An Overview of Graceful Labelings

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FAMUS

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Department of Mathematics and Statistics

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Presentation Outline

In this presentation we will:

1. Define graceful labelings.
2. Present known results on graceful labelings.
3. Look over past research from an NAU student.
4. Introduce the Ringel-Kotzig Conjecture and state recent progress.

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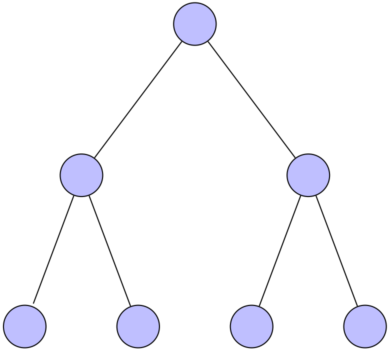
- *each edge label is the absolute difference of its vertex endpoint labels,*
- *vertex and edge labelings are unique.*

Vertex Labels: $\{0, 1, 2, 3, 4, 5, 6\}$
Edge Labels: $\{1, 2, 3, 4, 5, 6\}$



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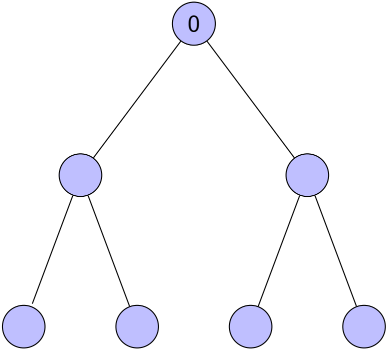

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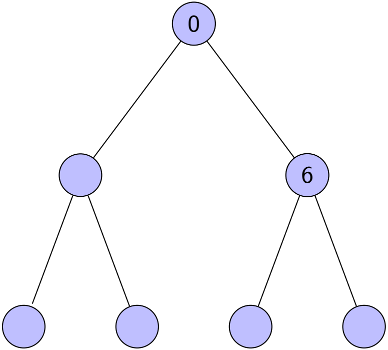

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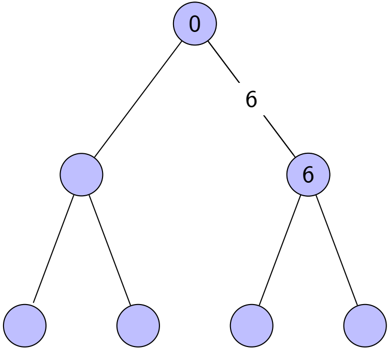

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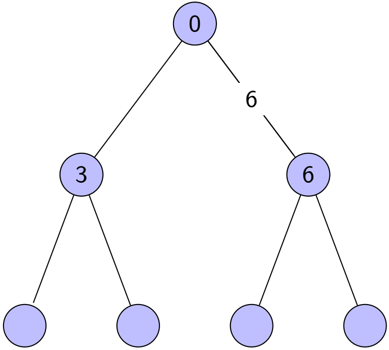

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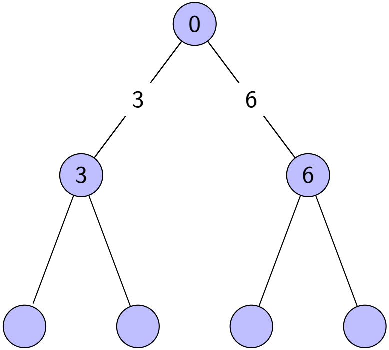

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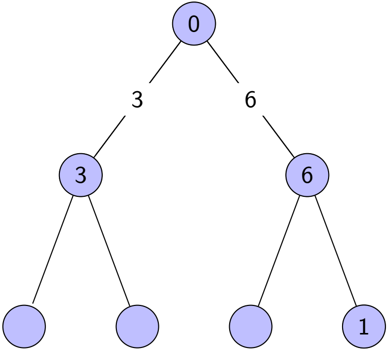

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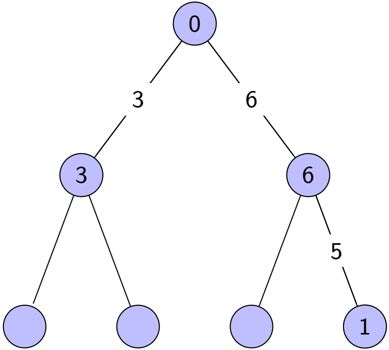

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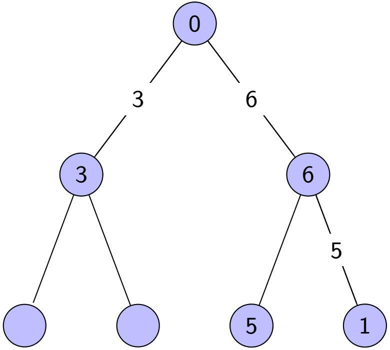

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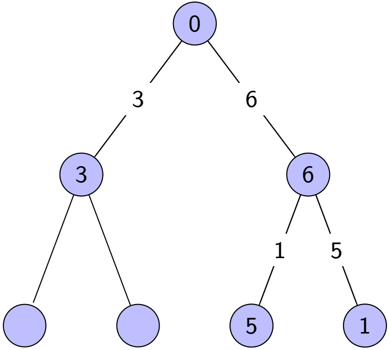

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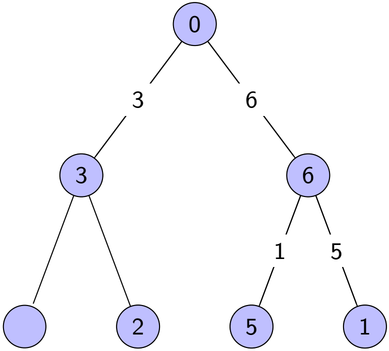

