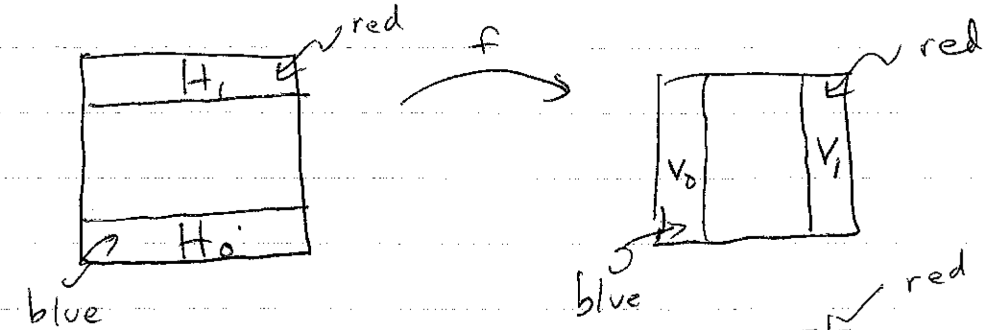
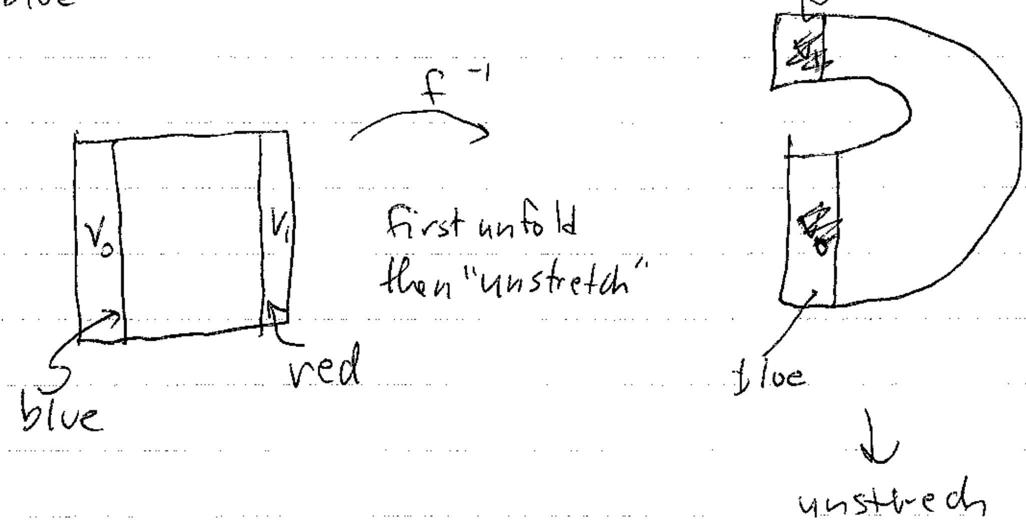
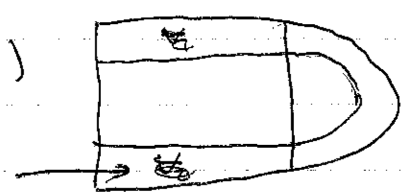
Smale horseshoe:



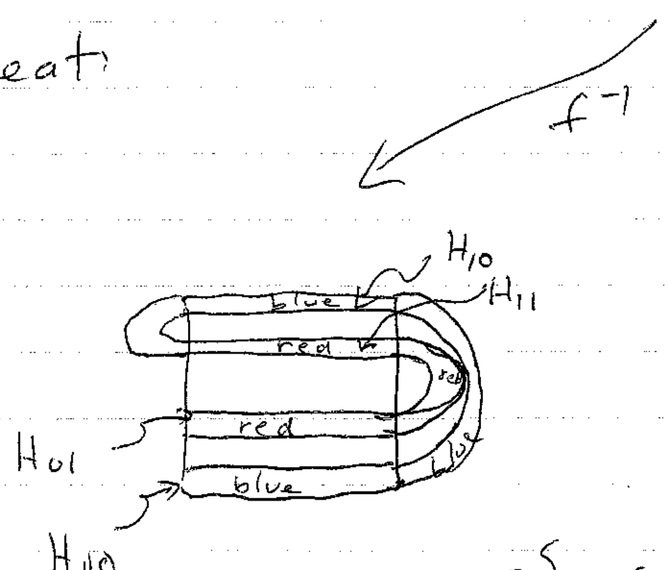


$H_1 = f^{-1}(V_1)$

$H_0 = f^{-1}(V_0)$



repeat:



$$D \cap f^{-1}(D) \cap f^{-2}(D)$$

$$= \bigcup_{s_i \in S} H_{s_0 s_1}$$

$$= \{ p \in H_{s_0}, f(p) \in H_{s_1}, \ s_i \in \{0,1\}, \ i = 0,1 \}$$

Λ = invariant set = $\bigcap_{n=-\infty}^{+\infty} f^n(D)$

= set of pts in intersection of vertical lines $V_{s_{-1}s_{-2}\cdots}$ & horizontal lines $H_{s_0 s_1 \cdots}$

$$p \xrightarrow{\phi} \underbrace{\cdots s_{-k} \cdots s_{-2}\, s_{-1}}_{\substack{\text{labels vertical strips} \\ \text{(f map, the past)}}} \cdot \underbrace{s_0\, s_1 \cdots s_k \cdots\cdots}_{\substack{\text{labels horizontal strips} \\ (f^{-1} \text{ map, the future})}}$$

if $s = \cdots s_{-k} \cdots s_{-1} \cdot s_0\, s_1 \cdots s_k \cdots\cdots$

then $\sigma(s) = \cdots s_{-k} \cdots s_{-1}\, s_0 \cdot s_1 \cdots s_k \cdots\cdots$

gives us $f(p)$

<u>Thm.</u> Shift map σ acting on space of bi-infinite sequence of 0's & 1's has

(i) countable set of periodic orbits of arbitrarily high periods ← repeating sequences

(ii) uncountably infinite set of nonperiodic orbits ← nonrepeating

(iii) a dense orbit

last lecture on ~~shift~~ map $x_{n+1} = 2x_n \pmod 1$

example from ODE's where smale horseshoe arises "homoclinic tangle" (p. 306 in text — quote from Poincaré)

$(x,y) \in \mathbb{R}^2$ but with time-dependent perturbation

$$\dot{x} = \frac{\partial H}{\partial y}(x,y) + \epsilon\, g_1(x,y,t;\epsilon)$$

$$\dot{y} = -\frac{\partial H}{\partial x}(x,y) + \epsilon\, g_2(x,y,t;\epsilon)$$

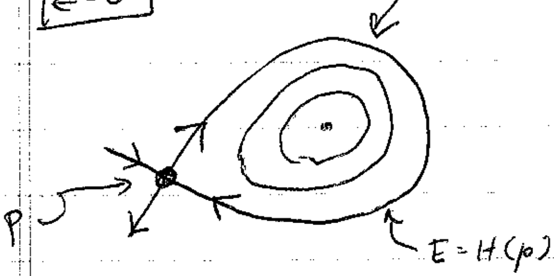
$\epsilon \ll 1$

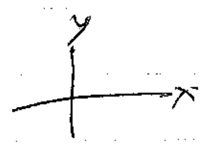
g_i is T-periodic

$\epsilon = 0 \Rightarrow$ Hamiltonian system with conserved quantity $H(x_0, y_0) = E_0$

assume for $\epsilon = 0$, that there is a saddle-pt p_s with a homoclinic orbit

$\boxed{\epsilon = 0}$





re-write nonautonomous ($\epsilon \neq 0$) problem as an autonomous one on $\mathbb{R}^2 \times S^1$

