STAT 31410: Homework 6

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November 27, 2023

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Problem 1

In a Science paper that appeared in 1999, Klausmeier proposed a simple model to explain properties of vegetation patterns that have been found in drylands of Africa, Australia and the Americas. The nondimensionalized version of his model (in one space dimension) is

$$\frac{\partial w}{\partial t} = p - w - wb^2 + v\frac{\partial w}{\partial x}$$

$$\frac{\partial b}{\partial t} = -mb + wb^2 + \frac{\partial^2 b}{\partial x^2}$$

This problem is concerned with bifurcations of the model with changes in the mean precipitation p, which we treat as the bifurcation parameter.

Problem 1, part a

Does $v > 0$ describe a hill with higher elevation to the left or to the right (where x increases as you move from left to right)? Briefly explain your reasoning.

Solution:

In the real world, I would imagine that water tends to accumulate at the point of least potential. If we are considering a hill with a steady incline (it basically looks like a door stopper), then I would imagine any water placed on that hill would slowly gather at the bottom. Let's investigate the first equation for a moment:

$$\frac{\partial w}{\partial t} = p - w - wb^2 + v\frac{\partial w}{\partial x}$$

For the sake of the hypothetical, let's imagine there's no input of rain into the system ($p = 0$), and the amount of both water and biomass density is constant $\frac{dw}{dt} = \frac{db}{dt} = 0$. We can then take the equation to be

$$\frac{\partial w}{\partial t} = k + v\frac{\partial w}{\partial x}$$

Where k soaks up all the constants into one term. If we imagine the hill in this scenario, then in reality, we would imagine over time the water would accumulate to the bottom of the hill, implying that for any point along the surface of the hill would have a flow of water opposite to the slope of the hill. Therefore, we can say that $\frac{\partial w}{\partial x} < 0$ in this scenario. since $\frac{dw}{dt} = 0$, then $\frac{\partial w}{\partial t} = 0$ (I'm implicitly stating x is not a function of time), therefore,

$$-k = v\frac{\partial w}{\partial x}.$$

Note that $k = -(w + wb^2)$, since $p = 0$. we can then note that $k \leq 0$, making $-k \geq 0$. Since $\frac{\partial w}{\partial x} < 0$, this would then require $v < 0$. Therefore, $v < 0$ for the scenario where the hill has higher elevation to the right. Doing this same analysis, except for the hill having higher elevation on the left, would return $\nu > 0$. Thus, I would say that $\nu > 0$ corresponds to a hill with higher elevation to the left. I have included an image (Figure 1) of this scenario, modeled in Microsoft Paint, for your enjoyment.

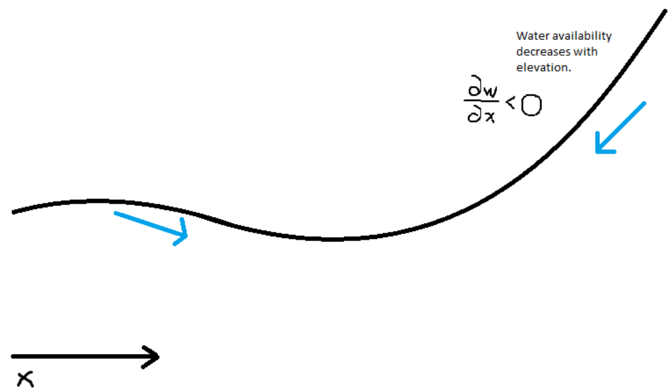

Figure 1: The scenario described in Problem 1, part a. I am imagining this model should resemble real life, so water should flow from high areas to low areas due to gravity. Here the blue arrows represent the flow of the water to lower elevations.

Problem 1, part b

Consider the mortality m to be a fixed positive number, and construct a qualitative bifurcation diagram that shows all spatially uniform, stationary solutions as a function of the mean annual precipitation level p (i.e. solutions with $\frac{\partial b}{\partial t} = \frac{\partial b}{\partial x} = 0$, etc.). For this bifurcation diagram, you should plot the biomass level b as a function of the precipitation level p qualitatively, and identify any precipitation threshold levels of p_c where there is a change in the number of stationary, uniform solutions. Include the stability of the solutions within this framework. Indicating stable solutions as solid lines and unstable ones as dashed lines.

Solution:

The stationary, spatially uniform solutions of our system are given as:

$$p - w - wb^2 = 0, \tag{1}$$

$$-mb + wb^2 = 0 \tag{2}$$

The second equation can be immediately solved for b to get that $b = 0, \frac{m}{w}$. The first solution gives us the "extinction" state, where plugging $b = 0$ into the original equations gives that $\frac{\partial w}{\partial t} = -(w + \frac{m^2}{w})$, which means w will decay exponentially over time, thus it is a stable solution. This is of course ignoring that $w = w(t)$, but I will assume the form will hold once solving the ODE. The second solution, when $b = \frac{m}{w}$, can be plugged into the first equation to get

$$p - w - w\frac{m^2}{w^2} = 0$$

Since w is a function of t, and is unknown, we should solve this polynomial for it. Since we are taking this to be the non-extinction solution ($b \neq 0 \neq w$), then we can multiply by w both sides to get

$$pw - w^2 - m^2 = 0$$

This is then a quadratic in w. Using the quadratic equation gives us

$$w = \frac{p \pm \sqrt{p^2 - 4m^2}}{2}$$

This then gives us two solutions for the system. We are assuming that the quantity under the radical is positive, since if it weren't then we would get an imaginary number for the amount of available water in our system, which makes no sense. Therefore, these two solutions are only valid for $p^2 \geq 4m^2$. The the point $p = 2m$ is the bifurcation point for our system. The positive branch is perfectly fine, which will give us stable fixed points (when plugging this back into the first equation, we get $b = 1$). The negative portion, however, needs to be positive (you can't have negative water in the system). This requires

$$p - \sqrt{p^2 - 4m^2} \geq 0.$$

Solutions under this regime will be unstable.I have included Figure 2 to describe the behavior of the bifurcation. Apologies for the hastily drawn figure and explanation.

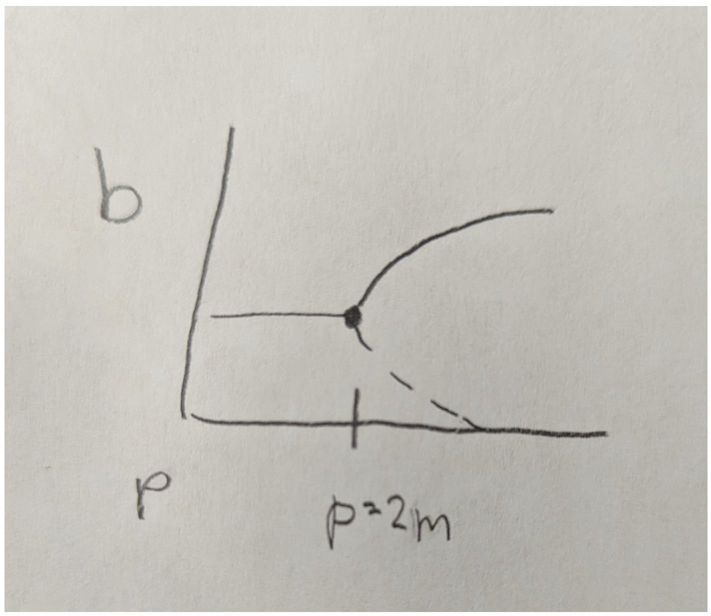

Figure 2: Qualitative Bifurcation plot for b as a function of p. The bifurcation appears when $p = 2m$.

Problem 1, part 3

We can convert the partial differential equations describing interactions between biomass and water to a system of ordinary differential equations in a traveling frame by introducing a variable $z = x - ct$ that will allow us to capture traveling wave solutions. This leads to a first order differential equation for w and a second order differential equation for b. Rewrite the differential equations in the z coordinate as a system of three ordinary differential equations by introducing a new variable $u = db/dz$.

Solution:

When going from $(x, t) \longrightarrow (z)$ via the transformation $z = x - ct$, we need to keep in mind the change in partials. When taking the partial derivatives of w and b, we need to apply the chain rule to each, i.e.,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial z}\frac{\partial z}{\partial t} = -c\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial}{\partial z}.$$

Note the partial with respect to z is just a total derivative. Therefore, the first equation can be rewritten as

$$-c\frac{dw}{dz} = p - w - wb^2 + v\frac{dw}{dz}$$

which, when rearranging yields

$$\frac{dw}{dz} = \frac{1}{v+c}\Big(- p + w + wb^2\Big)$$

This is then our first order ODE for the first equation. Moving into the second equation, we apply the same process to get

$$-c\frac{db}{dz} = -mb + wb^2 + \frac{d^2b}{dz^2}$$

This is a second order ODE, when solving for the second order derivative we get

$$\frac{d^2b}{dz^2} = c\frac{db}{dz} + mb - wb^2$$

If we let $u = db/dz$, we can decouple this into two first order equations. that is,

$$\frac{db}{dz} = u$$

$$\frac{du}{dz} = cu + mb - wb^2$$

Therefore, our initial system is transformed into the following:

$$\frac{dw}{dz} = \frac{1}{v + c}\left(- p + w + wb^2\right)$$

$$\frac{db}{dz} = u$$

$$\frac{du}{dz} = cu + mb - wb^2$$

Problem 1, part 4

The equilibrium solutions of the system of ODEs for $(b(z), u(z), w(z))$ are identical to those determined in your answer to question (2). It is possible to show that the equilibrium solution with the largest value of biomass b can undergo a Hopf bifurcation with a linear frequency ω under certain conditions on p and c. Describe how you would determine the conditions for a Hopf bifurcation and how you would interpret a periodic solution branch born from this bifurcation in the original setting of the vegetation model. What do the values of p and c indicate? How about the frequency ω? Here, I am not necessarily asking you to do the calculations, but for full credit you must carefully and accurately outline how you'd do them, and, importantly, how you would interpret your results, especially in light of what I've told you about these ecosystems.

