4-Output Games and 3-Colorings

Calder Evans and Taylor Smith

Northern Arizona University
Department of Mathematics and Statistics

Undergraduate Symposium
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NAU NORTHERN ARIZONA UNIVERSITY
Department of Mathematics and Statistics

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1. Share our research goals

Presentation Outline

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- built a graph based on the game algebra of the Mermin-Peres Magic Square Game
  - We chose the Mermin-Peres game because it is the smallest synchronous game that has a winning quantum strategy but no classical strategy
- encoded the graph into SageMath and ran a multitude of constructions to identify which vertices could be removed without changing the colorabilities of the graph

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  - i.e. $P_A(a|x) = \sum_{b \in \mathcal{O}} P(a, b|x, y)$ is independent of y, and $P_B(b|y) = \sum_{a \in \mathcal{O}} P(a, b|x, y)$ is independent of x.

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  - i.e. If they give different answers they lose

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Definition

The Mermin-Peres Magic Square Game is a synchronous non-local game. In this game, Alice and Bob will each receive one equation from the following list:

$$x_1 + x_2 + x_3 \equiv 0 \pmod 2$$

$$x_4 + x_5 + x_6 \equiv 0 \pmod 2$$

$$x_7 + x_8 + x_9 \equiv 0 \pmod 2$$

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Alice and Bob fill out their assigned equations with 0's and 1's simultaneously without knowing how the other fills theirs out. Their goal is to satisfy each of their own equations while producing the same value for any shared variable.

Magic Square Representation

The Magic Square game can be represented by a 3x3 grid by labeling each cell like so:

x_1	x_2	x_3	$x_1 + x_2 + x_3 = 0$
x_4	x_5	x_6	$x_4 + x_5 + x_6 = 0$
x_7	x_8	x_9	$x_7 + x_8 + x_9 = 0$
$x_1 + x_4 + x_7 = 0$	$x_2 + x_5 + x_8 = 0$	$x_3 + x_6 + x_9 = 1$	

