4-Output Games and 3-Colorings

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Abstract

Non-local games involve two players, Alice and Bob, who attempt to give correct answers to individual questions assigned to them by a referee. The players cannot communicate but can agree on a strategy beforehand. Whether they win or lose depends on the 4-tuple of questions and answers involving the players. We exhibit a new transformation of 4-output synchronous games to 3-coloring games. This transformation gives an improvement on a transformation of the faculty mentor in the 4-output setting. One can apply this transformation to games where the players cannot win with classical strategies, but can win 100% of the time using quantum strategies (i.e. with the resources of quantum mechanics). Applying this transformation to the Mermin-Peres magic square game yields a graph on at most 110 vertices that is quantum 3-colorable, but not classically 3-colorable, a significant reduction on previous work of the faculty mentor.

Definitions

Game

A game $\mathcal{G}$ is played by Alice and Bob. Alice and Bob have an output set $\mathcal{O}$ used to respond to questions the referee gives from the input set $\mathcal{I}$.

Winning the game is defined by a rule function $\lambda : \mathcal{O} \times \mathcal{O} \times \mathcal{I} \times \mathcal{I} \to \{0, 1\}$.

- If $\lambda(a, b, x, y) = 0$, then Alice and Bob lose.
- If $\lambda(a, b, x, y) = 1$, then Alice and Bob win.

Non-Local Game

We say a game $\mathcal{G}$ is non-local if Alice and Bob cannot communicate while playing. However, Alice and Bob can prepare a strategy before hand to maximize their chance of winning.

A non-local game is synchronous if whenever Alice and Bob are given the same question $x \in \mathcal{I}$ then, $\lambda(a, b, x, x) = 0$ if $a \neq b$.

Mermin-Peres Magic Square

The Mermin-Peres magic square game is a synchronous non-local game where the questions from $\mathcal{I}$ correspond to equations evaluated modulo 2. The answers in $\mathcal{O}$ correspond to valid solutions for each equation. Alice and Bob each receive one of six equations from a referee. Their goal is to assign 0's and 1's to satisfy equations and agree on shared variables.

Below are the six possible equations:

$$x_1 + x_2 + x_3 = 0$$
$$x_4 + x_5 + x_6 = 0$$
$$x_7 + x_8 + x_9 = 0$$
$$x_1 + x_4 + x_7 = 0$$
$$x_2 + x_5 + x_8 = 0$$
$$x_3 + x_6 + x_9 = 1$$

The game can be represented as a 3×3 grid, with each row and column representing an equation. To the right is a classical strategy: Alice and Bob pre-agree on variable values. But no assignment simultaneously satisfies all six equations.

