Lecture 7 Notes

Looked at fixed points - what about orbits?

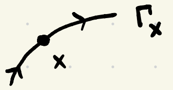
Want to see asymptotic behavior: what happens in limit as $t \to \infty$ or $-\infty$?

Sec 4.9

ODEs generate complete flows (mod a reparameterization of time as in HW 2)

Consider some flow $\varphi_t(x)$.

Orbit: $\Gamma_x = \{\varphi_t(x) : t \in \mathbb{R}\}$ (all points in forward and backward time)



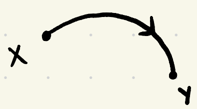
$\Gamma_x^+ = \{\varphi_t(x) : t \in \mathbb{R}^+\}$

$\Gamma_x^- = \{\varphi_t(x) : t \in \mathbb{R}^-\}$

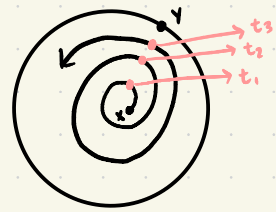
GOAL: In dynamical systems, we want a geometric classification of the types of orbits that occur in a flow.

Limit point

y is a limit point of Γ_x^+ if $\exists$ a sequence $t_1 < t_2 < \cdots < t_k < \cdots$ with $t_k \to \infty$ and $\varphi_{t_k}(x) \to y$ as $k \to \infty$



Why t_k, not t?



ω-limit set (for $t \to \infty$)

The collection of all limit points of Γ_x^+, denoted $\omega(x)$

Note: if $z \in \Gamma_x$ (a point in the orbit), then $\omega(z) = \omega(x)$ since they limit on the same set of points by uniqueness

So we can instead write $\omega(\Gamma_x)$ (for entire trajectory)

α-limit set (for $t \to -\infty$)

The collection of all limit points of Γ_x^-.

Examples

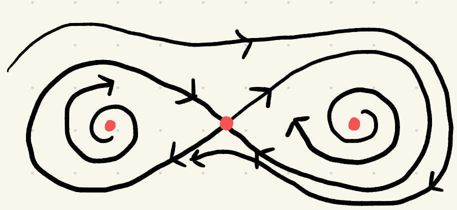
Let x^* be an equilibrium. Then $\varphi_t(x^*) = x^*$ $\forall t \in \mathbb{R}$. So $\omega(x^*) = \alpha(x^*) = x^*$

Let x^* be asymptotically stable. Recall what this means - $\exists$ neighborhood N s.t. if $x(0) \in N$, $x(t) \to x^*$

Then $\exists$ $N(x^*)$ s.t. $\omega(N(x^*)) = x^*$

↖ ω limit set of a set of points

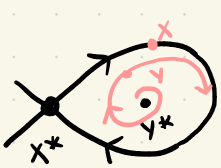
Note: ω- and α-limit sets need not be a single point



(saddle point, 2 unstable fixed points)

1. What are the α- and ω- limit sets of the 3 fixed points?

$\omega(x^*) = \alpha(x^*) = x^*$

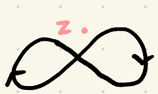


↙ homoclinic orbit γ (approach same point in forward + backward time)

2. Let $x \in \gamma$. $\omega(x) = \alpha(x) = x^*$

$\omega(y) = \gamma$, $\alpha(y) = y^*$

Note: Points in $\varphi_t(y)$ limit on γ as $t \to \infty$, but never reach γ.

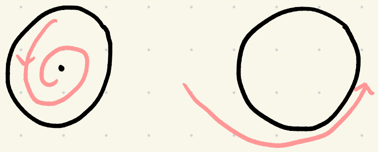
3. 

ω-limit set of z is the entire "figure 8"

Limit cycle

Motivated by $\omega(y) = \gamma$. A periodic orbit γ that is the ω- or α-limit cycle of a point $x \notin \gamma$.

Invariant loop with a nearby orbit spiraling away from or towards it.



limit cycles

not limit cycle

Properties of ω-limit sets: (proofs p. 139)

1. Closure: ω-limit set is closed. (contains all its limit points)
2. Invariance: If $y \in \omega(x)$ then $\varphi_t(y) \in \omega(x)$ $\forall t > 0$

Now suppose an orbit is bounded. Then

3. Existence: ω-limit set of a bounded orbit is nonempty (if not bounded, then some points are going to $\pm\infty$)
4. Compact + connected: ω-limit set of a bounded orbit is compact and connected

