Lectures 7 and 8: Max Flow & Min Cut

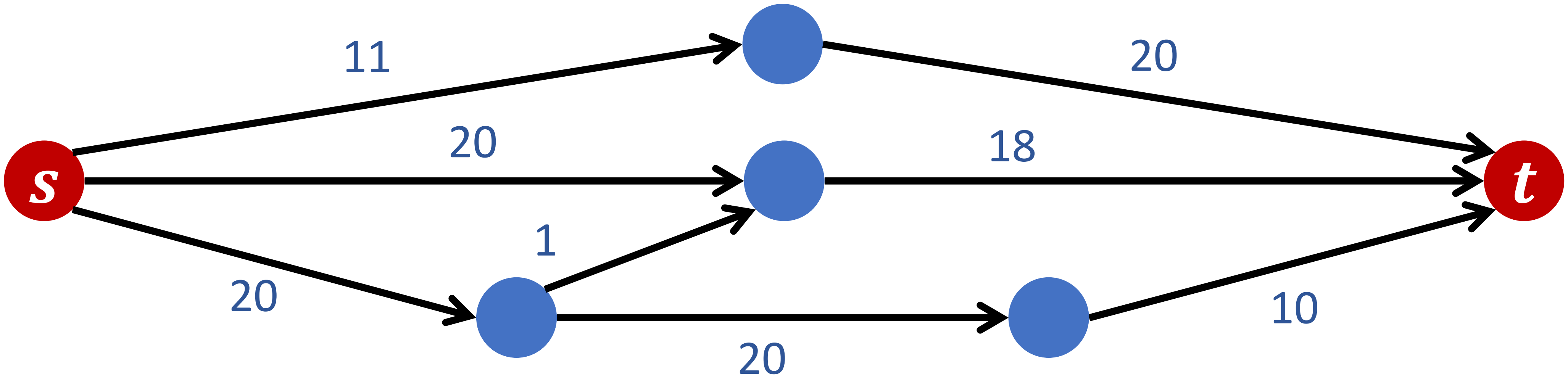
Yury Makarychev
TTIC and the University of Chicago

Maximum Flow & Minimum Cut

Flow Network

An (s, t)-flow network:

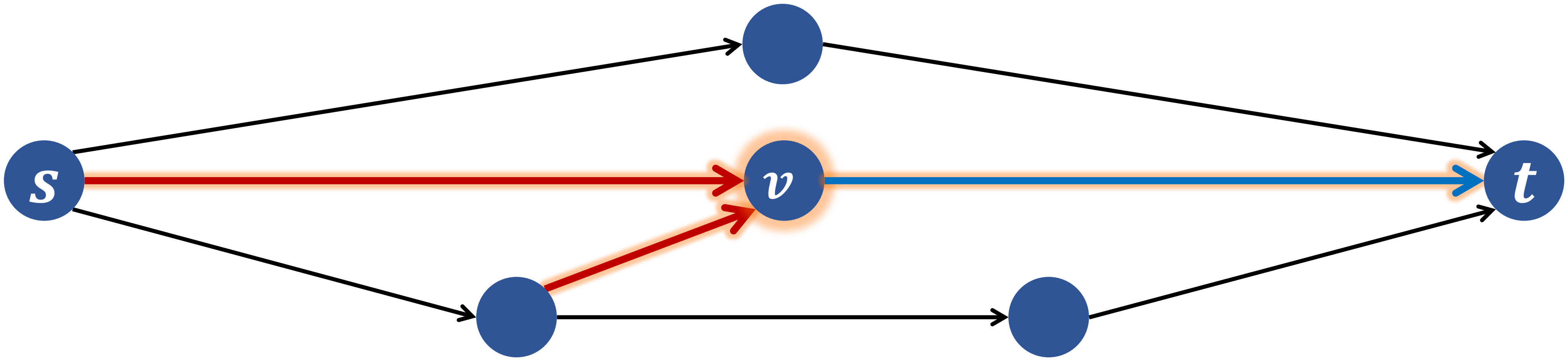
- Directed graph $G = (V, E)$.
- Two designated nodes (terminals): source s and sink t
- Every edge e has a capacity $c(e)$.



Flow Network

$in(v)$ is the set of edges incoming in v
$out(v)$ is the set of edges outgoing from v

Assume: $in(s) = out(t) = \emptyset$



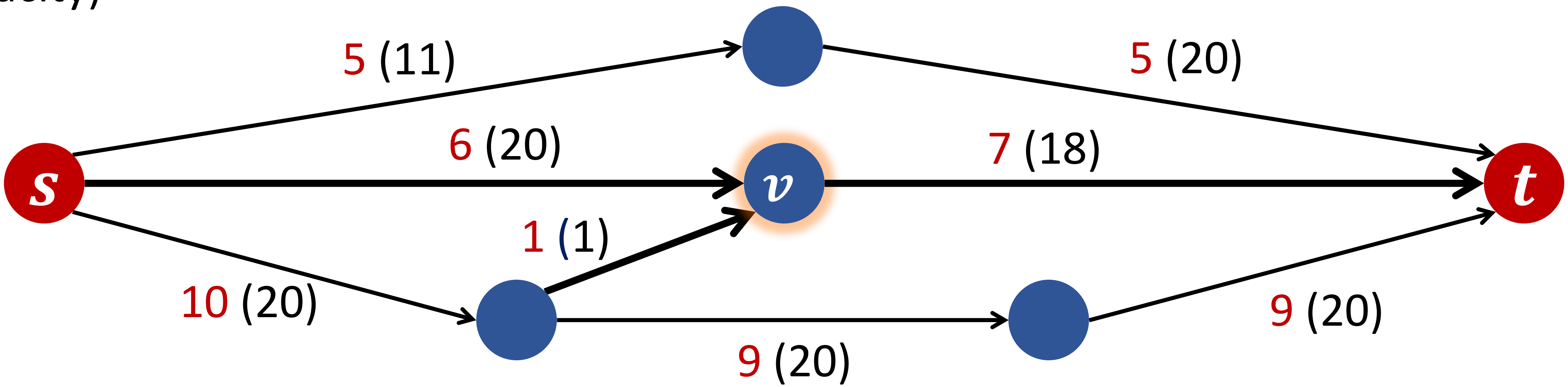
Feasible Flow

Flow $f: E \to \mathbb{R}_{\geq 0}$. We route $f(e)$ units of flow over each edge e.

a. Flow conservation constraints: for all vertices $v \notin \{s, t\}$,

$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

flow (capacity)



Feasible Flow

Flow $f: E \rightarrow \mathbb{R}_{\geq 0}$. We route $f(e)$ units of flow over each edge e.

a. Flow conservation constraints: for all vertices $v \notin \{s, t\}$,

$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

b. Capacity constraints: for all edges e:

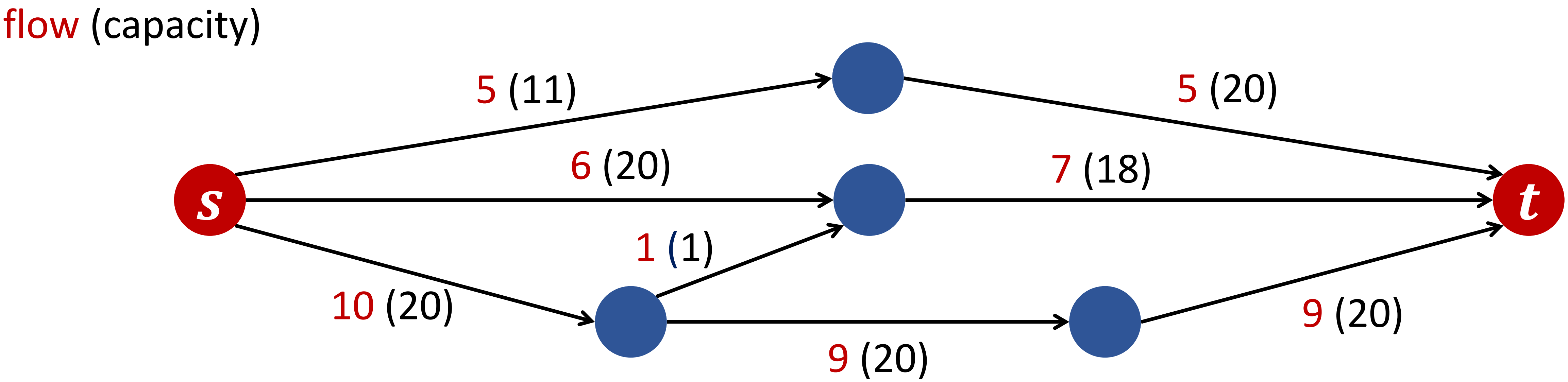
$$f(e) \leq c(e)$$

The value of the flow f equals

$$val(f) = \sum_{e \in out(s)} f(e)$$

Maximum Flow Problem

Given a network $G = (V, E)$, find the maximum feasible flow from s to t.



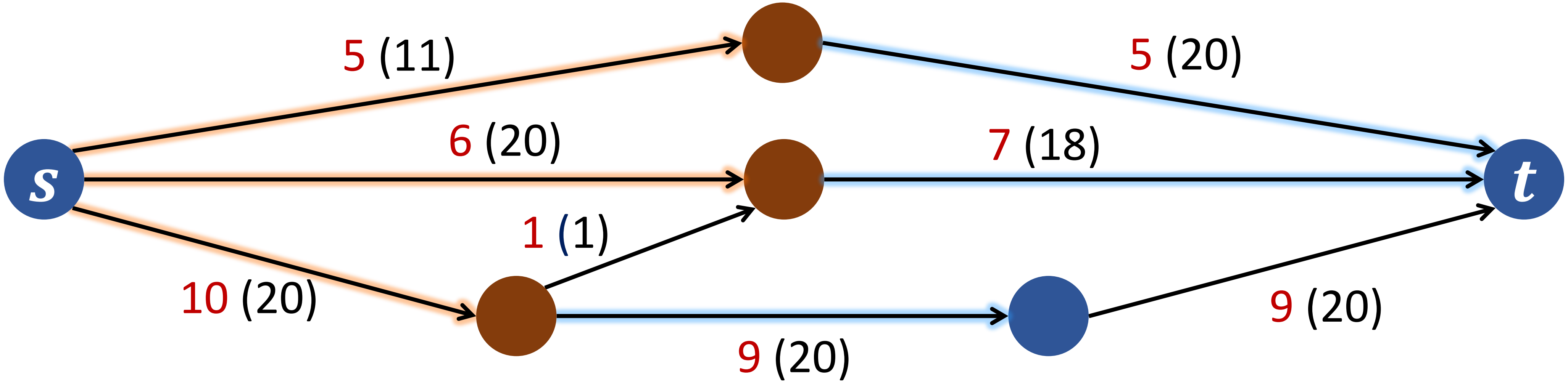
Flow Network

$$f_{in}(S) = \sum_{\substack{(u,v)\in E \\ u\notin S, v\in S}} f(u,v)$$

$$f_{out}(S) = \sum_{\substack{(u,v)\in E \\ u\in S, v\notin S}} f(u,v)$$

$$5 + 6 + 10 = 21$$

$$5 + 7 + 9 = 21$$



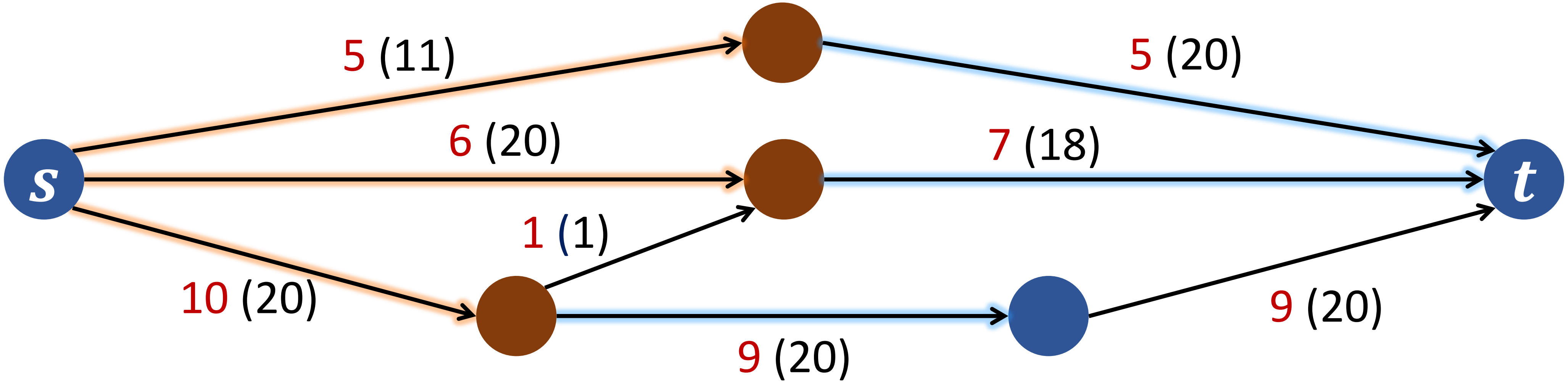
Flow Network

In this example $f_{in}(S) = f_{out}(S)$.

Q: is it always the case?

$5 + 6 + 10 = 21$

$5 + 7 + 9 = 21$



$val(f) = f_{out}(s)$

Flow Network

Claim

a) If $s, t \notin S$ then $f_{in}(S) = f_{out}(S)$

b) If $s, t \in S$ then $f_{in}(S) = f_{out}(S)$

c) If $s \in S$ and $t \notin S$ then $f_{in}(S) = f_{out}(S) - val(f)$

d) If $s \notin S$ and $t \in S$ then $f_{in}(S) = f_{out}(S) + val(f)$

Proof

The claim is very intuitive. E.g.,

(a) says: no flow originates or terminates in S if $s, t \notin S$

(c) says: all flow leaving S either originates at $s \in S$ or first enters S and then leaves it

Prove item (c).

$val(f) = f_{out}(s)$

Proof

If $s \in S$ and $t \notin S$ then $f_{in}(S) = f_{out}(S) - val(f)$

Proof

$$f_{in}(S) = \sum_{\substack{(u,v)\in E \\ u\notin S, v\in S}} f(u,v) = \sum_{u\in S} f_{in}(u) - \sum_{\substack{(u,v)\in E \\ u,v\in S}} f(u,v)$$

(A) the sum is over all edges entering S

(B) the sum is over all edges entering vertices in S. Those include (A) and edges between vertices in S.

$val(f) = f_{out}(s)$

Proof

If $s \in S$ and $t \notin S$ then $f_{in}(S) = f_{out}(S) - val(f)$

Proof

$$f_{in}(S) = \sum_{u \in S} f_{in}(u) - \sum_{\substack{(u,v) \in E \\ u,v \in S}} f(u,v) = \sum_{u \in S \setminus \{s\}} f_{in}(u) - \sum_{\substack{(u,v) \in E \\ u,v \in S}} f(u,v)$$

since $f_{in}(s) = 0$

$val(f) = f_{out}(s)$

Proof

If $s \in S$ and $t \notin S$ then $f_{in}(S) = f_{out}(S) - val(f)$

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the flow conservation constraint
(use that $s, t \notin S \setminus \{s\}$)

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Proof

$$f_{in}(S) = \sum_{u \in S \setminus \{s\}} f_{out}(u) - \sum_{\substack{(u,v) \in E \\ u,v \in S}} f(u,v)$$

$$= \left(\sum_{u \in S} f_{out}(u) - f_{out}(s) \right) - \sum_{\substack{(u,v) \in E \\ u,v \in S}} f(u,v)$$

$val(f) = f_{out}(s)$

Proof

If $s \in S$ and $t \notin S$ then $f_{in}(S) = f_{out}(S) - val(f)$

Proof

$$f_{in}(S) = \left(\sum_{u \in S} f_{out}(u) - \sum_{\substack{(u,v) \in E \\ u,v \in S}} f(u,v) \right) - val(f)$$

$$= f_{out}(S) - val(f)$$

■

Corollary

$$f_{in}(t) = f_{out}(V \setminus \{t\}) = f_{in}(V \setminus \{t\}) + val(f) = val(f)$$

why?

why?