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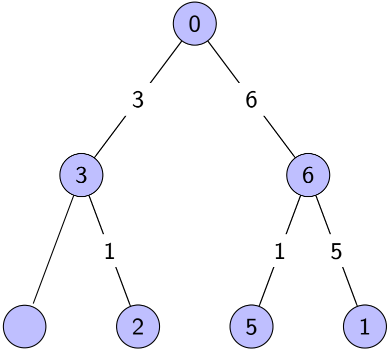

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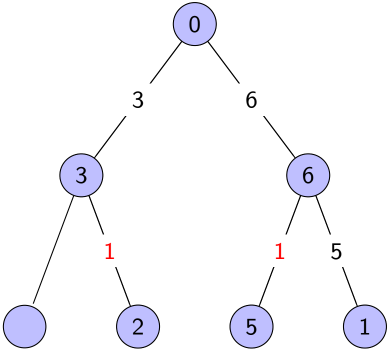

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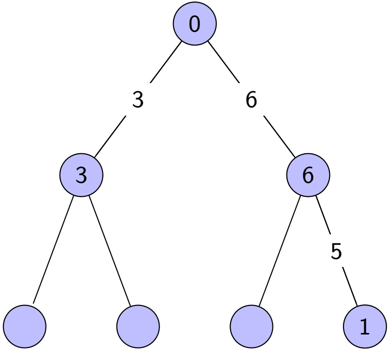

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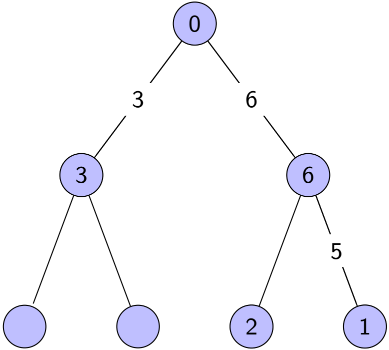

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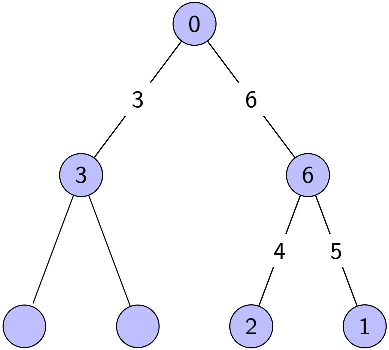

Graceful Tree Labeling

Vertex Labels: $\{\cancel{0}, \cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}\}$
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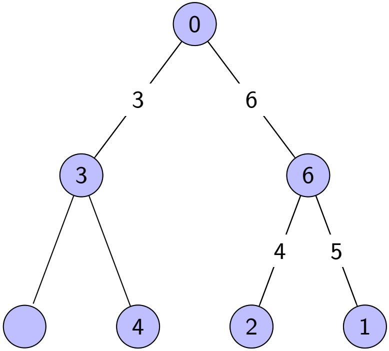

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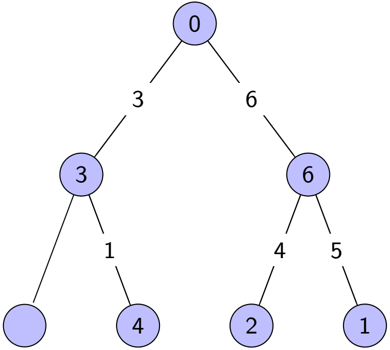

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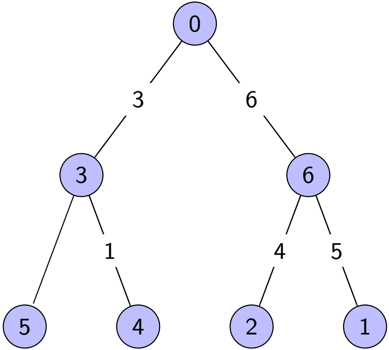

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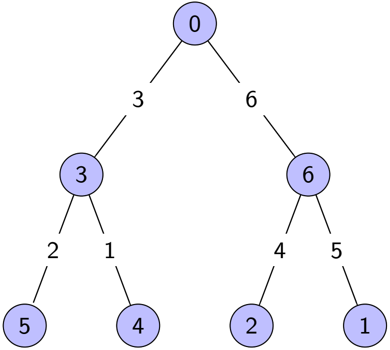

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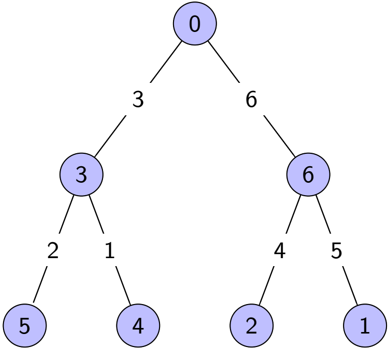

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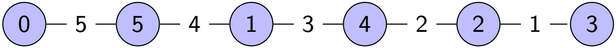

Figure: A graceful labeling of the path P_6.

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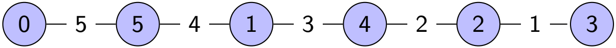

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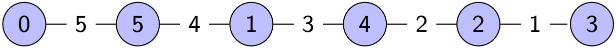

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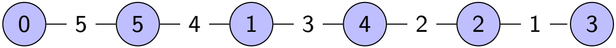

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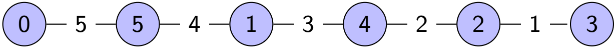

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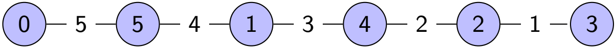

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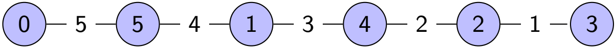

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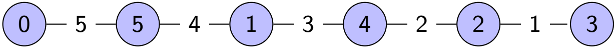

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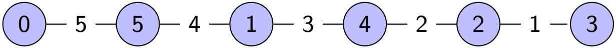

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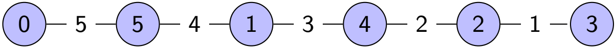

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Thus, P_6 has a graceful labeling.

