\mu \right) \\
&= \lambda \int_A f_+ \; d\mu - \lambda \int_A f_- \; d\mu \\
&= \int_A \lambda f_+ \; d\mu - \int_A \lambda f_- \; d\mu \\
&= \int_A \lambda f \; d\mu.
\end{aligned}$$

b) We find that

$$\begin{aligned}
\int_A f \; d\mu + \int_A g \; d\mu &= \sup \left\{ \int_A s \; d\mu \mid 0 \leq s \leq f \text{ simple measurable} \right\} \\
&\quad + \sup \left\{ \int_A t \; d\mu \mid 0 \leq t \leq g \text{ simple measurable} \right\} \\
&= \sup \left\{ \int_A s \; d\mu + \int_A t \; d\mu \mid 0 \leq s \leq f, 0 \leq t \leq g \text{ s.m.} \right\} \\
&= \sup \left\{ \int_A (s+t) \; d\mu \mid 0 \leq s + t \leq f + g \text{ s.m.} \right\} \\
&\geq \sup \left\{ \int_A s \; d\mu \mid 0 \leq s \leq f + g \text{ s.m.} \right\} \\
&= \int_A f + g \; d\mu.
\end{aligned}$$

2. For all $0 \leq s \leq f$, we have $0 \leq s \leq g$. Hence, by the supremum property, we find that

$$\int_A f \; d\mu \leq \int_A g \; d\mu.$$

3. We have

$$\begin{aligned}\left|\int_A f\ d\mu\right| &= \left|\int_A f_+\ d\mu - \int_A f_-\ d\mu\right| \\ &\leq \int_A f_+ d\mu + \int_A f_-\ d\mu = \int_A |f|\ d\mu.\end{aligned}$$

4. a) Let $0 \leq s \leq f$ be simple and measurable. Then,

$$\int_A s\ d\mu = \sum_{t \in s(X)} t \cdot \mu(f^{-1}(t) \cap A) = \sum_{t \in s(X)} t \cdot 0 = 0.$$

Hence,

$$\int_A f\ d\mu = \sup\{0\} = 0.$$

b) For any $C \in \mathcal{A}$, we have

$$\mu(C \cap B) = \mu(C \cap A) + \mu(C \cap B \setminus A) = \mu(C \cap A)$$

since $C \cap B \setminus A \subseteq B \setminus A$. Hence, for any simple measurable function $s \geq 0$,

$$\begin{aligned}\int_B s\ d\mu &= \sum_{t \in s(X)} t \cdot \mu(f^{-1}(t) \cap B) \\ &= \sum_{t \in s(X)} t \cdot \mu(f^{-1}(t) \cap A) \\ &= \int_A s\ d\mu.\end{aligned}$$

This implies that for a function $f \geq 0$,

$$\begin{aligned}\int_B f\ d\mu &= \sup\left\{\int_B s\ d\mu \mid 0 \leq s \leq f \text{ sm}\right\} \\ &= \sup\left\{\int_A s\ d\mu \mid 0 \leq s \leq f \text{ sm}\right\} \\ &= \int_A f\ d\mu.\end{aligned}$$

Hence, for an arbitrary function f,

$$\begin{aligned}\int_B f\ d\mu &= \int_B f_+\ d\mu - \int_B f_-\ d\mu \\ &= \int_A f_+\ d\mu - \int_A f_-\ d\mu \\ &= \int_A f\ d\mu.\end{aligned}$$

c) Assume first that $f \geq 0$. Let

$$B = \{x \in X \mid f(x) = g(x)\}.$$

We know that $\mu(B \setminus A) = 0$. Now, let $0 \leq s \leq f$ be simple measurable. Define the function $s' \colon X \to [0, \infty)$ by

$$s'(x) = \begin{cases} s(x) & x \in B \\ 0 & \text{otherwise.} \end{cases}$$

We have $s'(X) = s(X) \cup \{0\}$, meaning that s' is simple. Moreover, s' is measurable, since

$$(s')^{-1}(a, \infty) = \begin{cases} X & x < 0 \\ s^{-1}(a, \infty) \cap B & \text{otherwise.} \end{cases}$$

