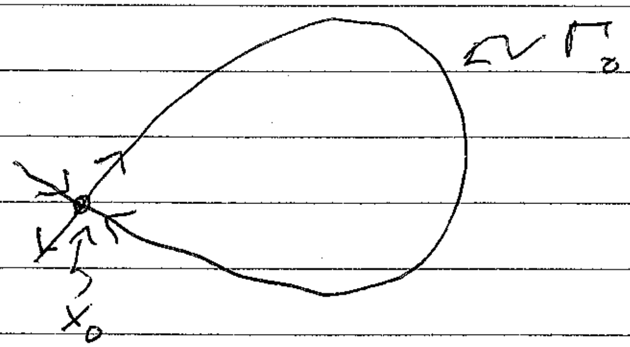
homoclinic bifurcation — a mechanism for creation of a large amplitude, long period limit cycle — this is an example of a "global bifurcation". There is no hint of it from a local analysis near an equilibrium. (Stay tuned to student presentation of the proof.)

$\dot{x} = f(x,\alpha)$, $x \in \mathbb{R}^2$, f is smooth, $\alpha \in \mathbb{R}$ is paramater

@ $\alpha = 0$: $\exists$ a homoclinic orbit Γ_0 to a saddle equilibrium x_0.
Let $\lambda_1(0) < 0 < \lambda_2(0)$ be eigenvalues of $D_x f(x_0; \alpha = 0)$

$\alpha = 0$

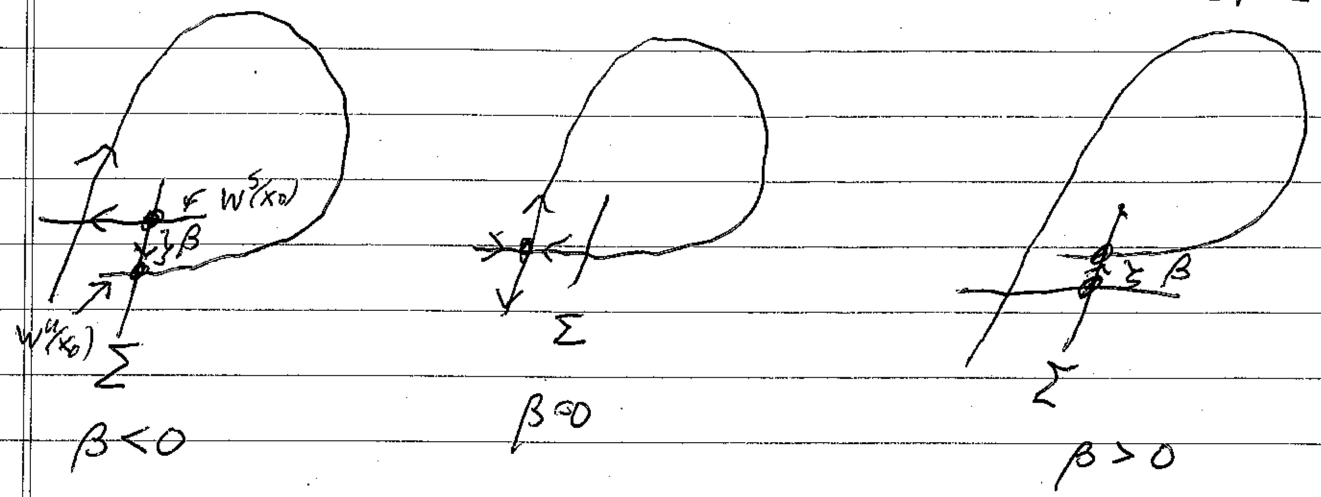


if $x \in \Gamma_0$ then $\alpha(x) = \omega(x) = x_0$

"Genericity Assumptions"

(H1): $\sigma_0 = \lambda_1(0) + \lambda_2(0) \neq 0$

(H2): $\beta'(0) \neq 0$ where $\beta(\alpha) =$ "splitting function"

