Meromorphic Functions on an Open Set

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Before we get started lets review some definitions

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> ### Definition 1
> Let $G \subseteq \mathbb{C}$ and $f : G \to \mathbb{C}$ be a function. We say that f is **holomorphic** at a point $a \in int(G)$ if $\exists\, r > 0$ such that $D(a, r) \subseteq G$ and f is differentiable at each $x \in D(a, r)$. We say that f is holomorphic on G if it is holomorphic at each point in G.

Definition 2

Let $f : G \to \mathbb{C}$ be a holomorphic function on the open set G. We say that f has an **isolated singularity** at a point $a \notin G$ if f is holomorphic on $D_p(a, r)$ for some $r > 0$, but not defined at a.

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Definition 3
Let $f : G \to \mathbb{C}$ be holomorphic, where G is open, and suppose that $a \notin G$ is an isolated singularity for f. We say that f has a **pole** at $z = a$ if $\lim_{z \to a} |f(z)| = \infty$.

Definition 4

If G is an open set in $\mathbb{C}$ and (Ω, d) is a complete metric space, we call the set of all continuous functions from G to Ω, $\mathbf{C(G, \Omega)}$. The set is always non-empty since it always contains at least the constant functions.

Definition 5

The family $\mathscr{F}$ is normal in G if every subsequence $(f_n) \subset \mathscr{F}$ has a subsequence that converges that converges locally uniformly on G.

Equivalently:

$\mathscr{F}$ is normal $\iff$ every sequence has a locally uniformly convergent subseq

Why do we need Meromorphic functions when we already have Holomorphic functions?

- Holomorphic functions are perfectly smooth.

- But real life mathematical functions often have several places where they blow up.

- Meromorphic functions capture this idea: smooth everywhere except at isolated, predictable singularities that we call poles.

Definition 6
If G is open and f is a defined holomorphic function in G except for poles, then f is a **meromorphic function** on G.

A meromorphic function in the complex plane that is either holomorphic at infinity or has a pole at infinity is said to be **meromorphic in the extended complex plane**.

We will denote the set of all meromorphic functions on G as $M(G)$.

- **The Tangent Function**:
$$\tan z = \frac{\sin z}{\cos z}$$

  Meromorphic on $\mathbb{C}$, poles wherever $\cos z = 0$.

- **Rational Functions**:
$$f(z) = \frac{P(z)}{Q(z)}$$

  These are meromorphic functions on $\mathbb{C}$; poles occur at zeros of Q

- $\frac{1}{z}$:
  Meromorphic on $\mathbb{C}\backslash\{0\}$. Has a pole at 0

- Specify the domain (open).

- Show f is holomorphic except at isolated points.

- Check each singularity: prove it is a pole.

- Use a Laurent series or express $f = g/h$, where g and f are holomorphic.

- Use closure properties to simplify the argument.

If G is a region and f is a meromorphic function on G, and if $f(z) = \infty$ whenever z is a pole of f, then $f : G \to \mathbb{C}_\infty$ is a continuous function. We can see that $M(G)$ is a subset of the set of all continuous functions from G to $\mathbb{C}_\infty$ designated by, so it also has the same metric as $C(G, \mathbb{C}_\infty)$.

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$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{[(1 + |z_1|^2) + (1 + |z_2|)^2]^{\frac{1}{2}}}$$

for z_1 and z_2, and

$$d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}$$

for $z \in \mathbb{C}$.

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$$d(z_1, z_2) = d\left(\frac{1}{z_1}, \frac{1}{z_2}\right) \text{ and } d(z, 0) = d\left(\frac{1}{z}, \infty\right)$$

These will be very useful when working with meromorphic functions.

The metric on $M(G)$ is

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1 + \rho_n(f,g)}$$

where $G = \bigcup_{n=1}^{\infty} K_n$, each K_n is compact, $K_n \subseteq \text{int}(K_{n+1})$, and

$$\rho_n(f,g) = \sup\{d(f(z),g(z)) | z \in K_n\}$$

The above metric makes $C(G, \mathbb{C}_\infty)$ a complete metric space and it induces locally uniform convergence.

Proposition 7
- *If $a \in \mathbb{C}$ and $r > 0$ then $\exists \rho > 0$ such that $D_\infty(a; \rho) \subseteq D(a; r)$.*
- *Conversely, if $\rho > 0$ and $a \in \mathbb{C}$ then $\exists r > 0$ such that $D(a; r) \subseteq D_\infty(a; \rho)$.*
- *If $\rho > 0$ then there is a compact set $K \subset \mathbb{C}$ such that $\mathbb{C}_\infty - K \subseteq D_\infty(\infty; \rho)$.*
- *Conversely, if a is a compact set $K \subseteq \mathbb{C}$, $\exists \rho > 0$ such that $D_\infty(\infty; \rho) \subseteq \mathbb{C}_\infty - K$.*

Corollary 8

$M(G) \cup \{\infty\}$ *is a complete metric space.*

Theorem 9

The only meromorphic functions in the extended complex plane ($\mathbb{C}_\infty$) are the rational functions.

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The proof is left up to Casey.