Definition

A cycle graph, C_n, is a graph with n vertices that has a path that starts and ends on the same vertex, in which every vertex has degree two.

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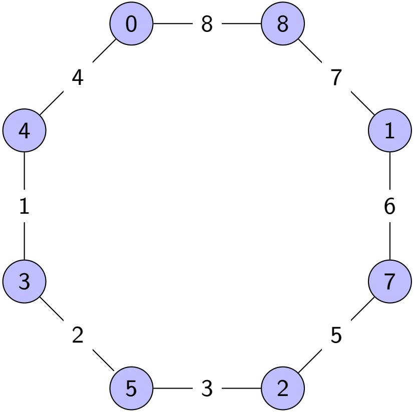

Figure: A graceful labeling of the cycle C_8.

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A cycle with n vertices is graceful if and only if $n \equiv 0$ or 3 (mod 4).

The labeling functions for graceful cycles are:

If $n \equiv 0 \pmod 4$:

$$f(v_i) = \begin{cases} \frac{i-1}{2}, & \text{if } i \text{ is odd,} \\ n + 1 - \frac{i}{2}, & \text{if } i \text{ is even and } i \leq \frac{n}{2}, \\ n - \frac{i}{2}, & \text{if } i \text{ is even and } i > \frac{n}{2}. \end{cases}$$

If $n \equiv 3 \pmod 4$:

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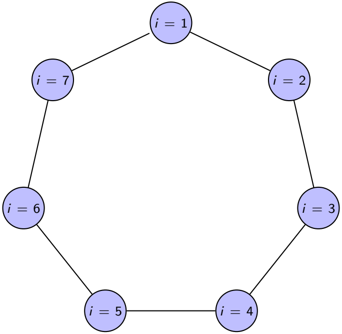
Here, the vertices of C_n are denoted $v_1, v_2, \ldots, v_n$, where each $\{v_i, v_{i+1}\}$ is an edge for all $1 \leq i \leq n-1$, as is $\{v_n, v_1\}$.

It is important to note that a given labeling function is not necessarily unique.

Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



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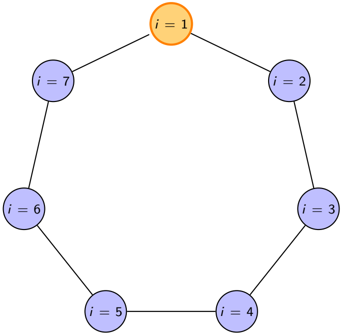

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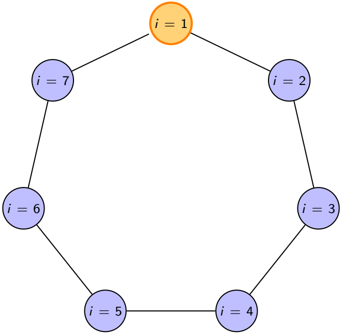
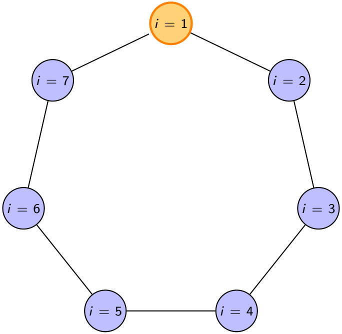


$i = 1 : 1$ is odd and $1 \leq 3$

$$7 \equiv 3 (\bmod 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



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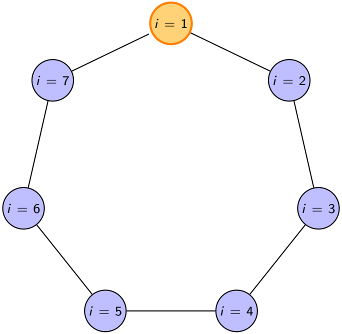
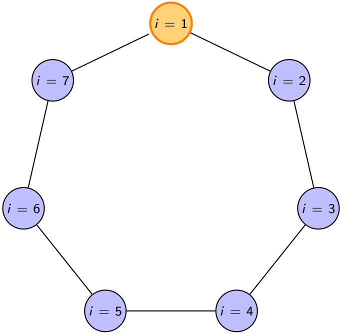


$$i = 1 : 1 \text{ is odd and } 1 \leq 3 \Rightarrow f(v_1) = \frac{1-1}{2}$$

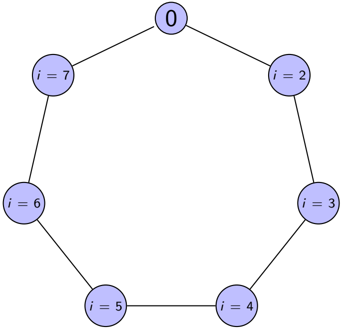
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



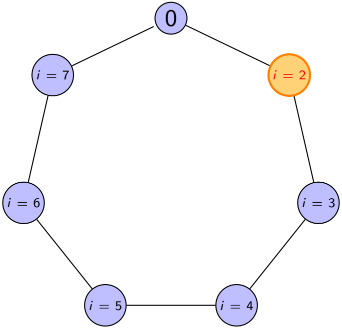
$$i = 1 : 1 \text{ is odd and } 1 \leq 3 \Rightarrow f(v_1) = \frac{1-1}{2} = 0$$

Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



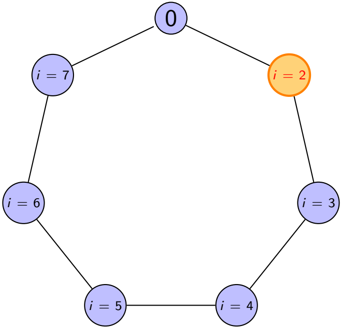
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 2 :$

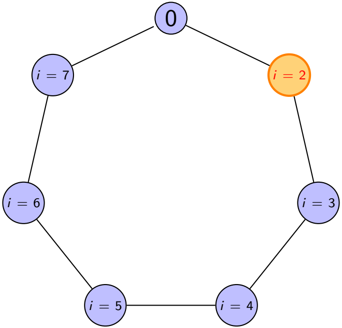
Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 2$: 2 is even

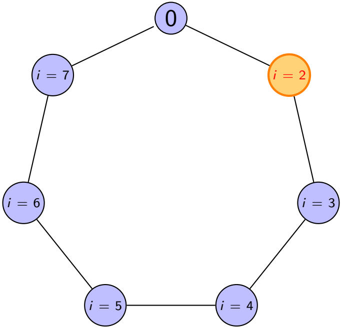
$$7 \equiv 3 (\mathrm{mod}\ 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even}, \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 2 : 2$ is even

Labeling Function Example

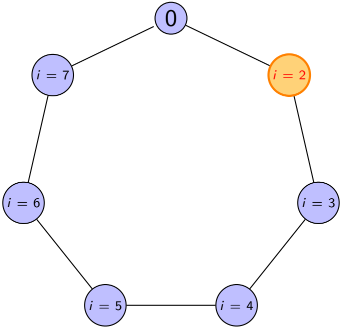
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 2 : 2 \text{ is even } \Rightarrow f(v_2) = 7 + 1 - \frac{2}{2}$$

Labeling Function Example

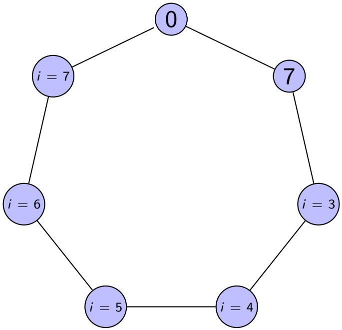
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even}, \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 2 : 2 \text{ is even } \Rightarrow f(v_2) = 7 + 1 - \frac{2}{2} = 7$$

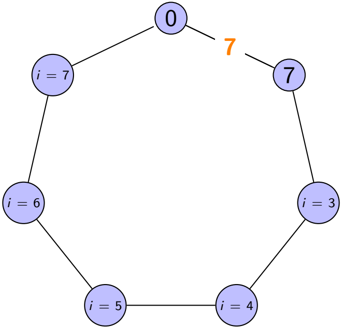
Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

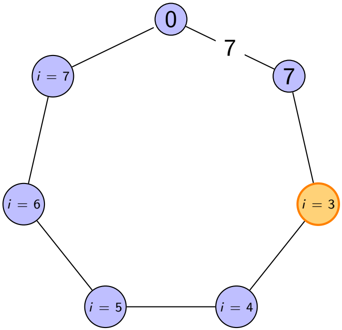


Labeling Function Example

$$7 \equiv 3 (\bmod\ 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



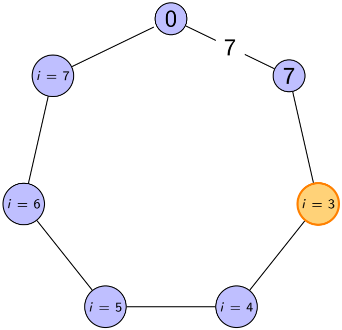
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 3 :$

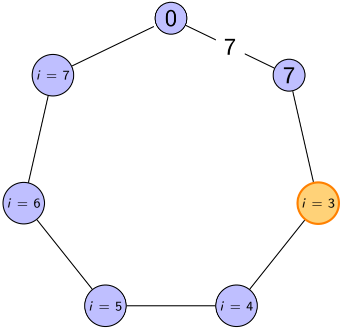
Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 3 : 3$ is odd and $3 \leq 3$

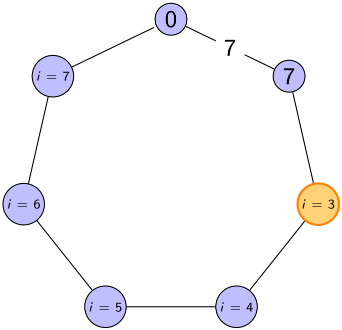
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 3 : 3$ is odd and $3 \leq 3$

Labeling Function Example

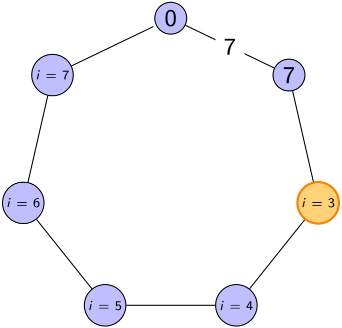
$$7 \equiv 3 (\bmod 4) : f(v_i) = \begin{cases} 7+1-\frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 3 : 3 \text{ is odd and } 3 \leq 3 \Rightarrow f(v_3) = \frac{3-1}{2}$$

Labeling Function Example

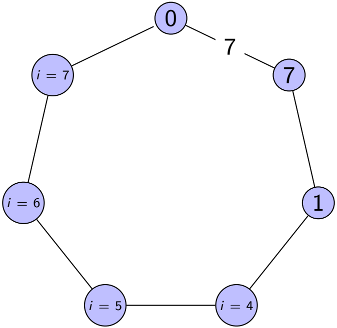
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