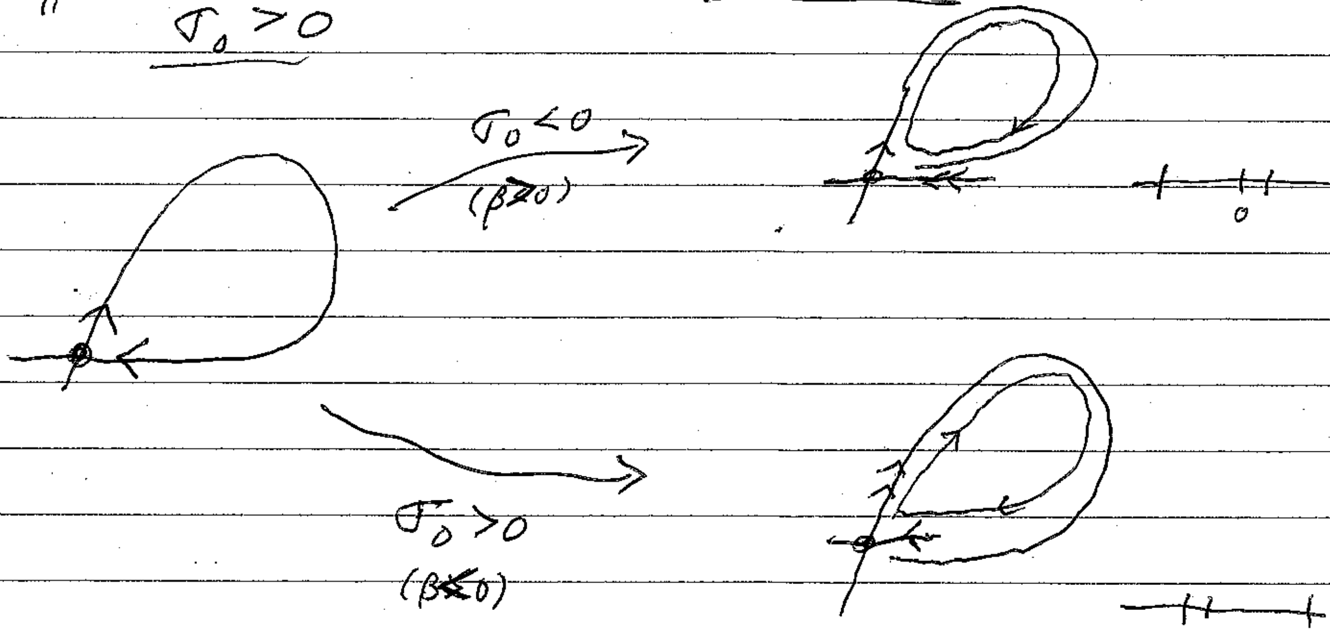


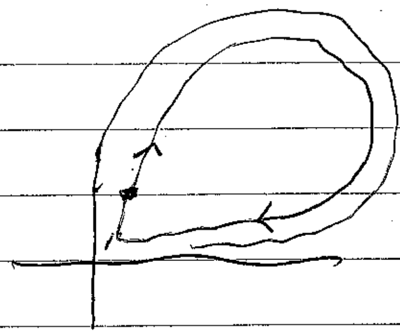
Then, for sufficiently small $|\alpha|$, there exists a neighborhood U_0 of $\Gamma_0 \cup \{x_0\}$ in which a unique limit cycle L_β bifurcates from Γ_0

