Linear Asymptotic Stability $\left(\begin{matrix}Re(\lambda)<0 \\ \text{for linear problem}\end{matrix}\right)$
$\Rightarrow$ Asymptotic Stability $\left(\begin{matrix}\exists N \text{ of } x^* \text{ s.t.} \\ \lim_{t\to\infty} x(t) = x^* \\ \text{for } x(0)\in N\end{matrix}\right)$

$\dot{x} = f(x)$, $f: \mathbb{R}^n \to \mathbb{R}^n$ is C^1

$f(x^*) = 0$

Re-write with equilibrium at origin:
$x = x^* + y$

$\dot{y} = Ay + g(y)$

$g(0) = 0$

Here $A = Df(x^*)$, which has eigenvalues λ for which $Re(\lambda) < 0$

and $g(y) = f(x^* + y) - Ay$ = nonlinear part of f.

From Taylor's theorem we know that

$g(y)$ is $o(y)$ ["little o of y"]

i.e. for all $\epsilon > 0$ there is a neighborhood $N(\epsilon)$ of $y = 0$, s.t. $|g(y)| < \epsilon |y|$, $\forall y \in N(\epsilon)$
[if f is C^2 then $g(y)$ is $\mathcal{O}(y^2)$; $|g(y)| < c|y^2|$]

$$\begin{cases} \dot{y} = Ay + g(y) \\ y(0) = y_0 \end{cases}$$

re-write this by integrating it, with an integrating factor:

$$\dot{y} - Ay = g(y)$$

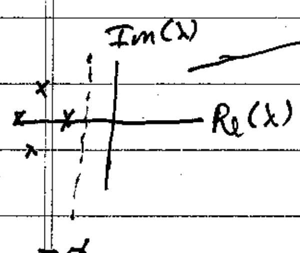
$$e^{-At}(\dot{y} - Ay) = e^{-At} g(y)$$

$$\int_0^t \frac{d}{ds}\left[e^{-As} y\right] ds = \int_0^t e^{-As} g(y(s))\, ds$$

$$e^{-At} y(t) = y_0 + \int_0^t e^{-As} g(y(s))\, ds$$

$$(*) \left\{ y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} g(y(s))\, ds \right.$$

$$|y(t)| \le |e^{At} y_0| + \int_0^t |e^{(t-s)} g(y(s))|\, ds$$



$(\mathrm{Re}(\lambda) < -\alpha < 0)$

Proved in book: If $\dot{y} = Ay$ then there is a $K \ge 1$ and $\alpha > 0$ s.t. $|e^{At} y_0| \le K e^{-\alpha t} |y_0|$ for any y_0 & $t \ge 0$

$$|y(t)| \le Ke^{-\alpha t}\delta + \int_0^t |e^{A\overbrace{(t-s)}^{t-s\ge 0}} \underbrace{g(y(s))}_{\substack{\text{bound this}\\ \text{using}\\ g(y) \text{ is } o(y)?}}| ds$$

(here $|y_0| < \delta$)

For any ϵ we can find a δ s.t. if $|y| < K\delta$ then $|g(y)| < \epsilon |y|$

Also, since $|y_0| < \delta$ and $y(s)$ is continuous, we can assume there is an interval $[0,\tau)$ where $|y(s)| < K\delta$ [we will later see we can let $\tau \to \infty$]

$$|e^{A(t-s)} g(y(s))| \le Ke^{-\alpha \overbrace{(t-s)}^{t-s\ge 0}} |g(y(s))|$$

$$\le Ke^{-\alpha(t-s)} \epsilon |y(s)|, \quad s\in[0,\tau)$$

$$\boxed{\underbrace{e^{\alpha t}|y(t)|}_{\rho(t)} \le K\delta + K\epsilon\int_0^t \underbrace{e^{\alpha s}|y(s)|}_{\rho(s)} ds \qquad s\in[0,\tau)}$$

(If it was an equality then it looks like ρ is soln. to $\dot{\rho} = K\epsilon\rho$, $\rho(0) = K\delta$ which has soln. $\rho = K\delta e^{K\epsilon t}$)

