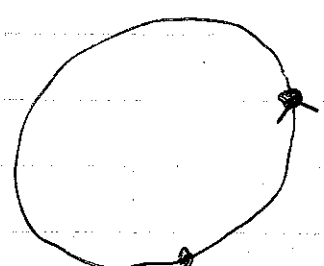
Limit cycles & their stability – 2 approaches

(1) Compute Floquet multipliers from Monodromy Matrix

(2) Compute Floquet multipliers by linearizing about a fixed pt. of a Poincaré return map.

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \text{ is } C^1, \quad n \geq 2$$

This is an autonomous problem. Assume it has a limit cycle $\gamma(t)$, i.e. it can produce "spontaneous oscillations" that have some period T



$\gamma(t) = \varphi_t(x_0)$, $t \in [0, T)$ could restrict

$x_0 = \varphi_T(x_0)$ ← true for every x_0 in γ.

Note: usually we don't know T, but we could try to estimate it.

For linear stability, let $x(t) = \gamma(t) + y(t)$

$$\Rightarrow \dot{x} = \cancel{\dot{\gamma}} + \dot{y} = f(\gamma + y) \approx \cancel{f(\gamma(t))} + D_x f(\gamma(t))\, y \cancel{+ \cdots}$$

linearized problem:

$$\dot{y} = \underbrace{D_x f(\gamma(t))}_{A(t)} y$$

$\gamma(t) = \gamma(t+T)$, so $A(t) = A(t+T)$

We've seen this before for the forced pendulum, but there we had a non-autonomous problem $\dot{x} = f(x,t) = f(x, t+T)$, where we examined the problem linearized about a fixed-pt.
Now we have an autonomous problem, linearized about a limit cycle.
We'll see there are some differences in computing the stability properties

$$\dot{y} = A(t)\, y, \quad A(t) = A(t+T)$$

recall how we calculated the Floquet multipliers as eigenvalues of the Monodromy matrix

$$\frac{d\Phi}{dt} = A(t)\,\Phi, \quad \Phi(0) = Id.$$

$\Phi(t)$ = Fundamental matrix solution
if $y(0) = y_0 \Rightarrow y(t) = \Phi(t)\, y_0$

Monodromy matrix $= M = \Phi(T)$

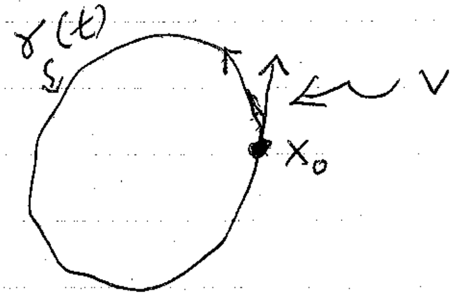
$$y_0 \xrightarrow{M} y_1 = My_0 \xrightarrow{M} y_2 = M^2 y_0 \longrightarrow$$

M advances the initial perturbation y_0 to its value time T later.

If any eigenvalue of M, μ, satisfies $|\mu| > 1$, then the perturbation is growing and the limit cycle is unstable.

Previously, we said if all eigenvalues μ of M satisfy $|\mu| < 1$ then ~~the limit cycles~~ perturbations decay & we have asymptotic stability.

However, we are guaranteed to have one eigenvalue $\mu = 1$, ie there exists a perturbation v s.t. $Mv = v$.
We can show this by an explicit computation of v



is tangent to the limit cycle at $t=0$

i.e. let $v = \dot{\gamma}(t)\big|_{t=0}$

& show $Mv = v$.

Proof

If $\gamma(t)$ solves $\dot{x} = f(x)$, then so does $\gamma(t+\tau)$ for any τ

given $\dot{\gamma}(t) = f(\gamma(t))$

$\Rightarrow \dot{\gamma}(t+\tau) = f(\gamma(t+\tau))$

$\gamma(t) = \gamma(t+T)$ because equation is autononomous

$$\frac{d}{d\tau}\left[\dot{\gamma}(t+\tau) = f(\gamma(t+\tau)) \right]\Bigg|_{\tau=0}$$

$$\frac{d\dot{\gamma}}{dt} = \underbrace{Df(\gamma(t))}_{A(t)}\,\dot{\gamma}(t)$$

Thus $\dot{\gamma}(t)$ satisfies $\dot{y} = A(t)y$

& we can write $\dot{\gamma}(T) = M\dot{\gamma}(0)$

but $\gamma(t)$ is T-periodic so $\dot{\gamma}(t)$ is also T periodic

$\dot{\gamma}(T) = \dot{\gamma}(0)$

& $\dot{\gamma}(T) = M\dot{\gamma}(0)$

so $M\dot{\gamma}(0) = \dot{\gamma}(0)$

& $\mu = 1$ is an eigenvalue of M.