Magic Square Walkthrough

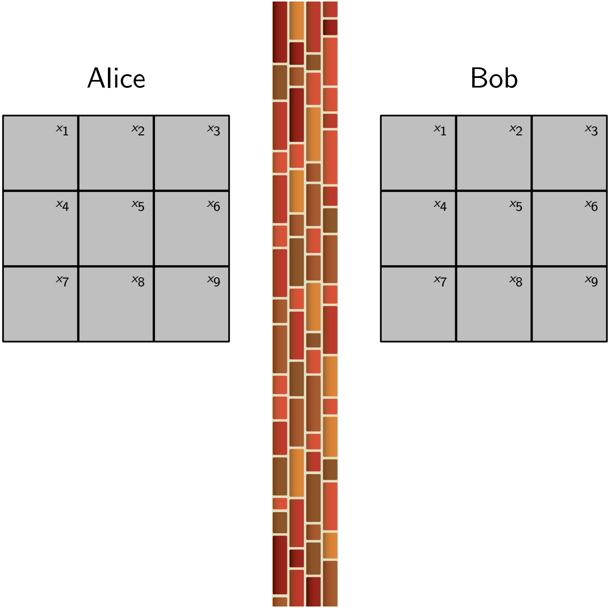
Alice

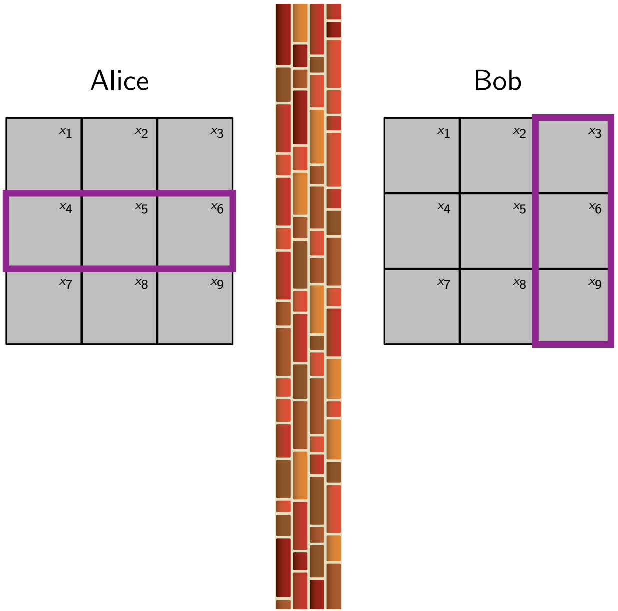
x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

Bob

x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

Magic Square Walkthrough




Alice
x_1
x_2
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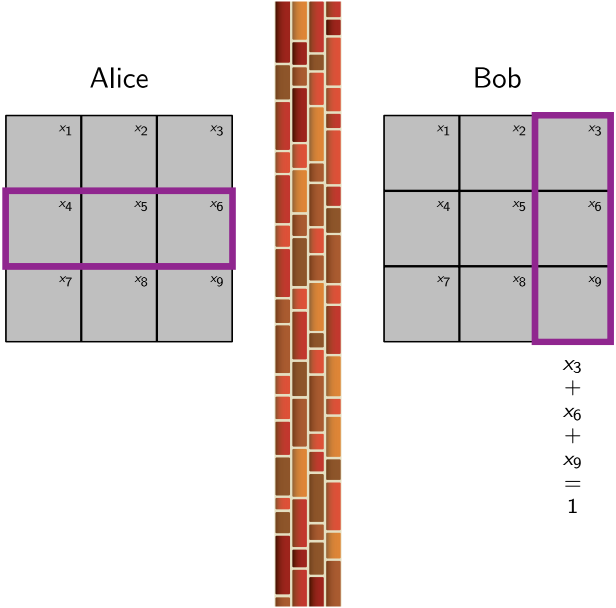
$0 = x_4 + x_5 + x_6$



$$x_3 + x_6 + x_9 = 1$$

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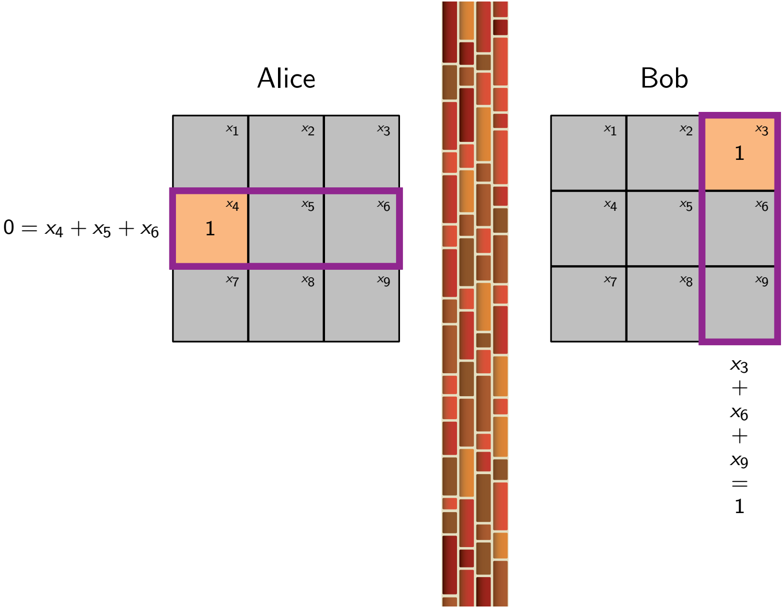

