Approximation Theory: Cheat Sheet for Midterm

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Contents

Introduction

- Errors for smooth functions accumulates and are due to the endpoints.
- If f is periodic as well as its derivatives and smooth, then the trapezoid rule if very good.

Schauder Basis:

A sequence $\{v_j\}_{j=1}^{\infty}$ in a Banach Space X is a Schauder basis if for all $v \in X$ there exists unique coefficients $\alpha_1, \alpha_2, ...$ (not finite!) such that

$$v = \sum_{j=1}^{\infty} \alpha_j v_j,$$

where implicitly we require the sum on the right hand side to converge.

Schauder Basis Bound:

Suppose $\{v_j\}_{j=1}^{\infty}$ is a Schauder basis of the Banach space X. Then there exists a constant M such that for all $v \in X$,

$$\left\| \sum_{j=1}^{N} \alpha_j v_j \right\| \leq M \|v\|$$

for all $N = 1, 2, ...$ The minimal such bound M, which should hold for all N is called the basis constant.

- A Schauder Basis for a Banach space X does not always exist! A necessary condition for a Schauder basis to exist is for the space to be separable.
- There is an isomorphism from the Banach Space X to a new space, the set of coefficients of the Schauder basis expansion. This is a Banach Space, denoted E.
- The N-th partial sum of the Schauder basis expansion, $S_N(v)$, is a projection onto the first N Schauder bases. Note that S_N is uniformly bounded.

Orthogonal Basis Properties for Hilbert Spaces:

Let $\{v_j\}_{j=1}^{\infty}$ be an orthogonal system. The following are equivalent:

1. $\sum_j \alpha_j v_j$ converges in $\mathcal{H}$.
2. $\sum_j |\alpha_j|^2 \|v_j\|^2 < \infty$.
3. $\sum_j \alpha_j v_j$ converges unconditionally
4. If there is convergence, then

$$\left\| \sum_j \alpha_j v_j \right\|^2 = \sum_j |\alpha_j|^2 \|v_j\|^2$$

Fourier Series are good:

Among all convergent sequences $s = \sum_j \alpha_j v_j$, $\|v - s\|$ is minimized by the Fourier series of v. The same is true of partial sums.

- Given a Banach Space E, the linear operator $P : E \to E$ is called a projection if it is bounded and idempotent.

Projections to Subspaces:

Suppose $P : E \to E$ is a projection and the range of P is a subspace V of E. Then V is a closed subspace of E, and $V = \ker(\mathbb{I}_E - P)$

- $\mathbb{I} - P$ and P^* are also projections, since they are bounded and idempotent.

Projection Shenanigans:

Let $P, Q : E \to E$ be two projections. Define the operator $P \oplus Q = P + Q - PQ$. Suppose $PQP = QP$. Then,

1. $P \oplus Q$ is a projection onto range(P) + range(Q).
2. QP is a projection onto range$(P) \cap$ range(Q).

General Principles

- If U is an n-dimensional subspace of a Banach space E, then there exists a projection $P : E \to U$ with norm at most n.

- Given $v \in X$, let $P_A(v)$ denote the set of nearest points to v in A, i.e., the set of best approximations to v in A.
$$P_A(v) := \{u_0 \in A : \|v - u_0\| = d(v, A) = \inf_{u \in A} \|v - u\|\}$$

- A set A is called an existence set if $P_A(v)$ is nonempty for all $v \in A$, and is a uniqueness set if $|P_A(v)| \leq 1$. If the set is both an existence and uniqueness set, then it is called a Chebyshev set.

- Existence sets are closed. Being closed does not imply existence.

If A is convex, then $P_A(v)$ is convex.:

Proof: Given $v \in X$, suppose $u_0, u_1 \in P_A(v)$. For any $\lambda \in [0, 1]$, set $u_\lambda = \lambda u_0 + (1 - \lambda)u_1$. Then,

$$\begin{aligned}
\|v - u_\lambda\| &= \|\lambda(v - u_0) + (1 - \lambda)(v - u_1)\| \\
&\leq \lambda\|v - u_0\| + (1 - \lambda)\|v - u_1\| \\
&= d(v, A)
\end{aligned}$$

Kolmogorov Criterion:

Suppose M is a convex set in a Banach space X, and $v \in X$, $v \notin \overline{M}$. Then u_0 is a best approximation to v in M if and only if there is a linear functional $\ell \in X^*$ satisfying the following conditions:

1. $\|\ell\|_{x^*} = 1$,
2. $\ell(v - u_0) = \|v - u_0\|$,
3. $\mathrm{Re}[\ell(u - u_0)] \leq 0$ for all $u \in M$.

Haar subspaces

- A Haar subspace M is an n-dimensional subspace of $C(\Omega)$ such that any $u \in M$ has at most $n-1$ zeros in Ω.
- M is an n dimensional Haar subspace if and only if for any n points $x_1, ..., x_n \in \Omega$ and $\beta_1, ..., \beta_n \in \mathbb{C}$, the interpolation problem:

  $$\text{Find } u \in M \text{ such that } u(x_i) = \beta_i, i = 1, ..., n$$

  always has a solution.
- Another equivalent phrasing is for any given basis $u_1, ..., u_n$, the $n \times n$ matrix Φ defined by $(\Phi)_{i,j} = u_j(x_i)$ is invertible for any distinct points $x_1, ..., x_n$.