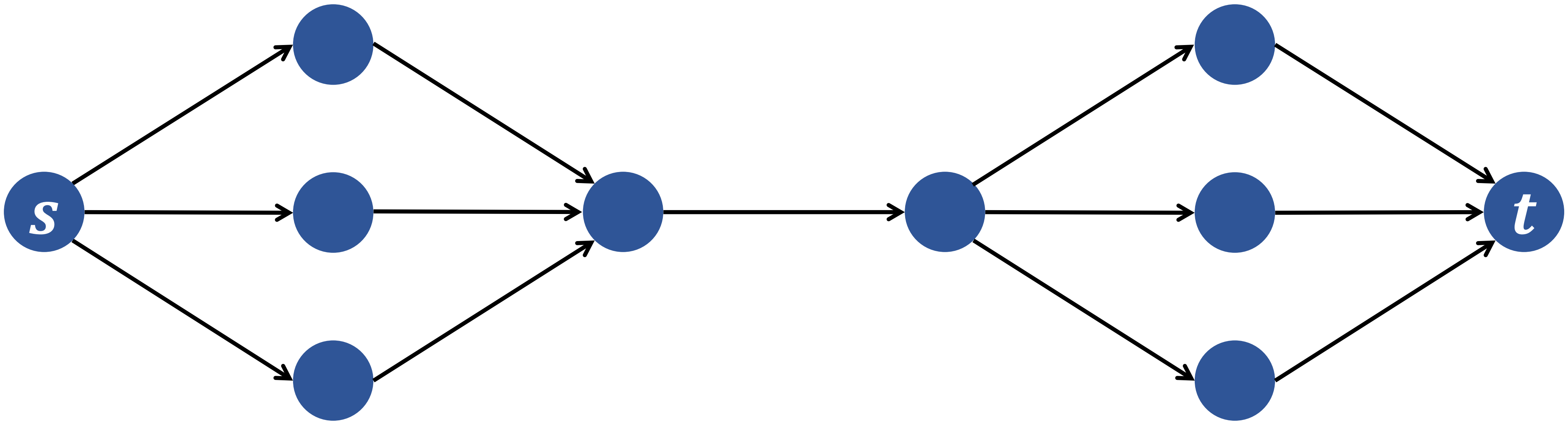
Flow Network

$$val(f) = f_{out}(s) = f_{in}(t) = f_{out}(S) - f_{in}(S)$$

for every set S that contains s but not t

What is the value of the maximum flow?

In this example, all edges have capacity 1.



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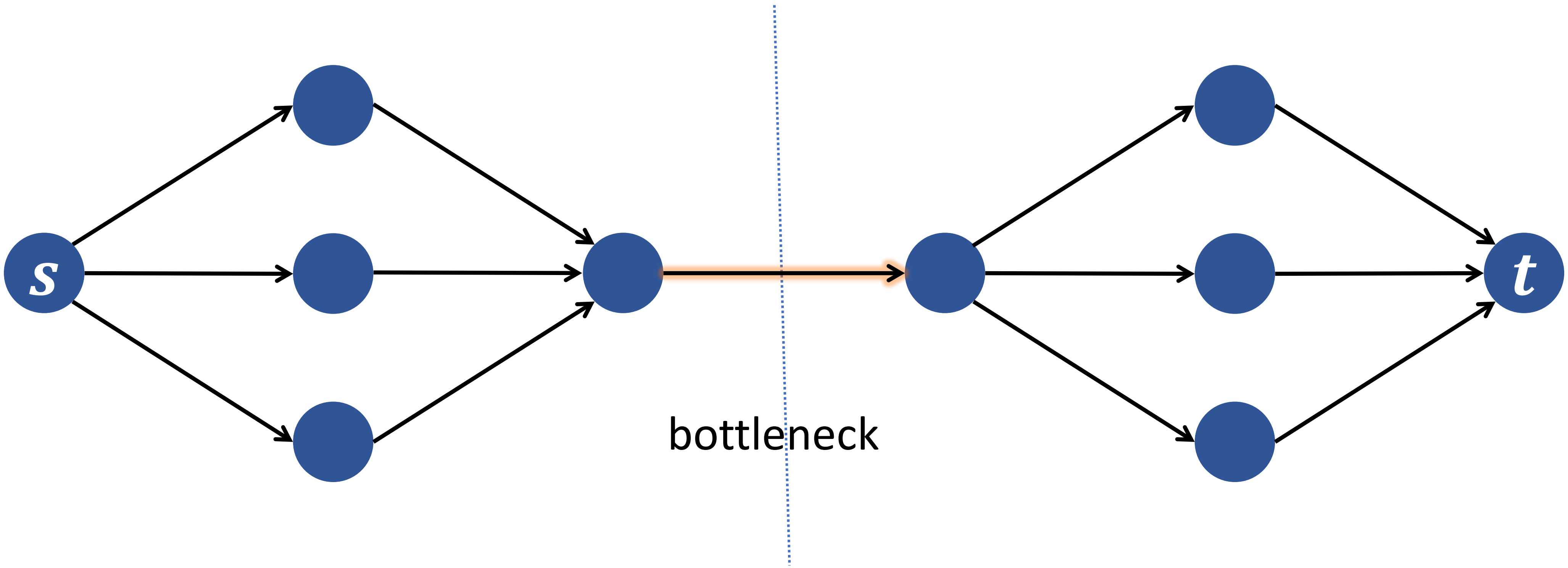

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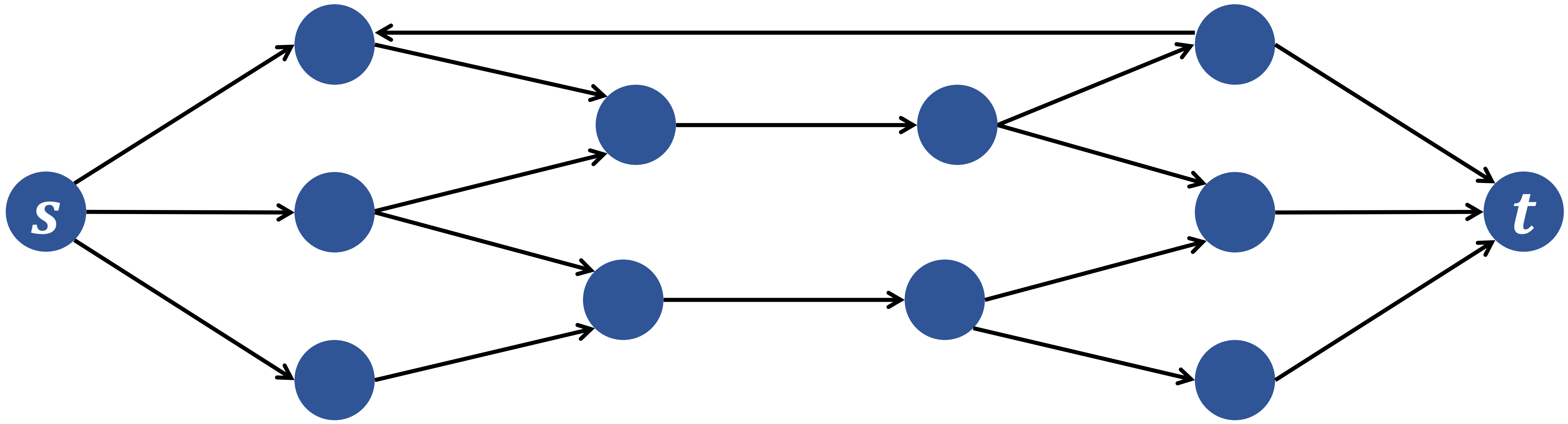

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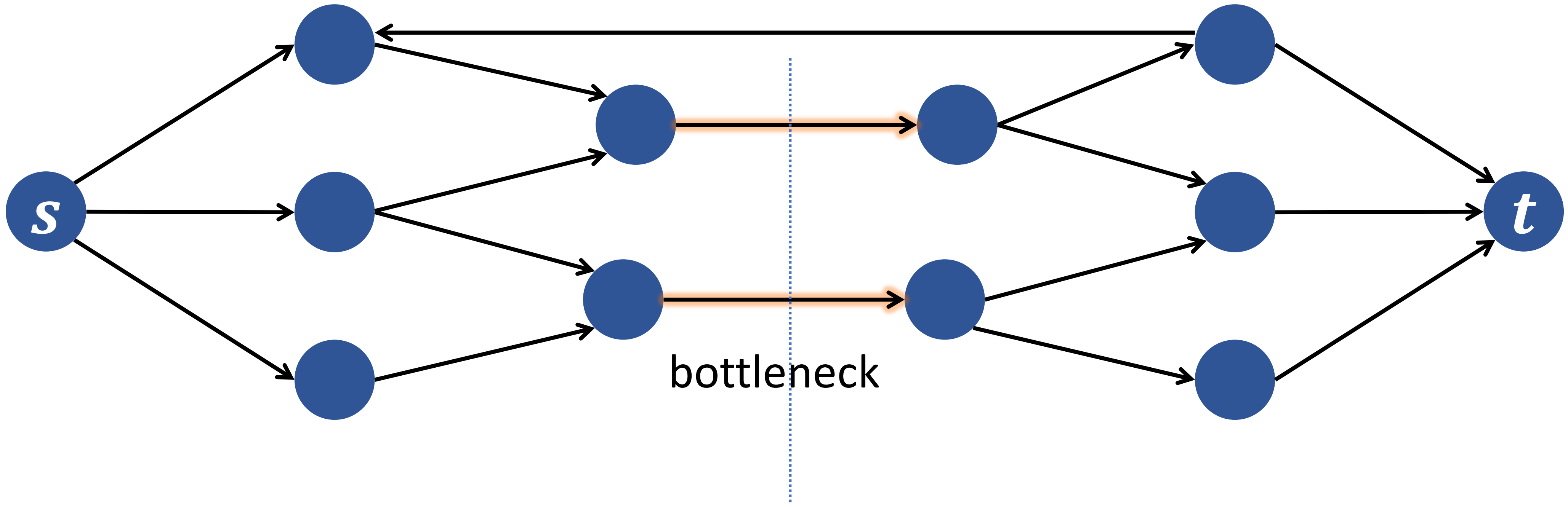


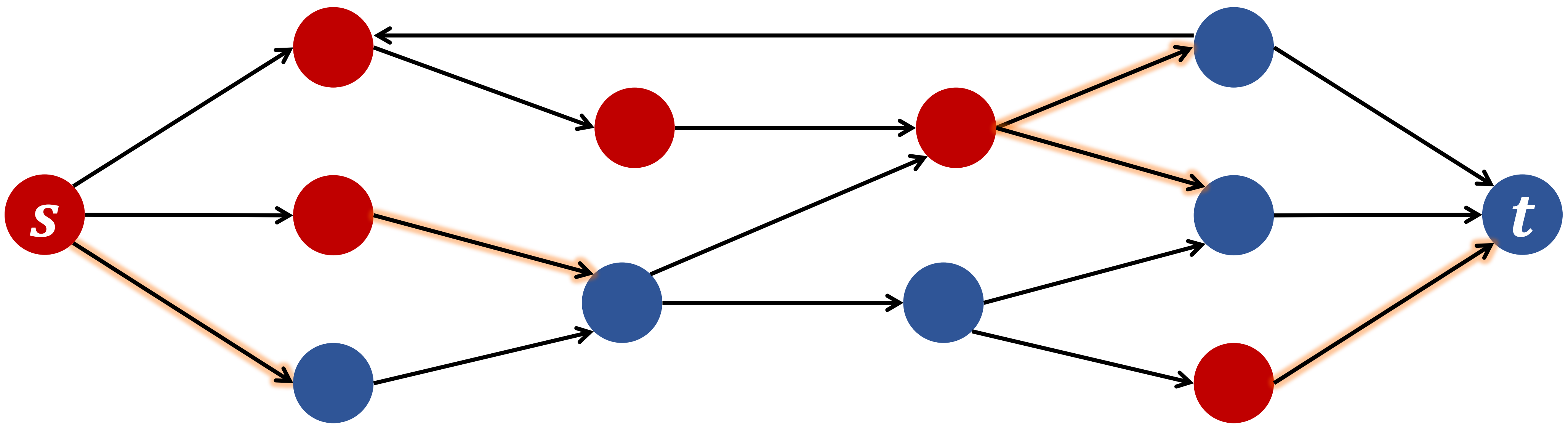
Minimum Cut

Directed Cut

A directed cut is a partition of V into two nonempty sets S and T:

$$V = S \cup T \text{ and } S \cap T = \emptyset$$

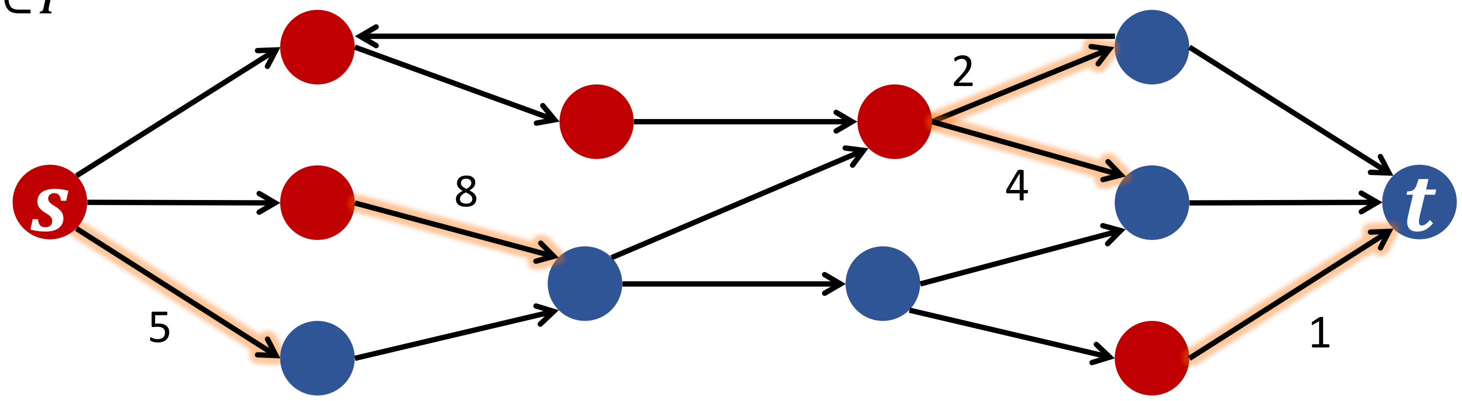
An edge (u, v) is cut if $u \in S$ and $v \in T$. Edges from T to S are not cut!



The capacity or cost of the cut is the total capacity of all cut edges.

The capacity/cost of a cut

$$cap(S,T) = \sum_{\substack{(u,v)\in E \\ u\in S, v\in T}} c(u,v)$$

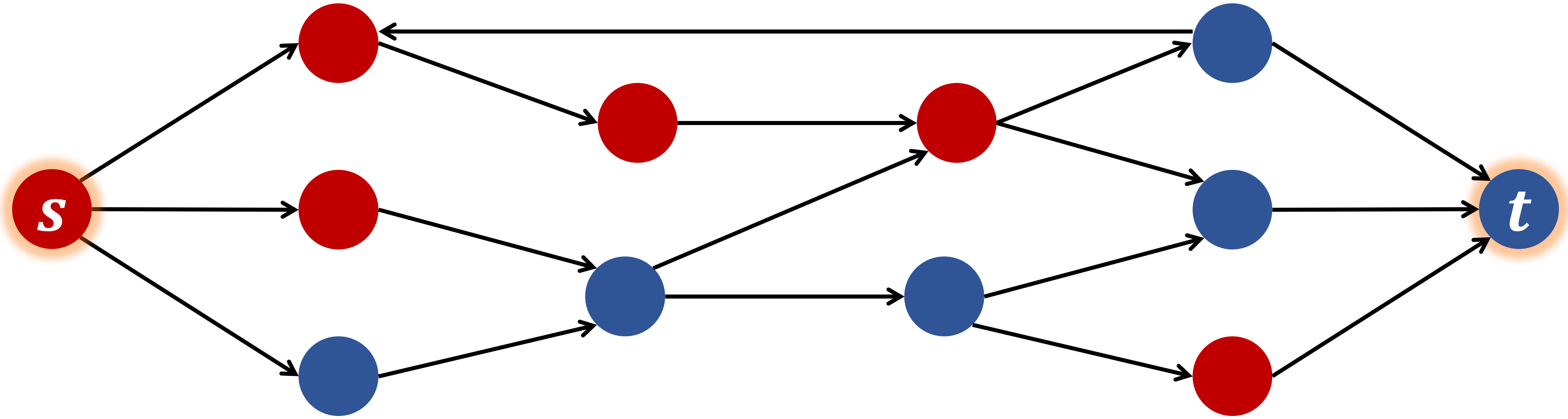


In this example, $cap(S,T) = 5 + 8 + 2 + 4 + 1 = 20.$

s-t Cut

(S, T) is an s-t cut if $s \in S$ and $t \in T$.