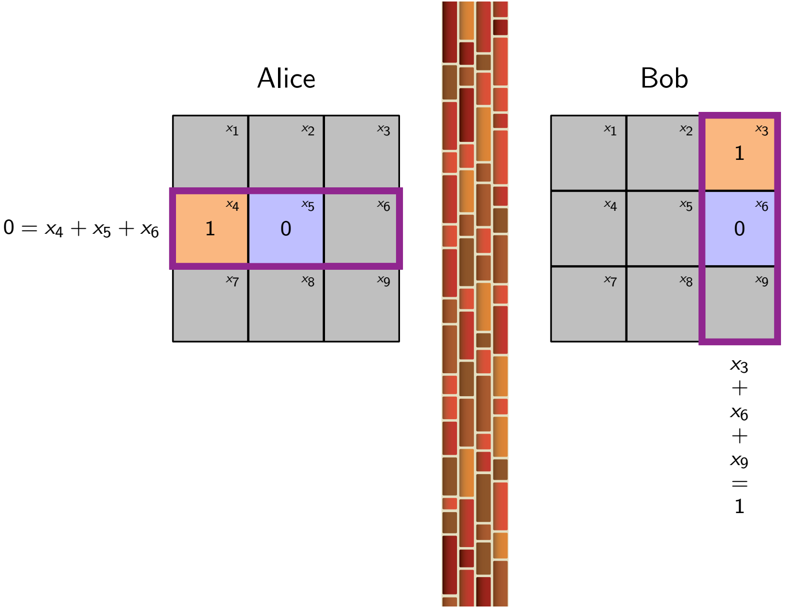
Alice
Bob
x_1 x_2 x_3
x_4 x_5 x_6
x_7 x_8 x_9
1
0
$0 = x_4 + x_5 + x_6$
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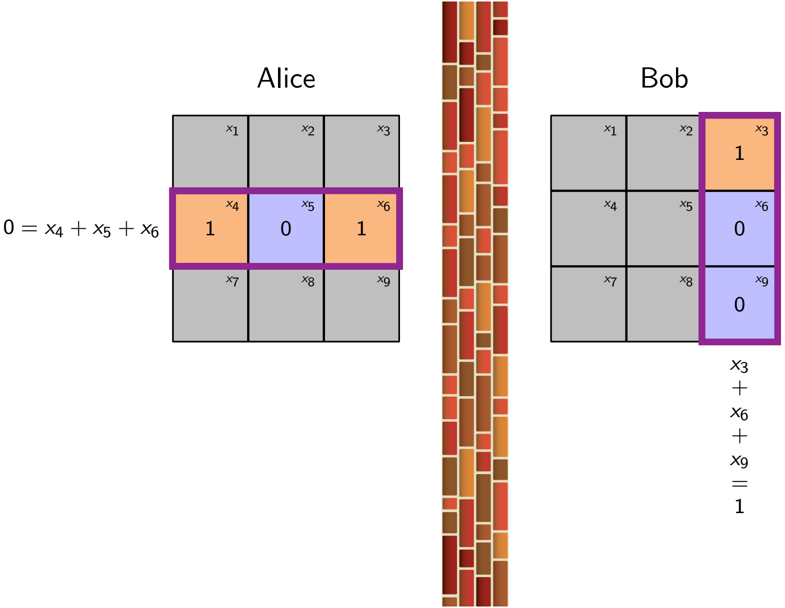
Magic Square Walkthrough



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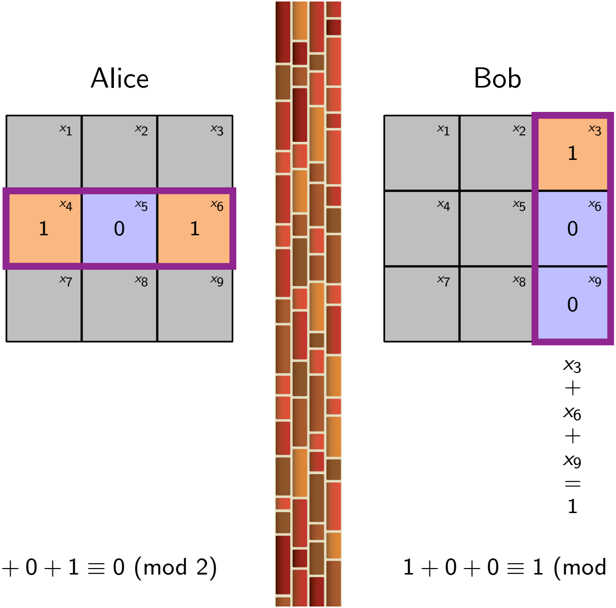

