The Tietze Extension Theorem

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Topology Final Project

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Theorem (The Tietze Extension Theorem)

Let X be a normal space and A be a closed subspace of X. Then the following properties hold:

(1) Any continuous map of A into $[a, b] \subseteq \mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

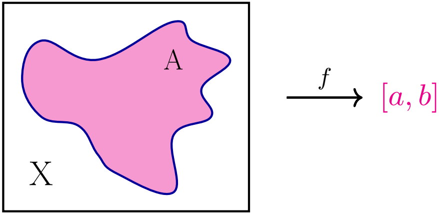
(2) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

Definition 1 (Normal Space)

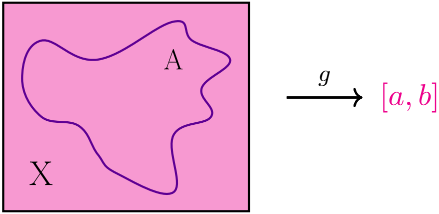
A space X is **normal** if, for every pair of disjoint closed sets of X, there exist disjoint open sets containing them.

Visualization for (1).

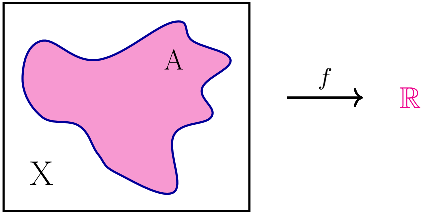
(1). Let X be a normal space and A be a closed subspace. If there exists a continuous function f such that:



then there exists a continuous function g such that:



(2). Let X be a normal space and A be a closed subspace. If there exists a continuous function f such that:



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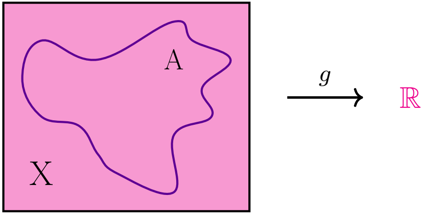

Timeline:

- The Tietze Extenstion theorem was originally proven for a specific case where X has to be a finite-dimensional real vector space by L.E.J. Brouwer and Henri Lebesgue in the early 1900s.
- In 1915 Heinrich Tietze expanded the proof to include all metric spaces.
- After proving his own lemma, Uryson's Lemma, in 1923, Pavel Urysohn used this lemma to further the proof to include all normal spaces.

Proof:

Theorem (**The Tietze Extension Theorem**)

Let X be a normal space and A be a closed subspace of X. Then the following properties hold:

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(2) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

Idea: Construct a sequence of continuous functions, $\{s_n\}_{n\in\mathbb{N}}$, defined on the entire space X such that the sequence s_n converges uniformly to a sequence such that the restriction of s_n to A approximates f as n goes to infinity.

$$(s_{n|A} \xrightarrow{n\to\infty} f)$$

Proof (continued): Step 1:

The goal of Step 1 is to construct a specific function g defined on all of X such that g approximates f.

Firstly, without loss of generality, let $f : A \to [-r, r]$. (We can do so since there exists a homeomorphism between $[-r, r]$ and $[a, b]$).

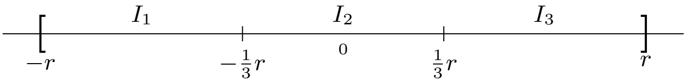
Claim: There exists a continuous function $g : X \to \mathbb{R}$ such that:

$$\begin{aligned} |g(x)| &\leq \tfrac{1}{3}r && \forall x \in X \\ |g(a) - f(a)| &\leq \tfrac{2}{3}r && \forall a \in A. \end{aligned}$$

Proof (continued): Step 1 (continued)

To construct this specific function g, first divide the interval $[-r, r]$ into three equal lengths:

$$I_1 = [-r, -\tfrac{1}{3}r], \qquad I_2 = [-\tfrac{1}{3}r, \tfrac{1}{3}r], \qquad I_3 = [\tfrac{1}{3}r, r]$$



Now, let the following subsets of A exist:

$$B = f^{-1}(I_1) \quad \& \quad C = f^{-1}(I_3)$$

Since f is continuous and I_1 & I_3 are closed and disjoint, B & C are closed disjoint subsets of A. And since A is closed in X, B & C are closed in X too.