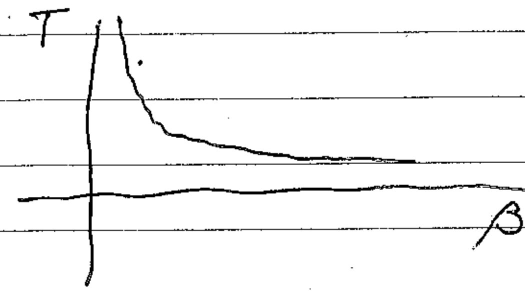
(a) cycle exists & is stable for $\beta>0$ if $\sigma_0<0$

or

(b) " " " " unstable " $\beta>0$ " $\sigma_0>0$

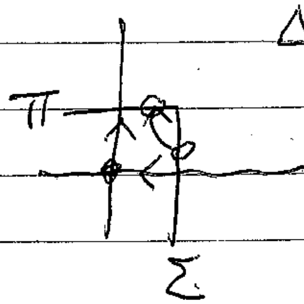




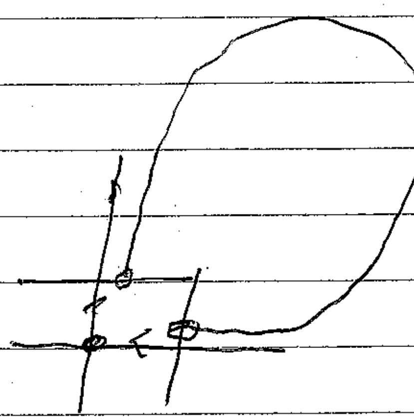


period diverges as $\beta \to 0$.
What do you expect?

Proof is based on constructing a Poincaré return map as a composition of two maps. One is local in phase space to the fixed pt. & the other is local in parameter space to the homoclinic connection



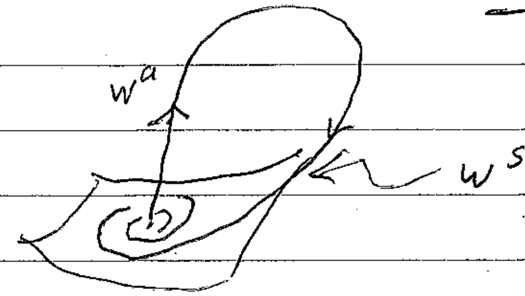
$\Delta : \Sigma^+ \to \Pi^+$
depends on σ_0



$Q : \Pi^+ \to \Sigma^+$
depends on β

$P = Q \circ \Delta$

Shilnikov Bifurcation:

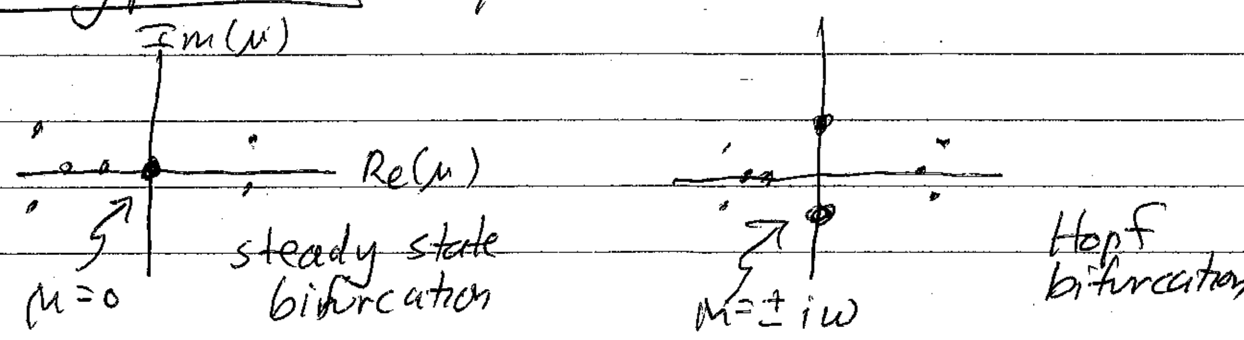


can produce ~~countable~~ infinitely many saddle limit cycles, ie. not a unique one.
This topic is available for presentation.

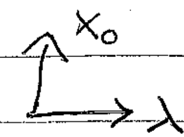
Back to local bifurcations (local to an equilibrium)

- If the equilibrium is hyperbolic, then small changes to parameters do not change $\dim(E^s)$ & $\dim(E^u)$. Equilibrium can also not be destroyed via a small change of parameters (implicit function theorem argument). Hyperbolic equilibria are robust "structurally stable" to perturbations of equations.

