STAT 30900: Homework 4

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Contents

Problem 1

Let $A \in \mathbb{R}^{m \times n}$ where $m \geq n$ and $\operatorname{rank}(A) = n$. Suppose GECP is performed on A to get

$$\Pi_1 A \Pi_2 = LU$$

where $L \in \mathbb{R}^{m \times n}$ is unit lower triangular, $U \in \mathbb{R}^{n \times n}$ is upper triangular, and $\Pi_1 \in \mathbb{R}^{m \times m}, \Pi_2 \in \mathbb{R}^{n \times n}$ are permutation matrices.

Problem 1, part a

Show that U is nonsingular and that L is of the form

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

where $L_1 \in \mathbb{R}^{n \times n}$ is nonsingular.

Solution:

Suppose false, that is, A is nonsingular, but U is singular. Then there exists $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ where $U\mathbf{v} = \mathbf{0}$. Then, the following steps are justified:

$U\mathbf{v} = \mathbf{0}$	(Assumed.)
$\implies LU\mathbf{v} = L\mathbf{0} = \mathbf{0}$	(Multiplying by L.)
$\implies \Pi_1 A \Pi_2 \mathbf{v} = \mathbf{0}$	(GECP.)
$\implies A\Pi_2\mathbf{v} = \Pi_1^\intercal \mathbf{0} = \mathbf{0}$	(Π_1 is orthogonal.)
$\implies A\mathbf{w} = \mathbf{0}$	($\mathbf{w} = \Pi_2\mathbf{v} \neq 0$.)
$\implies \mathbf{w} = \mathbf{0}$	(A is full rank.)

Therefore, we have found $\mathbf{w} = \Pi_1^\intercal \mathbf{v} = \mathbf{0}$. Since $\Pi_1^\intercal$ is full rank, this means $\mathbf{v} = \mathbf{0}$, which is a contradiction, since we assumed $\mathbf{v} \neq 0$. Then U must be nonsingular.

We next want to show that L is of the given form, where $L_1 \in \mathbb{R}^{n \times n}$ is nonsingular. Note that any $m \times n$ matrix with $m \geq n$ can be written as this form, where we separate the first n rows from the last $m - n$ rows. We are also given L is unit lower triangular, thus all elements in its diagonal are equal to one. Since $m \geq n$, then the diagonal of L will span the first n rows, and n columns, which is of the same shape as L_1. Thus, L_1 inherits the diagonal of L, which has entries one, thus L_1 is nonsingular.

Problem 1, part b

We will see how the LU factorization may be used to solve the least squares problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x}-\mathbf{b}\|_2.$$

Problem 1, part b, subpart i

Show that the problem may be solved via

$$U\tilde{\mathbf{x}} = \mathbf{y}, \qquad L^\top L\mathbf{y} = L^\top\tilde{\mathbf{b}},$$

where $\tilde{\mathbf{b}} = \Pi_1\mathbf{b}$ and $\tilde{\mathbf{x}} = \Pi_2^\top\mathbf{x}$.

Solution:

We will go straight into calculations.

$$\begin{aligned}
\|A\mathbf{x}-\mathbf{b}\|_2 & & \text{(Given.)}\\
&= \left\|\Pi_1^\top LU\Pi_2^\top\mathbf{x}-\mathbf{b}\right\| & \text{(GECP.)}\\
&= \left\|LU\Pi_2^\top\mathbf{x}-\Pi_1\mathbf{b}\right\|_2 & \text{(Unitary invariance.)}\\
&= \left\|LU\tilde{\mathbf{x}}-\tilde{\mathbf{b}}\right\|_2 & \text{(Given definitions.)}\\
\left\|LU\tilde{\mathbf{x}}-\tilde{\mathbf{b}}\right\|_2^2 &= (LU\tilde{\mathbf{x}}-\tilde{\mathbf{b}})^\top(LU\tilde{\mathbf{x}}-\tilde{\mathbf{b}}) & \text{(Squaring the norm.)}\\
&= \tilde{\mathbf{x}}^\top U^\top L^\top LU\tilde{\mathbf{x}} - \tilde{\mathbf{x}}^\top U^\top L^\top\tilde{\mathbf{b}} - \tilde{\mathbf{b}}^\top LU\tilde{\mathbf{x}} + \tilde{\mathbf{b}}^\top\tilde{\mathbf{b}} & \text{(Expanding.)}\\
&= \tilde{\mathbf{x}}^\top U^\top(L^\top LU\tilde{\mathbf{x}} - L^\top\tilde{\mathbf{b}}) + \tilde{\mathbf{b}}^\top(\tilde{\mathbf{b}} - LU\tilde{\mathbf{x}}) & \text{(Grouping.)}
\end{aligned}$$