Poincaré - Bendixson Thm

Thm: If the forward orbit $\Gamma^+(\underline{x})$ of some pt $\underline{x}$ remains in a compact (closed + bounded) set $K \subset \mathbb{R}^2$ that contains no FPs, then the $\omega(\underline{x})$ is a periodic orbit

Ex: $\dot{x} = x - y - 2x(x^2+y^2)$
$\dot{y} = y + x - y(x^2+y^2)$

$\exists$ FP at $\underline{0}$ (an unstable focus)
cubic nonlinear terms push inwards

Try for an annulus in polars as "trapping region"

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y}) = \frac{1}{r}(x[x-y-2x(x^2+y^2)] + y[y+x-y(x^2+y^2)])$$
$$= \frac{1}{r}(r^2 - 2x^2r^2 - y^2r^2)$$
$$= r(1 - x^2 - r^2)$$
$$= r(1 - r^2(1+\cos^2\theta))$$

$$1 - 2r^2 \le 1 - r^2(1+\cos^2\theta) \le 1 - r^2$$

So $\dot{r} > 0$ for $0 < r < \frac{1}{\sqrt{2}}$ and $\dot{r} < 0$ for $r > 1$

So consider $K = \{\frac{1}{\sqrt{2}} \le r \le 1\}$ — once inside K never leave

check for FPs: $\dot{\theta} = \frac{1}{r^2}(x\dot{y} - y\dot{x}) = 1 + \frac{1}{2}r^2\sin 2\theta > 0$ in K

so $\dot{\theta} \neq 0 \Rightarrow$ no FPs

So apply Poincaré - Bendixson $\Rightarrow \exists$ PO in K

Hamiltonian systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \partial H/\partial y \\ -\partial H/\partial x \end{pmatrix}$$ Hamiltonian $H(x,y)$ "energy"

$$\frac{dH}{dt} = \dot{x}H_x + \dot{y}H_y = 0$$ $\leftarrow$ H is a conserved quantity

trajectories on the contours of $H(x,y) = $ const.
$\Rightarrow$ many POs

Writing a system in Hamiltonian form with perturbations is a useful approach to finding conditions for POs

Suppose $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} H_y + g_1(x,y) \\ -H_x + g_2(x,y) \end{pmatrix}$$

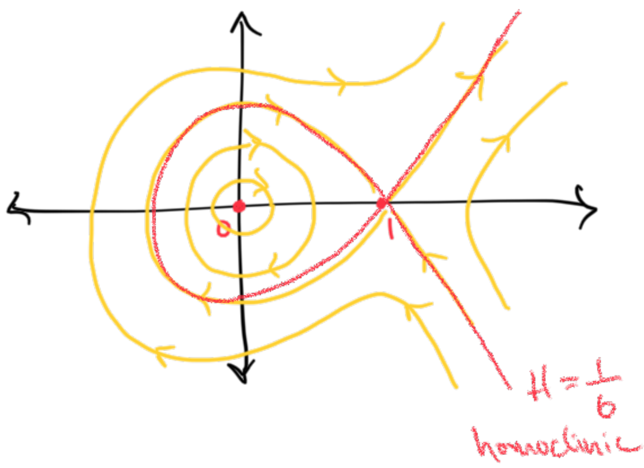
then $$\frac{dH}{dt} = g_2\dot{x} - g_1\dot{y}$$

$H_x\dot{x} + H_y\dot{y}$
$= (g_2 - \dot{y})\dot{x} + (\dot{x} - g_1)\dot{y}$

if there is a closed orbit c, $\oint_c dH = \oint_c g_2\dot{x} - g_1\dot{y} = 0$
when this line integral cannot vanish $\Rightarrow$ no POs

Ex: $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} y \\ -x + x^2 + \varepsilon y(a-x)\end{pmatrix}$ $\quad \left.\begin{aligned} H_y &= y \\ -H_x &= -x + x^2\end{aligned}\right\} H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$

FPs: $(0,0)$, $(1,0)$



a) $\varepsilon = 0$
POs exist for $0 < H < \frac{1}{6}$

b) $\varepsilon \neq 0$
1) $\dot{H} = \varepsilon y^2(a-x)$
so any PO must straddle $x = a$

Notes on Poincaré-Bendixson

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October 19, 2023

1 Planer Systems

Consider the ODE:

$$u'(t) = F(u(t)) \tag{1}$$

$F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 (Lipschitz is also fine), the Picard-Lindelöf theorem tells us that any initial condition, there exists a unique maximal solution (i.e biggest domain that this function is defined).

