STAT 31210: Homework 1

Caleb Derrickson

January 12, 2024

Collaborators: The TA's of the class, as well as Kevin Hefner, and Alexander Cram.

Contents

Problem 1.4

Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Prove that the Cartesian product $Z = X \times Y$ is a metric space with metric d defined by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2),$$

where $z_1 = (z_1, y_1)$ and $z_2 = (z_2, y_2)$.

Solution:

To show (Z, d) is a metric space, we just need to show that d is indeed a metric. This requires d to satisfy the three stated properties for a metric (Definition 1.1). These will be shown below.

- $d(z_1, z_2) \geq 0$ and $d(z_1, z_2) = 0 \iff z_1 = z_2$.

  Note since d_X and d_Y are both given as metrics on their respective spaces, they obey the given properties of a metric. Therefore, d_X and d_Y are both positive functions. Since the addition of two positive functions is positive, then $d_X + d_Y \geq 0$. But this is the definition for d, so $d \geq 0$.

  Next we need to show that $d(z_1, z_2) = 0 \iff z_1 = z_2$. Again, since d_X and d_Y are metrics, then $d_X(x_1, x_2) = 0 \iff x_1 = x_2$ (same for d_Y). Therefore, the sum $d_X(x_1, x_2) + d_Y(y_1, y_2)$ equals zero if and only if $x_1 = x_2$ and $y_1 = y_2$ ($d_X = -d_Y$ is only valid when $d_X = 0, d_Y = 0$.) Therefore, $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2) = 0$ if and only if $z_1 = z_2$.

- $d(z_1, z_2) = d(z_2, z_1)$.

  We will again use the fact that d_X and d_Y are metrics.

$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$	(Given.)
$= d_X(x_2, x_1) + d_Y(y_2, y_1)$	($d_Z, d_Y$are metrics.)
$= d(z_2, z_1)$	(Given.)

- $\underline{d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3).}$

  Defining $z_3 = (x_3, y_3)$, we can write the following.

$$
\begin{aligned}
d(z_1, z_3) &= d_X(x_1, x_3) + d_Y(y_1, y_3) && \text{(Given.)} \\
&\leq d_X(x_1, x_2) + d_X(x_2, x_3) + d_Y(y_1, y_3) && \text{(}d_X\text{ is a metric.)} \\
&\leq d_X(x_1, x_2) + d_X(x_2, x_3) + d_Y(y_1, y_2) + d_Y(y_2, y_3) && \text{(}d_Y\text{ is a metric.)} \\
&= d_X(x_1, x_2) + d_Y(y_1, y_2) + d_X(x_2, x_3) + d_Y(y_2, y_3) && \text{(Rearranging.)} \\
&= d(z_1, z_2) + d(z_2, z_3) && \text{(By definition.)}
\end{aligned}
$$

Problem 1.12

Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces and let $f : X \to Y$, and $g : Y \to Z$ be continuous functions. Show that the composition

$$h = g \circ f : X \to Z,$$

defined by $h(x) = g(f(x))$, is also continuous.

Solution:

We will follow Definition 1.26 in the book. Let $x_0 \in X, y_0 = f(x_0) \in Y$. Since f is continuous, $\forall\ \varepsilon_1 > 0\ \exists\ \delta_1 > 0$ such that for $d_X(x, x_0) < \delta_1 \implies d_Y(f(x), f(x_0)) < \varepsilon_1$. Furthermore, g is continuous, so $\forall\ \varepsilon_2 > 0\ \exists\ \delta_2 > 0$ such that for $d_Y(y, y_0) < \delta_2 \implies d_Z(g(y), g(y_0)) < \varepsilon_2$. If we let $\delta_2 = \varepsilon_1$, then $d_Y(f(x), f(x_0)) < \varepsilon_1 \implies d_Z(g(f(x)), g(f(x_0))) < \varepsilon_2$. Letting $\delta_1 = \varepsilon_1$, then $d_X(x, x_0) < \delta_1 \implies d_Y(f(x), f(x_0)) < \varepsilon_1 \implies d_Z(g(f(x)), g(f(x_0))) < \varepsilon_2$. Therefore, $h = g \circ f$ is continuous at x_0. Since x_0 was chosen arbitrarily, then h is continuous. $\square$

Problem 1.15

Prove that every compact subset of a metric space is closed and bounded. Prove that a closed subset of a compact space is compact.

