

# An Overview of Graceful Labelings

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Department of Mathematics and Statistics

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2. Present known results on graceful labelings.
3. Look over past research from an NAU student.
4. Introduce the Ringel-Kotzig Conjecture and state recent progress.

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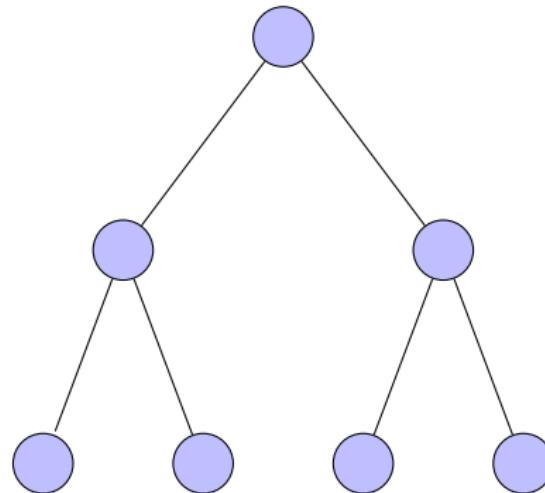
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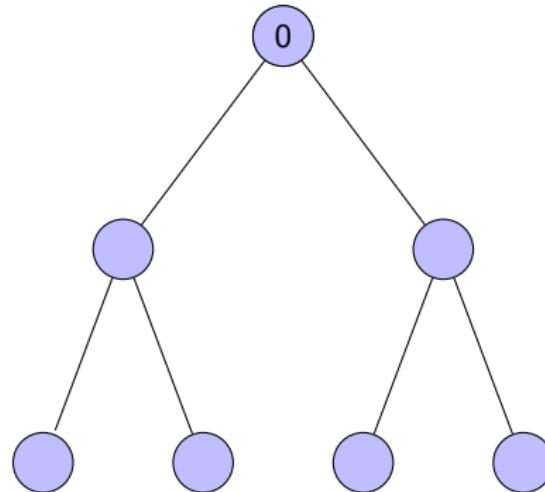
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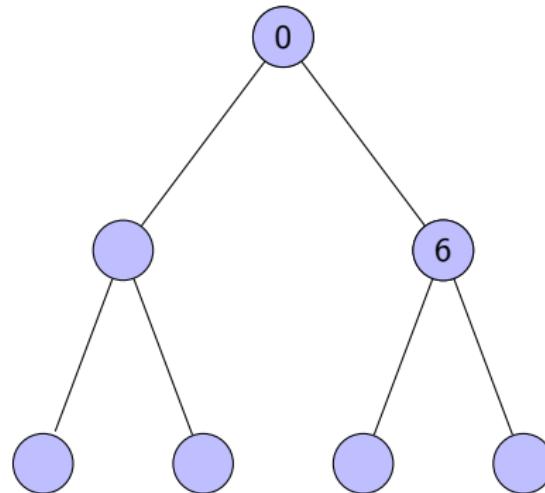
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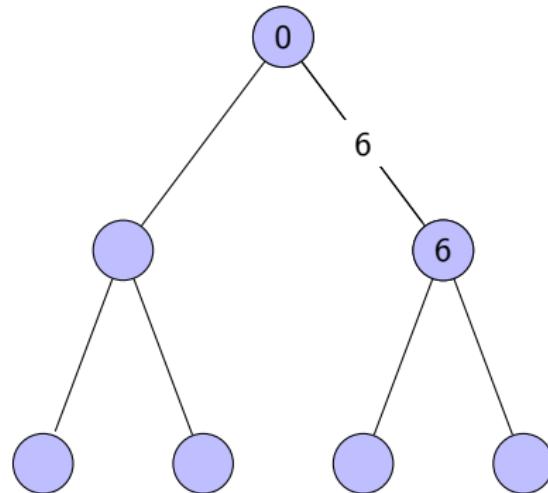
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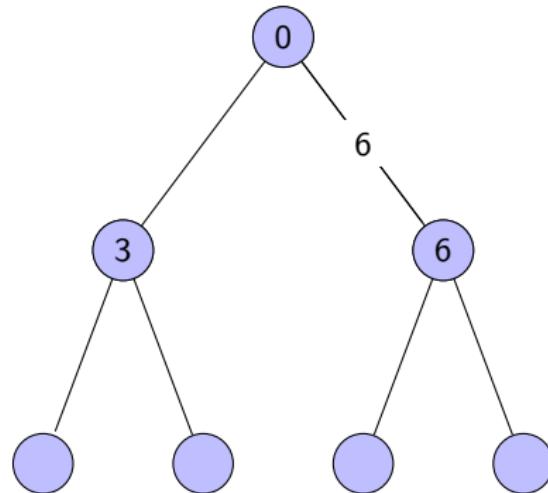
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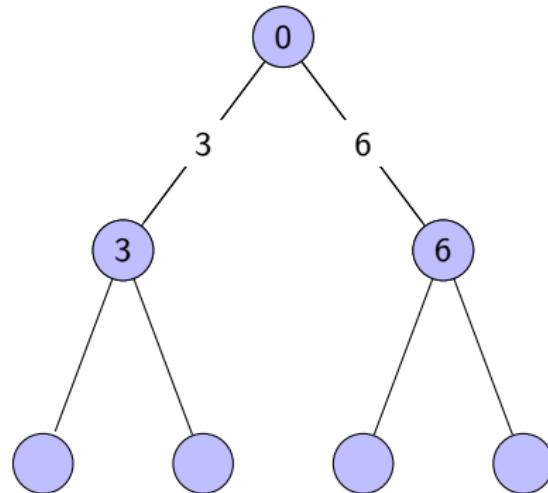
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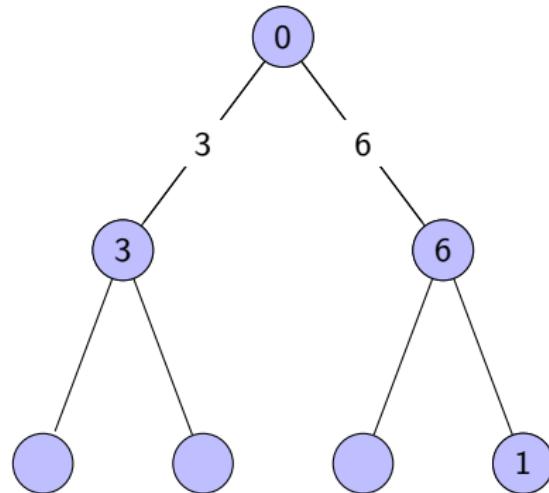
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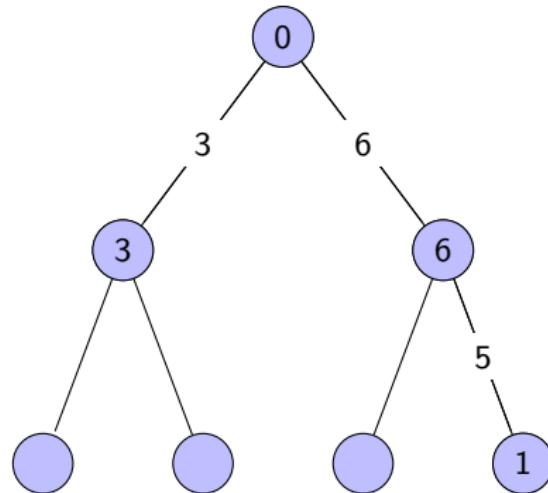
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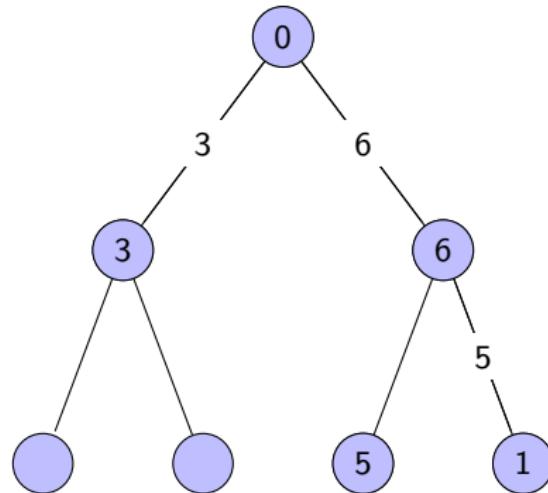
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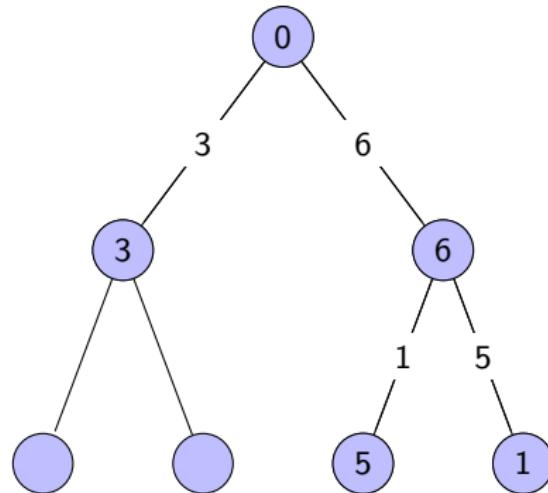
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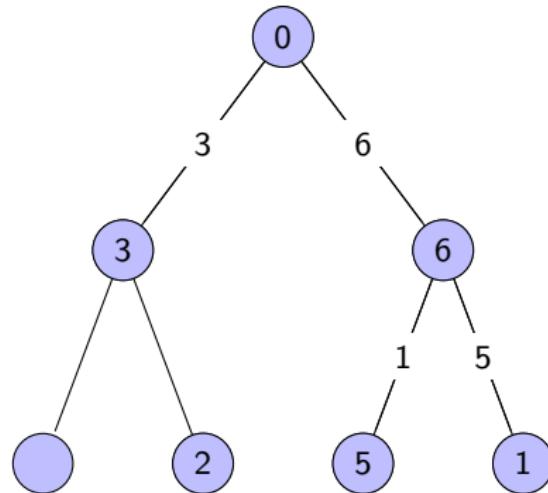
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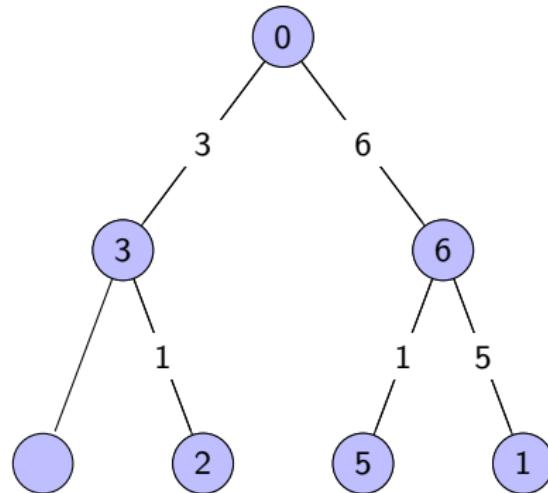
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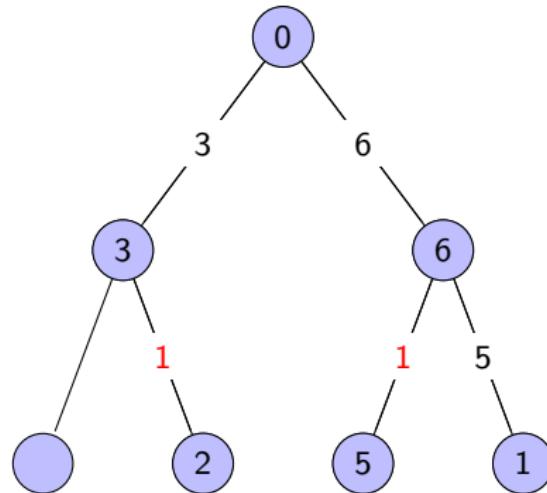


