STAT 31210: Homework 7

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Problem 8.12

Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint, linear operator such that there is a constant $c > 0$ with

$$c\|x\| \leq \|Ax\| \quad \text{for all } x \in \mathcal{H}.$$

Prove that there is a unique solution x of the equation $Ax = y$ for every $y \in \mathcal{H}$.

Solution:

For this proof, we have to show two things: that there is a solution for all $x \in \mathcal{H}$ and that solution is unique. First, we wish to show that range(A) is closed, which we can then apply Theorem 8.18, which will give us that Ax has a solution for y orthogonal to $\ker(A^*)$, which is $\ker(A)$ since A is self-adjoint. To show range(A) is closed, take a Cauchy sequence $Ax_n \in \mathcal{H}$. Then, by the property given of A, we have that

$$\|x_n - x_m\| \leq \frac{1}{c}\|Ax_n - Ax_m\|$$

Since Ax_n is Cauchy, we have that $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. Therefore, x_n is a Cauchy sequence in $\mathcal{H}$. Since $\mathcal{H}$ is complete, then $x_n \to x \in \mathcal{H}$, as $n \to \infty$. Note then that

$$\|x_n - x\| = \frac{\|A\|}{\|A\|}\|x_n - x\| \geq \frac{1}{\|A\|}\|Ax_n - Ax\|$$

The last step is an application of Cauchy Schwartz. The multiplication and division of the norm of A can be done since A is bounded (and assumed not the 0-matrix)[1]. The line above then implies that $\|Ax_n - Ax\| \to 0$ as $n \to \infty$, since it is bounded by $\|x_n - x\|$. Therefore, $x \in$ range(A), so it is closed. By Theorem 8.18, we have that $Ax = y$ as solutions for all y orthogonal to $\ker(A)$, since A is self adjoint. I claim that $\ker(A) = \{0\}$, since if it weren't, then there would exist some $x \in \mathcal{H}$ nonzero such that $Ax = 0$. But,

$$\|x\| \leq c\|Ax\| = 0 \implies \|x\| = 0 \iff x = 0.$$

Therfore, we see a contradiction on x, so $\ker(A) = \{0\}$. Therefore, $Ax = y$ has a solution for all $y \in \mathcal{H}$, since all of $\mathcal{H}$ is orthogonal to the zero set. Finally, we need to show that the solution is unique for all $x \in \mathcal{H}$. Suppose there were $x_1, x_2 \in \mathcal{H}$ such that $Ax_1 = Ax_2 = y$. But then $A(x_1 - x_2) = 0$, which implies that $x_1 - x_2 = 0$, since the kernel of A is the zero set. Therefore, $x_1 = x_2$, so the solution is unique for all $x \in \mathcal{H}$, proving the statement.

[1] Even if we assumed that A could be the zero-matrix, then the solutions of Ax would not be unique, therefore we exclude this case.

Problem 8.13

Prove that an orthogonal set of vectors $\{u_\alpha : \alpha \in \mathcal{A}\}$ in a Hilbert space $\mathcal{H}$ is an orthonormal basis if and only if

$$\sum_{\alpha \in \mathcal{A}} u_\alpha \otimes u_\alpha = \mathbb{I}.$$

Solution: I am most familiar with the notation used by physicists, so instead of $u_\alpha \otimes u_\alpha$, I will use $|u_\alpha\rangle\langle u_\alpha|$. For simplicity, denote A as the operator given.

$\underline{\implies}$:

Here we will assume that $\{u_\alpha\}$ is an orthonormal basis, and I will show the forward implication holds. Let $x \in \mathcal{H}$. Then by since $\{u_\alpha\}$ is an orthonormal subset of $\mathcal{H}$ (it is an orthonormal basis for $\mathcal{H}$ so this should hold), then by Theorem 6.26, we can write x as

$$|x\rangle = \sum_{\alpha \in \mathcal{A}} \langle u_\alpha|x\rangle \, |u_\alpha\rangle$$

By the definition of the orthogonal projection on page 190 in the book, for any $v, y \in \mathcal{H}$, $(|v\rangle\langle v|)\,|y\rangle = \langle v|y\rangle\,|v\rangle$. Applying this, we have that

$$|x\rangle = \sum_{\alpha \in \mathcal{A}} (|u_\alpha\rangle\langle u_\alpha|)\,|x\rangle$$

Therefore, in terms of A, we have that $x = Ax$, implying that $A = \mathbb{I}$. This then shows the forward implication.

