"Flows" Chapter 4 of textbook

A <u>complete flow</u> $\varphi_t(x)$ is a $\overbrace{\text{one-parameter}}^{\text{time } t}$ differentiable mapping $\varphi : \underset{\text{time}}{\mathbb{R}} \times M \to M$ with following properties:

phase space
manifold M

(a) $\varphi_0(x) = x$, ie. φ_0 is the identity

(b) $\varphi_t \circ \varphi_s = \varphi_{t+s}$ ie $\varphi_t \circ \varphi_s(x) = \varphi_t(\varphi_s(x)) = \varphi_{t+s}(x)$

defined for all $x \in M$, $t \in \mathbb{R}$

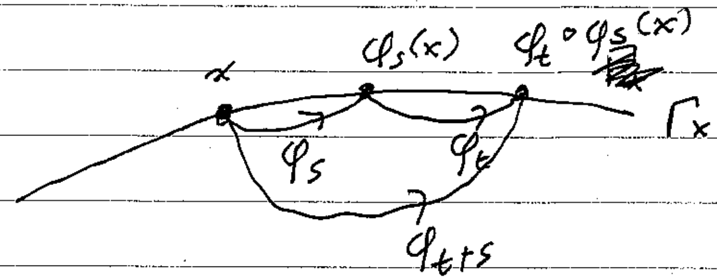
(1) complete: defined for all $t \in \mathbb{R}$, not just an interval

(2) $\varphi_t \circ \varphi_s = \varphi_{t+s}$ is called the group property for flows

(3) $\varphi_t \circ \varphi_{-t} = \varphi_0 = \text{identity}$ $(\varphi_t)^{-1} = \varphi_{-t}$

φ_t is an invertible map

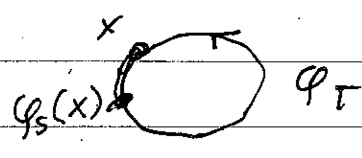
(4) $\varphi_t(x)$ defines a curve Γ_x in M as t varies over $\mathbb{R}$ which is called the <u>orbit</u> or trajectory through x



examples

What if $\varphi_t(x^*) = x^*$ for all $t \in \mathbb{R}$?
then x^* is a fixed-pt. of the flow

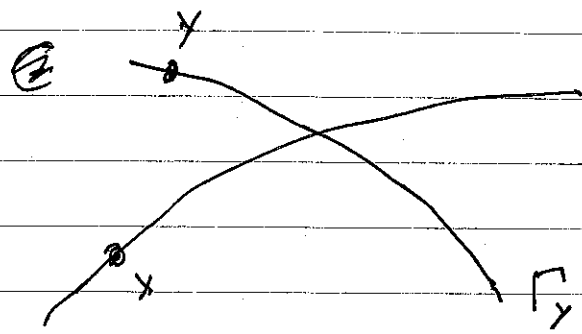
What if $\varphi_T(x) = x$ for some $T > 0$
& $\varphi_t(x) \neq x$ for all $t \in (0, T)$



x is a periodic pt. with period T

Show that $\varphi_s(x)$ is also a periodic pt. with period T: $\varphi_T \circ \varphi_s(x) = \varphi_{T+s}(x)$
$= \varphi_s \circ \varphi_T(x)$
$= \varphi_s(x)$ ✓

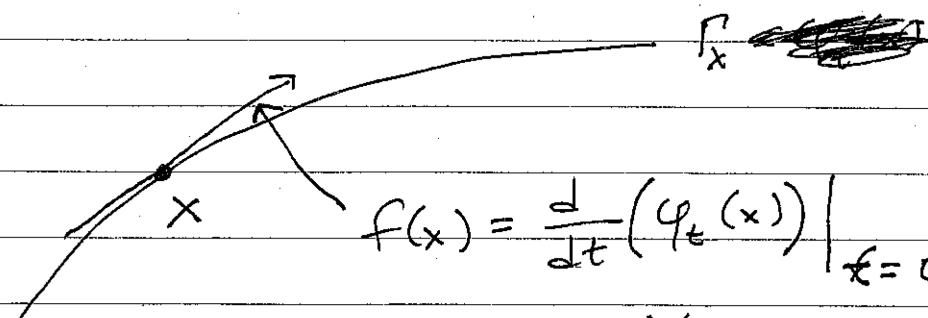
Group properpty $\varphi_t \circ \varphi_s = \varphi_{t+s}$ implies two distinct trajectories cannot cross



to see this <u>can't</u> happen, consider the pt. z where $\varphi_t(y) = \varphi_s(x) = z$

but then $\varphi_{t+r}(y) = \varphi_{s+r}(x)$ for all $r \in \mathbb{R}$ ~~So~~ that contradicts that they are distinct, ~~which is a con~~

Connection to ODEs: Flows are differentiable, ~~we~~ so we can't associate with a flow a vector field $f: M \to \mathbb{R}^n$, $n = \dim(M)$ as follows:



$$f(x) = \frac{d}{dt}\left(\varphi_t(x)\right)\Big|_{t=0}$$

$$= \lim_{\epsilon \to 0}\left(\frac{\varphi_\epsilon(x) - x}{\epsilon}\right)$$

$\varphi_t(x_0)$ solves the following initial value problem

$$(*) \begin{cases} \frac{d}{dt}\left(\varphi_t(x_0)\right) = f(\varphi_t(x_0)) \\ \varphi_0(x_0) = x_0 \end{cases}$$