(S, T) is a minimum s-t cut if $cap(S, T) \leq cap(S', T')$ for every s-t cut (S', T').



$f_{in}(S) = f_{out}(S) - val(f)$

Weak Duality

Let f be a feasible s-t flow and (S, T) be an s-t cut. Then

$$val(f) \leq cap(S, T)$$

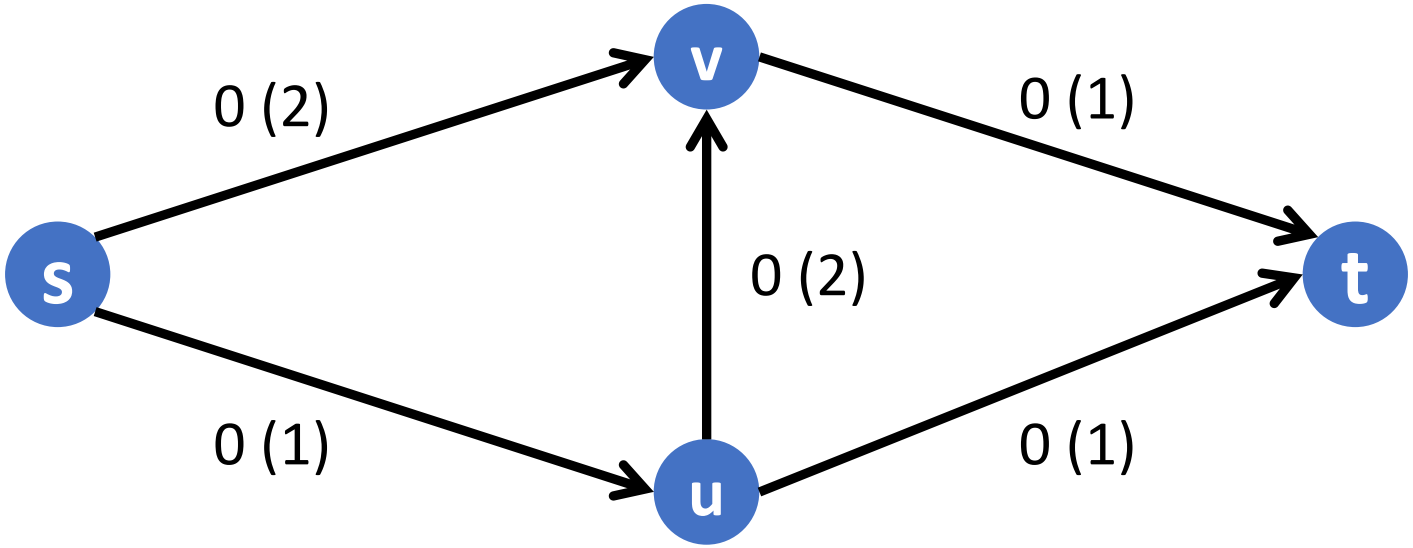
Proof: Since $s \in S$ and $t \notin S$,

$$val(f) = f_{out}(S) - f_{in}(S) \leq f_{out}(S) = \sum_{\substack{(u,v)\in E \\ u\in S, v\in T}} f(u, v)$$

$$\leq \sum_{\substack{(u,v)\in E \\ u\in S, v\in T}} cap(u, v) = cap\,(S, T)$$

Ford–Fulkerson Algorithm

Attempt #1: Naïve Greedy Algorithm



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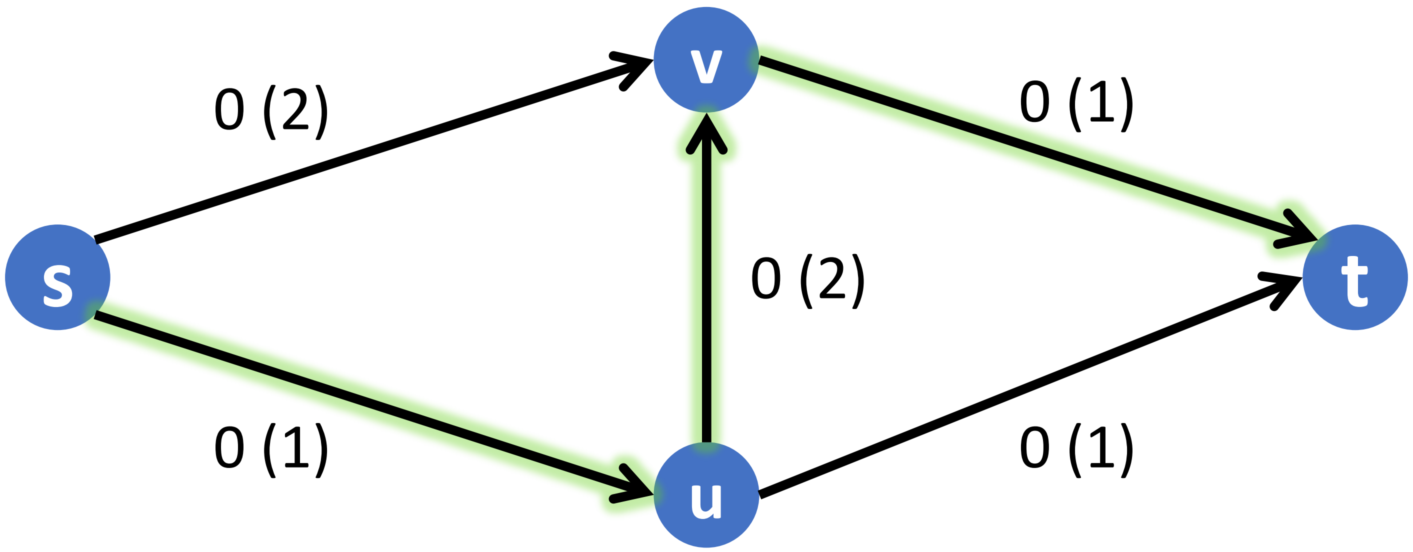

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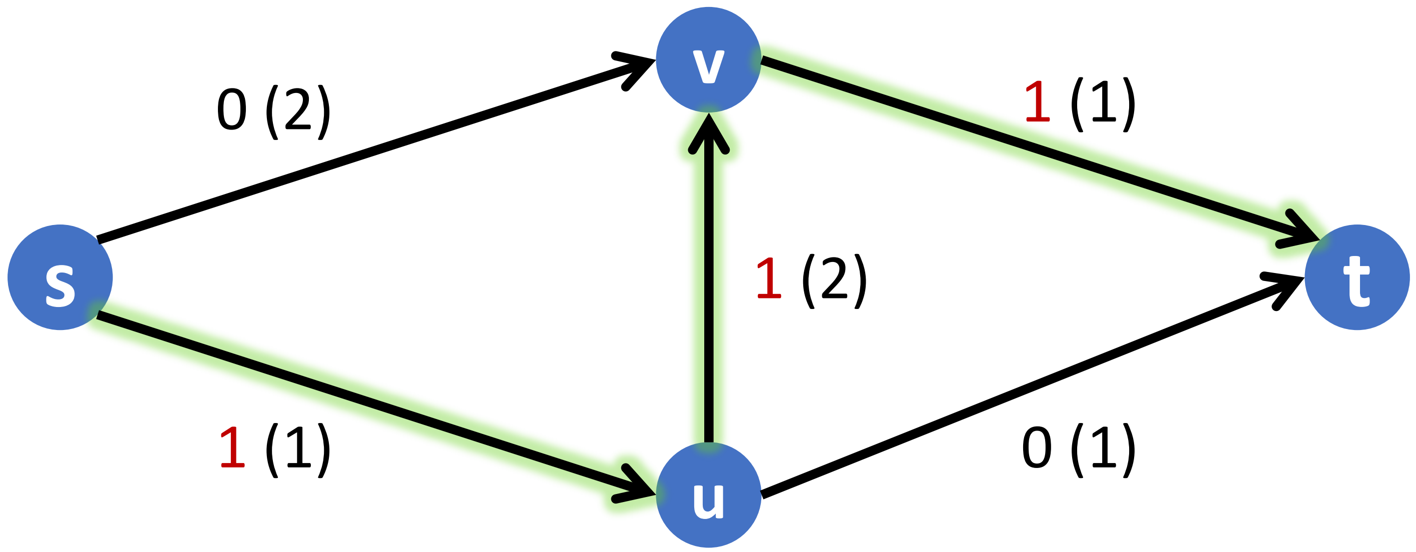
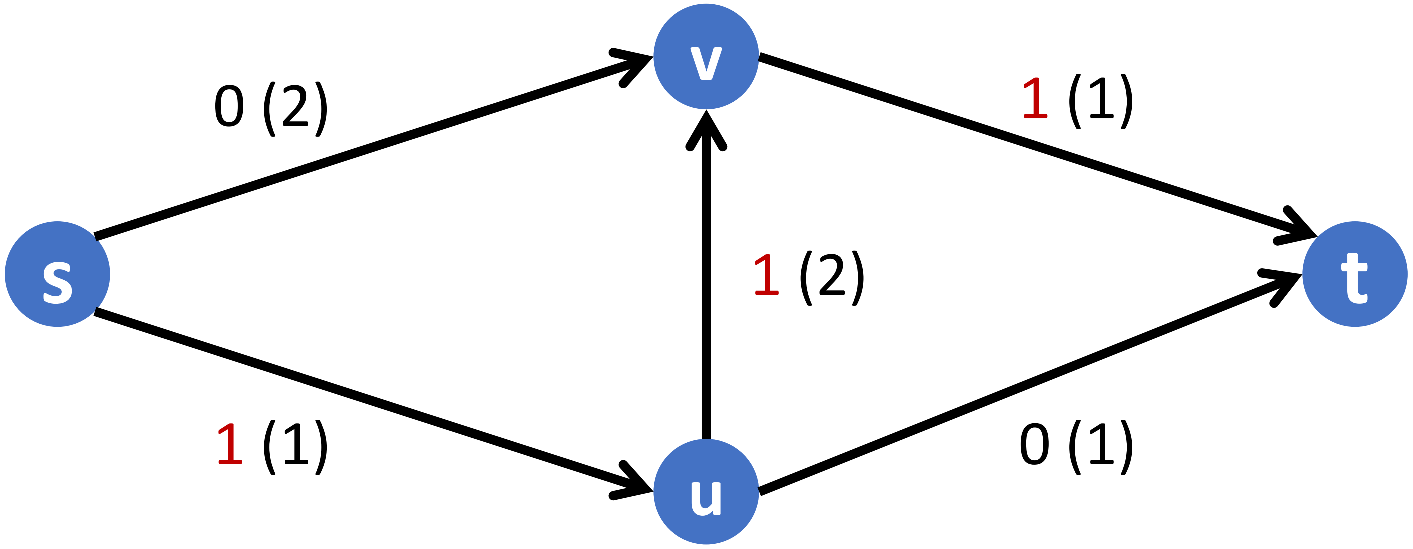
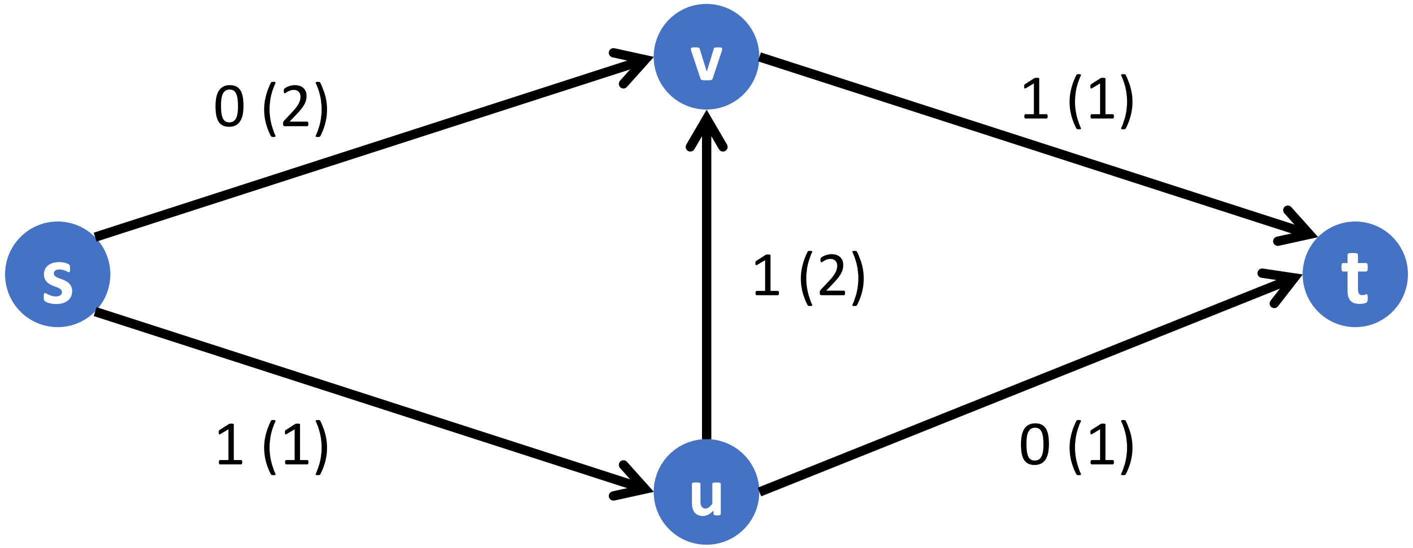


Two edges are at capacity and cannot route any more flow.
The algorithms cannot increase the flow.



Maximum flow: 2

Our algorithm found a flow of value 1



Given a flow f, define the residual network G_f

G_f is a directed graph on V with the following edges:

- edges $(u, v) \in E$ with $f(u, v) < c(u, v)$

  edge (u, v) is a forward edge of G_f

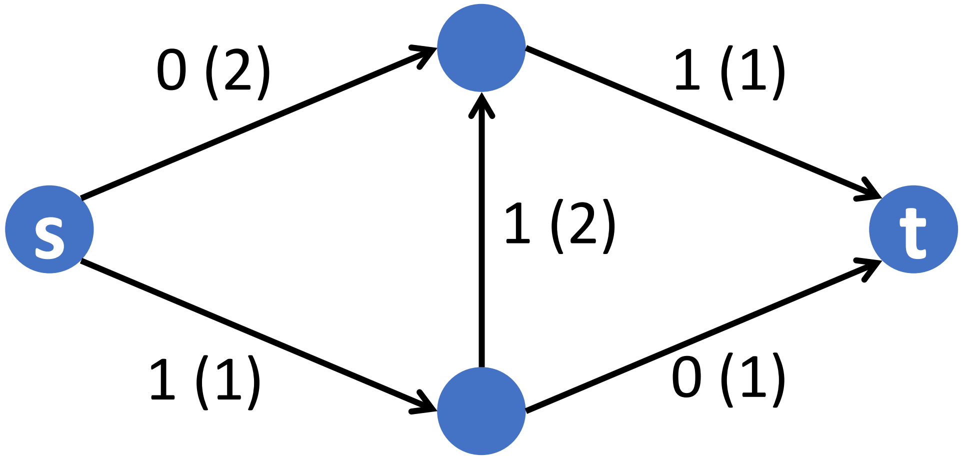
  its residual capacity $c_f(u, v) = c(u, v) - f(u, v)$

- edges $e' = (v, u)$ where $(u, v) \in E$ and $f(u, v) > 0$

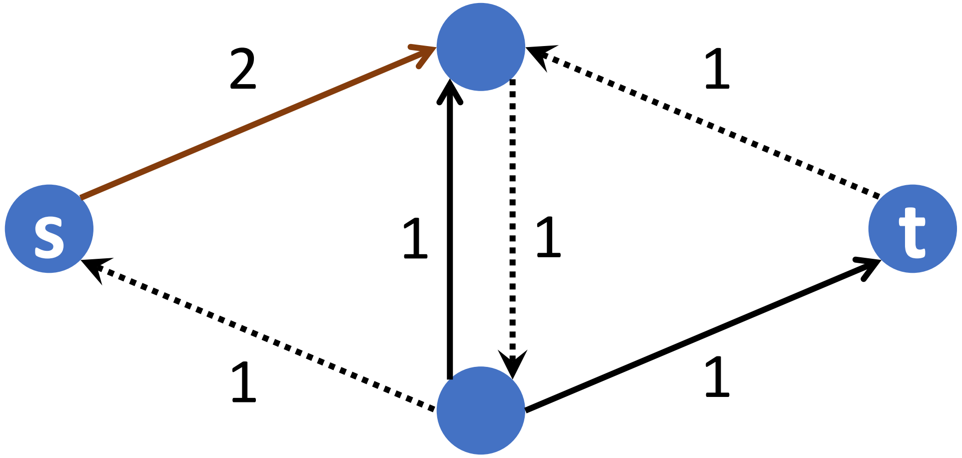
  edge (v, u) is a backward edge

  $c_f(v, u) = f(u, v)$

Residual Network



Flow network G



Residual network G_f

Forward edges – solid lines

Backward edges – dotted lines

Augmenting the flow along a path

Let f be a feasible s-t flow and P be a path in G_f.

