12.10.2021

Seminar 3 – Grammars

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1. Given the grammar $G = (N, \Sigma, P, S)$, $N = \{S, C\}$, $\Sigma = \{a, b\}$,

P: $S \to ab \mid aCSb$

$C \to S \mid bSb$

$CS \to b$,

prove that $w = ab(ab^2)^2 \in L(G)$.

Sol.: *@B David Turcas*

Obs.: (1)

P: (1) $S \to ab$

(2) $S \to aCSb$

(3) $C \to S$

(4) $C \to bSb$

(5) $CS \to b$

(2) $(ab)^2 = abab \neq aabb = a^2b^2$

$$S \underset{(2)}{\Rightarrow} aCSb \underset{(4)}{\Rightarrow} abSbSb \underset{\underbrace{}_{(1),(1)}}{\overset{2}{\Rightarrow}} ababbabb = ab(ab^2)^2 \;\Rightarrow\; S \overset{4}{\Rightarrow} w \;\Rightarrow\; S \overset{*}{\Rightarrow} w$$

=> $w \in L(G)$

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2. Given the grammar $G = (N, \Sigma, P, S)$, $N = \{S\}$, $\Sigma = \{a, b, c\}$, $P = \{S \to a^2S \mid bc\}$, find $L(G)$.

Sol.: *@B Andrada Tiutin*

#IW Alexandra Tudorescu

Let $L = \{a^{2k}bc \mid k \in \mathbb{N}\}$ and prove that $L = L(G)$ (by double inclusion).

I) ? $L \subseteq L(G)$ (**all** sequences of that shape are generated by G)

? $\forall k \in \mathbb{N}\ \ a^{2k}bc \in L(G)$

Consider $P(n)$: $a^{2n}bc \in L(G), n \in \mathbb{N}$

and prove that $P(n)$ is true $\forall n \in \mathbb{N}$ by math induction.

(i) verification step

? $P(0)$: $a^0bc = bc \in L(G)$

$S \underset{2}{\Rightarrow} bc$ => $bc \in L(G)$ => $P(0)$ is true

(ii) proof step

Assume $P(k)$ is true (for arbitrary $k \in \mathbb{N}$) and prove $P(k+1)$ is true

$P(k)$ is true => $a^{2k}bc \in L(G)$ => $S \overset{*}{\Rightarrow} a^{2k}bc$ (this is the induction hypothesis)

Then,

$S \underset{1}{\Rightarrow} a^2S \underbrace{\overset{*}{\Rightarrow}}_{ind.hyp} a^2a^{2k}bc = a^{2(k+1)}bc$ => $S \overset{*}{\Rightarrow} a^{2(k+1)}bc$ => $P(k+1)$ is true.

(i) + (ii) => $P(n)$ is true $\forall n \in \mathbb{N}$, thus I).

II) ? $L(G) \subseteq L$ (G generates **only** sequences of that shape)

Proof method: math. ind. on the number of derivation steps

Accepted alternative: build a tree-like structure covering everything G generates (method used when identifying the language)

$S \underset{2}{\Rightarrow} bc = \boldsymbol{a^0bc}$

$\underset{1}{\Rightarrow} a^2S \underset{2}{\Rightarrow} \boldsymbol{a^2bc}$

$\underset{1}{\Rightarrow} a^4S \underset{2}{\Rightarrow} \boldsymbol{a^4bc}$

$\underset{1}{\Rightarrow} a^6S \underset{2}{\Rightarrow} \boldsymbol{a^6bc}$

$\underset{1}{\Rightarrow} a^6S \underset{2}{\Rightarrow} \ldots$

$\underset{1}{\Rightarrow} \ldots$

We notice that, by using **all** productions of *G*, in **all** possible combinations, we get, as sequences of terminals, only sequences of the shape $a^{2k}bc, k \in \mathbb{N}$. Therefore, *G* does not generate anything else.

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3. Find a grammar that generates $L = \{0^n 1^n 2^m \mid n, m \in \mathbb{N}^*\}$

Sol.: *#V Andrada Tiutin + Alexandra Tudorescu*

Let $G = (N, \Sigma, P, S)$, $N = \{S, A, B\}$, $\Sigma = \{0,1,2\}$,

P: $S \to AB$

$A \to 0A1 \mid 01$

$B \to 2B \mid 2$

? *L= L(G)*

I) ? $L \subseteq L(G)$

? $\forall\, n, m \in \mathbb{N}^*,\ 0^n 1^n 2^m \in L(G)$

Let $n, m \in \mathbb{N}^*$. Then

$$S \underset{1}{\Rightarrow} AB \ \underbrace{\overset{n}{\Rightarrow}}_{by\,(a)}\ 0^n 1^n B \ \underbrace{\overset{m}{\Rightarrow}}_{by\,(b)}\ 0^n 1^n 2^m \quad => 0^n 1^n 2^m \in L(G)$$

(a) $A \overset{n}{\Rightarrow} 0^n 1^n,\ \forall n \in \mathbb{N}^*$

(b) $B \overset{m}{\Rightarrow} 2^m,\ \forall m \in \mathbb{N}^*$

HW: Prove properties (a) and (b), by math. ind.

II) ? $L(G) \subseteq L$

Hint: Show (by building the corresponding tree-like structures) that A can only generate sequences of the shape $0^n 1^n, n \in \mathbb{N}^*$ and B can only generate sequences of the shape $2^m, m \in \mathbb{N}^*$. Since S has only one production ($S \to AB$), it follows that it can only generate sequences that are concatenations of sequences generated by A with sequences generated by B.