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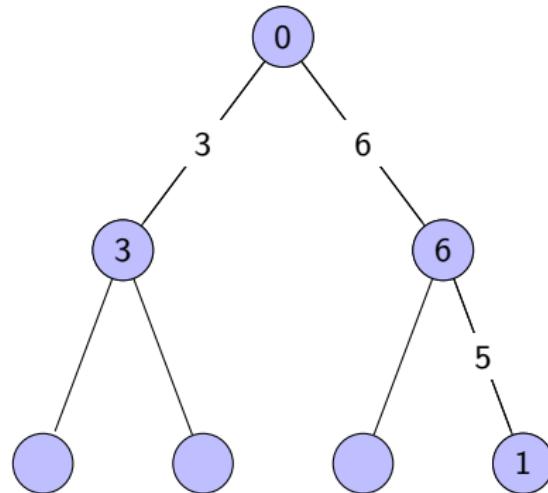


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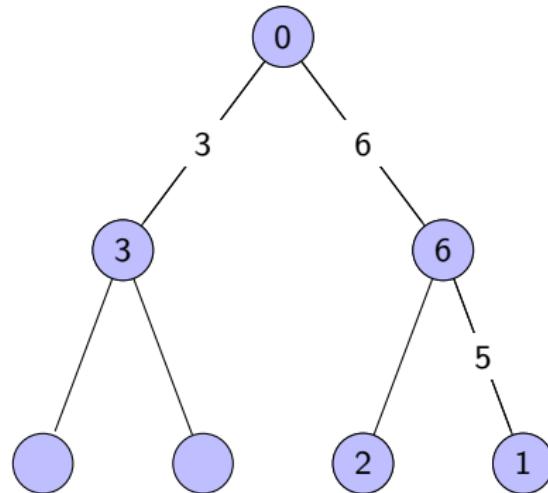
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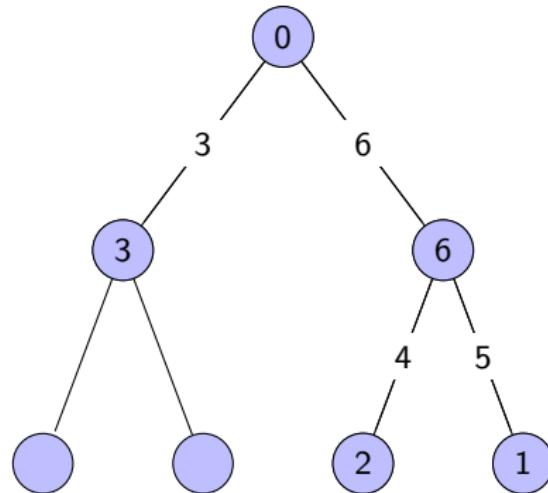
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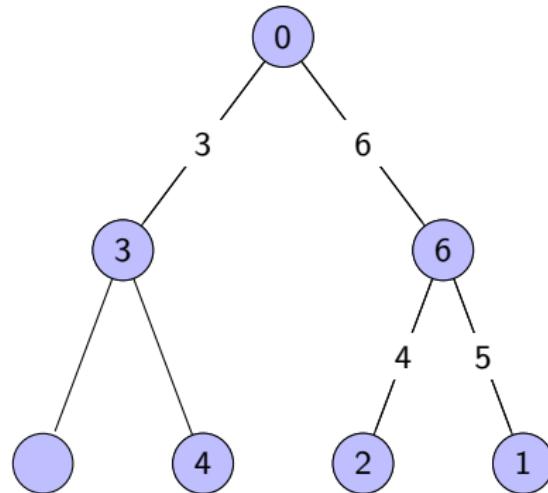
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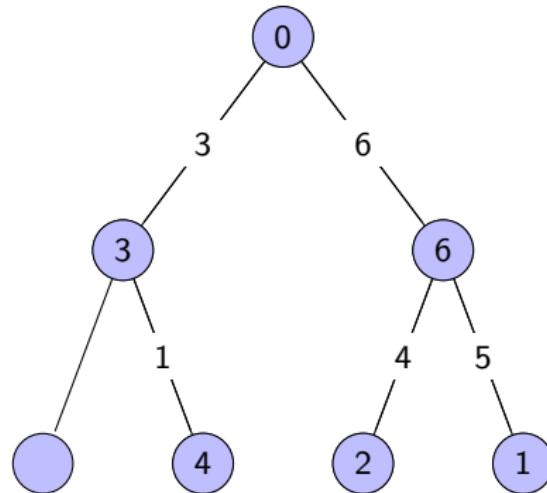


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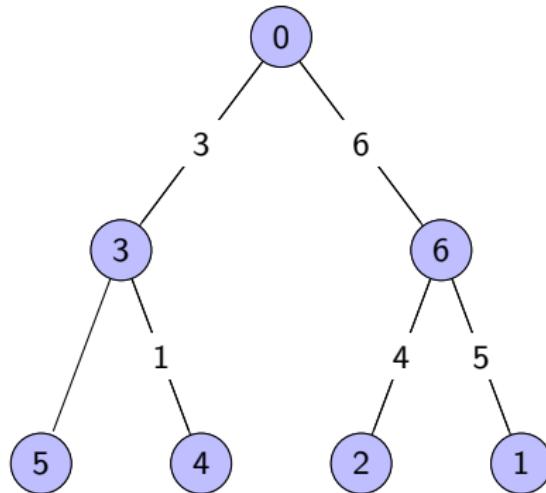


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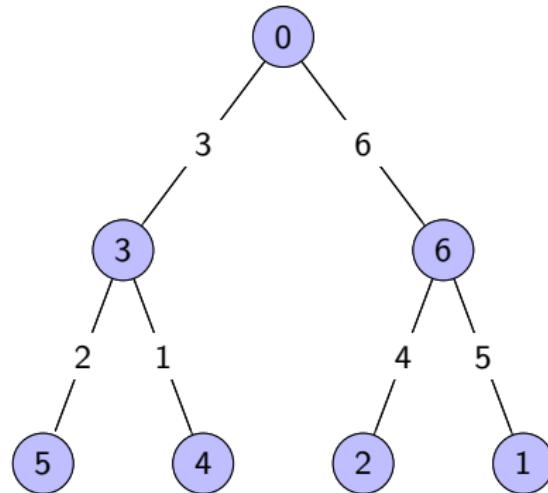


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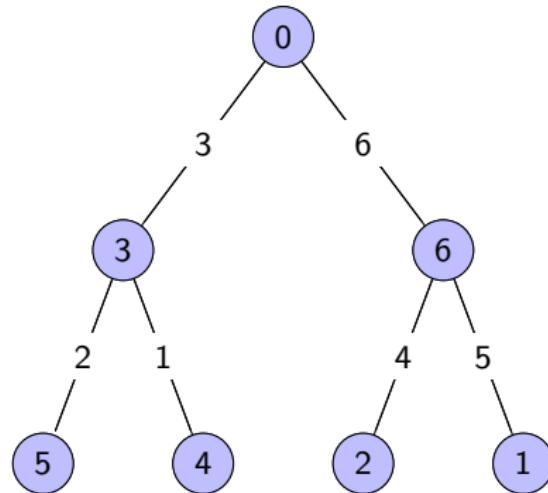
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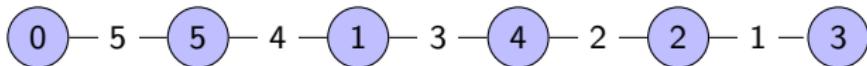