Let $\delta \leq c_f(e)$ for every $e \in P$.

- for every forward edge $(u, v) \in P$, increase $f(u, v)$ by δ

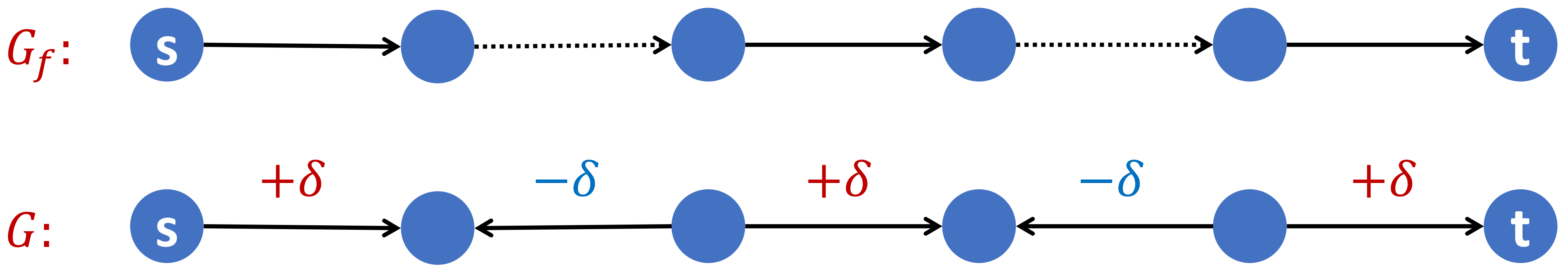
$$f'(u, v) = f(u, v) + \delta$$

- for every backward edge (v, u), decrease $f(u, v)$ by δ

$$f'(u, v) = f(u, v) - \delta$$

Augmenting the Flow

- $f'(u,v) = f(u,v) + \delta$ if (u,v) is a forward edge
- $f'(u,v) = f(u,v) - \delta$ if (v,u) is a backward edge



Augmenting the Flow

Claim: Obtained flow f' is a feasible flow.

Proof: Verify that f' satisfies capacity constraints.

If $(u, v) \in P$ and (u, v) is a forward edge,

$$f'(u,v) = f(u,v) + \delta \leq f(u,v) + c_f(u,v) = c(u,v)$$
$$\geq 0$$

If $(v, u) \in P$ and (v, u) is a backward edge,

$$f'(u,v) = f(u,v) - \delta \geq f(u,v) - c_f(v,u) = 0$$
$$\leq f(u,v) \leq c(u,v)$$

Otherwise, $f'(e) = f(e)$

Augmenting the Flow

Claim: Obtained flow f' is a feasible flow.

Proof: Verify that f' satisfies flow conservation constraints.

G_f: $v \to u \to w$	$f_{in}(u)$	$f_{out}(u)$
$\bullet \rightarrow u \rightarrow \bullet$	$+\delta$	$+\delta$
$\bullet \rightarrow u \leftarrow \bullet$	$+\delta + (-\delta) = 0$	0
$\bullet \leftarrow u \rightarrow \bullet$	0	$-\delta + \delta = 0$
$\bullet \leftarrow u \leftarrow \bullet$	$-\delta$	$-\delta$

Augmentation: Flow Value

Q: What is the value of the augmented flow f'?

$$val(f') = ?$$

Augmentation: Flow Value

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$$val(f') = val(f) + \delta$$

Q: What is the best choice of δ for a given path P?

Augmentation: Flow Value

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A: $\delta = \min_{e \in P} c_f(e)$

Q: What happens with the residual capacities of edges on P?

Augmentation: Flow Value

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A: $\delta = \min_{e \in P} c_f(e)$

Q: What happens with the residual capacities of edges on P?

Forward edge: $f'(e) = f(e) + \delta \quad \Rightarrow \quad c_{f'}(e) = c_e - f'(e) = c_f(e) - \delta$

Backward edge: $f'(e') = f(e') - \delta \Rightarrow \quad c_{f'}(e) = f'(e') = f(e') - \delta = c_f(e) - \delta$

here e is the backward edge for e'

Augmentation: Flow Value

Summary

A bottleneck edge on P is the edge with the least residual capacity δ.

- $val(f') = val(f) + \delta$
- The residual capacity of each edge on P decreases by δ.
- Bottleneck edges disappear.

Ford-Fulkerson Algorithm

- start with an empty flow f: $f(e) = 0$ for all $e \in E$. $G_f = G$
- while there is an s-t path P in G_f

  augment flow f along path P

  update G_f
- return f

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TODO items:

- Prove that the algorithm finds an optimal flow (if it stops)
- Does the algorithm stop? Find its running time.

Optimality

Need to prove: when the algorithm, f is a maximum flow.

The Max Flow / Min Cut Theorem:

1. If there is no s-t path P in G_f then f is a maximum flow.
2. (Strong Duality)

$$val(f) = cap(A, B)$$

where (A, B) is a minimum cut.

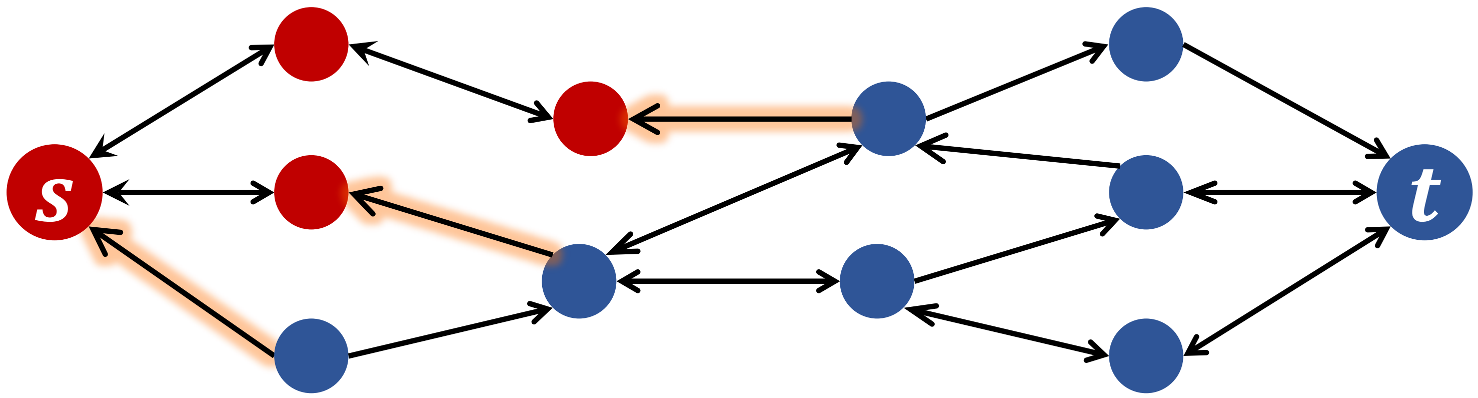
Optimality

Let $\quad A = \{u\text{: there is a path from } s \text{ to } u \text{ in } G_f\}$ (vertices reachable from s)

$B = V \setminus A$

Note that

- $s \in A$ (trivially)
- $t \notin A$, since there is no s-t path in G_f. Thus, $t \in B$.



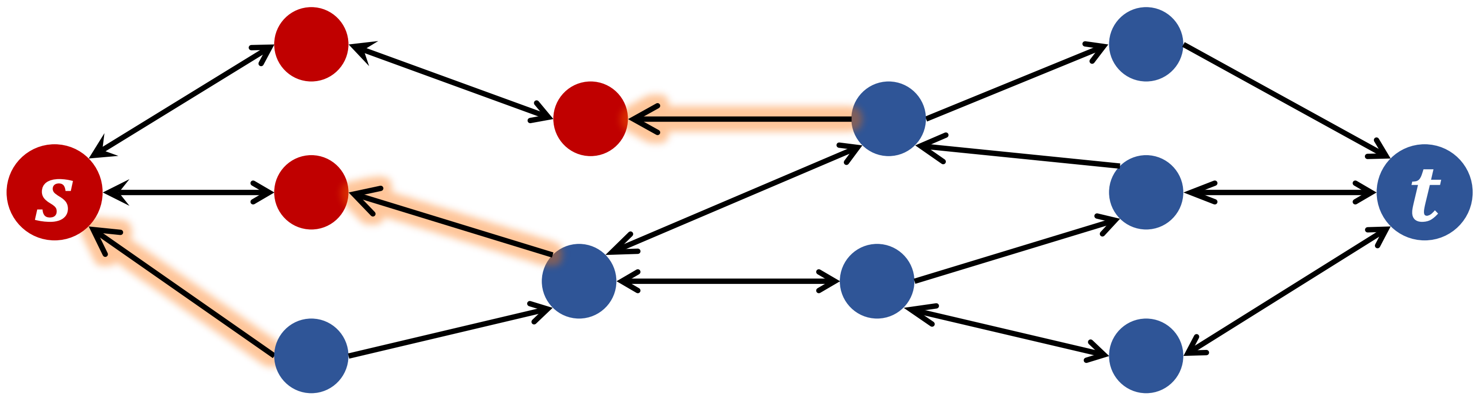
Optimality

Thus, (A, B) is an s-t cut

There are no edges from A to B in G_f. Why?

$\Rightarrow$ if $u \in A, v \in B, (u, v) \in E$, then $f(u, v) = c(u, v)$

$\Rightarrow$ if $u \in B, v \in A, (u, v) \in E$, then $f(u, v) = 0$



Optimality

$\Rightarrow$ if $u \in A$, $v \in B$, $(u, v) \in E$, then $f(u, v) = c(u, v)$

$\Rightarrow$ if $u \in B$, $v \in A$, $(u, v) \in E$, then $f(u, v) = 0$

$$cap(A, B) = \sum_{\substack{(u,v)\in E \\ u\in A, v\in B}} c(u, v) = \sum_{\substack{(u,v)\in E \\ u\in A, v\in B}} f(u, v) = f_{out}(A) = f_{in}(A) + val(f) = val(f)$$

Using weak duality:

$$val(f') \leq cap(A, B) = val(f) \leq cap(A', B')$$

Thus f is a maximum flow, (A, B) is a minimum cut, and $val(f) = cap\ (A, B)$.

Integrality

Assume that all capacities are integers.

Prove by induction that after each iteration:

- The flow will be integral.
- All residual capacities will be integral.
- The bottleneck capacity δ will be integral.

The algorithm returns an integral maximum flow.

Corollary: There is an integral maximum flow.

Note that there may be a fractional maximum flow as well.

Running Time

In each iteration the flow increases by 1.

Therefore, the algorithm stops in at most val^* iterations.

$$val^* \leq C = \sum_{e \in out(s)} c(e)$$

Each iteration can be implemented in $O(n)$ time (use BFS or DFS).

Running time: $O(val^* \cdot n)$.

If all capacities are rational numbers, the algorithm also always terminates.

Variants of Ford-Fulkerson

There are various rules for choosing path P.

The running time depends on the rule.

1. Scaling variant: $O((\log C + 1)\, m^2)$ (see the textbook)
2. Edmonds–Karp: $O(m^2 n)$ doesn't depend on the capacities (strongly polynomial-time algorithm).

   The algorithm uses BSF to find a shortest path P between s and t.

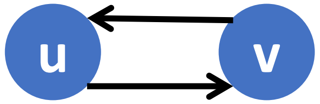
Finding a Minimum Cut

Q: Can we find a minimum cut (A, B) in G using the Ford-Fulkerson algorithm?

Max Flow & Min Cut in Undirected Graphs

Undirected Graphs

We can reduce the case of undirected graphs to that of directed.



The capacity of an s-t cut (S, T) is the total capacity of edges from S to T.

The Max Flow / Min Cut Theorem also holds or undirected graphs.

Applications of Max Flow and Min Cut

Routing

- Routing vehicles on the road.
- Routing electricity.
- Routing internet traffic.

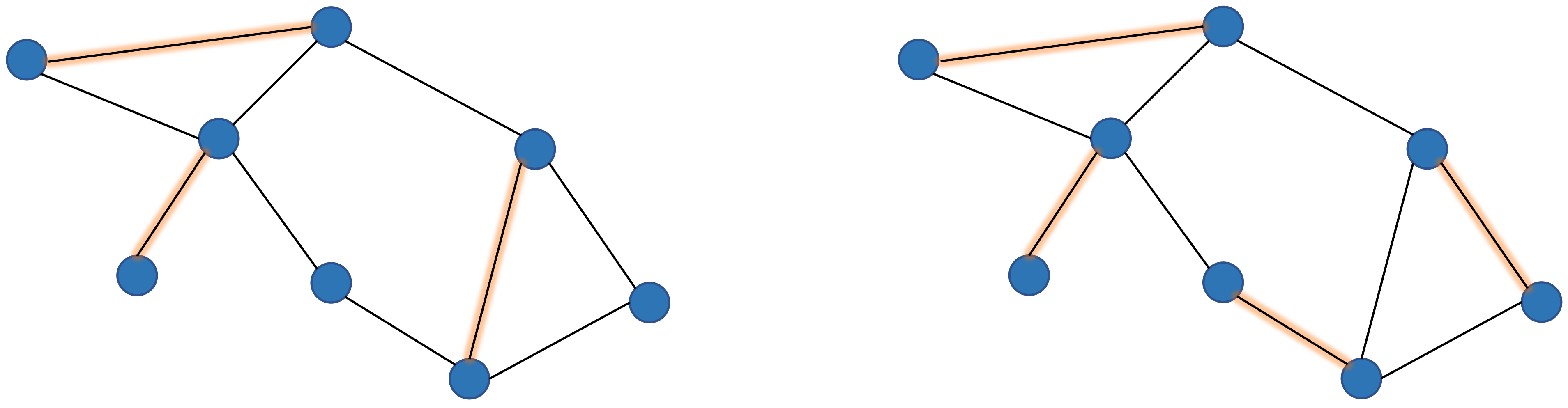
Bipartite Matching

Matching

Consider an undirected graph.

A subset of edges M is a matching if no two edges in M share a vertex.

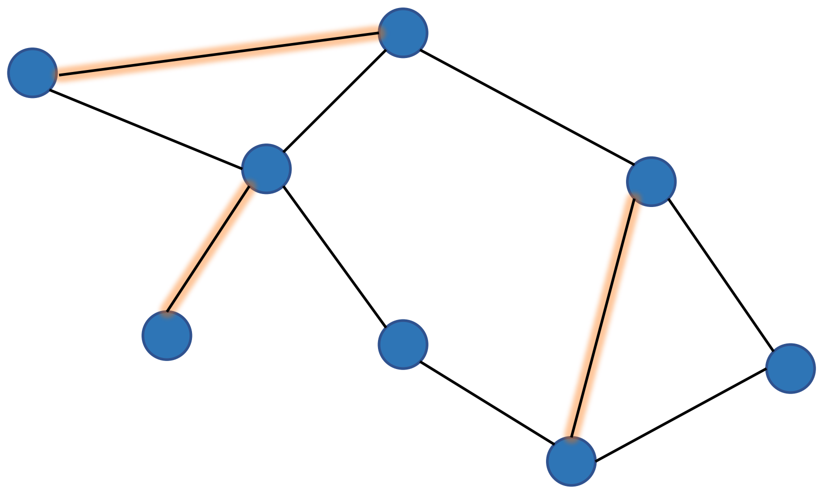
A vertex u is matched by M if there is an edges $(u, v) \in M$ for some v.