Solution:

I didn't have enough time to finish this one, I apologize. I would imagine you would take the given parameters we found for p in problem 2 and plug them into our new system. After that, you would take the Jacobian of the system and find the eigenvalues of that Jacobian. With some luck, you would then find a general form for the eigenvalues, and show where they cross the imaginary axis. From there we would get the limit cycle and approximate its frequency. I just didn't have enough time to show this thoroughly.

Problem 5.7.10

The three-dimensional system

$$\dot{x} = y + 2z + (x+z)^2 + xy - y^2,$$

$$\dot{y} = (x+z)^2,$$

$$\dot{z} = -2z - (x-z)^2 + y^2$$

has a nonhyperbolic equilibrium at the origin.

Problem 5.7.10, part a

Find a linear transformation to write this system into the form

$$\dot{x} = Cx + F(x,y,z)$$

$$\dot{y} = Sy + G(x,y,z)$$

$$\dot{z} = Uz + H(x,y,z)$$

Solution:

In order to find this transformation, we need to group the linear and non linear terms together in the system. This grouping will resemble:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + (x+z)^2 + xy - y^2$$

$$\dot{y} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + (x+z)^2$$

$$\dot{z} = \begin{bmatrix} 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - (x+z)^2 + y^2$$

The system is then of the form $\dot{\xi} = A\xi + g(\xi)$, where $g = o(\xi)$ is the nonlinear bits and ξ is some generalized coordinate space. A is then

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

This is not yet in normal form, so to do this, we need to find the mtrix P which has columns of the generalized eigenvectors of A. The normal form will then be $J = P^{-1}AP$. We can see that A has a nonzero kernel, which immediately tells us there is a non-empty stable manifold. Examining the eigenvalues of this matrix, we see it has two distinct values: 0 with multiplicity 2, and -2 with multiplicity 1. This then tells us there are two eigenvectors corresponding to the center manifold and one corresponding to the stable manifold. We can immediately pick out the eigenvector for the stable manifold: $v_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^\intercal$ (I am labelling this as the third eigenvector since the stable manifold's eigenvectors come after the center manifold ones). Getting one eigenvector corresponding to eigenvalues 0 is simple: it is just any scalar multiplication of the first column, so $v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. To find the second, we need to solve the generalized eigenvector problem

$$(A - 0\mathbb{I})^2 v_2 = v_1.$$

This is a relatively simple procedure from here. We then get the following system for v_2:

$$v_{22} - 2v_{23} = 1$$

$$0v_{12} + 0v_{22} + 0v_{23} = 0$$

$$-2v_{23} = 0$$

The third equation immediately tells us $v_{23} = 0$. This can be plugged back into the first equation to get $v_{22} = 1$. Next, both values can be plugged into the second equation to get that v_{12} is a free parameter - we'll set it to zero.

Therefore, our matrix P is then constructed as:

$$P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This matrix will now transform A into normal form.

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Therefore, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $S = \begin{bmatrix} -2 \end{bmatrix}$, $U = \phi$. The system is now

$$\dot{x} = Cx + F(x, y, z)$$

$$\dot{y} = Sy + G(x, y, z)$$

where the $\dot{z}$ has been omitted.

Problem 5.7.10, part b

Find the quadratic approximation for $W^c(0,0,0)$.

Solution:

Here, we restrict the center manifold to be tangent to the center eigenspace at the origin, so W^c will be of the form $\{(x, y, g(x, y))\}$ for some $g(x, y)$ to be determined via power series expansion of x and y $(g(x, y) = \alpha x^2 + \beta xy + \gamma y^2)$. Then when requiring that $z = g(x, y)$ is an invariant manifold (applying (5.35), we get

$$\dot{z} = Dg(x, y)\dot{\mathbf{x}} == \frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}$$

Before we plug in what we have for each expression, I would like to note we are only interested in terms up to quadratic order; that is, any term which we find of higher order should be banished. In particular, the term $(x+z)^2$ will then be compressed as

$$(x+z)^2 = (x + g(x, y))^2 = x^2 + O(3)$$

so we don't have to go through any unnecessary - and potentially erroneous - algebra. Therefore the following steps are justified (apologies for the formatting):

$$\dot{z} = Dg(x, y)\dot{\mathbf{x}} = \frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y} \qquad \text{(Given.)}$$

$$-2g(x, y) - (x + g(x, y))^2 + y^2 = \frac{\partial g}{\partial x}(y + 2g(x, y) + (x + g(x, y))^2 + xy - y^2)$$
$$+ \frac{\partial g}{\partial y}(x + g(x, y))^2 \qquad \text{(Plugging in.)}$$

$$\implies -2\alpha x^2 - 2\beta xy - 2\gamma y^2 - x^2 + y^2 = (2\alpha x + \beta y)(y + 2\alpha x^2 + 2\beta xy + 2\gamma y^2 + x^2 + xy - y^2)$$
$$+ (\beta x + 2\gamma y)(x^2) \qquad \text{(Plugging in, removing } O(3) \text{ terms.)}$$

$$\implies -(2\alpha + 1)x^2 - 2\beta xy + (1 - 2\gamma)y^2 = 2\alpha xy + \beta y \qquad \text{(Simplifying, removing } O(3) \text{ terms.)}$$

$$\implies -(2\alpha + 1)x^2 - 2(\alpha + \beta)xy + (1 - 2\gamma - \beta)y^2 = 0 \qquad \text{(Rearranging.)}$$

When comparing the left and right sides, we get the following:

$$\alpha = -\frac{1}{2}, \quad \beta = -\alpha = \frac{1}{2}, \quad \gamma = \frac{1-\beta}{2} = \frac{1}{4}$$

Therefore, the center manifold of the origin is approximated as:

$$W^c(0,0,0) = \{(x, y, -\frac{1}{2}x^2 + \frac{1}{2}xy + \frac{1}{4}y^2)\}$$

Problem 5.7.10, part c

Obtain the reduced dynamics (5.36) on W^c and use your favorite software package to study it. Is the origin stable or unstable?

Solution:

To obtain the reduced dynamics on the center manifold, we just plug in what we found for $g(x, y)$ into the system in place of z for $\dot{x}$ and $\dot{y}$. We then get

$$\dot{x} = \frac{1}{16}(4x^4 - 8x^3y - 16x^3 + 16x^2y + 4xy^3 + 8xy^2 + 32xy + y^4 - 8y^2 + 16y)$$

$$\dot{y} = \frac{1}{16}(4x^4 - 8x^3y - 16x^3 + 16x^2y + 16x^2 + 4xy^3 + 8xy^2 + y^4)$$

We can then analyze this system qualitatively. Figure 3 demonstrates the reduced dynamics of the system around the origin. We can note that the origin is still an equilibrium point in the reduced dynamics system, but there seems to be no solutions attracted to the origin. So we can say the origin is an unstable equilibrium point.

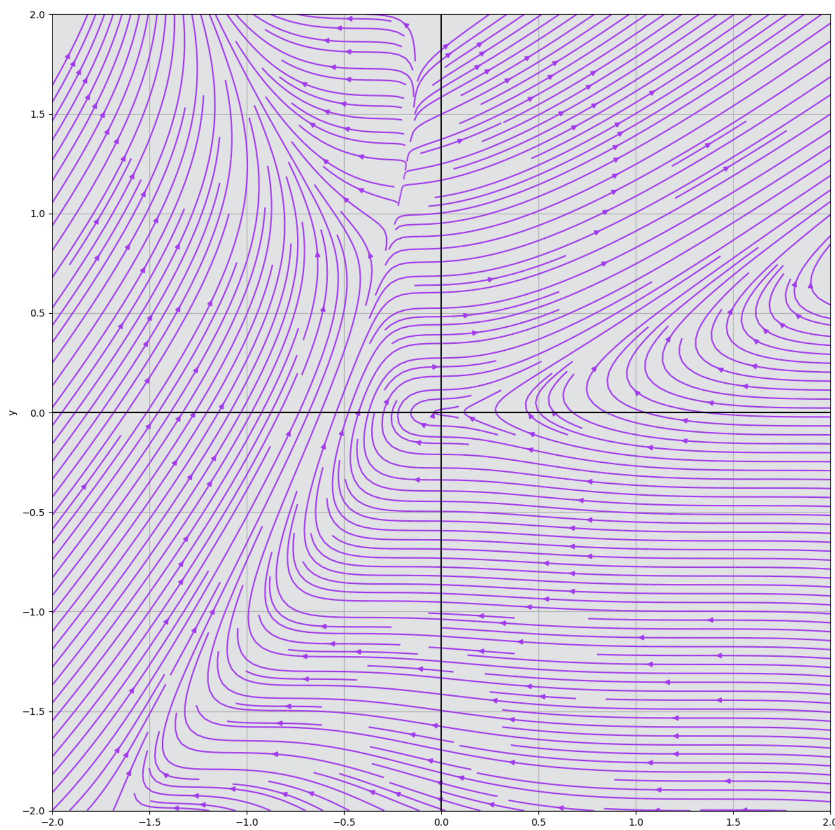

Figure 3: System in Problem 5.7.10. Note the origin seems to be unstable.

