Lecture Note

October 22, 2023

1 Exploring Iterates

1.1 Logistic maps

Consider the Logistic map given by $x_{n+1} = f(x) = rx_n(1 - \frac{x_n}{K})$, $r > 0$. This function can be seen in applications including population dynamics, where both exponential growth (with growth rate r) and carrying capacity (K) are taken into consideration. In this lecture, we consider the carrying capacity to be 1, in which case solutions of this ode reflect the proportion of the carrying capacity the population reaches at each t step. For this interpretation, we must have $0 \leq x(t) \leq 1$. However, for this to be true, we need $f(x) \in [0, 1]$ for all $x \in [0, 1]$, which is when $r \leq 4$. Thus, in summary, the system to be investigated is given by

$$x_{n+1} = f(x) = rx_n(1 - x_n), r \in (0, 4], \quad 0 \leq x \leq 1.$$

Check yourself that $f(x)$ is a parabola that is non-negative for $0 \leq x \leq 1$, meaning that for all non-zero initial conditions, the solution would monotonically increase. Furthermore, the rate of increase peaks at $x = 0.5$, at which place $f(x) = r/4$. Intuitively, this is when we have enough population for the exponential growth to reach a high growth rate, while not being heavily restrained by the carrying capacity. From the construction of $f(x)$, this is when $x(1-x)$ attains its maximum. In the first part of the lecture, we will focus on the dynamics of solutions of this system under the influence of the parameter value r.

1.2 Cobweb Graph

To obtain the trajectories of solutions in a discrete dynamic system, one needs to repeatedly apply $f(x)$ on the given initial condition x_0 to obtain $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_2))$, and so on. The Cobweb graph does exactly this. Notice the three main elements of a Cobweb graph: the line $y = x$, the curve $f(x)$, and the Cobweb lines. The Cobweb line portrays the dynamics of the solution by starting with the initial condition x_0 on the x-axis, finding $f(x_0)$ by moving upwards from the initial condition x_0 until we reach the curve $f(x)$ at $(x_0, f(x_0))$. Next, we notice that for the next iteration, we would have $x_1 = f(x_0)$, so we need $x_1 = f(x_0)$ to be our x-value. Thus, we move to the side until we reach the line $y = x$ at $(f(x_0), f(x_0)) = (f(x_1), f(x_1))$. From this point, we can move up to meet $f(x)$ again and so on. This iteratively gives us each point x_n in our solution. Given this construction, it is clear that the points where our Cobweb-line touches the line $y = x$ are at $(f(x_i), f(x_i))$ for each i, which is exactly our solution at each step $i + 1$. Thus, if we take the line $y = x$ together with all the points at which the Cobweb line touches it, we would have the dynamic of the solution as time steps move along.

Aside from providing information on the dynamics of x_n, the Cobweb graph also provides insights into the system's fixed points. In a discrete system, we define a fixed point to be the value of x such that $f(x) = x$. This can be directly observed from the Cobweb graph: whenever the curve $f(x)$ contacts the curve $y = x$, at this value x, we have $x = y = f(x)$, which matches our definition for a fixed point. Furthermore, fixed points are said to be stable if solutions near them converge to them, and unstable if solutions near them diverge from them. This can also be seen from a Cobweb graph.

Let x^* be a fixed point, let ϵ be a small perturbation and $x^* + \epsilon$ would be a solution near the fixed point. For x^* to be stable, we want

$$|f(x^* + \epsilon) - x^*| < |x^* + \epsilon - x^*| = |\epsilon|.$$

However, using the linear approximation of f, f maps $x^* + \epsilon$ to

$$f(x^* + \epsilon) \approx f(x^*) + \epsilon f'(x^*).$$

Since x^* is a fixed point, by definition $f(x^*) = x^*$. Thus, we have

$$|f(x^* + \epsilon) - x^*| \approx |f(x^*) + \epsilon f'(x^*) - x^*| = |\epsilon f'(x^*)|$$

so that the fixed point x^* is stable if $|f'(x^*)| < 1$, unstable if $|f'(x^*)| > 1$, we say x^*'s stability is inconclusive if $|f'(x^*)| = 1$.