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$1 + 0 + 1 \equiv 0 \pmod 2$

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Alice

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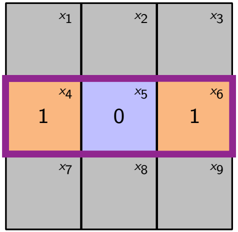

✓

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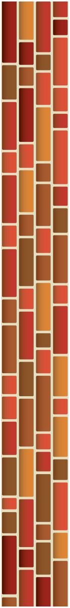
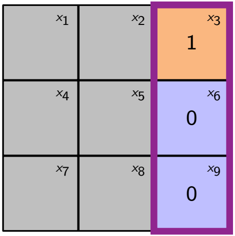


Bob



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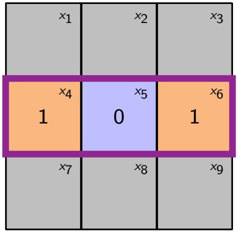

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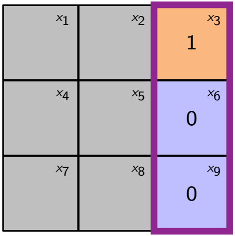

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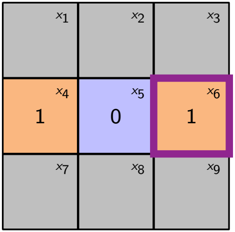

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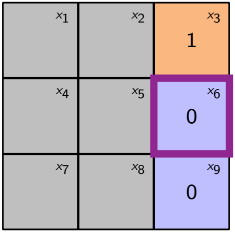

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Magic Square Walkthrough

Alice | Bob

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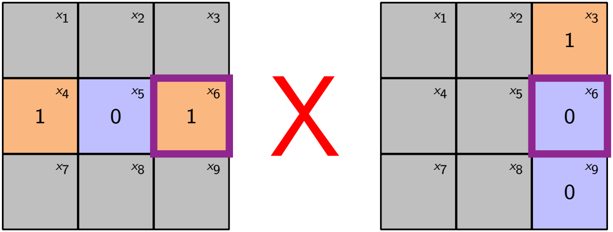

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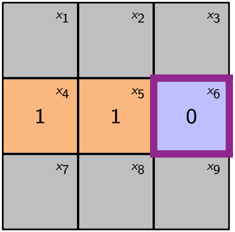

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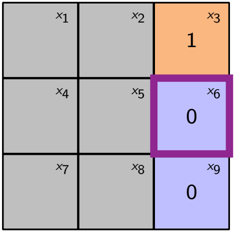

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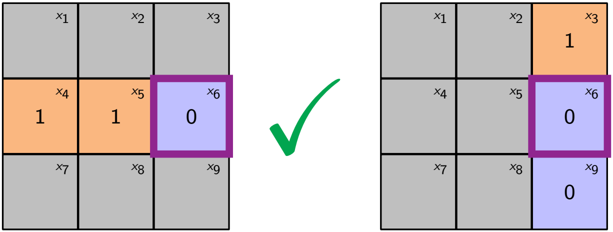

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Classical Strategy for Magic Square Game Strategy

Definition

A classical strategy for a Non-Local Game is represented by a function $f : \mathcal{I} \to \mathcal{O}$.

