Lecture ~~4~~
Oct. 5, 2023

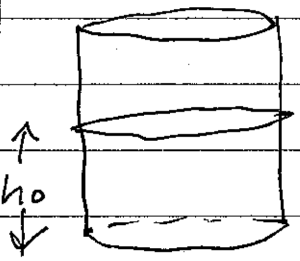
Existence & uniqueness Theorem for the initial value problem (*)

$$(*)\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \qquad \begin{matrix} f: \mathbb{R}^n \to \mathbb{R}^n \\ x_0 \in \mathbb{R}^n \end{matrix}$$

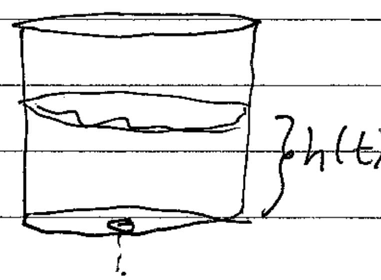
A soln. of (*) exists on some time interval containing $t=0$ provided f is continuous. (Does not need to exist for all time.)

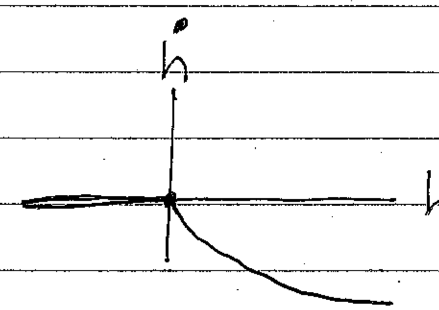
For uniqueness of the soln. we need more that just continuity of f.

<u>Example</u>: showing that lack of uniqueness can be "physical" – not necessarily a sign that the model is flawed.



at $t=0$, punch a hole in the bucket





$$\frac{dh}{dt} = -\sqrt{h} \qquad h \geq 0$$

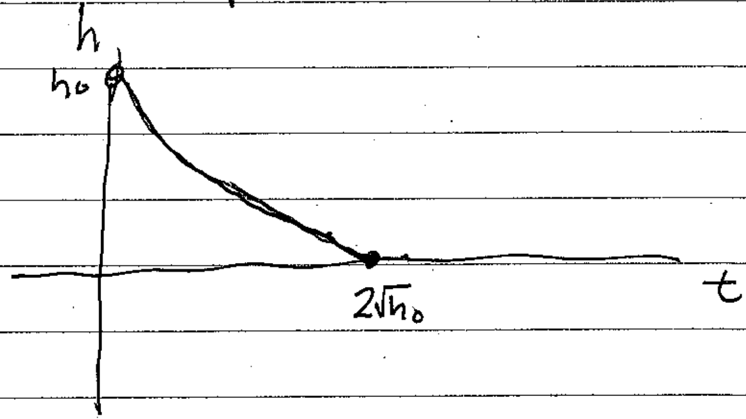
Construct soln. for $h_0 > 0$

$$\int_{h_0}^{h(t)} \frac{d\tilde{h}}{-\sqrt{\tilde{h}}} = \int_0^t d\tilde{t} = t$$

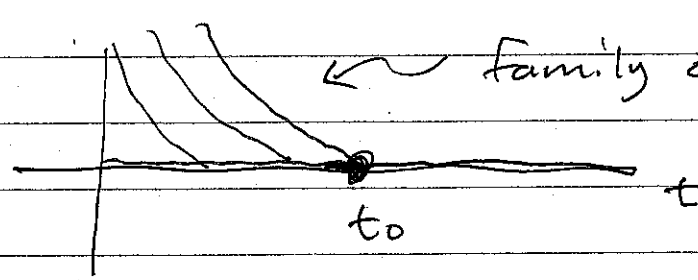
$$-2\sqrt{h(t)} + 2\sqrt{h_0} = t \quad , \qquad t \in [0, t_{max}]$$

$$t_{max} = 2\sqrt{h_0}$$

$$h(t) = \begin{cases} \left(\sqrt{h_0} - \frac{t}{2}\right)^2 & t < 2\sqrt{h_0} \\ 0 & t \geq 2\sqrt{h_0} \end{cases}$$