$q = (x,y) \in \mathbb{R}^2$

$\phi = \omega t$ $\quad \omega = \frac{2\pi}{T}$

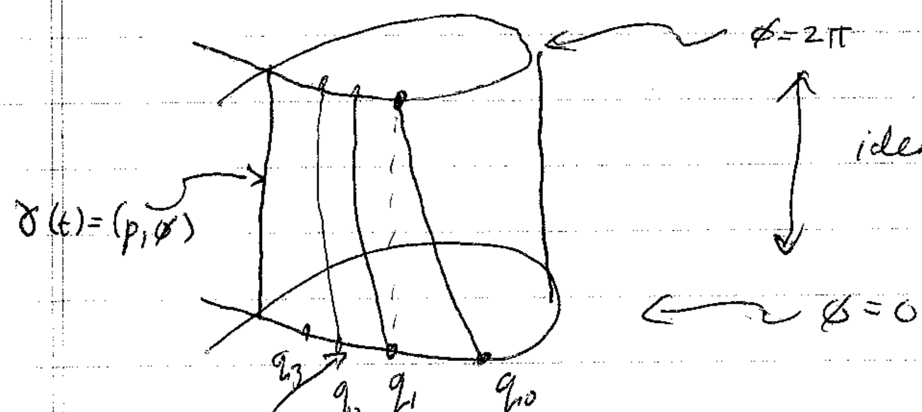
identify $\phi = 0$ & $\phi = 2\pi$

$$\dot{q} = J\nabla H + \epsilon g(q, \phi; \epsilon)$$

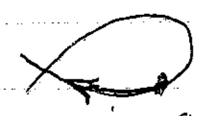
$$\dot{\phi} = \omega$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \nabla H = \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial y \end{pmatrix}$$

$\boxed{\epsilon = 0}$: $\left.\begin{array}{l} \dot{q} = J\nabla H \\ \dot{\phi} = \omega \end{array}\right\}$ ~~right~~ rhs doesn't depend on ϕ



2-dim homoclinic manifold to the periodic orbit $\gamma(t)$

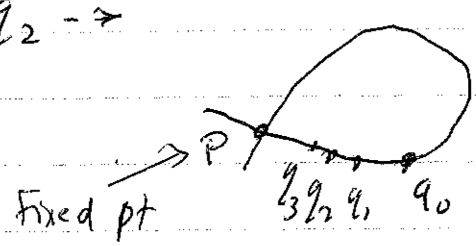


in q-plane with $\phi(t) = \omega t$ (mod 2π)

in terms of "stroboscopic map"

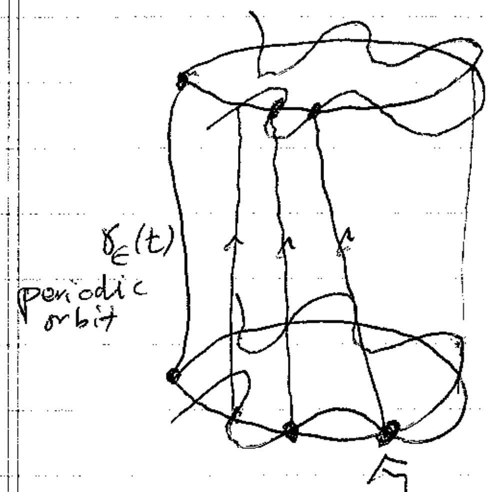
i.e. $(q_0, \phi_0 = 0) \longmapsto (q_1, \phi_1 = 2\pi)$

$q_0 \to q_1 \to q_2 \to$



iterates of 2-d map

$\epsilon \neq 0$: a generic possibility — the 2d stable & Unstable manifolds of $\gamma_\epsilon(t)$ have a 1-d intersection