Grönwall inequality

Suppose $g, K : [0,a] \to \mathbb{R}$ are continuous $a>0$, $K(t) \geq 0$, and

$$g(t) \leq G(t) \equiv c + \int_0^t K(s)\, g(s)\, ds$$

for all $0 \leq t \leq a$. Then for all $t \in [0,a]$

$$g(t) \leq c\, e^{\int_0^t K(s)\, ds}$$

Proof: $g, K \in C^0$, $G \in C^1$ with $G(0) = c$

$\Rightarrow$ differentiat $G(t)$

$$\dot{G} = K(t)\, g(t) \leq K(t)\, G(t)$$

$$\dot{G} - K(t)\, G(t) \leq 0$$

$$e^{-\int_0^t K(s)\, ds} \left[\dot{G}(t) - K(t)\, G(t)\right] \leq 0$$

$$\frac{d}{dt}\left[G(t)\, e^{-\int_0^t K(s)\, ds} \right] \leq 0$$

non-increasing

$$e^{-\int_0^t K(s)\, ds}\, G(t) \leq G(0) = c$$

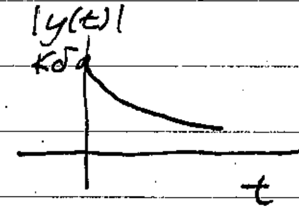
$$G(t) \leq c\, e^{\int_0^t K(s)\, ds}, \quad g(t) \leq G(t) \Rightarrow g(t) \leq c\, e^{\int_0^t K(s)\, ds}$$

$$\xi(t) = e^{\alpha t} |y(t)|$$

$$\xi(t) \le K\delta + K\epsilon \int_0^t \xi(s)\, ds$$

$$\Rightarrow \quad \xi(t) \le K\delta\, e^{K\epsilon t} \qquad \text{by Grönwall's}$$

$$|y(t)| \le K\delta\, e^{-(\alpha - K\epsilon)t} \qquad \text{for } t \in [0, \tau)$$



Now choose $\epsilon < \frac{\alpha}{K}$

Note $|y(t)| < K\delta$ for $t \in (0, \tau)$

We introduced τ because we wanted an interval where $|y(t)| < K\delta$, but now we've shown that $|y(t)|$ isn't in danger of reaching $K\delta$ at some finite time τ so can let $\tau \to \infty$

$$|y(t)| \le K\delta\, e^{-(\alpha - K\epsilon)t} \qquad t \in [0, \infty)$$

$$\lim_{t \to \infty} |y(t)| = 0$$

Alternative method to show that the stability of a hyperbolic equilibrium, (ones w/ $Re(\lambda) \neq 0$) can be determined by linearization is via the <u>Hartman-Grobman Thm</u>.

Let x^* be a hyperbolic equilibrium of a C^1 vector field $f(x)$ with flow $\varphi_t(x)$. Then there is a neighborhood N of x^* s.t. $\varphi_t(x)$ is <u>topologically conjugate</u> to its linearized flow $\psi_t(x)$ on N.

Two flows $\varphi_t : A \to A$ & $\psi_t : B \to B$ are <u>topologically conjugate</u> if there exists a <u>homeomorphism</u> $h : A \to B$ s.t. for each $x \in A$ & $t \in \mathbb{R}$, $h(\varphi_t(x)) = \psi_t(h(x))$

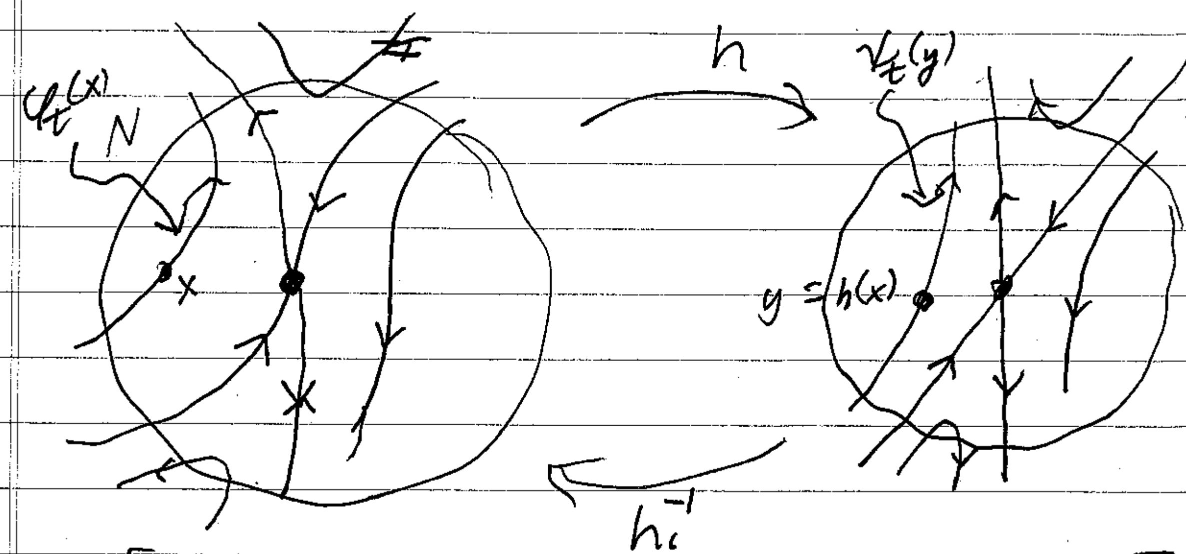
<u>homeomorphism</u> $h : A \to B$ is continuous.