Problem 8.15.8

Verify the calculations leading to the normal form (8.43)

$$\dot{x} = -y + (x^2 + y^2)(\alpha x - \beta y) + O(4)$$

$$\dot{y} = x + (x^2 + y^2)(\alpha y + \beta x) + O(4)$$

of the center in $\mathbb{R}^2$. In particular,

- Derive the homological operator L_A.
- Find its action on the standard bases of $\mathbb{H}_2^2$ and $\mathbb{H}_3^2$, and obtain the matrices L.
- Find the eigenvectors and eigenvalues of each L.
- Show that the null, left eigenvectors in $\mathbb{H}_3^2$ are given by

$$v_1 = \begin{pmatrix} (3x^2 + y^2)x \\ (x^2 + 3y^2)y \end{pmatrix}, \quad v_2 = \begin{pmatrix} (x^2 + 3y^2)y \\ -(3x^2 + y^2)x \end{pmatrix}$$

  but that the null, right eigenvectors, together with the range of $L_A(\mathbb{H}_3^2)$, do indeed span $\mathbb{H}_3^2$.

Solution:

- Derive the homological operator L_A.

  I will give the background which build up tot he derivation of the homological operator L_A.

  Suppose $x \in \mathbb{R}^n$ and, without loss of generality, the origin is an equilibrium point of the system $\dot{x} = f(x)$. Expanding f as a taylor series gives us

$$f(x) = \sum_{k=1}^{N} f_k(x) + O(N+1) \tag{8.19}$$

  Note that f_k is a vector of homogeneous polynomials of degree k in x, that is, $f_k(\alpha x) = \alpha^k f(x)$. The space of homogeneous polynomials is the denoted as $\mathbb{H}_k$. By this definition, $\mathbb{H}_k$ is a vector space. A basis for $\mathbb{H}_k$ is then the set of monomials $x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, where $m \in \mathbb{N}^n$ is a vector of size n which has entries only sourced from the natural numbers (the vector 1 norm of m is equal to k). As an extension, let $\mathbb{H}_k^n$ be the space of vectors of homogeneous polynomials on $\mathbb{R}^n$. We can then denote a set of basis vectors for $\mathbb{H}_k^n$ as $p_{m,i} \equiv x^m e_i$, where e_i is the standard basis on $\mathbb{R}^n$. Using this notation, the degree k terms in the power series can be written as

$$f_k = \sum_{i=1}^{n} \sum_{\|m\|_1 = k} f_{m,i} p_{m,i}.$$

With this, we wish to construct the "simplest: vector filed g that is conjugate to f by a near identity transformation. Let ξ represent the new variables, so that $\dot{\xi} = g(\xi)$ and $\xi = h(x) = x + h_2(x) + O(3)$. This then implies $g(h(x)) = Dh(x)f(x)$.

First, consider only the quadratic terms, $h(x) = x + h_2(x)$. We attempt to eliminate f_2 so that $g(\xi) = A\xi + O(3)$. Putting the expansion into the implication above gives the following:

$$\begin{aligned} Ax + Ah_2(x) + O(3) &= Dh(x)f(x) \\ &= f(x) + h_2(x)f(x) \\ &= Ax + f_2(x) + Dh_2(x)Ax + O(3) \end{aligned}$$

Collecting the quadratic terms then gives the homological operator,

$$L_A \equiv Dh_2(x)Ax - Ah_2(x) = -f_2(x).$$

- Find its action on the standard bases of $\mathbb{H}_2^2$ and $\mathbb{H}_3^2$, and obtain the matrices L.

  The standard bases of each will be given in their respective calculations for convenience.

  - $\underline{\mathbb{H}_2^2}$:

    The basis of $\mathbb{H}_2^2$ is given as the 6 monomials of all products of x, y which has degree equal 2. The basis is then given as follows:

$$\left\{\begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}\right\}$$

    Denote these basis vectors as $p_i, i \le 6$. We will then construct L by the action of L_A taken on them. Note that in this circumstance, a linear center on $\mathbb{R}^2$ has a real normal form given as $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$[L_A, p_1] = Dp_1L - DLp_1 = \begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2xy \\ -x^2 \end{pmatrix} = -2p_2 - p_4$$

$$[L_A, p_2] = Dp_2L - DLp_2 = \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} xy \\ 0 \end{pmatrix} = \begin{pmatrix} -y^2 + x^2 \\ -xy \end{pmatrix} = p_1 - p_3 - p_5$$

$$[L_A, p_3] = Dp_3L - DLp_3 = \begin{pmatrix} 0 & 2y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2xy \\ -y^2 \end{pmatrix} = 2p_2 - p_6$$

$$[L_A, p_4] = Dp_4L - DLp_4 = \begin{pmatrix} 0 & 0 \\ 2x & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^2 \\ -2xy \end{pmatrix} = p_1 - 2p_5$$

$$[L_A, p_5] = Dp_5L - DLp_5 = \begin{pmatrix} 0 & 0 \\ y & x \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ xy \end{pmatrix} = \begin{pmatrix} -xy \\ -y^2+x^2 \end{pmatrix} = p_2 - p_6 + p_4$$

$$[L_A, p_6] = Dp_6L - DLp_6 = \begin{pmatrix} 0 & 0 \\ 0 & 2y \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y^2 \end{pmatrix} = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix} = p_3 + 2p_5$$

The matrix L is then constructed as

$$L = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 2 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

- $\underline{\mathbb{H}_3^2}$:
  The basis is then given as:

$$\left\{\begin{pmatrix} x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2y \\ 0 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 \end{pmatrix}\right\}$$

Label these then as $p_i, i \le 8$. We can then repeat the calculations above, except there are even more.

$$[L_A, p_1] = Dp_1L - DLp_1 = \begin{pmatrix} 3x^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x32 \\ 0 \end{pmatrix} = \begin{pmatrix} -3x^2y \\ -x^3 \end{pmatrix} = -3p_2 - p_5$$

$$[L_A, p_2] = Dp_2L - DLp_2 = \begin{pmatrix} 2xy & x^2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^2y \\ 0 \end{pmatrix} = \begin{pmatrix} -2xy^2 + x^3 \\ -x^2y \end{pmatrix} = -2p_3 + p_1 - p_6$$

$$[L_A, p_3] = Dp_3L - DLp_3 = \begin{pmatrix} y^2 & 2xy \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} xy^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -y^3 + 2x^2y \\ -xy^2 \end{pmatrix} = -p_4 + 2p_2 - p_7$$

$$[L_A, p_3] = Dp_4L - DLp_4 = \begin{pmatrix} 0 & 3y^2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y^3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3y^2x \\ -y^3 \end{pmatrix} = 3p_3 - p_8$$

$$[L_A, p_5] = Dp_5L - DLp_5 = \begin{pmatrix} 0 & 0 \\ 3x^2 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^3 \\ -3x^2y \end{pmatrix} = p_1 - 3p_6$$

$$[L_A, p_6] = Dp_6L - DLp_6 = \begin{pmatrix} 0 & 0 \\ 2xy & x^2 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^2y \end{pmatrix} = \begin{pmatrix} x^2y \\ -2xy^2 + x^3 \end{pmatrix} = p_2 - 2p_7 + p_5$$

$$[L_A, p_7] = Dp_7L - DLp_7 = \begin{pmatrix} 0 & 0 \\ y^2 & 2xy \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ xy^2 \end{pmatrix} = \begin{pmatrix} xy^2 \\ -y^3 + 2xy^2 \end{pmatrix} = p_3 - p_8 + 2p_6$$

$$[L_A, p_8] = Dp_8L - DLp_8 = \begin{pmatrix} 0 & 0 \\ 0 & 3y^2 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y^3 \end{pmatrix} = \begin{pmatrix} y^3 \\ 3y^2x \end{pmatrix} = p_4 + 3p_7$$

The matrix L is then

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -3 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

- Find the eigenvectors and eigenvalues of each L.

  Hopefully I don't have to calculate these by hand, and I can just use numpy to calculate these. I will bullet the results similarly as above.

  - $\underline{\mathbb{H}_2^2}$: The eigenvalues and eigenvectors are given below.

Eigenvalue (λ)	Eigenvector ($\mathbf{v}$)
$-3i$	$\begin{bmatrix} -i & -2 & i & -1 & 2i & 1 \end{bmatrix}$
$3i$	$\begin{bmatrix} i & -2 & -i & -1 & -2i & 1 \end{bmatrix}$
i	$\begin{bmatrix} 0 & i & -1 & -i & 1 & 0 \end{bmatrix}$
i	$\begin{bmatrix} -i & 0 & -i & 1 & 0 & 1 \end{bmatrix}$
$-i$	$\begin{bmatrix} 0 & -i & -1 & i & 1 & 0 \end{bmatrix}$
$-i$	$\begin{bmatrix} i & 0 & i & 1 & 0 & 1 \end{bmatrix}$

- $\underline{\mathbb{H}_3^2}$:

  The eigenvalues and eigenvectors are given below. They do not look as good as the ones above, notably.

Eigenvalue (λ)	Eigenvector ($\mathbf{v}$)
0	$\begin{bmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix}$
0	$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$
$-4i$	$\begin{bmatrix} 1 & -3i & -3 & i & -i & -3 & 3i & 1 \end{bmatrix}$
$4i$	$\begin{bmatrix} 1 & 3i & -3 & -i & i & -3 & -3i & 1 \end{bmatrix}$
$-2i$	$\begin{bmatrix} 0 & 1 & -2i & -1 & -1 & 2i & 1 & 0 \end{bmatrix}$
$-2i$	$\begin{bmatrix} -1 & 0 & -3 & 2i & 2i & 3 & 0 & 1 \end{bmatrix}$
$2i$	$\begin{bmatrix} 0 & 1 & 2i & -1 & -1 & -2i & 1 & 0 \end{bmatrix}$
$2i$	$\begin{bmatrix} -1 & 0 & -3 & -2i & -2i & 3 & 0 & 1 \end{bmatrix}$

- Show that the null, left eigenvectors in $\mathbb{H}_3^2$ are given by

$$v_1 = \begin{pmatrix} (3x^2 + y^2)x \\ (x^2 + 3y^2)y \end{pmatrix}, \quad v_2 = \begin{pmatrix} (x^2 + 3y^2)y \\ -(3x^2 + y^2)x \end{pmatrix}$$

but that the null, right eigenvectors, together with the range of $L_A(\mathbb{H}_3^2)$, do indeed span $\mathbb{H}_3^2$.