Now onto perhaps the most important property of a Cobweb graph: periodic orbits of a Cobweb graph reflect periodic orbits of $f(x)$. We have seen periodic orbits of continuous dynamic system Tuesday but in a discrete system if we let $f^n(x)$ denote n copies of f compounded on x, an orbit of f is defined as the value x such that $f^n(x)$ for some $n \in \mathbb{N}$, in which case we call x, $f(x)$, ..., $f^{n-1}(x)$ an n-orbit. Orbits show up in Cobweb graphs as closed curves whose period can be counted by counting the number of times the closed curve intersects $y = x$. Furthermore, if x is such that $f^n(x) = x$ and $n = pq$, $g(x) = f^p(x)$ fulfills that $g^q(x) = x$. Lastly, the Cobweb graph can display chaotic behavior by displaying whether the system is sensitive to change in the initial condition. This can be directly observed by looking into how much the Cobweb graph changes as x_0 changes. You can find my Cobweb graph demo used in lecture at https://github.com/PeterXQC/Cobweb-demo

1.3 Period Doubling Bifurcation

For almost all initial conditions, when r is small, the initial conditions converge to some fixed points of the system. As r increases, at some point, the solution starts converging to two points on $\mathbb{R}$. This is what a period-doubling bifurcation looks like in terms of how it affects a given trajectory. To get an idea from a different perspective, let's look at the bifurcation diagram. This plot has the x axis to be the changing r values and draws all "attractors", which is generally defined as where solutions go to as t increase, or as more steps are taken in discrete systems. We can observe that as r increases, the stable attractor evolves from being 1-periodic to 2-periodic, then four, and so on. One might assume that this pattern would continue and we would have periodic orbits of all powers of 2, which would be correct. However, we also have orbits whose periods are not a power of 2, like period 3 orbits. Furthermore, as r increases, period-doubling bifurcations happen at a faster rate. In fact, the gap between two period-doubling bifurcations adjacent in r shortens at a constant ratio. Thus, if we zoom into a smaller branch of the bifurcation diagram, we would obtain a similar image. This is called the self-similarity of period-doubling bifurcations. We mentioned that period-doubling bifurcation leads to period 3 orbits. In fact, period 3 orbits mark the onset of chaotic behaviors. Let's look into why this is the case:

2 Sharkovskii's

Define an ordering on $\mathbb{N}$

$$\begin{aligned}
&3 \succ 5 \succ 7 \succ \cdots \\
&\succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ \cdots \\
&\succ 2^2 \times 3 \succ 2^2 \times 5 \succ 2^2 \times 7 \cdots \\
&\cdots \\
&\succ 2^k \times 3 \succ 2^k \times 5 \succ 2^k \times 7 \succ \\
&\cdots \\
&\succ 16 \succ 8 \succ 4 \succ 2 \succ 1
\end{aligned}$$

called Sharkovskii's ordering.

We then have the following theorem.

Theorem 1 (Sharkovskii)**.** *Let $f : J \to J$ be a continuous function from a closed interval $J \subset \mathbb{R}$. Assume f has a periodic point of (minimal) period p. Then f has a periodic point of (minimal) period q for all $q \prec p$.*

Note that if we have a period 3, then we have periods of all other orders. A further result, of Li and Yorke, is that f has uncountably many aperiodic points. We prove the special case of Sharkovskii's theorem for period three, illustrating the procedure for the general proof. This proof is adapted from Sharkovskii's paper "Coexistence of Cycles of a Continuous Map From the Line to Itself" and Li and Yorke's "Period Three Implies Chaos".

First we prove a few lemmas.

