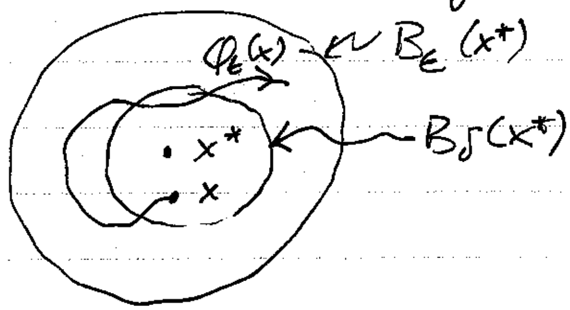
<u>Lyapunov Functions</u>: useful for determining stability of equilibria, when you can find or build them. Can sometimes be used for global stability

Recall <u>Lyapunov stability</u> for x^*:
For any $\epsilon > 0$, there is a $\delta \in (0, \epsilon)$ s.t $\varphi_t(x) \in B_\epsilon(x^*)$ for all $t > 0$ provided $x \in B_\delta(x^*)$



start in $B_\delta(x^*)$, stay in $B_\epsilon(x^*)$. True for any ϵ

A continuous function $L: \mathbb{R}^n \to \mathbb{R}$ is a (<u>strong</u>) Lyapunov function for any equilibrium x^* of the flow φ_t on $\mathbb{R}^n$ if

- $L(x^*) = 0$

- $\exists$ a neighborhood U of x^* s.t. $\forall x \in U$, $x \neq x^*$, $L(x) > 0$, and $L(\varphi_t(x)) < L(x)$ $\forall t > 0$

<u>weak</u> Lyapunov function: $L(\varphi_t(x)) \leq L(x)$

<u>Thm.</u> If L is a weak Lyapunov function for x^*, then x^* is Lyapunov stable. If L is a strong Lyapunov function for x^*, then x^* is asymptotically stable.

Note: If L is C^1 then we can check $L(\varphi_t(x)) \underset{(\leq)}{<} L(x)$, $\forall t > 0$ by computing

$$\dot{L} = \frac{d}{dt}\left(L(\varphi_t(x))\right) = \nabla L \cdot \frac{d}{dt}(\varphi_t(x)) = \nabla L \cdot f$$

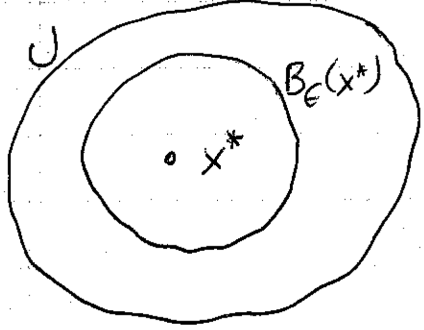
if $\nabla L \cdot f \underset{(\leq 0)}{< 0}$ then L is decreasing (non-increasing) on trajectory

∇L "opposes the flow" generated by f

L needs to be decreasing (or non-increasing) function along trajectories in neighborhood U of x^*.

<u>Idea of proof in weak case:</u>

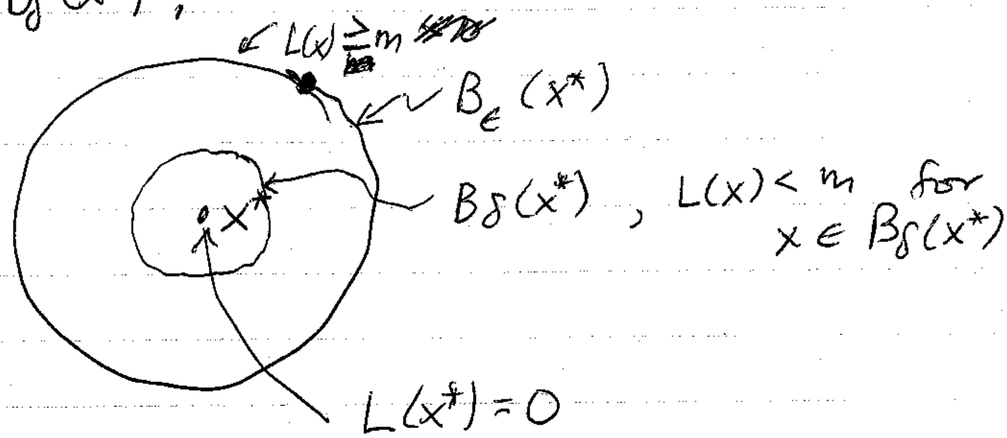
Choose ϵ so that $B_\epsilon(x^*) \subset U$



Let $m = \min\{L(x) : |x - x^*| = \epsilon\}$

$m > 0$

Since $L(x)$ is continuous and $L(x^*)=0$, there is a $\delta \in (0,\epsilon)$ s.t. $L(x) < m$ $\forall x \in B_\delta(x^*)$,



Since $L(\varphi_t(x))$ can't increase for $t>0$ to reach m, the trajectory can't escape from $B_\epsilon(x^*)$.