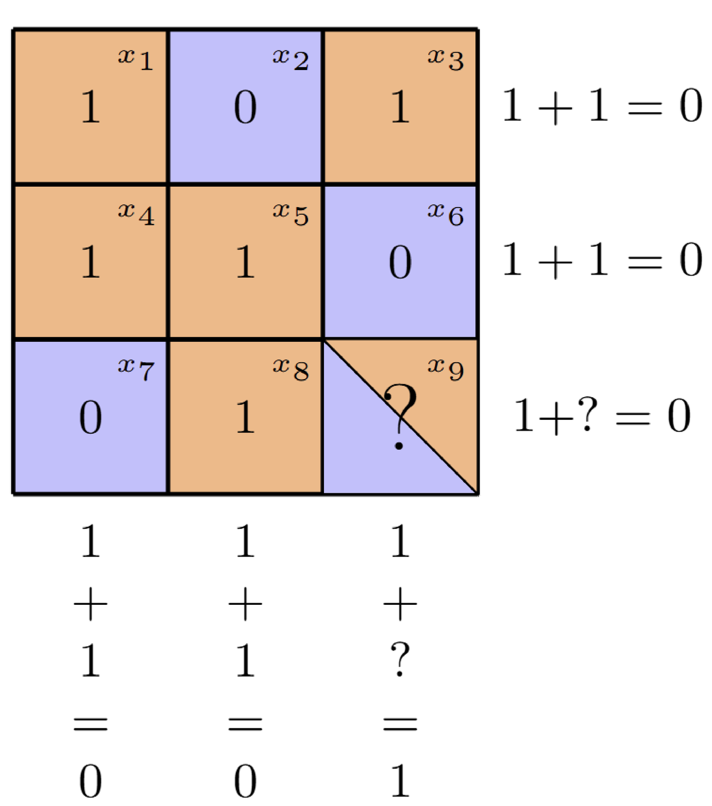

Figure: a classical strategy for Mermin-Peres

Winning the Mermin-Peres Magic Square Game

- The Mermin-Peres Magic Square game is the smallest synchronous non-local game with a winning quantum strategy and no winning classical strategy.
- It is known that there is a quantum solution to the Mermin-Peres magic square, but no classical solution. We use this to construct a graph that is quantum 3-colorable, but not classically 3-colorable.

Quantum Solutions

A quantum solution to Mermin-Peres is given by matrices $e_{a,x} \in M_d(\mathbb{C})$, for $a \in \mathcal{O}$ and $x \in \mathcal{I}$, satisfying:

- $e_{a,x}^2 = e_{a,x} = e_{a,x}^*$ for all $a \in \mathcal{O}$ and $x \in \mathcal{I}$,
- $\sum_{a \in \mathcal{O}} e_{a,x} = I_d$ for all $x \in \mathcal{I}$,
- Whenever $\lambda(a, b, x, y) = 0$ this implies $e_{a,x} \cdot e_{b,y} = 0$.

Matrices with these relations, called projections, allow us to create a quantum coloring for our graph representation.

Quantum 3-Coloring

A quantum 3-coloring of a graph is made when each vertex is assigned a 3-tuple of projections $p_1, p_2, p_3 \in \mathbf{M}_d(\mathbb{C})$ with their sum equal to 1; these come from our quantum solutions. These matrices are assigned in such a way that when two vertices $v_1, v_2 \in V(G)$ are adjacent we have that their assigned 3-tuples have orthogonal matrices element by element.

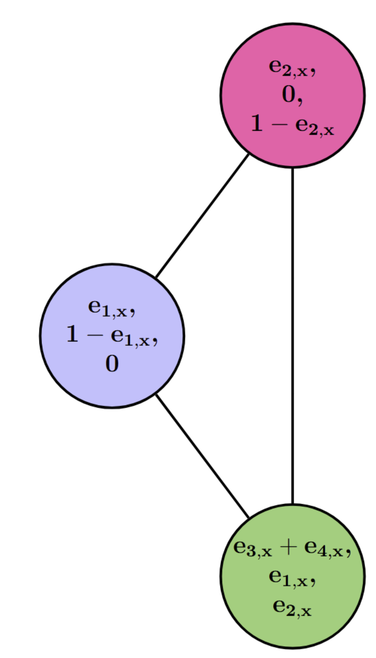

Figure: quantum colored 3-cycle

Example

$$(e_{2,x}, 0, 1 - e_{2,x}) \cdot (e_{3,x} + e_{4,x}, e_{1,x}, e_{2,x}) = (e_{2,x} \cdot (e_{3,x} + e_{4,x}), 0 \cdot e_{1,x}, (1 - e_{2,x}) \cdot e_{2,x}) = (0, 0, 0)$$