Lemma 1. *If $g : I \to \mathbb{R}$ is continuous, I some closed interval, $g(I) \supseteq I$, then g has a fixed point in I.*

Proof. Writing $I = [a, b]$, we can find $x_1, x_2 \in I$ such that $g(x_1) = a$ and $g(x_2) = b$. Let $h(x) = g(x) - x$. Then

$$h(x_1) = a - x_1 \leq 0,$$

and

$$h(x_2) = b - x_2 \geq 0.$$

Since h is continuous, by the intermediate value theorem, there exists $y \in I$ between x_1 and x_2 such that $h(y) = 0$, and thus that $g(y) = y$. $\square$

Lemma 2. *For a continuous function $f : [a, b] \to \mathbb{R}$, given any interval $[c, d] \subseteq f([a, b])$, there exists an interval $I \subseteq [a, b]$ such that $f(I) = [c, d]$.*

Proof. Take $p, q \in [a, b]$ such that $f(p) = c$, $f(q) = d$. Suppose that $p < q$. Let $r \in [p, q]$ be the greatest value such that $f(r) = c$, and let s be the first point to the right of r such that $f(s) = d$. Then $f([r, s]) = [c, d]$. The proof is similar for $p > q$. $\square$

We are now ready to prove that the existence of a period 3 orbit implies existence of orbits of any other period.

Proof of "period 3 implies chaos". Assume $f : J \to J$ continuous, $J \subset \mathbb{R}$ closed. Suppose that there exist $x_1, x_2, x_3 \in J$ with $x_1 < x_2 < x_3$, such that f has a period-3 orbit on the x_j. There are two cases:

1. $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1$
2. $x_1 \mapsto x_3 \mapsto x_2 \mapsto x_1$

First, consider case 1. Here $f(x_1) = x_2$, $f(x_2) = x_3$, and $f(x_3) = x_1$. By the continuity of f (specifically IVT), $f([x_1, x_2]) \supseteq [x_2, x_3]$. Similarly, $f([x_2, x_3]) \supseteq [x_1, x_3]$.

Define $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$. From the fact that $[x_1, x_3] \subseteq f(I_2)$, we see in particular that both $I_2 \subseteq f(I_2)$ and $I_1 \subseteq f(I_2)$. We also still have $f(I_1) \supseteq I_2$. Claim: there exists $J_n \subset I_1$ such that $f^n(J_n) = J_n$, for any $n \in \mathbb{N}$.

To show this, we will construct J_n iteratively. By lemma (2), since $f(I_1) \supseteq I_2$, there exists an interval $J_1 \subset I_1$ such that $f(J_1) = I_2$.

In the next step, since $f(f(J_1)) \supseteq I_2$, we can find an interval $J_2 \subset J_1$ such that $f(f(J_2)) = I_2$. [1]

From this last step, we may proceed inductively to find that for some interval, $f^{n-1}(J_{n-1}) = I_2$. Since $f^n(J_{n-1}) = f(I_2) \supseteq I_1$, there exists also $J_n \subset J_{n-1}$ such that $f^n(J_n) = I_1$. Since $J_i \supset J_n$ for all $i = 1, \ldots, n-1$, from the construction of the J_i, $f^i(J_n) \subset f^i(J_i) = I_2$.

Since f^n is a continuous function with $f^n(J_n) \supset J_n$, f^n has a fixed point $y \in J_n$, meaning y is an n-periodic point of f. Since $f_i(J_n) \subset I_2$ for $i = 1, \ldots, n-1$, y cannot be m-periodic for any $m < n$. Thus, f has periodic point of minimal period n.

The proof of case 2 is similar. In fact, the proof can be obtained from the proof of case 1 simply by making the change of variable $x \mapsto -x$. $\square$

Aside:

[1] The interval J_2 cannot contain x_2, since $f(f(x_2)) = x_1 \notin I_2$. A similar argument shows that $x_1 \notin J_3$ unless $n = 3$.