Solution:

- Every compact subset of a metric space is closed and bounded.

  Let (X, d) be a metric space and $K \subset X$ is a compact subset. By Theorem 1.62, this equivalent to saying K is sequentially compact. So let x_n be a converging sequence in $\overline{K}$[1] with $x_n \longrightarrow x$ as $n \longrightarrow \infty$. If K is closed, then $\overline{K} = K$. By Proposition 1.41, $x \in \overline{K}$. Let x_{n_α} be a subsequence of x_n. By definition, $x_{n_\alpha} \longrightarrow k \in K$ as $\alpha \longrightarrow \infty$. However, $x_{n_\alpha} \longrightarrow x$ as $\alpha \longrightarrow \infty$. Since the limit of a sequence is unique, then $k = x$. Therefore, $x \in K$, so K is closed.

  To show that K is bounded, we wish to find $r > 0$ such that for any $x, y \in K, d(x, y) < r$. Let $x \in K$, and consider taking open balls around x, $B_a(x), a > 0$. Then the set $\mathbb{O} = \{B_a(x) : a > 0\}$ defines an open cover over over K. Since K is compact, there exists a finite subcover of $\mathbb{O}$ which covers K. Suppose $B_{r_1}(x), ..., B_{r_n}(x)$ is one such subcover. Let $r = \max\{\, r_i : i \leq n\}$. Then for any $x, y \in K$, $d(x, y) < r$. Therefore, K is bounded.

- Prove that a closed subset of a compact space is compact.

  Suppose that K is a compact subset of a metric space (X, d), and $T \subseteq K$, where T is closed. We need to show that T is compact. Suppose x_n is a sequence in T such that $x_n \longrightarrow x \in X$. Since $T \subseteq K$ and K is compact (i.e. sequentially compact), then there exists a subsequence $x_{n_\alpha} \in K$ where $x_{n_\alpha} \longrightarrow x$ as $\alpha \longrightarrow \infty$. Note that T is closed, therefore, $x \in T$. Therefore, T is sequentially compact, therefore compact.

[1]The overline will denote the closure of a set.

Problem 1.16

Suppose that F and G are closed and open subsets of $\mathbb{R}^n$, respectively, such that $F \subset G$. Show that there is a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that:

(a) $0 \leq f(x) \leq 1$;

(b) $f(x) = 1$ for $x \in F$;

(c) $f(x) = 0$ for $x \in G^c$.

Solution:

Consider the function

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}$$

where $d(x, F) = \min\{d(x, y) : y \in F\}$, similarly for G^c. Explicitly showing this function is continuous, we would need to provide an $\varepsilon - \delta$ proof. However, since $f(x)$ is a ratio of two continuous functions[2], and the denominator is never zero, $f(x)$ does not limit to ∞ for any $x \in \mathbb{R}^n$, thus is continuous. Note that the denominator is never zero since, if it was, then there exists an $x \in \mathbb{R}^n$ where $x \in F \cap G^c$. However, since $F \subset G$, $F \not\subset G^c$, so $F \cap G^c = \phi$[3]. Therefore, $x \in \phi$, which is a contradiction. Therefore, the denominator is never zero. We just now need to show that f obeys the enumerated properties.

(a) Note that $f(x)$ can be rewritten as

$$f(x) = 1 - \frac{d(x, F)}{d(x, G^c) + d(x, F)}.$$

Since the second term is ≥ 0 (the denominator is > 0 and the numerator is ≥ 0), this implies that $f(x) \leq 1$. Furthermore, we can also ration out that $f(x)$ is bounded below by zero, since $f(x)$ (before rewriting) is a ratio of two functions which are positive. Therefore, $0 \leq f(x) \leq 1$.

