PCA

1. $X^T X$ eigenvalues:

$$[2.81 \times 10^2, 9.64 \times 10^{-1}, 0, 0, 0, 0]$$

2. $$X X^T = \begin{bmatrix} 112 & 137 \\ 137 & 170 \end{bmatrix}$$

$$\begin{bmatrix} 112-\lambda & 137 \\ 137 & 170-\lambda \end{bmatrix}$$

$$(112-\lambda)(170-\lambda) - 137^2 = 0$$

$$\lambda^2 - 282\lambda + 19040 - 18{,}769 =$$

$$\lambda^2 - 282\lambda + 271 = 0$$

$$\lambda^2 - 282\lambda + 141^2 = -271 + 141^2$$

$$\lambda^2 - 282\lambda + 19881 = 19610$$

$$(\lambda - 141)^2 = 19610$$

$$\lambda = \pm\sqrt{19610} + 141$$

$$\lambda = 2.81 \times 10^2, .964$$

3. Yes they are the same, X^TX and XX^T can be shown to have equal eigenvalues using some algebra. But essentially this makes sence because showing covariance of X_{ij} and X_{ji} is equivalent in both matrices.

4. I'm guessing that we are to reduce the dimensions + ugh PCA:

mean vector = $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Subtracted → $\begin{bmatrix} -2 & -1 & -1 & 0 & 1 & 3 \\ -3 & -1 & 0 & 0 & 1 & 3 \end{bmatrix}$

normalized → $\begin{bmatrix} .174 & 2.17 & 3.17 & 3.17 & 4.17 & 6.17 \\ .36 & 1.36 & 1.36 & 2.36 & 3.36 & 5.36 \end{bmatrix}$

Cov ($\cancel{XX}^T$) = $\begin{bmatrix} 77 & 114 \\ 114 & 168 \end{bmatrix}$

eigenvalues = $\begin{bmatrix} 77-\lambda & 114 \\ 114 & 168-\lambda \end{bmatrix}$

$(168-\lambda)(77-\lambda) - 114^2 = 0$

"equation solver": $\lambda = \frac{245}{2} \pm \frac{\sqrt{60265}}{2}$

we only want big eigenvalue, so,

$\lambda = \frac{245}{2} + \frac{\sqrt{60265}}{2}$

eigenvector = $\begin{bmatrix} 77-245 & 114 \\ 114 & 168-245 \end{bmatrix} v_1 = 0$

$-168 v_{11} - 114 v_{12} = 0$

$-\frac{114}{168} v_{12} = v_{11}$

top eigen → $\begin{bmatrix} \frac{114}{168} \\ -1 \end{bmatrix}$

newdata = $6 \times 2 \times 2 \times 1 = 6 \times 1$

new data $= X^T V_1 = \begin{bmatrix} -.53 & -1.62 & -2.45 & -1.84 & -2.15 & -2.69 \end{bmatrix}$

this is the same as through code.

2 EM

$P(A = \text{heads}) = \pi \longrightarrow h \to B$, $T \to C$; $h \to 1$, $T \to 0$

$P(B = h) = p$

$P(C = h) = q$

$[1,1,0,1,0,0,1,0,1,1]$

$$P(Y|\theta) = \sum_z P(y,z|\theta) = \sum_z P(z|\theta)P(y|z,\theta)$$

$$= \pi p^y(1-p)^{(1-y)} + (1-\pi)q^y(1-q)^{1-y}$$

$$P(Y|\theta) = \prod_{j=1}^{10}\left[\pi p^{y_j}(1-p)^{(1-y_j)} + (1-\pi)q^{y_j}(1-q)^{1-y_j}\right]$$

1. $$\hat{\theta} = \max_{\theta} P(Y|\theta)$$

This can be attempted using log: see eq 8 in PDF

This equation is not easily solvable, as we now have the log of a messy summation.

$y_j =$

2. $$\mu_j^{(i+1)} = \frac{\pi^{(i)}(p^{(i)})^{y_j}(1-p^{(i)})^{1-y_j}}{\pi^{(i)}(p^{(i)})^{y_j}(1-p^{(i)})^{1-y_j} + (1-\pi^{(i)})(q^{(i)})^{y_j}(1-q^{(i)})^{1-y_j}}$$

$i=0$ $$\mu^{(1)} = \frac{.5\,(.5)^1\,(1-.5)^{1-1}}{(.5)(.5)^1(1-.5)^{1-1} + (1-.5)(.5)^1(1-.5)^{1-1}} = [.5 \longrightarrow .5]$$

2.3 θ^1

$$\pi^2 = \frac{1}{10}\sum \underline{.5}$$

$$= .5$$

$$p^2 = \frac{.5 \cdot 6}{5} = \underline{.6}$$

$$q^2 = \underline{.6}$$

2.4 $$\mu^2 = \frac{.5\,(.6)^{y_j}(1-.6)^{1-y_j}}{.5(.6)^{y_j}(1-.6)^{1-y_j} + .5(.6)^{y_j}(.4)^{y_j}} = [.5 \longrightarrow .5]$$

2.5 θ^2

$$\pi^3 = .5$$

$$p^3 = .5$$

$$q^3 = .5$$

2.6 $$\theta^{(0)} = (\pi^0, p^0, q^0) = (.4, .6, .7)$$

$$\mu^1 = \frac{\pi^{(i)}(p^{(i)})^{y_j}(1-p^{(i)})^{1-y_j}}{\pi^{(i)}(p^{(i)})^{y_j}(1-p^{(i)})^{1-y_j} + (1-\pi^{(i)})(q^{(i)})^{y_j}(1-q^{(i)})^{1-y_j}} = \begin{array}{cc} y_j=1 & y_j=0 \\ .36 & .47 \end{array}$$

$$\mu_j^2 = \begin{cases} .36 & y_j = 1 \\ .47 & y_j = 0 \end{cases}$$

$\theta^1 =$

$$\pi^2 = \frac{6 \cdot .36 + 4 \cdot .47}{10} = .4$$

$$p^2 = .53$$

$$q^2 = .64$$

$$h^2 = \frac{\pi^{(i)}(p^{(i)})^{y_i}(1-p^{(i)})^{1-y_i}}{\pi^{(i)}(p^{(i)})^{y_i}(1-p^{(i)})^{1-y_i} + (1-\pi^{(i)})(q^{(i)})^{y_i}(1-q^{(i)})^{1-y_i}} = \qquad \mu_j = \begin{cases} .35 & y_j = 1 \\ .46 & y_j = 0 \end{cases}$$

$\theta^2 =$

$\pi^2 = .394$

$p^2 = .533$

$q^2 = .643$

these are different with varying initialization

2.7 No, the results are different.

Code for this assignment is available in the jupyter notebook pushed to github.