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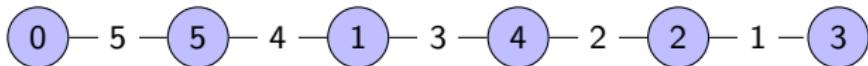


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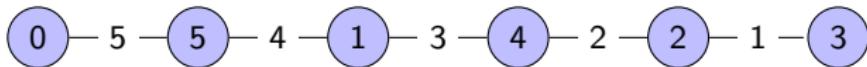


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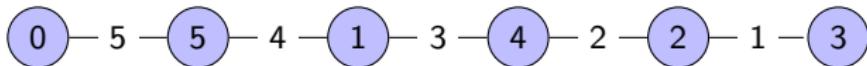


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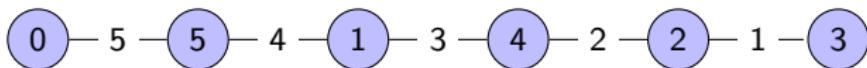


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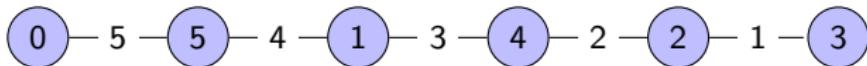


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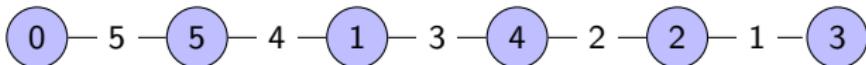


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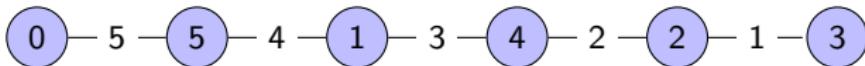


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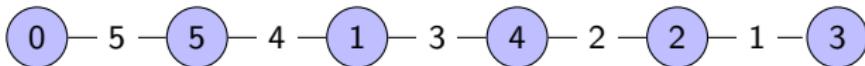


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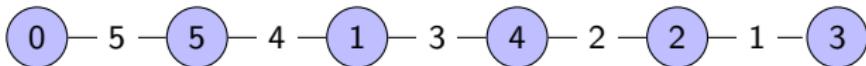


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Thus,  $P_6$  has a graceful labeling.

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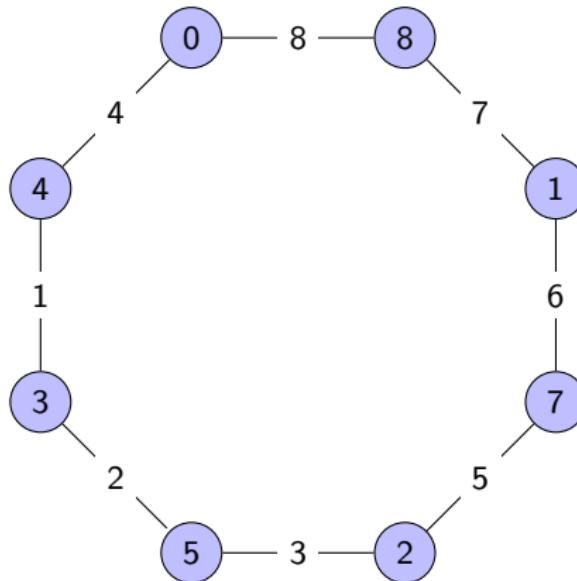


Figure: A graceful labeling of the cycle  $C_8$ .

## "The Father of Graceful-ness"

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*A cycle with  $n$  vertices is graceful if and only if  $n \equiv 0$  or  $3 \pmod{4}$ .*

The labeling functions for graceful cycles are:

If  $n \equiv 0 \pmod{4}$ :

$$f(v_i) = \begin{cases} \frac{i-1}{2}, & \text{if } i \text{ is odd,} \\ n+1 - \frac{i}{2}, & \text{if } i \text{ is even and } i \leq \frac{n}{2}, \\ n - \frac{i}{2}, & \text{if } i \text{ is even and } i > \frac{n}{2}. \end{cases}$$

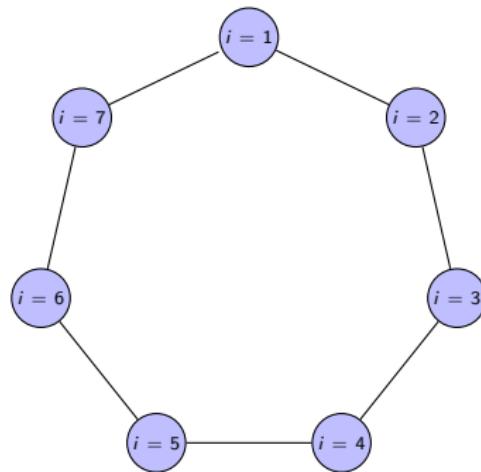
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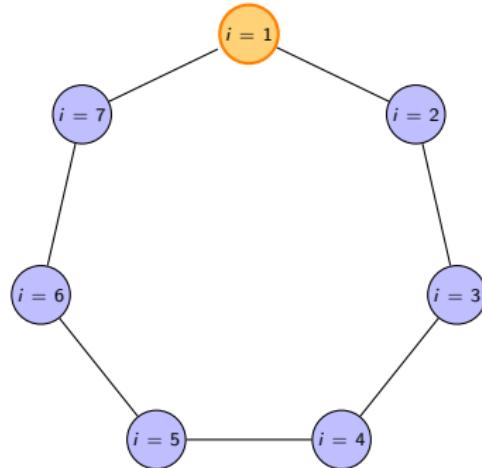
Here, the vertices of  $C_n$  are denoted  $v_1, v_2, \dots, v_n$ , where each  $\{v_i, v_{i+1}\}$  is an edge for all  $1 \leq i \leq n-1$ , as is  $\{v_n, v_1\}$ .

**It is important to note that a given labeling function is not necessarily unique.**

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

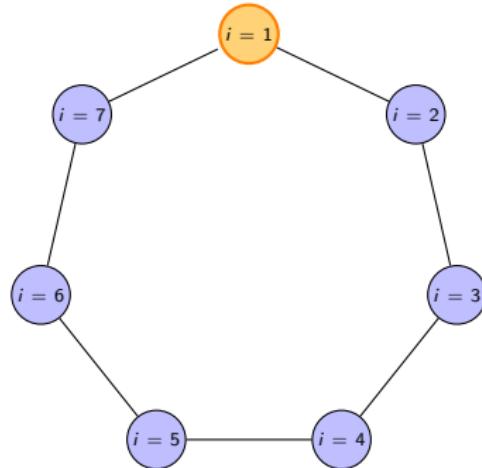


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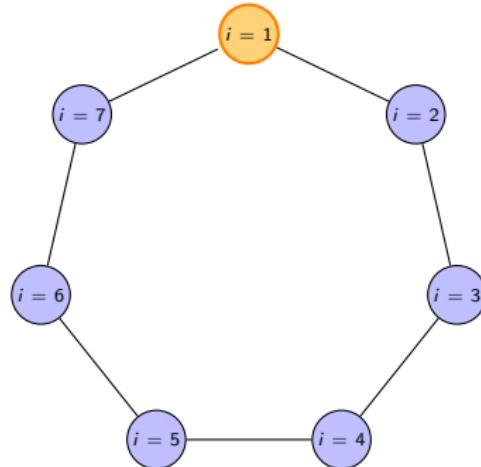
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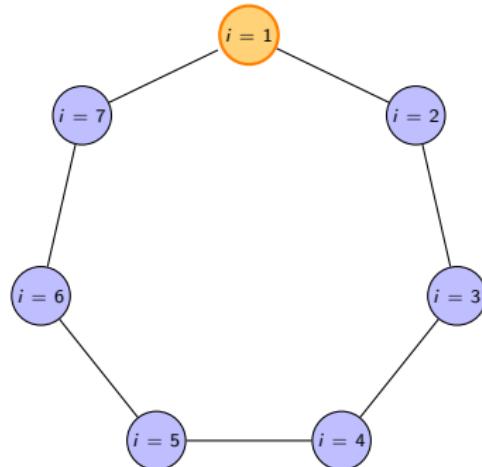
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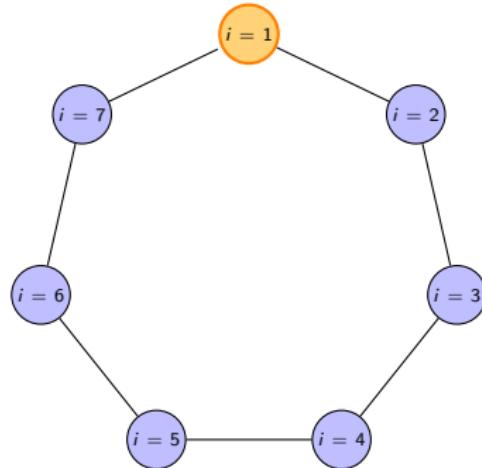
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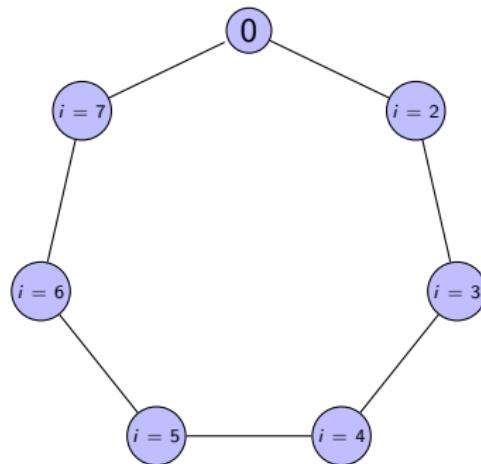
$$i = 1 : 1 \text{ is odd and } 1 \leq 3 \Rightarrow f(v_1) = \frac{1-1}{2}$$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