Proof (continued): Step 1 (continued)

Since B & C are disjoint, we can use Urysohn's Lemma to see that there exists a continuous function

$$g : X \to [-\tfrac{1}{3}r, \tfrac{1}{3}r]$$

such that

$$g(B) = \{-\tfrac{1}{3}r\} \qquad g(C) = \{\tfrac{1}{3}r\}.$$

And by definition, any $x \notin B, C$ is in $[-\frac{1}{3}r, \frac{1}{3}r]$.

Thus, $|g(x)| \leq \frac{1}{3}r \quad \forall x \in X$.

(This was the first part of our claim!)

Proof (continued): Step 1 (continued)

Remember:

$$\boxed{\begin{array}{c} B = f^{-1}(I_1) \quad \text{and} \quad C = f^{-1}(I_3) \\ \& \\ g(B) = \{-\frac{1}{3}r\} \quad \text{and} \quad g(C) = \{\frac{1}{3}r\}. \end{array}}$$

The second part of our claim stated $|g(a) - f(a)| \leq \frac{2}{3}r \quad \forall a \in A$.

There are three possibilities for a based on our definitions of B & C:

- if $a \in B$: then $f(a) \in I_1$, and $g(a) = -\frac{1}{3}r$ since $g(B) = \{-\frac{1}{3}r\} \in I_1$. And since $f(a)$,$g(a) \in I_1$ and the length of $I_1 = \frac{2}{3}r$, $|g(a) - f(a)| \leq \frac{2}{3}r$.
- if $a \in C$: then the case is similar to when $a \in B$.
- if $a \notin B, C$: then by definition, $f(a), g(a) \in I_2$ and since the length of $I_2 = \frac{2}{3}r$, $|g(a) - f(a)| \leq \frac{2}{3}r$.

Thus, $|g(a) - f(a)| \leq \frac{2}{3}r \quad \forall a \in A$.

So both parts of our claim have been proven, and we are done with Step 1.

Proof (continued): Step 2: Property (1) of our proof

Reminder: (1) Any continuous map of A into $[a, b] \subseteq \mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

Without loss of generality we will replace the arbitrary closed interval $[a, b] \in \mathbb{R}$ with the interval $[-1, 1] \in \mathbb{R}$.

Let $f : X \to [-1, 1]$ be a continuous map. By step 1, we know that there must exist a $g_1 : X \to \mathbb{R}$ with $r = 1$ such that:

$$\begin{aligned} |g_1(x)| &\leq \tfrac{1}{3}(1) && \forall x \in X \\ |f(a) - g_1(a)| &\leq \tfrac{2}{3}(1) && \forall a \in A. \end{aligned}$$

Proof (continued): Step 2 (continued)

Now consider the map $f - g_1 : A \to [-\frac{2}{3}, \frac{2}{3}]$. So again by step 1, with $r = \frac{2}{3}$ now, there exists a $g_2 : X \to \mathbb{R}$ such that:

$$|g_2(x)| \leq \tfrac{1}{3}(\tfrac{2}{3}) \qquad \forall x \in X$$

$$|(f(a) - g_1(a)) - g_2(a)| \leq \tfrac{2}{3}(\tfrac{2}{3}) \quad \forall a \in A.$$

Now if repeat this process a couple times to infinity for $g_1, \ldots g_n$, we can see that $\quad |f(a) - g_1(a) - \ldots - g_n(a)| \leq (\frac{2}{3})^n \quad \forall a \in A$.

Now, get ready Dr. Harris, we process by induction:

For the $n + 1^{th}$ time, apply step 1. This time with the function g_{n+1} such that: $f - g_1 - \ldots - g_n$, with $r = (\frac{2}{3})^n$, so there exists a g_{n+1} such that:

$$|g_{n+1}(x)| \leq \tfrac{1}{3}(\tfrac{2}{3})^n \qquad \forall x \in X$$

$$|f(a) - g_1(a) - \ldots - g_n(a) - g_{n+1}(a)| \leq (\tfrac{2}{3})^{n+1} \qquad \forall a \in A$$

Proof (continued): Step 2 (continued)

Now define $g(x) := \sum_{n=1}^{\infty} g_n(x)$. Since each $g_n(x) \leq \frac{1}{3}(\frac{2}{3})^{n-1}$, we can see that

$$\sum_{n=1}^{\infty} g_n(x) \leq \sum_{n=1}^{\infty} \frac{1}{3}(\frac{2}{3})^{n-1}.$$

And $\sum_{n=1}^{\infty} \frac{1}{3}(\frac{2}{3})^{n-1}$ is a convergent geometric series. So by the comparison test, $\sum_{n=1}^{\infty} g_n(x)$ converges. Calc 2!