Thus, $\|A\mathbf{x}\| - \mathbf{b}_2$ is minimized when $L^\top LU\tilde{\mathbf{x}} = L^\top\tilde{\mathbf{b}}$. Via the substitution $U\tilde{\mathbf{x}} = \mathbf{y}$, we can recover the first equation, and write $L^\top L\mathbf{y} = L^\top\tilde{\mathbf{b}}$. This minimization can be achieved since $L^\top L$ is invertible for any matrix L with nonzero eigenvalues (this was shown in the first part to not be the case). Note this holds for any choice of $\mathbf{b}$ since if $\mathbf{b}\in\operatorname{im}(A)$, then the obvious minimization would be when $\tilde{\mathbf{b}} = LU\tilde{\mathbf{x}}$, which would then imply the first term in the final calculation above would also be minimized.

Problem 1, part b, subpart ii

Describe how you would compute the solution $\mathbf{y}$ in

$$L^\top L\mathbf{y} = L^\top \tilde{\mathbf{b}}.$$

Solution:

Well, I would just invert the matrix $L^\top L$ to get $\mathbf{y} = (L^\top L)^{-1}\tilde{\mathbf{b}}$.[1] If I couldn't take advantage of that, say in the scenario of literally calculating the solution in some software, I would take advantage of the triangularity of L. Denote $\mathbf{c} = L^\top \tilde{\mathbf{b}}$, which is known and can be computed by Matrix vector multiplication, since $L^\top$ would be an upper triangular matrix. Then we need to solve $L^\top L\mathbf{y} = \mathbf{c}$. This can be solved incurring minimal errors via the following system:

$$\begin{cases} L\mathbf{y} = \mathbf{z} \\ L^\top \mathbf{z} = \mathbf{c} \end{cases}$$

where the first equation is solved by forward substitution, and the last is solved by backward substitution.

[1] WARNING: THIS IS A JOKE.

Problem 2

Let $\varepsilon > 0$. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix}.$$

Probem 2, part a

Why is it a bad idea to solve the normal equation associated with A, i.e.

$$A^\top A\mathbf{x} = A^\top \mathbf{b}$$

when ε is small?

Solution:

It is in general a bad idea to solve the normal equation, since the errors incurred scale as the square of the condition number of A. If we take the condition number to be defined as $\kappa(A) = \left\|A^\dagger\right\|\|A\|$, then we can note the pseudoinverse of A is

$$A^\dagger = \begin{bmatrix} \frac{1}{3} & \frac{2\varepsilon-3}{6\varepsilon} & \frac{2\varepsilon+3}{6\varepsilon} \\ 0 & \frac{1}{2\varepsilon} & -\frac{1}{2\varepsilon} \end{bmatrix}$$

which its norm would explode as $\varepsilon \to 0$. This was calculated prior to seeing part d, via Wolfram Alpha.

Problem 2, part b

Show that the condensed LU factorization of A is

$$A = LU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}$$

Solution:

We will solve this naively. Note that A is full column rank, thus A can be written in the same shape as in problem 1. Since this is a small system, we can directly solve for the elements of L and U.

$$\implies \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{12} & \ell_{22} \\ \ell_{31} & \ell_{32} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

Multiplying this out, we get the following equations:

$$\ell_{11}u_{11} = 1 \tag{1}$$

$$\ell_{11}u_{12} = 1 \tag{2}$$

$$\ell_{12}u_{11} = 1 \tag{3}$$

$$\ell_{21}u_{21} + \ell_{22}u_{22} = 1 + \varepsilon \tag{4}$$

$$\ell_{31}u_{11} = 1 \tag{5}$$

$$\ell_{31}u_{12} + \ell_{32}u_{22} = 1 - \varepsilon \tag{6}$$

Equations 1, 2, and 3 imply $u_{11} = u_{12}$ and $\ell_{11} = \ell_{12}$, so these can be arbitrarily set to 1. Plugging these into the fourth equation get us $\ell_{22}u_{22} = \varepsilon$, and $\ell_{32}u_{22} = -\varepsilon$ for the sixth equation. Letting $\ell_{22} = 1$ and $\ell_{32} = -1$ gives us $u_{22} = \varepsilon$. Finally, since u_{11} was set to equal 1, the fifth equation implies $\ell_{31} = 1$. Thus, plugging in what we found, we get the following:

$$LU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix},$$

which is what we wanted to find.

Problem 2, part c

Why is it a much better idea to solve the normal equation associated with L, i.e.,

$$L^\intercal L\mathbf{y} = L^\intercal \tilde{\mathbf{b}}?$$

This shows that the method in Problem 1 is a more stable method than using the normal equation in (a) directly.

Solution:

This is a much better idea since we can employ the same methods used in Problem 1, part b, subpart ii, meaning we can exploit the triangularity of L to minimize the errors incurred in inverting the normal equation.