R_x Gadgets

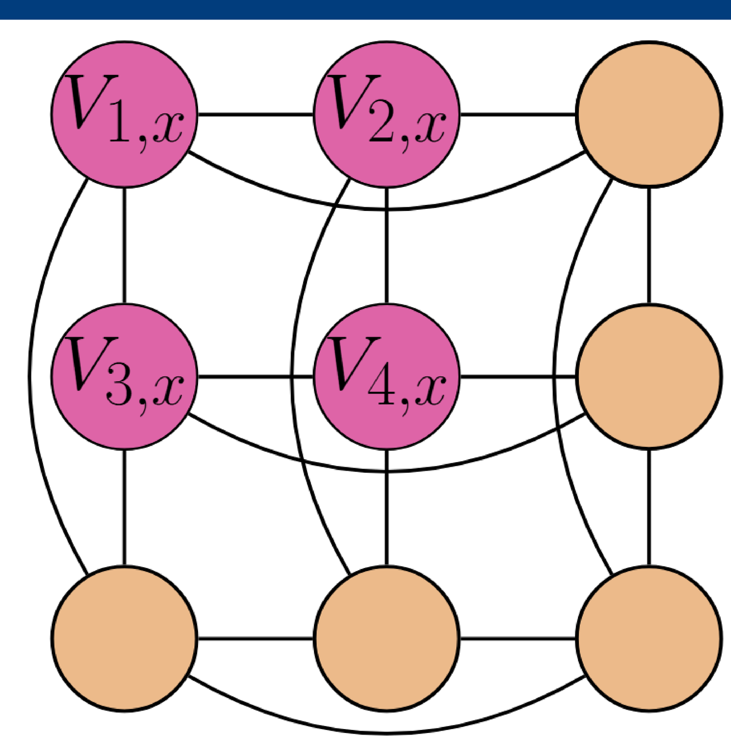

Figure: R_x Gadget with highlighted V vertices

Gadgets are subgraphs that we use to construct the graph representation of the Mermin-Peres game. We have three types of gadgets, R_x, E_λ, and F_λ. For each $x \in \mathcal{I}$ we create a gadget R_x, the answers associated with the question x are represented by $V_{a,x}$ with $a \in \mathcal{O}$. These vertices represent the question and answer pairs in the Mermin-Peres game.

We call $V_{a,x}$ in R_x and $V_{b,y}$ in R_y incompatible whenever $\lambda(a, b, x, y) = 0$.

E_λ and F_λ Gadgets

Whenever $\lambda(a, b, x, y) = 0$ we say that the question answer pairs (a, x) and (b, y) are incompatible. Depending on their colorings they either end up being adjacent if their projections are orthogonal, or we create an E_λ subgraph or an F_λ subgraph as illustrated on the right.

