Meromorphic Functions on an Open Set

Calder Evans, Joshua Martey

MAT 535: Complex Analysis
Final Presentation

4 December 2025

NORTHERN ARIZONA UNIVERSITY
Department of Mathematics and Statistics

Before we get started lets review some definitions

Before we get started lets review some definitions

Definition 1

Let $G \subseteq \mathbb{C}$ and $f : G \to \mathbb{C}$ be a function. We say that f is **holomorphic** at a point $a \in int(G)$ if $\exists\ r > 0$ such that $D(a, r) \subseteq G$ and f is differentiable at each $x \in D(a, r)$. We say that f is holomorphic on G if it is holomorphic at each point in G.

Definition 2

Let $f : G \to \mathbb{C}$ be a holomorphic function on the open set G. We say that f has an **isolated singularity** at a point $a \notin G$ if f is holomorphic on $D_p(a, r)$ for some $r > 0$, but not defined at a.

Definition 2

Let $f : G \to \mathbb{C}$ be a holomorphic function on the open set G. We say that f has an **isolated singularity** at a point $a \notin G$ if f is holomorphic on $D_p(a, r)$ for some $r > 0$, but not defined at a.

Definition 3

Let $f : G \to \mathbb{C}$ be holomorphic, where G is open, and suppose that $a \notin G$ is an isolated singularity for f. We say that f has a **pole** at $z = a$ if $\lim_{z \to a} |f(z)| = \infty$.

Definition 4

If G is an open set in $\mathbb{C}$ and (Ω, d) is a complete metric space, we call the set of all continuous functions from G to Ω, $\mathbf{C(G,\Omega)}$. The set is always non-empty since it always contains at least the constant functions.

Definition 5

The family $\mathscr{F}$ is normal in G if every subsequence $(f_n) \subset \mathscr{F}$ has a subsequence that converges that converges locally uniformly on G.

Equivalently:

$\mathscr{F}$ is normal $\iff$ every sequence has a locally uniformly convergent subseq

Why do we need Meromorphic functions when we already have Holomorphic functions?

- Holomorphic functions are perfectly smooth.
- But real life mathematical functions often have several places where they blow up.
- Meromorphic functions capture this idea: smooth everywhere except at isolated, predictable singularities that we call poles.

Definition 6

If G is open and f is a defined holomorphic function in G except for poles, then f is a **meromorphic function** on G.

A meromorphic function in the complex plane that is either holomorphic at infinity or has a pole at infinity is said to be **meromorphic in the extended complex plane**.

We will denote the set of all meromorphic functions on G as $M(G)$.

- **The Tangent Function**:
$$\tan z = \frac{\sin z}{\cos z}$$
Meromorphic on $\mathbb{C}$, poles wherever $\cos z = 0$.

- **Rational Functions**:
$$f(z) = \frac{P(z)}{Q(z)}$$
These are meromorphic functions on $\mathbb{C}$; poles occur at zeros of Q

- $\mathbf{\frac{1}{z}}$:
Meromorphic on $\mathbb{C}\backslash\{0\}$. Has a pole at 0

Some Steps to Show a Function is Meromorphic

- Specify the domain (open).
- Show f is holomorphic except at isolated points.
- Check each singularity: prove it is a pole.
- Use a Laurent series or express $f = g/h$, where g and f are holomorphic.
- Use closure properties to simplify the argument.

If G is a region and f is a meromorphic function on G, and if $f(z) = \infty$ whenever z is a pole of f, then $f : G \to \mathbb{C}_\infty$ is a continuous function. We can see that $M(G)$ is a subset of the set of all continuous functions from G to $\mathbb{C}_\infty$ designated by, so it also has the same metric as $C(G, \mathbb{C}_\infty)$.