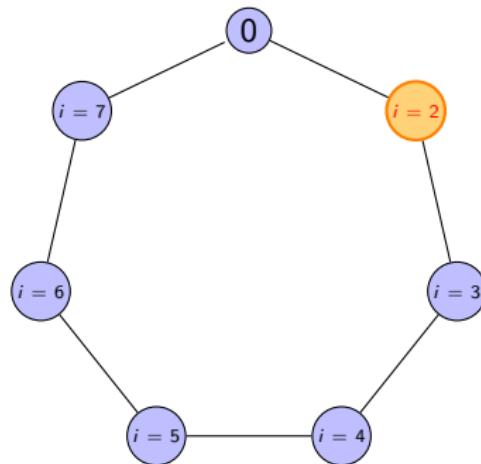


$$i = 1 : 1 \text{ is odd and } 1 \leq 3 \Rightarrow f(v_1) = \frac{1-1}{2} = 0$$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

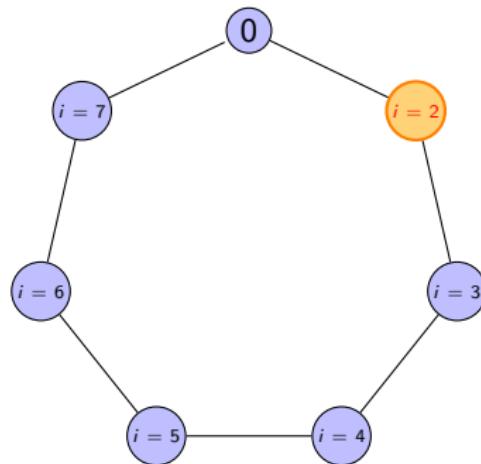


$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



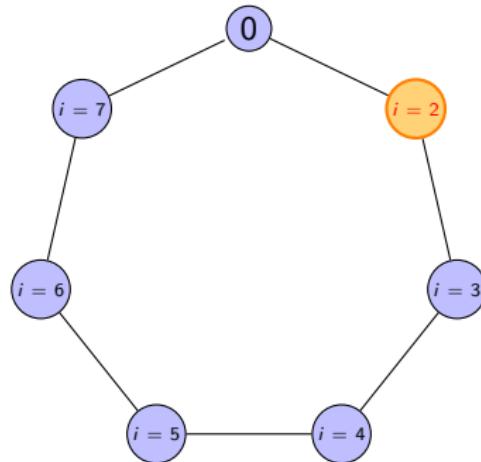
$i = 2 :$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



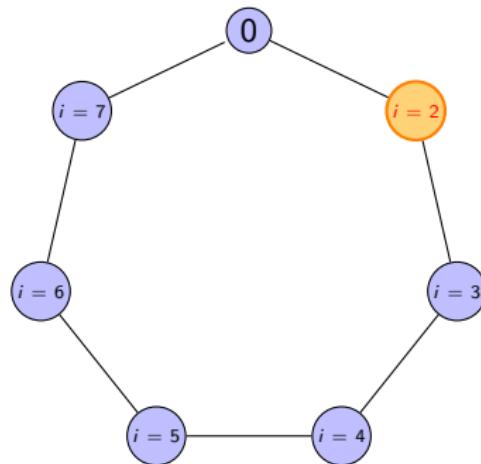
$i = 2 : 2$  is even

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



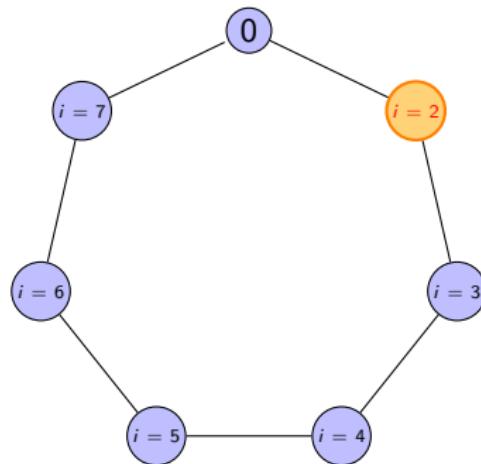
$i = 2 : 2$  is even

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 2 : 2 \text{ is even} \Rightarrow f(v_2) = 7 + 1 - \frac{2}{2}$$

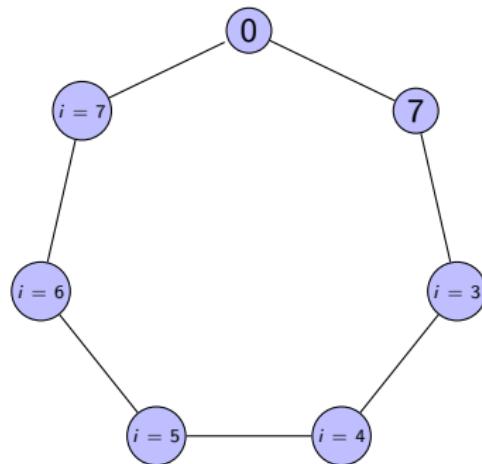
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



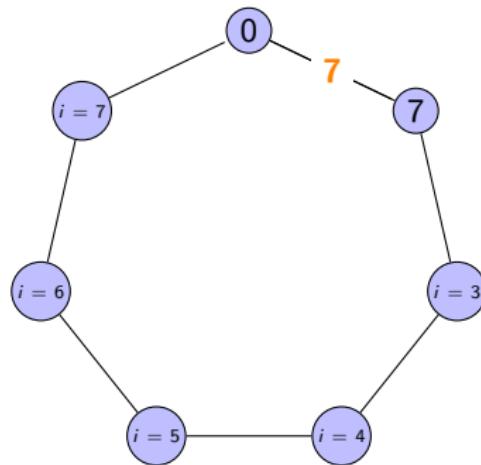
$$i = 2 : 2 \text{ is even} \Rightarrow f(v_2) = 7 + 1 - \frac{2}{2} = 7$$

## Labeling Function Example

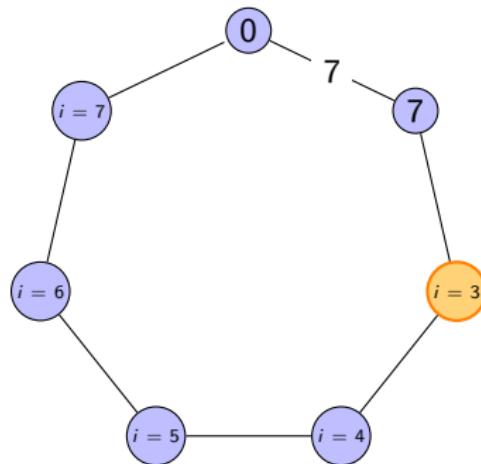
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

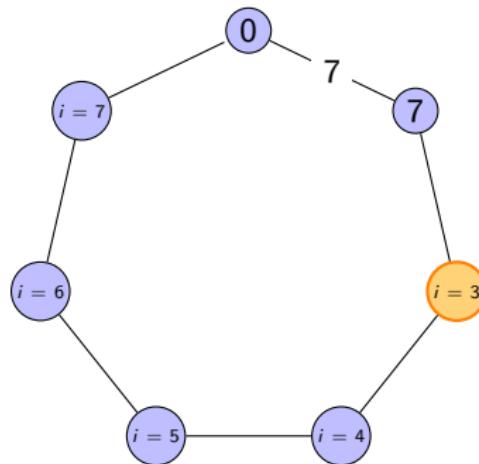


$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



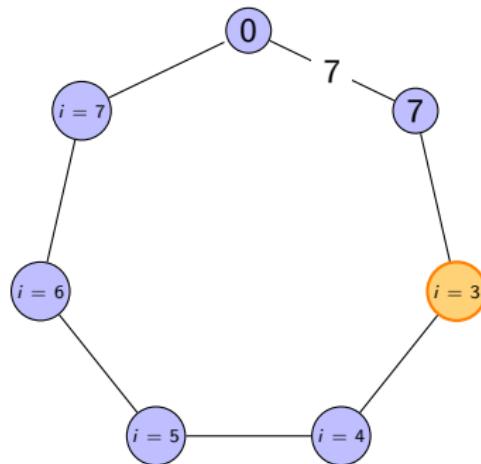
$i = 3 :$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



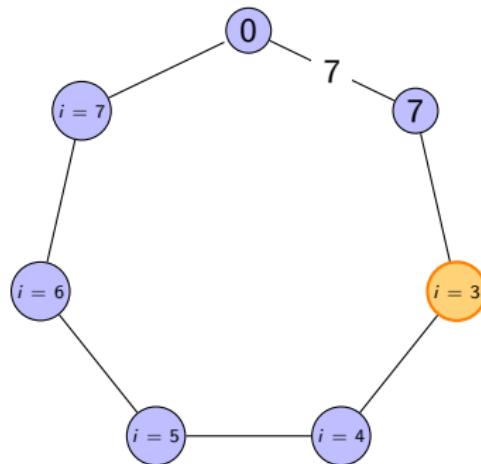
$i = 3 : 3$  is odd and  $3 \leq 3$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



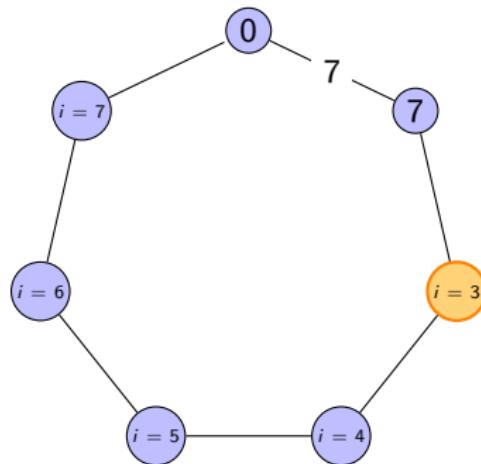
$i = 3 : 3$  is odd and  $3 \leq 3$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 3 : 3 \text{ is odd and } 3 \leq 3 \Rightarrow f(v_3) = \frac{3-1}{2}$$