$\underline{\impliedby}$:

Suppose false. That is, $\{u_\alpha\}$ is not an orthonormal set (still orthogonal, just $\|u_\alpha\| \neq 1$), yet

$$\sum_{\alpha \in \mathcal{A}} |u_\alpha\rangle\langle u_\alpha| = \mathbb{I}.$$

In terms of matrix language, this would imply that for any $\alpha \in \mathcal{A}$,

$$|u_\alpha\rangle\langle u_\alpha| = |e_\alpha\rangle\langle e_\alpha|\,,$$

where e_α is the standard basis vector. This implies the rank one matrix generated by $|u_\alpha\rangle\langle u_\alpha|$ is the same as the rank one matrix generated by $|e_\alpha\rangle\langle e_\alpha|$. Note that both have only one nonzero element, the index (α, α), since both are orthogonal. This implies that $\|u_\alpha\|^2 = \|e_\alpha\|^2 = 1$, therefore, $\|u_\alpha\|^2 = \langle u_\alpha|u_\alpha\rangle = 1$, which is a contradiction. Therefore, $\{u_\alpha\}$ is an orthonormal basis for $\mathcal{H}$.

Problem 8.14

Suppose that $A, B \in \mathfrak{B}(\mathcal{H})$ satisfy

$$\langle x|Ay\rangle = \langle x|By\rangle \qquad \text{for all } x, y \in \mathcal{H}.$$

Prove that $A = B$. Use a polarization type identity to prove that if $\mathcal{H}$ is a complex Hilbert space and

$$\langle x|Ax\rangle = \langle x|Bx\rangle \qquad \text{for all } x \in \mathcal{H},$$

then $A = B$. What can you say about A and B for real Hilbert spaces?

Solution:

We will show that $A = B$ when $\langle x|Ay\rangle = \langle x|By\rangle$. Note that this implies $\langle x|(A-B)y\rangle = 0$, therefore, $x \perp \text{range}(A-B)$ for all $x \in \mathcal{H}$. Since this is for all $x \in \mathcal{H}$, then it must mean $\text{range}(A-B) = \{0\}$, therefore, $A - B$ is the zero mapping. This implies that $A - B = 0$, so $A = B$.

In using a polarization identity, we can see when expanding out the inner product that

$$\langle x|Ax\rangle = \frac{1}{4}\left[\langle x+Ax|x+Ax\rangle - \langle x-Ax|x-Ax\rangle + i\langle x+Ax|x+Ax\rangle - i\langle x-Ax|x-Ax\rangle\right].$$

Similarly, for B,

$$\langle x|Bx\rangle = \frac{1}{4}\left[\langle x+Bx|x+Bx\rangle - \langle x-Bx|x-Bx\rangle + i\langle x+Bx|x+Bx\rangle - i\langle x-Bx|x-Bx\rangle\right].$$

Since $\langle x|Ax\rangle = \langle x|Bx\rangle$ for all x, we have that

$$\begin{aligned}
&\langle x|Ax\rangle - \langle x|Bx\rangle = 0 \\
\implies &\langle x+Ax|x+Ax\rangle + \langle x-Ax|x-Ax\rangle - \langle x+Bx|x+Bx\rangle - \langle x-Bx|x-Bx\rangle \\
&= i\langle x-Ax|x-Ax\rangle - i\langle x+Ax|x+Ax\rangle - i\langle x+Bx|x+Bx\rangle + i\langle x-Bx|x-Bx\rangle
\end{aligned}$$

Since the right side is complex and the left side is real valued, we have that

$$\langle x+Ax|x+Ax\rangle + \langle x-Ax|x-Ax\rangle - \langle x+Bx|x+Bx\rangle - \langle x-Bx|x-Bx\rangle = 0$$

$$\langle x-Ax|x-Ax\rangle - \langle x+Ax|x+Ax\rangle - \langle x+Bx|x+Bx\rangle + \langle x-Bx|x-Bx\rangle = 0$$

From the first line, we can group terms together to get

$$\langle x+Ax|x+Ax\rangle - \langle x+Bx|x+Bx\rangle + \langle x+Ax|x+Ax\rangle - \langle x-Bx|x-Bx\rangle = 0$$

$$\langle (A-B)x | (A-B)x \rangle + \langle (B-A)x | (B-A)x \rangle = 0$$

Therefore, $\|(A-B)x\| = -\|(B-A)x\|$ for all x. Since we are free to multiply by a negative 1 inside the norm, we have that

$$\|(A-B)x\| = -\|(A-B)x\|$$

Therefore, $(A-B)x = 0$ for all x, meaning $A-B$ is the zero mapping, so $A = B$. This result only hinged on the real part, where the complex part is not necessary. So the A and B for real Hilber spaces must equal as well.

Problem 8.17

Prove that strong convergence implies weak convergence. Also prove that strong and weak convergence are equivalent in a finite-dimensional Hilbert space.

Solution:

Strong $\implies$ weak:

Let $x, y \in \mathcal{H}$, and suppose there exists a sequence x_n which converges to x strongly. Then, $\lim_{n\to\infty} \|x_n - x\|$ = 0. Let $\varepsilon > 0$; there exists an $N \in \mathbb{N}$ such that when $n \geq N \implies \|x_n - x\| < \frac{\varepsilon}{\|y\|}$ for $y \neq 0$. [2] Therefore, when analyzing the definition of the weak topology, we see that

$$|\langle x_n|y\rangle - \langle x|y\rangle| = |\langle x_n - x|y\rangle| \leq \|x_n - x\|\|y\| < \frac{\varepsilon}{\|y\|}\|y\| = \varepsilon.$$

This will hold for any $y \in \mathcal{H}$, therefore, if $x_n \to x$ strongly, then $x_n \to x$ weakly.

