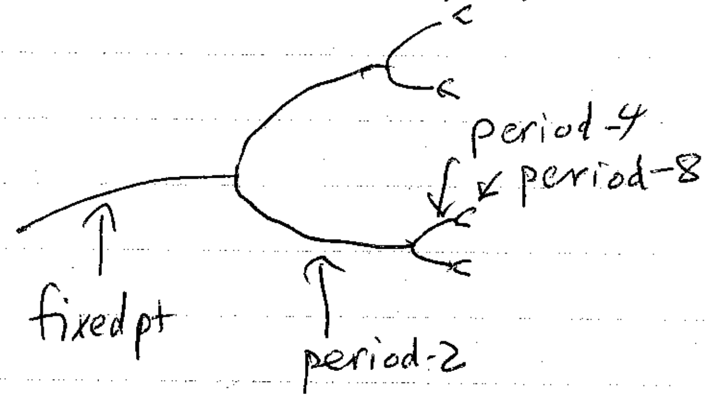
Some background to the Rössler problem on PS#4

Recall logistic map $x_{n+1} = f(x_n) = \lambda x_n (1-x_n)$
$x \in [0,1]$, $\lambda \in (0,4]$

"Orbit diagram"



This shows "period-doubling" cascade. Note: only attractors are plotted, not the unstable orbits.

For example, the nontrivial fixed-pt. $x^* \neq 0$ exists for $\lambda > 1$, but it is not stable for all $\lambda \geq 1$.

$$x^* = \lambda x^* (1-x^*) \Rightarrow 1-x^* = \frac{1}{\lambda} \Rightarrow x^* = 1 - \frac{1}{\lambda}$$

$x^* \in (0,4]$
$\lambda > 1$

Stability? $x_n = x^* + y_n$
$x_{n+1} = x^* + y_{n+1}$

$x_{n+1} = f(x_n)$

linearized about x^*: $y_{n+1} = \underbrace{f'(x^*)}_{\mu} y_n$

$$\mu = \lambda(1-2x^*) = \lambda\left(1-2+\frac{2}{\lambda}\right) = \lambda\left(-1+\frac{2}{\lambda}\right)$$
$= 2-\lambda$, $\lambda \in (0,4]$, x^* is stable if $|\mu| < 1$

$|\mu| < 1$ for $\lambda \in (1,3)$

at $\lambda = 3$, $\mu = -1$

What happens at $\lambda = 3$?

Period-2 cycle is born

Period-2 orbit: $f^2 = f \circ f$ develops a pair of fixed-pts

$a \to b \to a \to b$

$$x_{n+1} = \lambda x_n (1 - x_n)$$
$$x_{n+2} = \lambda x_{n+1} (1 - x_{n+1})$$
$$= \lambda^2 x_n (1 - x_n)(1 - \lambda x_n (1 - x_n))$$

$$x^* = \lambda^2 x^* (1 - x^*)(1 - \lambda x^* + \lambda x^{*2})$$

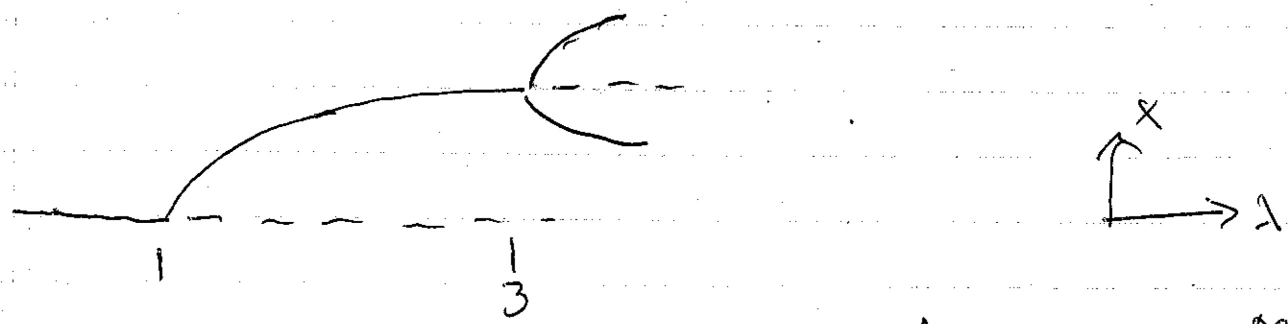
quartic, where we know 2 of the roots: $x^* = 0$, $x^* = 1 - \frac{1}{\lambda}$.

