Lecture 1
Sept. 26, 2023
Ryerson 277

Dynamical system

- an evolution rule that defines a trajectory as a function of a single parameter (time) on a set of states (the phase space).

rule is determinstic – no randomness

Primary focus of course is on evolution rule via ODEs:

$$\dot{x} = f(x), \quad \text{where} \quad \dot{x} = \frac{dx}{dt}, \quad x \in \mathbb{R}^n \text{ (say)}$$

Secondary focus on

maps

$x_{n+1} = f(x_n)$

e.g. discrete time
$x_n = x(t_n)$
$t_n = nT$
"stroboscopic map"

PDEs

in case of traveling wave solns. in 1-space dimension (z)

$\xi = z - ct$

← periodic in ξ spatially
→ periodic traveling wave

ξ is "time" variable
& c is then a parameter

ODEs: often interested in parameterized families
e.g.

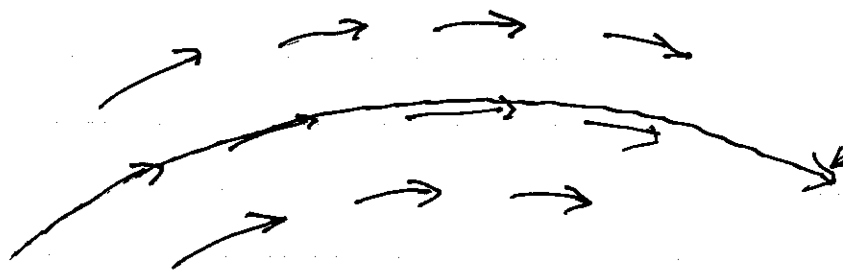
$$\dot{x} = f(x; \mu)$$ ← vector of parameters of the model

how does qualitative behavior change with changes of parameters and initial conditions?

$x \in M$, M = "state space" or "phase space", e.g. $M = \mathbb{R}^n$

$x(t)$ = state at time t
$x(0) = x_0$ = initial condition

Typically $f \in C^k$, $k \geq 1$, continuously differentiable vector field (existence & uniqueness ensured)



More terminology

$$\dot{x} = f(x; \mu)$$ ← "autonomous" vector field doesn't depend on t.

vs.

$$\dot{x} = f(x, t; \mu)$$ ← "non-autonomous" vector field depends on t.

We could convert non-autonomous to autonomous:

$$\dot{x} = f(x;\theta;\mu)$$
$$\dot{\theta} = 1 \qquad \text{with } \theta(0) = 0 \Rightarrow \theta = t$$

if $x \in \mathbb{R}^n$, then we converted our n-dim. non-autonomous ODE to an $(n+1)$-dim autonomous one

In practice, this is not ~~so~~ always helpful, so the distinction between autonomous & non-autonomous cases remains.

What about initial condition? What about higher order eqns?

higher order: e.g. $m\ddot{x} = f(x)$
can convert to 2 first-order eqns

$$\left.\begin{aligned} \dot{x} &\equiv v \\ \dot{v} &= \frac{1}{m} f(x) \end{aligned}\right\}$$

if $x \in \mathbb{R}^3$ then our phase space is $\mathbb{R}^6$

initial conditions: existence-uniqueness is in terms of initial value problem

$$\dot{x} = f(x)$$
$$x(0) = x_0$$

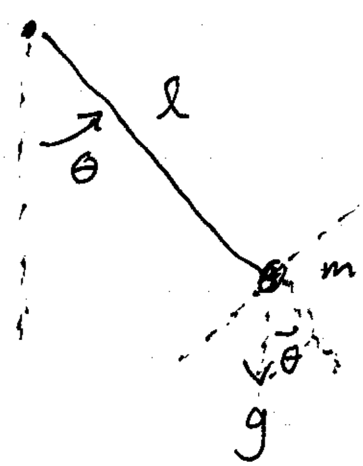
Ideally, we could determine behavior as functions of the initial condition, in all regions of phase-space

For example, in bi-stable systems where asymptotic behavior depends on where you start

e.g. $$\lim_{t\to\infty} x(t) = \begin{cases} x_1^* & \text{for all } x(0) \in S_1 \subset M \\ x_2^* & \text{for all } x(0) \in S_2 \subset M \end{cases}$$

here S_1 & S are "basins of attraction" for the equilibria/fixed-pts. x_1^* & x_2^*