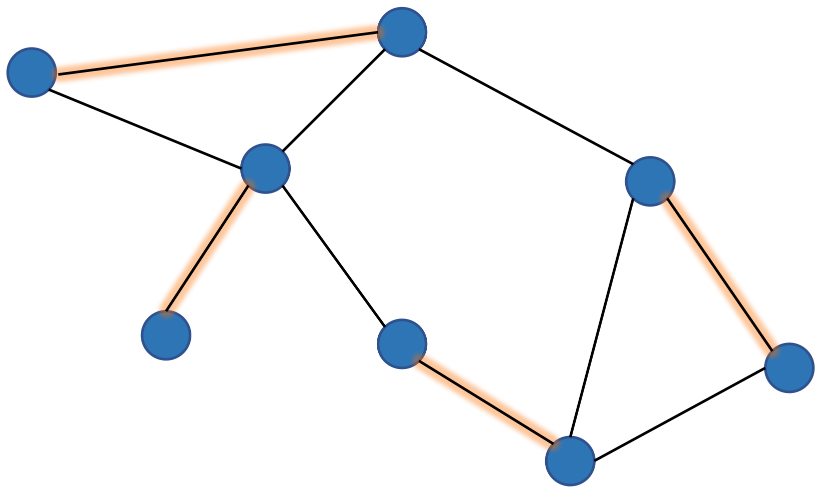
Matching

A matching is a perfect matching in G if all vertices are matched.

not a perfect matching



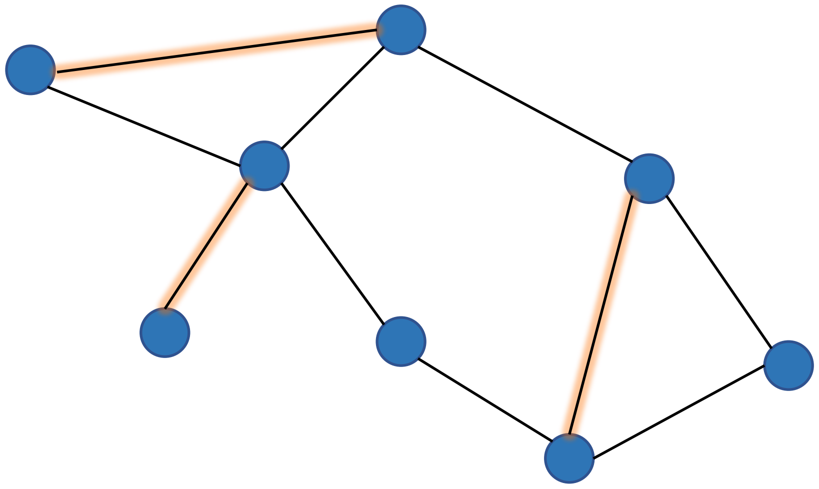
perfect matching



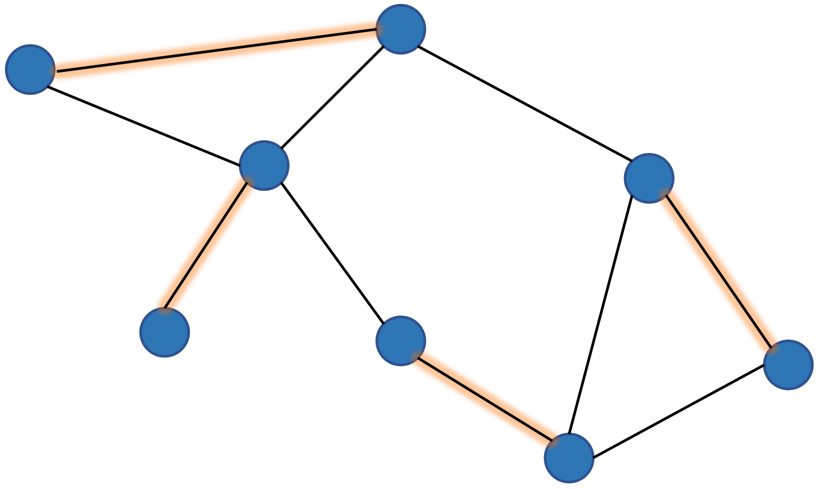
Matching

M is a maximum matching if $|M| \geq |M'|$ for every matching M'.
M is a maximal matching if $M \cup \{e\}$ is not a matching for every $e \notin M$.

maximal matching
(not maximum)



maximum matching

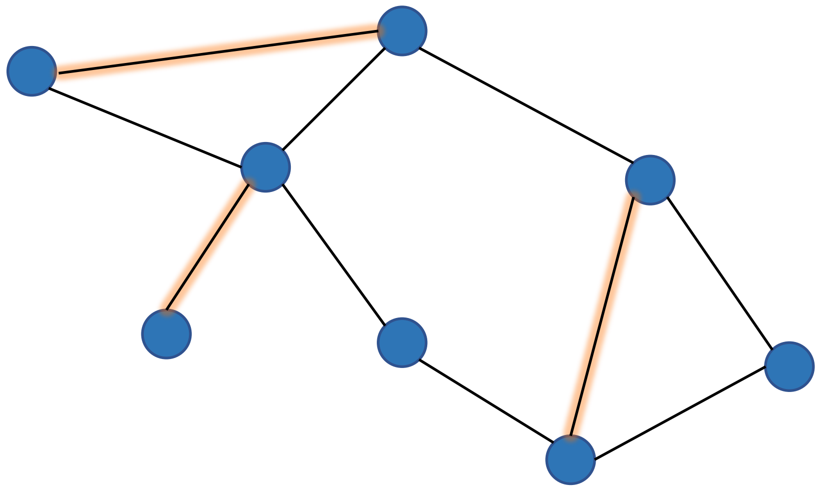


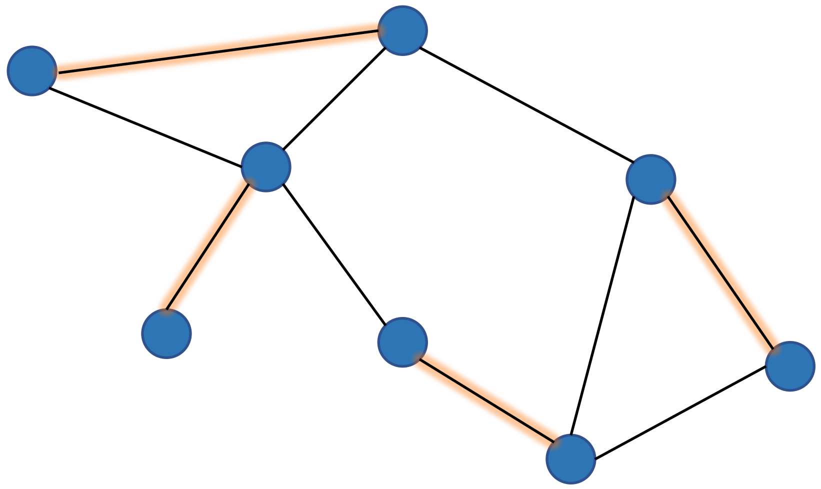
Greedy Algorithm for Finding Matchings

Consider a greedy algorithm:

- start with $M = \emptyset$
- while there is an edge $e \notin M$ s.t. $M \cup \{e\}$ is a matching
  add e to M

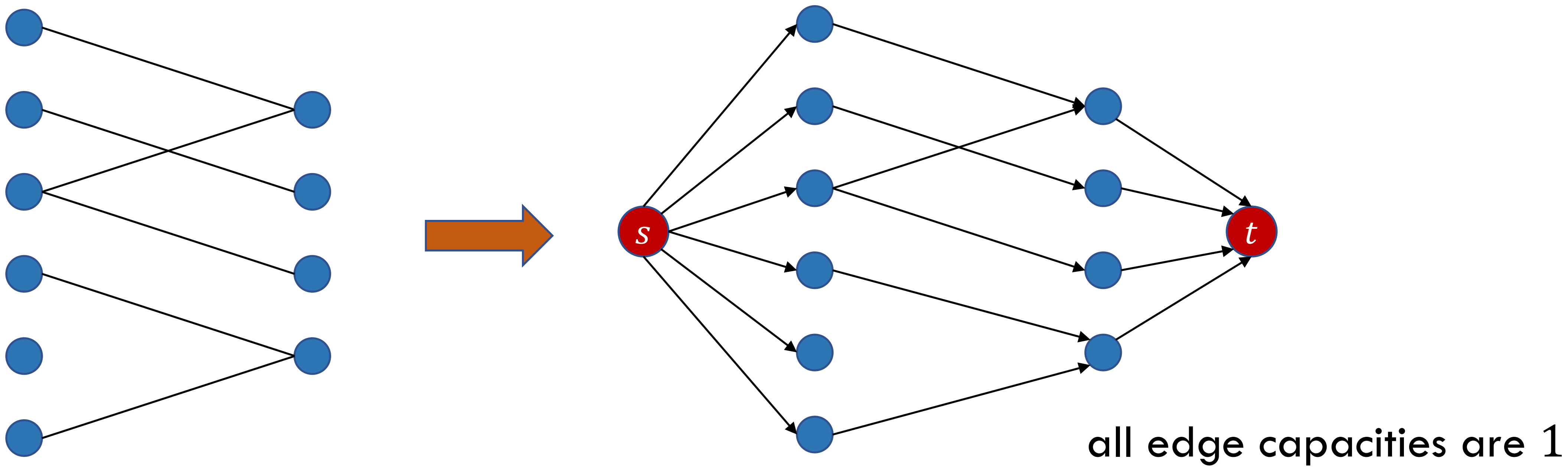
Q: What kind of matching will this algorithm find?





Matching in Bipartite Graphs

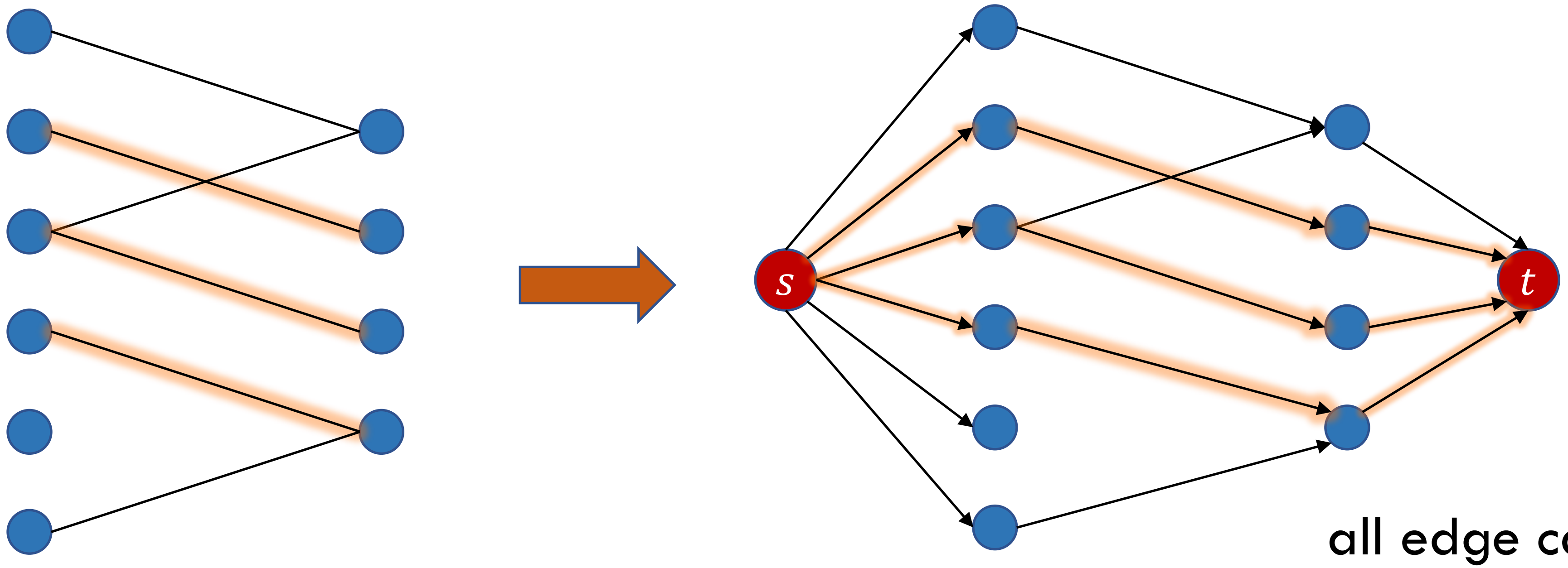
Assume that $G = (L \cup R, E)$ is a bipartite graph. Transform G to an s-t flow network G'. Then the size of the maximum matching in G equals the value of the maximum s-t flow in G'.



Matching in Bipartite Graphs

If M is a matching, then there is a flow f in G' of value $|M|$.

$$val(f^*) \geq \max_{\mathrm{M}} |M|$$



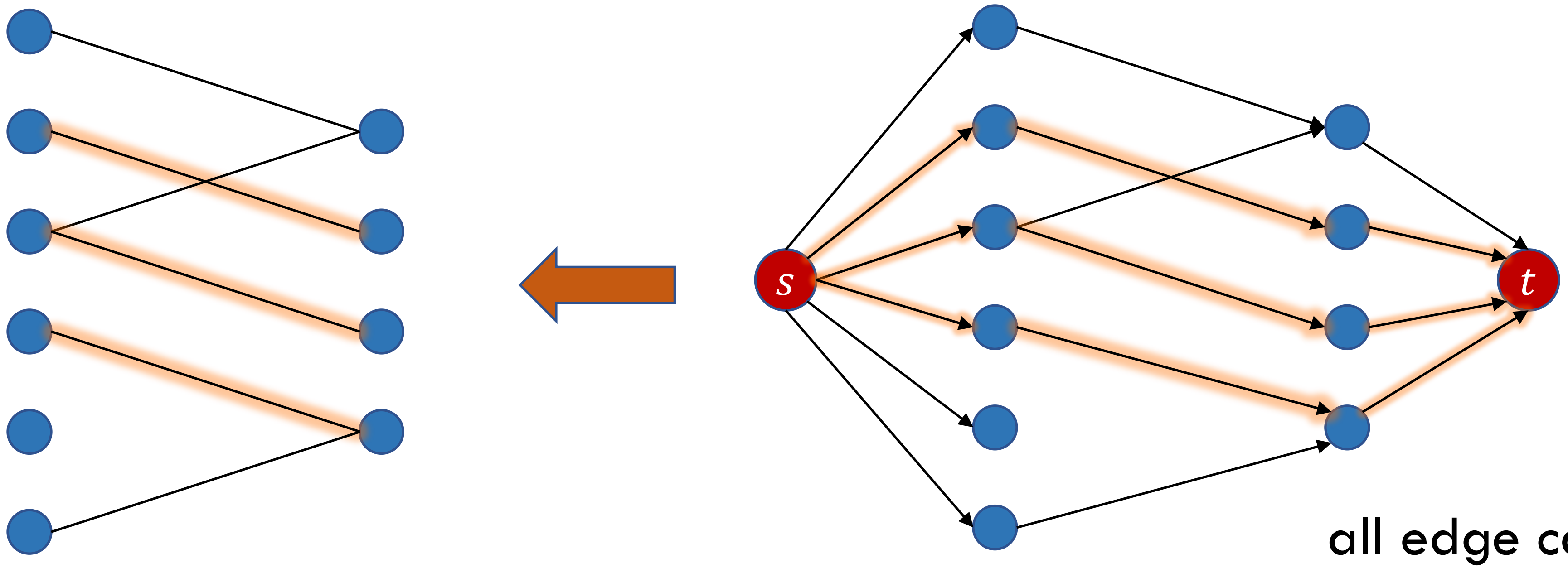
all edge capacities are 1

Matching in Bipartite Graphs

Let f^* be an integral maximum flow in G'. Note $f(e) \in \{0,1\}$ for every $e \in E$.