Taking the nullspace of $L^\intercal$ for $\mathbb{H}_3^2$, we see that it is spanned by the vectors

$$\ker(L(\mathbb{H}_3^2)^\intercal) = \mathrm{span}\Big((0, -1, 0, -3, 3, 0, 1, 0)^\intercal, (1, 0, 1/3, 0, 0, 1/3, 0, 1)^\intercal\Big)$$

These are written in the basis vectors $p_i, i \leq 8$. Retrieving what these equate to gives

$$v_1 = \begin{pmatrix} (x^2 + 1/3y^2)x \\ (1/3x^2 + y^2)y \end{pmatrix}, \quad v_2 = \begin{pmatrix} -(x^2 + 3y^2)y \\ (3x^2 + y^2)x \end{pmatrix}$$

Since any scaling of these vectors will be in the kernel of $L(\mathbb{H}_3^2)^\intercal$, we can scale the first one by 3 and the second by -1 to get back the ones given.

To show that the right eigenvectors of $L_A(\mathbb{H}_3^2)$ with eigenvalue 0, along with the image of $L(\mathbb{H}_3^2)$, span the entirety of $\mathbb{H}_3^2$, we can create a new matrix M with columns containing them. If M has full rank (i.e., $\mathrm{rank}(M) = 8$), then it spans the entirety of $\mathbb{H}_3^2$. As to avoid showing this by hand, I wrote a simple program to demonstrate this. If you run it, it will produce the desired results.

```
import sympy as sm
import numpy as np

#Creating Matrix \H_3^2
h2 = sm.Matrix([[0, 1, 0, 0, 1, 0, 0, 0],
              [-3, 0, 2, 0, 0, 1, 0, 0],
              [0, -2, 0, 3, 0, 0, 1, 0],
              [0, 0, -1, 0, 0, 0, 0, 1],
              [-1, 0, 0, 0, 0, 1, 0, 0],
              [0, -1, 0, 0, -3, 0, 2, 0],
              [0, 0, -1, 0, 0, -2, 0, 3],
              [0, 0, 0, -1, 0, 0, -1, 0]])

#Extracting eigenvectors and image
evects = (h2.T).eigenvects()
colsp = h2.columnspace()

#Populating M
M = np.zeros((8, 8))
for i, vect in enumerate(colsp):
    M[i, :] = np.array(vect.T)[0]

M[6, :] = np.array(evects[0][2][0].T)[0]
M[7, :] = np.array(evects[0][2][1].T)[0]

#Checking rank of M
rank = np.linalg.matrix_rank(M)
if (rank, rank) == h2.shape:
    print(f'Matrix rank of M: {rank}')
else:
    print("Bad, wrong.")
```

Problem 8.15.13

Consider the system

$$\dot{x} = \mu x - y + ay^2 + x^3$$

$$\dot{y} = x + \mu y + xy^2 + y^2$$

Problem 8.15.13, part a

Determine $\alpha(a)$ using (8.59).

Solution:

(8.59) gives us the following formula for α:

$$\alpha = \frac{1}{16}(p_{xxx} + p_{xyy} + q_{xxy} + q_{yyy}) + \frac{1}{16}(q_{xy}(q_{xx} + q_{yy}) - p_{xy}(p_{xx} + p_{yy}) + p_{xx}q_{xx} - p_{yy}q_{yy})$$

where the indices are given as partials of the functions p, q, evaluated at the origin. Note that I have already replaced $\omega = 1$. By the form of the problem, we have $p = \mu x + ay^2 + x^3$ and $q = \mu y + xy^2 + y^2$. The rest of this part is just an exercise in taking partial derivatives.

$$p_{xxx} = \frac{\partial^3}{\partial x^3}\left(\mu x + ay^2 + x^3\right)|_{\mathbf{0}} = 6 \qquad p_{xy} = \frac{\partial^2}{\partial x \partial y}\left(\mu x + ay^2 + x^3\right)|_{\mathbf{0}} = 0$$

$$p_{xx} = \frac{\partial^2}{\partial x^2}\left(\mu x + ay^2 + x^3\right)|_{\mathbf{0}} = 6x|_{\mathbf{0}} = 0 \qquad p_{yy} = \frac{\partial^2}{\partial y^2}\left(\mu x + ay^2 + x^3\right)|_{\mathbf{0}} = 2a$$

$$p_{xyy} = \frac{\partial^3}{\partial x \partial y^2}\left(\mu x + ay^2 + x^3\right)|_{\mathbf{0}} = 0 \qquad q_{yy} = \frac{\partial^2}{\partial y^2}\left(\mu y + xy^2 + y^2\right)|_{\mathbf{0}} = 2x + 2|_{\mathbf{0}} = 2$$

$$q_{xxy} = \frac{\partial^3}{\partial x^2 \partial y}\left(\mu y + xy^2 + y^2\right)|_{\mathbf{0}} = 0 \qquad q_{xy} = \frac{\partial^2}{\partial x \partial y}\left(\mu y + xy^2 + y^2\right)|_{\mathbf{0}} = 0$$

$$q_{yyy} = \frac{\partial^3}{\partial y^3}\left(\mu y + xy^2 + y^2\right)|_{\mathbf{0}} = 0$$

Note some evaluations were omitted since their value would be multiplied by zero. We then get the following value for $\alpha(a)$:

$$\alpha(a) = \frac{2a + 3}{8}$$

Problem 8.15.13, part b

Find the set on which this system has an Andronov-Hopf bifurcation. Is the bifurcation supercritical or subcritical?

Solution:

Theorem 8.21 (Andronov-Hopf bifurcation) gives us a restriction on which the system permits a limit cycle. For any parameters that satisfy α Re $(\lambda) < 0$, then the limit cycle will exist. Based on our findings, the parameters a, μ should satisfy

$$\frac{2a+3}{8}(\mu) < 0$$

which means $2a+3$ and μ should have different signs. This then breaks down into two cases: where $\mu < 0$ and where $\mu > 0$. • If $\mu < 0$, then $2a+3 > 0$, so $a > -3/2$. • If $\mu > 0$, then $2a+3 < 0$, so $a < -3/2$. Therefore, the set of parameters on which a limit cycle would exist is $\{(a, \mu) : (\mu > 0, a < -3/2) \text{ or } (\mu < 0, a > -3/2)\}$. The system could then be supercritical or subcritical, depending if $a < -3/2$ (supercritical) or $a > -3/2$ (subcritical). Figure 4 shows the system for various μ, under the subcritical regime. In this case, limit cycles should exist when $\mu < 0$. The code that generated these plots will follow.