In fact, the converse of Sharkovskii's theorem is also true. Every upper set of the Sharkovskii ordering, that is, every set $\{n \in \mathbb{N} : m \succ n\}$ for some $m \in \mathbb{N}$, corresponds to the set of orders of all periodic points of some function. These sets of periods are achieved by the following class of maps:

$$\mathfrak{T} = \left\{ T_h : [0,1] \to [0,1],\ x \mapsto \min\left(h, 1 - 2\left|x - \frac{1}{2}\right|\right), h \in [0,1] \right\}.$$

This class achieves every upper set other than the empty set, but the assumption that we start with a continuous function mapping a closed interval to itself implies the existence of a fixed point, i.e., a periodic point of order 1.

3 Conjugacy

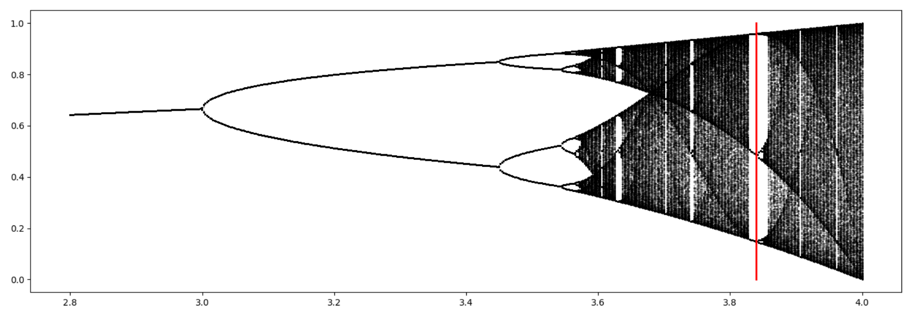


Figure 1: Bifurcation diagram of the Logistic Map. Note the stable 3-periodic orbit around $\lambda = 3.839$

Coming back to the Logistic map, we see a clear stable 3-periodic orbit in the bifurcation diagram 1 around $\lambda = 3.839$. The immediate question that comes to mind in the context of the 'period three implies chaos' theorem is: what happened to the infinite number of periodic orbits implied by Sharkovskii's theorem? In this tallk I wish to convince you that they are there, albeit hidden to our computer simulations. I will introduce the concept of Symbolic Dynamcis, to make our analysis easier, and as a nontrivial application use it to count the number of period orbits of all orders.

First consider the Logistic map $f(x) = 3.839$, with the paramter $\lambda = 3.839$ fixed. We are looking for all the 'invisible' periodic orbits. To do so, we first study the clear periodic orbit. Note that 3-periodic points are fixed points of $f \circ f \circ f$. Recall that a fixed point x_* of a discrete dynamical system $x_{n+1} = g(x_n)$ is stable when $|g'(x_*)| < 1$. This motivates us to find the stable fixed points of $f \circ f \circ f = f^{(3)}$. Looking at the graph of $f^{(3)}$ in Figure 2 shows the three stable points $a_1 \approx 0.1499, a_2 \approx, 0.4892, a_3 \approx 0.9593$ we expect. We are also interested in finding all the points that eventually tend to the orbit $\{a_1, a_2, a_3\}$. Thus, we may ask for maximal open intervals A_i around each a_i such that $f^{(3)}(A_i) \subset A_i$ and $f^{3n}(A_i) \to_n a_i$. These intervals are illustrated in 3a.

We may eliminate anything that eventually reaches the A_i in our search for the other periodic orbits. We decompose

$$[0,1] = I_0 \cup A_1 \cup I_1 \cup A_2 \cup I_2 \cup A_3 \cup I_3,$$

where the I_k are closed intervals.

Claim 1. *All periodic orbits other than a_1, a_2, a_3, and 0 lie in $I_1 \cup I_2$.*

Proof.
- If x eventually reaches A_i then $f^k(x) \to \{a_1, a_2, a_3\}$.
- $f(I_1 \cup I_2) \subset I_1 \cup A_2 \cup I_2$ (see Figure 3b.
- If $0 < x \in I_0$, $f^n(x)$ eventually reaches $I_1 \cup A_2 \cup I_2$ since the origin is unstable.