Definition 1.1 (Flow Map)**.** Let $u_0 \in \mathbb{R}^n$ and u(t) be the unique maximal solution to (1), t be the maximal interval of existence and initial condition u_0.

$$\phi_t(u_0) = u(t)$$

From Picard-Lindelöf we know that $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, for maximal solution domain t.

Definition 1.2 (ω limit set)**.** Let u_0 be such that $\phi_t(u_0)$ is defined for all $t > 0$ then its ω limit set is just

$$\Omega := \{x \in \mathbb{R}^n : \phi_{t_k}(u_0) \to x \text{ for some sequence } t_k \to \infty\}$$

Proposition 1.1. Let Ω be an ω - limit set of the flow $\phi_t(u_0)$

1. Ω is closed.
2. If $x \in \Omega$, then $\phi_t(x) \in \Omega$.
3. If Ω is bounded, then it is connected.

Proof. (1) By definition $\Omega = \cap_{\tau \geq 0} \overline{\{u(t) : t \geq \tau\}}$, intersection of closed set is closed.
(2) If $x \in \Omega$, then $\exists\ s_k \to \infty$ such that $x = \lim_{k\to\infty} \phi_{s_k}(u_0)$,

$$\phi_t(x) = \phi_t(\lim_{k\to\infty} \phi_{s_k}(u_0)) = \lim_{k\to\infty} \phi_{t+s_k}(u_0)$$

therefore $\phi_t(x) \in \Omega$
(3) Exercise to reader. □

Some trivial example of ω limit sets:

1. u_0 is an equilibrium point(i prefer to call it fixed point), then it is its own ω limit sets
2. u^* is an equilibrium point then $\phi_t(u_0) \to u^*$ as $t \to \infty$, ω limit sets is just u^*

3. Let $\phi_t(u_0)$ be a periodic solution, ω limit sets is the $\text{Im}(\phi_t(u_0))$.

Theorem 1.1 (Poincaré-Bendixson)**.** *For a planer ODE, let Ω be a non-empty bounded and compact (need for BolzanoWeierstrass later) ω limit set. Then either Ω contains at least one equilibrium point, or there exists a periodic solution $u(t)$ such that $u(t)$ is exactly Ω.*

Remark 1.1. *This is false for $\mathbb{R}^n$ for $n > 2$, (i.e Lorenz attractor), Theorem 1.1 is equivalent to saying there is no "chaos" in planer dynamics.*

1.1 Transversal Line

Consider the ODE system (1) in $\mathbb{R}^2$.

Definition 1.3 (Transversal lines)**.** A *transversal* of ϕ-flow is a line segment $S = \lambda x_0 + (1-\lambda)x_1 : \lambda \in (0,1)$, defined by two points x_0 and x_1 such that RHS- $F(x)$ of the differential equation is nowhere parallel and non-vanishing ($Fx \neq 0$) to S.

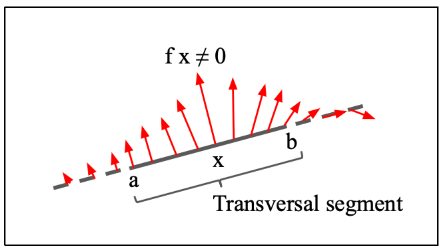

Figure 1: Transversal segment from [2]

Lemma 1.1. *Let S be a transversal line and $x_0 \in S$, then $\exists U$ an open set, $x \in U$ and C^1 function $\tau : U \to \mathbb{R}$ such that $\phi_{\tau(x)}(x) \in S$ for all $x \in U$.*

Proof. (y, z) be the coordinate in $\mathbb{R}^2$. W.L.O.G., assume $x_0 = (0,0)$ and S is a subset of $y = 0$. For $\epsilon > 0$, $\Phi : B_{\mathbb{R}}(0,\epsilon) \times B_{\mathbb{R}}(0,\epsilon) \to \mathbb{R}$ by

$$\Phi(t, x) = \pi\phi_t(x),$$

where $\pi = \mathbb{R}^2 \to \mathbb{R}$ is the map $(y, z) \mapsto y$ $\square$

1.2 Monotonicity Property

Proposition 1.2. Let S be a transversal line and $u(t)$ be the solution to the ODE. Suppose $u(t_0)$, $u(t_1)$ and $u(t_2)$ be three points on S with order $t_0 < t_1 < t_2$ such that uniquessness hold $u(t_0) \neq u(t_1)$, then they are monotone on S. To put it differently, successive intersections of a trajectory with transversal line is are ordered increasingly (or decreasingly) along the segment.

