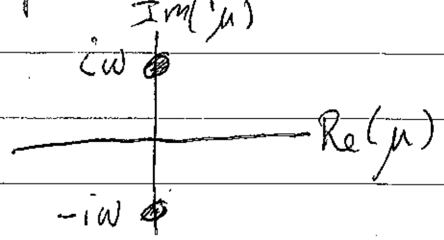
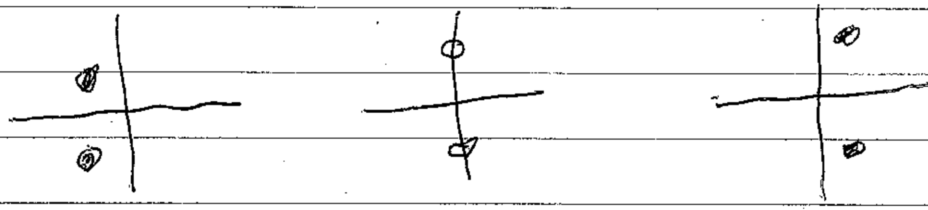
Simple example of Hopf bifurcation in the phase plane



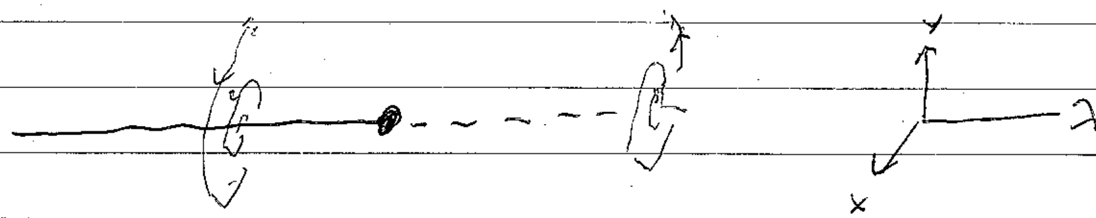
$D_x f$ has purely imaginary eigenvalues at $\lambda = 0$



$\lambda < 0$ | $\lambda = 0$ | $\lambda > 0$

stable spiral

unstable spiral



Note: there is no change in # of equilibria at Hopf bifurcation $\mathrm{Det}(D_x f) = \omega^2 > 0$ & implicit function theorem applies

The "warm-up" example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}}_{\text{eigenvalues } \lambda \pm i\omega} \begin{pmatrix} x \\ y \end{pmatrix} + a(x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix} + b(x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix}$$

Example constructed to be simple in polar coordinates

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad \begin{array}{l} x^2+y^2 = r^2 \\ \dfrac{y}{x} = \tan\theta \end{array}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} (x^2+y^2)x \\ (x^2+y^2)y \end{pmatrix}$$

Eqns are rotationally symmetric

If $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a soln. then so is

$$R_\theta\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$\dot{X} = F(X)$ & $R_\theta F(X) = F(R_\theta(X))$

$\Rightarrow R_\theta \dot{X} = F(R_\theta X)$ for all θ.

Follows since $R_\theta \begin{pmatrix} \mu & -\Omega \\ \Omega & \mu \end{pmatrix} = \begin{pmatrix} \mu & -\Omega \\ \Omega & \mu \end{pmatrix} R_\theta$