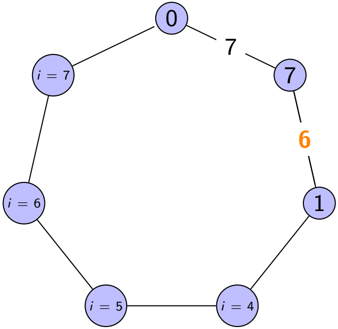
$$i = 3 : 3 \text{ is odd and } 3 \leq 3 \Rightarrow f(v_3) = \frac{3-1}{2} = 1$$

Labeling Function Example

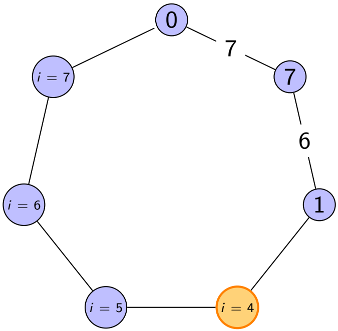
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7+1-\frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$

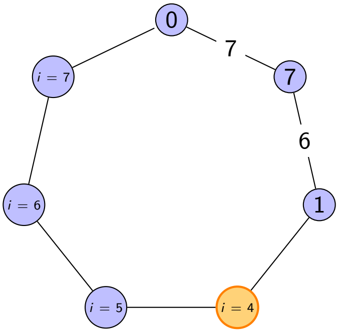


$$7 \equiv 3 (\bmod 4) : f(v_i) = \begin{cases} 7+1-\frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



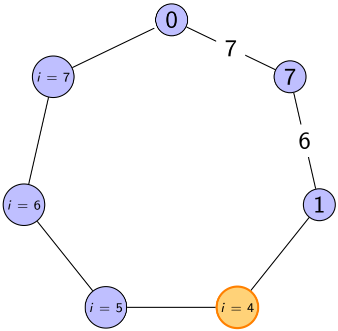
$i = 4 :$

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 4 : 4$ is even

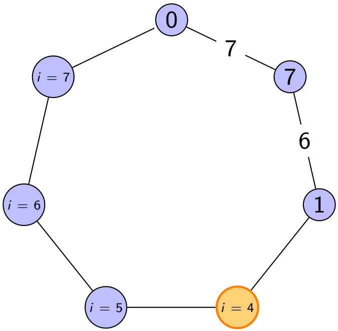
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even}, \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 4 : 4$ is even

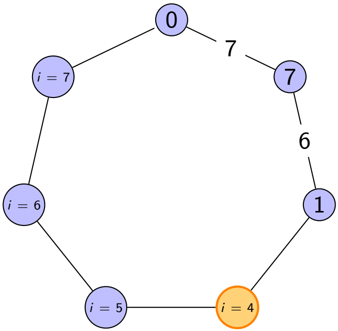
Labeling Function Example

$$7 \equiv 3 (\bmod\ 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$i = 4 : 4 \text{ is even } \Rightarrow f(v_4) = 7 + 1 - \frac{4}{2}$$

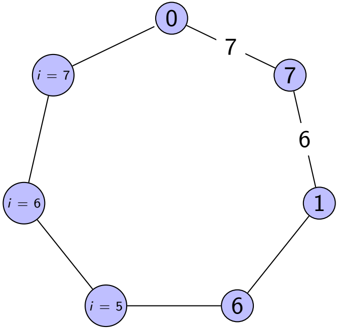
$$7 \equiv 3 (\mathrm{mod}\ 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even}, \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



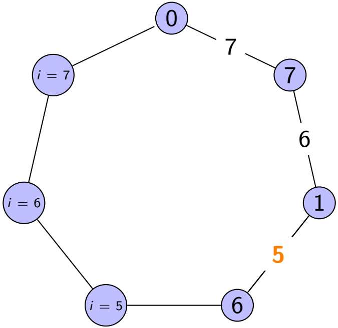
$$i = 4 : 4 \text{ is even } \Rightarrow f(v_4) = 7 + 1 - \frac{4}{2} = 6$$

Labeling Function Example

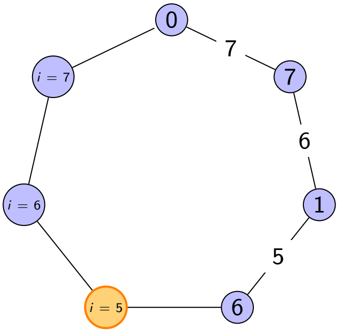
$$7 \equiv 3 (\bmod\ 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \le \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7+1-\frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



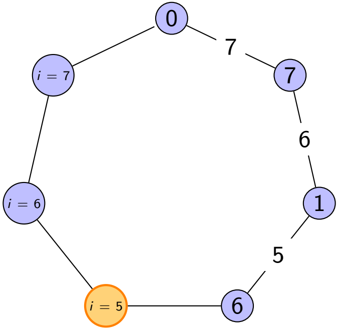
$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 5 :$

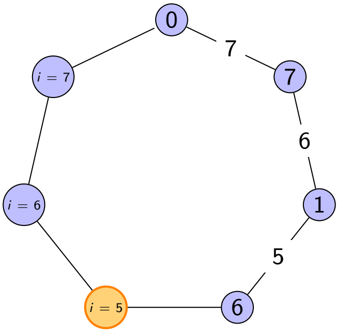
Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 5$: 5 is odd and $5 > 3$

