Tutorial 7: Linear Programming

Problem 1. *Consider the following problem. We are given two sequences $c_1, \ldots, c_n$ and $w_1, \ldots, w_n$, where $c_i \in \{0, \ldots, n\}$ and $w_i \geq 0$ for every i. A subset $S \subset \{1, \ldots, n\}$ is feasible if $|S \cap \{1, \ldots, i\}| \leq c_i$ (that is, S contains at most c_i numbers from 1 to i). The weight of a subset $S \subset \{1, \ldots, n\}$ is $w(S) = \sum_{i \in S} w_i$. The goal is to find the heaviest feasible solution.*

Consider the following LP relaxation for the problem.

LP variables: $x_1, \ldots, x_n$

$$\textbf{maximize} \quad \sum_{i=1}^{n} w_i x_i$$

s.t.

$$\begin{aligned}
&\sum_{i=1}^{j} x_i \leq c_j && \text{for every } j \in \{1, \ldots, n\} \\
&x_i \leq 1 && \text{for every } i \in \{1, \ldots, n\} \\
&x_i \geq 0 && \text{for every } i \in \{1, \ldots, n\}
\end{aligned}$$

Let OPT be the optimal value of the problem and LP be the value of this linear program.

- *Prove that $LP \geq OPT$.*
- *Prove that $LP \leq OPT$. You may assume that there is an optimal vertex solution $\hat{x}$ for the LP.*

Hint: *Let $A = \{i : 0 < \hat{x}_i < 1\}$ (note that the inequalities are strict). Consider three cases.*

- *$|A| = 0$. In this case, prove that $LP \leq OPT$.*
- *$|A| = 1$. In this case, find two feasible LP solutions x' and x'' such that $\hat{x} = (x' + x'')/2$ ($x' \neq x''$) and conclude that $\hat{x}$ is not a vertex solution.*
- *$|A| \geq 2$. In this case, find the two smallest elements i_1 and i_2 in A and then appropriately define feasible LP solutions x' and x'' ($x' \neq x''$) so that $\hat{x} = (x' + x'')/2$.*

Solution. Let S be a feasible subset of weight W. Setting

$$x_i := \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise} \end{cases}$$

we get a feasible solution for the LP of value W. Therefore $LP \geq OPT$.

Now let $\hat{x}$ be an optimal vertex solution for the LP and let $A = \{i : 0 < \hat{x}_i < 1\}$.

- If $|A| = 0$, then $\hat{x}_i \in \{0, 1\}$ for every i and $S := \{i : \hat{x}_i = 1\}$ is a feasible subset of weight LP. Therefore $LP \leq OPT$.

- If $|A| = 1$, then let i be the index for which $0 < \hat{x}_i < 1$. Set $\delta := \min\{\hat{x}_i, 1 - \hat{x}_i\}$ and consider the vectors x' and x'' with

$$x'_j = \begin{cases} \hat{x}_j & \text{if } j \neq i, \\ \hat{x}_i + \delta & \text{if } j = i; \end{cases}$$

$$x''_j = \begin{cases} \hat{x}_j & \text{if } j \neq i, \\ \hat{x}_i - \delta, & \text{if } j = i. \end{cases}$$

  x' and x'' are feasible solutions for the LP and $\hat{x} = (x' + x'')/2$, therefore $\hat{x}$ is not a vertex solution. We conclude that $|A|$ cannot be 1.

- Suppose $|A| \geq 2$. Let i_1 and i_2 be the two smallest elements in A. Set $\delta := \min\{\hat{x}_{i_1}, \hat{x}_{i_2}, 1 - \hat{x}_{i_1}, 1 - \hat{x}_{i_2}\}$ and consider the vectors x' and x'' with

$$x'_j = \begin{cases} \hat{x}_j & \text{if } j \notin \{i_1, i_2\}, \\ \hat{x}_i + \delta & \text{if } j = i_1, \\ \hat{x}_j - \delta & \text{if } j = i_2; \end{cases}$$

$$x''_j = \begin{cases} \hat{x}_j & \text{if } j \notin \{i_1, i_2\}, \\ \hat{x}_i - \delta & \text{if } j = i_1, \\ \hat{x}_j + \delta & \text{if } j = i_2; \end{cases}$$

  Again, x' and x'' are feasible solutions for the LP and $\hat{x} = (x' + x'')/2$, therefore $\hat{x}$ is not a vertex solution. We conclude that $|A|$ cannot be at least 2. Thus a vertex solution must be a 0-1 vector and must correspond to a feasible subset.

Problem 2. *Design a DP algorithm for Problem 1.*

Solution. Let $\mathrm{DP}[i, k]$ be the maximum weight of a feasible subset S for the sequences $c_1, \ldots, c_i$ and $w_1, \ldots, w_i$ such that $|S| \leq k$. We need to compute $\mathrm{DP}[n, c_n]$. The recursion for $\mathrm{DP}[i, k]$ is:

$$\mathrm{DP}[1, k] = \begin{cases} w_1 & \text{if } k \geq 1, \\ 0 & \text{if } k < 1; \end{cases}$$

$$\mathrm{DP}[i, k] = \max\left\{w_i + \mathrm{DP}[i-1, \min\{k-1, c_{i-1}\}], \mathrm{DP}[i-1, \min\{k, c_{i-1}\}]\right\}.$$

There are $n \cdot c_n$ subproblems and we need constant time for each. Hence the running time will be $O(n \cdot c_n) = O(n^2)$.