

CHAPTER 1
CONTINUITY AND LIMITS

1.1 Complex Numbers

We start by defining the complex numbers.

Definition 1.1.1. The complex numbers \mathbb{C} is a vector space of $M_{2 \times 2}(\mathbb{R})$ given by

$$\mathbb{C} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

We denote this by $a + bi$.

By construction, it follows that $i^2 = -1$. Moreover, \mathbb{C} forms a field over (matrix) addition and multiplication. We now construct the inverse element for a non-zero complex number.

Proposition 1.1.2. Let $z \in \mathbb{C}$, with $z = a + bi$ for $a, b \in \mathbb{Z}$. If $z \neq 0$, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2}.$$

Proof. Since $z \neq 0$, we know that $a^2 + b^2 > 0$. Moreover,

$$\frac{(a + bi)(a - bi)}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

Therefore,

$$z^{-1} = \frac{a - bi}{a^2 + b^2}.$$

□

Next, we now define certain values for the complex numbers.

Definition 1.1.3. Let $z \in \mathbb{C}$, with $z = a + bi$ for $a, b \in \mathbb{R}$. Then,

- its conjugate is $\bar{z} = a - bi$;
- its modulus is $|z| = \sqrt{a^2 + b^2}$;
- the real part of z , $\text{Re}(z) = a$ and the imaginary part of z , $\text{Im}(z) = b$.

We can define the modulus, real part and the imaginary part in the following way as well.

Proposition 1.1.4. Let $z \in \mathbb{C}$. Then,

- $|z| = \sqrt{z\bar{z}}$;
- $\text{Re}(z) = \frac{z + \bar{z}}{2}$; and

- $\text{Im}(z) = \frac{z - \bar{z}}{2i}$.

Proof. Let $z = a + bi$. We know that $\bar{z} = a - bi$.

- We have

$$\sqrt{z\bar{z}} = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2 + b^2} = |z|.$$

- We find that

$$\frac{z + \bar{z}}{2} = \frac{a + bi + a - bi}{2} = a = \text{Re}(z).$$

- We find that

$$\frac{z - \bar{z}}{2i} = \frac{a + bi - a - bi}{2i} = b = \text{Im}(z).$$

□

We will now establish an important property of the modulus.

Proposition 1.1.5. *Let $z_1, z_2 \in \mathbb{C}$. Then, $|z_1 z_2| = |z_1| |z_2|$.*

Proof. Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then,

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1).$$

In that case,

$$\begin{aligned} |z_1|^2 |z_2|^2 &= (a_1^2 + b_1^2)(a_2^2 + b_2^2) \\ &= a_1^2 a_2^2 + a_1^2 b_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2 \\ &= a_1^2 a_2^2 - 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + a_1^2 b_2^2 + 2a_1 a_2 b_1 b_2 + b_1^2 a_2^2 \\ &= (a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 \\ &= |z_1 z_2|^2. \end{aligned}$$

So, $|z_1| |z_2| = |z_1 z_2|$.

□

Next, we prove the triangle inequality for \mathbb{C} . We start by proving a lemma.

Lemma 1.1.6. *Let $z \in \mathbb{C}$. Then, $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$.*

Proof. Let $z = a + bi$. We have

$$|\text{Re}(z)|^2 = a^2 \leq a^2 + b^2 = |z|^2,$$

and

$$|\text{Im}(z)|^2 = b^2 \leq a^2 + b^2 = |z|^2.$$

□

Now, we prove the triangle inequality.

Proposition 1.1.7 (Triangle Inequality in \mathbb{C}). *Let $z_1, z_2 \in \mathbb{C}$. Then, $|z_1 + z_2| \leq |z_1| + |z_2|$.*

Proof. We find that

$$\begin{aligned}
 |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
 &= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
 &= |z_1|^2 + 2\operatorname{Re}(z_1\overline{z_2}) + |z_2|^2 \\
 &\leq |z_1|^2 + 2|z_1z_2| + |z_2|^2 \\
 &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
 &= (|z_1| + |z_2|)^2.
 \end{aligned}$$

Therefore, $|z_1 + z_2| \leq |z_1| + |z_2|$. \square

Finally, we will show that there is no way to order the complex numbers. We first define what we mean by an order.

Definition 1.1.8 (Order). Let R be a ring, and let $>$ be a relation on R . Then, we say that $(R, >)$ is an *order* if:

- for all $x \in R$, either $x = 0$, $x > 0$ or $0 > x$, and precisely one of these is true;
- for all $x, y \in R$, $x > y$ if and only if $x - y > 0$;
- for all $x, y \in R$, if $x, y > 0$, then $x + y > 0$;
- for all $x, y \in R$, if $x, y > 0$, then $xy > 0$; and

If $(R, >)$ is an order, then we can define the following relations on R :

- The relation \geq is defined on R by $x \geq y$ if and only if $x = y$ or $x > y$;
- The relation $<$ is defined on R by $x < y$ if and only if $y > x$;
- The relation \leq is defined on R by $x \leq y$ if and only if $x = y$ or $x < y$.

To prove that there is no order on \mathbb{C} , we need to build from the axioms some familiar properties of an ordering. We start by proving that the negative numbers are less than 0.

Lemma 1.1.9. *Let R be a ring, and let $(R, >)$ be an order, and let $x \in R$. Then, $x > 0$ if and only if $-x < 0$.*

Proof.

- First, assume that $x > 0$. In that case, we know that $x \neq 0$, so either $-x > 0$ or $-x < 0$. If $-x > 0$, then $0 = x - x > 0$, which is a contradiction. Therefore, we must have $-x < 0$.
- Now, assume that $-x < 0$. In that case, $0 > -x$. This implies that $x = 0 + x > 0$.

Therefore, $x > 0$ if and only if $-x < 0$. \square

Next, we prove that the product of a positive and a negative number is negative.

Lemma 1.1.10. *Let R be a ring, and let $(R, >)$ be an order, and let $x > 0, y < 0$. Then, $xy < 0$.*

Proof. Since $y < 0$, we know that $-y > 0$. In that case, $-xy = x(-y) > 0$. Therefore, $xy < 0$. \square

Finally, we show that there is no order on \mathbb{C} .

Proposition 1.1.11. *There is no order relation on \mathbb{C} .*

Proof. Assume that $(\mathbb{C}, >)$ is an order. We know that $i \neq 0$. So, either $i < 0$ or $i > 0$:

- If $i < 0$, then $-i > 0$. In that case, $-1 = (-i)^2 > 0$, as well. Moreover,

$$1 = i \cdot (-i) < 0 \quad \text{and} \quad 1 = (-1)^2 > 0.$$

This is a contradiction.

- Instead, if $i > 0$, then $-i < 0$. In that case, $-1 = i^2 > 0$. Moreover,

$$1 = i \cdot (-i) < 0 \quad \text{and} \quad 1 = (-1)^2 > 0.$$

This is a contradiction.

Therefore, $>$ cannot define an order on \mathbb{C} . \square

1.2 Definition of Continuity

We start by reviewing the definition of continuity.

Definition 1.2.1 (Continuity). Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function and let $c \in \mathbb{R}$. Then, f is *continuous at c* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. If f is continuous at c for every $c \in \mathbb{R}$, then we say that f is *continuous*.

We start by proving the sequential characterisation of continuity.

Proposition 1.2.2 (Sequential Characterisation of Continuity). *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function and let $c \in \mathbb{R}$. Then, f is continuous at c if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in \mathbb{R} , if $x_n \rightarrow c$, then $f(x_n) \rightarrow f(c)$.*

Proof.