Problem 2, part d

Show that the Moore-Penrose pseudoinverse of A is

$$A^{\dagger} = \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix}$$

Solution:

We just need to show the properties of the pseudoinverse are obeyed. If we can do this, noting that the pseudoinverse is unique, the this must be $A^{\dagger}$.

1. $\underline{AA^{\dagger}A = A}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix} \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix}$$

$$= \left(\frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}\right) \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix}$$

2. $\underline{A^{\dagger}AA^{\dagger} = A^{\dagger}}$:

$$\frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix} \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix}$$

3. $\underline{(AA^{\dagger})^{\intercal} = AA^{\dagger}}$: Note in the first part, I wrote parenthesis around the product $AA^{\dagger}$. Noting that it's symmetric this property is obeyed.

4. $\underline{(A^{\dagger}A)^{\intercal} = A^{\dagger}A}$: I also wrote parenthesis around the product $A^{\dagger}A$ in the second part. This is obviously symmetric, so this is obeyed.

Problem 2, part e

Describe a method to compute $A^\dagger$ given L and U. Verify that your method is correct by checking it against the expression in part d.

Solution:

To get a formula for the pseudoinverse of a matrix this particular form (full column rank), we will take the general formula and apply $A = LU$ to it.

$$\begin{aligned}
A^\dagger &= (A^\intercal A)^{-1} A^\intercal && \text{(Full column rank pseudo inverse.)} \\
&= (U^\intercal L^\intercal L U)^{-1} U^\intercal L^\intercal && \text{(Applying } A = LU\text{.)} \\
&= U^{-1} (L^\intercal L)^{-1} U^{-\intercal} U^\intercal L^\intercal && \text{(Applying inverse.)} \\
&= U^{-1} (L^\intercal L)^{-1} L^\intercal && \text{(Simplifying.)}
\end{aligned}$$

We can then apply what we found for L and U. Note that I am taking the inverse of matrices, which in the normal case I would instantly fail the course and become a laugingstock to all known living creatures. This however will be a 2×2 matrix, so I should be fine. The only danger is taking the inverse of U, since $L^\intercal L$ will just be a diagonal matrix. Thus, when plugging in our L and U,

$$\begin{aligned}
A^\dagger &= \begin{bmatrix} 1 & 0 \\ 1 & \varepsilon \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} && \text{(Plugging in.)} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & \varepsilon \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} && \text{(Matrix multiplication.)} \\
&= \begin{bmatrix} 1 & -1/\varepsilon \\ 0 & 1/\varepsilon \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} && \text{(Taking inverses.)} \\
&= \begin{bmatrix} 1/3 & -1/(2\varepsilon) \\ 0 & 1/(2\varepsilon) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} && \text{(Matrix multiplication.)} \\
&= \begin{bmatrix} 1/3 & 1/3 - 1/(2\varepsilon) & 1/3 + 1/(2\varepsilon) \\ 0 & 1/(2\varepsilon) & -1/(2\varepsilon) \end{bmatrix} && \text{(Matrix multiplication.)} \\
&= \frac{1}{6} \begin{bmatrix} 2 & 2 - 3\varepsilon^{-1} & 2 + 3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix} && \text{(Simplifying.)}
\end{aligned}$$

This is exactly what we see in part (d).

Problem 3

We will now discuss an alternative method to solve the least squares problem in Problem 1 that is more efficient when $m - n < n$.

Problem 3, part a

Show that the least squares problem in Problem 1 is equivalent to

$$\min_{\mathbf{z}\in\mathbb{R}^n} \left\| \begin{bmatrix} \mathbb{I}_n \\ S \end{bmatrix} \mathbf{z} - \tilde{\mathbf{b}} \right\|_2$$

where $S = L_2 L_1^{-1}$ and $L_1 \mathbf{y} = \mathbf{z}$. Here and below, $\mathbb{I}_n$ denotes the $n \times n$ identity matrix.

Solution:

We will go straight into calculations:

$$
\begin{aligned}
&\|A\mathbf{x} - \mathbf{b}\| && \text{(Given.)} \\
&= \left\| \Pi_1^\intercal \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} U \Pi_2^\intercal \mathbf{x} - \mathbf{b} \right\|_2 && \text{(GECP.)} \\
&= \left\| \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} U \tilde{\mathbf{x}} - \tilde{\mathbf{b}} \right\|_2 && \text{(Definition, unitary invariance.)} \\
&= \left\| \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \mathbf{y} - \tilde{\mathbf{b}} \right\|_2 && \text{(Definition of } \mathbf{y}\text{.)} \\
&= \left\| \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} L_1^{-1} L_1 \mathbf{y} - \tilde{\mathbf{b}} \right\|_2 && \text{(Multiplying by } \mathbb{I}_n\text{.)} \\
&= \left\| \begin{bmatrix} \mathbb{I}_n \\ S \end{bmatrix} \mathbf{z} - \tilde{\mathbf{b}} \right\|_2 && \text{(Simplifying, definition.)}
\end{aligned}
$$

Therefore, the two are shown to be equivalent minimum least squares, aside from minimizing over different variables.