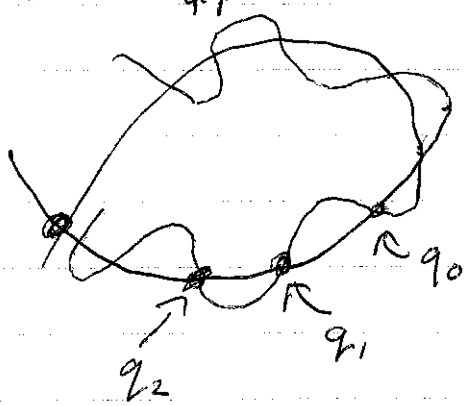


one soln. in the 1-dim homoclinic manifold

$W^s(\gamma_\epsilon^\epsilon(t))$ blue

$W^u(\gamma_\epsilon(t))$ red

as 2-d map



$q_{n+1} = F(q_n)$

$\Rightarrow$ Wiggins

homoclinic tangle

2-d map

p = fixed point

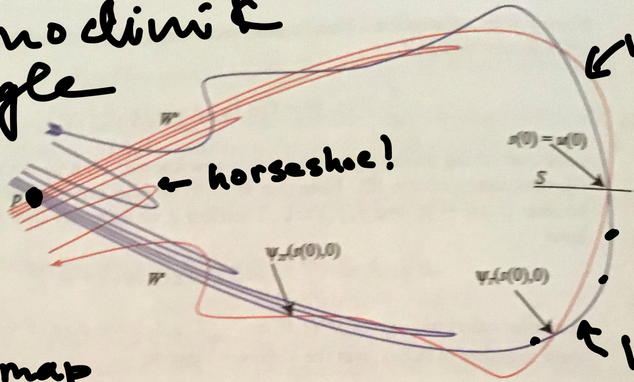


Figure 8.22. *Sketch of a cross section $\bar{S}$ for $\theta_o = 0$ of the stable and unstable manifolds. Here we suppose that $s(0) = u(0) = 0$, so that the crossing takes place on the section $\bar{S}$. The next crossing on the orbit of $s(0)$ occurs at time T, the period of g.*

infinitely many times. The resulting picture is extremely intricate, and only a brief indication of the complexity is sketched in Figure 8.22. Indeed, when Poincaré discovered the possibility of the transverse crossing of stable and unstable manifolds, he said the following (in our translation from the French):

> *When one tries to depict the figure formed by these two curves and their infinity of intersections, each of which corresponds to a doubly asymptotic solution, these intersections form a kind of net, web or infinitely tight mesh; neither of the two curves can ever cross itself, but must fold back on itself in a very complex way in order to cross the links of the web infinitely many times. One is struck by the complexity of this figure that I am not even attempting to draw.*) !
> (Henri Poincaré, *New Methods in Celestial Mechanics*, 1892, Vol. 3, §397)

Since Poincaré, there have been many attempts to sketch this figure—many are incorrect!

Example 8.34. To compute the Melnikov function for the example Hamiltonian (5.3), we first need to find the solution on the unperturbed homoclinic orbit, defined by

$$\dot{x} = y = \pm x\sqrt{1-2ax}. \tag{8.91}$$

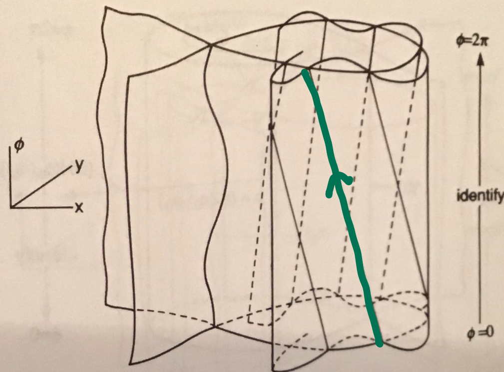
Choosing a point on the unperturbed manifold, $q = (1/2a, 0)$, we must find $\varphi_t(q)$. Since (8.91) is separable, it can be integrated to obtain