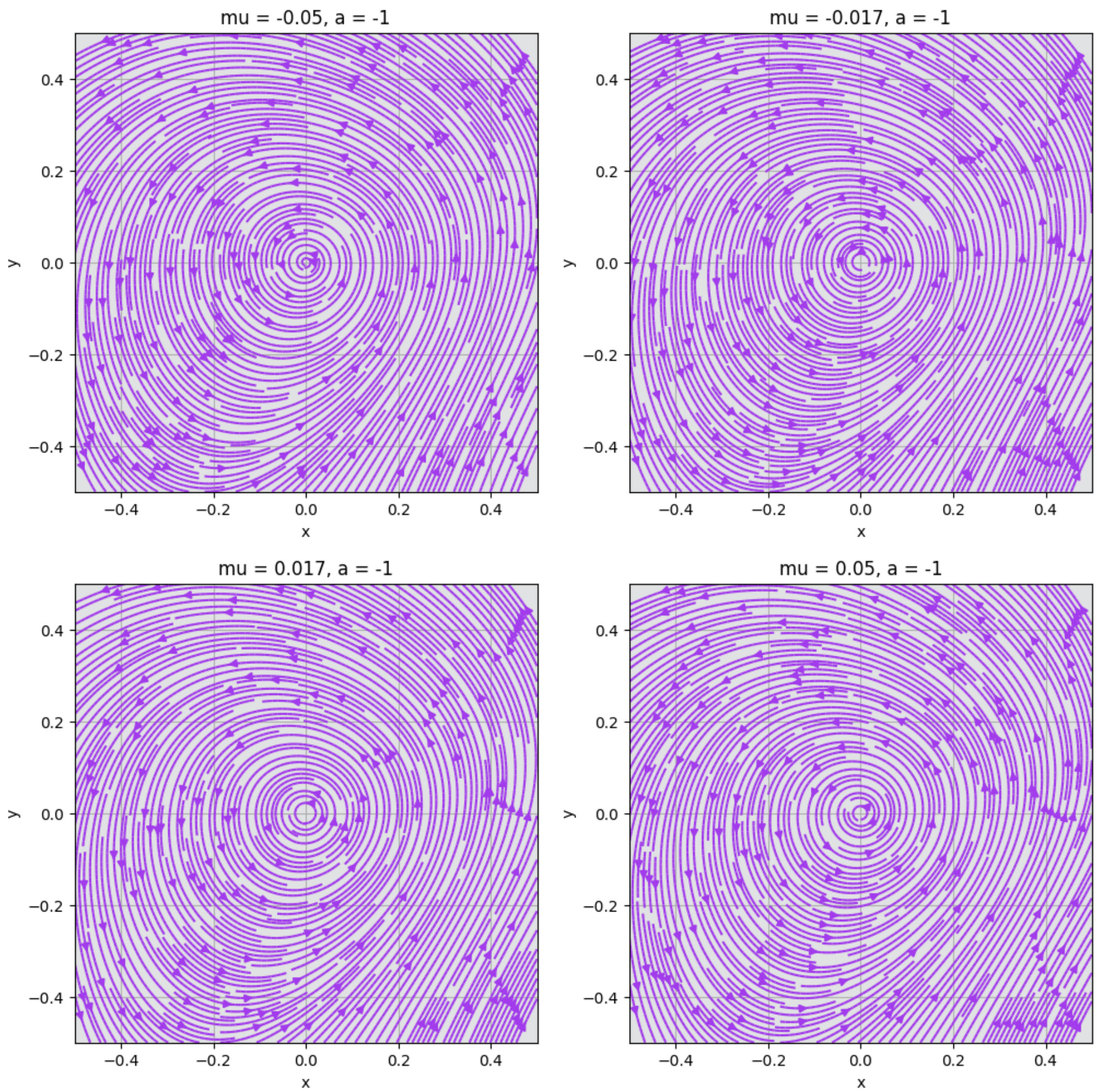

Figure 4: The system in 8.15.13 for small μ 's crossing zero. Note the limit cycle only exists for positive μ, since $\alpha > 0$.

```python
import matplotlib.pyplot as plt
import numpy as np

fig, axs = plt.subplots(2, 2, figsize =(12, 12))
axs = axs.flat

bound = 0.5
#Vector field
xvect, yvect= np.meshgrid(np.linspace(-bound, bound, 40),
                np.linspace(-bound, bound, 40))

a = -1
mus = np.linspace(-0.05, 0.05, 4)
for i, mu in enumerate(mus):
    #Update vector field
    u = mu * xvect - yvect + a*yvect**2 + xvect**3
    v = xvect + mu*yvect + xvect*yvect**2 + yvect**2

    #Plotting stream plot
    axs[i].streamplot(xvect, yvect, u, v, density = 3, linewidth = None, color='#A23BEC')
    axs[i].grid(True)
    axs[i].set_facecolor("#e1e2e3")
    axs[i].set_xlabel("x")
    axs[i].set_ylabel("y")
    axs[i].set_xlim(-bound, bound)
    axs[i].set_ylim(-bound, bound)
    axs[i].set_title(f'mu = {round(mu, 3)}, a = {a}')
```

Problem 8.15.13, part c

Investigate numerically the behavior for values of a such that $\alpha > 0$, $\alpha < 0$, and $\alpha = 0$.

Solution:

Figure 5 gives us a image of the system for changing $a's$ with fixed $\mu = 0.05$. In this instance, $\alpha < 0$, so the limit cycle is supercritical, thus stable.

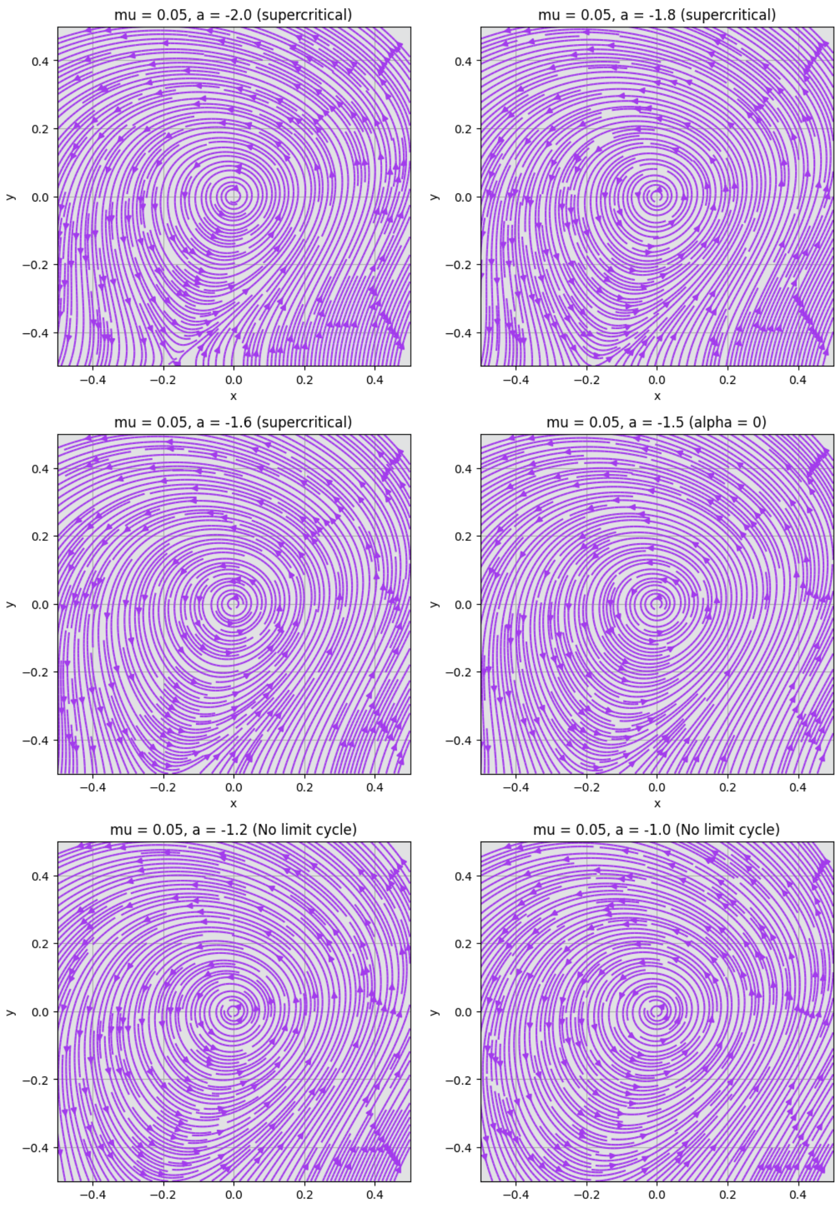

Figure 5: The system in Problem 8.15.13 for changing a values. The values which determine the sign of α , thus existence of the limit cycle, have been marked.

Problem 8.15.21

Consider the system

$$\dot{x} = y + \varepsilon x$$

$$\dot{y} = x - xy - x^3$$

Problem 8.15.21, part a

Find the fixed points and characterize their stability.

Solution:

The fixed points will be found when the system is equal zero, that is,

$$y + \varepsilon x = 0 \tag{3}$$

$$x - xy - x^3 = 0 \tag{4}$$

This is simple to find - the first equation implies $y = -\varepsilon x$ for fixed points. This turns the second equation into a third degree polynomial in x, which when solving we get $x = 0, \frac{1}{2}(\varepsilon \pm \eta)$ for fixed points, where $\eta = \sqrt{\varepsilon^2 + 4}$. We can then look at the stability of these fixed points for some ϵ. Note the Jacobian of our system is given as

$$Df(x, y) = \begin{bmatrix} \varepsilon & 1 \\ 1 - y - 3x^2 & -x \end{bmatrix}.$$

---

1. $\underline{x = 0, y = 0}$:

   Then we seek to find the eigenvalues of the matrix $Df(0,0) = \begin{bmatrix} \varepsilon & 1 \\ 1 & 0 \end{bmatrix}$. This is a simple matrix, which when plugging into Wolfram Alpha[1], we get $\lambda = \frac{1}{2}(\varepsilon \pm \eta)$. This will then be strictly real and nonzero for $\varepsilon \in \mathbb{R}$, so the origin will be considered a hyperbolic stable point.

2. $\underline{x = \frac{1}{2}(\varepsilon + \eta), y = -\frac{\varepsilon}{2}(\varepsilon + \eta)}$:

   Performing the same calculations from above we get these horrendous values for the eigenvalues, $\lambda = \frac{1}{4}(\varepsilon - \eta \pm \sqrt{5\varepsilon^2 - 10\varepsilon\eta - 11\eta^2 + 16})$. These eigenvalues are imaginary, provided $5\varepsilon^2 - 10\varepsilon\eta - 11\eta^2 + 16 \leq 0$. When solving this for ε, we get that this stable permits limit cycles for $\varepsilon \gtrapprox -1.74$. For values less, the eigenvalues collapse to strictly real values, which we will see in a later plot.

[1] I am assuming I don't have to find these by hand, since this exercise isn't focused on their calculation

3. $x = \frac{1}{2}(\varepsilon - \eta), y = -\frac{\varepsilon}{2}(\varepsilon - \eta)$:

   We see similar calculations as in the case above, $\lambda = \frac{1}{4}(\varepsilon + \eta \pm \sqrt{5\varepsilon^2 + 10\varepsilon\eta - 11\eta^2 + 16})$. These eigenvalues are imaginary, provided the discriminant is less than zero, which happens for $\varepsilon \lessapprox 1.74$.

I went ahead and generated some plots for your viewing pleasure. You can see in Figure 6 the calculated values of both eigenvalues for each stable point. Also, we see the system for 4 interesting values of ε ($\varepsilon = 0.5$ was thrown in as a bonus) in Figure 7. The code which generates these plots are given afterwards.