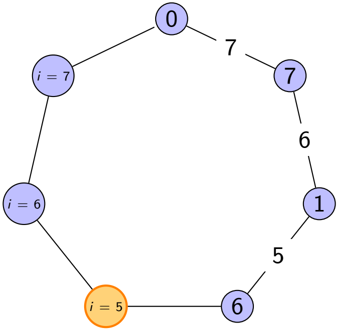
$$7 \equiv 3(\text{mod } 4) : f(v_i) = \begin{cases} 7+1-\frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 5 : 5$ is odd and $5 > 3$

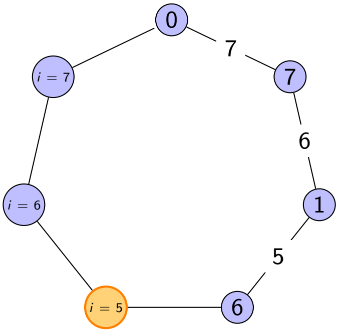
Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



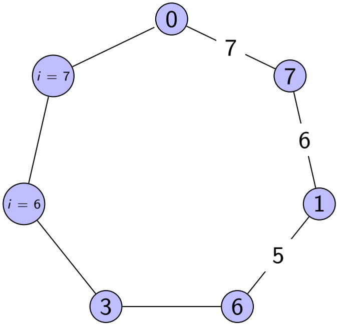
$$i = 5 : 5 \text{ is odd and } 5 > 3 \Rightarrow f(v_5) = \frac{5+1}{2}$$

$$7 \equiv 3 (\bmod 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



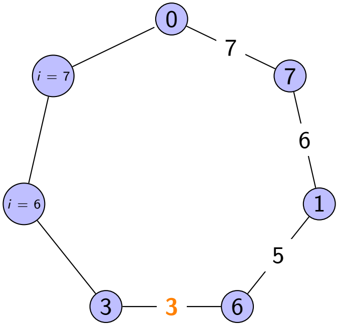
$$i = 5 : 5 \text{ is odd and } 5 > 3 \Rightarrow f(v_5) = \frac{5+1}{2} = 3$$

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7+1-\frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



Labeling Function Example

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



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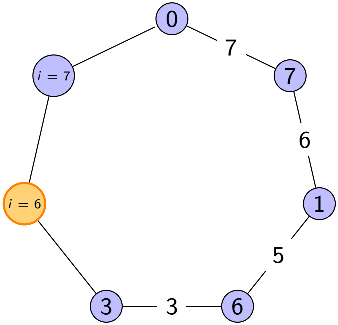
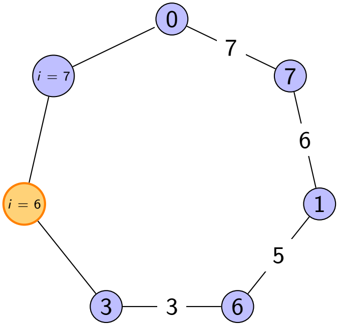


$i = 6 :$

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



$i = 6$: 6 is even

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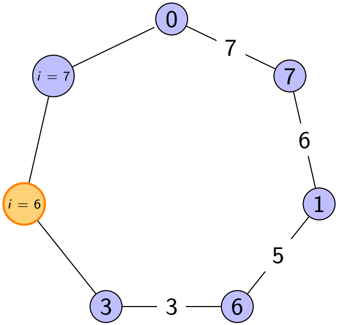

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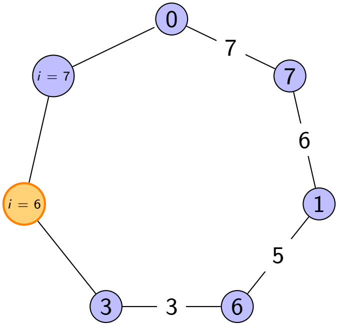

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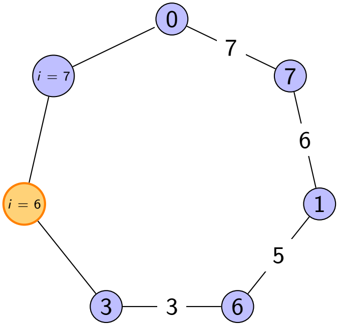
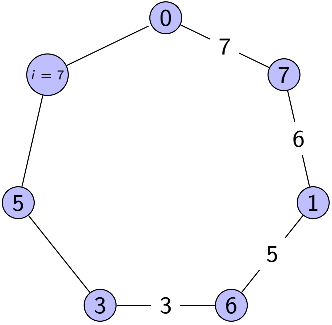


$$i = 6 : 6 \text{ is even } \Rightarrow f(v_6) = 7 + 1 - \frac{6}{2} = 5$$

$$7 \equiv 3 (\text{mod } 4) : f(v_i) = \begin{cases} 7 + 1 - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \text{ is odd and } i \leq \frac{7-1}{2} = 3, \\ \frac{i+1}{2}, & \text{if } i \text{ is odd and } i > \frac{7-1}{2} = 3. \end{cases}$$



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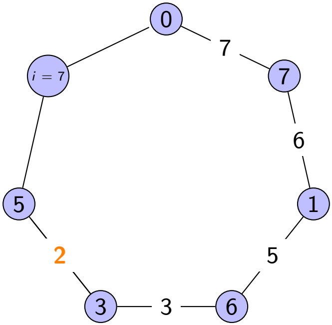

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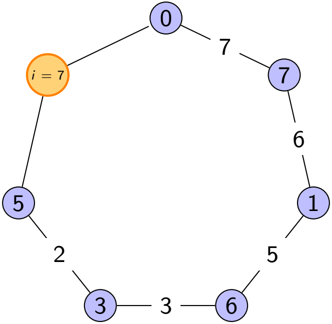