Proof. Consider the curve $\Gamma \subset \mathbb{R}^2$ given by

$$\Gamma := \{u(t) : t \in [t_0, t_1]\} \cup \{x \in S : x \text{ is between } u(t_0) \text{ and } u(t_1)\}.$$

By Jordan curve theorem, Γ divides $\mathbb{R}^2$ into two components, D_1 and D_2. Since $F(u(t_1))$ is transversal to S, u must either enter D_1 or D_2 after t_1. Suppose that u enters D_1 after t_1 (i.e there exist $\epsilon > 0$ such that $u(t) \in D_1$ for $t \in (t_1, t_2 + \epsilon)$) We claim that $u(t) \in D_1$ for all $t > t_1$, assume not then there exits a time $t_* > t_1$ such that $u(t_*) \in \Gamma$, this is impossible since this contradicts the uniqueness of solution. This implies that $u(t_2) \in int(D_1)$ it follows that the points $u(t_0)$, $u(t_1)$ and $u(t_2)$ are monotone along S.

$\square$

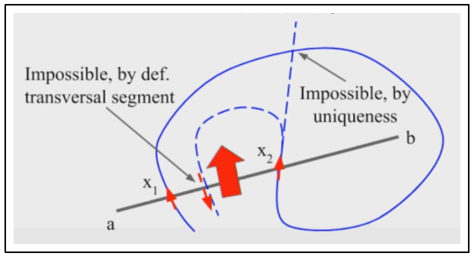

Figure 2: Monotonicity argument from [2]

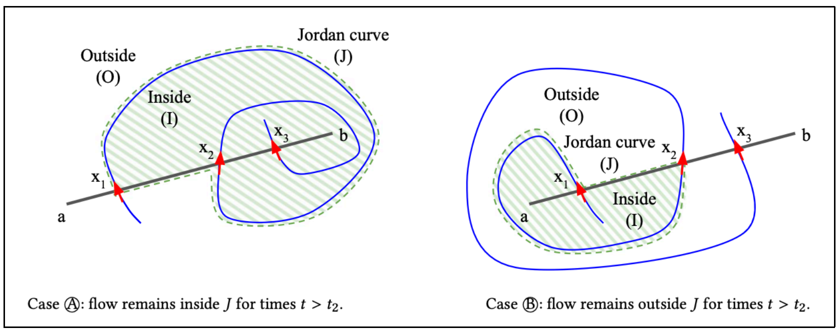

Figure 3: Applying Jordan Curve Theorem from [2]

Proposition 1.3. Let $u(t)$ be a solution to the ODE system (1) and Ω be the corresponding ω limit set. Let S be a transversal line. Then $S \cap \Omega$ has at most one point.

Proof. Suppose not, assume $x_1, x_2 \in S \cap \Omega$ are distinct, by Lemma 1.1 ... left as an exercise to the reader. $\square$

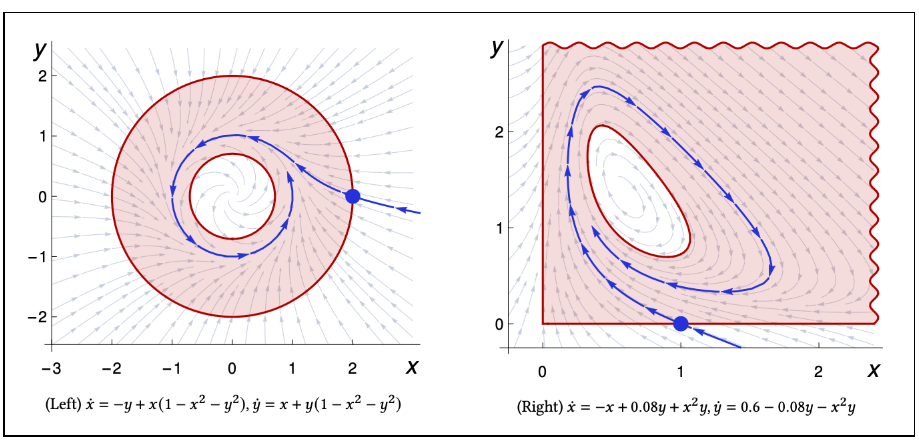

Figure 4: Trapping Region from [2]

1.3 Proof of Poincaré-Bendixson

For the proof of Poincaré-Bendixson, let Ω be a non-empty, bounded ω limit set with no equilibrium points. The objective is to establish that Ω corresponds to a periodic solution. This is achieved through two distinct steps. In the initial step, it is demonstrated that the flow originating from any point within Ω is periodic. Subsequently, in the second step, prove that this periodic solution encompasses the entirety of Ω.