Weak $\implies$ strong (in finite dimensions.):

Suppose $|\mathcal{H}| = n$. Since any Hilbert space has an orthonormal basis, let $\{u_j\}_{j=1}^n$ be such basis. Let $\varepsilon > 0$; there exists an $N \in \mathbb{N}$ such that when $k \geq N \implies |\langle x_k|y\rangle - \langle x|y\rangle| < \frac{\varepsilon}{\sqrt{n}}$ for any $y \in \mathcal{H}$. By linearity of the inner product, we have

$$|\langle y|x_k - x\rangle| < \frac{\varepsilon}{\sqrt{n}}.$$

Since this should hold for any $y \in \mathcal{H}$, it should hold then for the bases. Then $|\langle u_j|x_k - x\rangle| < \frac{\varepsilon}{\sqrt{n}}$ for any $u_j \in \{u_j\}_{j=1}^n$. By Theorem 6.26, we have that

$$\|x_k - x\|^2 = \sum_{j=1}^n |\langle u_j|x_k - x\rangle|^2.$$

But since each term is less than $\frac{\varepsilon}{n}$,

$$\|x_k - x\|^2 = \sum_{j=1}^n |\langle u_j|x_k - x\rangle|^2 < \sum_{j=1}^n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Therefore, $\|x_k - x\| < \varepsilon$ for sufficiently large k, implying weak convergence is equivalent to strong convergence in a finite dimensional Hilbert space.

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[2] If $y = 0$, then weak topology is already satisfied, since $\langle x_n|y\rangle - \langle x|y\rangle = 0 < \varepsilon$.

Problem 8.18

Let u_n be a sequence of orthonormal vectors in a Hilbert space. Prove that $u_n \rightharpoonup 0$ weakly.

Solution:

Let $y \in \mathcal{H}$. To show that $u_n \rightharpoonup 0$ weakly, we need to show that $|\langle u_n | y \rangle| \to 0$ as $n \to \infty$. By Bessel's Inequality, we have that

$$\sum_{n=1}^{\infty} |\langle u_n | y \rangle|^2 \leq \|y\|^2$$

Since this is then a converging sum, we have that $|\langle u_n | y \rangle|^2 \to 0$ as $n \to \infty$. This then implies that $|\langle u_n | y \rangle| \to 0$ as $n \to \infty$, showing weak convergence.

Problem 8.20

Let $\mathcal{H}$ be a real Hilbert space and $\varphi \in \mathcal{H}^*$. Define the quadratic functional $f : \mathcal{H} \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}\|x\|^2 - \varphi(x).$$

Prove that there is a unique element $\bar{x} \in \mathcal{H}$ such that

$$f(\bar{x}) = \inf_{x \in \mathcal{H}} f(x).$$

Solution:

Suppose $\bar{x}$ is the value at which the minimum is obtained. By the Riesz Representation Theorem, (Theorem 8.12), there exists a unique vector $x_0 \in \mathcal{H}$ such that $\varphi(x) = \langle x|x_0\rangle$ for all $x \in \mathcal{H}$. Then, since $\bar{x}$ is the minimum of f, we then have, for any $x \in \mathcal{H}$,

$$f(\bar{x}) \leq f(\bar{x} + x) \implies \frac{1}{2}\|\bar{x}\|^2 - \langle \bar{x}|x_0\rangle \leq \frac{1}{2}\|\bar{x} + x\|^2 - \langle \bar{x} + x|x_0\rangle.$$

By rearranging, we have that

$$\frac{1}{2}\|\bar{x}\|^2 \leq \frac{1}{2}\|\bar{x} + x\|^2 - \langle x|x_0\rangle$$

By using the definition of the norm on a Hilbert space, we can expand the norm to get

$$\frac{1}{2}\|\bar{x}\|^2 \leq \frac{1}{2}(\|x\|^2 + 2\langle \bar{x}|x\rangle + \|\bar{x}\|^2) - \langle x|x_0\rangle.$$

Simplifying, we get

$$\langle x_0 - \bar{x}|x\rangle \leq \frac{1}{2}\|x\|^2 \; \forall \, x \in \mathcal{H}.$$

Set $x = x_0 - \bar{x}$. Then,

$$\langle x_0 - \bar{x}|x_0 - \bar{x}\rangle \leq \frac{1}{2}\|x_0 - \bar{x}\| \implies \frac{1}{2}\|x_0 - \bar{x}\| \leq 0$$

Since the norm function is positive, we have that $x_0 - \bar{x} = 0$, therefore, $x_0 = \bar{x}$. Since the function given is quadratic, its minimum is then uniquely attained.