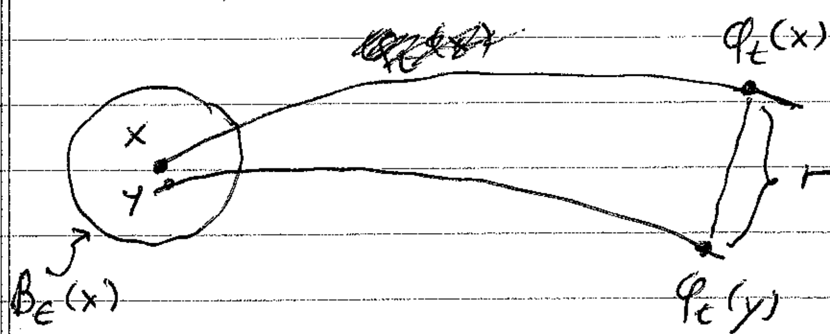
Last topic: chaotic dynamics

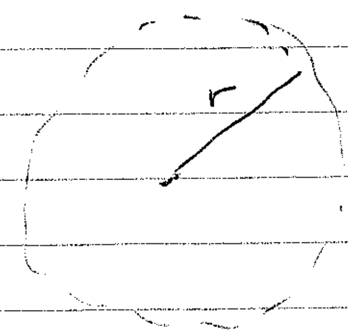
Hallmark: "Sensitive dependence on initial conditions" on a bounded invariant set

↖ excludes linear exponential growth

existence of chaos makes long-term prediction impossible.

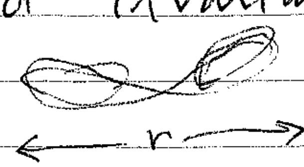
Def. Flow φ_t exhibits sensitive dependence on initial conditions on invariant set X if there is a fixed $r>0$ s.t. for each $x \in X$ and any $\epsilon > 0$, there is a nearby $y \in B_\epsilon(x) \cap X$ s.t. $|\varphi_t(x) - \varphi_t(y)| > r$ for some $t > 0$.





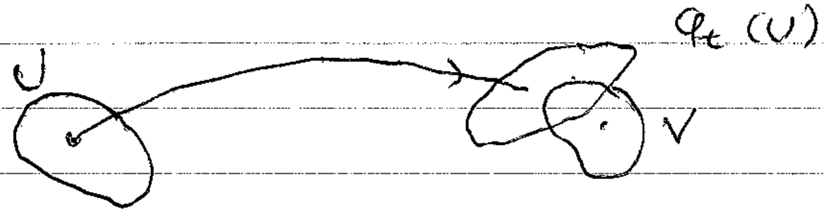
If ϵ = precision with which you can specify x, then inevitably you only know $\varphi_t(x)$ to be within some radius r, which may be quite large, e.g. size of invariant set

(decreasing ϵ may increase time horizon for knowing $\varphi_t(x)$ to some precision)



<u>Additional ingredients</u>: we want some recurence (aperiodic) – an orbit that visits everywhere on X, the invariant set.

<u>Def.</u> A flow φ_t is <u>topologically transitive</u> on an invariant set X if for each pair of non-empty sets $U, V \in X$ there is a $t > 0$ s.t. $\varphi_t(U) \cap V \neq \emptyset$



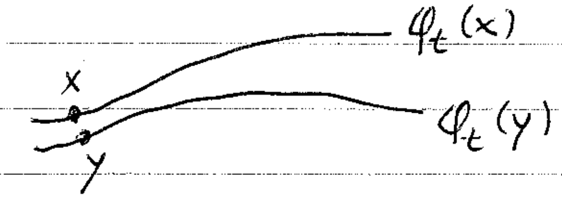
<u>Thm</u> (Birkhoff transitivity): A flow φ_t is transitive on X if and only if φ_t has an orbit that is dense on X.

dense orbit $\varphi_t(x) \in X$: for each $\epsilon > 0$, and $y \in X$, there is a $T > 0$ s.t. $d(\varphi_T(x), y) < \epsilon$

distance function

orbit comes arbitrarily close to each pt. $y \in X$.