& (x^2+y^2) is rotationally invariant function

rotational symmetry means that

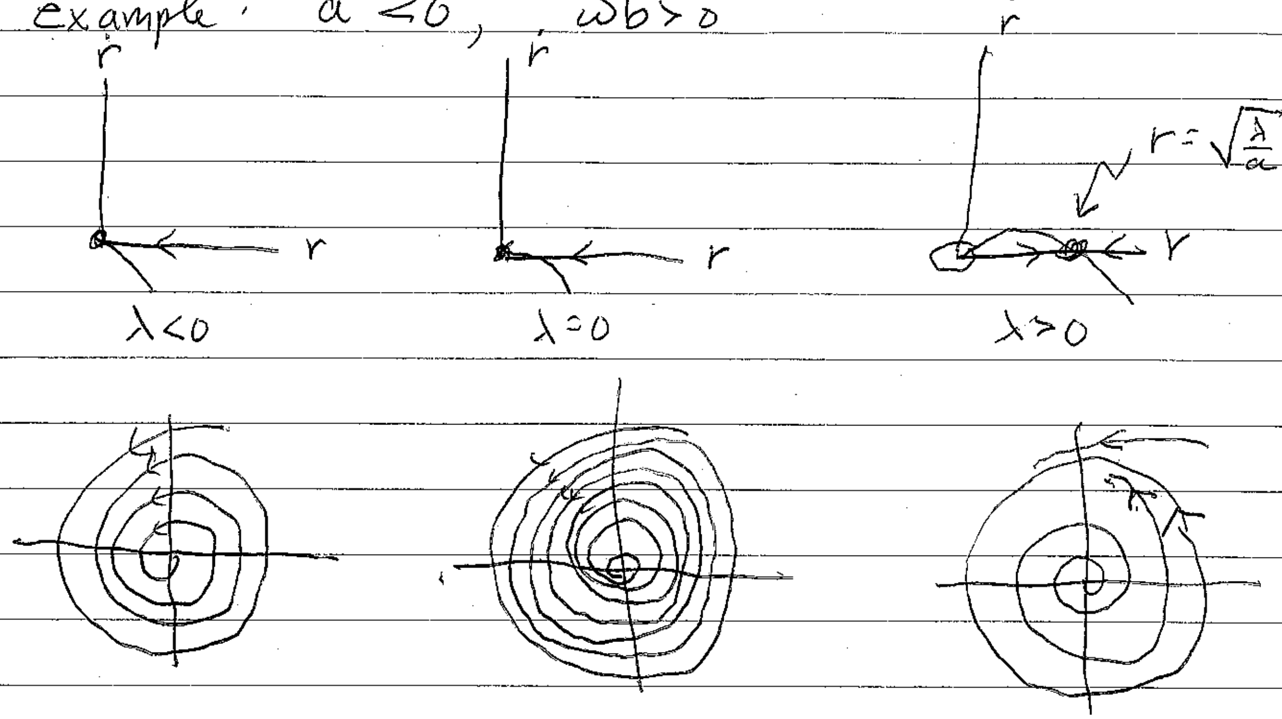
$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} \lambda r + ar^3 \\ \omega + br^2 \end{pmatrix}$$

RHS doesn't depend on θ

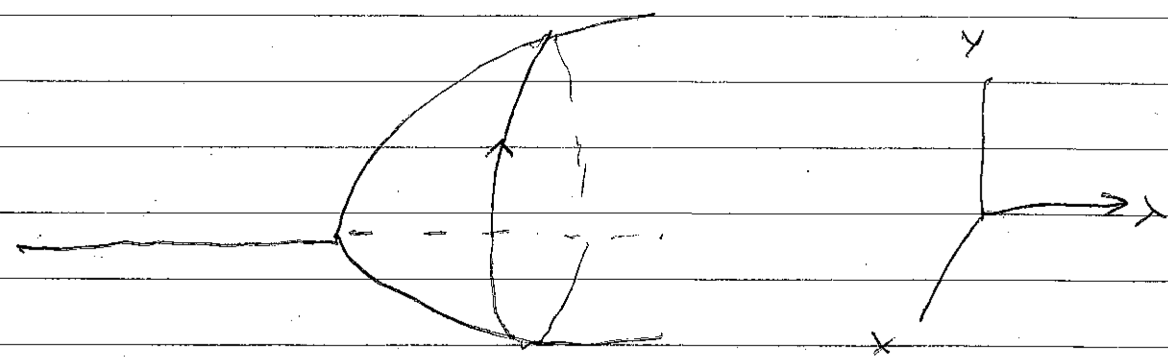
$\dot{r} = \lambda r + a r^3$ ← like pitchfork — $a<0$ supercrit ; $a>0$ subcrit

$\dot{\theta} = \omega + b r^2$

example: $a<0$, $\omega b>0$



Supercritical Hopf bifurcation



produces stable (in CM) small-amplitude limit cycle $\sim\sqrt{\lambda}$ with angular frequency $\sim\omega$ in neighborhood of fixed-pt. & exists for $\lambda>0$ small enough.

Our analysis relied on rotational symmetry of my example. What if we didn't have that?

Idea: perform a "normal form transformation". In the new coordinates, we can ensure (approximate) rotational symmetry.

$$\dot{X} = \underbrace{AX}_{\text{linear}} + \underbrace{g_2(X)}_{\text{quadratic}} + \underbrace{g_3(X)}_{\text{cubic}} + \underbrace{o(X^3)}_{\text{remainder}}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$AX = L = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{DL} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \qquad (\omega = 1)$$

let $X = Y + \underbrace{P_2(Y)}_{O(Y^2)}$, first in a sequence of "near-identity" transformations

Choose $P_2(Y)$ so that $\dot{Y}$ is "simpler" than $\dot{X}$ eqn.