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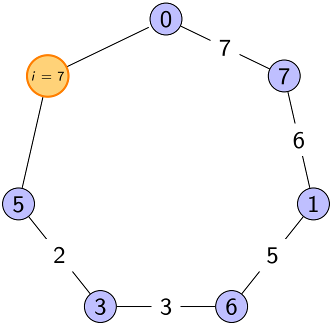

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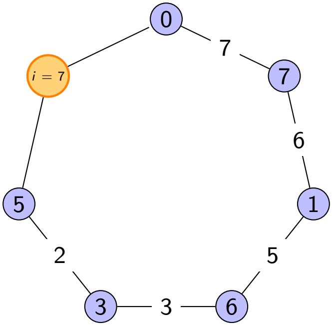

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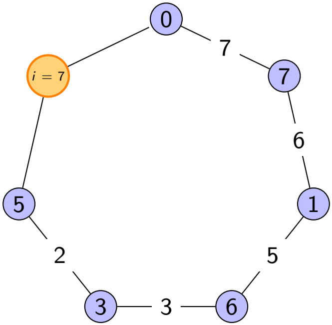

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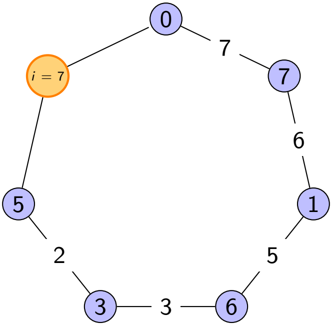

$$i = 7 : 7 \text{ is odd and } 7 > 3 \Rightarrow f(v_7) = \frac{7+1}{2} = 4$$

Labeling Function Example

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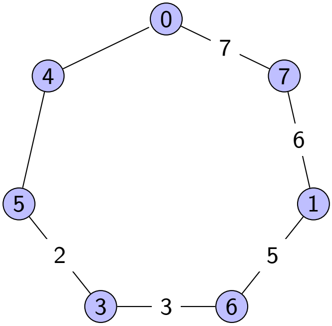

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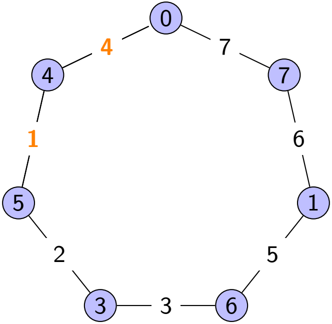

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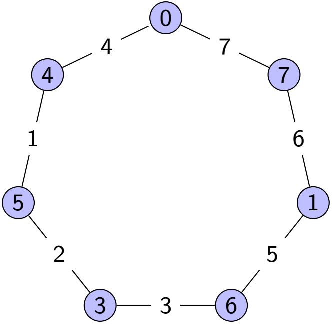

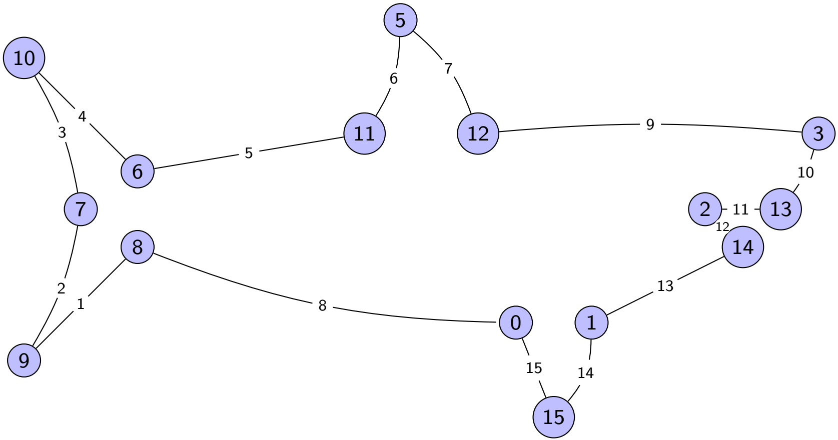
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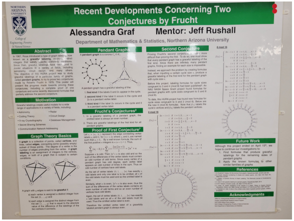
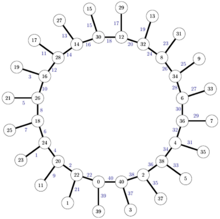
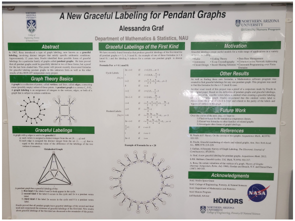
Definition

A pendant graph is an n-cycle graph with one pendant vertex attached to each cycle vertex.

Dr. Alessandra Graf researched graceful labelings of pendant graphs for several semesters at NAU starting in 2013. She was able to prove that all pendant graphs that have labels 0 and $2n$, where $n \equiv 4$ or $8 \pmod 8$ and $n \equiv 0 \pmod{16}$ can be gracefully labeled, while also finding the labeling functions for them.







Since Rosa first introduced the idea of graceful labeling, many mathematicians (including Dr. Alessandra Graf) have proved that certain graphs can always be gracefully labeled.

Miscellaneous Graph Families With Graceful Labelings

Since Rosa first introduced the idea of graceful labeling, many mathematicians (including Dr. Alessandra Graf) have proved that certain graphs can always be gracefully labeled.

Let's look at some of them...

Theorem

All wheel, web, helm, gear, grid, star, complete, banana tree, and caterpillar tree graphs are graceful. [Various mathematicians]

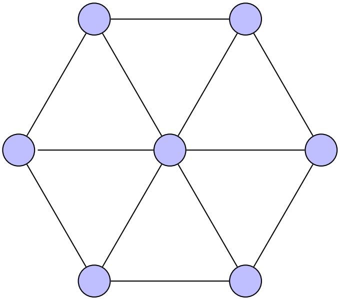


Figure: The wheel graph W_7.