A couple of side observations:

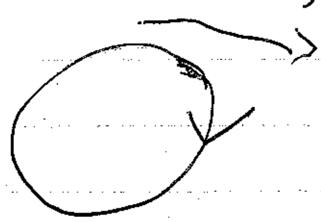
(1) A limit cycle in the phase plane

$\mu_1 = +1$, μ_2

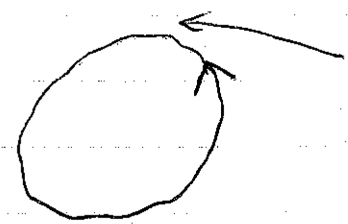
Abel's Thm. $\quad \mu_2 = \mathrm{Det}\,(M) = \exp\left[\int_0^T \mathrm{Tr}(A(s))\,ds\right]$

where $A(s) = Df(\gamma(s))$

(2) An unstable limit cycle in $\mathbb{R}^2$ $(\mu_2 > 1)$ for $\dot{x} = f(x)$ is a stable one for $\dot{x} = -f(x)$, i.e., let $t \to -t$



$\dot{x} = f(x)$

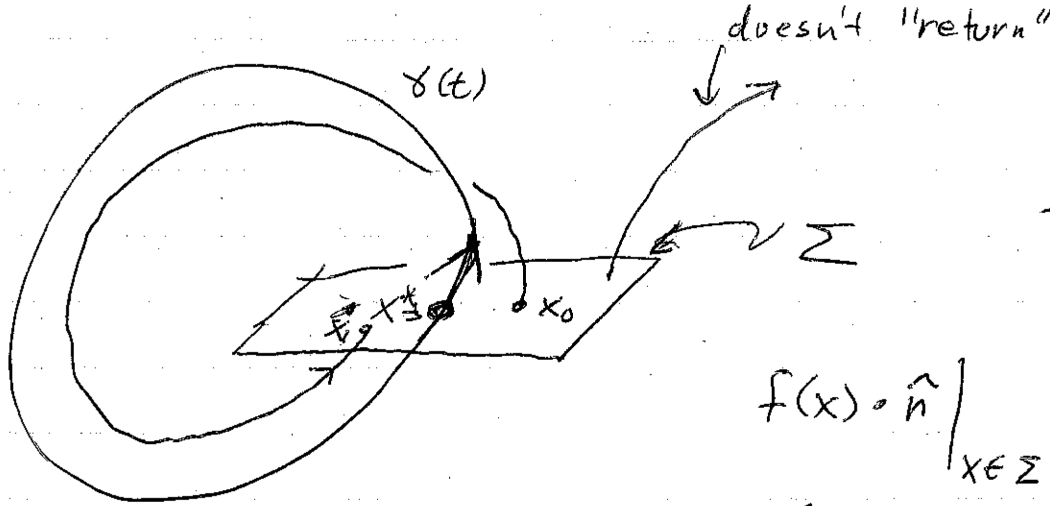


$\dot{x} = -f(x)$

Second approach to examining stability:
Poincaré return maps.

Introduce a "surface of section" Σ that's transverse to the flow near $\gamma(t)$.

(Σ may only be defined in some small region.)

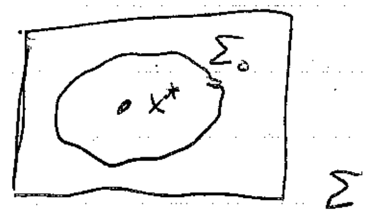


$$f(x)\cdot\hat{n}\Big|_{x\in\Sigma} > 0$$

$\hat{n}$ = surface normal

Poincaré return map P

$P: \Sigma_0 \to \Sigma\ ,\quad \Sigma_0 \subset \Sigma$



$P(x^*) = x^*$, which is the pt. on the ~~periodic~~ limit cycle $\gamma(t)$ that is in Σ

$P(x_0) = x_1$

$P(x) = \varphi_{\tau(x)}(x)$, where $\tau(x)$ = first return time to Σ

$\tau(x) > 0$

$P(x)$ is an $(n-1)$-dim map

$x_{k+1} = P(x_k)$, $P(x^*) = x^*$

linearize about $x = x^*$: $\quad x_k = x^* + y_k$

$x_{k+1} = x^* + y_{k+1}$

$$x^* + y_{k+1} = P(x^* + y_k)$$

$$= \underbrace{P(x^*)}_{=x^*} + D_x P(x^*)\, y_k \cancel{+ \cdots}$$

linearized map is

$$y_{k+1} = D_x P(x^*)\, y_k$$

can determine whether perturbations y_0 are growing or decaying via eigenvalues of $D_x P(x^*)$. How are these related to the Floquet multipliers of M?