Notice that the soln. is <u>not</u> unique on a neighborhood of $t = t_0$ $(t > 2\sqrt{h_0})$, where $h(t) = 0$.



family of solns. parameterized by h_0 up to some largest h_0 where bucket empties at t_0

Existence-uniqueness thm. (p. 82, Chapter 3)

$$(*) \begin{cases} \dot{x} = f(x) & f: \mathbb{R}^n \to \mathbb{R}^n \\ x(0) = x_0 \in \mathbb{R}^n \end{cases}$$

Suppose there are $a, b > 0$ s.t. $f: B_b(x_0) \to \mathbb{R}^n$ is Lipschitz with constant K. Then $(*)$ has a unique soln. $x(t)$ for $t \in J = [-a, a]$ provided $a = b/M$, where $M = \max_{x \in B_b(x_0)} |f(x)|$

$B_b(x_0)$ = ball of radius b about x_0
$= \{x : |x - x_0| \leq b\}$

$t \in J = [-a, a]$, $a = b/M$
this ensures that the soln. doesn't escape the ball in that time interval

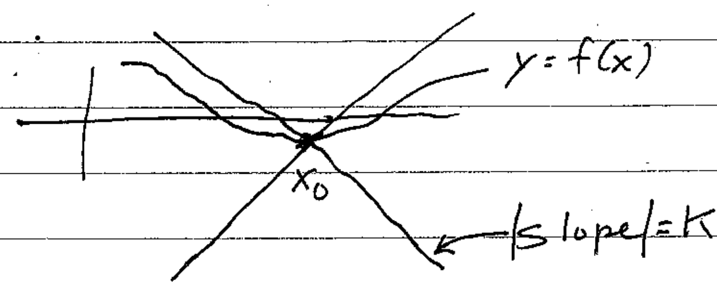
Lipschitz with constant K:

$$|f(x_2) - f(x_1)| \leq K |x_2 - x_1| \qquad \forall x_1, x_2 \in B_b(x_0)$$

ie $$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \leq K \qquad (x_1 \neq x_2)$$

slope of secant line

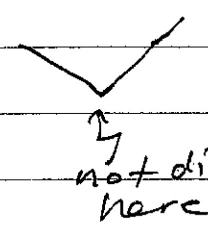
single variable case :



Note that this condition doesn't hold for our bucket problem if $x_0 = 0$, ie. there is no neighborhood of $x_0 = 0$ where we can bound the slope of the secant line since $f'(x)$ diverges as $x \to 0$.

Note: if a function is differentiable (C^1) then it is Lipschitz.
if it is Lipschitz then it is continuous (C^0), but it doesn't have to be differentiable

example $f(x) = |x|$
use $K = 1$

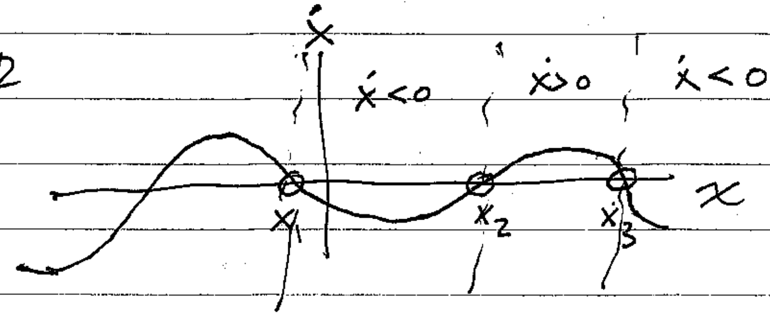


Before mentioning elements of proof
~~(Picard iteration, contraction mapping thm.)~~

