1.)

Applicative order

$$(\lambda f x. f(fx))(\lambda f x. f(fx))\, f x$$

$$= (\lambda f.(\lambda x. f(fx)))(\lambda f.(\lambda x. f(fx)))\, f x$$

$$\Rightarrow_\alpha (\lambda f.(\lambda x. f(fx)))(\lambda f_1.(\lambda x_1. f_1(f_1 x_1))\, f' x'$$

$$\Rightarrow_\beta (\lambda x.(\lambda f_1.(\lambda x_1. f_1(f_1 x_1)))((\lambda f_1.(\lambda x_1. f_1(f_1 x_1))) x))\, f' x'$$

$$\Rightarrow_\beta (\lambda f_1.(\lambda x_1. f_1(f_1 x_1)))((\lambda f_1.(\lambda x_1. f_1(f_1 x_1)))f')\, x$$

$$\Rightarrow_\beta (\lambda f_1.(\lambda x_1. f_1(f_1 x_1)))\ (\lambda x_1 f'(f' x_1))\, x'$$

$$\Rightarrow_\alpha (\lambda f_1.(\lambda x_1 f_1.(f_1 x_1)))\ (\lambda x_2. f'(f' x_2))\, x'$$

$$\Rightarrow_\beta (\lambda x_1.(\lambda x_2. f'(f' x_2)))((\lambda x_2. f'(f' x_2)) x_1))\, x'$$

$$\Rightarrow_\beta (\lambda x_2. f'(f' x_2))((\lambda x_2. f'(f' x_2)) x')$$

$$\Rightarrow_\alpha (\lambda x_2. f'(f' x_2))((\lambda x_3. f'(f' x_3)) x')$$

$$\Rightarrow_\beta (\lambda x_2. f'(f' x_2))\ (f'(f' x'))$$

$$\Rightarrow_\beta (f'(f'(f'(f' x'))))$$

$$= (f(f(f(fx))))$$

1ii) Normal

$$(\lambda f x. f(fx))(\lambda f x. f(fx))\ fx$$

$$= (\lambda f. (\lambda x. (f(fx))))(\lambda f. (\lambda x. (f(fx))))\ fx$$

$$\Rightarrow_\alpha (\lambda f. (\lambda x. (f(fx))))(\lambda f_1. (\lambda x_1. (f_1(f_1 x_1))))\ f' x'$$

$$\Rightarrow_\beta (\lambda x. ((\lambda f_1. (\lambda x_1. (f_1(f_1 x_1))))((\lambda f_1. (\lambda x_1 (f_1(f_1 x_1))))x)))f' x'$$

$$\Rightarrow_\alpha (\lambda x. ((\lambda f_1. (\lambda x_1. (f_1(f_1 x_1))))((\lambda f_2. (\lambda x_2 (f_2(f_2 x_2))))x)))f' x'$$

$$\Rightarrow_\beta (\lambda f_1. (\lambda x_1. (f_1(f_1 x_1))))((\lambda f_2. (\lambda x_2 (f_2(f_2 x_2))))f'))x'$$

$$\Rightarrow_\beta (\lambda x_1 (((\lambda f_2. (\lambda x_2. (f_2(f_2 x_2))))f'))(((\lambda f_2. (\lambda x_2. (f_2(f_2 x_2))))f')x_1)x'$$

$$\Rightarrow_\alpha (\lambda x_1. (((\lambda f_2. (\lambda x_2. (f_2(f_2 x_2))))f'))(((\lambda f_3. (\lambda x_3. (f_3(f_3 x_3))))f')x_1)x'$$

$$\Rightarrow_\beta (((\lambda f_2. (\lambda x_2. (f_2(f_2 x_2))))f'))(((\lambda f_3. (\lambda x_3. (f_3(f_3 x_3))))f')x')$$

$$\Rightarrow_\beta ((\lambda x_2 (f'(f' x_2))))(((\lambda f_3. (\lambda x_3 (f_3(f_3 x_3))))f')x')$$

$$\Rightarrow_\beta (f'(f'((\lambda f_3. (\lambda x_3 (f_3(f_3 x_3))))f')x'))$$

$$\Rightarrow_\beta (f'(f'((\lambda x_3. (f'(f' x_3)))x')))$$

$$\Rightarrow_\beta (f'(f'(f'(f' x'))))$$

$$= (f(f(f(fx))))$$

iii) The program's ouput was

'(f (f (f (f #<procedure: x>))))

This certifies that both answers are correct as the reductions reduce to the same regardless which order you do. Scheme itself is an applicative order language meaning that this reduction is proved true for the applicative order reductions, its true because scheme evaluates arguments when the procedure is applied (applicative) therefore the reduction for applicative order is proven true.