- local bifurcations are associated with non-hyperbolic equilibrium.



Steady state bifurcations can lead to changes in # of equilibria:



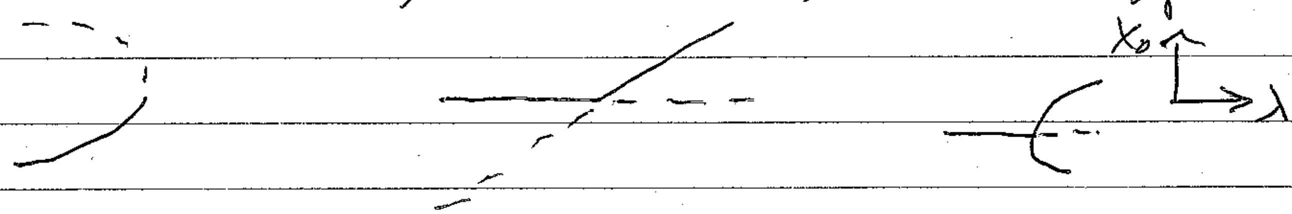
$\dot{x} = f(x; \lambda)$

equilibria satisfy $f(x_0; \lambda) = 0$

at steady state bifurcation $\text{Det}\left[D_x f(x_0; \lambda_0)\right] = 0$

i.e. $\mu = 0$ is an eigenvalue of $D_x F(x_0)$ at $\lambda = \lambda_0$

We will study saddle-node, transcritical, pitchfork

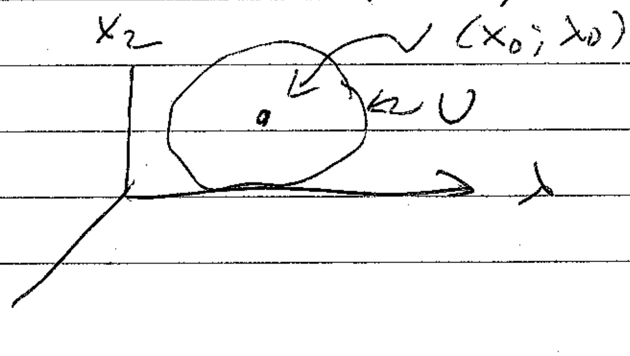


each has a change in # of equilibria in neighborhood of (x_0, λ_0).

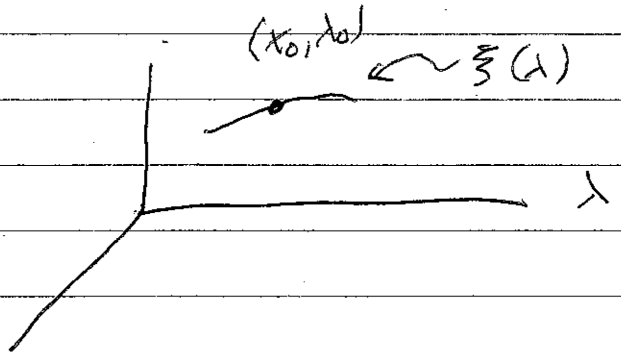
Consider $\dot{x} = f(x; \lambda)$, $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^k$, $F \in C^r(U, \mathbb{R}^n)$, $r \geq 1$

U = open set in $\mathbb{R}^n \times \mathbb{R}^k$ that contains equilibrium (x_0, λ_0)

i.e. $F(x_0; \lambda_0) = 0$



Implicit function theorem: If $D_x F(x_0;\lambda_0)$ is non-singular ($\mathrm{Det} \neq 0$), then there are open sets $V \subset \mathbb{R}^n$ containing x_0, $W \subset \mathbb{R}^k$ containing λ_0, and a unique C^r function $\cancel{\Phi}$ $\xi(\lambda) : W \to V$ for which $F(\xi(\lambda);\lambda) = 0$, $x_0 = \xi(\lambda_0)$