- Assume that f is continuous at c , and let $(x_n)_{n=1}^{\infty}$ be a sequence in \mathbb{R} such that $x_n \rightarrow c$. Now, let $\varepsilon > 0$. Since f is continuous at c , there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Moreover, since $x_n \rightarrow c$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \geq N$, $|x_n - c| < \delta$. In that case, for $n \geq N$, $|f(x_n) - f(c)| < \varepsilon$. Therefore, $f(x_n) \rightarrow f(c)$.
- Assume that f is not continuous at c . In that case, there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists an $x \in \mathbb{R}$ such that although $|x - c| < \delta$, $|f(x) - f(c)| \geq \varepsilon$. So, define the sequence $(x_n)_{n=1}^{\infty}$ by $0 \leq |x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \varepsilon$. By Sandwich Theorem, we find that $|x_n - c| \rightarrow 0$. Therefore, $x_n \rightarrow c$. However, by construction, we find that $|f(x_n) - f(c)| \geq \varepsilon$ for all $n \in \mathbb{Z}_{\geq 1}$. This implies that $f(x_n) \not\rightarrow f(c)$.

□

The sequential characterisation of continuity is not that different (than proving from the definition) when it comes to proving a function is continuous at a point, however it is much more natural to use this when proving a function is not continuous. First, we compare how we can use both the definition and the characterisation to prove a function is continuous.

Example 1.2.3. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be the function $f(x) = \sqrt{x}$. Then, f is continuous.*

Proof by the definition. Let $\varepsilon > 0$. Set $\delta = \varepsilon\sqrt{c}$. Then, if $|x - c| < \delta$, we find that for $x \in [0, \infty)$,

$$\begin{aligned} |f(x) - f(c)| &= |\sqrt{x} - \sqrt{c}| \\ &= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\ &\leq \frac{|x - c|}{\sqrt{c}} \\ &< \frac{\delta}{\sqrt{c}} = \varepsilon. \end{aligned}$$

Therefore, f is continuous for all $c \in (0, \infty)$.

□

Proof using sequential characterisation. Let $c \in (0, \infty)$, and let $(x_n)_{n=1}^\infty$ be a sequence in \mathbb{R} such that $x_n \rightarrow c$. We shall prove that $f(x_n) \rightarrow f(c)$. So, let $\varepsilon > 0$. Since $x_n \rightarrow c$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then $|x_n - c| < \varepsilon\sqrt{c}$. In that case, for $n \in \mathbb{Z}_{\geq 1}$, if $n \geq N$, then

$$\begin{aligned} |f(x_n) - f(c)| &= |\sqrt{x_n} - \sqrt{c}| \\ &= \frac{|x_n - c|}{\sqrt{x_n} + \sqrt{c}} \\ &< \frac{|x_n - c|}{\sqrt{c}} \\ &= \frac{\varepsilon\sqrt{c}}{\sqrt{c}} = \varepsilon. \end{aligned}$$

Therefore, f is continuous for all $c \in (0, \infty)$. \square

As we can see, proving continuity using the definition is quite similar to proving continuity using the sequential characterisation.

However, using sequential characterisation is much more natural at showing that a function is not continuous at a point- we simply have to find a sequence $(x_n)_{n=1}^\infty$ in \mathbb{R} such that $x_n \rightarrow c$ but $f(x_n) \not\rightarrow f(c)$. Like before, we shall prove that a function is not continuous at a point using both the definition and the characterisation.

Example 1.2.4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Then, f is not continuous at 0.

Proof by the definition. Let $\delta > 0$. Choose a natural number $n \in \mathbb{Z}_{\geq 1}$ such that $n > \frac{1}{\delta}$. In that case, set $x = \frac{1}{2n\pi + \frac{\pi}{2}}$. Then, although $0 < |x - 0| = x < \delta$,

$$|f(x) - f(0)| = f(x) = \sin(2n\pi + \frac{\pi}{2}) = 1 \geq 1.$$

This implies that f is not continuous at 0. \square

Proof using sequential characterisation. Define the sequence $(x_n)_{n=1}^\infty$ by

$$x_n = \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Then, we have $0 < x_n < \frac{1}{2n\pi}$, so $x_n \rightarrow 0$ by Sandwich Theorem. Moreover, for all $n \in \mathbb{Z}_{\geq 1}$,

$$f(x_n) = \sin(2n\pi + \frac{\pi}{2}) = 1,$$

so $f(x_n) \rightarrow 1 \neq f(0)$. By the sequential characterisation of continuity, this implies that f is not continuous at 0. \square

Although the two proofs have a similar structure, the proof using the definition essentially defines a sequence. Therefore, it is more natural to use sequential characterisation when showing a function is not continuous at a point.

Now, we will consider two main theorems of continuity- intermediate and extreme value theorem. We start by proving the intermediate value theorem.

Theorem 1.2.5 (Intermediate Value Theorem). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function that is continuous on $[a, b]$, and let $f(a) < c < f(b)$ or $f(b) < c < f(a)$. Then, there exists a $d \in [a, b]$ such that $f(d) = c$.*

Proof. Without loss of generality, assume that $f(a) < c < f(b)$ ¹. Define the set

$$S = \{x \in [a, b] \mid f(x) \leq c\}.$$

We know that S is bounded above by b , and $a \in S$. So, the completeness axiom tells us that $d = \sup(S)$ exists. We will show that $f(d) = c$.²

We know that there exists a sequence $(x_n)_{n=1}^\infty$ in S such that $x_n \rightarrow d$. Since f is continuous at d , we find that $f(x_n) \rightarrow f(d)$. By construction, we know that for all $n \in \mathbb{Z}_{\geq 1}$, $f(x_n) \leq c$. So, the order property of limits tells us that $f(d) \leq c$. Since $c < f(b)$, this implies that $d \neq b$. Since $S \subseteq [a, b]$, this implies that $d < b$.

Now, define the sequence $(y_n)_{n=1}^\infty$ in $[a, b]$ by

$$y_n = d + \frac{b - d}{n}.$$

Then, $y_n \rightarrow d$. By continuity, we know that $f(y_n) \rightarrow f(d)$. Moreover, for all $n \in \mathbb{Z}_{\geq 1}$, $y_n > d = \sup(S)$. Therefore, $y_n \notin S$, i.e. $f(y_n) > c$. By the order property of limits, this implies that $f(d) \geq c$.

This implies that $f(d) = c$. \square

Using this theorem, we can show that a function with a ‘jump’ is not continuous. We will consider a simple version of this- if a continuous function maps to integers, it can only map to one integer.

Corollary 1.2.6. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$, $f(x) \in \mathbb{Z}$. Then, f is a constant.*

Proof. Assume that f is not constant. In that case, there exist $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) \neq f(b)$. We know that $f(a), f(b) \in \mathbb{Z}$, so $|f(a) - f(b)| \geq 1$. Without loss of generality, assume that $f(a) < f(a) + 1 \leq f(b)$. In that case, the Intermediate Value Theorem tells us that there exists a $c \in \mathbb{R}$ such that $f(c) = f(a) + \frac{1}{2}$. Since $f(a) \in \mathbb{Z}$, we find that $f(a) + \frac{1}{2} \notin \mathbb{Z}$. This is a contradiction. Therefore, f must be a constant. \square

We can also use this theorem to show that a continuous function that is never zero is either always positive or always negative.

Corollary 1.2.7. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$, $f(x) \neq 0$. Then, either for all $x \in \mathbb{R}$, $f(x) > 0$ or for all $x \in \mathbb{R}$, $f(x) < 0$.*

Proof. Assume that there exist $x, y \in \mathbb{R}$ such that $f(x) > 0$ and $f(y) < 0$. Since $f(y) < 0 < f(x)$, the intermediate value theorem tells us that there exists a $c \in \mathbb{R}$ such that $f(c) = 0$. This is a contradiction. Therefore, either for all $x \in \mathbb{R}$, $f(x) > 0$ or for all $x \in \mathbb{R}$, $f(x) < 0$. \square

Using this result, we can prove the following.