Moreover, $0 \leq s' \leq g$ by construction with

$$\begin{aligned} \int_B s \; d\mu &= \sum_{t \in s(X)} t \cdot \mu(s^{-1}(t) \cap B) \\ &= \sum_{t \in s'(X)} t \cdot \mu((s')^{-1}(t) \cap B) \\ &= \int_B s' \; d\mu. \end{aligned}$$

Hence,

$$\left\{ \int_B s \; d\mu \mid 0 \leq s \leq f \text{ sm} \right\} = \left\{ \int_B s \; d\mu \mid 0 \leq s \leq g \text{ sm} \right\},$$

meaning that

$$\int_A f \; d\mu = \int_B f \; d\mu = \int_B g \; d\mu = \int_A g \; d\mu.$$

Hence, for an arbitrary measurable f,

$$\begin{aligned} \int_A f \; d\mu &= \int_A f_+ \; d\mu - \int_A f_- \; d\mu \\ &= \int_A g_+ \; d\mu - \int_A g_- \; d\mu \\ &= \int_A g \; d\mu. \end{aligned}$$

$\square$

2.2 Convergence Theorems

Theorem 2.2.1 (Lebesgue's Montone Convergence Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $f_n \colon X \to [0, \infty]$ such that $0 \leq f_1 \leq f_2 \leq \dots$. Define the function $f \colon X \to [0, \infty]$ by $f(x) = \lim_{n \to \infty} f_n(x)$. Then, f is measurable, with*

$$\lim_{n \to \infty} \int_A f_n \, d\mu = \int_A f \, d\mu = \int_A \lim_{n \to \infty} f_n \, d\mu.$$

In particular, if the sequence $(\int_A f_n \, d\mu)$ is bounded above, then f is integrable.

Proof. By monotonicity, we know that $f_n \leq f$. Hence,

$$\int_A f_n \, d\mu \leq \int_A f \, d\mu,$$

meaning that

$$\lim_{n \to \infty} \int_A f_n \, d\mu \leq \int_A f \, d\mu.$$

We now show that

$$\lim_{n \to \infty} \int_A f_n \, d\mu \geq \int_A f \, d\mu.$$

□

Proposition 2.2.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $0 \leq f_1 \leq f_2 \leq \dots$ almost everywhere, i.e. for all $i \in \mathbb{Z}_{\geq 1}$,*

$$\mu(\{x \in X \mid f_i(x) > f_{i+1}(x)\}) = 0.$$

Then, f is measurable, with

$$\lim_{n \to \infty} \int_A f_n \, d\mu = \int_A f \, d\mu = \int_A \lim_{n \to \infty} f_n \, d\mu.$$

Proof. □

Proposition 2.2.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $(g_n)_{n=1}^{\infty}$ be a sequence of measurable functions that are non-negative almost everywhere. Then,*

$$\sum_{n=1}^{\infty} \int_A g_n \, d\mu = \int_A \sum_{n=1}^{\infty} g_n \, d\mu.$$

Proof. Define the sequence $(f_n)_{n=1}^{\infty}$ by

$$f_n = \sum_{k=1}^{n} g_k.$$

Since g_n are non-negative, we know that (f_n) is non-decreasing. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \int_A g_n \, d\mu &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_A g_k \, d\mu \\ &= \lim_{n\to\infty} \int_A \sum_{k=1}^{n} g_k \, d\mu \\ &= \int_A \lim_{n\to\infty} \sum_{k=1}^{n} g_k \, d\mu \\ &= \int_A \sum_{n=1}^{\infty} g_n \, d\mu.\end{aligned}$$

□

Proposition 2.2.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to [0,\infty]$ be measurable. Then, $f = 0$ almost everywhere if and only if*

$$\int_A f \, d\mu = 0.$$

for all $A \in \mathcal{A}$.