Proposition 1.4. Let Ω be the ω-limit set associated with the solution $\phi_t(u_0)$. Suppose Ω is non-empty, bounded, compact, and has no equilibrium points. If $y \in \Omega$, then $\phi_t(y)$ is a periodic solution.

Proof. By Proposition 1.1, $\phi_t(y) \in \Omega$ as long as it is defined. In particular, by the extension theorem, this implies $\phi_t(y)$ is defined for all $t \geq 0$. Since $\{\phi_t(y)\}_{t\geq 0}$ is bounded, by the Bolzano-Weierstrass theorem, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \infty$ and a point z such that $\varphi_{t_k}(y) \to z$. Since $\phi_{t_k}(y) \in \Omega$ for all $k \in \mathbb{N}$ and Ω is closed (by Proposition 1.1), this implies $z \in \Omega$. In particular, z is not an equilibrium point. Therefore, we can find a transversal line S passing through z.

Since $\phi_t(y) \to z$, by Lemma 1.2, there exists $\tilde{t}_k \to \infty$ such that $\phi_{\tilde{t}_k}(y) \to z$ and $\phi_{\tilde{t}_k}(y) \in S$. Since $t_k \neq \tilde{t}_k$, $\phi_{\tilde{t}_k}(y) \in S \cap \Omega$, by Proposition 2.2, they must, in fact, be the same point for all $k \in \mathbb{N}$.

In particular, there exist $\tilde{t}_{k_1} \neq \tilde{t}_{k_2}$ such that $\phi_{\tilde{t}_{k_1}}(y) = \phi_{\tilde{t}_{k_2}}(y)$. This implies that $\phi_t(y)$ is a periodic solution. $\square$

Proposition 1.5. Let Ω be the ω-limit set associated with the solution $\phi_t(u_0)$. Suppose Ω is non-empty, bounded, and has no equilibrium points, and let $y \in \Omega$. Then $\Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\} = \emptyset$.

Proof. It suffices to show that $\Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\}$ is closed. Proposition 1.1 ensures the connectedness of Ω, enabling us to decompose Ω as the union of two disjoint closed sets: $\Omega = \left(\bigcup_{t\geq 0}\{\phi_t(y)\}\right) \cup \left(\Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\}\right)$. One of these sets must be empty, and since $\bigcup_{t\geq 0}\{\phi_t(y)\} \neq \emptyset$, it follows that $\Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\} = \emptyset$.

Consider a sequence of points $\{z_k\}_{k=1}^{\infty} \subset \Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\}$ with $z_k \to z$. We aim to show that $z \in \Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\}$. This implies $\Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\}$ is closed.

Since Ω is closed, $z \in \Omega$; notably, z cannot be an equilibrium point. Therefore, a transversal line S containing z can be found. For every $k \in \mathbb{N}$, since $z_k \in \Omega$, there exists a sequence of times $\{t_{k,l}\}_{l=1}^{\infty}$ (with $t_{k,l} \to \infty$ as $l \to \infty$) such that $\phi_{t_{k,l}}(u_0) \to z_k$ as $l \to \infty$. By Lemma 1.1, for sufficiently large k, we can identify $\{\tilde{t}_k\}_{l=1}^{\infty}$ (with $\tilde{t}_k \to \infty$ as $l \to \infty$) such that $\phi_{\tilde{t}_k}(u_0) \to z$ as $l \to \infty$, and furthermore, $\phi_{\tilde{t}_k}(u_0) \in S$ for all $l \in \mathbb{N}$. Proposition 2.2 asserts that all these points coincide, implying $\phi_{\tilde{t}_k}(u_0) = z$ for all k sufficiently large and for all $l \in \mathbb{N}$. Consequently, $z_k = z$ for all k sufficiently large. Given that $z_k \in \Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\}$, it follows that $z \in \Omega \setminus \bigcup_{t\geq 0}\{\phi_t(y)\}$. $\square$

Combining Proposition 1.4 and 1.5 yields the proof of Poincaré-Bendixson.

References

[1] Morris W. Hirsch, Stephen Smale, Robert L. Devaney, *Differential Equations, Dynamical Systems,and an Introduction to Chaos (Third Edition)*, Acadamic Press, 2013.

[2] Immler, Fabian and Tan, Yong Kiam *The Poincaré-Bendixson Theorem in Isabelle/HOL*, Association for Computing Machinery, 2020.

[3] Jonathan Luk, *Notes on The Poincaré-Bendixson Theorem*, Stanford, 2023.