2. $M = \lambda a \lambda b\ (a(a \cdots\cdots (ab) \cdots)$

$N = \lambda a \lambda b\ (a(a \cdots\cdots (ab) \cdots)$

i) $[M+N] = \lambda a.\lambda b.((Ma)((Na)b))$

$= \lambda a.\lambda b.((\underbrace{\lambda f.\lambda c.(f(f\cdots(fc)\cdots)}_{M})a)((\underbrace{\lambda f.\lambda c.(f(f\cdots(fc)\cdots)}_{N})a)b)$

$\Rightarrow_\beta \lambda a.\lambda b.(\underbrace{\lambda c.(a(a\cdots(ac)\cdots))}_{M_x}(\underbrace{(\lambda f.\lambda c(f(f\cdots(fc)\cdots))a)b}_{N_x})$

$\Rightarrow_\beta \lambda a.\lambda b(\lambda c.(\underbrace{a(a\cdots(ac)\cdots)}_{M_x})(\lambda c.(\underbrace{a(a(\cdots(ac)\cdots)}_{N_x})b)$

$\Rightarrow_\beta \lambda a.\lambda b.(\underbrace{a(a\cdots\cdots(a(}_{M}\lambda c.(\underbrace{a(a\cdots(ac}_{N}))\cdots)b)\cdots)$

$\Rightarrow_\beta \lambda a.\lambda b.(\underbrace{a(a\cdots(a}_{M}\underbrace{(a\cdots(ab}_{N})\cdots)$

$= \lambda a.\lambda b.(\underbrace{a(a\cdots(ab}_{M+N}))\cdots)$

2 ii) $M \times N = \lambda a (M (N a))$

$= \lambda a. ((\lambda f. \lambda c. (\overbrace{f \cdots\cdots (fc)}^{M} \cdots)) (\lambda f. \lambda c (\overbrace{f \cdots\cdots (fc)}^{N} \cdots\cdots) a))$

$\Rightarrow_\beta \lambda a. ((\lambda f. \lambda c (\overbrace{f \cdots (fc)}^{M} \cdots)) (\lambda c (\overbrace{a \cdots\cdots (ac)}^{N} \cdots))$

$\Rightarrow_\beta \lambda a. (\lambda c. \underbrace{(\lambda c. (\underbrace{a \cdots\cdots (ac)}_{N} \cdots) \cdots\cdots ((\lambda c (\underbrace{a \cdots (ac)}_{N})) c) \cdots)}_{M}$

$\Rightarrow_\beta \lambda a. (\lambda c. (\lambda c. \underbrace{(\overbrace{a \cdots\cdots (ac)}^{N} \cdots) \overset{N}{\cdots\cdots} (\overbrace{a c \cdots\cdots (ac)}^{N} \cdots)}_{M}$

$\Rightarrow_\beta \lambda a. (\lambda c. (a \cdots\cdots \underbrace{(a (a (\cdots (ac)}_{M \times N} \cdots\cdots))$

Since c is an arbitrary value c = b and the reduction is true

2iii) $M^N = (N\ M)$

$= \left((\lambda f.\lambda c.\overset{N}{f}(f \cdots (fc)(\lambda f.\lambda c.f(f \cdots (fc))))\right)$

$\Rightarrow_\beta \lambda c.(\lambda f.\lambda c.\overset{N_1}{f} \cdots (fc)\ \overset{N\cdots}{\cdots\cdots}\ ((\overset{N_n}{\lambda f}.\lambda c.f(\cdots(fc)))c\cdots)$

$\Rightarrow_\alpha \lambda c.(\lambda f.\lambda b.\overset{N_1}{f} \cdots (fb)\ \overset{N_{2\cdots N-1}}{\cdots\cdots}\ ((\lambda f.\lambda b.f(\cdots(fb)))c\cdots)$

$\Rightarrow_\beta \lambda c.(\lambda f.\lambda b.\overset{N_1}{f} \cdots (fb))\ \cdots\ \overset{N-1}{(\lambda f.\lambda b.f(fb)\cdots)}\ \overset{N_n}{(\lambda b.c(\cdots(cb)\cdots)}$

$\Rightarrow_\beta \lambda c.(\lambda fb.\overset{N_1}{(f}\cdots(fb))(\cdots\overset{N_{n-2}}{(\lambda fb.f}\cdots(fb)\cdots)(\lambda b.(\lambda b(c\cdots(cb)(\cdots(\lambda b.c\cdots(cb)(b)))$

β red on N_{n-1} by N_n $(M \times M)$

$\Rightarrow_\beta \lambda c.(\lambda fb.\overset{N_1}{f}\cdots(fb)\cdots)(\cdots(\lambda b.(\lambda b.c(c(c\cdots(cb)(\lambda b.c\cdots(cb)(\cdots\underbrace{c(c\cdots c((cb)\cdots)}_{M\times M\times M}$

$\Rightarrow_\beta \lambda c(\lambda fb.f\cdots(fb)\cdots)(\cdots(\lambda b.(\lambda b.c\cdots(cb))(c\cdots(c(c\cdots(cb))\cdots)$

$\Rightarrow_\beta \lambda c(\lambda fb.f\cdots(fb))\cdots(\lambda b.(c\cdots(c(c\cdots(cb)\cdots)$

$\Rightarrow_\beta \lambda c(\lambda b.(\lambda b.c(\cdots(cb)\cdots)(\lambda b.(\cdots(cb)\cdots)b)\cdots)$

$\Rightarrow_\beta \lambda c.\lambda b.(\lambda b.c(\cdots(cb)\cdots)(c\cdots(cb))$

$\Rightarrow_\beta \lambda c.\lambda b(\overbrace{c\cdots(c(c\cdots(cb)\cdots)}^{M^n}$