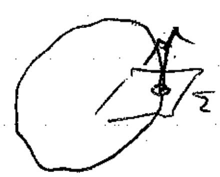
<u>Thm.</u> 4.55 in text (p.151)

$$\operatorname{spec}(M) = \operatorname{spec}(D_x P(x^*)) \cup \{1\}$$

n Floquet multipliers on $n \times n$ Monodromy matrix

$(n-1)$ eigenvalues of linearized Poincaré return map $(n-1) \times (n-1)$-dim matrix

associated with perturbations along γ



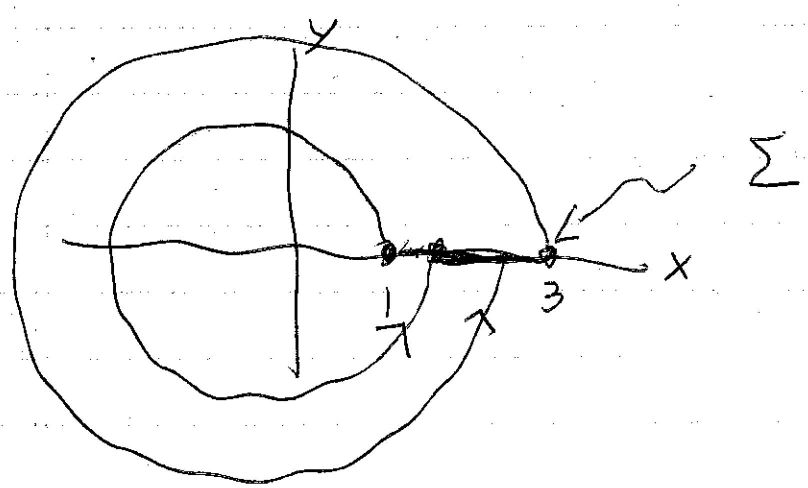
Problem on PS #4

$\dot{x} = 2y$
$\dot{y} = -2x + \frac{1}{2}(1-x^2)y$

has a unique fixed pt. $(x,y) = 0$

$$Df = \begin{bmatrix} 0 & 2 \\ -2-xy & \frac{1}{2}(1-x^2) \end{bmatrix} \qquad Tr = \frac{1}{2}(1-x^2)$$

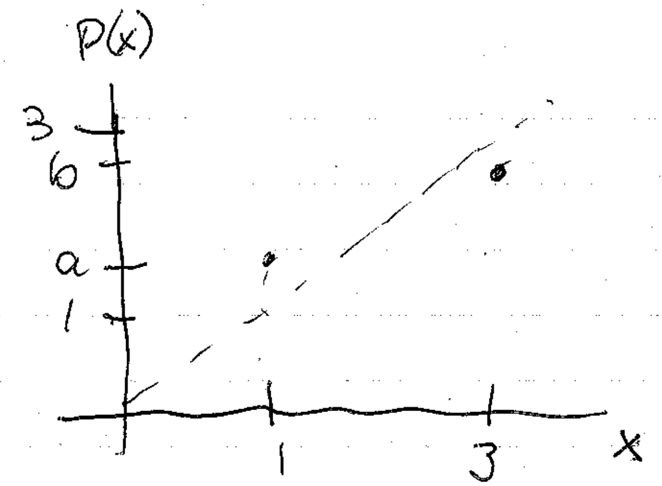
$$Df(0,0) = \begin{bmatrix} 0 & 2 \\ -2 & \frac{1}{2} \end{bmatrix} \qquad \mu_{\pm} = \frac{1}{4} \pm \frac{3}{4} i\sqrt{7}$$



$P(1) = a$
$P(3) = b$

$P[1,3] = [a,b]$

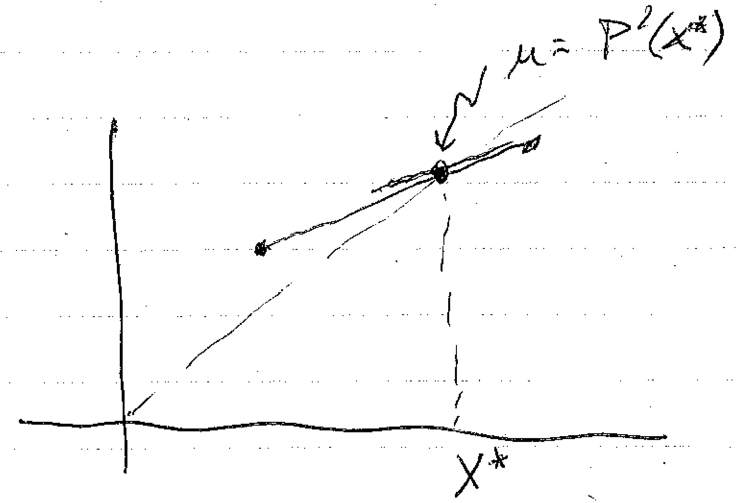
every pt. in Σ returns to Σ



must have at least one fixed pt by intermediate value thm.

What do we know about possible shape of $P(x)$?

monotonically increasing function?
or non-decreasing?



Monodromy matrix approach
Find period T & pt. on limit cycle