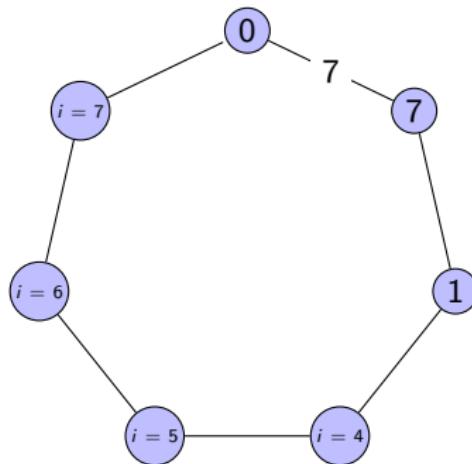
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



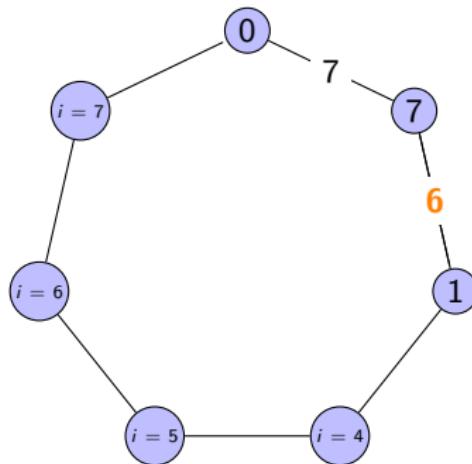
$$i = 3 : 3 \text{ is odd and } 3 \leq 3 \Rightarrow f(v_3) = \frac{3-1}{2} = 1$$

## Labeling Function Example

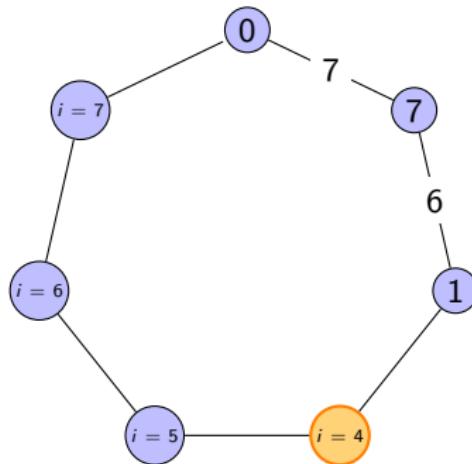
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

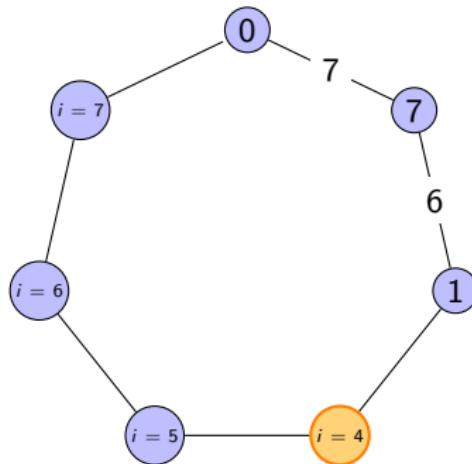


$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



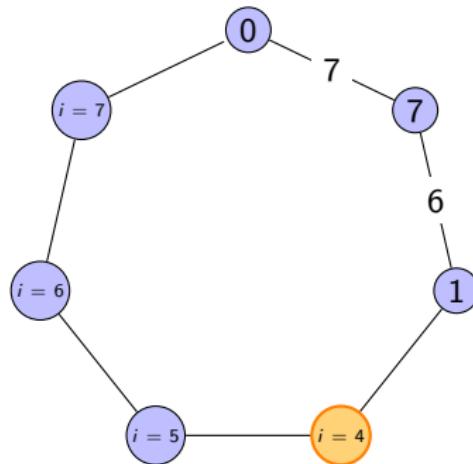
$i = 4 :$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



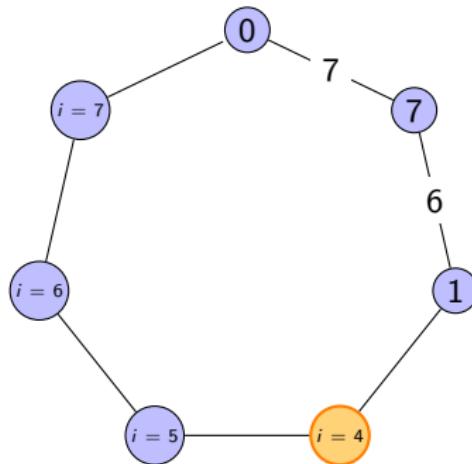
$i = 4 : 4$  is even

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



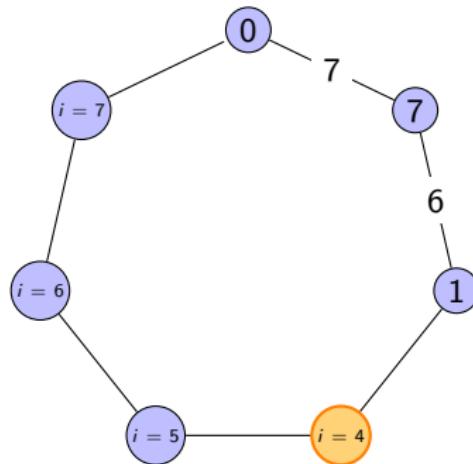
$i = 4 : 4$  is even

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 4 : 4 \text{ is even} \Rightarrow f(v_4) = 7 + 1 - \frac{4}{2}$$

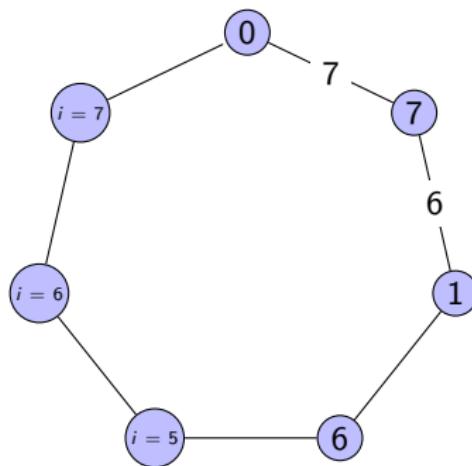
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



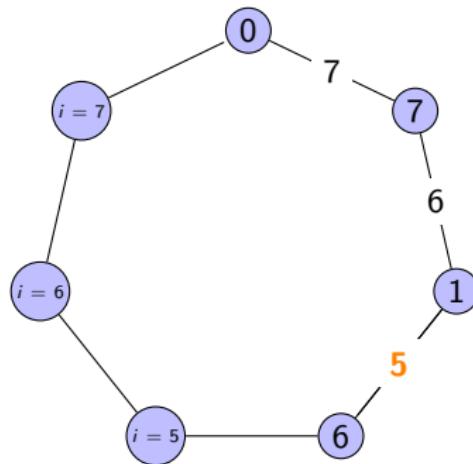
$$i = 4 : 4 \text{ is even} \Rightarrow f(v_4) = 7 + 1 - \frac{4}{2} = 6$$

## Labeling Function Example

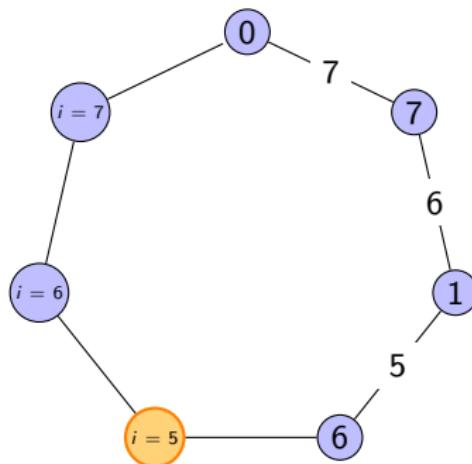
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

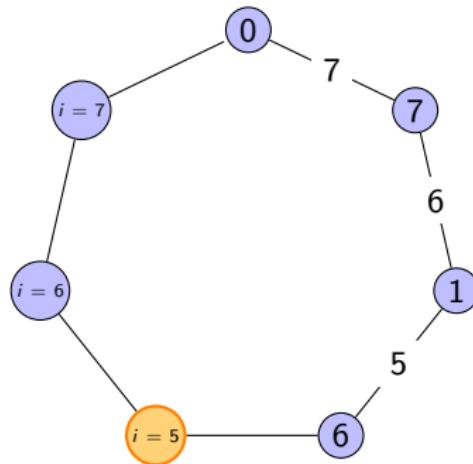


$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



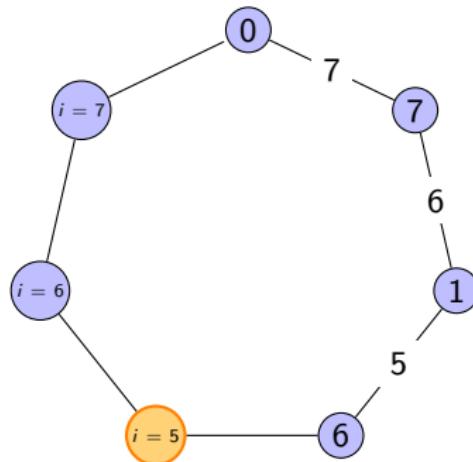
$i = 5 :$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