Theorem

All wheel, web, helm, gear, grid, star, complete, banana tree, and caterpillar tree graphs are graceful. [Various]

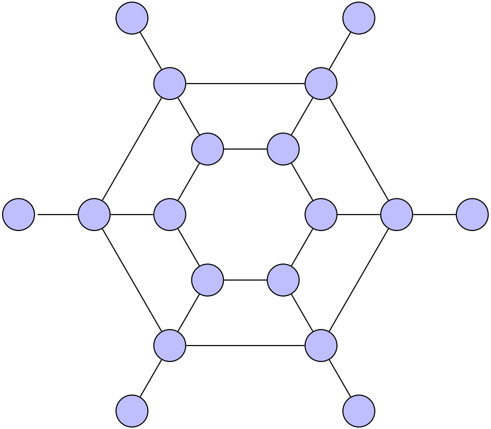


Figure: The web graph W_6.

Theorem

All wheel, web, helm, gear, grid, star, complete, banana tree, and caterpillar tree graphs are graceful. [Various]

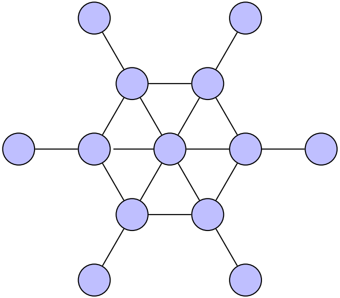


Figure: The helm graph H_6.

Theorem

All wheel, web, helm, gear, grid, star, complete, banana tree, and caterpillar tree graphs are graceful. [Various]

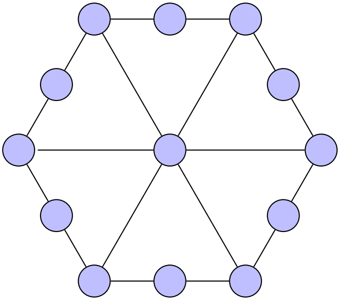


Figure: The gear graph G_6.

Theorem

All wheel, web, helm, gear, grid, star, complete, banana tree, and caterpillar tree graphs are graceful. [Various]

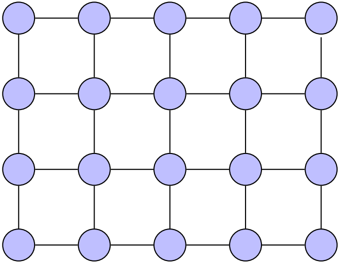


Figure: The grid graph $P_5 \times P_4$.

Theorem

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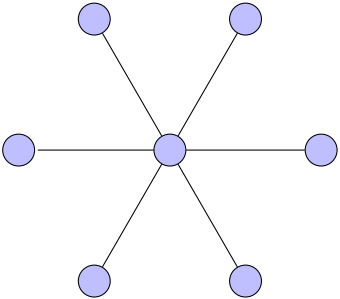

Figure: The star graph S_7.

Theorem

All wheel, web, helm, gear, grid, star, complete, banana tree, and caterpillar tree graphs are graceful. [Various]

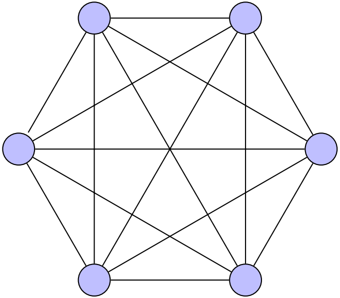

Figure: The complete graph K_6.

Theorem

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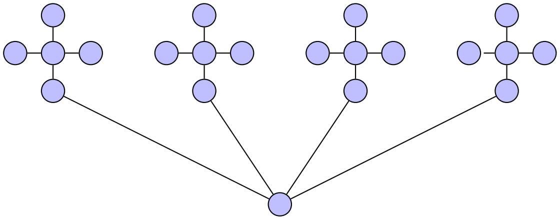

Figure: The banana tree graph $B_{4,5}$.

Miscellaneous Graph Families With Graceful Labelings

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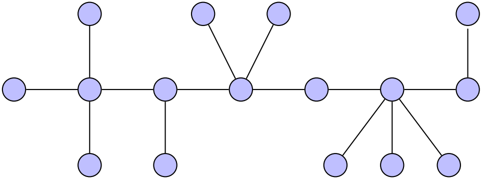
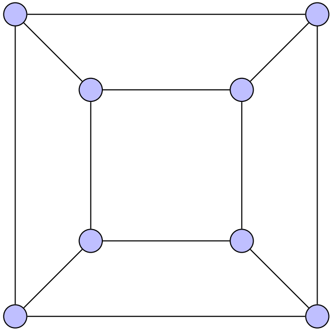


Figure: A caterpillar tree graph.

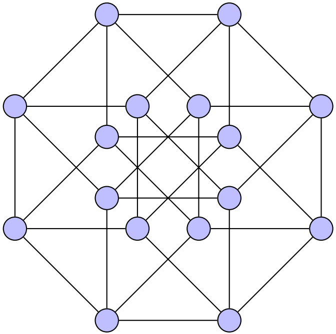
Miscellaneous Graph Families With Graceful Labelings

Theorem

All n-dimension hypercubes are graceful. [Kotzig, 1981]



3-dimensional hypercube



4-dimensional hypercube

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- This is a major open conjecture in graph theory.
- Kotzig commented that trying to prove this conjecture was a "disease".
- It has several related (yet weaker) conjectures, many of which have proven partial results.

The Ringel-Kotzig Conjecture – Recent Progress

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Questions?