$$\pm t + c = \int \frac{dx}{x\sqrt{1-2ax}} = -2\int \frac{du}{\ldots}$$

S. Wiggins

Texts in Applied Mathematics 2

Introduction to Applied Nonlinear Dynamical Systems and Chaos



Γ_γ intersects of manifolds

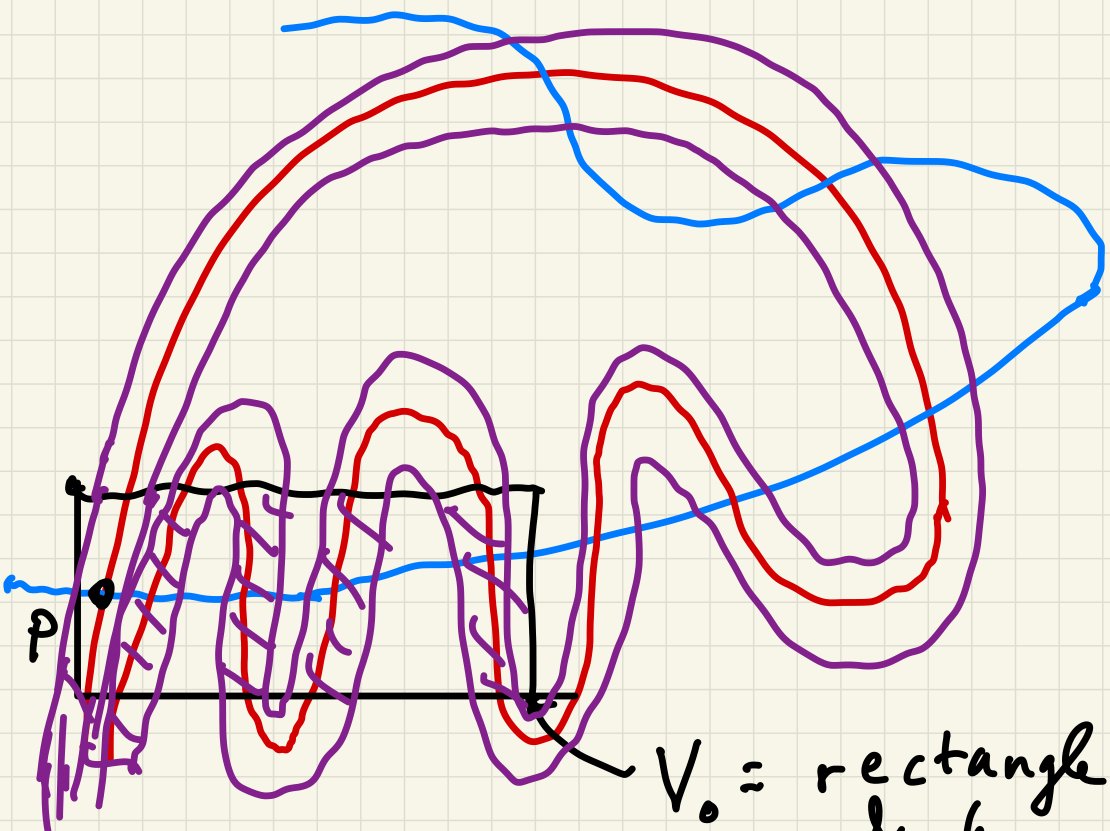
FIGURE 4.5.3.

the distance between the perturbed stable and unstable manifolds along the direction normal to Γ_γ. Evidently, this measurement will vary from point-to-point on Γ_γ so we first need to describe a parametrization of Γ_γ.

Parametrization of Γ_γ: Homoclinic Coordinates. Every point on Γ_γ can be represented by

$$(q_0(-t_0), \phi_0) \in \Gamma_\gamma \tag{4.5.9}$$

...tation of t_0 is the time of flight from ...rturbed homoclinic



$W^u(p)$

$W^s(p)$

V_0 = rectangle containing p
& homoclinic pts

image of V_0 after some n
iterations of map, n large enough

From wiggins Book

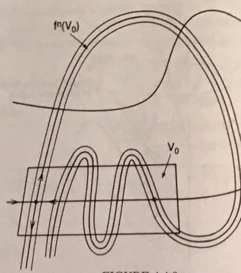


FIGURE 4.4.9.