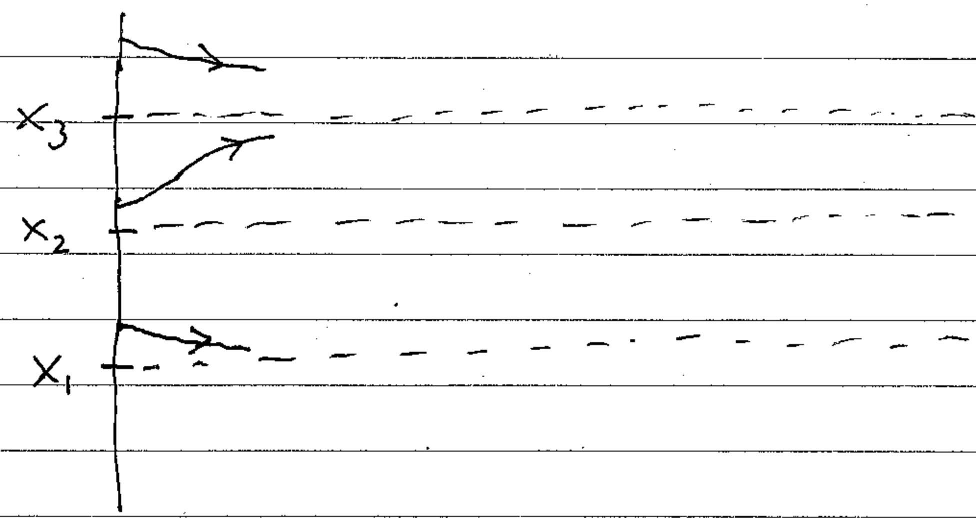
let's give a simple example of exploiting uniqueness to provide a qualitative analysis of solns,

$\dot{x} = f(x) \qquad x \in \mathbb{R}$

graph of $f(x)$ →



Assume $x_0 > 0$ on above interval of x



f is smooth $\Rightarrow$ uniqueness of solns holds

$x(t) \to x^*$, where x^* is one of x_1, x_2, x_3

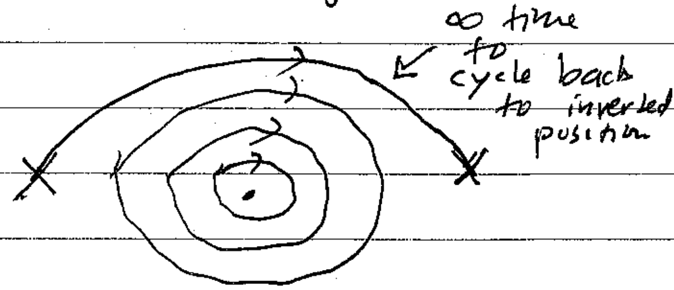
How long does it take to reach equilibrium? (assuming it doesn't start at one)

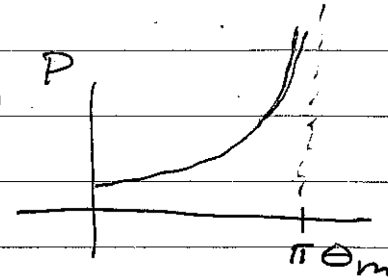
It <u>must</u> take an infinite amount of time $\quad \lim_{t\to\infty} x(t) = x^*, \quad x(t) \neq x^*$ for any t

can't have this

because it violates uniqueness thm.

This is how we know that the period of the pendulum diverges as $\theta_m \to \pi^-$





In this course we almost always assume $f(x)$ is C^1 & uniqueness thm applies.

Elements of proof (You are asked to identify role of K, Lipschitz constant, in proof on homework.)

$$(*) \begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \implies \int_0^t \dot{x}\, dt = \int_0^t f(x)\, dt$$

$$(**) \quad \boxed{x(t) = x_0 + \int_0^t f(x(s))\, ds}$$

Soln. of $(**)$ satisfies $(*)$. We'll use $(**)$, i.e. we're interested in a function $x(t)$ for which $(**)$ holds.