Lyapunov exponents: quantify sensitive dependence on initial conditions



(initially) find exponential separation

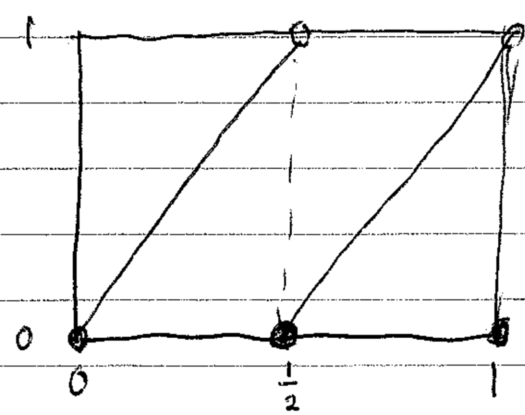
$$\delta(t) = |\varphi_t(y) - \varphi_t(x)| \sim e^{\lambda t}, \quad \lambda > 0 \text{ for chaotic system}$$

λ = largest Lyapunov exponent

HW#7: estimate λ for Lorenz eqns in chaotic regime with 2 methods & share result on canvas.

Example in terms of a map:

$$x_{n+1} = 2x_n \pmod 1, \quad x \in [0,1)$$



idea for analysing this: write x_n, x_{n+1} in terms of their binary representation in binary:

$$x_n = 0.a_1 a_2 a_3 \ldots \qquad a_j \in \{0,1\}$$

can convert to decimal form

$$X = \sum_{j=1}^{\infty} a_j \left(\tfrac{1}{2}\right)^j$$

map: $X_{n+1} = 2 X_n \pmod 1$

$$= \sum_{j=1}^{\infty} a_j \left(\tfrac{1}{2}\right)^{j-1} \pmod 1$$

$$= a_1 + a_2\left(\tfrac{1}{2}\right) + a_3\left(\tfrac{1}{2}\right)^2 + \ldots \pmod 1$$

$$= \sum_{j=1}^{\infty} a_{j+1} \left(\tfrac{1}{2}\right)^j$$

$$\left.\begin{array}{l} x_n = 0.a_1 a_2 a_3 a_4 \ldots \\ x_{n+1} = 0.a_2 a_3 a_4 \ldots \end{array}\right\} \text{"shift map on 2 symbols" } \{0,1\}$$

<u>Sensitive dependence on initial conditions</u>!

Given any $\epsilon > 0$, where $|x_0 - y_0| < \epsilon$ is initial separation, find N & y_0 s.t. $|x_N - y_N| \geq r = \frac{1}{2}$

$$x_0 = 0.a_1 a_2 a_3 \cdots a_N a_{N+1} a_{N+2} \cdots$$
$$y_0 = 0.a_1 a_2 a_3 \cdots a_N b_{N+1} b_{N+2} \cdots$$

let $b_j = \begin{cases} 1 & \text{if } a_j = 0 \\ 0 & \text{if } a_j = 1 \end{cases} \quad j = N+1, \ldots$

may need to fix this as discussed in class.

$$|x_0 - y_0| = \left| \sum_{j=N+1}^{\infty} (a_j - b_j) \left(\tfrac{1}{2}\right)^j \right|$$

$$\leq \left(\tfrac{1}{2}\right)^N \sum_{j=1}^{\infty} \underbrace{|a_{N+j} - b_{N+j}|}_{=1} \left(\tfrac{1}{2}\right)^j$$

$$\leq \left(\tfrac{1}{2}\right)^N = \epsilon$$

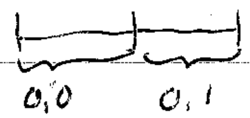
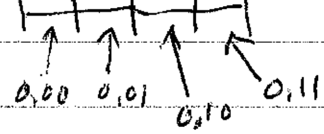
after N iterations

$$x_N = 0.a_{N+1} a_{N+2} a_{N+3} \cdots$$
$$y_N = 0.b_{N+1} b_{N+2} b_{N+3} \cdots$$

$|x_N - y_N| > \frac{1}{2}$ for suitably chosen $a_{N+2}, \ldots$

We can also construct an initial condition for a dense orbit.

Given any $\epsilon > 0$, there exists an N s.t. we come within ϵ of any pt. in our interval.

distance(ϵ)		approximation of x_0
$\frac{1}{2}$		$x_0 = 0.01$
$\frac{1}{4}$		~~$x_0 = 0.0001$~~ $x_0 = 0.\underbrace{01}_{2}\underbrace{00011011}_{8}$
$\frac{1}{8}$		$x_0 = 0.\underbrace{01}_{=2\times 1}\underbrace{00011011}_{\overset{8}{=4\times 2}}\underbrace{000001\ldots}_{=8\times 3}$

and so on.