Problem 3, part b

Write

$$\tilde{\mathbf{b}} = \begin{bmatrix} \tilde{\mathbf{b}}_1 \\ \tilde{\mathbf{b}}_2 \end{bmatrix}$$

where $\tilde{\mathbf{b}}_1 \in \mathbb{R}^n$ and $\tilde{\mathbf{b}}_2 \in \mathbb{R}^{m-n}$. Show that the solution $\mathbf{z}$ is given by

$$\mathbf{z} = \tilde{\mathbf{b}}_1 + S^\top (\mathbb{I}_{m-n} + SS^\top)^{-1} (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1).$$

Solution:

We can first rewrite the system via the substitution $\mathbf{u} = \mathbf{z} - \tilde{\mathbf{b}}_1$. This then changes the system to

$$\min_{\mathbf{z}\in\mathbb{R}^n} \left\| \begin{bmatrix} \mathbf{u} \\ S(\mathbf{u} + \tilde{\mathbf{b}}_1) - \tilde{\mathbf{b}}_2 \end{bmatrix} \right\|_2 = \min_{\mathbf{z}\in\mathbb{R}^n} \left\| \begin{bmatrix} \mathbb{I}_n \\ S \end{bmatrix} \mathbf{u} - \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1 \end{bmatrix} \right\|_2 .$$

This system is uniquely solved via taking the pseudoinverse, so $\mathbf{u} = \begin{bmatrix} \mathbb{I}_n \\ S \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1 \end{bmatrix}$. Since the matrix in question has full column rank ($\mathbb{I}_n$ spans the columns), its pseudoinverse is given by

$$\begin{bmatrix} \mathbb{I}_n \\ S \end{bmatrix}^\dagger = \left(\begin{bmatrix} \mathbb{I}_n & S^\top \end{bmatrix} \begin{bmatrix} \mathbb{I}_n \\ S \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbb{I}_n & S^\top \end{bmatrix} = (\mathbb{I}_n + S^\top S)^{-1} \begin{bmatrix} \mathbb{I}_n & S^\top \end{bmatrix}.$$

Plugging this in, we see

$$\mathbf{u} = (\mathbb{I}_n + S^\top S)^{-1} S^\top (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1)$$

Using the Sherman-Woodbury-Morrison formula, we can transform the inverse into a more suitable form

$$\implies (\mathbb{I}_n + S^\top S)^{-1} = \mathbb{I}_n - S^\top (\mathbb{I}_{m-n} + SS^\top)^{-1} S$$

Therefore, the following steps can be taken:

$$\begin{aligned}
\mathbf{u} &= (\mathbb{I}_n + S^\top S)^{-1} S^\top (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) && \text{(Given.)} \\
&= \left[\mathbb{I}_n - S^\top (\mathbb{I}_{m-n} + SS^\top)^{-1} S \right] S^\top (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) && \text{(Plugging in above.)} \\
&= \left[S^\top - S^\top (\mathbb{I}_{m-n} + SS^\top)^{-1} SS^\top \right] (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) && \text{(Rearranging.)} \\
&= S^\top \left[\mathbb{I}_n - (\mathbb{I}_{m-n} + SS^\top)^{-1} SS^\top \right] (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) && \text{(Rearranging.)} \\
&= S^\top \left[(\mathbb{I}_{m-n} + SS^\top)^{-1} (\mathbb{I}_{m-n} + SS^\top) - (\mathbb{I}_{m-n} + SS^\top)^{-1} SS^\top \right] (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) && \text{(Rewriting identity.)}
\end{aligned}$$

$$= S^\top \left[(\mathbb{I}_{m-n} + SS^\top)^{-1} (\mathbb{I}_{m-n} + SS^\top - SS^\top) \right] (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) \qquad \text{(Grouping.)}$$

$$\mathbf{u} = S^\top (\mathbb{I}_{m-n} + SS^\top)^{-1} (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) \qquad \text{(Simplifying.)}$$

$$\Longrightarrow \mathbf{z} = \tilde{\mathbf{b}}_1 + S^\top (\mathbb{I}_{m-n} + SS^\top)^{-1} (\tilde{\mathbf{b}}_2 - S\tilde{\mathbf{b}}_1) \qquad \text{(Substitution.)}$$

Which is what we wanted.

Problem 3, part c

Explain why when $m - n < n$, the method in (a) is much more efficient than the method in problem 1. For example, what happens when $m = n + 1$?