"Picard Iteration" — consider the following map T:

$$Tu = x_0 + \int_0^t f(u(s))\, ds$$

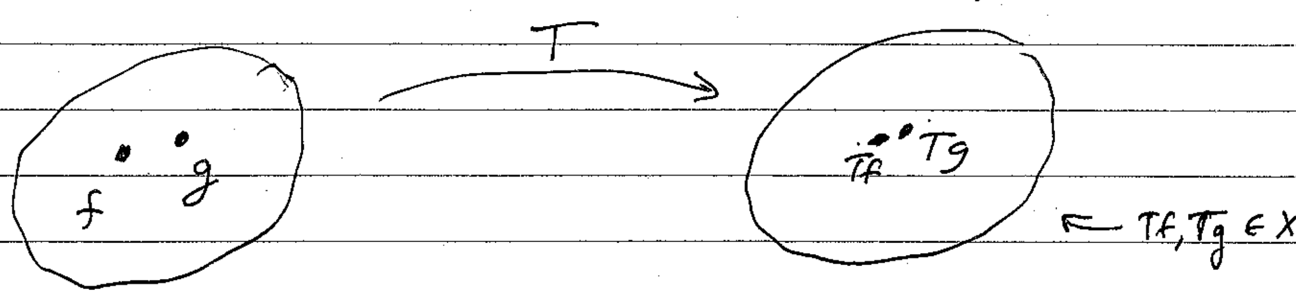
For example, you could start with some guess of the soln. $u_0(t)$ & iterate from there:

$$u_0(t) \xrightarrow[T]{} u_1(t) \xrightarrow[T]{} u_2(t) \longrightarrow \cdots$$

Note that a "fixed-pt." of this map, ~~is~~ ~~Tf~~ $Tx(t) = x(t)$ ~~is our~~ solves $(**)$

$$x(t) = x_0 + \int_0^t f(x(s))\,ds$$

Proof of existence-uniqueness relies on showing that T is a <u>contraction mapping</u> in which case it has a <u>unique</u> fixed pt



$f, g \in X$

distance between Tf & Tg is smaller than distance between f & g

X = "metric space"

Contraction Mapping Thm

Let $T: X \to X$ be a map on a complete metric space X. The map is a contraction if there exists $c<1$ s.t. for all $f, g \in X$,

$$\rho(T(f), T(g)) \leq c\rho(f,g)$$

In this case there is a unique fixed-pt.

We need to have a distance function $\rho(\cdot,\cdot)$ ~~and define~~ for our space X and then make sure there is a $c<1$.

$$X = C^0(J, B_b(x_0)) = \text{set of continuous functions } u(t) \in B_b(x_0) \text{ for } t \in J = [-a, a]$$

$$\rho(u_1(t), u_2(t)) = \sup_{t \in J} |u_1(t) - u_2(t)|$$

sup = supremum = least upper bound

(Note: $Tu(t)$ is continuous & stays in $B_b(x_0)$ via our restriction of $t \in J$)

need to find $c < 1$ s.t.

$$\rho\left(T(u_1(t)), T(u_2(t))\right) \leq c\, \rho\left(u_1(t), u_2(t)\right)$$

$\rho\left(T(u_1(t)), T(u_2(t))\right) = \sup_{t \in J} \left|T(u_1(t)) - T(u_2(t))\right|$

$$\left|T(u_1(t)) - T(u_2(t))\right|$$

$$= \left| \left(x_0 + \int_0^t f(u_1(s))\,ds\right) - \left(x_0 + \int_0^t f(u_2(s))\,ds\right) \right|$$

$$= \left| \int_0^t f(u_1(s))\,ds - \int_0^t f(u_2(s))\,ds \right|$$

$$\leq \int_0^t \underbrace{\left| f(u_1(s)) - f(u_2(s)) \right|}_{\leq K |u_1(s) - u_2(s)|}\,ds$$

~~by Lipschitz~~ assumption

$$\leq K a\, \rho(u_1(t), u_2(t))$$

Let $Ka = c < 1$, $a < 1/K$
& $a < b/M$

Note: this additional restriction on a isn't needed. See proofs in text to eliminate it.

~~note $\rho(T(u_1(t)), T(u_2(t))) = \sup |T(u_1(t)) - T(u_2(t))|$~~

~~$|T(u_1(t)) - T(u_2(t))|$~~ $\leq \rho(T(u_1(t)), T(u_2(t)))$