Example of phase-space representation of solns. for the idealized pendulum



$$ml \frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\underbrace{\frac{g}{l}}_{\equiv\,\omega^2}\sin\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\omega^2\sin\theta$$

non-dimensionalize time:

let $\tau = \omega t$

$$\Rightarrow \quad \frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \omega \frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2} \quad \Rightarrow \quad \cancel{\omega^2}\frac{d^2\theta}{d\tau^2} = -\cancel{\omega^2}\sin\theta$$

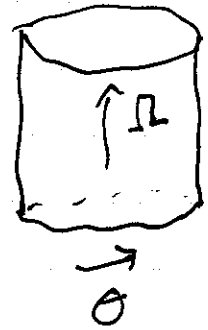
no parameters, every pendulum is the same as every other one

$$\theta(\tau) \xrightarrow[\tau = \omega t]{} \theta(t)$$

Phase-space: (θ, Ω), where $\Omega = \frac{d\theta}{d\tau}$

$$\Rightarrow \quad \left.\begin{aligned} \dot{\theta} &= \Omega \\ \dot{\Omega} &= -\sin\theta \end{aligned}\right\} \quad \begin{aligned} &\theta \in (-\pi, \pi] \\ &\Omega \in \mathbb{R} \end{aligned}$$

M = cylinder = $S^1 \times \mathbb{R}$



eqns. of the form

$$\ddot{x} = f(x) \qquad x \in \mathbb{R}$$

have a "conserved quantity"

let $f(x) = -\frac{dV}{dx}$, i.e. introduce $V(x) = -\int f(x)\,dx$

$$\underbrace{\dot{x}\,\ddot{x}}_{\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right)} = \underbrace{-\dot{x}\frac{dV}{dx}}_{\frac{d}{dt}(V(x))}$$

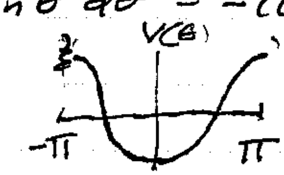
$$\frac{d}{d\tau}\left(\frac{1}{2}\dot{x}^2 + V(x)\right) = 0$$

kinetic energy

potential energy

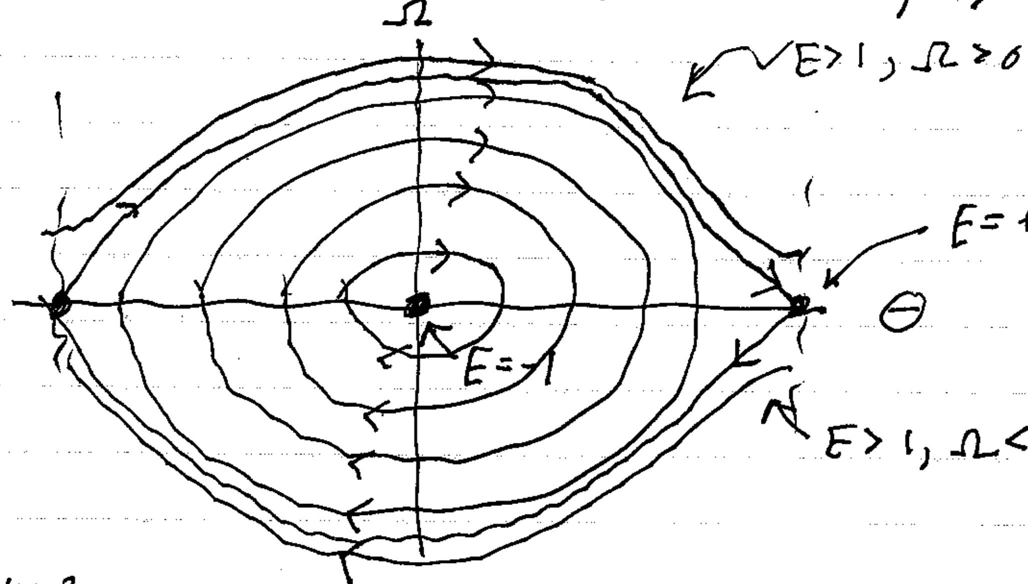
For our example $\ddot{\theta} = -\sin\theta$

$$V(\theta) = \int \sin\theta \, d\theta = -\cos\theta$$



$$E = \underbrace{\frac{1}{2}\Omega^2}_{\geq 0} - \underbrace{\cos\theta}_{\in[-1,1]} \geq -1$$

Plot of the level sets of $E(\theta,\Omega)$:

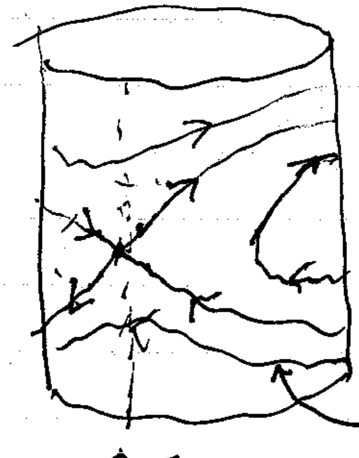


$E=\frac{1}{2}\Omega^2-\cos\theta$

if $E\in(-1,1)$ ~~$E=\frac{1}{2}\Omega^2-\cos\theta$~~ $\Rightarrow$ there is a max displacement angle $\theta_m\in(0,\pi)$ (where $\cos\theta=-E$, $\Omega=0$)

if $E>1 \Rightarrow \Omega^2>0$ $\begin{cases}\Omega>0\\ \Omega<0\end{cases}$

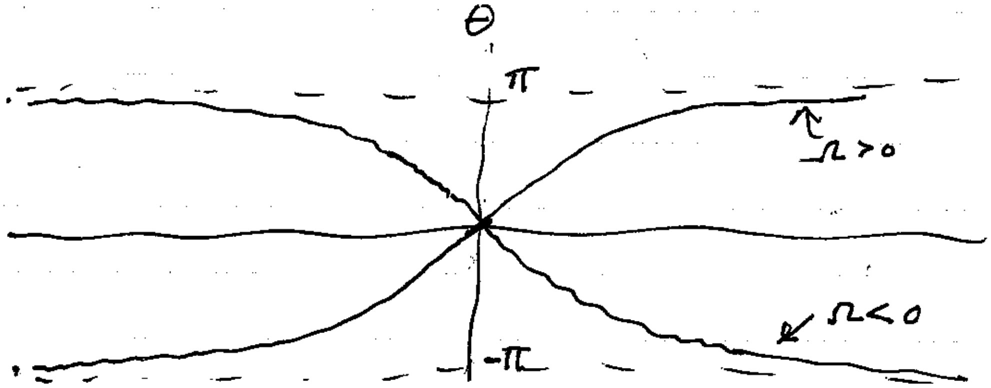
back to plotting on the cylinder:

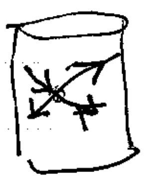


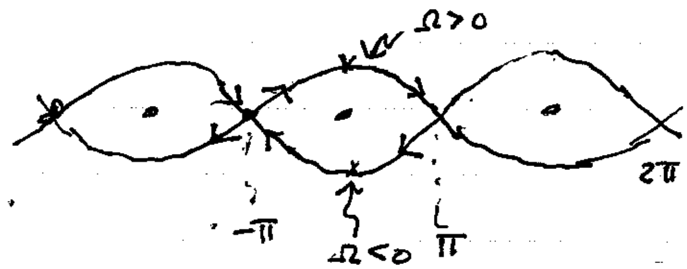
periodic solns. by virtue of cylinder's topology

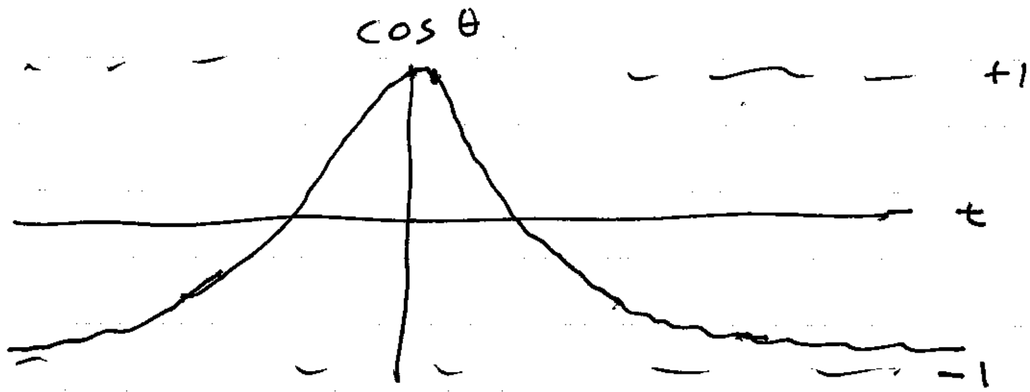
$\theta=\pi,\ \Omega=0$ is a "saddle" that has an associated "homoclinic orbit" "homoclinic orbit"

$$\lim_{t\to\pm\infty} x(t) = x^*, \text{ where } f(x^*)=0$$









$$\lim_{t\to\pm\infty} \cos\theta = -1$$