Haar's uniqueness theorem (Not Important):

If Ω is locally compact, then a finite dimensional subspace M of $C(\Omega)$ has a unique best approximation in M, i.e., M is a Chebyshev set.

Marihuber-Curtis Theorem:

Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 2$ contains an interior points. Then there is no Haar subspace of $C(\Omega)$ of dimension ≥ 2.

Continuous Functions

- Do not need to regurgitate the following proof:

Stone-Weierstrass:

Let X be a compact metric space and $\mathcal{A}$ be a subalgebra of $C(X)$. If $\mathcal{A}$ separates points in X and vanishes at no point in X, then $\mathcal{A}$ is dense in $C(X)$. Here $\mathcal{A}$ and $C(X)$ consists of real valued functions.

- Basically, all we need to know is that $\mathcal{A}$ separates points, vanishes nowhere, and is closed under addition and multiplication.
- Also, helps to know the general layout. Argument by compactness, and reduces to approximating the absolute value function.
- $\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$.

Weierstrass Approximation Theorem:

For any $f \in C([0, 1])$ and $\varepsilon > 0$ there exists a polynomial p such that $\|f - p\| < \varepsilon$.

- Bernstein Polynomials for $1, x$ are fair game. Compute them for practice.

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

The Equioscillation Theorem:

The space of polynomials of degree at most n on $[-1,1]$, which we denote by P_n is a Chebyshev set in $C[-1,1]$, i.e., a best approximation always exitst and is always unique. Moreover, if f is real, the best approximation p^* is real, and $f - p^*$ equioscillate in at least $n+2$ extreme points. That is to say, there exists ordered points $x_0, x_1, ..., x_{n+1} \in [-1,1]$ such that

$$p^*(x_i) - f(x_i) = -[p^*(x_{i+1}) - f(x_{i+1})], \quad i = 0, ..., n$$

and $|p^*(x_i) - f(x_i)| = \|p^* - f\|_\infty$ for all $i = 0, ..., n+1$.

- The only thing should know to prove is uniqueness. It goes like this: Suppose p and q are both optimal polynomials. Set $r = (p+q)/2$. By the equioscillation characterization, there exists points $-1 \leq x_0 < x_1 < ... < x_{n+1} \leq 1$ for with $(f-r)(x_i) = (-1)^{i-1}\|f-r\|$ for all $i = 0, ..., n+1$. But,

$$\begin{aligned} |f(x_i) - r(x_i)| &= |\frac{1}{2}(f(x_i) - p(x_i)) + \frac{1}{2}(f(x_i) - q(x_i)| \\ &\leq \frac{1}{2}|f(x_i) - p(x_i)| + \frac{1}{2}|f(x_i) - q(x_i)| \\ &\leq \frac{1}{2}\|f-p\| + \frac{1}{2}\|f-q\|. \end{aligned}$$

Also, $|f(x_i) - r(x_i)| = \|f-p\| = \|f-q\|$ so it follows that

$$|f(x_i) - p(x_i)| = \|f-p\| = \|f-q\| = |f(x_i) - q(x_i)|$$

and the sign of $(f(x_i) - p(x_i))$ is the same as $(f(x_i) - q(x_i))$. Thus $p(x_i) = q(x_i)$ for $i = 0, ..., n+1$. This implies that $p = q$. [1]

de la Vallée Poussin :

Given $f \in C[a,b]$, suppose $p \in P_n$ and $f(x_i) - q(x_i)$ alternates in sign at $a \leq x_0 < ... < x_{n+1} \leq b$. If $E_n(f)$ denotes the error of the best approximation in P_n, then

$$E_n(f) \geq \min_{i=0,...,n+1} |f(x_i) - q(x_i)|$$

[1]NOTE: FOR A POLYNOMIAL OF DEGREE n, NEED ONLY TO SET $n+1$ POINTS TO DEFINE IT UNIQUELY.

General Equioscillaiton Theorem:

Suppose V is an $(n+1)$-dimensional Haar subspace of $C[-1,1]$. Given $f \in C[-1,1]$, let u^* denote the best approximation to f from V. Then u^* is unique and $f - u^*$ equioscillates in at least $n+2$ extreme points.

Chebyshev Polynomials

- The n-th Chebyshev polynomial is defined as $T_n(x) = \cos(n \arccos(x))$.
- The recurrence formula is $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, setting $T_0(x) = 1, T_1(x) = x$.
- Another formula, which Hoskins gave but might not be true is $T_{n+1} + T_{n-1} = 2T_n$.