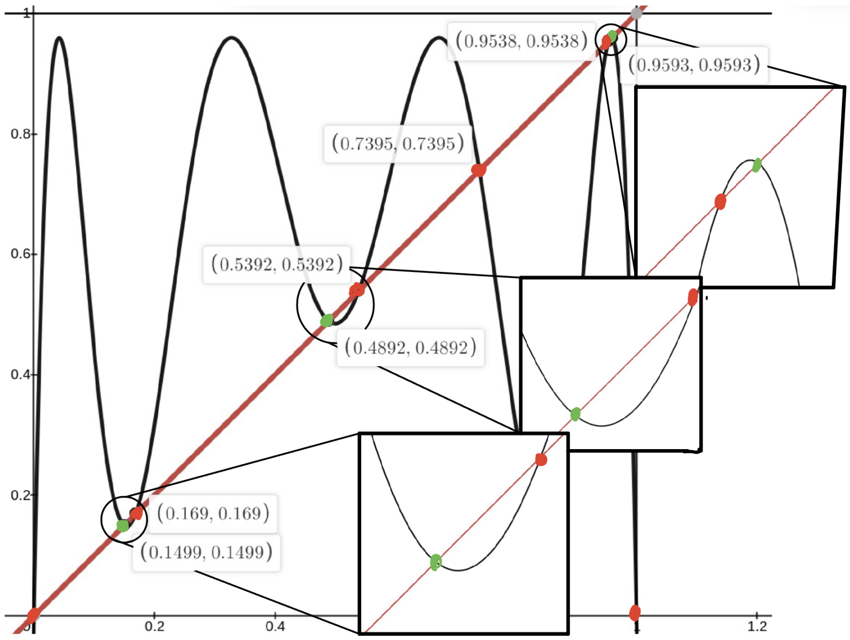
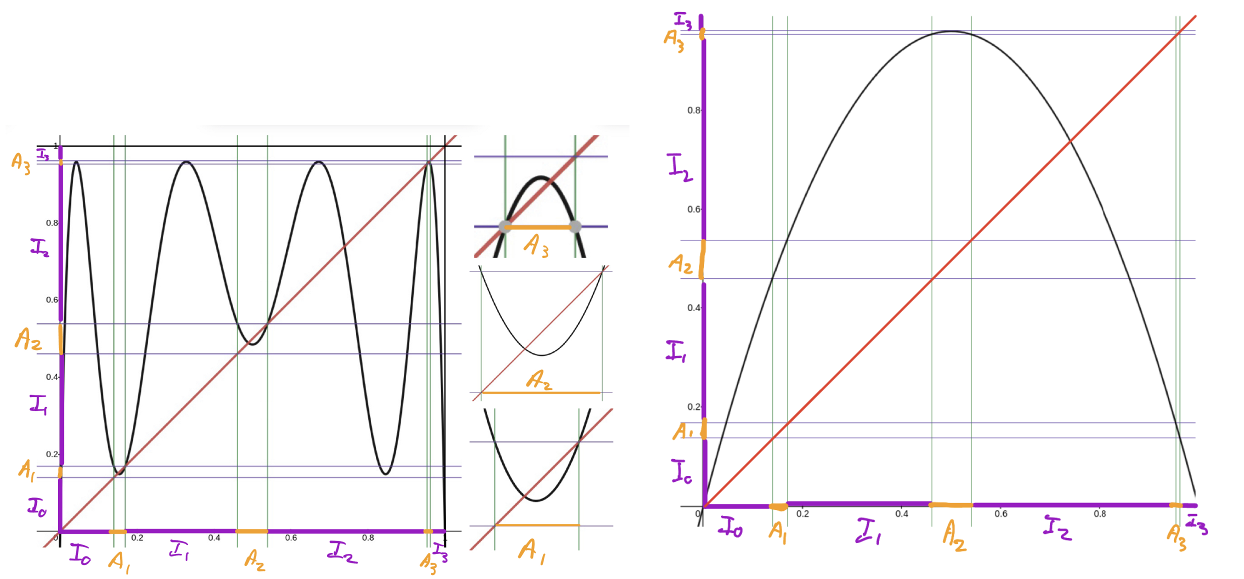


Figure 2: The graph of $f^{(3)}$. The green points are the stable fixed points, and the red are the unstable.