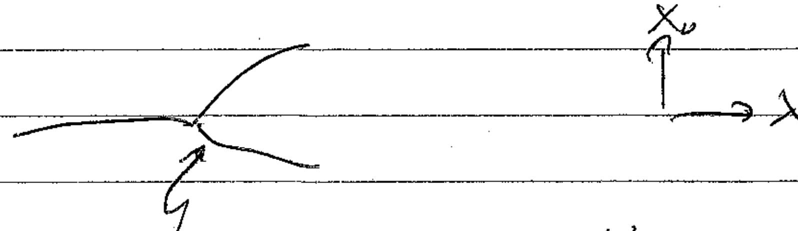


Side: if $D_x f(x_0;\lambda_0)$ is non-singular then we could estimate $\xi(\lambda)$ in neighborhood of $(x_0;\lambda_0)$ as follows:

$$0 = f(x;\lambda) = \cancelto{0}{f(x_0;\lambda_0)} + D_x f(x_0;\lambda_0)(x - x_0) + D_\lambda f(x_0;\lambda_0)(\lambda-\lambda_0) + o(|x-x_0|, |\lambda-\lambda_0|)$$

$$x \approx x_0 - \left(D_x f(x_0,\lambda_0)\right)^{-1}\left(D_\lambda f(x_0;\lambda_0)\right)(x - x_0)$$

Bifurcations:



implicit function theorem <u>must</u> fail here

Simplest steady state bifurcation: $\mu = 0$ has multiplicity one (not repeated).

$$\dot{x} = \underset{\nwarrow \mu=0}{0}\, x + F(x, g(x), h(x))$$
$$\left.\begin{aligned} \dot{y} &= Sy \\ \dot{z} &= Uz \end{aligned}\right\} \text{eigenvalues with } Re(\mu) \neq 0$$

Focus here on 1-d problem of form

$$\left.\begin{aligned} \dot{x} &= f(x;\lambda) \\ \dot{\lambda} &= 0 \end{aligned}\right\} \quad x, \lambda \in \mathbb{R}$$

2-d center manifold after reduction

Assume bifurcation occurs at $x=0, \lambda=0$, so that for $\lambda = 0$, $x=0$ equilibrium is non-hyperbolic