Convergence of Chebyshev series:

Suppose we are given a function f which is Lipschitz continuous on $[-1,1]$. Then f has a unique representation as a Chebyshev series,

$$f(x) = \sum_{k=0}^{\infty} \alpha_k T_k(x),$$

which is absolutely convergent and uniformly continuous. Moreover, the coefficients are given by the following formula,

$$\alpha_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\, dx, k > 0$$

$$\alpha_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx$$

Chebyshev and best approximation:

Suppose f is bounded and continuous on $[-1,1]$ and let p^* denote its best polynomial approximation from P_n. Let p denote its truncated Chebyshev expansion (truncated after $n+1$ terms). Then

$$\frac{\|f-p\|_\infty}{\|f-p^*\|_\infty} \sim \frac{4}{\pi^2} \log(n), \quad \text{as } n \to \infty.$$

All need to know in this proof is up to bound calculation on the projection. For a m-dimension subspace M of E, define $P_M(u) = \sum_{i=1}^{m} \phi_i \langle \varphi_i | u \rangle$. Then,

$$f - p = f - p^* - P_M(f - p).$$

Implying, $\|f-p\|_E \leq \|f-p^*\|_E + \|P_M\| \|f-p^*\|_E$, therefore, $\frac{\|f-p\|}{\|f-p^*\|} \leq 1 + \|P_M\|$.

- Corollary: Truncated Chebyshev expansion:

$$\|f - f_n\| \leq \frac{2V}{\pi(n-p)^p}$$

Regularity and Decay

- :)

> **Regularity**:
>
> For an integer $p \geq 0$, let $f, f', ..., f^{(p-1)}$ be absolutely continuous and suppose $f^{(p)}$ is of bounded variation with total variation V. This is to say
>
> $$\int_{-1}^{1} \frac{|f^{(p)}(x)|}{\sqrt{1-x^2}}\, dx = V < \infty.$$
>
> Then, for $k \geq p+1$, the Chebyshev coefficients α_k of f satisfy
>
> $$|\alpha_k| \leq \frac{2V}{\pi(k-p)^{(p+1)}}$$

Analyticity

- The Joukowsky map, $x = \frac{1}{2}(z + 1/z)$. For $\rho > 1$, the image of the circle of radius ρ, under this map, is a ellipse with foci at ± 1. these ellipses are called Bernstein Ellipses and we will denote their interior by E_ρ.

> **Bernstein Ellipses**:
>
> Suppose f is defined on $[-1, 1]$ and is analytically continuable (to $\mathbb{C}$) to the open ellipse E_ρ and $|f(x)| \leq M$ on that ellipse. If α_k denotes its Chebyshev coefficients, then
>
> $$|\alpha_0| \leq M$$
>
> $$|\alpha_k| \leq 2M\rho^{-k}$$

- Corollary: If f satisfies the conditions of the previous theorem, then

$$\|f - f_n\| \leq \frac{2M\rho^{-n}}{\rho - 1}$$

Interpolation

Lagrange Interpolation

- Lagrange interpolant:

$$\ell_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}, \quad i = 0, ..., n$$

- Given $x_1, ..., x_n$ the Lebesgue Constant Λ is defined via

$$\Lambda = \sup_{f \in C([a,b])} \frac{\|p_f\|_{L^\infty}}{\|\mathbf{f}\|_{\ell^\infty}} = \sup_{f \in C([a,b])} \frac{\|p_f\|_{L^\infty}}{\|f\|_{L^\infty}},$$

  where $\mathbf{f} = (f_0, ..., f_n)^\intercal$ and

$$p_f = \sum_{j=0}^{n} f_j \ell_j(x)$$

- The function

$$\lambda(x) = \sum_{j=0}^{n} |\ell_j(x)|,$$

  is called the Lebesgue function. Note that here we assume the x_i's are in an interval $[a, b]$ and the infinity norms are taken over that interval. In the first equality, the supremum should be taken over all **f** which do not vanish at the x_j's, and in the second, over all nonzero continuous functions.

- If all x_i's are distinct, then $1 \leq \Lambda < \infty$.

- Suppose we have our $n + 1$ evaluation points and the Lebesgue constant Λ. Given some $f \in C([a, b])$, let p be its Lagrange approximation and p^* be its best polynomial approximation to f in P_n. Then,

$$\|f - p\| \leq (1 + \Lambda)\|f - p^*\|$$

  Lebesgue Approximation:
  On the interval $[-1, 1]$, if Λ_n is the Lebesgue constant for any set of $n + 1$ distinct points in $[-1, 1]$, then
  1. $\Lambda_n \geq \frac{2}{\pi} \ln(n + 1) + \frac{2}{\pi} \left(\gamma + \ln\left(\frac{4}{\pi}\right)\right)$
  2. For Chebyshev roots,
  $$\Lambda_n \leq \frac{2}{\pi} \ln(n + 1) + 1$$

- For equispaced points, the Lebesgue constant is bounded below by

$$\Lambda_n \geq \frac{2^{n-2}}{n^2}, \quad \Lambda_n \sim \frac{2^{n+1}}{en \log n}$$

The witch of Agnesi

Consider the function

$$f(x) = \frac{1}{1 + a^2 x^2}$$

$$|f(x) - p_f(x)| \leq\sim \frac{1}{n^{3/2}} \sqrt{8\pi} \left(\frac{2a}{e}\right)^n$$

So $\|f - p_f\|$ converges if $2a/e < 1$.