Let M be the set of edges between L and R used by the flow.

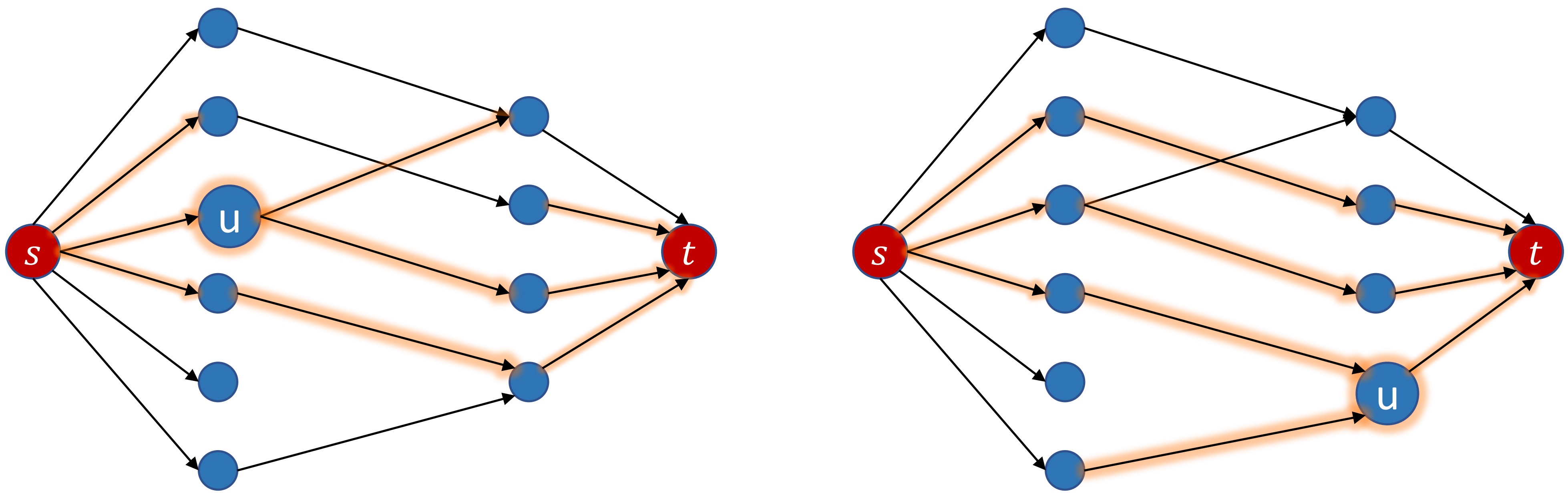
Is M a matching?



all edge capacities are 1

Matching in Bipartite Graphs

Is M a matching? Is it possible that a vertex u in incident to two edges in M?



all edge capacities are 1

Matching in Bipartite Graphs

- Transform the graph into a flow network
- Find an integral maximum flow using Ford-Fulkerson
- Return the corresponding matching

Running time: $O(m\, f^*) = O(mn)$.

Matching in Bipartite Graphs

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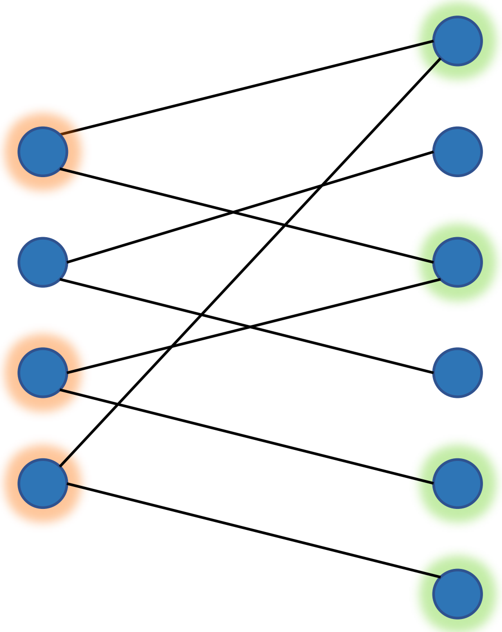
Hall's theorem

Consider a bipartite graph $G = (L \cup R, E)$. For every subset $A \subset L$, let $N(A)$ be the set of neighbors of A in R.



Hall's theorem

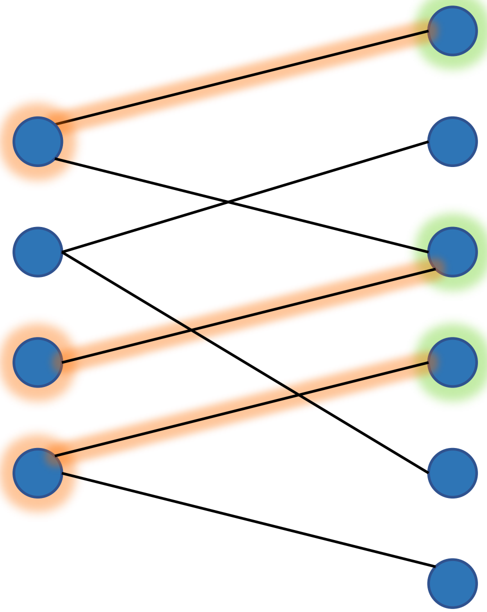
Consider a bipartite graph $G = (L \cup R, E)$. For every subset $A \subset L$, let $N(A)$ be the set of neighbors of A in R.

Assume that there is a matching M of size $|L|$.

I.e., every vertex L is matched.

Then every $A \subset L$ is matched with exactly $|A|$ vertices in R. All of them are neighbors of A.

Thus, $|N(A)| \geq |A|$.



Hall's theorem

Hall's Theorem

There is a matching M of size $|L|$ if and only if

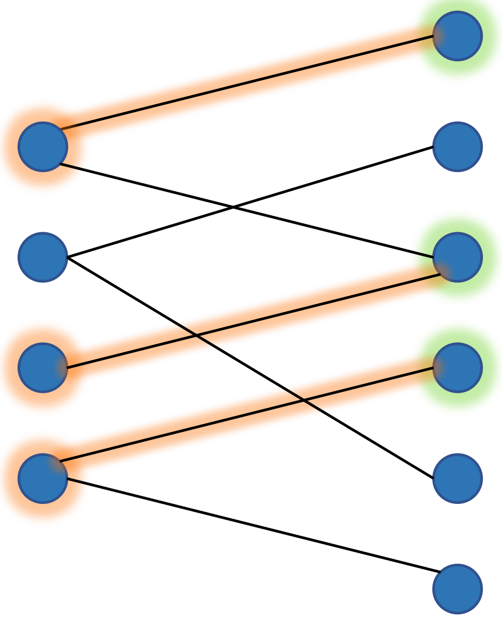
$$|N(A)| \geq |A|$$

for every $A \subset L$.

Proof:

We showed that "$\Rightarrow$" holds.

Now we prove "$\Leftarrow$".

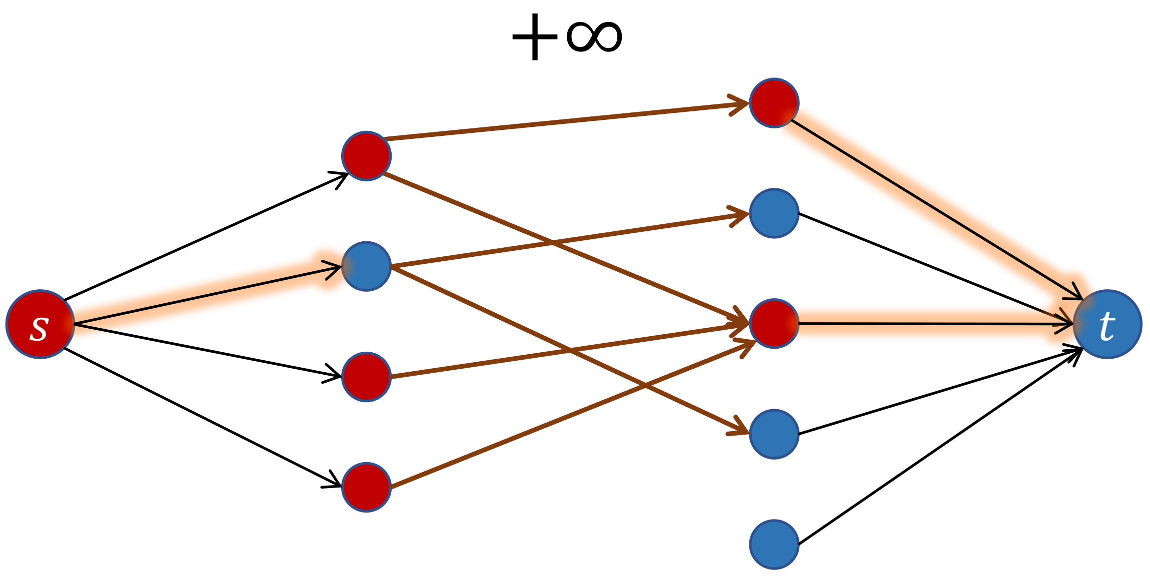


Hall's theorem

Assume that $|M| < |L|$ for a maximum matching M

We will show that $|N(A)| < |A|$ for some set $A \subset L$.

Max Flow / Min Cut Thm $\Rightarrow$ There is a cut (P, Q) of size $|M| < |A|$ in G'.



What edges are cut?

- edges from s to $Q \cap L$
- edges from $P \cap R$ to t

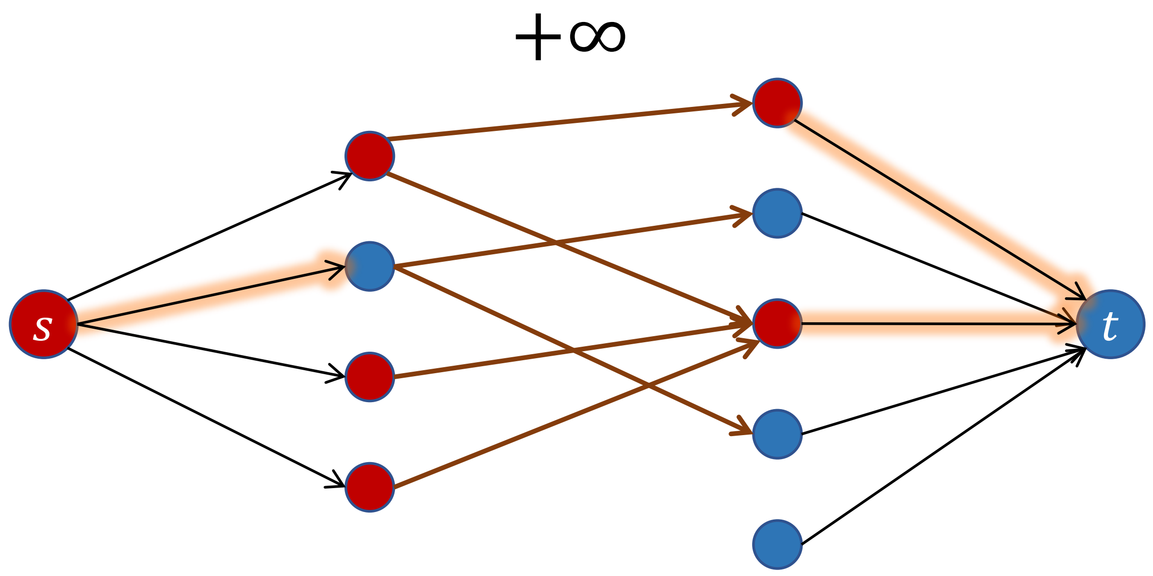
? edges from $P \cap L$ to $Q \cap R$

Hall's theorem

Thus, $|\mathrm{L}| > cap(P, Q) = |Q \cap L| + |P \cap R|$.

That is,

(#blue vertices on the left) + (#red vertices on the right) $< |L|$



Hall's theorem

Thus, $|\mathrm{L}| > cap(P, Q) = |Q \cap L| + |P \cap R|$.

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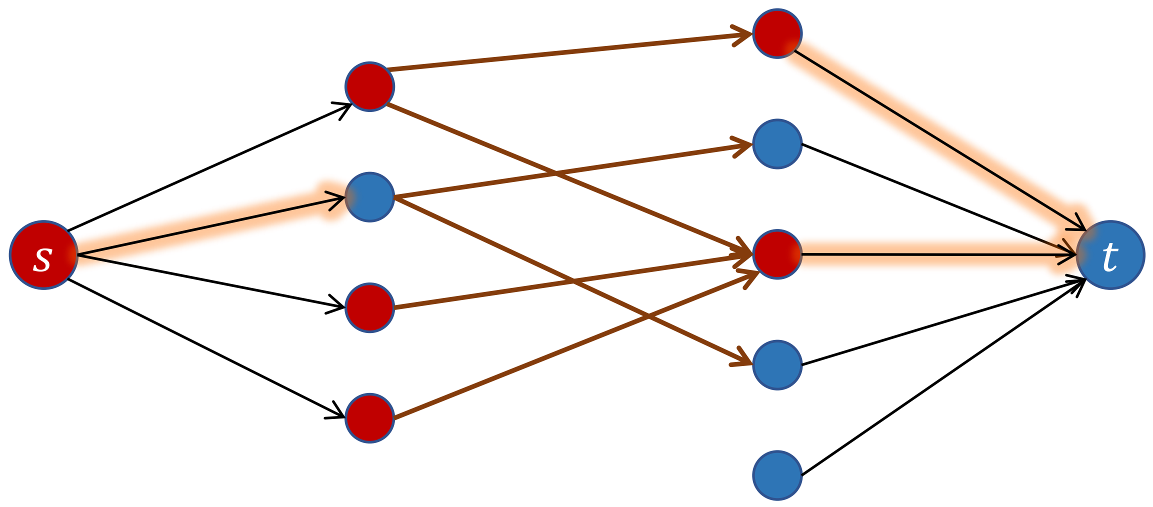
(#red vertices on the right) < (#red vertices on the left)

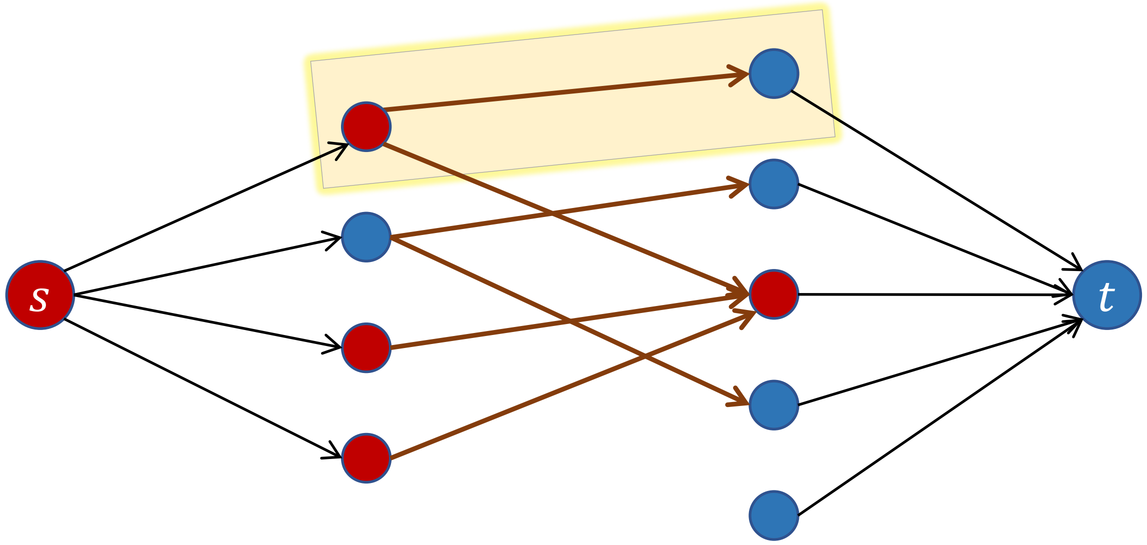


Hall's theorem

Let $A = P \cap L$ (red vertices on the left). What is the size of $N(A)$?