For the Magic Square Game, we can define a classical strategy using the grid below. The best strategy has them agree upon a set table of answers, this allows them to win most of the time. It is impossible to create a solution set that satisfies all nine equations at the same time, otherwise they would be able to win every single game.

x_1: 1	x_2: 0	x_3: 1	$1+1=0$
x_4: 1	x_5: 1	x_6: 0	$1+1=0$
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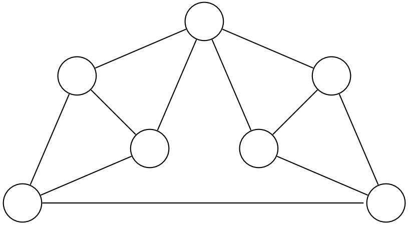

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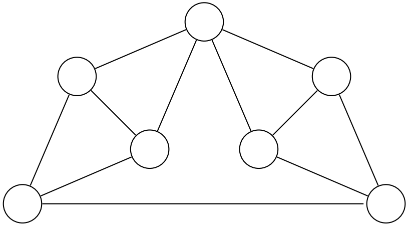

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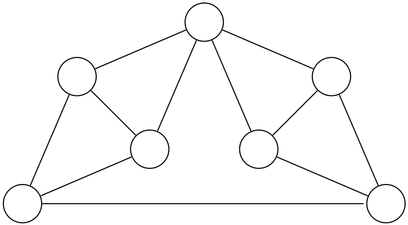

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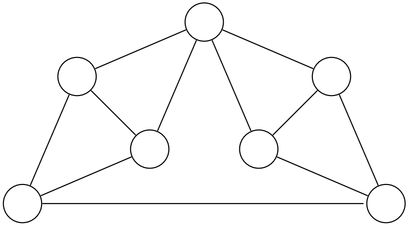

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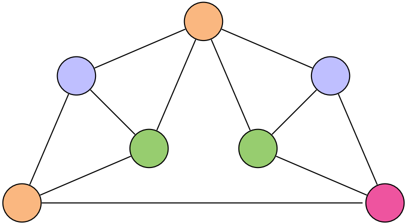
- To color a graph, we assign colors to each vertex of a graph such that no adjacent vertices share the same color.
- We say that a graph is k-colorable if the minimum amount of colors needed to color the graph is at most k.



Game Algebra

- We can represent the rules of the Mermin-Peres Magic Square Game using an algebra $\mathcal{A}(\mathcal{G})$. We use $a \in \mathcal{O}$ and $x \in \mathcal{I}$ to index our elements. $\mathcal{A}(\mathcal{G})$ has elements that satisfy the following.

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$$e_{1,2} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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- For all $x \in \mathcal{I}$ and $a \in \mathcal{O}$ we have $e_{a,x}^2 = e_{a,x} = e_{a,x}^*$

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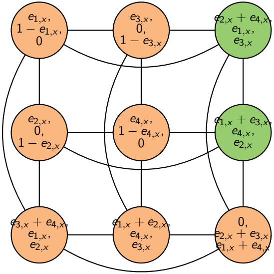
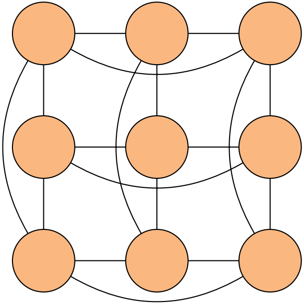
Example

$$(e_{2,x} + e_{4,x}, e_{1,x}, e_{3,x}) \cdot (e_{1,x} + e_{3,x}, e_{4,x}, e_{2,x})$$
$$= ((e_{2,x} + e_{4,x})(e_{3,x} + e_{1,x}), e_{1,x}e_{4,x}, e_{3,x}e_{2,x})$$
$$= (0, 0, 0).$$



Definitions

- Gadgets are subgraphs that encode the game rules into our graph.