If G is a region and f is a meromorphic function on G, and if $f(z) = \infty$ whenever z is a pole of f, then $f : G \to \mathbb{C}_\infty$ is a continuous function. We can see that $M(G)$ is a subset of the set of all continuous functions from G to $\mathbb{C}_\infty$ designated by, so it also has the same metric as $C(G, \mathbb{C}_\infty)$. This metric being:

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{[(1 + |z_1|^2) + (1 + |z_2|)^2]^{\frac{1}{2}}}$$

for z_1 and z_2, and

$$d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}$$

for $z \in \mathbb{C}$.

If G is a region and f is a meromorphic function on G, and if $f(z) = \infty$ whenever z is a pole of f, then $f : G \to \mathbb{C}_\infty$ is a continuous function. We can see that $M(G)$ is a subset of the set of all continuous functions from G to $\mathbb{C}_\infty$ designated by, so it also has the same metric as $C(G, \mathbb{C}_\infty)$. This metric being:

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{[(1 + |z_1|^2) + (1 + |z_2|)^2]^{\frac{1}{2}}}$$

for z_1 and z_2, and

$$d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}$$

for $z \in \mathbb{C}$. An interesting consequence of this space and its metric is that

If G is a region and f is a meromorphic function on G, and if $f(z) = \infty$ whenever z is a pole of f, then $f : G \to \mathbb{C}_\infty$ is a continuous function. We can see that $M(G)$ is a subset of the set of all continuous functions from G to $\mathbb{C}_\infty$ designated by, so it also has the same metric as $C(G, \mathbb{C}_\infty)$. This metric being:

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{[(1 + |z_1|^2) + (1 + |z_2|)^2]^{\frac{1}{2}}}$$

for z_1 and z_2, and

$$d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}$$

for $z \in \mathbb{C}$. An interesting consequence of this space and its metric is that

$$d(z_1, z_2) = d\left(\frac{1}{z_1}, \frac{1}{z_2}\right) \text{ and } d(z, 0) = d\left(\frac{1}{z}, \infty\right)$$

These will be very useful when working with meromorphic functions.

The metric on $M(G)$ is

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$$

where $G = \bigcup_{n=1}^{\infty} K_n$, each K_n is compact, $K_n \subseteq \operatorname{int}(K_{n+1})$, and

$$\rho_n(f,g) = \sup\{d(f(z),g(z)) | z \in K_n\}$$

The above metric makes $C(G, \mathbb{C}_\infty)$ a complete metric space and it induces locally uniform convergence.

Proposition 7

- *If* $a \in \mathbb{C}$ *and* $r > 0$ *then* $\exists\ \rho > 0$ *such that* $D_\infty(a;\rho) \subseteq D(a;r)$.
- *Conversely, if* $\rho > 0$ *and* $a \in \mathbb{C}$ *then* $\exists\ r > 0$ *such that* $D(a;r) \subseteq D_\infty(a;\rho)$.
- *If* $\rho > 0$ *then there is a compact set* $K \subset \mathbb{C}$ *such that* $\mathbb{C}_\infty - K \subseteq D_\infty(\infty;\rho)$.
- *Conversely, if a is a compact set* $K \subseteq \mathbb{C}$, $\exists\ \rho > 0$ *such that* $D_\infty(\infty;\rho) \subseteq \mathbb{C}_\infty - K$.

Corollary 8

$M(G) \cup \{\infty\}$ is a complete metric space.

Theorem 9

The only meromorphic functions in the extended complex plane ($\mathbb{C}_\infty$) are the rational functions.

Corollary 8

$M(G) \cup \{\infty\}$ is a complete metric space.

Theorem 9

The only meromorphic functions in the extended complex plane ($\mathbb{C}_\infty$) are the rational functions.

The proof is left up to Casey.

Theorem 10

Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is holomorphic then either f is holomorphic or $f \equiv \infty$.

Proof.

- We have meromorphic functions f_n on G converging to f in the spherical metric.
- Goal: show that f is either meromorphic or identically ∞.