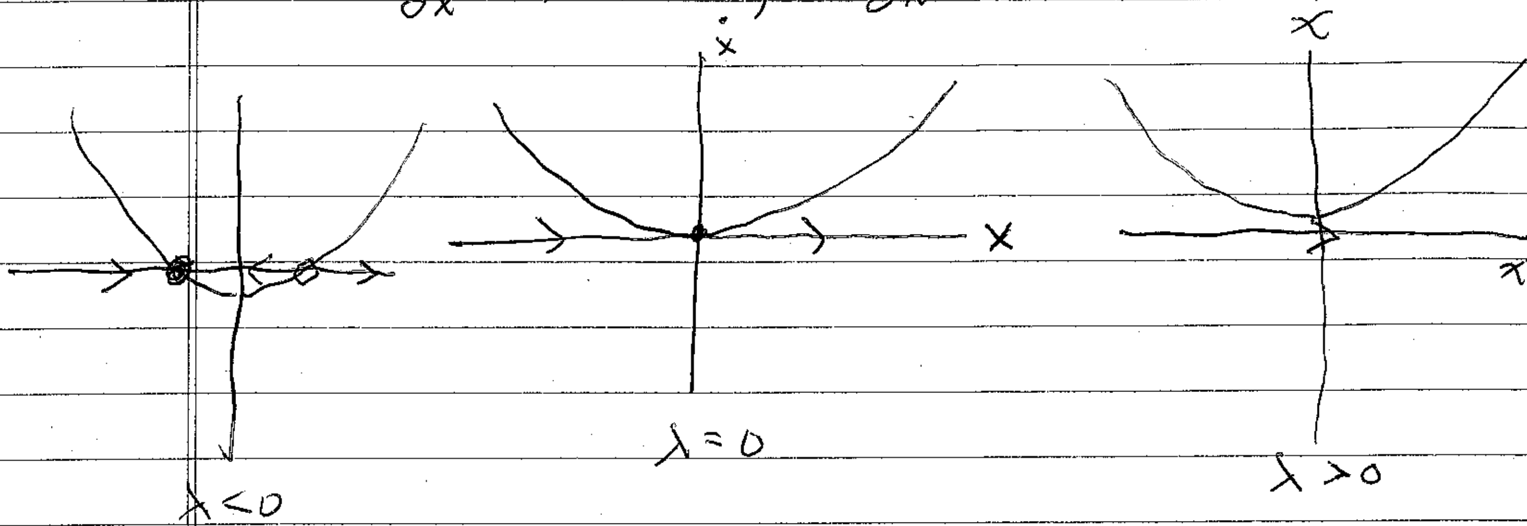
$$\dot{x} = f(x, \lambda)$$

$$\left.\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0 \end{aligned}\right\} \text{ defining conditions for bifurcation to occur at } x=0,\ \lambda=0$$

If $\underbrace{\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0, \quad \frac{\partial f}{\partial \lambda}(0,0) \neq 0}_{}$

non-degeneracy/ ~~conditions~~
transversality } conditions

If $\frac{\partial^2 f}{\partial x^2}(0,0) > 0$, $\frac{\partial f}{\partial \lambda}(0,0) > 0$:

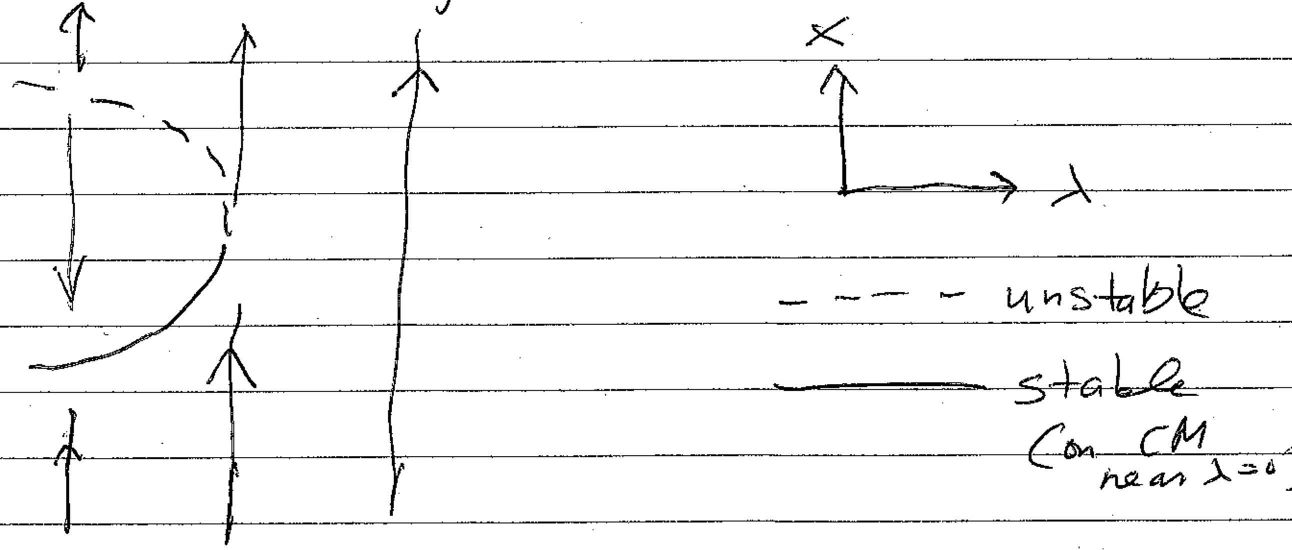


2 equilibria
1 - stable
1 - unstable

1 equilibrium
unstable

no
equilibria

Bifurcation diagram



This is a "saddle-node" or "fold" bifurcation