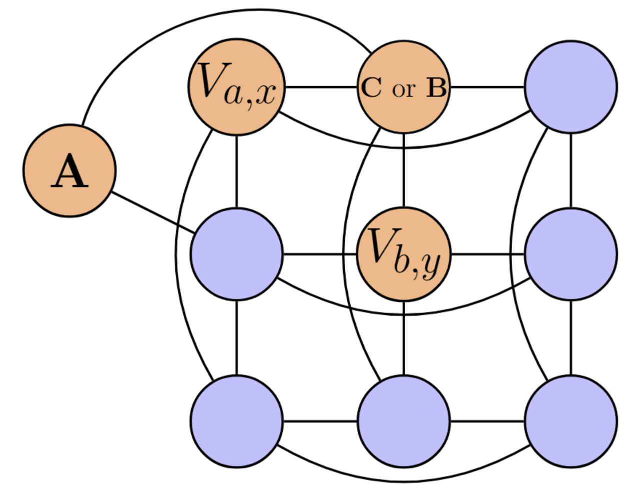

Figure: example of an E_λ or F_λ gadget

Mermin-Peres Graph Representation

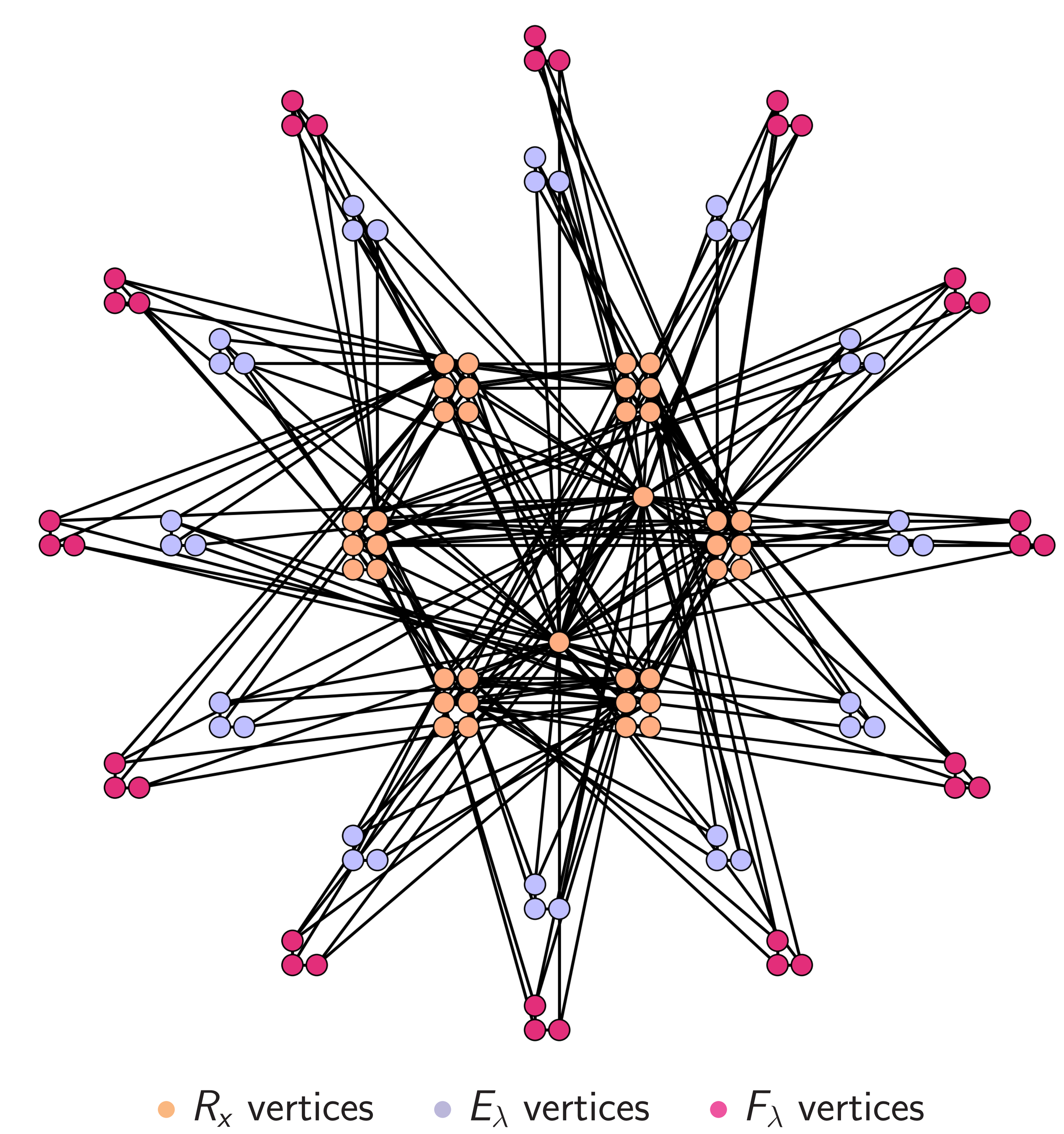

Figure: a 110 vertex graph that is quantum 3-colorable but not classically 3-colorable

Future Work

- We hope to continue trying different constructions to find smaller graph representations.
- These transformations could be applied to other games to see different relationships between quantum and classical colorings.

References

- S. J. Harris, *Universality of graph homomorphism games and the quantum coloring problem*, Annales Henri Poincaré, **25** (2024), 4321–4356.

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