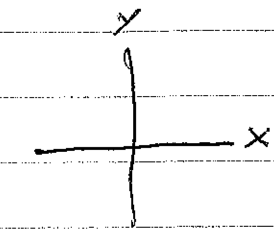
Andranov-Hopf bifurcation Thm. (Section 8.8)

$$\dot{x} = -\omega y + p(x,y)$$
$$\dot{y} = \omega x + q(x,y)$$



re-write in complex coordinates

$$\left.\begin{aligned} z &= x+iy \\ \bar{z} &= x-iy \end{aligned}\right\} \quad x^2+y^2 = |z|^2$$

after normal form transformation

bifurcation parameter

$$\dot{z} = \underbrace{\lambda(\mu)}_{\text{eigenvalue}} z + z\left(c(\mu)|z|^2 + d(\mu)|z|^4 + \cdots \right)$$

$\lambda(0) = i\omega$ for Hopf at $\mu = 0$

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y}$$

$$\dot{z} = \dot{r}\cancel{e^{i\theta}} + ir\dot{\theta}\cancel{e^{i\theta}} = \lambda(\mu) r\cancel{e^{i\theta}} + r\cancel{e^{i\theta}}(c(\mu) r^2 + d(\mu) r^4 + \ldots)$$

$\dot{r} = Re(\quad)$ $\quad \cancel{\dot{r} = \lambda(\mu) + Re\, c(\mu) r^3 + \ldots}$

$r\dot{\theta} = Im(\quad)$ $\quad \cancel{\dot{\theta} =}$

0 at $\mu = 0$

$$\dot{r} = Re(\lambda(\mu) r + c(\mu) r^3 + d(\mu) r^4 + \ldots)$$
$$\dot{\theta} = Im(\lambda(\mu) + c(\mu) r^2 + d(\mu) r^4 + \ldots)$$

ω @ $\mu = 0$

Hopf Bifurcation Thm (8.21 in text)

Let $f(x;\mu)$ be a C^3 vector field in $\mathbb{R}^n$, $n \geq 2$, such that

$f(0;0) = 0$
$\text{spec}(D_x f(0;0)) = \{(i\omega, -i\omega, \lambda_3, \ldots, \lambda_n), \text{Re}(\lambda_k) \neq 0, k \geq 3)$

The normal form on the center manifold of f_0 has an unfolding of the form

$$\dot{z} = \lambda(\mu) z + z(c(\mu)|z|^2) + \cdots$$

Assume $\alpha(0) = \text{Re}(c(0)) \neq 0$

and that ~~as~~ a parameter μ causes the eigenvalues to cross the imaginary axis, ie.

$$\frac{d}{d\mu}\left[\text{Re}(\lambda(\mu))\right]_{\mu=0} \neq 0$$

Then there is a Hopf bifurcation that gives birth to a limit cycle in the center manifold at $\mu = 0$.