¹If instead $f(b) < c < f(a)$, then we can apply the intermediate value theorem to the function $-f$ since $-f(a) < -c < -f(b)$.

²By construction, d will be the ‘last time’ $f(d) = c$ in the interval $[a, b]$.

Example 1.2.8. *Let*

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

and let $f : S^1 \rightarrow \mathbb{R}$ such that $h : \mathbb{R} \rightarrow \mathbb{R}$ given by $h(t) = f(\sin t, \cos t)$ is continuous. Then, there exists a $t \in \mathbb{R}$ such that $f(\sin t, \cos t) = f(-\sin t, -\cos t)$.

Proof. Assume that $f(\sin t, \cos t) \neq f(-\sin t, -\cos t)$ for all $t \in \mathbb{R}$. Then, define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(t) = f(\sin t, \cos t) - f(-\sin t, -\cos t)$. Since h is continuous, we find that g too is continuous. Moreover, we know that $g(t) \neq 0$ for all $t \in \mathbb{R}$. So, either for all $t \in \mathbb{R}$, $g(t) > 0$ or for all $t \in \mathbb{R}$, $g(t) < 0$.

- First, assume that for all $t \in \mathbb{R}$, $g(t) > 0$. In that case, $f(\sin t, \cos t) > f(-\sin t, -\cos t)$ for all $t \in \mathbb{R}$. This implies that

$$\begin{aligned} f(\sin t, \cos t) &> f(-\sin t, -\cos t) \\ &= f(\sin(t + \pi), \cos(t + \pi)) \\ &> f(-\sin(t + \pi), -\cos(t + \pi)) \\ &= f(\sin(t + 2\pi), \cos(t + 2\pi)) \\ &= f(\sin t, \cos t). \end{aligned}$$

This is a contradiction.

- Instead, assume that for all $t \in \mathbb{R}$, $g(t) < 0$. In that case, $f(\sin t, \cos t) < f(-\sin t, -\cos t)$ for all $t \in \mathbb{R}$. This implies that

$$\begin{aligned} f(\sin t, \cos t) &< f(-\sin t, -\cos t) \\ &= f(\sin(t + \pi), \cos(t + \pi)) \\ &< f(-\sin(t + \pi), -\cos(t + \pi)) \\ &= f(\sin(t + 2\pi), \cos(t + 2\pi)) \\ &= f(\sin t, \cos t). \end{aligned}$$

This is a contradiction.

Therefore, there exists a $t \in \mathbb{R}$ such that $g(t) = 0$. So, $f(\sin t, \cos t) = f(-\sin t, -\cos t)$. \square

One big application of the intermediate value theorem is finding a value that a function must equal.

Example 1.2.9. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^{27} - 52x^{15} + 7x^2 + 2$. Then, there exists an $x \in \mathbb{R}$ such that $f(x) = 42$.*

Proof. We have $f(0) = 2$ and $f(-1) = 60$. So, the intermediate value theorem tells us that there exists an $x \in (-1, 0)$ such that $f(x) = 42$. \square

We now show that an injective continuous function must be monotonic.

Corollary 1.2.10. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous injection. Then, f is strictly monotonic.*

Proof. Since f is injective, we find that $f(a) \neq f(b)$. Without loss of generality, assume that $f(a) < f(b)$. Now, let $x, y \in (a, b)$ with $x < y$, and assume that $f(x) > f(y)$.

- If $f(a) < f(y) < f(x)$, then there exists a $c \in (a, x)$ such that $f(c) = f(y)$. This is not possible since f is injective. So, we must have $f(y) < f(a)$.
- Now, if $f(y) < f(a) < f(x)$, then there exists a $c \in (x, y)$ such that $f(c) = f(a)$. This is not possible since f is injective. So, we must have $f(x) < f(a)$.
- Finally, if $f(x) < f(a) < f(b)$, then there exists a $c \in (x, b)$ such that $f(c) = f(a)$. This is not possible since f is injective.

Therefore, we must have $f(x) \leq f(y)$. Since f is injective, we have $f(x) < f(y)$ - f is strictly increasing. \square

Using this result, we can show that inverse of a bijective function is continuous.

Proposition 1.2.11. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bijection that is continuous at $c \in \mathbb{R}$. Then, the inverse function f^{-1} is continuous at $f(c)$.*

Proof. Without loss of generality, assume that f is strictly increasing. Let $\varepsilon > 0$. We know that $c - \varepsilon < c + \varepsilon$, so $f(c - \varepsilon) < f(c + \varepsilon)$. In that case, for all $y \in \mathbb{R}$ with $c - \varepsilon < y < c + \varepsilon$, the Intermediate Value Theorem tells us that there exists an $x \in (f(c - \varepsilon), f(c + \varepsilon))$ such that $f(x) = y$. Now, let $\delta > 0$ such that $(f(c) - \delta, f(c) + \delta) \subseteq (f(c - \varepsilon), f(c + \varepsilon))$. In that case, for $y \in \mathbb{R}$, if $|y - f(c)| < \delta$, then we know that $y \in (f(c) - \delta, f(c) + \delta) \subseteq (f(c - \varepsilon), f(c + \varepsilon))$. So, there exists an $x \in (c - \varepsilon, c + \varepsilon)$ such that $f(x) = y$. Therefore, if $|y - f(c)| < \delta$,

$$|f^{-1}(y) - c| = |x - c| < \varepsilon.$$

So, f^{-1} is continuous at $f(c)$. \square

We will now prove the extreme value theorem.

Theorem 1.2.12 (Extreme Value Theorem). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function that is continuous on $[a, b]$. Then, there exist $u, v \in [a, b]$ such that for all $x \in [a, b]$, $f(u) \leq f(x) \leq f(v)$.*

Proof.

- First, we show that there exists a $v \in [a, b]$ such that for all $x \in [a, b]$, $f(x) \leq f(v)$. Assume that f is not bounded above. In that case, the set

$$S = \{f(x) \mid x \in [a, b]\}$$

is not bounded above. Therefore, we can define the sequence $(x_n)_{n=1}^{\infty}$ in $[a, b]$ such that $f(x_n) > n$. We know that (x_n) is a bounded sequence, so there exists a subsequence $(x_{k_n})_{n=1}^{\infty}$ such that $x_{k_n} \rightarrow x$, for some $x \in [a, b]$. Since f is continuous, we find that $f(x_{k_n}) \rightarrow f(x)$. Therefore, the sequence $(f(x_{k_n}))$ is bounded. This is a contradiction since $f(x_{k_n}) > k_n \geq n$ for all $n \in \mathbb{Z}_{\geq 1}$. So, the set S is bounded above.

Since we have $f(a) \in S$, S is non-empty. So, let $w = \sup(S)$. By the supremum property, we can find a sequence $(y_n)_{n=1}^\infty$ in S such that $y_n \rightarrow w$. By construction, there exists a corresponding sequence $(x_n)_{n=1}^\infty$ in $[a, b]$ such that $y_n = f(x_n)$. Since (x_n) is a bounded sequence, there exists a convergent subsequence $(x_{k_n})_{n=1}^\infty$ with $x_{k_n} \rightarrow v$. Since (x_{k_n}) is a sequence in $[a, b]$, $v \in [a, b]$. Moreover, since f is continuous, $f(x_{k_n}) = y_{k_n} \rightarrow f(v)$. Since $y_n \rightarrow w$, we find that $y_{k_n} \rightarrow w$. Since limits are unique, $w = f(v)$. Therefore, for all $x \in [a, b]$, $f(x) \leq f(v)$.

- Since f is continuous, we know that $-f$ is continuous. From the result above, we know that there exists a $u \in [a, b]$ such that for all $x \in [a, b]$, $-f(x) \leq -f(u)$. In that case, $f(u) \leq f(x)$.