Now, to show that g is continuous, we will show that the sequence s_n converges to g uniformly. To do this we can see that for all $k > n$:

$$|s_k(x) - s_n(x)| = |\sum_{i=n+1}^{k} g_i(x)| \leq \frac{1}{3} \sum_{i=n+1}^{k} (\frac{2}{3})^{i-1} < \frac{1}{3} \sum_{i=n+1}^{\infty} (\frac{2}{3})^{i-1} = (\frac{2}{3})^n$$

Now letting $k \to \infty$, we can see that $|g(x) - s_n(x)| \leq (\frac{2}{3})^n$.

Therefore, by the definition of uniform continuity, s_n converges to g uniformly.

Proof (continued): Step 2 (continued)

Now we want to show that $g_{|A} = f$. To do so let $s_n(x) = \sum_{i=1}^{\infty} g_i(x)$, which is the n^{th} partial sum of the series. Then we can see that:

$$|f(a) - \sum_{i=1}^{\infty} g_i(a)| = |f(a) - s_n(a)| \leq (\frac{2}{3})^n \quad \forall a \in A$$

So, as $n \to \infty$, $|f(a) - s_n(a)| = 0$.

Therefore, $f(a) = g(a)$ for all $a \in A$.

Lastly, we want to show that g maps X into the interval $[-1, 1]$. To do so let the map $r : \mathbb{R} \to [-1, 1]$ exist such that $r \circ g$ is the map where

$$\begin{aligned} (r \circ g)(y) &= y \quad &&\text{if } |y| \leq 1 \\ (r \circ g)(y) &= \tfrac{y}{|y|} \quad &&\text{if } |y| \geq 1 \end{aligned}$$

Then $r \circ g$ would be an extension of $f : X \to [-1, 1]$, so we have finished step 2!

Proof (continued): Step 3: Property (1) of our proof

Reminder: (2) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

To prove (2) we will actually prove that it works for the open interval $(-1,1)$, we can do this since $(-1,1)$ is homeomorphic to $\mathbb{R}$. So, let $f : A \to (-1,1)$ be a continuous map. By the first two parts of the proof we know that we can extend f to the continuous map $g : X \to [-1,1]$, but this is closed, so we want to find a continuous map $h : X \to (-1,1)$.

Proof (continued): Step 3 (continued)

So, let's define $D \subseteq X$ by the equation

$$D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\}).$$

(union of the pre-images of the extrema)

Since g is continuous, D is closed in X. We know that $g(A) = f(A) \subseteq (-1, 1)$. So A is clearly disjoint from D. Thus by Urysohn's Lemma, there exists a continuous function $w : X \to [0, 1]$ such that:

$$w(D) = \{0\}$$
$$w(A) = \{1\}.$$

Then let $h(x) = w(x) \cdot g(x)$. Clearly $h(x)$ is continuous and we can see that $h_{|A} = f$ since $h(A) = w(A) \cdot g(A) = 1 \cdot g(A) = g(A) = f(A)$.

Lastly, we can verify that h maps all of X into the open interval $(-1, 1)$ using 2 cases.

If $x \in D$, then $h(x) = 0 \cdot g(x) = 0 \in (-1, 1)$.

If $x \notin D$, then $|g(x)| < 1$, and since $|h(x)| < |g(x)|$, $|h(x)| < 1$.

Therefore, $h : X \to (-1, 1)$.

Thus Step 3 in complete, and so if the whole proof.

$\blacksquare$

Consequences of the Tietze Extension Theorem

You prove Urysohn's Metrization Theorem by combining the Tietze Extension Theorem and Urysohn's Lemma.

Bibliography

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