Solution:

The benefit we see is the dimensionality of our problem is dropped significantly when approaching this problem from the one seen in part (a). In particular, when $m = n+1$, L_2 will have size $1 \times n$, which means S will have size $1 \times n$, so the product $SS^\intercal$ is simply a scalar. If we only look at the normal equation approach, the one seen in problem 1, we are stuck with solving a system with size $n + 1 \times n + 1$, which is far more computationally expensive, and is also more susceptible to instability. Therefore it is a lose-lose situation under the Problem 1 approach when $m - n < n$.

Problem 4

Let $\mathbf{c} \in \mathbb{R}^n$ and consider the linearly constrained least squares problem/minimum norm linear system

$$\begin{aligned} &\text{Minimize} \quad \|\mathbf{w}\|_2 \\ &\text{Subject to} \quad A^\top \mathbf{w} = \mathbf{c} \end{aligned}$$

Problem 4, part a

If we write $\tilde{\mathbf{c}} = \Pi_2^\top$ and $\tilde{\mathbf{w}} = \Pi_1 \mathbf{w}$, show that

$$\tilde{\mathbf{w}} = L(L^\top L)^{-1} U^{-\top} \tilde{\mathbf{c}}.$$

Solution:

The solution to this system is $\mathbf{w} = (A^\dagger)^\top \mathbf{c}$, since $A^\dagger \mathbf{c}$ is the minimum length solution to the least squares problem $\min \|A^\top \mathbf{w} - \mathbf{c}\|$. The calculation of $(A^\dagger)^\top$ is inspired by my calculation in Problem 2, part e, with some slight modifications. I will assume we are under the same conditions as in problem 2, since we are supposed to take its solution. Note L is unit lower triangular, so $L^\top L$ is invertible. Furthermore, since $L^\top L$ is symmetric, its inverse is also symmetric.

$$\begin{aligned}
A^\dagger &= (A^\top A)^{-1} A^\top && \text{(Full column rank pseudoinverse.)} \\
&= (\Pi_1 U^\top L^\top \Pi_1 \Pi_1^\top L U \Pi_2^\top)^{-1} (\Pi_2 U^\top L^\top \Pi_1) && \text{(GECP.)} \\
&= (\Pi_1 U^\top L^\top L U \Pi_2^\top)^{-1} (\Pi_2 U^\top L^\top \Pi_1) && \text{(Π_1 orthogonal.)} \\
&= \Pi_2 U^{-1} (L^\top L)^{-1} U^{-\top} \Pi_2^\top \Pi_2 U^\top L^\top \Pi_1 && \text{(Taking inverse.)} \\
&= \Pi_2 U^{-1} (L^\top L)^{-1} U^{-\top} U^\top L^\top \Pi_1 && \text{(Π_2 orthogonal.)} \\
&= \Pi_2 U^{-1} (L^\top L)^{-1} L^\top \Pi_1 && \text{($U^{-\top} U^\top = \mathbb{I}$.)} \\
\implies (A^\dagger)^\top &= \Pi_1^\top L (L^\top L)^{-\top} U^{-\top} \Pi_2 && \text{(Taking transpose.)}
\end{aligned}$$

Then, when plugging in, we see that

$$\begin{aligned} \mathbf{w} &= (A^\dagger)^\top \mathbf{c} = \Pi_1^\top L (L^\top L)^{-1} U^{-\top} \Pi_2^\top \mathbf{c} \\ \implies \tilde{\mathbf{w}} &= L (L^\top L)^{-1} U^{-\top} \tilde{\mathbf{c}} \end{aligned}$$

Which is what we wanted to find.

Problem 4, part b

Write

$$\tilde{\mathbf{w}} = \begin{bmatrix} \tilde{\mathbf{w}}_1 \\ \tilde{\mathbf{w}}_2 \end{bmatrix}$$

where $\tilde{\mathbf{w}}_1 \in \mathbb{R}^n$ and $\tilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}$. Show that

$$\tilde{\mathbf{w}}_1 = L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} - S^\top \tilde{\mathbf{w}}_2$$

Solution:

We will go straight into calculations.