Therefore, there exist $u, v \in [a, b]$ such that for all $x \in [a, b]$, $f(u) \leq f(x) \leq f(v)$. \square

We finish this section with the following corollary.

Corollary 1.2.13. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function such that for all $x \in [a, b]$, $f(x) > 0$. Then, there exists an $\varepsilon > 0$ such that for all $x \in [a, b]$, $f(x) \geq \varepsilon$.*

Proof. By the extreme value theorem, there exists a $u \in [a, b]$ such that for all $x \in [a, b]$, $f(u) \leq f(x)$. By definition, we have $f(u) > 0$. So, let $\varepsilon = f(u)$. Then, for all $x \in [a, b]$, $f(x) \geq \varepsilon$. \square

1.3 Uniform Continuity

In this section, we will consider uniform continuity and some properties of it. We start by defining uniform continuity.

Definition 1.3.1 (Uniform Continuity). Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function. Then, f is *uniformly continuous* if for $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

The difference between continuity and uniform continuity is that the value of δ can depend on c if f is continuous, but not if f is uniformly continuous. This condition is very limiting- it turns out that the function $f(x) = x^n$ is only uniformly continuous if $n = 1$, even though it is continuous for all $n \in \mathbb{Z}_{\geq 1}$.

Proposition 1.3.2. Let $n \in \mathbb{Z}_{\geq 1}$, and define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^n$. Then, f is uniformly continuous if and only if $n = 1$.

Proof.

- First, assume that $n = 1$, and let $\varepsilon > 0$. Set $\delta = \varepsilon$. In that case, for $x, y \in \mathbb{R}$, if $|x - y| < \delta$,

$$|f(x) - f(y)| = |x - y| < \delta = \varepsilon.$$

Therefore, f is uniformly continuous if $n = 1$.

- Now, assume that $n \neq 1$. Let $\delta > 0$, and set $x = \max(1, \frac{1}{\delta})$ and $y = x + \frac{\delta}{2}$. Then, although $|x - y| = \frac{\delta}{2} < \delta$,

$$\begin{aligned} |f(x) - f(y)| &= |x^n - y^n| \\ &= |x - y| \cdot |x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}| \\ &\geq \frac{\delta}{2} \cdot (x^{n-1} + y^{n-1}) \\ &\geq \frac{\delta}{2} \cdot (x + y) \\ &= \frac{\delta}{2} \cdot \left(2x + \frac{\delta}{2}\right) \\ &= x \cdot \delta + \frac{\delta^2}{4} \\ &\geq x \cdot \delta \geq 1. \end{aligned}$$

This implies that f is not uniformly continuous if $n \neq 1$.

Therefore, f is uniformly continuous if and only if $n = 1$. □

Although not every continuous function is uniformly continuous, every uniformly continuous function is continuous.

Proposition 1.3.3. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be uniformly continuous. Then, f is continuous.

Proof. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Since f is uniformly continuous, we can find a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. In that case, for $x \in \mathbb{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. So, f is continuous at c . Therefore, f is continuous. \square

There is one instance when uniform continuity matches continuity. This is when the domain of the function is bounded.

Proposition 1.3.4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be continuous. Then, f is uniformly continuous on $[a, b]$.*

Proof by contradiction. Assume that f is not uniformly continuous on $[a, b]$. In that case, there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exist $x, y \in [a, b]$ such that although $|x - y| < \delta$, $|f(x) - f(y)| \geq \varepsilon$. So, define the sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Since (x_n) is a bounded sequence, we know that there exists a convergent subsequence (x_{k_n}) with $x_{k_n} \rightarrow x$. Since $|x_{k_n} - y_{k_n}| < \frac{1}{k_n} < \frac{1}{n}$, we find that $y_{k_n} \rightarrow x$ as well. Since f is continuous on $[a, b]$, and $x \in [a, b]$, we find that $f(x_{k_n}) \rightarrow f(x)$ and $f(y_{k_n}) \rightarrow f(x)$. Therefore, the sequence $f(x_{k_n}) - f(y_{k_n}) \rightarrow 0$. In that case, there exists an $n \in \mathbb{Z}_{\geq 1}$ such that $|f(x_{k_n}) - f(y_{k_n})| < \varepsilon$. This is a contradiction. So, f is uniformly continuous on $[a, b]$. \square

Direct Proof. Let $\varepsilon > 0$, and $c \in [a, b]$. Since f is continuous at c , we know that the set

$$S_c = \{\delta \in (0, 1] \mid \text{for all } x \in [a, b], \text{ if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon\}$$

is non-empty and bounded above. So, set $\delta_c = \sup(S_c) > 0$. Then, define

$$S = \{\delta_c > 0 \mid c \in [a, b]\},$$

and set $\delta = \inf(S)$.

If $\delta = 0$, then there exists a sequence $(x_n)_{n=1}^\infty$ in S such that $x_n \rightarrow 0$. By the definition of the set S , we can find a corresponding sequence $(c_n)_{n=1}^\infty$ in $[a, b]$ such that $x_n = \sup(S_{c_n})$. By Bolzano-Weierstrass, we know that (c_n) has a convergent subsequence (c_{k_n}) . Since $[a, b]$ is closed, we also know that $c_{k_n} \rightarrow c \in [a, b]$. Also, $x_n \rightarrow 0$, so we know that the subsequence $x_{k_n} \rightarrow 0$ as well.

Now, consider $c \in [a, b]$. Since f is continuous at c , we can find a $\delta_0 > 0$ such that if $|x - c| < \delta_0$, then $|f(x) - f(c)| < \frac{\varepsilon}{2}$. Moreover, since $c_{k_n} \rightarrow c$, we can find an $N_1 \in \mathbb{Z}_{\geq 1}$ such that for $n \geq N_1$, $|c_{k_n} - c| < \frac{\delta_0}{2}$. Finally, since $x_{k_n} \rightarrow 0$, we can find an $N_2 \in \mathbb{Z}_{\geq 1}$ such that for $n \geq N_2$, $0 < x_{k_n} < \frac{\delta_0}{3}$. Now, let $N = \max(N_1, N_2)$, and fix $n \geq N$. In that case, for $|c_{k_n} - x| < \frac{\delta_0}{2}$, we find that

$$\begin{aligned} |x - c| &= |x - c_{k_n} + c_{k_n} - c| \\ &\leq |x - c_{k_n}| + |c_{k_n} - c| \\ &< \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0. \end{aligned}$$

So, we have both $|c_{k_n} - c| < \delta_0$ and $|x - c_{k_n}| < \delta_0$. Therefore, if $|c_{k_n} - x| < \frac{\delta_0}{2}$,

$$\begin{aligned} |f(c_{k_n}) - f(x)| &= |f(c_{k_n}) - f(c) + f(c) - f(x)| \\ &\leq |f(c_{k_n}) - f(c)| + |f(c) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

In that case, $\frac{\delta_0}{2} \in S_{c_{k_n}}$. However, we know that $\sup(S_{c_{k_n}}) = x_{k_n} < \frac{\delta_0}{3}$ - this is a contradiction. Therefore, we must have $\sup(S_c) = \delta \neq 0$.

Finally, since S is a set of non-negative reals with $\inf(S) \neq 0$, we find that $\inf(S) = \delta > 0$. Now, let $x, y \in [a, b]$. Since $\delta = \inf(S)$, we find that $\delta_x \geq \delta$. Moreover, if $|x - y| < \delta \leq \delta_x$, then we can find a $\delta' \in S_x$ such that $|x - y| < \delta' \leq \delta_x$ - this is because $\sup(S_x) = \delta_x$. In that case, if $|x - y| < \delta$, then $|x - y| < \delta' \in S_x$, which implies that $|f(x) - f(y)| < \varepsilon$. Therefore, f is uniformly continuous on $[a, b]$. \square

It turns out that a multiple of a uniformly continuous function is uniformly continuous.