After that, let $Y = Z + \underbrace{P_3(Z)}_{O(Z^3)}$ & choose $P_3(Z)$

so that $\dot{Z}$ eqn is simpler than the $\dot{Y}$ eqn, & so on.

$$X = Y + P_2(Y)$$

LHS: $\dot{X} = AX + g_2(X) + g_3(X) + o(X^3)$

RHS: $\dot{Y} + DP_2(Y)\dot{Y} = (I + DP_2)\dot{Y}$

$$= A(Y + P_2(Y)) + g_2(Y + P_2(Y)) + \cdots$$

$$\dot{Y} = \underbrace{(I + DP_2)^{-1}}_{I - DP_2 + O(Y^2)} \left[AY + AP_2(Y) + g_2(Y) + O(Y^3) \right]$$

$$= AY \underbrace{- DP_2 AY + AP_2(Y) + g_2(Y)}_{O(Y^2)\text{ terms}} + \underbrace{\tilde{g}_3(Y)}_{\text{new } O(Y^3) \text{ terms}} + o(Y^3)$$

We have the freedom to pick $P_2(Y)$; $g_2(Y)$ is given.

How about

$$DP_2 AY - AP_2(Y) = g_2(Y) \ ?$$

Can we always do this?

If yes, then $\dot{Y} = AY + \tilde{g}_3(Y) + o(Y^3)$

no quadratic terms!

Q: Can we choose P_2 so that

$$\underbrace{DP_2 \underbrace{AY}_{L} - \underbrace{A}_{DL} P_2}_{[L, P_2] = DP_2 L - DL P_2 = \text{"Lie Bracket"}} = g_2(Y) \quad , \text{ for any } g_2 ?$$

of vector fields L & P_2

Approach based in linear algebra

Consider space of all 2^{nd} order vector monomials spanned by

$$\underbrace{\begin{pmatrix} y_1^2 \\ 0 \end{pmatrix}}_{=Y_1}, \underbrace{\begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix}}_{=Y_2}, \underbrace{\begin{pmatrix} y_2^2 \\ 0 \end{pmatrix}}_{=Y_3}, \underbrace{\begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}}_{=Y_4}, \underbrace{\begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix}}_{=Y_5}, \underbrace{\begin{pmatrix} 0 \\ y_2^2 \end{pmatrix}}_{=Y_6}$$

$$P_2 = \sum_{j=1}^{6} a_j Y_j = a_1 Y_1 + a_2 Y_2 + \cdots + a_6 Y_6$$

$$g_2 = \sum_{j=1}^{6} b_j Y_j = b_1 Y_1 + \cdots + b_6 Y_6$$

Compute $[L, Y_j] = DY_j L - DL Y_j \quad j = 1, \ldots, 6$

to determine $[L, P_2] = \sum_{j=1}^{6} a_j [L, Y_j]$

$$[L, Y_1] = DY_1\, L - DL\, Y_1 \quad ; \quad Y_1 = \begin{bmatrix} y_1^2 \\ 0 \end{bmatrix}, \; L = \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

$$= \begin{pmatrix} 2y_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2y_1 y_2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix} = -2Y_2 - Y_4$$

$[L, Y_1] = -2Y_2 - Y_4$ $\qquad$ $[L, Y_4] = Y_1 - 2Y_5$

$[L, Y_2] = Y_1 - Y_3 - Y_5$ $\qquad$ $[L, Y_5] = Y_2 + Y_4 - Y_6$

$[L, Y_3] = 2Y_2 - Y_6$ $\qquad$ $[L, Y_6] = Y_3 + 2Y_5$

$$\underbrace{\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 2 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{array}\right]}_{\text{columns from } [L, Y_j]} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}}_{\substack{\text{from} \\ \text{choice} \\ P_2}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix}}_{\substack{\text{given} \\ \text{by} \\ g_2}}$$

Det $\neq 0$ so ~~uni~~ invertible to get unique P_2, given g_2.

Now let $Y = Z + P_3(Z)$ & repeat with

$$\dot{Y} = AY + g_3(Y) + o(Y^3)$$

$$\begin{pmatrix} z_1^3 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1^2 z_2 \\ \end{pmatrix}, \ldots$$

Turns out we <u>cannot</u> remove

$$(z_1^2 + z_2^2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad (z_1^2 + z_2^2)\begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}$$

but those are the terms with rotational symmetry in our toy example problem.

Source of this issue:

$$\left[L, \begin{pmatrix} (x^2+y^2)x \\ (x^2+y^2)y \end{pmatrix} \right] = \left[L, \begin{pmatrix} -(x^2+y^2)y \\ (x^2+y^2)x \end{pmatrix} \right] = 0$$

So resulting matrix cannot be inverted.