Case 1: $f(a) \neq \infty$ for some point a

- Convergence in the spherical metric implies local equicontinuity near a.
- For all large n, the values $f_n(a)$ stay close to the finite number $f(a)$.
- Thus the tail of the sequence has no poles near a.
- Hence f_n is analytic near a for all sufficiently large n.
- Uniform convergence then implies that f is analytic near a.
- Since this holds at every finite point of f, the function f is meromorphic on G.

□

Case 2: $f(a) = \infty$ for some point a

- Consider the reciprocals $1/f_n$.
- Each $1/f_n$ is meromorphic and converges to $1/f$ in the spherical metric.
- Apply the same argument as in Case 1 to the sequence $1/f_n$.
- Either $1/f$ is analytic near a (so f has a pole at a), or $1/f \equiv 0$ (so $f \equiv \infty$).

Conclusion

- Therefore f is either meromorphic or identically ∞.
- If each f_n is analytic:
  - Case 1 implies f is analytic.
  - Case 2 can only give $f \equiv \infty$ (analytic limits cannot develop poles).
- Hence f is analytic or identically ∞ in this case.

Another interesting thing we can discuss is the normality in $M(G)$. To do so we have to look at the quantity $\frac{2|f'(z)|}{1+|f(z)|^2}$ (derivative of the Riemann Sphere) for each $f \in M(G)$ and what happens if z is a pole of f.

If z is a pole, then $f'(z)$ is meaningless, so we want to try to look at $\lim\limits_{z \to a}$ (where a is our pole). To show that the limits exist, let a have order $m \geq 1$ and let

$$f(z) = g(z) + \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)}$$

where g is holomorphic in $D(a, r)$ and $z \in D(a, r)$. Then

$$f'(z) = g'(z) - \left[\frac{mA_m}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2}\right]$$

Thus,

Thus, obviously,

Thus, obviously,

$$\frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2\left|g'(z) - \frac{mA_m}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2}\right|}{1+\left|g(z) + \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)}\right|^2}$$
$$= \frac{2|z-a|^{m-1} \cdot |mA_m + \cdots + A_1(z-a)^{m-1} - g'(z)(z-a)^{m+1}|}{|z-a|^{2m} + |A_m + \cdots + A_1(z-a)^{m-1} + g(z)(z-a)^m|^2}$$

So, if $m \geq 2$, $\lim\limits_{z\to a} \frac{2|f'(z)|}{1+|f(z)|^2} = 0$ and if $m = 1$, $\lim\limits_{z\to a} \frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2}{A_1}$

Thus, obviously,

$$\frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2\left|g'(z) - \frac{mA_m}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2}\right|}{1+\left|g(z) + \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)}\right|^2}$$

$$= \frac{2|z-a|^{m-1} \cdot |mA_m + \cdots + A_1(z-a)^{m-1} - g'(z)(z-a)^{m+1}|}{|z-a|^{2m} + |A_m + \cdots + A_1(z-a)^{m-1} + g(z)(z-a)^m|^2}$$

So, if $m \geq 2$, $\lim\limits_{z \to a} \frac{2|f'(z)|}{1+|f(z)|^2} = 0$ and if $m = 1$, $\lim\limits_{z \to a} \frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2}{A_1}$

This fact will tell us something really interesting when combined with an upcoming theorem.

Definition 11

If f is a meromorphic function on the region G then define $\mu(f) : G \to \mathbb{R}$ by

$$\mu(f)(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}$$

whenever z is not the pole of f, and

$$\mu(f)(a) = \lim_{z \to a} \frac{2|f'(z)}{1 + |f(z)|^2}$$

if a is a pole of f. It follows that $\mu(f) \in C(G, \mathbb{C})$.

Theorem 12

A family $\mathscr{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathscr{F}) \equiv \{\mu(f) : f \in \mathscr{F}\}$ is locally bounded.

Theorem 12
A family $\mathscr{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if $\mu(\mathscr{F}) \equiv \{\mu(f) : f \in \mathscr{F}\}$ is locally bounded.