Again, as a result of the lambda lemma, for N_0 sufficiently large, $|\eta_{f^T(z_0)}|$ can be made arbitrarily large, $|\xi_{f^T(z_0)}|$ can be made arbitrarily small, and by transversality of the intersection of $W^u(0)$ and $W^s(0)$ at q, $d \neq 0$ (with $\phi_{1\overline{y}}$ small compared to d). Thus, (4.4.29) can be made as large as we desire by choosing N_0 big enough.

The Smale–Birkhoff Homoclinic Theorem

The Smale–Birkhoff homoclinic theorem is very similar to Moser's theorem. We will state the theorem and describe briefly how it differs. The assumptions and set-up are the same as for Moser's theorem.

Theorem 4.4.3 (Smale [1963]) *There exists an integer $n \geq 1$ such that f^n has an invariant Cantor set on which it is topologically conjugate to a full shift on N symbols.*

Proof: We will only give the barest outline in order to show the difference between the Smale–Birkhoff homoclinic theorem and Moser's theorem and leave the details as an exercise for the reader (Exercise 4.8).

Choose a "rectangle," V_0, containing a homoclinic point *and* the hyperbolic fixed point as shown in Figure 4.4.9. Then, for n sufficiently large, $f^n(V_0)$ intersects V_0 a finite number of times as shown in Figure 4.4.9. Now, one can find μ_h-horizontal strips in V_0 that map over themselves in μ_v-vertical strips such that Assumptions 1 and 3 of Section 4.3 hold; see Figure 4.4.10. The details needed to prove these statements are very similar to those needed for the proof of Moser's theorem, and it will be an instructive exercise for the reader to give a rigorous proof. □

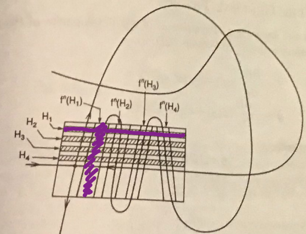
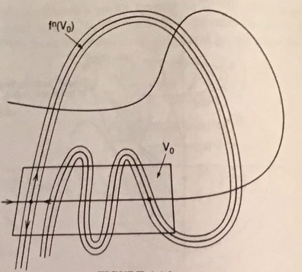


FIGURE 4.4.10. Horizontal strips $H_1, \cdots, H_4$ and their image under f^n.

From the outline of the proof of the Smale–Birkhoff homoclinic theorem, one can see how it differs from Moser's theorem. In both cases the invariant Cantor set is constructed near a homoclinic point sufficiently close to the hyperbolic fixed point. However, in the Smale–Birkhoff theorem, all points leave the Cantor set and return *at the same time* (i.e., after n iterates of f); in Moser's construction, points leave the Cantor set and may return *at different times* (recall the definition of f^T). What are the dynamical consequences of the two different constructions (see Exercise 4.9)?

4.5 Melnikov's Method for Homoclinic Orbits in Two-Dimensional, Time-Periodic Vector Fields

We have seen that transverse homoclinic orbits to hyperbolic periodic points of two-dimensional maps gives rise to chaotic dynamics in the sense of Theorems 4.4.2 and 4.4.3. We will now develop a perturbation method originally due to Melnikov [1963] for proving the existence of transverse homoclinic orbits to hyperbolic periodic orbits in a class of two-dimensional, time-periodic vector fields; then by considering a Poincaré map, Theorems 4.4.2 and 4.4.3 can be applied to conclude that the system possesses chaotic dynamics.



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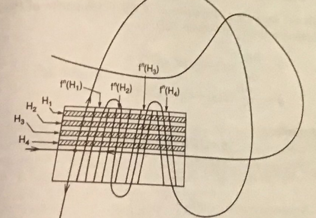

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