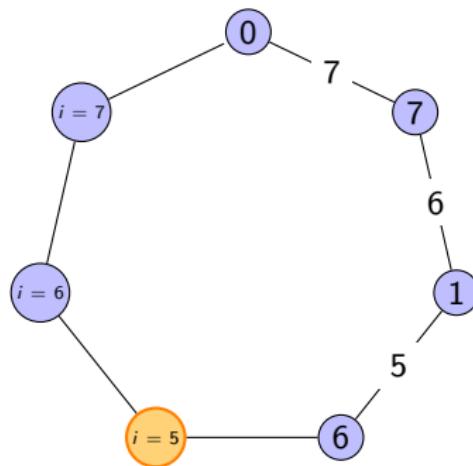
$i = 5 : 5$  is odd and  $5 > 3$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



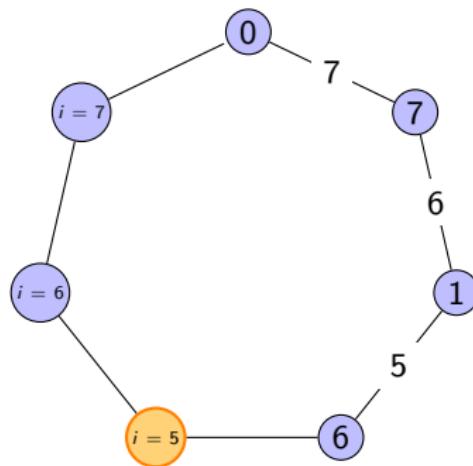
$i = 5 : 5$  is odd and  $5 > 3$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 5 : 5 \text{ is odd and } 5 > 3 \Rightarrow f(v_5) = \frac{5+1}{2}$$

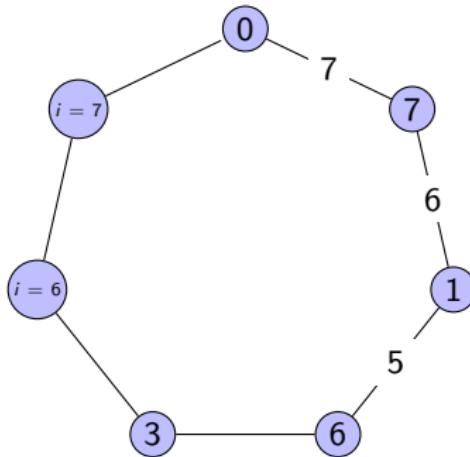
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



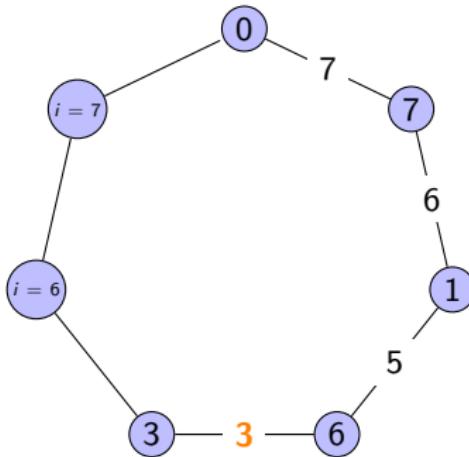
$$i = 5 : 5 \text{ is odd and } 5 > 3 \Rightarrow f(v_5) = \frac{5+1}{2} = 3$$

## Labeling Function Example

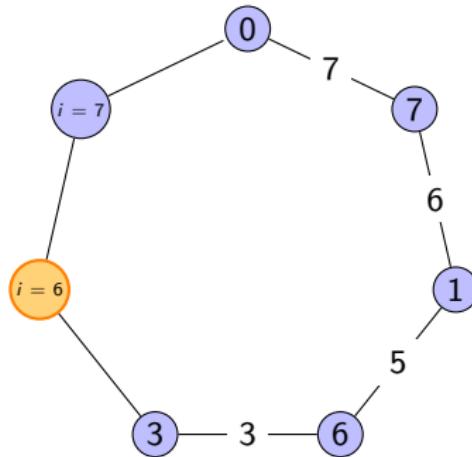
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

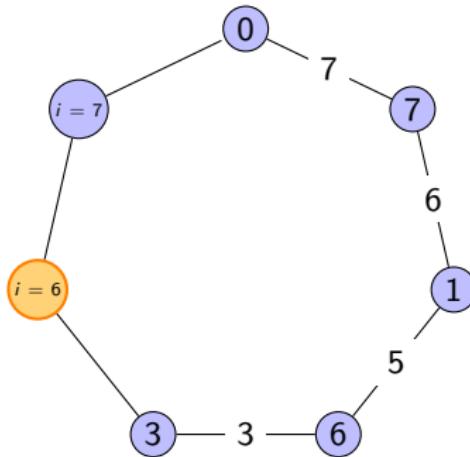


$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



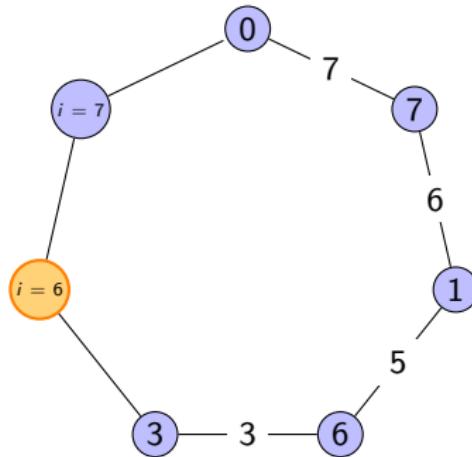
$i = 6 :$

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



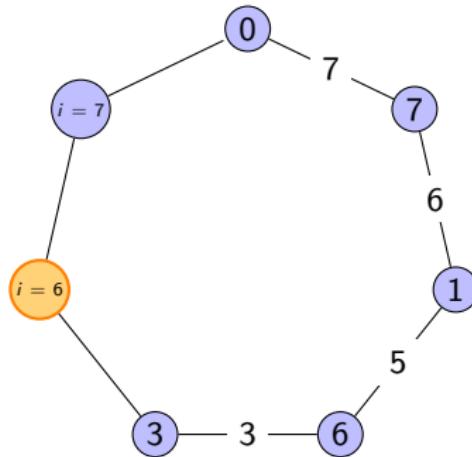
$i = 6 : 6$  is even

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



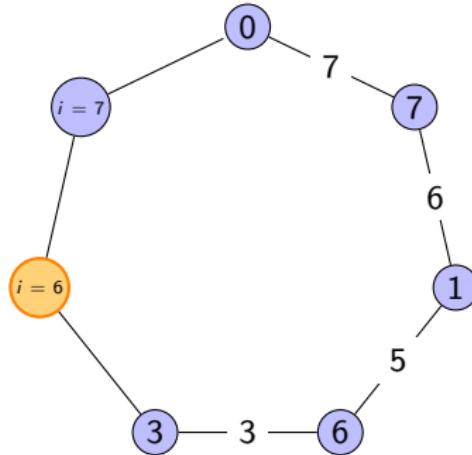
$i = 6 : 6$  is even

$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 6 : 6 \text{ is even} \Rightarrow f(v_6) = 7 + 1 - \frac{6}{2}$$

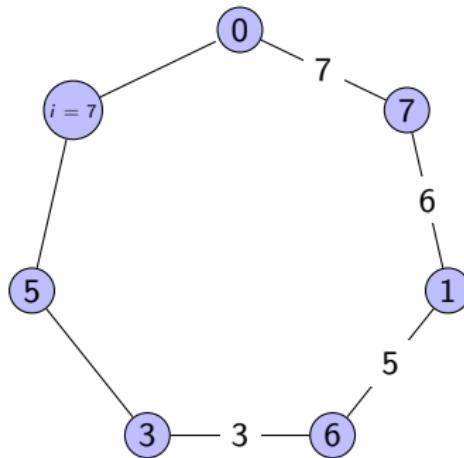
$$7 \equiv 3 \pmod{4} : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



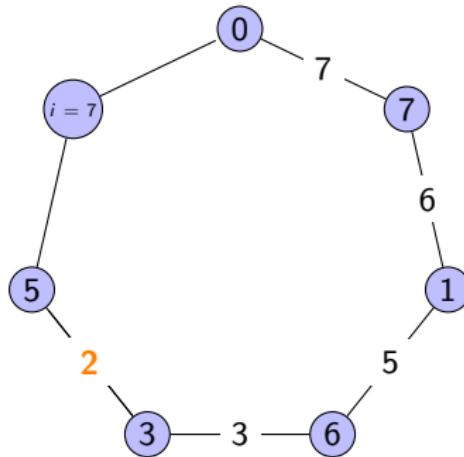
$$i = 6 : 6 \text{ is even} \Rightarrow f(v_6) = 7 + 1 - \frac{6}{2} = 5$$

## Labeling Function Example

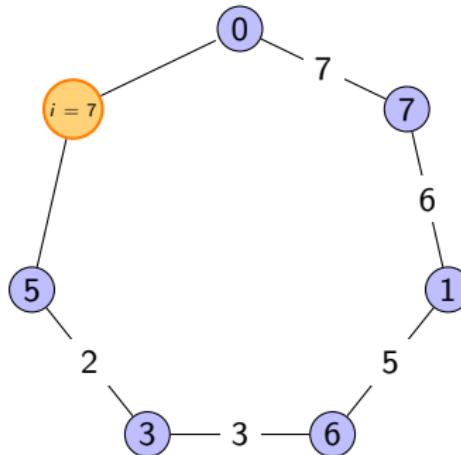
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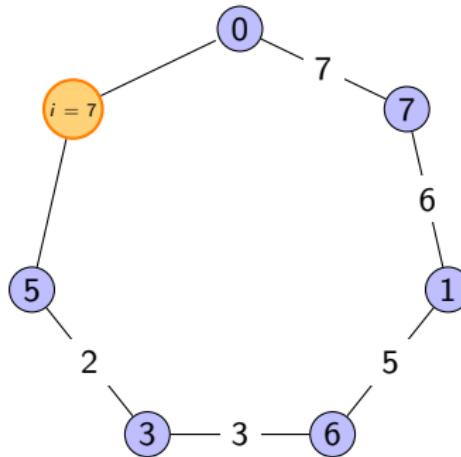