$$\begin{aligned}
\begin{bmatrix} \tilde{\mathbf{w}}_1 \\ \tilde{\mathbf{w}}_2 \end{bmatrix} &= L(L^\top L)^{-1} U^{-\top} \tilde{\mathbf{c}} && \text{(Given.)} \\
&= L\left[\begin{bmatrix} L_1^\top & L_2^\top \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\right]^{-1} U^{-\top} \tilde{\mathbf{c}} && \text{(Form of } L.\text{)} \\
&= L(L_1^\top L_1 + L_2^\top L_2)^{-1} U^{-\top} \tilde{\mathbf{c}} && \text{(Matrix multiplication.)} \\
&= L(L_1^\top L_1 + L_1^\top S^\top S L_1)^{-1} U^{-\top} \tilde{\mathbf{c}} && (S = L_2 L_1^{-1}.) \\
&= L[L_1^\top (\mathbb{I}_n + S^\top S) L_1]^{-1} U^{-\top} \tilde{\mathbf{c}} && \text{(Rearranging.)} \\
&= L L_1^{-1} (\mathbb{I}_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} && \text{(Taking inverse.)} \\
\implies \tilde{\mathbf{w}}_1 &= L_1 L_1^{-1} (\mathbb{I}_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \tilde{\mathbf{c}}, \\
\tilde{\mathbf{w}}_2 &= L_2 L_1^{-1} (\mathbb{I}_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} && \text{(Breaking up.)} \\
\implies \tilde{\mathbf{w}}_1 &= L_1 L_1^{-1} (\mathbb{I}_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \tilde{\mathbf{c}}, \\
\tilde{\mathbf{w}}_2 &= S L_1 L_1^{-1} (\mathbb{I}_n + S^\top S)^{-1} L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} && (S L_1 = L_2.) \\
\implies \tilde{\mathbf{w}}_1 + S^\top \tilde{\mathbf{w}}_2 &= (\mathbb{I} + S^\top S)(\mathbb{I} + S^\top S)^{-1} L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} && \text{(Combining terms.)} \\
&= L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} && \text{(Simplifying.)} \\
\implies \tilde{\mathbf{w}}_1 &= L_1^{-\top} U^{-\top} \tilde{\mathbf{c}} - S^\top \tilde{\mathbf{w}}_2 && \text{(Rearranging.)}
\end{aligned}$$

Problem 4, part c

Write $\mathbf{d} = L_1^{-\top} U^{-\top} \tilde{\mathbf{c}}$. Deduce that $\tilde{\mathbf{w}}_2$ may be obtained either as a solution to

$$\min_{\tilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} S^\top \\ \mathbb{I}_{m-n} \end{bmatrix} \tilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2$$

or as

$$\tilde{\mathbf{w}}_2 = (\mathbb{I}_{m-n} + SS^\top)^{-1} S\mathbf{d}.$$

Note that when $m - n < n$, this method is advantageous for the same reason in Problem 3.

Solution:

Note our original goal was to minimize $\|\tilde{\mathbf{w}}\|_2$ subject to $A^\top \mathbf{w} = \mathbf{c}$. If $\tilde{\mathbf{w}}_1 = d - S^\top \tilde{\mathbf{w}}_2$, then the problem is equivalent to

$$\min_{\tilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} d - S^\top \tilde{\mathbf{w}}_2 \\ \tilde{\mathbf{w}}_2 \end{bmatrix} \right\|_2$$

we are free to multiply by -1 in the norm, as well as pulling out the $\tilde{\mathbf{w}}_2$ term from both the top and the bottom to get

$$\min_{\tilde{\mathbf{w}}_2 \in \mathbb{R}^{m-n}} \left\| \begin{bmatrix} S^\top \\ \mathbb{I}_{m-n} \end{bmatrix} \tilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2$$

which is what we wanted. We then need to show that the given $\tilde{\mathbf{w}}_2$ is truly the minimum of this least squares problem. We can see the following steps hold:

$$\begin{aligned}
& \left\| \begin{bmatrix} S^\top \\ \mathbb{I}_{m-n} \end{bmatrix} \tilde{\mathbf{w}}_2 - \begin{bmatrix} \mathbf{d} \\ \mathbf{0} \end{bmatrix} \right\|_2 && \text{(Given.)} \\
& = (S^\top \tilde{\mathbf{w}}_2 - \mathbf{d})^\top (S^\top \tilde{\mathbf{w}}_2 - d) + \tilde{\mathbf{w}}_2^\top \tilde{\mathbf{w}}_2 && \text{(Expanding, 2-norm definition.)} \\
& = \tilde{\mathbf{w}}_2^\top SS^\top \tilde{\mathbf{w}}_2 - \tilde{\mathbf{w}}_2^\top S\mathbf{d} - \mathbf{d}^\top S^\top \tilde{\mathbf{w}}_2 + \mathbf{d}^\top \mathbf{d} + \tilde{\mathbf{w}}_2^\top \tilde{\mathbf{w}}_2 && \text{(Expanding.)} \\
& = \tilde{\mathbf{w}}_2^\top \left[(\mathbb{I}_{m-n} + SS^\top) \tilde{\mathbf{w}}_2 - S\mathbf{d} \right] + \mathbf{d}^\top (\mathbf{d} - S^\top \tilde{\mathbf{w}}_2) && \text{(Grouping.)} \\
& = \tilde{\mathbf{w}}_2^\top \left[(\mathbb{I}_{m-n} + SS^\top) \tilde{\mathbf{w}}_2 - S\mathbf{d} \right] + \mathbf{d}^\top \tilde{\mathbf{w}}_1 && \text{(By definition.)} \\
\implies & (\mathbb{I}_{m-n} + SS^\top) \tilde{\mathbf{w}}_2 - S\mathbf{d} = 0 && \text{(Minimizing condition.)} \quad (7) \\
\iff & \tilde{\mathbf{w}}_2 = (\mathbb{I}_{m-n} + SS^\top)^{-1} S\mathbf{d} && \text{(Rearranging.)}
\end{aligned}$$