Proposition 1.3.5. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a uniformly continuous function, and let $\lambda \in \mathbb{C}$. Then, the function λf is uniformly continuous.*

Proof. If $\lambda = 0$, then $\lambda f = 0$ for all $x \in \mathbb{R}$. Therefore, λf is uniformly continuous.

Instead, if $\lambda \neq 0$, then let $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that for $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{|\lambda|}$. In that case, for $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then

$$|(\lambda f)(x) - (\lambda f)(y)| = |\lambda| \cdot |f(x) - f(y)| < |\lambda| \cdot \frac{\varepsilon}{|\lambda|} = \varepsilon.$$

So, λf is uniformly continuous. \square

Also, the sum of two uniformly continuous functions is uniformly continuous.

Proposition 1.3.6. *Let $f, g : \mathbb{R} \rightarrow \mathbb{C}$ be uniformly continuous functions. Then, the function $f + g$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous, we can find a $\delta_1 > 0$ such that for $x, y \in \mathbb{R}$, if $|x - y| < \delta_1$, then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Moreover, since g is uniformly continuous, we can find a $\delta_2 > 0$ such that for $x, y \in \mathbb{R}$, if $|x - y| < \delta_2$, then $|g(x) - g(y)| < \frac{\varepsilon}{2}$. So, set $\delta = \min(\delta_1, \delta_2)$. In that case, for $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then

$$\begin{aligned} |(f + g)(x) - (f + g)(y)| &= |f(x) + g(x) - f(y) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that $f + g$ is uniformly continuous. \square

However, a product of two uniformly continuous functions is not uniformly continuous- we know that $f(x) = x$ is uniformly continuous, but $f(x) \cdot f(x) = x^2$ is not uniformly continuous. Nonetheless, the composition of two uniform continuous functions is uniformly continuous.

Proposition 1.3.7. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly continuous functions. Then, the composition $f \circ g : \mathbb{R} \rightarrow \mathbb{C}$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Since g is uniformly continuous, there exists a $\delta_1 > 0$ such that for $x, y \in \mathbb{R}$, if $|x - y| < \delta_1$, then $|g(x) - g(y)| < \varepsilon$. Moreover, since f is uniformly continuous, there exists a $\delta_2 > 0$ such that for $x, y \in \mathbb{R}$, if $|x - y| < \delta_2$, then $|f(x) - f(y)| < \delta_1$. In that case, for $x, y \in \mathbb{R}$,

$$|x - y| < \delta_2 \implies |f(x) - f(y)| < \delta_1 \implies |g(f(x)) - g(f(y))| < \varepsilon.$$

This implies that $f \circ g$ is uniformly continuous. \square

1.4 Function Limits

Now, we consider function limits.

Definition 1.4.1. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}$ and $L \in \mathbb{C}$. Then, we say that the *limit*

$$\lim_{x \rightarrow c} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.³

As we can see, the definition is very close to the definition of continuity. In fact, they generalise the concept of continuity. We can use limits to prove continuity.

Proposition 1.4.2 (Limit Characterisation of Continuity). *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}$. Then, f is continuous at c if and only if the limit*

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Proof.

- Assume that $\lim_{x \rightarrow c} f(x) = f(c)$, and let $\varepsilon > 0$. If $x = c$, then we know that

$$|f(x) - f(c)| = 0 < \varepsilon.$$

Moreover, since $\lim_{x \rightarrow c} f(x) = f(c)$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Therefore, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Therefore, f is continuous at c .

- Instead, assume that f is continuous at c , and let $\varepsilon > 0$. In that case, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Therefore, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. So, the limit

$$\lim_{x \rightarrow c} f(x) = f(c).$$

□

Moreover, there is a sequential characterisation of limits which is very similar to the sequential characterisation of continuity.

Proposition 1.4.3 (Sequential Characterisation of Limits). *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}$ and $L \in \mathbb{C}$. Then, the limit $\lim_{x \rightarrow c} f(x) = L$ if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R} \setminus \{c\}$, if $x_n \rightarrow c$, then $f(x_n) \rightarrow L$.⁴*

Proof.

- Assume that the limit $\lim_{x \rightarrow c} f(x) = L$, and let $(x_n)_{n=1}^{\infty}$ be a sequence in \mathbb{R} such that $x_n \rightarrow c$. Now, let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow c} f(x) = L$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. Moreover, since $x_n \rightarrow c$, there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $n \geq N$, $|x_n - c| < \delta$. In that case, for $n \geq N$, $|f(x_n) - L| < \varepsilon$. Therefore, $f(x_n) \rightarrow L$.

³Note that we do not care what happens if $x = c$!

⁴Here as well, we do not care about a sequence (x_n) if it satisfies $x_n = c$ for some $n \in \mathbb{Z}_{\geq 1}$!

- Assume that the limit $\lim_{x \rightarrow c} f(x) \neq L$. In that case, there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists an $x \in \mathbb{R}$ such that although $0 < |x - c| < \delta$, $|f(x) - L| \geq \varepsilon$. So, define the sequence $(x_n)_{n=1}^\infty$ by $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon$. By Sandwich Theorem, we find that $|x_n - c| \rightarrow 0$. Therefore, $x_n \rightarrow c$. However, by construction, we find that $|f(x_n) - L| \geq \varepsilon$ for all $n \in \mathbb{Z}_{\geq 1}$. This implies that $f(x_n) \not\rightarrow L$.

□

We will now compute use the definition to compute limits.

Example 1.4.4. *The limit*

$$\lim_{x \rightarrow 1} \frac{x^2 - 5}{x + 2} = -\frac{4}{3}.$$

Proof. Let $\varepsilon > 0$. For $x \in \mathbb{R}$,

$$|x - 1| < 1 \iff x \in (0, 2) \iff x + 2 \in (2, 3).$$

Therefore, if $|x - 1| < 1$, then $|x + 2| \geq 2$. Moreover, for $x \in \mathbb{R}$,

$$|x - 1| < 1 \iff x \in (0, 2) \iff 3x \in (0, 6) \iff 3x + 7 \in (7, 13).$$

Therefore, if $|x - 1| < 1$, then $|3x + 7| \leq 13$. In that case, let $\delta = \min(1, \frac{2}{13}\varepsilon)$. Then, for $x \in \mathbb{R}$, if $0 < |x - 1| < \delta$, we have $|x - 1| < 1$ and $|x - 1| < \frac{2}{13}$. In that case,

$$\begin{aligned} \left| \frac{x^2 - 5}{x + 2} + \frac{4}{3} \right| &= \left| \frac{3x^2 + 4x - 7}{3(x + 2)} \right| \\ &= \left| \frac{(3x + 7)(x - 1)}{3(x + 2)} \right| \\ &\leq \frac{13}{2} |x - 1| \\ &< \frac{13}{2} \cdot \frac{2}{13} \varepsilon = \varepsilon. \end{aligned}$$

This implies that the limit

$$\lim_{x \rightarrow 1} \frac{x^2 - 5}{x + 2} = -\frac{4}{3}.$$

□

Instead of proving by the definition, we could have noted that the function $f : \mathbb{R} \setminus \{-2\} \rightarrow \mathbb{R}$ given by $f(x) = \frac{x^2 - 5}{x + 2}$ is continuous, and so the limit

$$\lim_{x \rightarrow 1} \frac{x^2 - 5}{x + 2} = \lim_{x \rightarrow 1} f(x) = f(1) = -\frac{4}{3}.$$

Next, we compute the following limit.