> **Theorem 10**
> Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is holomorphic then either f is holomorphic or $f \equiv \infty$.

Proof.

- We have meromorphic functions f_n on G converging to f in the spherical metric.
- Goal: show that f is either meromorphic or identically ∞.

Case 1: $f(a) \neq \infty$ **for some point** a

- Convergence in the spherical metric implies local equicontinuity near a.
- For all large n, the values $f_n(a)$ stay close to the finite number $f(a)$.
- Thus the tail of the sequence has no poles near a.
- Hence f_n is analytic near a for all sufficiently large n.
- Uniform convergence then implies that f is analytic near a.
- Since this holds at every finite point of f, the function f is meromorphic on G.

$\square$

Case 2: $f(a) = \infty$ for some point a

- Consider the reciprocals $1/f_n$.
- Each $1/f_n$ is meromorphic and converges to $1/f$ in the spherical metric.
- Apply the same argument as in Case 1 to the sequence $1/f_n$.
- Either $1/f$ is analytic near a (so f has a pole at a), or $1/f \equiv 0$ (so $f \equiv \infty$).

Conclusion

- Therefore f is either meromorphic or identically ∞.
- If each f_n is analytic:
    - Case 1 implies f is analytic.
    - Case 2 can only give $f \equiv \infty$ (analytic limits cannot develop poles).
- Hence f is analytic or identically ∞ in this case.

Another interesting thing we can discuss is the normality in $M(G)$. To do so we have to look at the quantity $\frac{2|f'(z)|}{1+|f(z)|^2}$ (derivative of the Riemann Sphere) for each $f \in M(G)$ and what happens if z is a pole of f.

If z is a pole, then $f'(z)$ is meaningless, so we want to try to look at $\lim_{z \to a}$ (where a is our pole). To show that the limits exist, let a have order $m \geq 1$ and let

$$f(z) = g(z) + \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)}$$

where g is holomorphic in $D(a, r)$ and $z \in D(a, r)$. Then

$$f'(z) = g'(z) - \left[\frac{mA_m}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2} \right]$$

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\begin{aligned}
\frac{2|f'(z)|}{1+|f(z)|^2} &= \frac{2\left|g'(z) - \frac{mA_m}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2}\right|}{1+\left|g(z) + \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)}\right|^2} \\
&= \frac{2|z-a|^{m-1} \cdot |mA_m + \cdots + A_1(z-a)^{m-1} - g'(z)(z-a)^{m+1}|}{|z-a|^{2m} + |A_m + \cdots + A_1(z-a)^{m-1} + g(z)(z-a)^m|^2}
\end{aligned}
$$

So, if $m \geq 2$, $\lim\limits_{z \to a} \frac{2|f'(z)|}{1+|f(z)|^2} = 0$ and if $m = 1$, $\lim\limits_{z \to a} \frac{2|f'(z)|}{1+|f(z)|^2} = \frac{2}{A_1}$

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This fact will tell us something really interesting when combined with an upcoming theorem.

Definition 11

If f is a meromorphic function on the region G then define $\mu(f) : G \to \mathbb{R}$ by

$$\mu(f)(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}$$

whenever z is not the pole of f, and

$$\mu(f)(a) = \lim_{z \to a} \frac{2|f'(z)}{1 + |f(z)|^2}$$

if a is a pole of f. It follows that $\mu(f) \in C(G, \mathbb{C})$.

Theorem 12

A family $\mathscr{F} \subset M(G)$ is normal in $C(G, \mathbb{C}_\infty)$ if and only if
$\mu(\mathscr{F}) \equiv \{\mu(f) : f \in \mathscr{F}\}$ is locally bounded.

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Since we just showed that $M(G)$ is locally bounded when poles have an order
$m \geq 1$, we know that poles do not prevent normality.

$$H(G) \subset M(G) \subset C(G, C_\infty)$$

- $H(G)$

  This is the space of holomorphic functions on G. A function $f \in H(G)$ is complex-differentiable at every point of G (no singularities).

- $M(G)$

  This is the space of meromorphic functions on G. A function $f \in M(G)$ is holomorphic on G except at isolated poles.

  Since every holomorphic function is automatically meromorphic (with no poles), we have:

  $$H(G) \subset M(G).$$

- $C(G, C_\infty)$

  This is the set of continuous functions from G into the extended complex plane:

  $$C_\infty = \mathbb{C} \cup \{\infty\}.$$

  The value ∞ is allowed because meromorphic functions may take the value ∞ at their poles (in the Riemann sphere sense).

Example 13

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(d) (**5 points**) Assume that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a homeomorphism (i.e. f is bijective, continuous and f^{-1} is continuous). Suppose that $z_0 \in \mathbb{C}_\infty$ satisfies $f(z_0) = \infty$, and assume that f is holomorphic on $\mathbb{C} \setminus \{z_0\}$. Prove that f is a Möbius map. (**Hint:** The case when $z_0 = \infty$ follows from (b), so assume $z_0 \in \mathbb{C}$. Show that f has a simple pole at z_0 and that $g(z) = f(1/z)$ has a removable singularity at $z = 0$.)

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Questions?