Note step (7) is justified as the minimum since we are subject to $A^\top \mathbf{w} = \mathbf{c}$, which means $\tilde{\mathbf{w}}_1$ should be in general nonzero.

Problem 5

So far we have assumed that A has full column rank. Suppose now that $\text{rank}(A) = r \leq \min\{m, n\}$.

Problem 5, part a

Show that the LU factorization obtained using GECP is of the form

$$\Pi_1 A \Pi_2 = LU = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

where $L_1, U_1 \in \mathbb{R}^{r \times r}$ are triangular and nonsingular.

Solution:

In the notes, we see that GECP yields, for $\text{rank}(A) = r \leq \min\{m, n\}$,

$$\Pi_1 A \Pi_2 = LU = \begin{bmatrix} L_{11} & 0 \\ L_{21} & \mathbb{I}_{m-r} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & 0 \end{bmatrix}$$

Note that the form of L and U should be lower and upper triangular, respectively. This implies the zero entries on the super and sub diagonals of L and U. Note that $r < m$ - so the row-reduced echelon form of A will contain only zeros below the r-th row, hence the form of U. Furthermore, since L is unit lower triangular, the diagonal of L should contain only zeros, hence the identity matrix. The entries below the diagonal beyond the r-th column of L are necessarily zero, since the entries are given by $\ell_{ik} = a_{ik}^{(k)} / A_{kk}^{(k)}$, which, since $r < n$, we see that the k-th iteration ($k > r$) of A will be zero, due to row operations. Pivoting is used to enforce the unity triangularity of L. Note that L_1, U_1 correspond to the first r rows and columns of A's LU factorization, respectively. Since these matrices have full rank and are square, they are indeed nonsingular.

Problem 5, part b

Show that the above equation may be rewritten in the form

$$\Pi_1 A \Pi_2 = \begin{bmatrix} \mathbb{I}_r \\ S_1 \end{bmatrix} L_1 U_1 \begin{bmatrix} \mathbb{I}_r & S_2^\intercal \end{bmatrix}$$

for some matrices S_1 and S_2.

Solution:

We can work backwards to show this holds.

$$\begin{aligned}
\Pi_1 A \Pi_2 &= \begin{bmatrix} \mathbb{I}_r \\ S_1 \end{bmatrix} L_1 U_1 \begin{bmatrix} \mathbb{I}_r & S_2^\intercal \end{bmatrix} && \text{(Working backwards.)} \\
&= \begin{bmatrix} L_1 \\ S_1 L_1 \end{bmatrix} \begin{bmatrix} U_1 & U_1 S_2^\intercal \end{bmatrix} && \text{(Moving } L_1, U_1 \text{ inside.)} \\
&= \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix} && \text{(Want to relate.)}
\end{aligned}$$

If wet want the second line to equal the third line above, we require $SL_1 = L_2$ and $U_1 S_1^\intercal = U_2$. These then imply $S_1 = L_2 L_1^{-1}, S_2 = U_2^\intercal U_1^{-\intercal}$. Then taking the above steps in reverse shows us these two expressions for the LU GECP of A are equivalent.

Problem 5, part c

Hence show that the Moore-Penrose pseudoinverse of A is given by

$$A^{\dagger} = \Pi_2 \begin{bmatrix} \mathbb{I}_r & S_2^{\intercal} \end{bmatrix}^{\dagger} U_1^{-1} L_1^{-1} \begin{bmatrix} \mathbb{I}_r \\ S_1 \end{bmatrix}^{\dagger} \Pi_1$$

Solution:

To show this is the pseudoinverse of A, we can show it holds the four properties for a pseudoinverse. Note the first pseudoinverse inside $A^{\dagger}$ is full row rank and the second is full column rank, thus they will act like weak inverses on each side of their respective matrix. For simplicity in reading an typing, let $C^{\intercal} = \begin{bmatrix} \mathbb{I}_r & S_1^{\intercal} \end{bmatrix}^{\intercal}$ and $D = \begin{bmatrix} \mathbb{I}_r & S_2^{\intercal} \end{bmatrix}$. Then $DD^{\dagger} = \mathbb{I}, C^{\dagger}C = \mathbb{I}$.