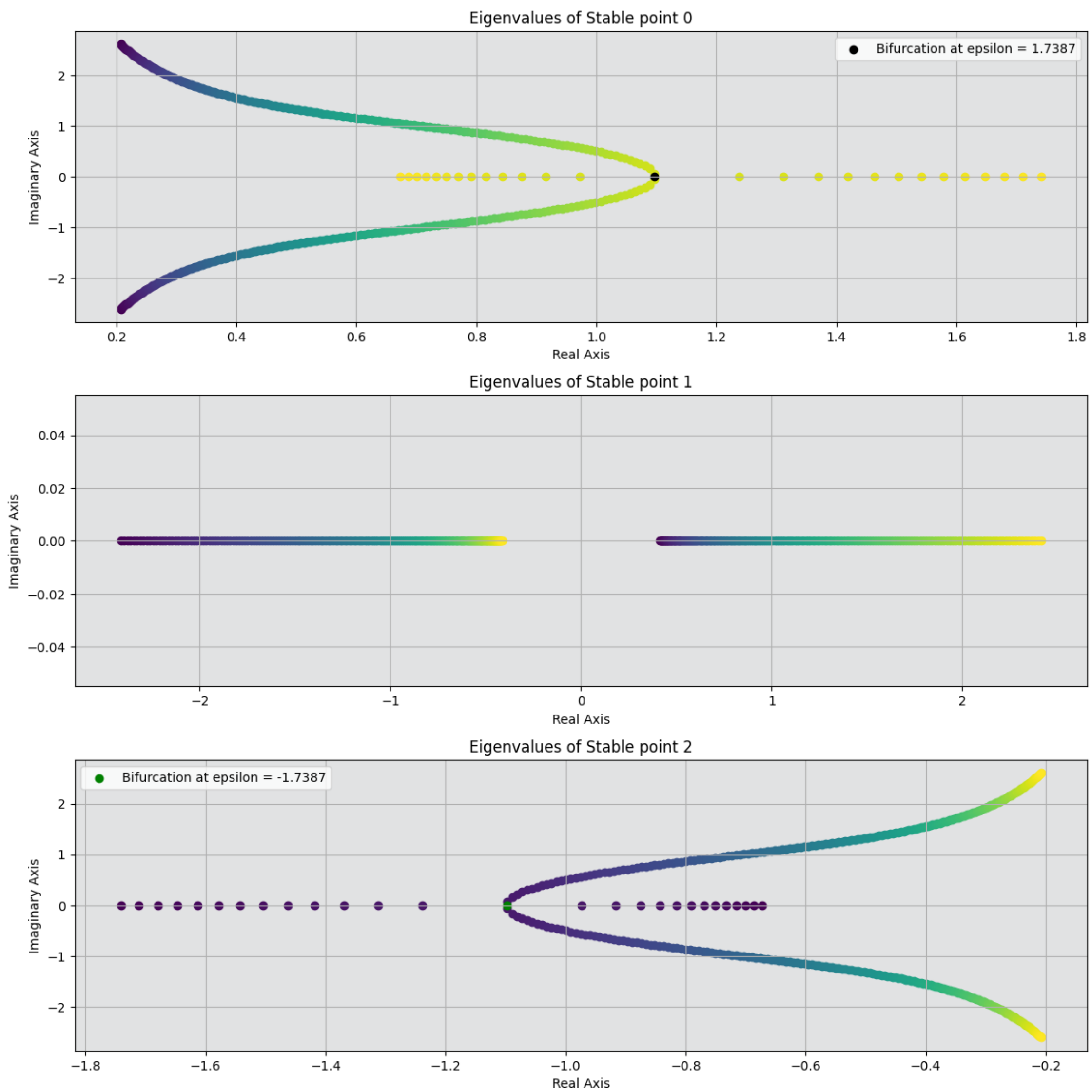

Figure 6: The eigenvalues for the above system at the stable points. Note the stable points are labelled from left to right, so Stable point 0 corresponds to the third case above, The values where ε bifurcates the system is also given in the relevant plots. Further note lighter colors correspond to greater values of ε.

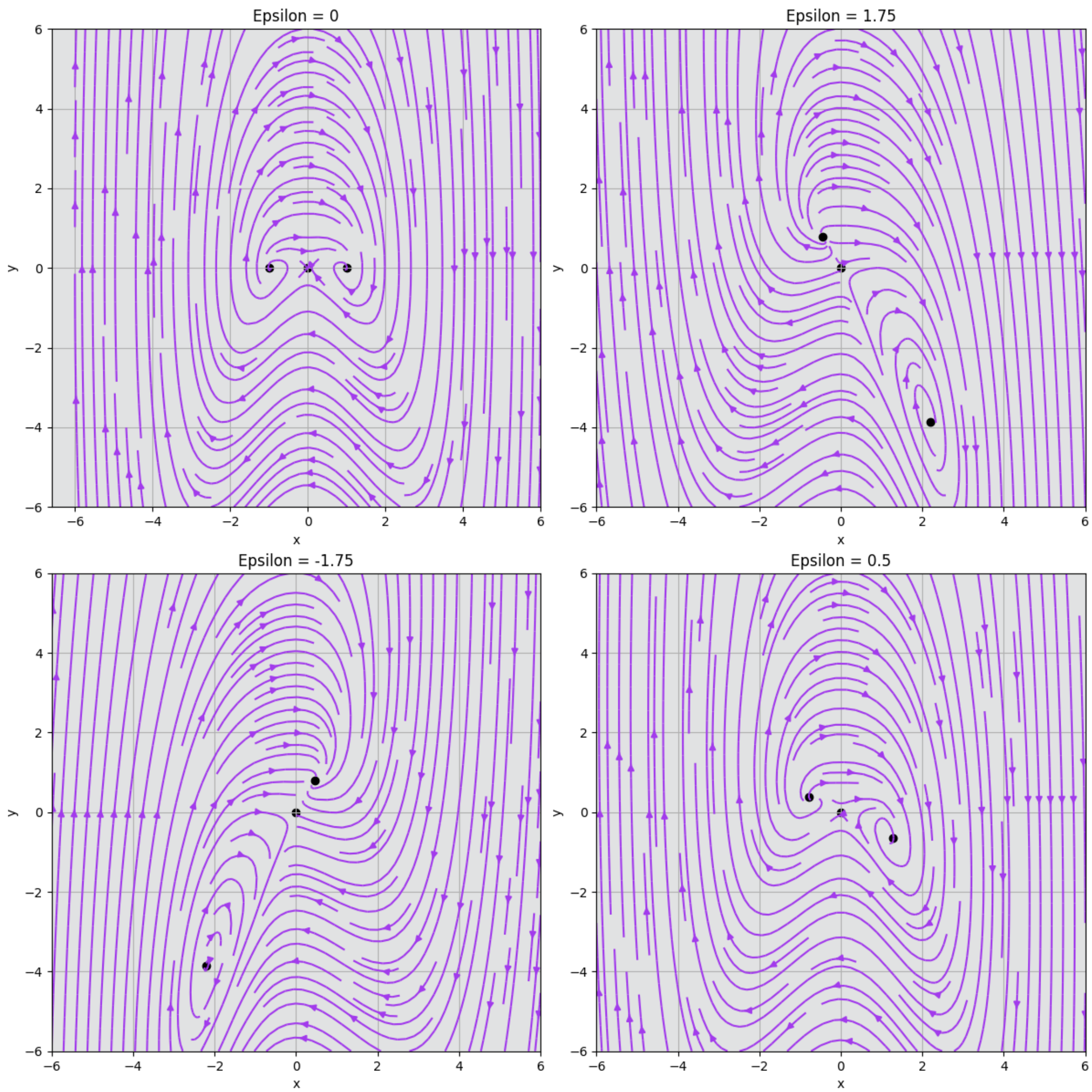

Figure 7: The system in Problem 8.15.21 for 4 interesting values of ε. We can see the two non zero stable points lose their limit cycles in respective $\varepsilon = \pm 1.75$ plots.

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve
import sympy as sm

def xdot(state, epsilon):
    x, y = state
    xdot = y + epsilon*x
    ydot = x - x*y - x**3
    return xdot, ydot

def find_stable_pts(xsm, ysm, eps):
    poly_y = sm.poly(xsm - xsm*(-eps * xsm) - xsm**3)
    x_stable = poly_y.all_roots()
    y_stable = [-eps * x for x in x_stable]

    return list(zip(x_stable, y_stable))

#System plots
eps = [0, 1.75, -1.75, 0.5]
fig, axs = plt.subplots(2, 2, figsize =(12, 12))
axs = axs.flat
ysm = sm.var("y")
xsm = sm.var("x")

bound = 6
#Vector field
xvect, yvect= np.meshgrid(np.linspace(-bound, bound, 20),
                np.linspace(-bound, bound, 20))
for i, ep in enumerate(eps):

    #Update vector field
    u = yvect + ep * xvect
    v = xvect - xvect * yvect - xvect**3

    stable_pts = find_stable_pts(xsm, ysm, ep)

    #Plotting stream plot
    axs[i].streamplot(xvect, yvect, u, v, density = 1.4, linewidth = None, color='#A23BEC')

    #Plotting Equilibrium points
```

```
    x_eq, y_eq = zip(*stable_pts)
    axs[i].scatter(x_eq, y_eq, color='k')

    #Plotting x any y axes
    axs[i].grid(True)
    axs[i].set_facecolor("#e1e2e3")

    axs[i].set_xlabel("x")
    axs[i].set_ylabel("y")

    axs[i].set_title(f'Epsilon = {ep}')

    plt.tight_layout(pad=0.5)

plt.show()

# Getting much more eps values
eps = np.linspace(-2, 2, 200)

# List of stable point and eigenvalues associated
# In form j, eps, eigenvalue
eig_stable = {}

# Getting Eigenvalues of Stable Points
for i, ep in enumerate(eps):
    stable_pts = find_stable_pts(xsm, ysm, ep)
    x_stable, y_stable = zip(*stable_pts)

    for j in range(len(stable_pts)):
        # Setting up system in Sympy
        state = sm.Matrix([xsm, ysm])
        xdotsm = sm.Matrix([ysm + ep*xsm, xsm - xsm*ysm - xsm**3])

        # Create a new DF object for each stable point
        DF = xdotsm.jacobian(state).subs({xsm: x_stable[j], ysm: y_stable[j]})

        eigenvals = DF.eigenvals().keys()

        if j not in eig_stable.keys():
            eig_stable[j] = [(ep, [e.evalf() for e in eigenvals])]
        else:
            eig_stable[j].append((ep, [e.evalf() for e in eigenvals]))
```