$\Rightarrow$ remaining roots solve

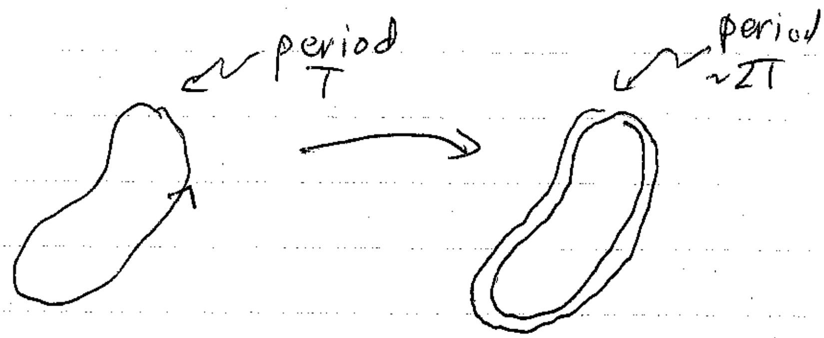
$$1 + \lambda - \lambda x - \lambda^2 x + \lambda^2 x^2 = 0$$

$$\lambda^2 x^2 - \lambda x (1 + \lambda) + (1 + \lambda)$$

$$\left. x^* = \frac{\lambda(1+\lambda) \pm \sqrt{\lambda^2 (1+\lambda)^2 - 4\lambda^2 (1+\lambda)}}{2\lambda^2} \right\} \; x^* \in \mathbb{R} \text{ provided } \lambda > 3 \checkmark$$



Rössler problem



Computing Floquet multipliers numerically — need to linearize about the limit cycle which solves

$$\boxed{\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= b + z(x-c)\end{aligned}}$$

$a = b = 0{,}2$

$c = 2{,}5$

$$Df(\gamma) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{bmatrix}$$

$$\boxed{\begin{aligned}\delta\dot{x} &= -\delta y - \delta z\\ \delta\dot{y} &= \delta x + a\,\delta y\\ \delta\dot{z} &= z\,\delta x + (x-c)\,\delta z\end{aligned}}$$

M obtained by solving with

$$\begin{bmatrix}\delta x_0\\ \delta y_0\\ \delta z_0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Next Topic: local stable manifold thm.

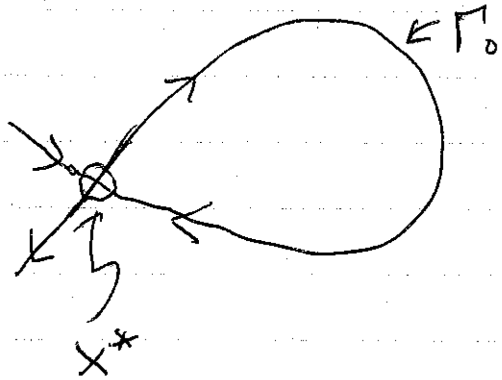
Stable & Unstable sets of Λ, where Λ is an invariant set of flow φ_t.
(if $x \in \Lambda$, then $\varphi_t(x) \in \Lambda$)

$$W^s(\Lambda) = \{ x \notin \Lambda : \lim_{t \to \infty} \rho(\varphi_t(x), \Lambda) = 0 \}$$

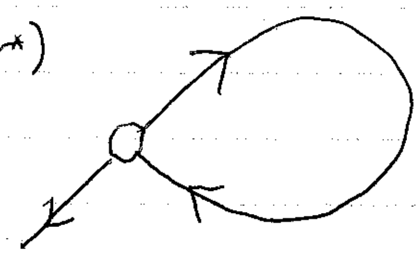
ρ — distance function

$$W^u(\Lambda) = \{ \quad " \quad \lim_{t \to -\infty} \quad " \quad \}$$

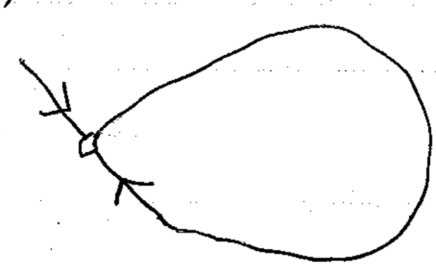
homoclinic orbit: $\Gamma_0 = W^s(x^*) \cap W^u(x^*)$
where $x^* = \varphi_t(x^*)$ $\forall t$
is a fixed pt of flow.