Proof. If $f = 0$ almost everywhere, then for all $A \in \mathcal{A}$,

$$\int_A f \, d\mu = \int_A 0 \, d\mu = 0.$$

Now, assume that $f \neq 0$ almost everywhere. In that case, let $A = f^{-1}(0,\infty]$, with $\mu(A) = \varepsilon > 0$. Now, define $A_n = f^{-1}(\frac{1}{n},\infty]$. We know that $A_n \in \mathcal{A}$ for all $n \in \mathbb{Z}_{\geq 1}$, since f is measurable, with

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Hence,

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) = \varepsilon.$$

This implies that there exists an $n \in \mathbb{Z}_{\geq 1}$ such that $\mu(A_n) > \frac{\varepsilon}{2} > 0$. Now, define the function $g\colon X \to [0,\infty)$ by $g = \frac{1}{n}\chi_{A_n}$. For $x \in X$, if $x \notin A_n$, then $g(x) = 0 \leq f(x)$, and if $x \in A_n$, then $g(x) = \frac{1}{n} < f(x)$. So, $g \leq f$, meaning that

$$\int_X f \, d\mu \geq \int_X g \, d\mu = \frac{1}{n}\mu(A_n) > 0.$$

□

Lemma 2.2.5 (Fatou's Lemma)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(f_n)_{n=1}^{\infty}$ be a sequence on non-negative measurable functions on X. Then,*

$$\int_A \liminf_{n\to\infty} f_n(x) \, d\mu \leq \liminf_{n\to\infty} \left(\int_A f_n \, d\mu \right).$$

Proof. □

Theorem 2.2.6 (Lebesgue's Dominated Convergence Theorem). *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions $f_n \colon X \to \mathbb{R}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. If there exists a $g \colon X \to [0, \infty]$ such that $|f_n| \leq g$ for all $n \in \mathbb{Z}_{\geq 1}$, then*

1. *f_n and f are integrable for all $n \in \mathbb{Z}$;*
2. *$\int_X |f_n - f| \; d\mu \to 0$ as $n \to \infty$;*
3. *$\lim_{n\to\infty} \int_X f_n \; d\mu = \int_X \lim_{n\to\infty} f_n \; d\mu = \int_X f \; d\mu$.*

Proof.

1. Since g is integrable with $|f_n| \leq g$ for all $n \in \mathbb{Z}_{\geq 1}$, we find that
$$\int_X |f_n| \; d\mu \leq \int_X g \; d\mu < \infty.$$
So, $|f_n|$ is integrable, meaning that f_n is integrable. Moreover,
$$\int_X f \; d\mu \leq \int_X g \; d\mu < \infty,$$
meaning that f is also integrable.
2. For $n \in \mathbb{Z}_{\geq 1}$, we have
$$|f_n - f| \leq |f_n| + |f| \leq g + |f| < 2g,$$
meaning that $h := g + |f|$ is integrable. Moreover,
$$\begin{aligned} \int_X h \; d\mu &= \int_X \lim_{n\to\infty} h - |f_n - f| \; d\mu \\ &= \int_X \liminf_{n\to\infty} h - |f_n - f| \; d\mu \\ &\leq \liminf_{n\to\infty} \int_X h - |f_n - f| \; d\mu \\ &= \int_X h \; d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \; d\mu. \end{aligned}$$
Hence, we find that
$$0 \leq \liminf \int_X |f_n - f| \; d\mu \leq \limsup \int_X |f_n - f| \; d\mu \leq 0.$$
This implies that $\int_X |f_n - f| \; d\mu \to 0$ as $n \to \infty$.
3. We find that for all $n \in \mathbb{Z}_{\geq 1}$,
$$\left| \int_X f_n \; d\mu - \int_X f \; d\mu \right| \leq \int_X |f_n - f| \; d\mu,$$
meaning that
$$\lim_{n\to\infty} \int_X f_n \; d\mu - \lim_{n\to\infty} \int_X f \; d\mu = 0.$$