<u>2-examples:</u> $x \in \mathbb{R}^n$

$$\dot{x} = -\nabla V(x)$$

"gradient flow"

try $L = V(x)$

$$\ddot{x} = -\nabla V(x)$$

"F=ma", where F comes from potential.

try $L = \frac{1}{2}p^2 + V(x)$

$$p = \frac{dx}{dt}$$

$$\dot{x} = p$$
$$\dot{p} = -\nabla V(x)$$

assume there is a neighborhood U of x^* where it's the ~~unique fixed pt.~~ unique critical pt. for $V(x)$

$$\nabla V(x^*) = 0$$
$$\nabla V(x) > 0 \quad \text{for } x \neq x^*, \ x \in U$$
$$V(x^*) = 0 \quad \text{(w.l.o.g.)}$$

Note that x^* is unique local min of $V(x)$ so $V(x) > 0$ for $x \neq x^*, x \in U$

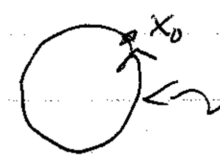
Ex. 1 $\dot{x} = -\nabla V$ "gradient flow"

$$L(t) = V(x(t))$$

$$\dot{L} = \nabla V(x(t)) \cdot \dot{x} = -|\nabla V|^2 < 0 \quad \text{for } x \neq x^*$$

$V(x)$ is a strong Lyapunov function for $x^* \Rightarrow x^*$ is asymptotically stable

Side note: gradient flows can't have periodic ~~closed~~ orbits since if they did we'd have $\varphi_T(x_0) = x_0$ which would contradict $V(\varphi_T(x_0)) < V(x_0)$



period T

Ex.2 $\ddot{x} = -\nabla V$, mechanical system
$F = ma$, where $F = -\nabla V$

re-write as first order eqns.

$$\left.\begin{aligned} \dot{x} &= p \\ \dot{p} &= -\nabla_x V \end{aligned}\right\} \qquad p \in \mathbb{R}^n,\ x \in \mathbb{R}^n$$

2n-dim phase space

∇_x ← n-dim gradient

We can write this in terms of a "Hamiltonian" & Hamilton's eqns.

$$H(x,p) = \tfrac{1}{2}|p|^2 + V(x)$$

$$(*)\quad \begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial p_i} \\[2mm] \dot{p}_i = -\dfrac{\partial H}{\partial x_i} \end{cases}$$

- under our assumptions we have that $(x,p) = (x^*, 0)$ is a critical pt. of H since

$$\left.\frac{\partial H}{\partial p_i}\right|_{p=0} = \left.\frac{\partial H}{\partial x_i}\right|_{x=x^*} = 0 \qquad i = 1, \dots, n$$

where $(x^*, 0)$ is ~~the~~ an equilibrium for $(*)$

$$H(x^*, 0) = 0$$

We assumed that x^* is a local min of $V(x)$ so $D^2V(x^*)$ is a positive definite matrix $\left[x^T D^2 V x > 0 \text{ for } x \neq x^*, x \in U' \right]$

$$D^2V = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \\ & & & \\ & & & \\ & & & \frac{\partial^2 V}{\partial x_n^2} \end{bmatrix}\Bigg|_{x=x^*} = \text{Hessian matrix}$$

It follows that Hessian of H is also positive definite at $(x,p)=(x^*,p)$

$$D^2H(x^*,0) = \left[\begin{array}{c|c} D^2V & 0 \\ \hline 0 & I \end{array}\right]$$

$(x^*,0)$ is a local min of H

Use H to obtain a Lyapunov function

H is constant on trajectories so we obtain that $(x^*, 0)$ is Lyapunov stable using it as a Lyapunov function since it is a weak Lyapunov function $[\dot{H} = 0]$

Let's see what happens if we add linear dumping. Asymptotic stability?

$$\dot{x} = p$$
$$\dot{p} = -\nabla_x V - \gamma p \qquad \gamma \geq 0 \quad (\gamma = 0 \text{ is undamped})$$

let $L(x, p) = H(x, p) - H(x^*, 0)$ ← energy at fixed pt. $(x^*, 0)$

- $L(x^*, 0) = 0$
- $L(x, p) > 0$ for $(x, p) \neq (x^*, 0)$ in neighborhood U of $(x^*, 0)$ since it's a local min.
- $$\frac{dL}{dt} = \sum \underbrace{\frac{\partial H}{\partial x_i}}_{\frac{\partial V}{\partial x_i}} \underbrace{\dot{x}_i}_{p_i} + \underbrace{\frac{\partial H}{\partial p_i}}_{p_i} \underbrace{\dot{p}_i}_{-\frac{\partial V}{\partial x_i} - \gamma p_i} = -\gamma |p|^2 \leq 0$$

Thus $(x^*, 0)$ is Lyapunov stable.

~~Note: we can't get as~~

~~Note: Lyapo~~ note: Hamiltonian only serves as a weak Lyaponov function since $-\gamma|p|^2 = 0$ ~~for~~ whenever $p=0$.

However, if we restrict to $p=0$, we find $\dot{p} = -\nabla_x V < 0$ for $x \neq x^*$, so p doesn't stay on subspace $p=0$ of our phase space – it is not an invariant subspace.

Can use "La Salle's Invariance Principle" to prove asymptotic stability in the case $\gamma > 0$.

<u>LaSalle's Invariance Principle</u>: Suppose x^* is a ~~an equilibrium~~ fixed pt. of $\varphi_t(x)$ & L is a weak Lyaponov function ($\dot{L} \leq 0$) of x^* on some compact forward invariant neighborhood U of x^*. Let $Z = \{x \in U : \frac{dL}{dt} = 0\}$. If $\{x^*\}$ is the largest forward invariant subset of Z, is is asymptotically stable & attracts every pt. in U

Forward invariant neighborhood U of x^*: if $x \in U$, then $\varphi_t(x) \in U$ $\forall t > 0$

U = subset of Euclidean space, then compact means closed & bounded.

in damped Hamiltonian problem
$Z = \{(x,p) \in U : p = 0\}$

$$\dot{p} = -\gamma p - \nabla_x V \Rightarrow \dot{p}\big|_Z = -\nabla_x V$$

which is nonzero for $x \neq x^*$, so largest ~~flow~~ forward invariant subset of Z is just $(x^*, 0)$, which is asymptotically stable.