Correlation of the Jacobian and Probability Flow

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Abstract

1 Computing the Jacobian

Suppose we have a Neural Network classifier $f : \mathbb{R}^d \to \mathbb{R}^k$, where d is the dimensionality of the input and k is the number of classes. We define the Jacobian matrix of the Neural Network as:

$$J(\mathbf{x}) = \nabla_{\mathbf{x}} f(\mathbf{x}) \in \mathbb{R}^k \times \mathbb{R}^d \tag{1}$$

Or the derivative of the Neural Network with respect to the input. Suppose we have a training dataset of size n, with examples given as $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$, where $\mathbf{x}_i \in \mathbb{R}^d$. We can concatenate the $J(\mathbf{x})$ for each example into a vector:

$$\mathcal{J}_i = \text{flatten}(J(\mathbf{x}_i)) \in \mathbb{R}^{k \times d} \tag{2}$$

We can then form a matrix of these $\mathcal{J}_i$'s into a matrix $\mathcal{J} \in \mathbb{R}^n \times \mathbb{R}^{k \times d}$, which we will call the Jacobian matrix of the training dataset. Each row corresponds to a training example and the columns are a concatenation of the derivatives of each dimension of the output with each dimension of the input.

As a classifier trained with a cross-entropy loss and with a softmax as the final output of the network, we can interpret the output of the network as a vector of probabilities:

$$f(\mathbf{x}) = (p(y_1|\mathbf{x}), p(y_2|\mathbf{x}), \cdots, p(y_k|\mathbf{x})) \in \mathbb{R}^k \tag{3}$$

The Jacobian matrix of the neural network then becomes:

$$J(\mathbf{x}) = \nabla_{\mathbf{x}} f(\mathbf{x}) = (\nabla_{\mathbf{x}} p(y_1|\mathbf{x}), \nabla_{\mathbf{x}} p(y_2|x), \cdots, \nabla_{\mathbf{x}} p(y_k|\mathbf{x})) \in \mathbb{R}^k \times \mathbb{R}^d \tag{4}$$

For conciseness (and abuse) of notation, we can write the the flattened $\mathcal{J}_i$ in vector notation as:

$$\mathcal{J}_i = \nabla_{\mathbf{x}_i} p(\mathbf{y}|\mathbf{x}_i) \in \mathbb{R}^{k \times d} \tag{5}$$

We can then build the correlation matrix of the Jacobian over the training dataset as:

$$M_{ij} = (\mathcal{J}\mathcal{J}^T)_{ij} = \nabla_{\mathbf{x}_i} p(\mathbf{y}|\mathbf{x}_i) \cdot \nabla_{\mathbf{x}_j} p(\mathbf{y}|\mathbf{x}_j) \in \mathbb{R}^{n \times n} \tag{6}$$

Theorem 1. *testing*

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Suppose we have a random matrix, M, where each matrix element M_{ij} is the covariance of two random variables, $\mathrm{Cov}(X_i, X_j)$.

References