Check:

$$\frac{d}{dt}\left(\varphi_t(x_0)\right) = \lim_{\epsilon\to 0}\left(\frac{\varphi_{t+\epsilon}(x_0) - \varphi_t(x_0)}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\varphi_\epsilon(\varphi_t(x_0)) - \varphi_t(x_0)}{\epsilon}\right)$$

$$= f(\varphi_t(x_0))$$

<u>Bounded Global existence</u>: If $f: \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz <u>and</u> bounded, then soln. to $(*)$ $\dot{x} = f(x)$, $x(0) = x_0$ exists ~~and~~ for all $t \in \mathbb{R}$.

$(*)$ generates a complete flow

<u>locally Lipschitz</u>: for every $x \in \mathbb{R}^n$ $\exists$ neighborhood N of x s.t. f restricted to N is Lipschitz continuous.

There are vector fields where we don't have global existence, e.g. $\dot{x} = x^2$, $x(0) = x_0 > 0$ solution only exists on an interval $t \in (-\infty, T)$ $\lim_{t \to T} x(t) = +\infty$. (See homework)

However, for such cases we can find an <u>equivalent</u> complete flow by re-parameterizing time

<u>Thm.</u> If $f(x)$ is locally Lipschitz on $\mathbb{R}^n$, then $\dot{x} = f(x)$, $x(0) = x_0$ is <u>equivalent</u> to

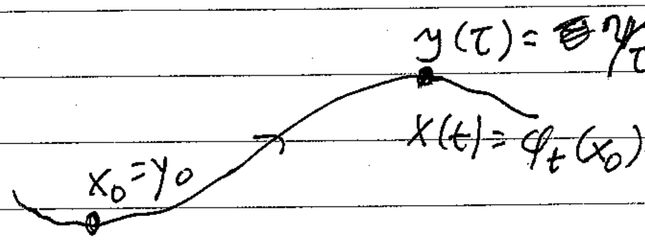
$$\left.\begin{aligned} \frac{dy}{d\tau} &= F(y) = \frac{f(y)}{1 + |f(y)|} \\ y(0) &= x_0 \end{aligned}\right\}$$

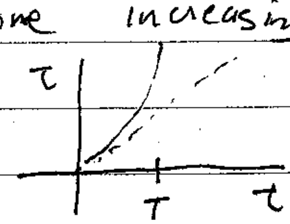
bounded & locally Lipschitz & generates a complete flow $\psi_\tau(x_0)$

Equivalence is by reparameterizing time as follows:

$$\text{let} \quad \tau = \int_0^t 1 + |f(x(s))| \, ds$$

which is strictly monotone increasing





$\tau(t) \Rightarrow t(\tau)$
(invertible)

Return to linearization at a fixed pt. x^*

Simplest solns. $\varphi_t(x^*) = x^*$ for all t

$\dot{x} = f(x)$, f is C^1

$f(x^*) = 0$

Associated linear problem

$$\frac{dy}{dt} = Ay \qquad A = Df(x^*) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & \\ & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=x^*}$$

Let E = (complex) vector space associated with $Df(x^*)$

$= E^u \oplus E^c \oplus E^s$

E^u = "unstable eigenspace" = span $\{u_j, w_j \mid Re(\lambda_j) > 0\}$
E^c = "center eigenspace" = " $\{$ " $Re(\lambda_j) = 0\}$
E^s = "stable eigenspace" = " $\{$ " $Re(\lambda_j) < 0\}$

here $v_j = u_j + i w_j$ is a (generalized) eigenvector associated with λ_j

generalized eigenvectors show up if $Df(x^*)$ is defective:

λ has algebraic multiplicity $k \geq 2$
& geometric multiplicity $l < k$

generalized eigenvector of rank m satisfies

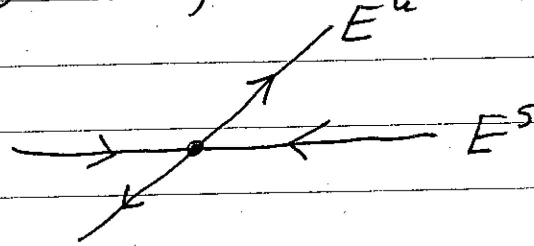
$$(A - \lambda I)^m v = 0$$
$$(A - \lambda I)^{m-1} v \neq 0$$

($m = 1 \Rightarrow$ eigenvector)

Every $n \times n$ matrix has n linearly independent ~~eigen~~ generalized eigenvectors

Each of E^u, E^c, E^s ~~are~~ are invariant under the flow generated by
$$\dot{y} = Df(x^*)\, y$$

If $y_0 \in E^u$, then $\varphi_t(y_0) \in E^u$, etc.



Invariant: a set Λ is invariant under φ_t if $\varphi_t(\Lambda) = \Lambda \quad \forall t \in \mathbb{R}$

$\forall\, y \in \Lambda,\ \varphi_t(y) \in \Lambda$