Since we just showed that $M(G)$ is locally bounded when poles have an order $m \geq 1$, we know that poles do not prevent normality.

$$H(G) \subset M(G) \subset C(G, C_\infty)$$

- $H(G)$
  This is the space of holomorphic functions on G. A function $f \in H(G)$ is complex-differentiable at every point of G (no singularities).
- $M(G)$
  This is the space of meromorphic functions on G. A function $f \in M(G)$ is holomorphic on G except at isolated poles.
  Since every holomorphic function is automatically meromorphic (with no poles), we have:
  $$H(G) \subset M(G).$$
- $C(G, C_\infty)$
  This is the set of continuous functions from G into the extended complex plane:
  $$C_\infty = \mathbb{C} \cup \{\infty\}.$$
  The value ∞ is allowed because meromorphic functions may take the value ∞ at their poles (in the Riemann sphere sense).

Example 13

If you have a bijective meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we know that it can be expressed as a rational function.

Example 13

If you have a bijective meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we know that it can be expressed as a rational function. Since every pole occurs when $f = \infty$ and we are bijective, we can only have 1 pole. Therefore, the order has to be 1 (simple pole).

Example 13

If you have a bijective meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we know that it can be expressed as a rational function. Since every pole occurs when $f = \infty$ and we are bijective, we can only have 1 pole. Therefore, the order has to be 1 (simple pole). Combining this with the knowledge that our function is rational, means both our numerator and denominator need to 1.

Example 13

If you have a bijective meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we know that it can be expressed as a rational function. Since every pole occurs when $f = \infty$ and we are bijective, we can only have 1 pole. Therefore, the order has to be 1 (simple pole). Combining this with the knowledge that our function is rational, means both our numerator and denominator need to 1. Meaning our function will look something like $f(z) = \frac{az+b}{cz+d}$,

Example 13

If you have a bijective meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we know that it can be expressed as a rational function. Since every pole occurs when $f = \infty$ and we are bijective, we can only have 1 pole. Therefore, the order has to be 1 (simple pole). Combining this with the knowledge that our function is rational, means both our numerator and denominator need to 1. Meaning our function will look something like $f(z) = \frac{az+b}{cz+d}$, which wouldn't you know it is the definition of a Möbius Map.

Example 13

If you have a bijective meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we know that it can be expressed as a rational function. Since every pole occurs when $f = \infty$ and we are bijective, we can only have 1 pole. Therefore, the order has to be 1 (simple pole). Combining this with the knowledge that our function is rational, means both our numerator and denominator need to 1. Meaning our function will look something like $f(z) = \frac{az+b}{cz+d}$, which wouldn't you know it is the definition of a Möbius Map.

(d) **(5 points)** Assume that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a homeomorphism (i.e. f is bijective, continuous and f^{-1} is continuous). Suppose that $z_0 \in \mathbb{C}_\infty$ satisfies $f(z_0) = \infty$, and assume that f is holomorphic on $\mathbb{C} \setminus \{z_0\}$. Prove that f is a Möbius map. (**Hint:** The case when $z_0 = \infty$ follows from (b), so assume $z_0 \in \mathbb{C}$. Show that f has a simple pole at z_0 and that $g(z) = f(1/z)$ has a removable singularity at $z = 0$.)

[1] John B. Conway, *Functions of One Complex Variable*, 2nd ed.. Springer- Verlag, New York, New York, 1978. Chapter 7.3.

[2] Elias M. Stein & Rami Shakarchi, *Complex Analysis.* Princeton University Press, Princeton, New Jersey, 2003. Chapter 3.3.

[3] Bob Gardner, *Spaces of Meromorphic Functions*, Online Notes, 13 September 2025. Chapter 7.3. Available at `https://faculty.etsu.edu/gardnerr/5510/notes/VII-3.pdf`

[4] Samuel J. Harris, *MAT 535 Part 8: Isolated Singularities; Residue Theorem*, Class Notes, November 2025.

Questions?