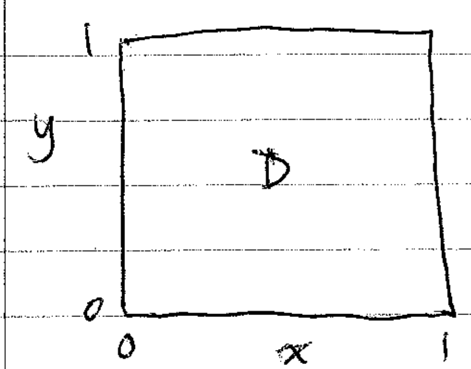
~~There are also periodic pts of all periods & aperiodic pts.~~

Shift map on 2-symbols is relevant to Smale horseshoe, which is relevant to the homoclinic tangle that arises in periodically-forced ode in 2+1 dimensions

<u>Smale horseshoe</u>: elements of chaos —
"stretching" & "folding"

gives exponential separation of neighboring trajectories

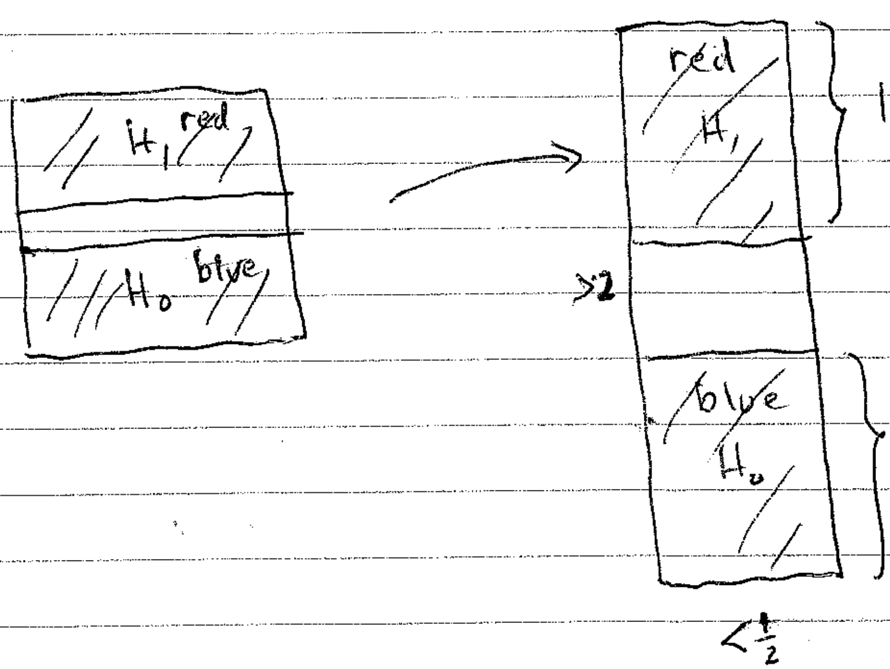
keeps it bounded.



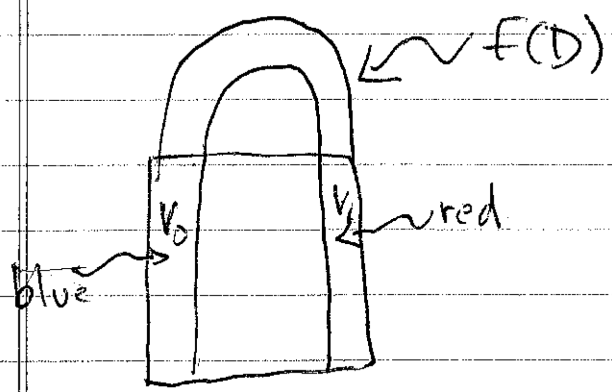
D = unit square

$x, y \in [0,1]$

<u>Step 1</u>: Stretch by factor > 2 in y-direction, Compress by factor $< \frac{1}{2}$ in x-direction



Step 2: Fold & overlay it on D



f = map

$f(H_0) = V_0$

$f(H_1) = V_1$

$$D \cap f(D) = V_0 \cup V_1$$

$$= \bigcup_{s_{-1} \in S} V_{s_{-1}}$$

$$S = \{0, 1\}$$

Idea: keep iterating map (forward & backward) to determine the invariant set & examine its dynamics. We'll see it can be related to the shift on 2-symbols..