1. $\underline{AA^{\dagger}A = A}$:

$$\begin{aligned}
&\Pi_1^{\intercal} C L_1 U_1 D \Pi_2^{\intercal} \Pi_2 D^{\dagger} U_1^{-1} L_1^{-1} C^{\dagger} \Pi_1 \Pi_1^{\intercal} C L_1 U_1 D \Pi_2^{\intercal} && \text{(Given.)} \\
=&\Pi_1^{\intercal} C L_1 U_1 D D^{\dagger} U_1^{-1} L_1^{-1} C^{\dagger} C L_1 U_1 D \Pi_2^{\intercal} && (\Pi_i^{\intercal} \Pi_i = \mathbb{I}.) \\
=&\Pi_1^{\intercal} C L_1 U_1 U_1^{-1} L_1^{-1} L_1 U_1 D \Pi_2^{\intercal} && \text{(Explained above.)} \\
=&\Pi_1^{\intercal} C L_1 U_1 D \Pi_2^{\intercal} && \text{(Inverses cancel.)} \\
=&A
\end{aligned}$$

---

2. $\underline{A^{\dagger}AA^{\dagger} = A^{\dagger}}$:

$$\begin{aligned}
&\Pi_2 D^{\dagger} U_1^{-1} L_1^{-1} C^{\dagger} \Pi_1 \Pi_1^{\intercal} C L_1 U_1 D \Pi_2^{\intercal} \Pi_2 D^{\dagger} U_1^{-1} L_1^{-1} C^{\dagger} \Pi_1 && \text{(Given.)} \\
=&\Pi_2 D^{\dagger} U_1^{-1} L_1^{-1} C^{\dagger} C L_1 U_1 D D^{\dagger} U_1^{-1} L_1^{-1} C^{\dagger} \Pi_1 && (\Pi_i^{\intercal} \Pi_i = \mathbb{I}.) \\
=&\Pi_2 D^{\dagger} U_1^{-1} L_1^{-1} L_1 U_1 U_1^{-1} L_1^{-1} C^{\dagger} \Pi_1 && \text{(Explained above.)} \\
=&\Pi_2 D^{\dagger} U_1^{-1} L_1^{-1} C^{\dagger} \Pi_1 && \text{(Inverses cancel.)} \\
=&A^{\dagger}
\end{aligned}$$

---

3. $\underline{(AA^\dagger)^\intercal = AA^\dagger}$:

$$\left(\Pi_1^\intercal C L_1 U_1 D \Pi_2^\intercal \Pi_2 D^\dagger U_1^{-1} L_1^{-1} C^\dagger \Pi_1\right)^\intercal \qquad \text{(Given.)}$$

$$= \left(\Pi_1^\intercal C L_1 U_1 D D^\dagger U_1^{-1} L_1^{-1} C^\dagger \Pi_1\right)^\intercal \qquad (\Pi_i^\intercal \Pi_i = \mathbb{I}.)$$

$$= \left(\Pi_1^\intercal C L_1 U_1 U_1^{-1} L_1^{-1} C^\dagger \Pi_1\right)^\intercal \qquad \text{(Explained above.)}$$

$$= \left(\Pi_1^\intercal C C^\dagger \Pi_1\right)^\intercal \qquad \text{(Inverses cancel.)}$$

$$= \Pi_1^\intercal (C C^\dagger)^\intercal \Pi_1 \qquad \text{(Taking transpose.)}$$

$$= \Pi_1^\intercal C C^\dagger \Pi_1 \qquad \text{(Pseudoinverse property.)}$$

---

4. $\underline{(A^\dagger A)^\intercal = A^\dagger A}$:

$$\left(\Pi_2 D^\dagger U_1^{-1} L_1^{-1} C^\dagger \Pi_1 \Pi_1^\intercal C L_1 U_1 D \Pi_2^\intercal\right)^\intercal \qquad \text{(Given.)}$$

$$= \left(\Pi_2 D^\dagger U_1^{-1} L_1^{-1} C^\dagger C L_1 U_1 D \Pi_2^\intercal\right)^\intercal \qquad (\Pi_i^\intercal \Pi_i = \mathbb{I}.)$$

$$= \left(\Pi_2 D^\dagger U_1^{-1} L_1^{-1} L_1 U_1 D \Pi_2^\intercal\right)^\intercal \qquad \text{(Explained above.)}$$

$$= \left(\Pi_2 D^\dagger D \Pi_2^\intercal\right)^\intercal \qquad \text{(Inverses cancel.)}$$

$$= \Pi_2 (D^\dagger D)^\intercal \Pi_2^\intercal \qquad \text{(Taking transpose.)}$$

$$= \Pi_2 D^\dagger D \Pi_2^\intercal \qquad \text{(Pseudoinverse property.)}$$