```
#Getting Bifuraction point
bifurcation = (max([sm.re(pt[1][0]) for pt in eig_stable[0]]), 0)
for pt in eig_stable[0]:
    if sm.re(pt[1][0]) == bifurcation[0]:
        bifurcation_ep = pt[0]
        break

#Plotting
fig, axs = plt.subplots(3, 1, figsize=(12, 12))

for j in range(3):
    for ep, evals in eig_stable[j]:
        real_parts = [sm.re(z) for z in evals]
        imaginary_parts = [sm.im(z) for z in evals]
        axs[j].scatter(real_parts, imaginary_parts, color=plt.cm.viridis(ep/4 + 0.5))
        axs[j].grid(True)
        axs[j].set_xlabel("Real Axis")
        axs[j].set_ylabel("Imaginary Axis")
        axs[j].set_facecolor("#e1e2e3")
        axs[j].set_title(f'Eigenvalues of Stable point {j}')


#Plotting Bifurcation points
axs[0].scatter(bifurcation[0], bifurcation[1], c = 'k', label = f'Bifurcation at epsilon =
    {round(bifurcation_ep, 4)}')
axs[2].scatter(-bifurcation[0], bifurcation[1], c = 'g', label = f'Bifurcation at epsilon =
    {round(-bifurcation_ep, 4)}')
axs[0].legend()
axs[2].legend()
plt.tight_layout(pad = 1)
plt.show()
```

Problem 8.15.21, part b

Show that when $\varepsilon = 0$, this system is time reversible but not Hamiltonian.

Solution:

When $\varepsilon = 0$, the system then turns into

$$\dot{x} = y$$

$$\dot{y} = x - xy - x^3$$

Note that a system is time reversible if there exists a diffeomorphism S, which acts on the vector field f such that

$$-f(S(z)) = DS(z)f(z).$$

This is given as (6.27) in the book. So we then need to find such a diffeomorphism, granted the problem is well-posed. By inspection, we can take $S(x, y) = (x, -y)$. This will then mean $-f(S(x, y)) = -f(x, -y) = (y, -x + xy + x^3)$. Taking the Jacobian of S, we see

$$DS(z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, $DS(x, y)f(x, y) = (y, -x + xy + x^3)$, thus the system is time reversible. However, for the system to be Hamiltonian, we require

$$\frac{\partial p}{\partial x} = -\frac{\partial q}{\partial y},$$

where $p = \dot{x}$, $q = \dot{y}$ (This is (6.22) in the book). Taking these partials, we then require $0 = -x$, which is not true in general. Thus the system is not Hamiltonian.

Problem 8.15.21, part c

Prove that when $\varepsilon = 0$, the origin has a homoclinic orbit.

Solution:

To explicitly show that there exists a homoclinic orbit somewhat evades me: I will show a numerical proof that should be somewhat convincing. We should first check that the origin is indeed a saddle point for the system, when $\varepsilon = 0$.

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 1 - y - 3x^2 & -x \end{bmatrix}_{x=y=0} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are then ± 1, which satisfies the conditions for $x = 0 = y$ to be a saddle point for the system. Zooming into the origin, we see that in Figure 8 shows us a clear division of the half plane $y < 0$. Inspired by this, let us take an initial condition centered sufficiently close to the origin, with $x_0 = y_0 = -0.01$ and see how the system evolves. Since the system has time symmetry, we can also investigate its "conjugate" initial conditions, $x_0^* = 0.01$, $y_0^* = -0.01$, where the initial conditions have been transformed under the diffeomorphism S found in the previous part. Running the initial conditions in forward time and the conjugate initial conditions in reverse time should give us the same plots. This is shown in Figure 9. To sufficiently demonstrate that the origin is the limit points for both initial conditions, we can examine the difference in solution positions for all times. This is shown in Figure 10. The backward time solution is plotted in negative time and vise versa for the forward solution. From this, we see that the position differences are approaching zero for later times in each coordinate, implying the origin is indeed the limit point for the two solutions. Thus, we have found a homoclinic orbit for the system. The code which generated these plots is given afterwards.

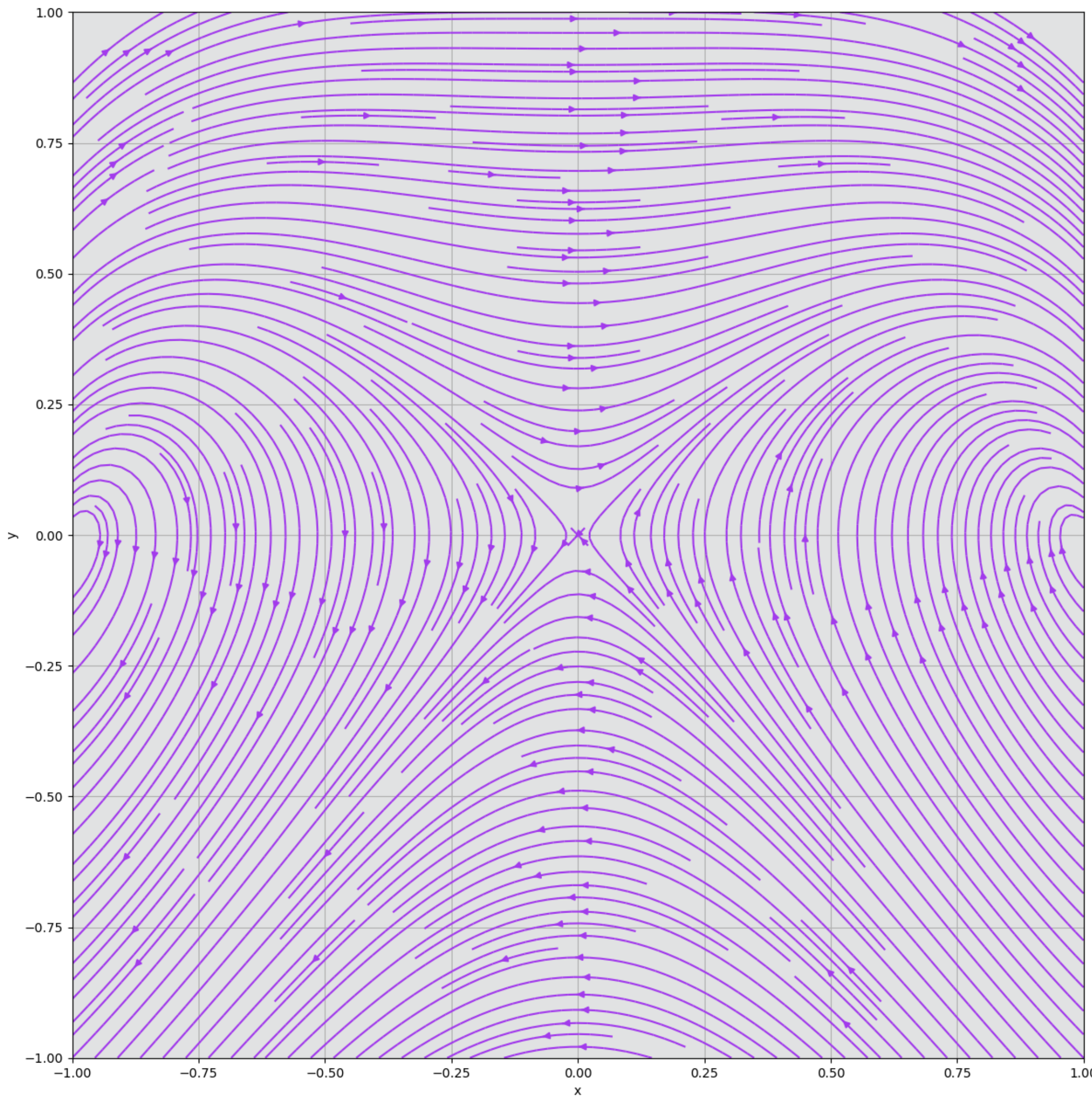

Figure 8: A zoom in on the origin for the system in Problem 8.15.21 when $\varepsilon = 0$. Note the clear directions on which solutions will take on $x = y < 0$. Solutions are repelled for $x < 0$ and attracted for $x > 0$.

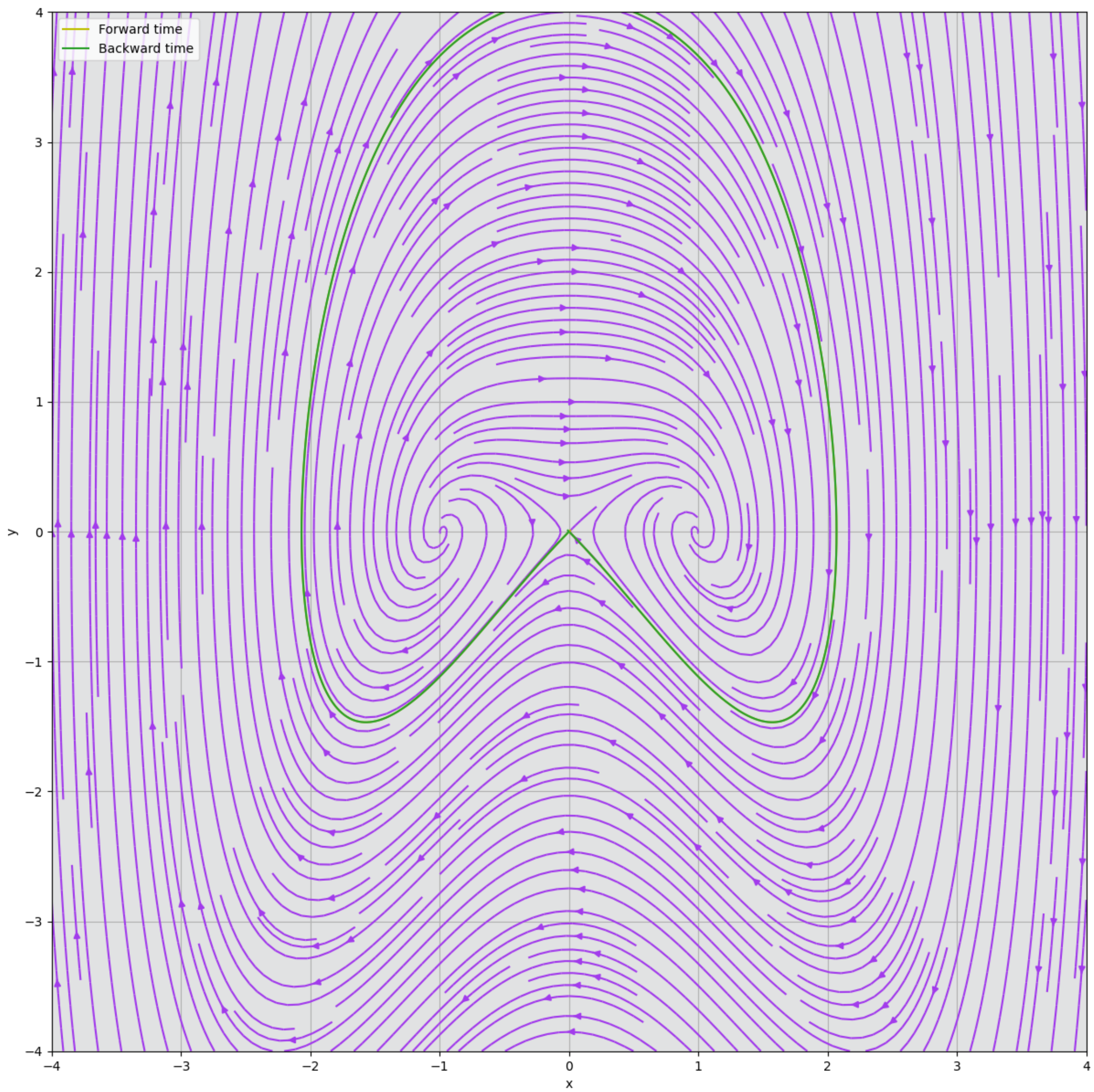

Figure 9: Numerical Solutions for $(x_0, y_0) = (-0.01, -0.01)$ and $S(x_0, y_0) = (x_0^*, y_0^*) = (0.01, -0.01)$. This isn't a terribly interesting plot, since the two solutions are - as they should be - overlapping at each point in time.

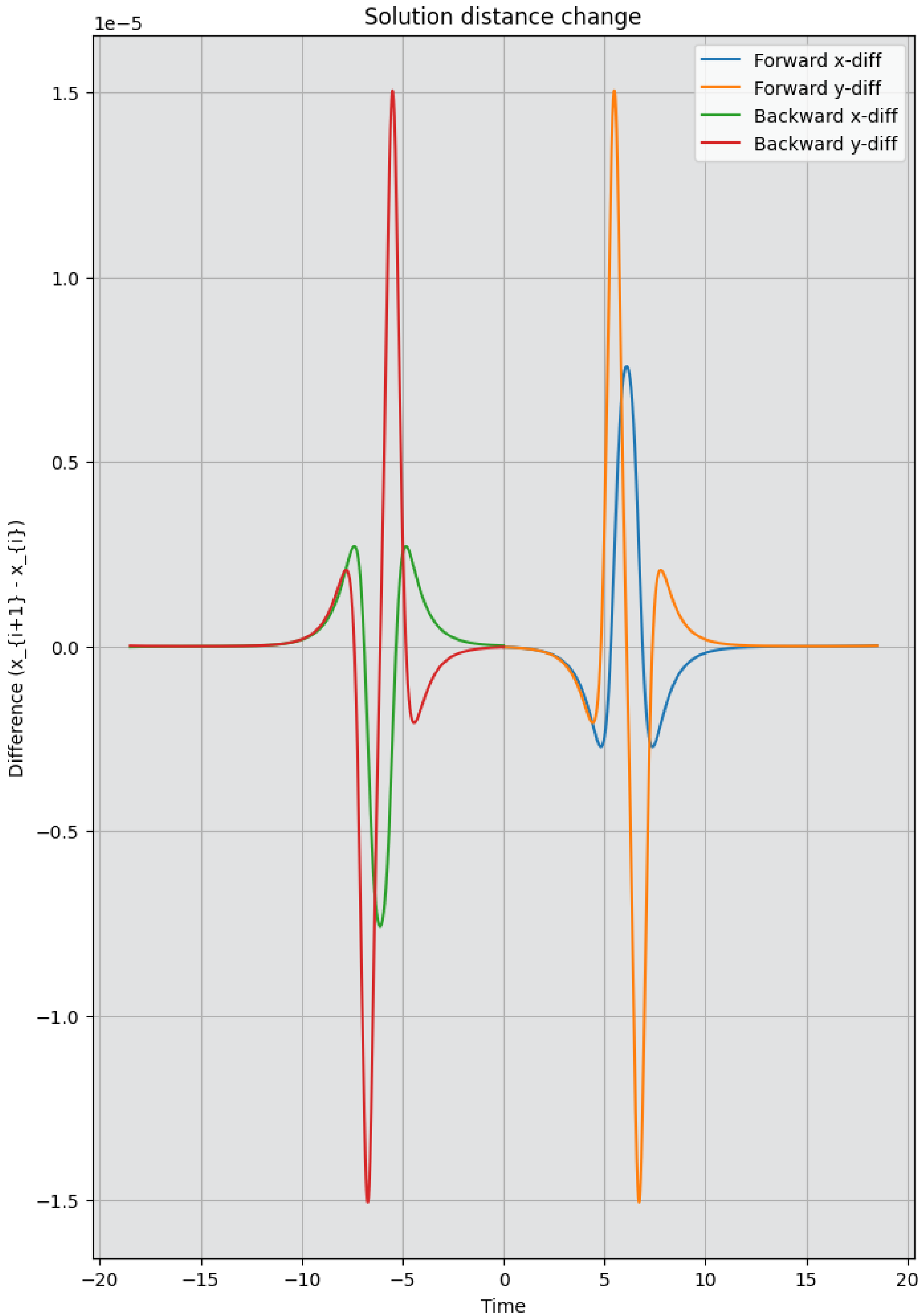

Figure 10: The incremental position differences for the x and y coordinates of the initial conditions $(x_0, y_0) = (-0.01, -0.01)$ and $S(x_0, y_0) = (x_0^*, y_0^*) = (0.01, -0.01)$. Note that numerical precision only allows me to go so far in plotting, as one timestep over the origin and we are no longer limiting to it.