□

Proposition 2.2.7. *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Then, f is Lebesgue integrable, with*

$$\int_a^b f(x)\ dx = \int_{[a,b]} f\ d\lambda.$$

Proof. Without loss of generality, assume that $a = 0, b = 1$. Define the sequence of partitions $(P_n)_{n=1}^{\infty}$ by

$$P_n = \{0, \tfrac{1}{2^n}, \dots, 1 - \tfrac{1}{2^n}, 1\}.$$

Define the functions $(g_n)_{n=1}^{\infty}, (h_n)_{n=1}^{\infty}$ by

$$g_n = \sum_{k=1}^{2^n} m_k \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}, \qquad h_n = \sum_{k=1}^{2^n} M_k \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]},$$

where $m_k = \inf f(-\frac{k-1}{2^n}, \frac{k}{2^n}]$ and $M_k = \sup f(-\frac{k-1}{2^n}, \frac{k}{2^n}]$. For $n \in \mathbb{Z}_{\geq 1}$, the functions g_n and h_n are simple measurable, with

$$\int_{[0,1]} g_n\ d\mu = \sum_{k=1}^{2^n} m_k \mu\left(\tfrac{k-1}{2^n}, \tfrac{k}{2^n}\right] = L(f, P_n)$$

$$\int_{[0,1]} h_n\ d\mu = \sum_{k=1}^{2^n} M_k \mu\left(\tfrac{k-1}{2^n}, \tfrac{k}{2^n}\right] = U(f, P_n).$$

Moreover, for all $n \in \mathbb{Z}_{\geq 1}$, $g_n \leq f \leq h_n$. Since f is Riemann integrable, we find that

$$L(f, P_n) \to \int_0^1 f(x)\ dx, \qquad U(f, P_n) \to \int_0^1 g(x)\ dx.$$

Since f is Riemann integrable, it is bounded by some $C > 0$. Hence, we find that $|g_n| \leq C$ and $|h_n| \leq C$ for all $n \in \mathbb{Z}_{\geq 1}$. Since the constant function C is measurable, the Dominated Convergence Theorem tells us that

$$\lim_{n\to\infty} \int_{[0,1]} g_n\ d\mu = \int_{[0,1]} \lim_{n\to\infty} g_n\ d\mu = \int_0^1 f(x)\ dx,$$

$$\lim_{n\to\infty} \int_{[0,1]} h_n\ d\mu = \int_{[0,1]} \lim_{n\to\infty} h_n\ d\mu = \int_0^1 f(x)\ dx.$$

Since

$$\int_{[0,1]} g\ d\mu = \int_{[0,1]} h\ d\mu,$$

we find that $g = h$ almost everywhere. Since $g \leq f \leq h$, this implies that f is measurable, with $g = f = h$ almost everywhere and

$$\int_{[0,1]} f\ d\mu = \int_{[0,1]} g\ d\mu = \int_0^1 f(x)\ dx.$$

□

2.3 $\mathcal{L}^p$ spaces

Definition 2.3.1. Let V be a real vector space. A map $\|.\|\colon V \to [0, \infty)$ if for $u, v \in V$ and $\lambda \in \mathbb{R}$,

- $\|u + v\| \leq \|u\| + \|v\|$;
- $\|\lambda v\| = |\lambda| \|v\|$;
- $\|v\| = 0$ if and only if $v = 0$.

Definition 2.3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,

$$\mathcal{L}^p(X, \mu) = \mathcal{L}^p(X) = \left\{ f\colon X \to \mathbb{R} \mid f \text{ measurable and } \int_X |f|^p \, d\mu < \infty \right\},$$

where $p \in [1, \infty)$. If $f \in \mathcal{L}^p(X)$, we define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

Proposition 2.3.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f, g \in \mathcal{L}^1(X)$. Then,*

$$\|f + g\|_1 \leq \|f\|_1 + \|g\|_1,$$

and

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2.$$

Proof. □

Proposition 2.3.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define the relation $\sim$ on functions $X \to \mathbb{R}$ by*

$$f \sim g \iff \mu(\{x \in X \mid f(x) \neq g(x)\}) = 0.$$

Then, $\sim$ is an equivalence relation.

Theorem 2.3.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then, the vector space $\mathcal{L}^p(X)$ is complete.*