one-to-one mapping with a continuous inverse

$$\begin{tikzcd} x \arrow[r, "\varphi_t"] \arrow[d, "h"'] & \varphi_t(x) \arrow[d, "h"] \\ y \arrow[r, "\psi_t"'] & \psi_t(y) \end{tikzcd}$$

$\psi_t(h(x)) = h(\varphi_t(x))$

In a picture:



$$\varphi_t(x) \quad \left[\begin{array}{ll} \dot{x} = f(x) & \qquad\qquad \dot{y} = Df(0)y \\ f(0) = 0 \quad (\text{w.l.o.g.}) & \end{array} \right] \quad \psi_t(y)$$

In homework you use this to show linear asymptotic stability implies asymptotic stability

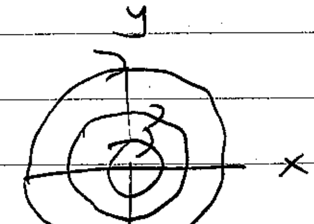
Nonhyperbolic equilibria with $Re(\lambda) \leq 0$: linearization is inconclusive

example:

$$(*)\begin{cases} \dot{x} = y + a(x^2+y^2)x \\ \dot{y} = -x + a(x^2+y^2)y \end{cases}$$

linearization $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\Rightarrow$ $\lambda = \pm i$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$$



linear center

$(E = E^c)$

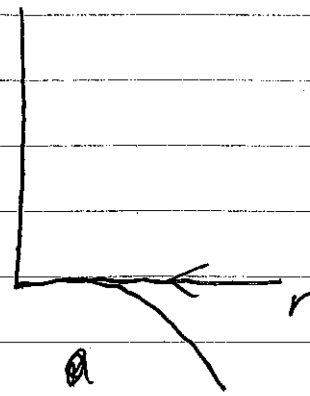
Lyapunov stable?

Re-write $(*)$ in polar coordinates

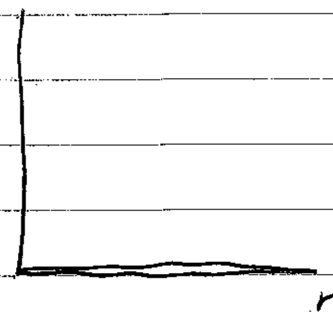
$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \begin{aligned} x^2+y^2 &= r^2 \\ \tan\theta &= y/x \end{aligned}$$

$$\left.\begin{aligned} 2r\dot{r} &= 2x\dot{x} + 2y\dot{y} \\ \sec^2\theta\ \dot{\theta} &= \frac{\dot{y}x - x\dot{y}}{x^2} \end{aligned}\right\} \quad \begin{aligned} \dot{r} &= ar^3 \\ \dot{\theta} &= -1 \end{aligned}$$

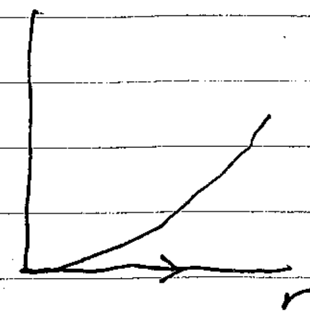
$\dot{r} = ar^3$



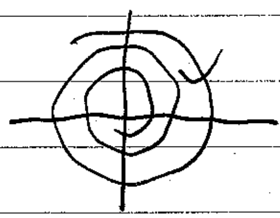
$a<0$



$a=0$

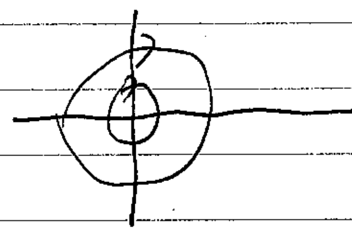


$a>0$



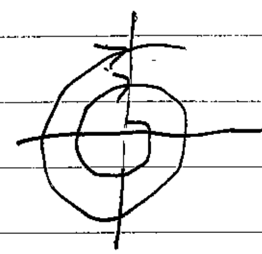
asymptotically stable

$a<0$



Lyapunov stable

$a=0$



unstable

$a>0$