Example 1.4.5. *The limit*

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4.$$

Proof. Let $\varepsilon > 0$. Set $\delta = \varepsilon$. In that case, for $x \in \mathbb{R}$, if $0 < |x - 2| < \delta$, then

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| = \left| \frac{(x - 2)(x + 2)}{x - 2} - 4 \right| = |x - 2| < \delta = \varepsilon.$$

□

Note that we needed $x \neq 2$ for the proof above to make sense- the denominator cannot be 0. However, what we essentially did was

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} x + 2 = 4.$$

We will establish this result below.

Proposition 1.4.6. *Let $c \in \mathbb{R}, L \in \mathbb{C}$ and $f : \mathbb{R} \rightarrow \mathbb{C}$ and $g : \mathbb{R} \setminus \{c\} \rightarrow \mathbb{C}$ be functions such that there exists a $\delta_1 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_1$, then $f(x) \neq 0$.⁵ Then,*

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c} \frac{f(x)g(x)}{g(x)} = L.$$

Proof.

- First, assume that the limit $\lim_{x \rightarrow c} f(x) = L$. By definition, this implies that there exists a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \varepsilon$. So, set $\delta = \min(\delta_1, \delta_2)$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$\left| \frac{f(x)g(x)}{g(x)} - L \right| = |f(x) - L| < \varepsilon.$$

Therefore, the limit $\lim_{x \rightarrow c} \frac{f(x)g(x)}{g(x)} = L$.

- Now, assume that the limit $\lim_{x \rightarrow c} \frac{f(x)g(x)}{g(x)} = L$. In that case, we can find a $\delta > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$\left| \frac{f(x)g(x)}{g(x)} - L \right| = |f(x) - L| < \varepsilon.$$

Therefore, the limit $\lim_{x \rightarrow c} f(x) = L$.

□

We could generalise this even further!

Proposition 1.4.7. *Let $c \in \mathbb{R}, L \in \mathbb{C}$ and $f : \mathbb{R} \rightarrow \mathbb{C}$ and $g : \mathbb{R} \setminus \{c\} \rightarrow \mathbb{C}$ be functions such that there exists a $\delta_1 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_1$, then $f(x) = g(x)$.⁶ Then,*

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c} g(x) = L.$$

Proof.

⁵We require this condition to ensure that we can divide by $g(x)$ - the function has to be non-zero around some interval/neighbourhood of c .

⁶So, there is a neighbourhood around c where f and g are essentially the same function.

- First, assume that the limit $\lim_{x \rightarrow c} f(x) = L$. In that case, there exists a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \varepsilon$. So, set $\delta = \min(\delta_1, \delta_2)$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$|g(x) - L| = |f(x) - L| < \varepsilon.$$

This implies that the limit $\lim_{x \rightarrow c} g(x) = L$.

- Now, assume that the limit $\lim_{x \rightarrow c} g(x) = L$. In that case, there exists a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|g(x) - L| < \varepsilon$. So, set $\delta = \min(\delta_1, \delta_2)$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$|f(x) - L| = |g(x) - L| < \varepsilon.$$

This implies that the limit $\lim_{x \rightarrow c} f(x) = L$.

□

Now, we will consider how to show a limit does not exist. Like in the case of continuity, we want to use the sequential characterisation to show this. However, there is a difference here- to show that the limit $\lim_{x \rightarrow c} f(x)$ does not exist, we need to either find a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R} \setminus \{c\}$ such that $f(x_n)$ does not converge, or we need to find two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $\mathbb{R} \setminus \{c\}$ such that $f(x_n)$ and $f(y_n)$ converge to different values. We illustrate both the possibilities with an example.

Example 1.4.8. Let $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be the function given by $f(x) = \sin(1/x)$. Then, the limit $\lim_{x \rightarrow 0} f(x)$ does not exist.

Proof using one sequence. Define the sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R} \setminus \{0\}$ by

$$x_n = \frac{1}{n\pi + \frac{\pi}{2}}.$$

We have $0 < x_n < \frac{1}{n\pi}$ for all $n \in \mathbb{Z}_{\geq 1}$, so Sandwich Theorem tells us that $x_n \rightarrow 0$. Moreover, for $n \in \mathbb{Z}_{\geq 1}$,

$$f(x_n) = f(n\pi + \frac{\pi}{2}) = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd.} \end{cases}$$

Since $(f(x_n))$ is not a Cauchy sequence, it cannot converge. Therefore, the sequential characterisation of limits tells us that the limit $\lim_{x \rightarrow 0} f(x)$ does not exist. □

Proof using two sequences. Define the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $\mathbb{R} \setminus \{0\}$ by

$$x_n = \frac{1}{2n\pi + \frac{\pi}{2}}, \quad y_n = \frac{1}{2n\pi + \frac{3\pi}{2}}.$$

We have $0 < y_n < x_n < \frac{1}{2n\pi}$, so the Sandwich Theorem tells us that both $x_n \rightarrow 0$ and $y_n \rightarrow 0$. Moreover, for $n \in \mathbb{Z}_{\geq 1}$,

$$f(x_n) = f(2n\pi + \frac{\pi}{2}) = 1, \quad f(y_n) = f(2n\pi + \frac{3\pi}{2}) = -1.$$

Therefore, $f(x_n) \rightarrow 1$ and $f(y_n) \rightarrow -1$. Since $1 \neq -1$, the sequential characterisation of limits tells us that the limit $\lim_{x \rightarrow 0} f(x)$ does not exist. □

We will now prove some properties of limits. We first show that the limit of a multiple of a function is the multiple of the limit.

Proposition 1.4.9. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}$ and $L, \lambda \in \mathbb{C}$, and assume that the limit $\lim_{x \rightarrow c} f(x) = L$. In that case, the limit*

$$\lim_{x \rightarrow c} \lambda f(x) = \lambda L.$$

Proof.

- If $\lambda = 0$, then we know that $\lambda f(x) = 0$ for all $x \in \mathbb{R}$. Therefore, the limit

$$\lim_{x \rightarrow c} \lambda f(x) = 0 = 0 \cdot L = \lambda L.$$

- Otherwise, $\lambda \neq 0$. So, let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow c} f(x) = L$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \frac{\varepsilon}{|\lambda|}$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$|\lambda f(x) - \lambda L| = |\lambda| \cdot |f(x) - L| = |\lambda| \cdot \frac{\varepsilon}{|\lambda|} = \varepsilon.$$

Therefore, the limit $\lim_{x \rightarrow c} \lambda f(x) = \lambda L$.

This implies that the limit $\lim_{x \rightarrow c} \lambda f(x) = \lambda L$. □

Next, we show that the limit of the sum of two functions is the sum of the two limits.

Proposition 1.4.10. *Let $f, g : \mathbb{R} \rightarrow \mathbb{C}$ be functions, and let $c \in \mathbb{R}$ and $L_1, L_2 \in \mathbb{C}$, and assume that the limits $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$. In that case, the limit*

$$\lim_{x \rightarrow c} f(x) + g(x) = L_1 + L_2.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{x \rightarrow c} f(x) = L_1$, there exists a $\delta_1 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_1$, then $|f(x) - L_1| < \frac{\varepsilon}{2}$. Moreover, since $\lim_{x \rightarrow c} g(x) = L_2$, there exists a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|g(x) - L_2| < \frac{\varepsilon}{2}$. In that case, set $\delta = \min(\delta_1, \delta_2)$. Then, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$\begin{aligned} |(f(x) + g(x)) - (L_1 + L_2)| &= |f(x) - L_1 + g(x) - L_2| \\ &\leq |f(x) - L_1| + |g(x) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that the limit $\lim_{x \rightarrow c} f(x) + g(x) = L_1 + L_2$. □

Next, we will show that the limit of the product of two functions is the product of the two limits. We start with a lemma.