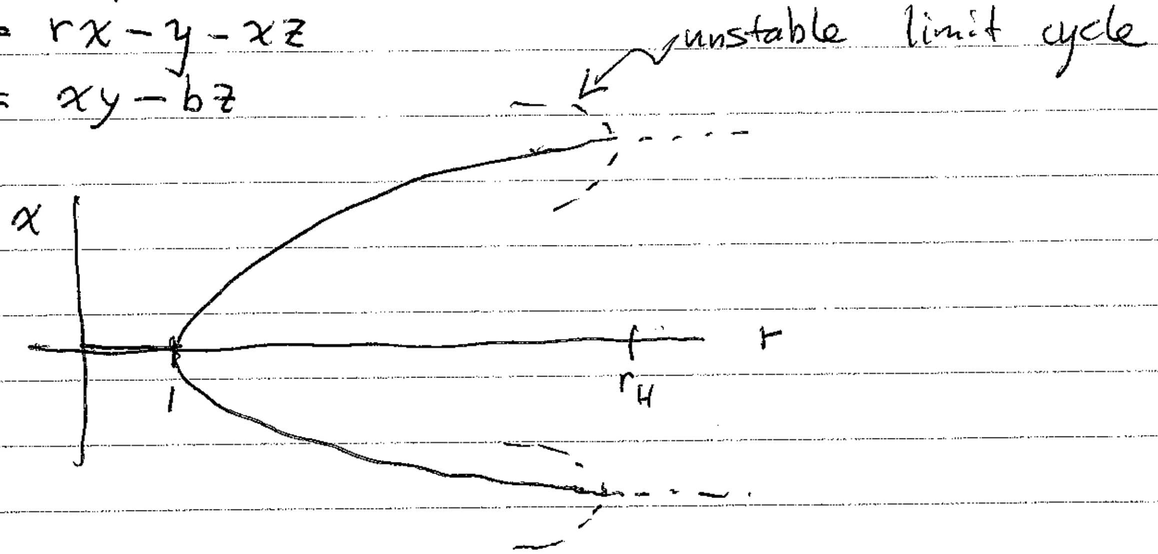
super vs. sub crit

The limit cycle exists when $\alpha \text{Re}(\lambda) < 0$ & is stable in center manifold if $\text{Re}(\lambda) > 0$ & unstable if $\text{Re}(\lambda) < 0$.

Claim: There is a subcritical Hopf bifurcation for standard parameters of the Lorenz system 3, $\sigma = 10$, $b = 8/3$ at

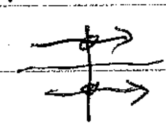
$$r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \approx 24.74\ldots$$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$



How would you prove this claim?

note: eigenvalues move with "nonzero speed" as $r \uparrow$



center manifold is 2-d at $r = r_H$ & stable manifold is 1-d.

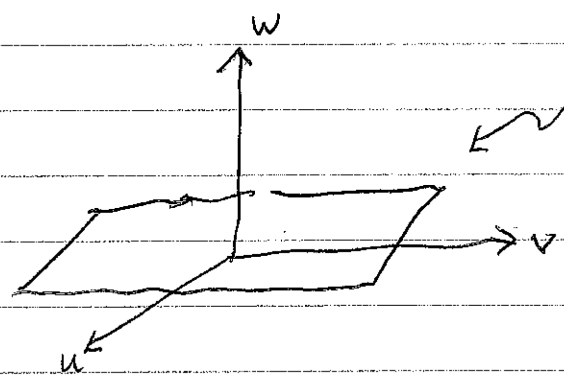
Vector field is C^∞

just need $\alpha(0)$... $\mu = r - r_H$ ...

① Move fixed-pt. to the origin of new coordinates system, and choose coordinates so that at $r = r_H$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \underbrace{\left(\begin{array}{cc|c} 0 & -\omega & 0 \\ \omega & 0 & \\ \hline 0 & & -\lambda_3 \end{array}\right)}_{\text{linear part}} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \underbrace{F(u,v,w)}_{\text{the rest}}$$

② Find approximation to the center manifold at $r = r_H$



$W^c_{loc}(0)$:

$w = h(u,v)$

$= \alpha u^2 + \beta uv + \gamma v^2 + \cdots$

③ Restrict $(\dot{u}, \dot{v})$ eqns to the center manifold at $r = r_H$ by setting $w = h(u,v)$ there.

$$\dot{u} = -\omega v + \underbrace{F_1(u,v,h(u,v))}_{p} = -\omega v + p(u,v)$$

$$\dot{v} = \omega u + \underbrace{F_2(u,v,h(u,v))}_{q} = \omega u + q(u,v)$$

④ We need the normal form coefficient $\alpha = \mathrm{Re}(c)$

Use formula (8.59) in the textbook

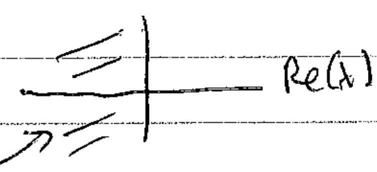
$$\alpha = \frac{1}{16}\left(p_{xxx} + p_{xyy} + q_{xxy} + q_{yyy}\right)$$
$$- \frac{1}{16\omega}\Big(q_{xy}\left(q_{xx} + q_{yy}\right) - p_{xy}\left(p_{xx} + p_{yy}\right)$$
$$+ p_{xx}\, q_{xx} - p_{yy}\, q_{yy}\Big)$$

Here $p_{xxx} = \left.\dfrac{\partial^3 p}{\partial x^3}\right|_{x=y=0}$, etc

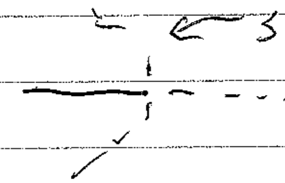
Sign of α determines whether Hopf bifurcation is sub- or super-critical.

Claim (for Lorenz, for standard parameters σ, b)

$\underline{\alpha > 0}$ $\Rightarrow$ limit cycle exist when
$\alpha \, Re(\lambda) < 0 \Rightarrow Re(\lambda) < 0$



exists in region where the equilibrium is stable.



subcritical & limit cycle unstable