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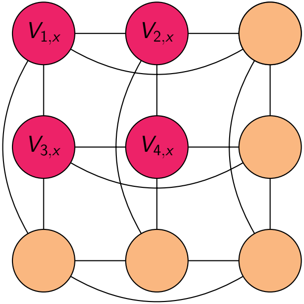
- The vertices in red we denote as $V_{a,x}$, with $a \in \mathcal{O}$ and $x \in \mathcal{I}$. These sets are the same as our input and output sets from the Mermin-Peres Magic Square Game.



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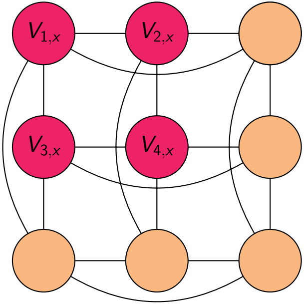

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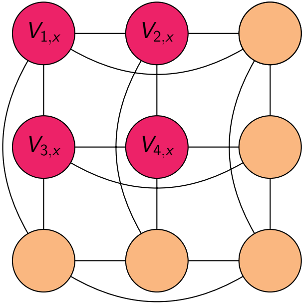
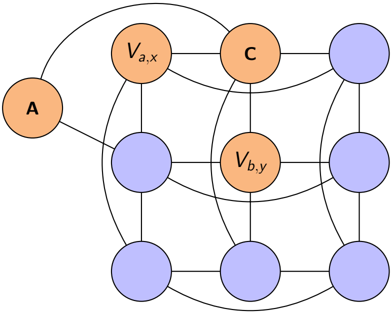
- We call $V_{a,x}$ and $V_{b,y}$ incompatible whenever $\lambda(a, b, x, y) = 0$.

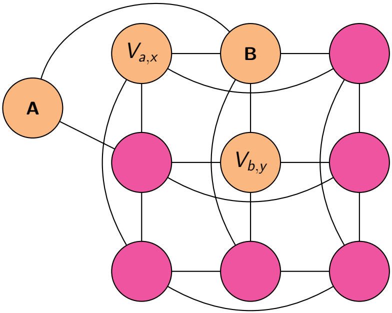


For vertices $V_{a,x}$ and $V_{b,y}$ who are incompatible, $\lambda(a, b, x, y) = 0$, and are not orthogonal to one another we create one of the subgraphs,

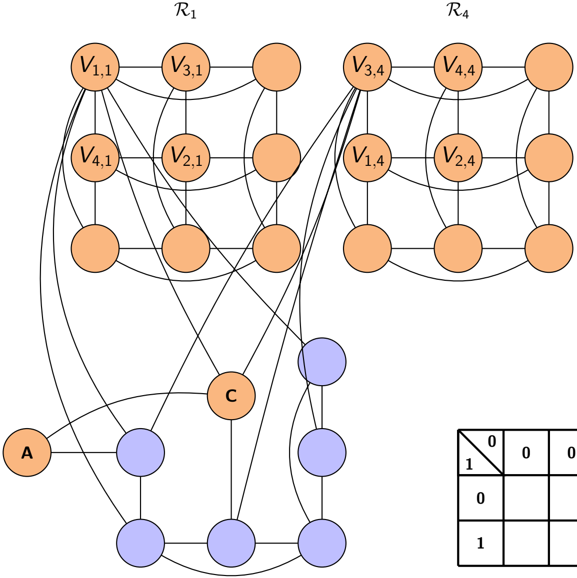
if $(a, b, x, y) \in \mathcal{E}_\lambda$