(a) The maximal intervals of attraction around each fixed point a_i.

(b) The intervals of attraction overlayed onto the graph of f.

Figure 3: The decomposition $[0,1] = I_0 \cup A_1 \cup I_1 \cup A_2 \cup I_2 \cup A_3 \cup I_3$.

- $f(I_3) = I_0$.

$\square$

This motivates the definition of the set

$$\Lambda := \bigcap_{n=0}^{\infty} f^{-n}(I_1 \cup I_2) = \{ \text{ the set of all points that start and stay in } I_1 \cup I_2 \}.$$

This set Λ is some sort of Cantor set, and by my claim, it contains all the periodic points we are looking for. However, it is hard to study Λ directly, and so we rather map Λ into the set

$$\Sigma_+ := \{s_0 s_1 s_3 \cdots : s_k \in \{1, 2\}\},$$

the set of sequences on two symbols. We define the map $\iota : \Lambda \to \Sigma_+$ by

$$\iota(x) := s_0 s_1 s_2 \cdots \tag{1}$$

$$s_k := \begin{cases} 1 \text{ if } f^k(x) \in I_1 \\ 1 \text{ if } f^k(x) \in I_2. \end{cases} \tag{2}$$

The map ι simply records which of I_1, I_2 we visit at time-step k.

Claim 2. *The map ι is injective.*

Proof. Assume $x, y \in \Lambda$ with $\iota(x) = \iota(y)$.

- $|f^{(3)\prime}| \geq (1+\delta) > 1$ on Λ (see Figure 4).
- Since $\iota(x) = \iota(y)$, we have $f^k(x)$, and $f^k(y)$ are btoh in the same interval I_{s_k}.
- Using the mean value we have $|f^{3k}(x) - f^{3k}(y)| \geq (1+\delta)|f^{3(k-1)}(x) - f^{3(k-1)}(y)| \geq \cdots (1+\delta)^k |x - y|$.
- This is only possible if $x = y$.

$\square$

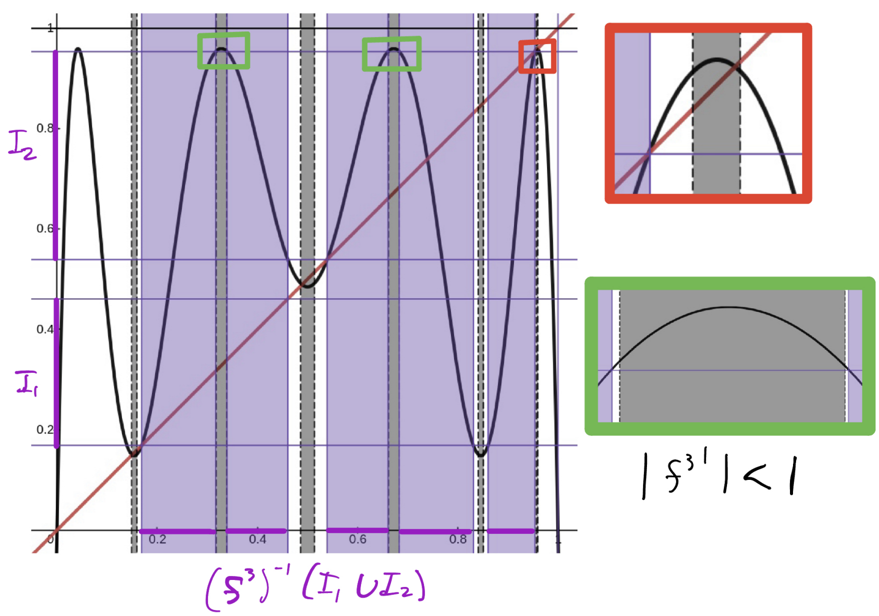


Figure 4: The gray area indicates regions where $|f^{(3)\prime}| \leq 1$ and the purple is a set strictly containing Λ. Note that the gray region is compactly contained in the complement of the purple, showing $|f^{(3)\prime}| \geq (1+\delta) > 1$ in the purple region.