```python
import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import odeint

def xdot(state, t):
    x, y = state
    xdot = y
    ydot = x - x*y - x**3
    return xdot, ydot

#Numerical Solution Plots!!!
fig, axs = plt.subplots(1, 1, figsize =(12, 12))

bound = 4
#Vector field
xvect, yvect= np.meshgrid(np.linspace(-bound, bound, 40),
                np.linspace(-bound, bound, 40))

#Update vector field
u = yvect
v = xvect - xvect * yvect - xvect**3

#Plotting stream plot
axs.streamplot(xvect, yvect, u, v, density = 3, linewidth = None, color='#A23BEC')
axs.grid(True)
axs.set_facecolor("#e1e2e3")
axs.set_xlabel("x")
axs.set_ylabel("y")
axs.set_xlim(-bound, bound)
axs.set_ylim(-bound, bound)

#Setting up solution along x0 = y0

x0 = -0.01
y0 = -0.01
timespan = np.linspace(0, 18.5, 10000000)
init_condit = [x0, y0]

sol = odeint(xdot, init_condit, t = timespan )
```

```python
fxsol, fysol = sol.T
plt.plot(fxsol, fysol, color='y', label="Forward time")

x0 = 0.01
y0 = -0.01
timespan = np.linspace(0, -18.5, 10000000)
init_condit = [x0, y0]

sol = odeint(xdot, init_condit, t = timespan )

bxsol, bysol = sol.T
plt.plot(bxsol, bysol, color='tab:green', label="Backward time")

plt.tight_layout(pad=0.5)
plt.legend()
plt.show()


#Difference plots!!!
fxdiff = np.diff(fxsol)
fydiff = np.diff(fysol)

bxdiff = np.diff(bxsol)
bydiff = np.diff(bysol)


fig, axs = plt.subplots(1, 1, figsize = (8, 12))

timespan = np.linspace(0, 18.5, 10000000)
axs.plot(timespan[:-1], fxdiff, label = "Forward x-diff")
axs.plot(timespan[:-1], fydiff, label = "Forward y-diff")
timespan = np.linspace(0, -18.5, 10000000)
axs.plot(timespan[:-1], bxdiff, label = "Backward x-diff")
axs.plot(timespan[:-1], bydiff, label = "Backward y-diff")
axs.grid(True)


axs.set_title("Solution distance change")
axs.set_xlabel("Time")
axs.set_ylabel("Difference (x_{i+1} - x_{i})")
axs.set_facecolor("#e1e2e3")
plt.legend()
```

```
plt.show()
```

Problem 8.15.21, part d

Study the system for small ε. Is there a homoclinic bifurcation?

Solution:

I tried applying the homoclinic bifurcation theorem (Theorem 8.28) from the book, but since the parameter $\tau = \mathrm{tr}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0$ when applying the conditions $\varepsilon = (x, y) = 0$, the nondegeneracy condition is violated. It might be possible to numerically show this; but with the small tweaking of ε that I did, I could not find one.