$W^u(x^*)$



$W^s(x^*)$



local stable manifold:

$\dot{x} = Ax + g(x)$

$x=0$ is a hyperbolic equilibrium

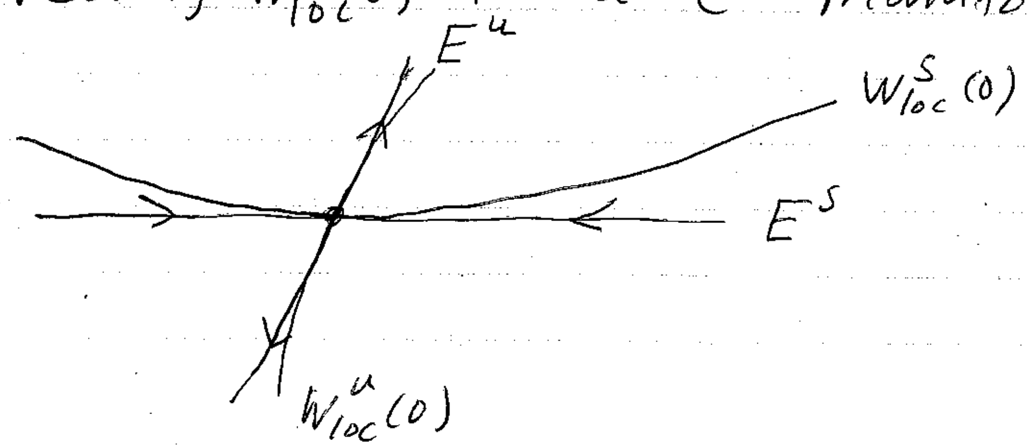
$g(x) \in C^k(U)$, $k \geq 1$, for some neighborhood U of 0. $g(x)$ is $o(x)$ as $x \to 0$

Denote linear eigenspaces of A by E^s & E^u (s for stable directions, u for unstable ones)

Then there is a $\tilde{U} \subset U$ s.t. the local stable manifold ~~W^s_{loc} to~~

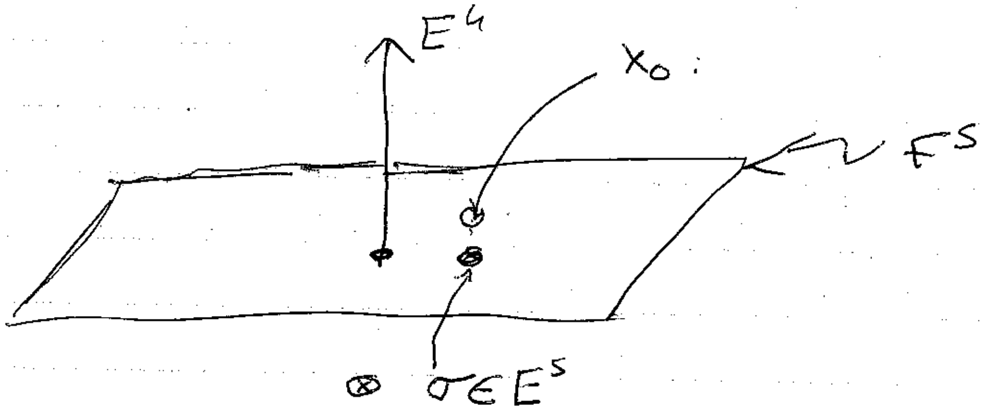
$$W^s_{loc}(0) = \{ x \in W^s(0), \varphi_t(x) \in \tilde{U}, t \geq 0 \}$$

is a Lipschitz graph over E^s that is tangent to E^s at 0. Moreover, $W^s_{loc}(0)$ is a C^k manifold.



Proof follows 3 steps in textbook

(1) Show that for each $\sigma \in E^s$, close enough to the origin, there is a unique, forward-bounded soln. associated with it,



$\varphi_t(x_0)$ bounded for $t > 0$

$\pi_s x_0 = \sigma$ $\qquad\qquad$ Π_s = projection onto E^s

unique: need contraction mapping thm.

(2) Show that these bounded solns. are asymptotic to $x=0$ as $t \to \infty$, i.e.
$\lim_{t\to\infty} \varphi_t(x_0) = 0$

Thus they lie in the stable manifold (uses a generalized Grönwall inequality)

(3) Show that soln. lies on a smooth Lipschitz graph over the stable eigenspace