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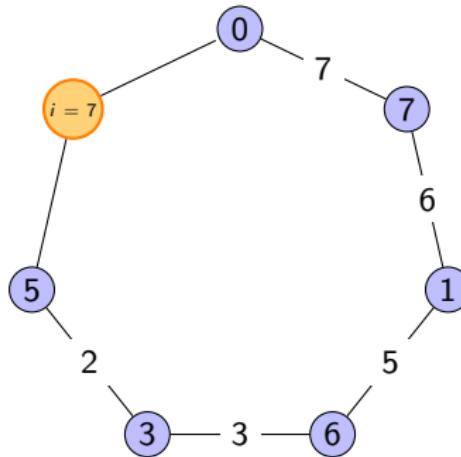
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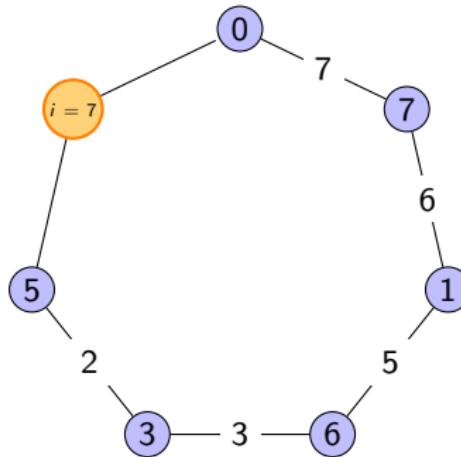
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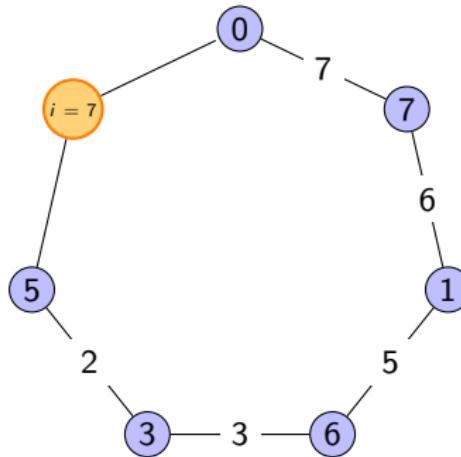
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$$i = 7 : 7 \text{ is odd and } 7 > 3 \Rightarrow f(v_7) = \frac{7+1}{2}$$

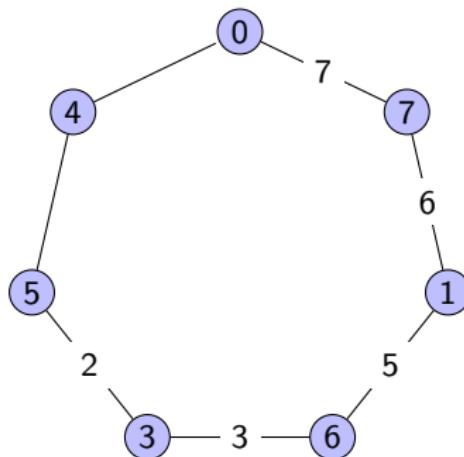
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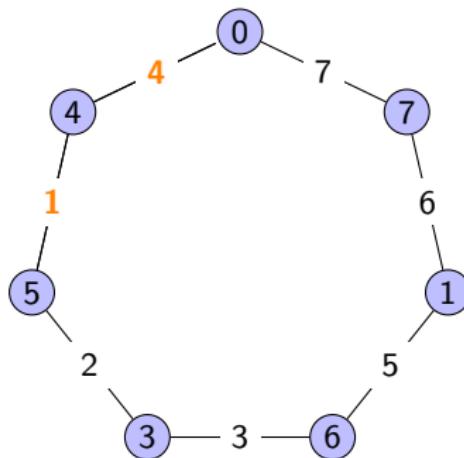
$$i = 7 : 7 \text{ is odd and } 7 > 3 \Rightarrow f(v_7) = \frac{7+1}{2} = 4$$

## Labeling Function Example

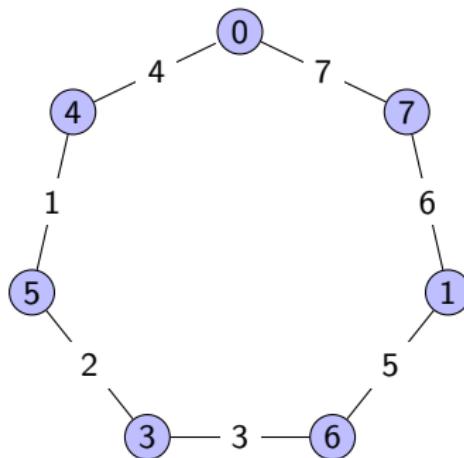
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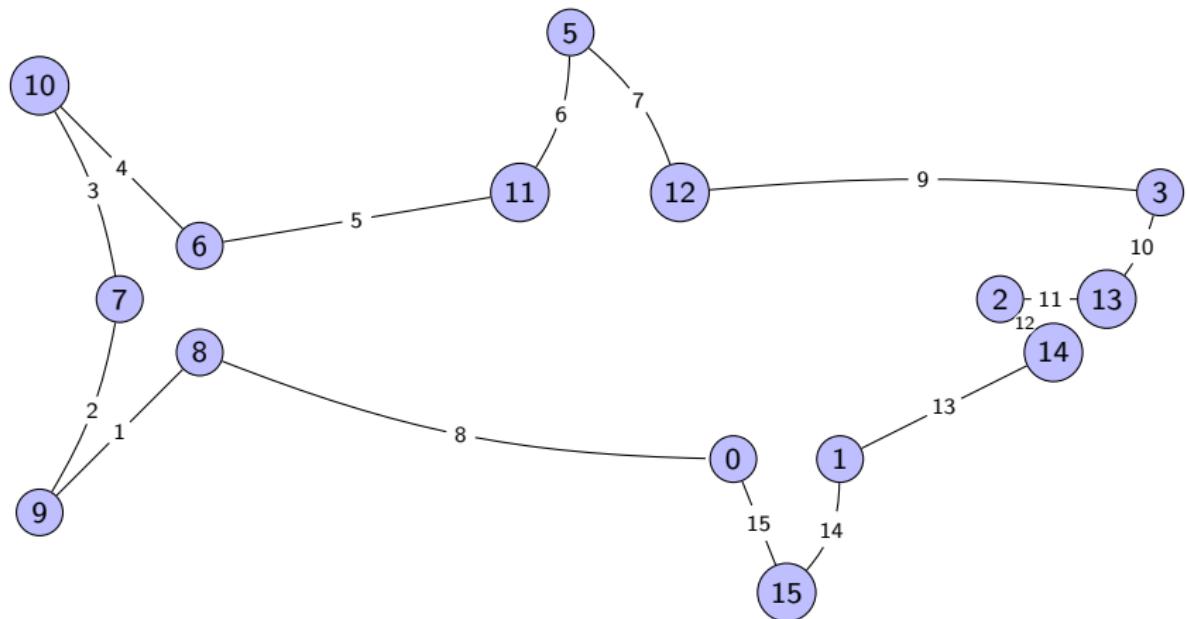


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## Shark $n$ -Cycles with $n \equiv 3 \pmod{4}$ are Graceful

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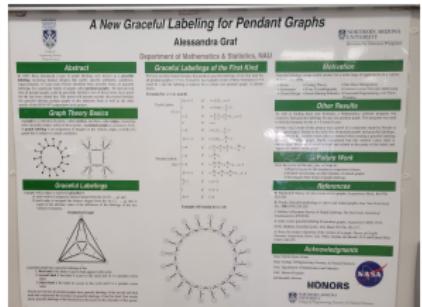
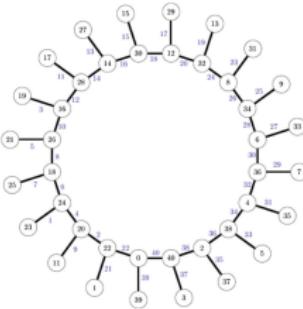
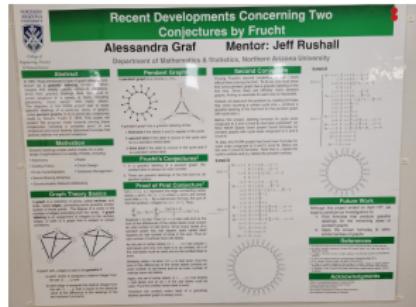


# Graceful Work from an NAU student

## Definition

A *pendant graph* is an  $n$ -cycle graph with one pendant vertex attached to each cycle vertex.

Dr. Alessandra Graf researched graceful labelings of pendant graphs for several semesters at NAU starting in 2013. She was able to prove that all pendant graphs that have labels 0 and  $2n$ , where  $n \equiv 4$  or  $8 \pmod{8}$  and  $n \equiv 0 \pmod{16}$  can be gracefully labeled, while also finding the labeling functions for them.



## Miscellaneous Graph Families With Graceful Labelings

Since Rosa first introduced the idea of graceful labeling, many mathematicians (including Dr. Alessandra Graf) have proved that certain graphs can always be gracefully labeled.

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Let's look at some of them...