(a) Note that if $x \in F$, $d(x, F) = 0$. Therefore, $f(x)$ simplifies to

$$f(x \in F) = \frac{d(x, G^c)}{d(x, G^c)} = 1.$$

(a) If $x \in G^c$, $d(x, G^c) = 0$. Therefore, $f(x)$ simplifies to

$$f(x \in G^c) = \frac{0}{d(x, F)} = 0.$$

[2]Here I am taking for granted that d is a continuous function. This follows from the definition of a metric.
[3]I denote ϕ as the empty set.

Problem 1.20

Let X be a normed linear space. A series $\sum x_n$ in X is *absolutely convergent* if $\sum \|x_n\|$ converges to a finite value in $\mathbb{R}$. Prove that X is a Banach space if and only if every absolutely convergent series converges.

Solution:

$\Longrightarrow$: Suppose X is a Banach space. Then X is complete with respect to the metric $d(x, y) = \|x - y\|$. We wish to show that every absolutely convergent series converges.

Let $\sum x_n \in X$ be an absolutely convergent series. Denote $s_n = \sum_{i=1}^{n} x_n$. Note that since $\sum \|x_n\| < \infty$, there then exists $N \in \mathbb{N}$ such that for any $\varepsilon > 0$, $\sum_{i=N}^{\infty} \|x_n\| < \varepsilon$. Notice that $\|s_n - s_m\| = \left\|\sum_{i=m+1}^{n} x_i\right\| \leq \sum_{i=m+1}^{n} \|x_i\|$ for any n, m. If we choose $n, m \geq N$, then $\|s_n - s_m\| < \varepsilon$, by the bound shown above. Thus s_n is a Cauchy sequence in X, which by assumption means that s_n converges. Therefore, $\sum x_n$ is a convergent sum.

$\Longleftarrow$: Here we assume that every absolutely convergent series converges in X, we then need to show that X is a Banach space. Let x_n be a Cauchy sequence in X, and a subsequence x_{n_k} for $k \geq 1$. Since x_n is Cauchy, we can find a subsequence for which $\left\|x_{n_{k+1}} - x_{n_k}\right\| < \varepsilon$, namely $\left\|x_{n_{k+1}} - x_{n_k}\right\| \leq 2^{-k}$. Note that

$$\sum_{k \geq 1} \left\|x_{n_{k+1}} - x_{n_k}\right\| \leq \sum_{k \geq 1} 2^{-k} = 1.$$

Define a new sequence y_k such that $y_1 = x_{n_1}$, $y_{k+1} = x_{n_{k+1}} - x_{n_k}$. Then by above,

$$\sum_{k \geq 1} \|y_k\| = \left\|x_{n_1}\right\| + \sum_{k \geq 1} \left\|x_{n_{k+1}} - x_{n_k}\right\| \leq \left\|x_{n_1}\right\| + 1.$$

Thus $\sum y_k$ is an absolutely convergent series in X, thus is a convergent series. Since x_n has a convergent subsequence (y_k), then x_n converges in X, thus X is a Banach space.

Problem 1.27

Suppose that (x_n) is a sequence in a compact metric space with the property that every convergent subsequence has the same limit x. Prove that $x_n \to x$ as $n \to \infty$.

Solution:

Suppose False. That is, $x_n \nrightarrow x$ as $n \to \infty$. Then there exists an $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ with $n \geq N \implies d(x_n, x) > \varepsilon$. Define a sequence y_m as the elements of x_n where $m > n$. Then $m > N$, and inherits the property given by $x_n : d(y_m, x) > \varepsilon \; \forall \; N \in \mathbb{N}$.

Note that y_m is a subsequence of x_n, thus by assumption there exists an $N' \in \mathbb{N}$ such that for $m \geq N'$ and $\varepsilon' > 0$, $d(y_m, x) \leq \varepsilon'$. This is then a contradiction, since we cannot choose any $N' \in \mathbb{N}$ with this property, by the inheritance of the requirement above. Thus, $x_n \to x$ as $n \to \infty$.