All vertices in $\mathrm{N}(A)$ are on the right.

Q: Can a vertex in $N(A)$ be blue?



(#red vertices on the right)
$<$ (#red vertices on the left)

Hall's theorem

Let $A = P \cap L$ (red vertices on the left). What is the size of $N(A)$?

$$\begin{aligned} A \quad &= P \cap L = \{\text{red vertices on the left}\} \\ N(A) &\subseteq P \cap R = \{\text{red vertices on the right}\} \end{aligned}$$

$$|N(A)| < |A|$$

$\blacksquare$

Hall's theorem

Corollary

Let G be a bipartite graph with $|L| = |R|$.

There exists a perfect matching in G

if and only if

$|N(A)| \geq |A|$ for every $A \subset L$

Assignment Problems

Basic Assignment Problem

There are m people and $n < m$ jobs.

- For every job i, we are given the list of people S_i that can perform it.
- We may assign only one to each person.

Find an assignment of jobs to people (if it exists).

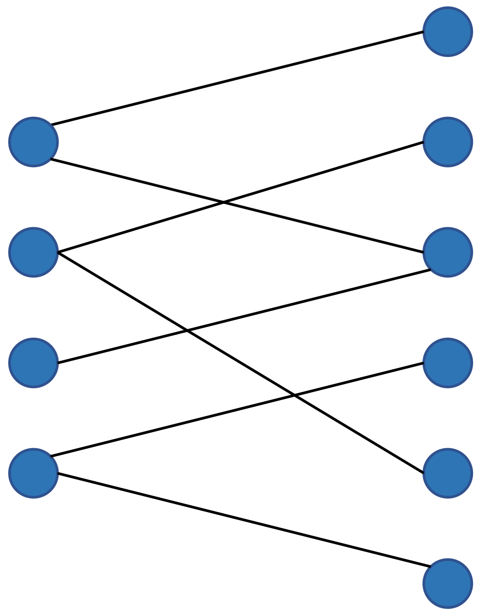
Solution:

Consider a bipartite graph with

$L = \{1, \ldots, m\}$ representing jobs and

$R = \{1, \ldots, n\}$ representing people

connect job i with people $j \in S_i$



Job Assignment Problem

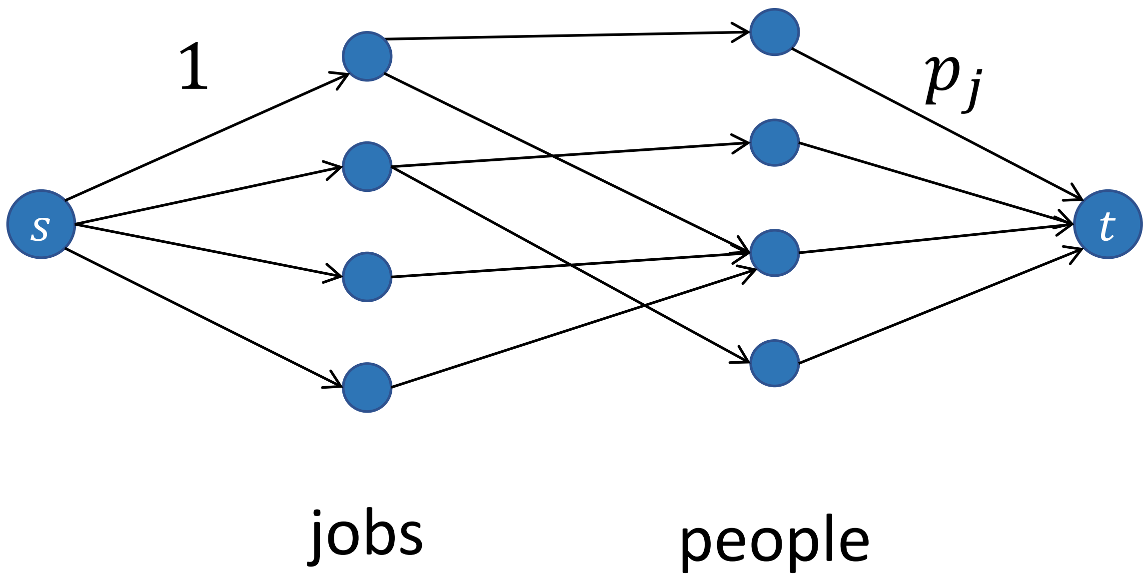
There are m people and m jobs.

- For every job i, we are given the list of people S_i that can perform it.
- We may assign p_j jobs to person j.

Find an assignment of jobs to people subject to the constraints above so as to maximize the number of assigned jobs.

Q: Suggestions?

Job Assignment Problem



Project Scheduling

There are n projects: $1, \dots, n$

We are given a set of dependencies between projects of the following form

in order to complete project i we need to complete project j first

E.g.,

1 depends on 3, 2 depends on 3 and 4, 4 depends on 6, etc

A feasible schedule A is a subset of projects s.t. if $i \in A$ then all projects j that i depends on are also in A.

Project Scheduling

Some projects are profitable and some are not.

For each project i, we are given p_i.

- If $p_i \geq 0$, project i has profit p_i.
- If $p_i < 0$, project i has cost $-p_i$.

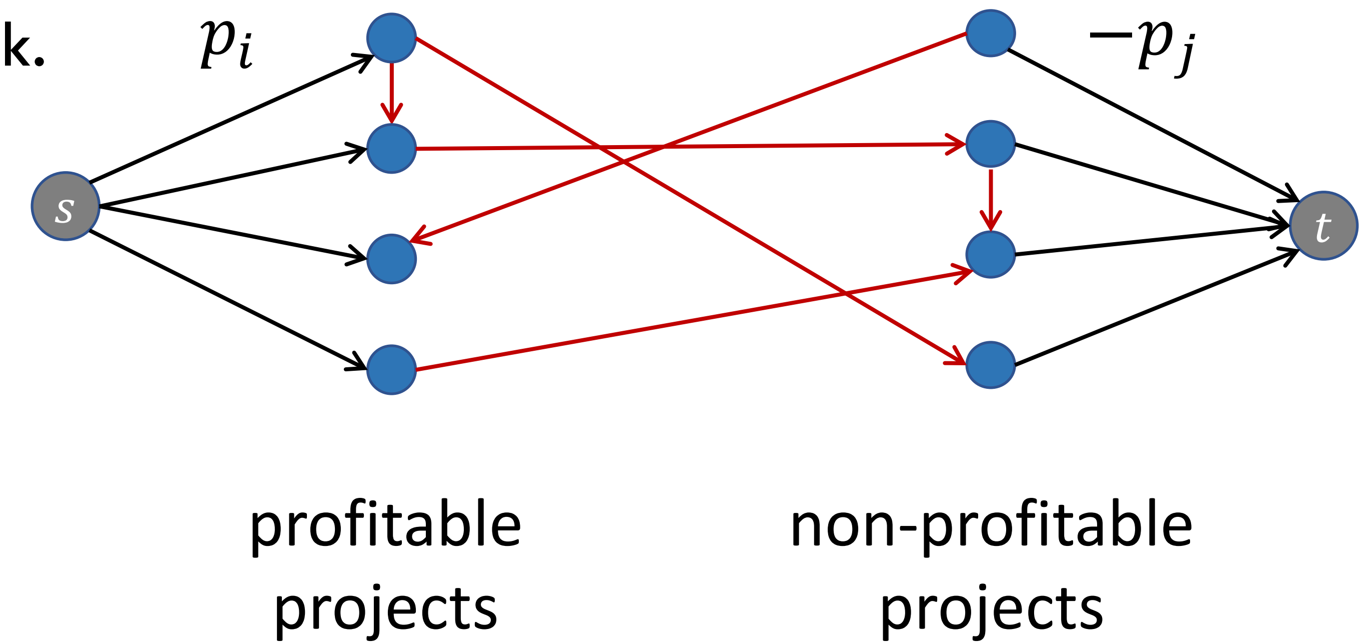
Out goal is to find a feasible schedule A that maximizes our profit

$$\sum_{i \in A} p_i$$

Project Scheduling

Construct the following s-t network.

If i depends on j, we have an edge (i, j) of ∞ capacity.



Observation:

S is a feasible schedule

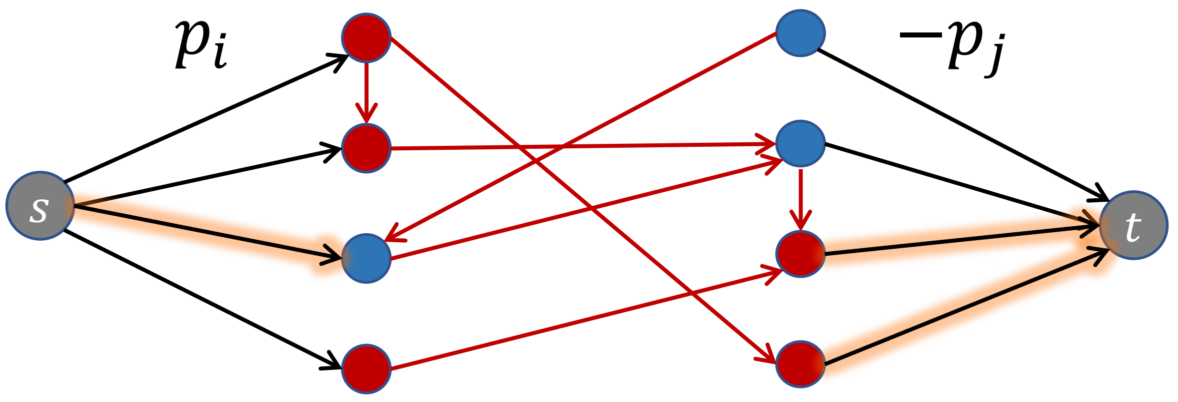
$\Updownarrow$

cut $(A \cup \{s\}, B \cup \{t\})$ has finite cost where $B = \{1, \ldots, n\} \setminus A$

Project Scheduling

There is a one-to-one correspondence between cuts of finite capacity and feasible schedules!

How are their costs & profits related?



$cap(A \cup \{s\}, B \cup \{t\})$ **equals**

the profit of non-scheduled jobs on the left

$+$

the cost of scheduled jobs on the right

Project Scheduling

$\boldsymbol{cap(A \cup \{s\}, B \cup \{t\})}$ **equals**

the profit of non-scheduled jobs on the left

$$+$$

the cost of scheduled jobs on the right

$$cap(A \cup \{s\}, B \cup \{t\}) = \sum_{\substack{i \notin A \\ p_i \geq 0}} p_i + \sum_{\substack{i \in A \\ p_i < 0}} (-p_i) = \left(\sum_{p_i \geq 0} p_i \right) - \sum_{i \in A} p_i$$

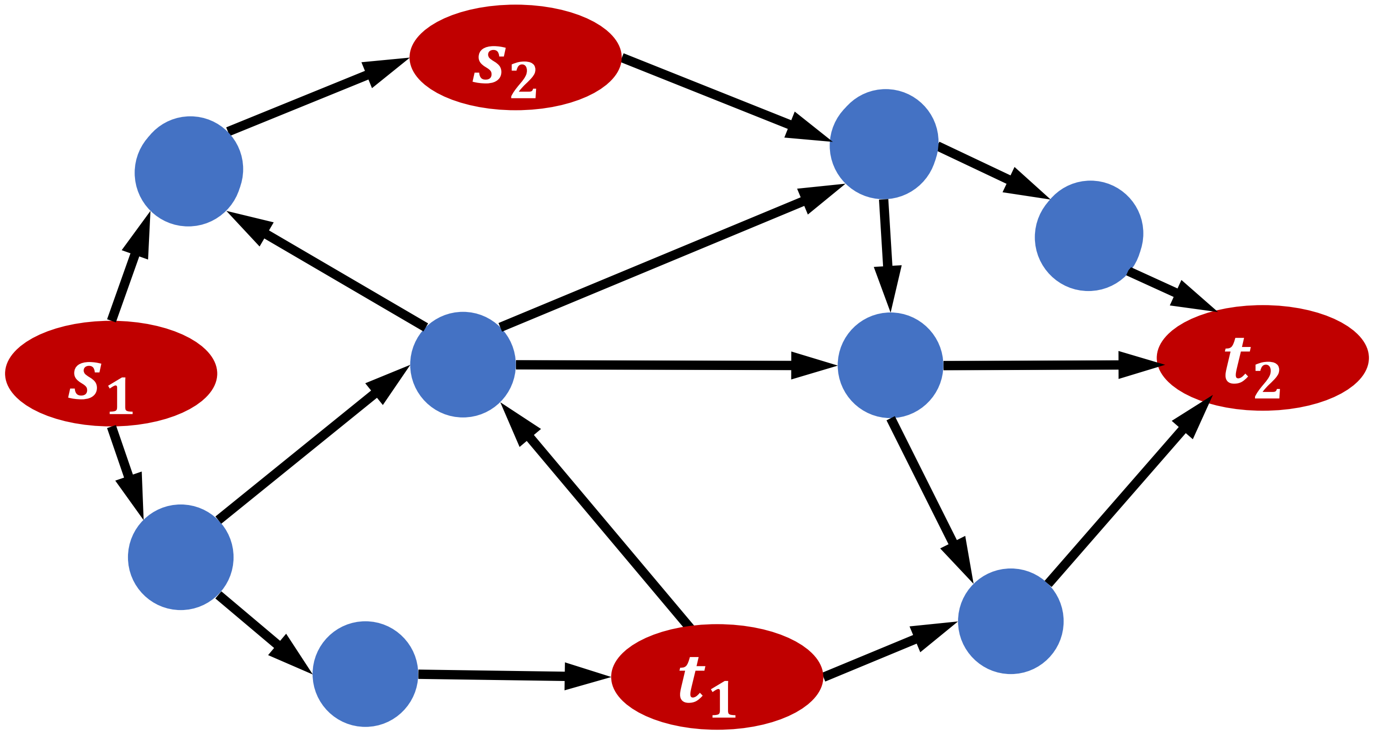
Q: How can we solve the problem now?

Multicommodity Flow

Multiple Sources & Sinks

Given: Flow network w/ many sources: $s_1, \dots, s_{k'}$ and
many sinks: $t_1, \dots, t_{k''}$.

Goal: Maximize the total amount of flow from all sources to all sinks.