if $(a, b, x, y) \in \mathcal{F}_\lambda$

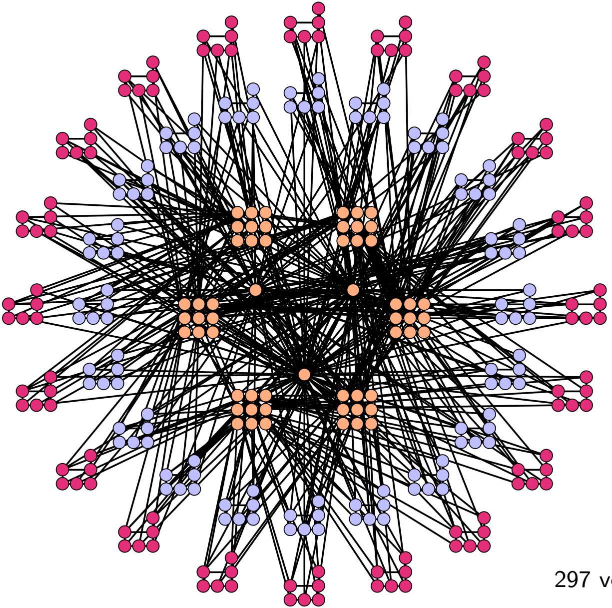


$\mathbf{E}_\lambda$ *Subgraphs*



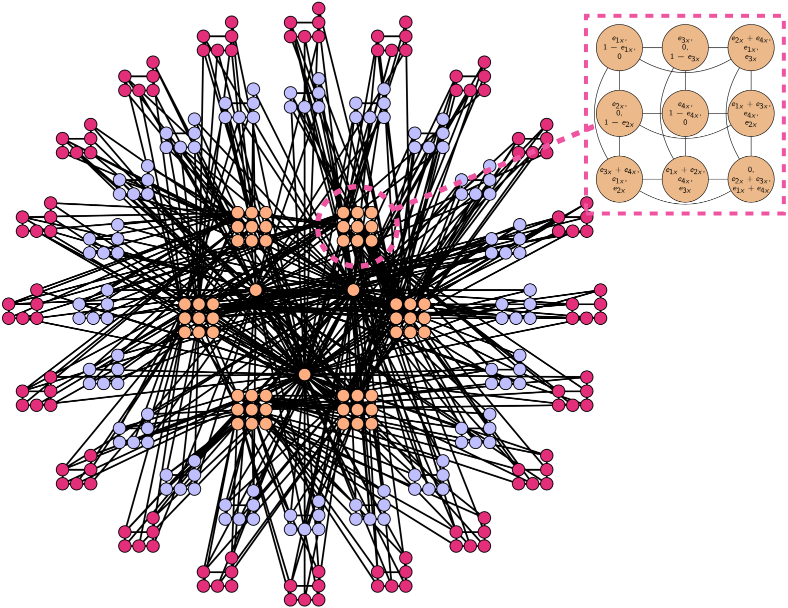
1 \ 0	0	0
0		
1		

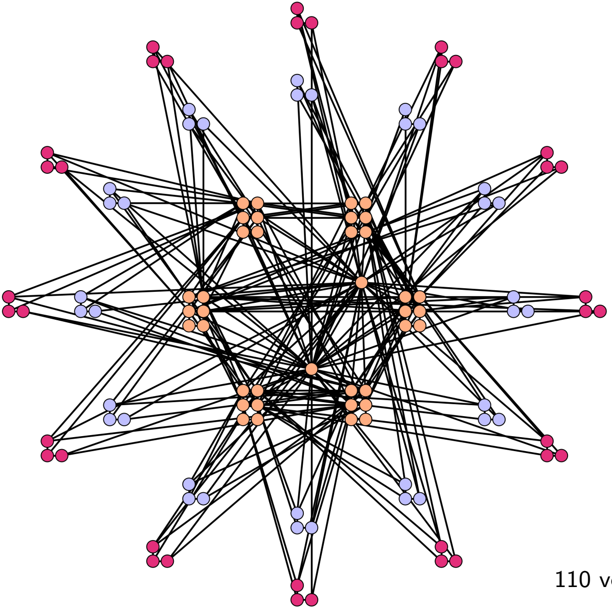
Mermin-Peres Graph



297 vertices

Mermin-Peres Graph





110 vertices

Thank you!

Questions?