Lemma 1.4.11. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}$ and $L \in \mathbb{C}$ such that the limit $\lim_{x \rightarrow c} f(x) = L$. In that case, there exists a $\delta > 0, K > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x)| < K$.*

Proof. Since the limit $\lim_{x \rightarrow c} f(x) = L$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < 1$. In that case,

$$|f(x)| \leq |f(x) - L| + |L| < 1 + |L|.$$

So, set $K = 1 + |L| \geq 1 > 0$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x)| < K$. \square

We now prove the result.

Proposition 1.4.12. *Let $f, g : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}$ and $L_1, L_2 \in \mathbb{C}$ such that the limits $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$. In that case, the limit $\lim_{x \rightarrow c} f(x)g(x) = L_1L_2$.*

Proof. Let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow c} f(x) = L_1$, we can find a $\delta_1 > 0$ and a $K > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_1$, then $|f(x)| < K$. Since the limit $\lim_{x \rightarrow c} g(x) = L_2$, there exists a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|g(x) - L_2| < \frac{\varepsilon}{2K}$.

- First, assume that $L_2 = 0$. In that case, set $\delta = \min(\delta_1, \delta_2)$. So, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$|f(x)g(x) - L_1L_2| = |f(x)| \cdot |g(x)| < K \cdot \frac{\varepsilon}{2K} = \frac{\varepsilon}{2} < \varepsilon.$$

- Next, assume that $L_2 \neq 0$. Then, we can find a $\delta_3 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_3$, then $|f(x) - L_1| < \frac{\varepsilon}{2|L_2|}$. In that case, set $\delta = \min(\delta_1, \delta_2, \delta_3)$. So, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$\begin{aligned} |f(x)g(x) - L_1L_2| &= |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1L_2| \\ &\leq |f(x)g(x) - f(x)L_2| + |f(x)L_2 - L_1L_2| \\ &= |f(x)| \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\ &< K \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\ &< K \cdot \frac{\varepsilon}{2K} + |L_2| \cdot \frac{\varepsilon}{2|L_2|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that the limit $\lim_{x \rightarrow c} f(x)g(x) = L_1L_2$. \square

Now, we show that the limit of the quotient of two functions is the quotient of the two limits. We start with a lemma.

Lemma 1.4.13. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$, $c \in \mathbb{R}$ and $L \in \mathbb{C}$ such that the limit $\lim_{x \rightarrow c} f(x) = L$, with $L \neq 0$. In that case, the limit $\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{1}{L}$.*

Proof. Since $L \neq 0$, we can find a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \frac{|L|}{2}$. In that case,

$$|f(x)| \in (\frac{1}{2}|L|, \frac{3}{2}|L|),$$

and so $|f(x)| \geq \frac{1}{2}|L|$ and $f(x) \neq 0$. This implies that the expression $\frac{1}{f(x)}$ is well defined for $0 < |x - c| < \delta_1$.

Now, let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow c} f(x) = L$, there exists a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \frac{\varepsilon \cdot |L|^2}{2}$. So, set $\delta = \min(\delta_1, \delta_2)$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$\left| \frac{1}{f(x)} - \frac{1}{L} \right| = \left| \frac{f(x) - L}{L \cdot f(x)} \right| \leq \frac{2|f(x) - L|}{|L|^2} < \frac{2\varepsilon \cdot |L|^2}{2|L|^2} = \varepsilon.$$

This implies that the limit $\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{1}{L}$. \square

Using this result, we can easily prove the quotient rule for limits.

Proposition 1.4.14. *Let $f, g : \mathbb{R} \rightarrow \mathbb{C}$, and let $c \in \mathbb{R}$ and $L_1, L_2 \in \mathbb{C}$ such that the limits $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$, with $L_2 \neq 0$. In that case, the limit $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$.*

Proof. Since $L_2 \neq 0$, we find that the limit

$$\lim_{x \rightarrow c} \frac{1}{g(x)} = \frac{1}{L_2}.$$

Therefore,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} \frac{1}{g(x)} = \frac{L_1}{L_2}.$$

\square

The final property we look at is the conjugate function.

Proposition 1.4.15. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$, and let $c \in \mathbb{R}$ and $L \in \mathbb{R}$ such that the limit $\lim_{x \rightarrow c} f(x) = L$. In that case, the limit $\lim_{x \rightarrow c} \overline{f(x)} = \overline{L}$.*

Proof. Let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow c} f(x) = L$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$|\overline{f(x)} - \overline{L}| = \overline{|f(x) - L|} = |f(x) - L| < \varepsilon.$$

This implies that the limit $\lim_{x \rightarrow c} \overline{f(x)} = \overline{L}$. \square

All of these properties are also true if f is continuous. That is, if $f, g : \mathbb{R} \rightarrow \mathbb{C}$ are continuous at $c \in \mathbb{R}$, then

- λf is continuous at c for any $\lambda \in \mathbb{C}$;
- $f + g$ is continuous at c ;
- fg is continuous at c ;
- if $g(c) \neq 0$, $\frac{f}{g}$ is continuous at c ; and
- \overline{f} is continuous at c .

We refer to all of these properties as *algebraic properties* of limits/continuity. There is one final algebraic property of continuity.

Proposition 1.4.16. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be functions, and let $c \in \mathbb{R}$ such that g is continuous at c and f is continuous at $g(c)$. In that case, the composition $f \circ g : \mathbb{R} \rightarrow \mathbb{C}$ is continuous at c .*

Proof. Let $\varepsilon > 0$. Since g is continuous, there exists a $\delta_1 > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta_1$, then $|g(x) - g(c)| < \varepsilon$. Moreover, since f is uniformly continuous, there exists a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $|x - c| < \delta_2$, then $|f(x) - f(c)| < \delta_1$. In that case, for $x \in \mathbb{R}$,

$$|x - c| < \delta_2 \implies |f(x) - f(c)| < \delta_1 \implies |g(f(x)) - g(f(c))| < \varepsilon.$$

This implies that $f \circ g$ is continuous. \square

The corresponding property for limits would be: if we have functions $f : \mathbb{R} \rightarrow \mathbb{C}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ and $c, L \in \mathbb{R}$ and $M \in \mathbb{C}$ such that

$$\lim_{x \rightarrow c} g(x) = L \quad \text{and} \quad \lim_{x \rightarrow L} f(x) = M,$$

then the limit

$$\lim_{x \rightarrow c} f(g(x)) = M.$$

However, this is not true. For example, consider the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise,} \end{cases} \quad g(x) = 0.$$

Then, $f(g(x)) = 1$ for all $x \in \mathbb{R}$, meaning that the limit $\lim_{x \rightarrow 0} f(g(x)) = 1$. However, the limit $\lim_{x \rightarrow 0} g(x) = 0$ and the limit $\lim_{x \rightarrow 0} f(x) = 0$. Therefore, we require g to be continuous at c and f to be continuous at $g(c)$ for the limit

$$\lim_{x \rightarrow c} f(g(x)) = f(\lim_{x \rightarrow c} g(x)).$$

The following property of limits tells us how limits behave with inequalities.

Proposition 1.4.17 (Order Property of Limits). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, and let $c, L, M \in \mathbb{R}$ and assume that there exists a $\delta_1 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_1$, then $f(x) \leq M$. Further, suppose that the limit $\lim_{x \rightarrow c} f(x) = L$. In that case, $L \leq M$.⁷*

Proof. Assume that $L > M$. In that case, set $\varepsilon = L - M > 0$. We can find a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \varepsilon$. Now, set $\delta = \min(\delta_1, \delta_2) > 0$. We know that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then both $f(x) \leq M$ and

$$f(x) > L - \varepsilon = M.$$

This is a contradiction. Therefore, $L \leq M$. \square

Finally, we look at the sandwich theorem.