Sadly though, ι is not surjective. However, this is easily remedied. Look again at Figure 3b and notice that $f(I_1) \subset I_2$, and so sequences in Σ_+ of the form $\ldots 11 \ldots$ are not possible. Define the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

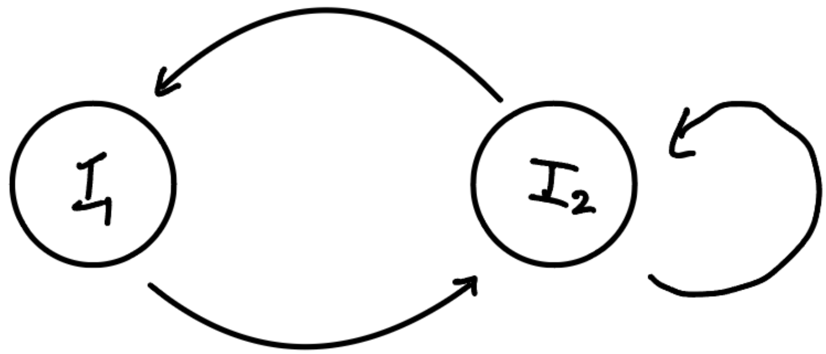


Figure 5: The graph indicating what f does to points in I_1 and I_2.

and the set Σ_A by

$$\Sigma_A := \{s_0 s_1 s_2 \cdots \in \Sigma_+ : a_{s_k s_{k+1}} = 1\}.$$

Note that Σ_A records all the valid paths that can be walked on the directed graph in Figure 5. Furthermore, our map ι is surjective as a map $\iota : \Lambda \to \Sigma$, with

$$\iota^{-1}(s_0 s_1 s_2 \cdots) = I_{s_0} \cap f^{-1}(I_{s_1}) \cap f^{-2}(I_{s_2}) \cdots . \tag{3}$$

Since each $f^{-1}(I_{s_1})$ is a compact set, the intersection is non-empty, and by the injectivity of ι the intersection contains a single element. Hence $\iota : \Lambda \to \Sigma_A$ is a bijection.

On Σ_A we can define the dynamical system $\sigma : \Sigma_A \to \Sigma_A$ by

$$\sigma(s_0 s_1 s_2 \cdots) := s_1 s_2 s_3 \cdots .$$

The map σ 'forgets' the first symbol in our sequence, and so $\iota \circ f(x) = \sigma \circ \iota(x)$. The dynamics of σ thus makes the following diagram commute, showing that the dynamical systems (Λ, f) and (Σ_A, σ) are the same:

$$\begin{tikzcd}
(\Lambda, f) \arrow[r, "f"] \arrow[d, "\iota"] & (\Lambda, f) \arrow[d, "\iota"] \\
(\Sigma_A, \sigma) \arrow[r, "\sigma"] & (\Sigma_A, \sigma)
\end{tikzcd}$$

Thus in order to find periodic points in Λ, it is sufficient to find periodic points in Σ_A. Listing the periodic points in Σ_A is simple.

Period	Point
1	222222 ...
2	212121 ... 121212 ...
3	221221221 ... 212212212 ... 122122122 ...

Note that A is the adjacency matrix of the graph in Figure 5. It is a fact from graph theory that $\mathrm{Tr}(A^k)$ counts all the valid cycles of length k in the graph defined by A. Hence, in our case, $\mathrm{Tr}(A^k)$ counts the number of periodic orbits of length k.

k	$\mathrm{Tr}(A^k)$
1	1
2	3
3	4