Multiple Sources & Sinks

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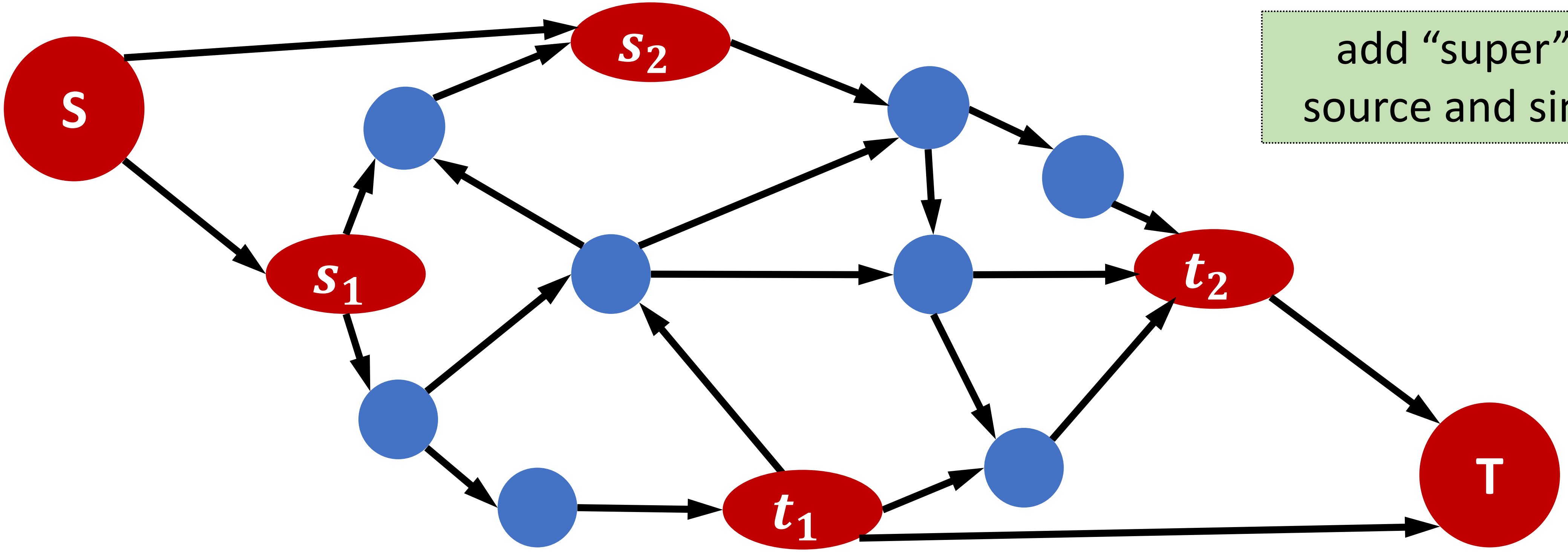


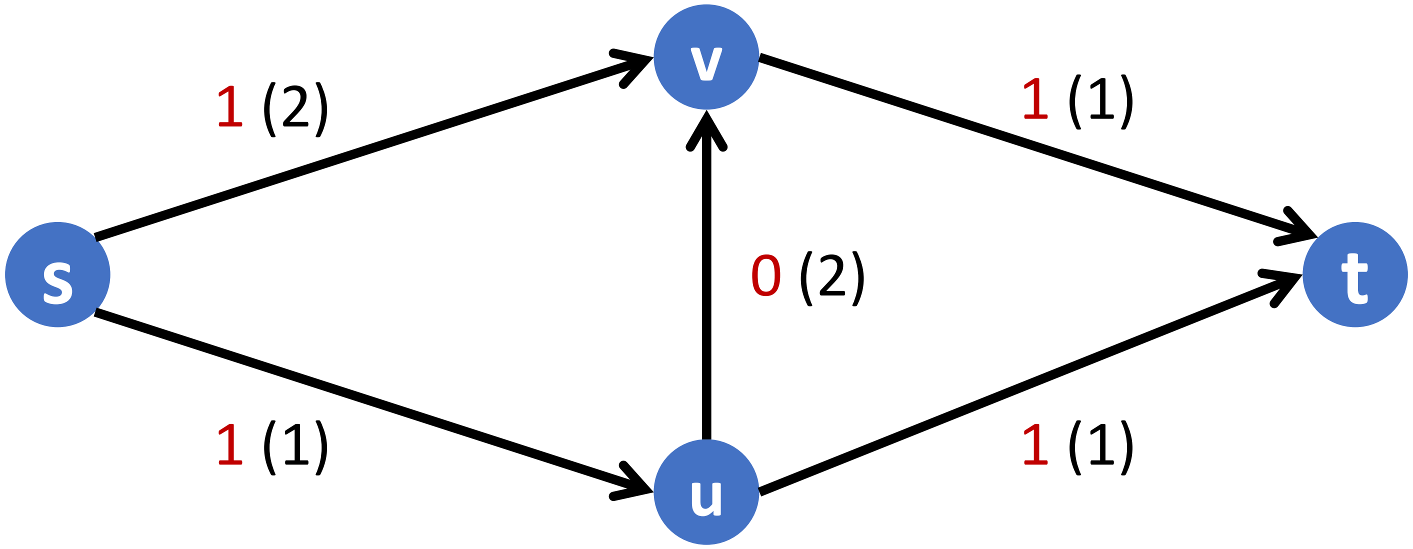
add "super" source and sink

Flow-Path Decomposition

Flow-Path Decomposition

Imagine that we are solving a truck routing problem.

We want to find a route for each truck.

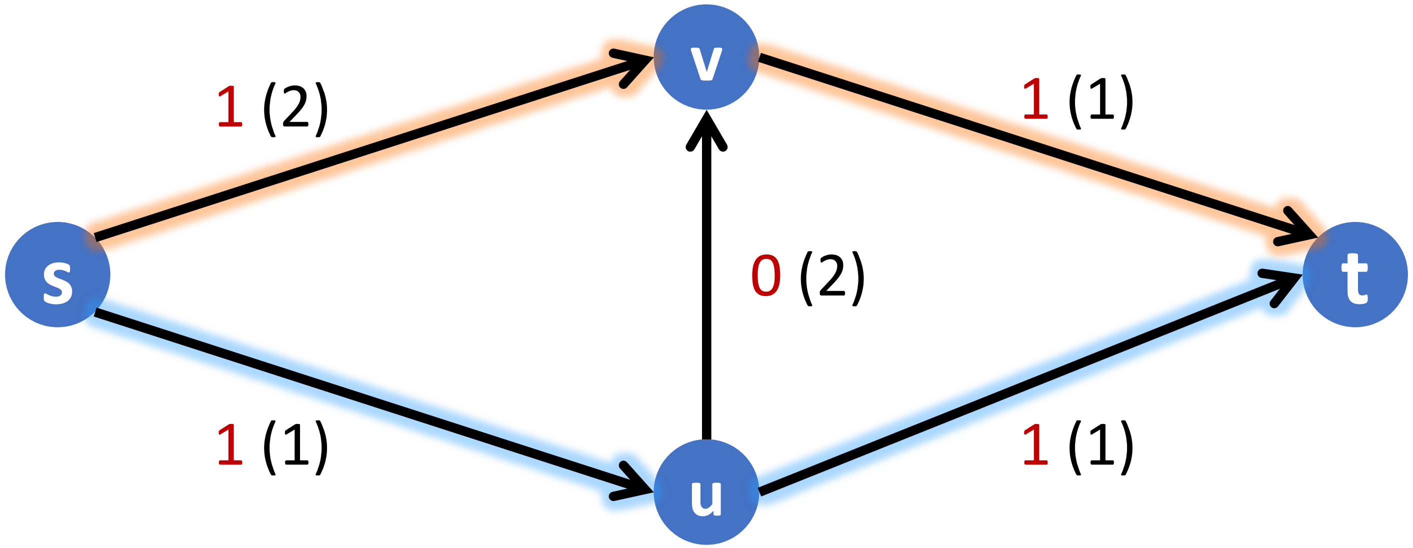


Flow-Path Decomposition

Imagine that we are solving a truck routing problem.

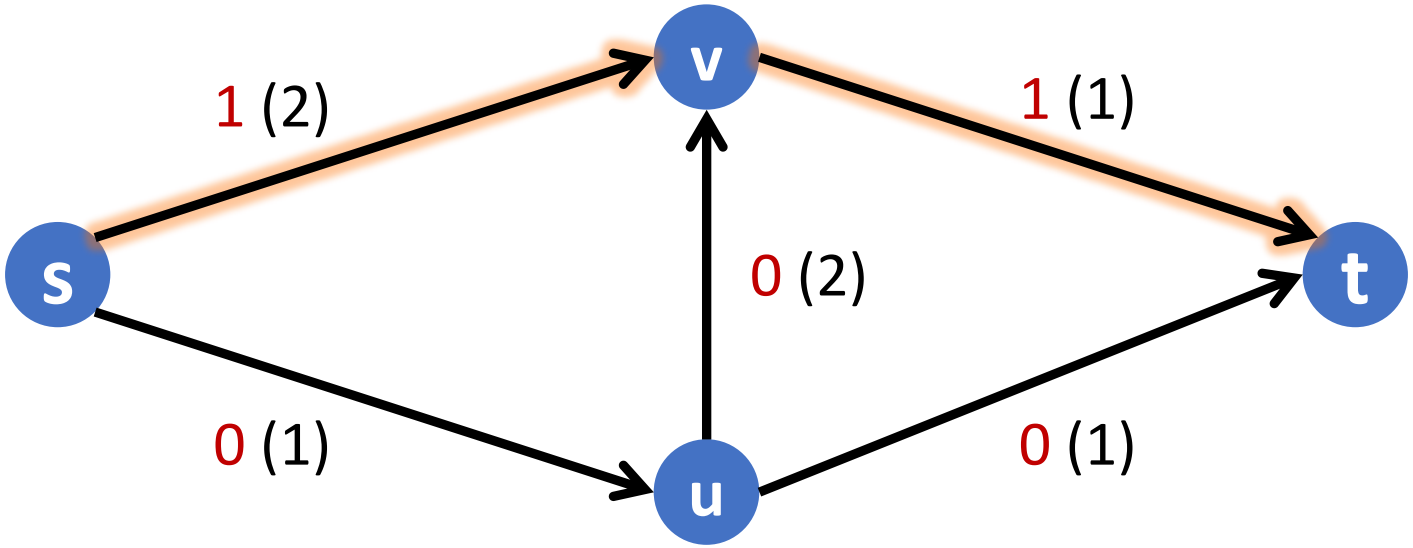
We want to find a route for each truck.

In this example we have 2 routes.



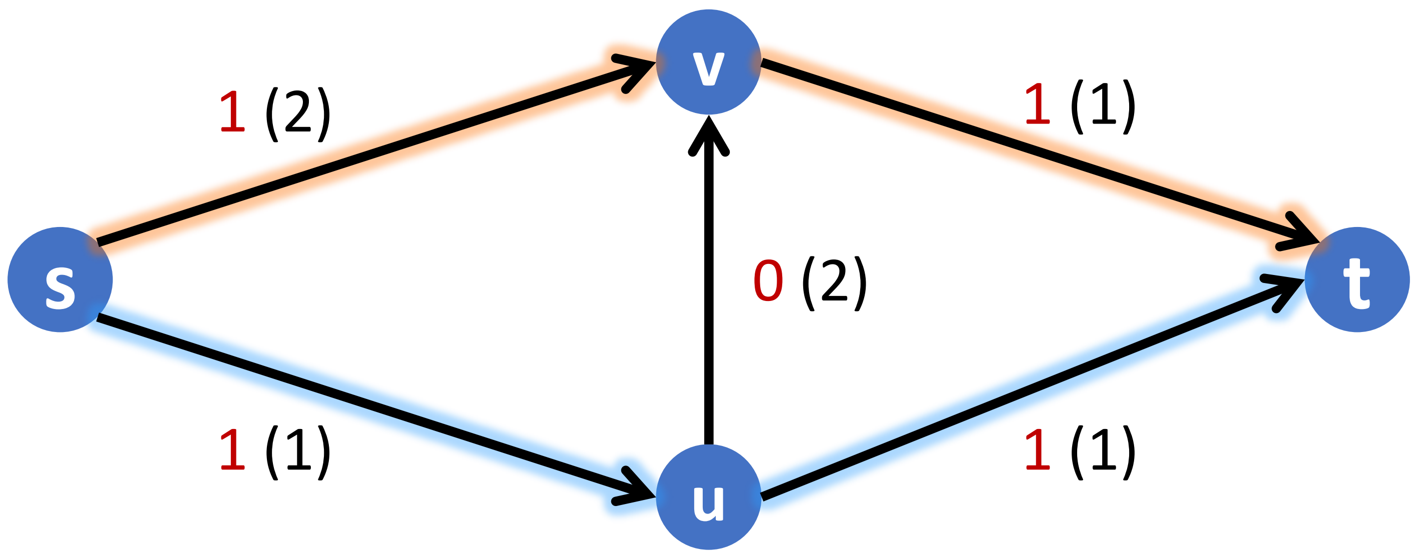
Flow-Path Decomposition

A flow f_i is a path flow if it uses a single s-t path.



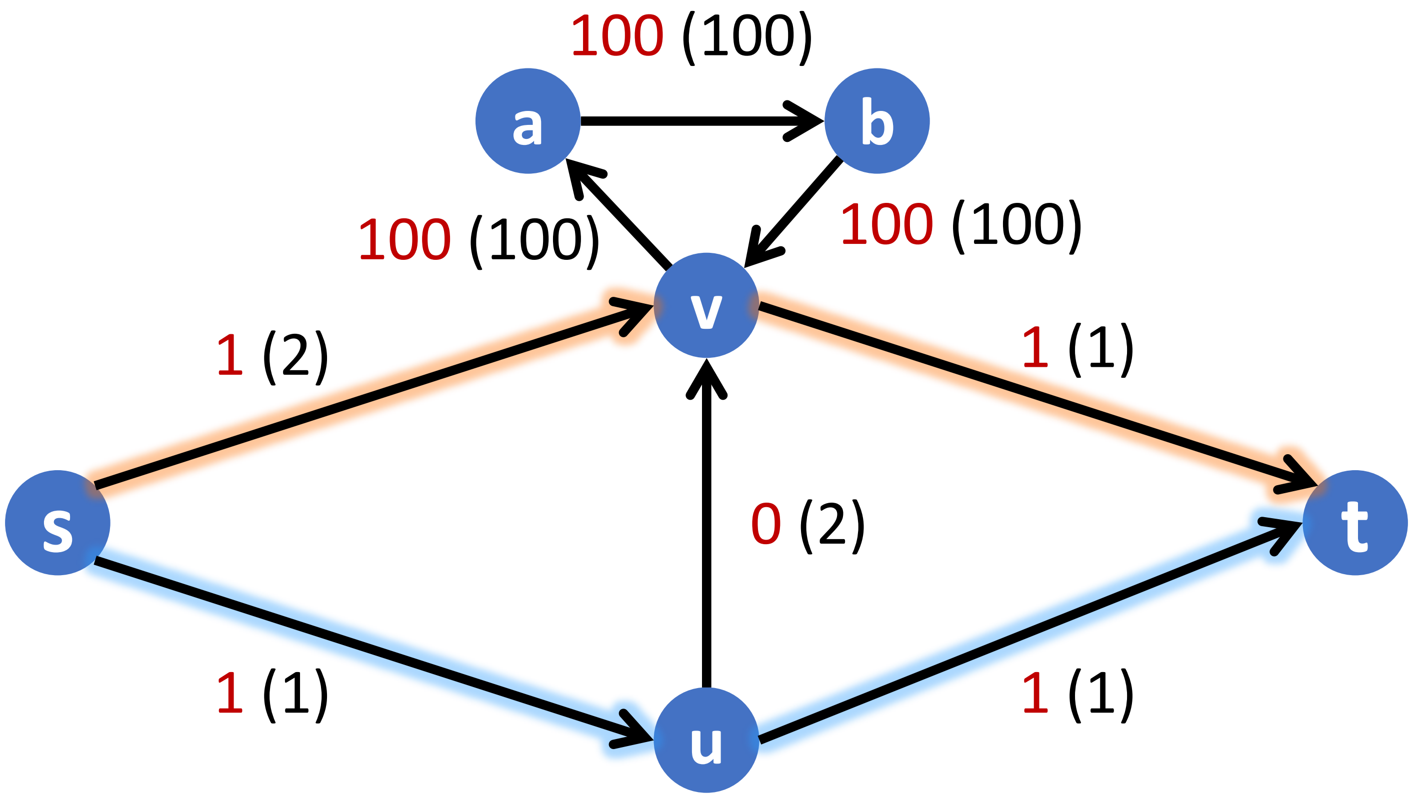
Flow-Path Decomposition

This flow is a sum of two path flows f_1 and f_2 of value 1 each.



Flow-Path Decomposition

Q: Can we represent every flow as a sum of path flows?



Flow-Path Decomposition

A flow f_i is a path flow if it uses a single s-t path.

A flow g_j is a cycle flow if it uses a single cycle.

Theorem

For every flow f there exist path flows $f_1, \ldots, f_a$ and cycle flows $g_1, \ldots, g_b$ s.t.

$$f(e) = \sum_i f_i(e) + \sum_j g_j(b)$$

Note $val(f) = \sum_i val\big(f_i(e)\big)$. If we remove the cyclic component, we will not change the value of the flow.

Edge and Vertex Disjoint Paths

Edge Disjoint Paths: **Menger's Theorem**

Consider a directed or undirected graph G and two vertices s and t.

We say that s-t paths $P_1, \ldots, P_k$ are edge disjoint if no two of them share a common edge.

Q: What is the maximum number of edge disjoint paths between s and t?

A: It is equal to the size of the minimum cut between s and t (in which all edges have capacity 1).

Proof: use path flow decomposition.

Vertex Disjoint Paths:
Menger's Theorem

Consider a directed or undirected graph G and two vertices s and t.

We say that s-t paths $P_1, \dots, P_k$ are vertex disjoint if no two of them share a common edge.

Q: What is the maximum number of vertex disjoint paths between s and t?

A: It is equal to the size of the minimum vertex separator between s and t.

Exercise: Prove this.