Theorem 1.4.18 (Sandwich Theorem). *Let $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ be functions, and let $c, L \in \mathbb{R}$, and assume that there exists a $\delta_1 > 0$ such that for all $x \in \mathbb{R}$, if $0 < |x - c| < \delta_1$, then $f(x) \leq g(x) \leq h(x)$. Further, suppose that the limits $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$. In that case, the limit $\lim_{x \rightarrow c} g(x) = L$.*

⁷If we know that for all $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $f(x) < M$ instead, then our conclusion would still be the same: $L \leq M$. It does not follow that $L < M$.

Proof. Let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow c} f(x) = L$, we can find a $\delta_2 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \varepsilon$. Similarly, since the limit $\lim_{x \rightarrow c} h(x) = L$, we can find a $\delta_3 > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta_3$, then $|h(x) - L| < \varepsilon$. So, set $\delta = \min(\delta_1, \delta_2, \delta_3)$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon.$$

This implies that $|g(x) - L| < \varepsilon$. So, the limit $\lim_{x \rightarrow c} g(x) = L$. \square

We finish this section by looking at left and right limits. First, we state the definition.

Definition 1.4.19. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}, L \in \mathbb{C}$. We say that the limit

$$\lim_{x \rightarrow c^-} f(x) = L$$

if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $x \in (c - \delta, c)$, then $|f(x) - L| < \varepsilon$. Also, we say that the limit

$$\lim_{x \rightarrow c^+} f(x) = L$$

if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $x \in (c, c + \delta)$, then $|f(x) - L| < \varepsilon$.

Now, we show that a limit can only exist when the left and the right limit exist, and are equal.

Proposition 1.4.20. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $c \in \mathbb{R}, L \in \mathbb{C}$. Then, the limit

$$\lim_{x \rightarrow c} f(x) = L$$

if and only if

$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

Proof.

- Assume that the limit $\lim_{x \rightarrow c} f(x) = L$, and let $\varepsilon > 0$. We know that there exists a $\delta > 0$ such that for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. In that case, for both $x \in (c - \delta, c)$ and $x \in (c, c + \delta)$, $|f(x) - f(c)| < L$. This implies that the limits

$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

- Instead, assume that the limits

$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

Let $\varepsilon > 0$. Then, we can find $\delta_1, \delta_2 > 0$ such that for $x \in \mathbb{R}$, if $x \in (c - \delta_1, c)$, then $|f(x) - L| < \varepsilon$, and if $x \in (c, c + \delta_2)$, then $|f(x) - L| < \varepsilon$. So, set $\delta = \min(\delta_1, \delta_2)$. In that case, for $x \in \mathbb{R}$, if $0 < |x - c| < \delta$, then either $x \in (c - \delta_1, c)$ or $x \in (c, c + \delta_2)$. Therefore, we have $|f(x) - L| < \varepsilon$. This implies that the limit $\lim_{x \rightarrow c} f(x) = L$. \square

1.5 Infinite Limits

We now define infinite limits. We start by defining a limit at infinity.

Definition 1.5.1. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $L \in \mathbb{C}$.

- We say that the limit

$$\lim_{x \rightarrow \infty} f(x) = L$$

if for every $\varepsilon > 0$, there exists an $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

- We say that the limit

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if for every $\varepsilon > 0$, there exists an $M > 0$ such that if $x < -M$, then $|f(x) - L| < \varepsilon$.

We will show by definition that the function $\frac{1}{x}$ converges to 0 as x goes to ∞ .

Example 1.5.2. *The limit*

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Proof. Let $\varepsilon > 0$. Set $M = \frac{1}{\varepsilon}$. In that case, if $x > M$, then

$$f(x) = \frac{1}{x} < \frac{1}{M} = \varepsilon.$$

So, the limit

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

□

Next, we look at limits to infinity.

Definition 1.5.3. Let $f : [a, b) \rightarrow \mathbb{R}$ be a function.

- We say that the limit

$$\lim_{x \rightarrow b^-} f(x) = \infty$$

if for every $N > 0$, there exists a $\delta > 0$ such that if $x \in (b - \delta, b)$, then $f(x) > N$.

- We say that the limit

$$\lim_{x \rightarrow b^-} f(x) = -\infty$$

if for every $N > 0$, there exists a $\delta > 0$ such that if $x \in (b - \delta, b)$, then $f(x) < -N$.

Note that the function does not converge to a limit in this case- it is just a special case of divergence. Next, we define limits to infinity.

Definition 1.5.4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function.

- We say that the limit

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

if for every $N > 0$, there exists an $M > 0$ such that if $x > M$, then $f(x) > N$.

- We say that the limit

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

if for every $N > 0$, there exists an $M > 0$ such that if $x > M$, then $f(x) < -N$.

- We say that the limit

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

if for every $N > 0$, there exists an $M > 0$ such that if $x < -M$, then $f(x) > N$.

- We say that the limit

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

if for every $N > 0$, there exists an $M > 0$ such that if $x < -M$, then $f(x) < -N$.

We will illustrate this definition with an example.

Example 1.5.5. *The limit*

$$\lim_{x \rightarrow \infty} \log(x) = \infty.$$

Proof. Let $N > 0$. Set $M = e^N$. In that case, for $x > M$,

$$\log(x) > \log(M) > \log(e^N) = N.$$

□

We finish by connecting these infinite limits.

Proposition 1.5.6. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function, and let $L \in \mathbb{C}$. Then, the following are equivalent:*

- (1) *the limit $\lim_{x \rightarrow \infty} f(x) = L$;*
- (2) *the limit $\lim_{x \rightarrow 0^+} f(1/x) = L$;*
- (3) *the limit $\lim_{x \rightarrow 0^-} f(-1/x) = L$; and*
- (4) *the limit $\lim_{x \rightarrow -\infty} f(-x) = L$.*

Proof.

- Assume that the limit

$$\lim_{x \rightarrow \infty} f(x) = L.$$

Let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow \infty} f(x) = L$, there exists an $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$. In that case, set $\delta = \frac{1}{M}$. Then, for $x \in (0, \frac{1}{M})$, we find that $\frac{1}{x} > M$, and so

$$|f(1/x) - L| < \varepsilon.$$

In that case, (1) implies (2).

- Now, assume that the limit

$$\lim_{x \rightarrow 0^+} f(1/x) = L.$$

Let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow 0^+} f(1/x) = L$, there exists a $\delta > 0$ such that for $x \in (0, \delta)$, $|f(1/x) - L| < \varepsilon$. In that case, for $x \in (-\delta, 0)$, we have $-x \in (0, \delta)$. This implies that

$$|f(-1/x) - L| < \varepsilon.$$

In that case, (2) implies (3).

- Next, assume that the limit

$$\lim_{x \rightarrow 0^-} f(-1/x) = L.$$

Let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow 0^-} f(-1/x) = L$, there exists a $\delta > 0$ such that for $x \in (-\delta, 0)$, $|f(-1/x) - L| < \varepsilon$. So, set $M = \frac{1}{\delta}$. In that case, if $x > M$, then $\frac{1}{x} < M = \frac{1}{\delta}$, and so $\frac{-1}{x} \in (-\delta, 0)$. Therefore,

$$|f(-1/x) - L| < \varepsilon.$$

In that case, (3) implies (4).

- Finally, assume that the limit

$$\lim_{x \rightarrow -\infty} f(-x) = L.$$

Let $\varepsilon > 0$. Since the limit $\lim_{x \rightarrow -\infty} f(-x) = L$, there exists an $M > 0$ such that if $x < -M$, then $|f(-x) - L| < \varepsilon$. In that case, for $x > M$, $-x < -M$, and so

$$|f(x) - L| < \varepsilon.$$

In that case, (4) implies (1).

So, the statements are equivalent. \square