# Miscellaneous Graphs With Graceful Labelings

## Theorem

All *wheel*, *web*, *helm*, *gear*, *grid*, *star*, *complete*, *banana tree*, and *caterpillar tree* graphs are graceful. [Various mathematicians]

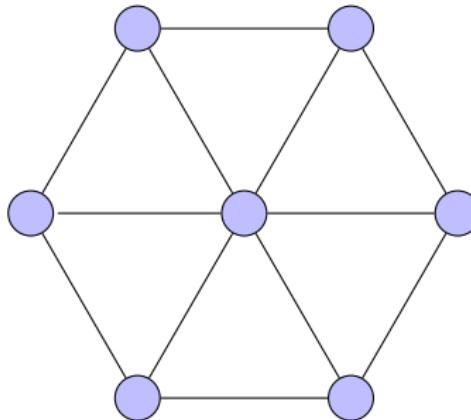


Figure: The wheel graph  $W_7$ .

# Miscellaneous Graph Families With Graceful Labelings

## Theorem

All wheel, *web*, helm, gear, grid, star, complete, banana tree, and caterpillar tree graphs are graceful. [Various]

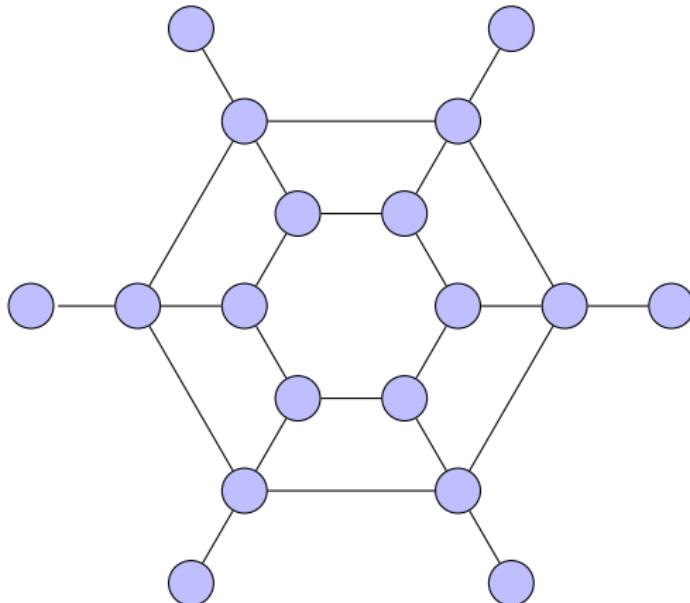


Figure: The web graph  $W_6$ .

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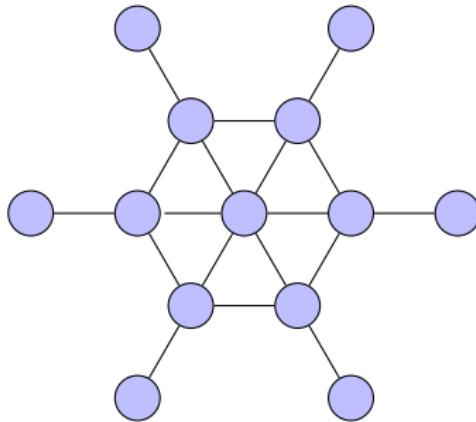


Figure: The helm graph  $H_6$ .

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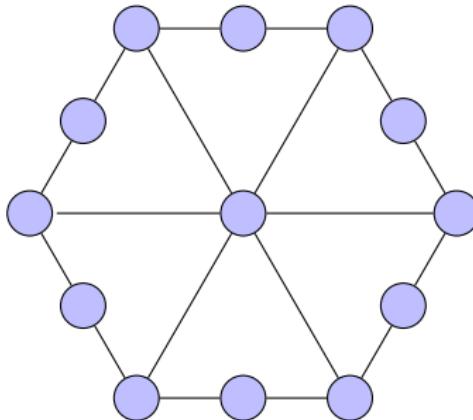


Figure: The gear graph  $G_6$ .

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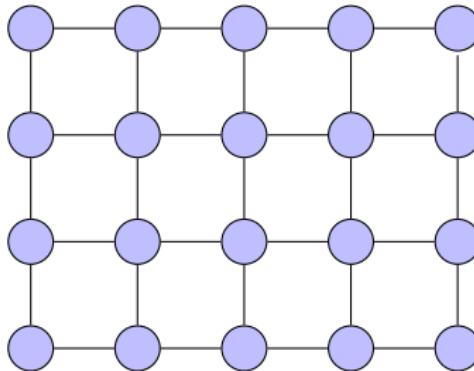


Figure: The grid graph  $P_5 \times P_4$ .

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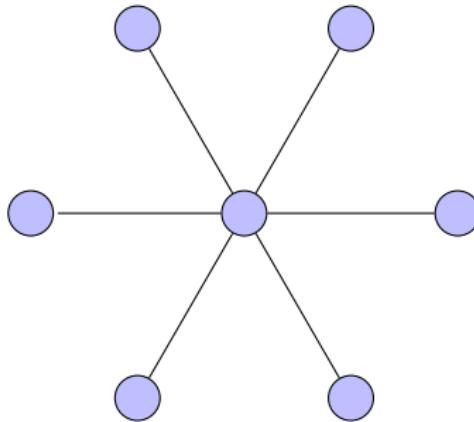


Figure: The star graph  $S_7$ .

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All wheel, web, helm, gear, grid, star, **complete**, banana tree, and caterpillar tree graphs are graceful. [Various]

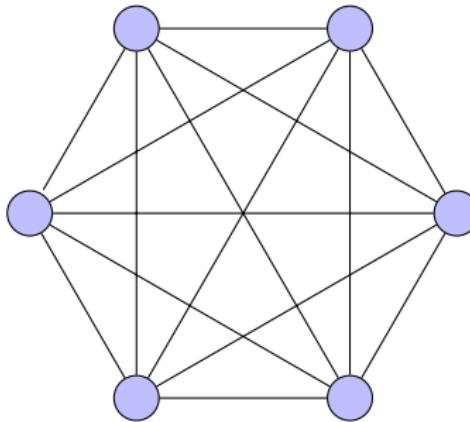


Figure: The complete graph  $K_6$ .

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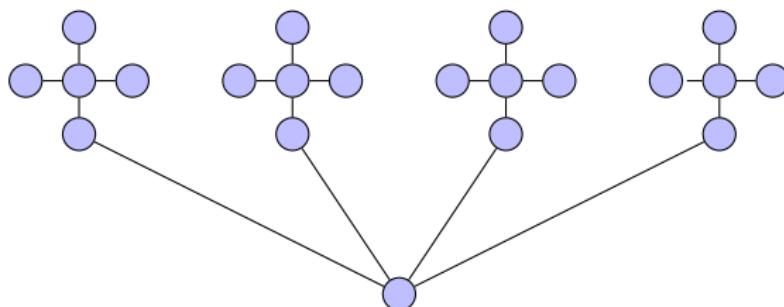


Figure: The banana tree graph  $B_{4,5}$ .

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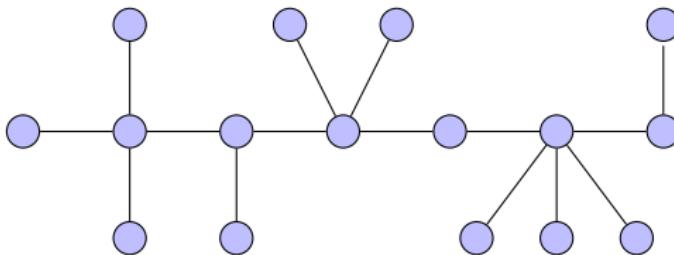
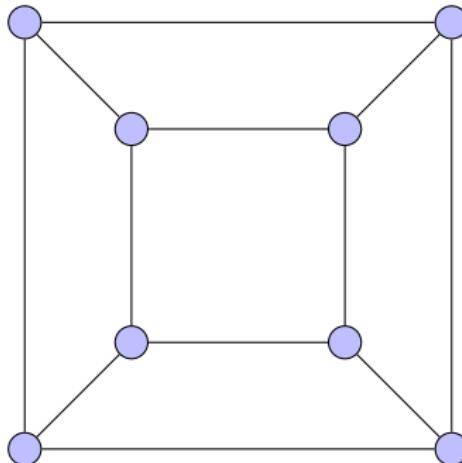


Figure: A caterpillar tree graph.

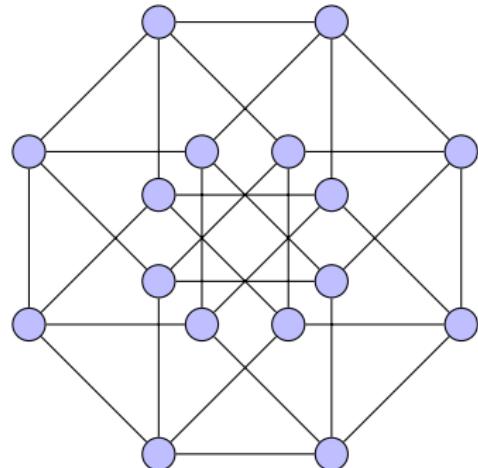
# Miscellaneous Graph Families With Graceful Labelings

## Theorem

All *n*-dimension hypercubes are graceful. [Kotzig, 1981]



3-dimensional hypercube



4-dimensional hypercube

The conjecture that all trees are graceful was proposed by Gerhard Ringel and Anton Kotzig, often identified as the [Ringel-Kotzig Conjecture](#).

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- Kotzig commented that trying to prove this conjecture was a “disease” .
- It has several related (yet weaker) conjectures, many of which have proven partial results.

# The Ringel-Kotzig Conjecture – Recent Progress

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Questions?