Review of linear homogeneous ODEs (review, Chapter 2)

(*) $\dot{X} = A(t) X$ $\qquad X \in \mathbb{R}^n$

$A(t) = n \times n$ (continuous) matrix

$X = 0$ is a soln. (homogeneous)

if $X_1(t)$ & $X_2(t)$ satisfy (*), then so does $c_1 X_1(t) + c_2 X_2(t)$ for any $c_1, c_2 \in$ (linear superposition principle)

If we can find n linearly independent solns. $X_1(t), X_2(t), \ldots X_n(t)$ then we have found the general soln. to (*), which can be written as

$$X = \sum_{k=1}^{n} c_k X_k(t)$$

Special case A = constant matrix

- Approach 1

compute eigenvalues of A as roots of the characteristic eqn. $\mathrm{Det}(A - \lambda I) = 0$
if none are repeated, then we know we can write the general soln. as

$$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

where $v_1, \dots, v_n$ are linearly independent eigenvectors $A v_j = \lambda_j v_j$

i.e. $$X(t) = \begin{bmatrix} | & | & & | \\ e^{\lambda_1 t} v_1 & e^{\lambda_2 t} v_2 & \cdots & e^{\lambda_n t} v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

initial condition $X(0) = X_0$ determines $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

ie. $$X_0 = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^{-1} X_0$$

guaranteed to be invertible since eigenvectors are linearly independent

$$\Phi(t) = \begin{bmatrix} | & | & & | \\ v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & \cdots & v_n e^{\lambda_n t} \\ | & | & & | \end{bmatrix}$$

= "Fundamental Matrix Soln." $\circledast$

$\Phi(t)$ satisfies:

$$\dot{\Phi} = A \Phi$$

[nice convention to add to this: $\Phi(0) = Id.$
then soln. to $\dot{X} = AX$, $X(0) = X_0$ is
$X = \Phi(t) X_0$]

side note:

if any eigenvalue λ is complex, ~~e.g.~~ ~~so that~~
$\lambda = \alpha + i\beta \Rightarrow \lambda^* = \alpha - i\beta$ is also an eigenvalue
$Av = \lambda v$, $Av^* = \lambda^* v^*$
then we can construct a real soln. from these as

$$c e^{\lambda t} v + c^* e^{\lambda^* t} v^*$$

& can rewrite as

$$e^{\alpha t} (a \cos(\beta t) + b \sin(\beta t))$$

or

$$e^{\alpha t} (A \cos(\beta t + \emptyset))$$

What if there is a repeated eigenvalue?
then it could be that

$$\underbrace{\text{geometric multiplicity}}_{\text{dim. of eigenspace}} < \text{algebraic multiplicity}$$

$\Rightarrow$ not enough eigenvectors to construct n linearly independent solns.

Additional soln(s) obtained in terms of "generalized eigenvectors" & have form $te^{\lambda t}$, $t^2 e^{\lambda t}$, etc.

example:

v is eigenvector of A w/ eigenvalue λ (multiplicity 2)

$$(A-\lambda I)v = 0$$

- w is generalized eigenvector

$$\left.\begin{aligned}(A-\lambda I)^2 w &= 0\\ (A-\lambda I) w &\neq 0\end{aligned}\right\} \quad (A-\lambda I)w = v$$

$$\Downarrow$$

$$Aw = v + \lambda w$$

$$X_1 = e^{\lambda t} v$$
$$X_2 = e^{\lambda t}(tv + w)$$

$$\dot{X}_2 = \lambda e^{\lambda t}(tv+w) + e^{\lambda t} v = e^{\lambda t}(\lambda t v + \lambda w + v)$$

$$AX_2 = e^{\lambda t}(t\lambda v + v + \lambda w) \qquad \boxed{\checkmark}$$

if multiplicity 3: $(A-\lambda I)v = 0$ $\qquad (A-\lambda I)^3 u = 0$

$$X_3 = e^{\lambda t}\left(\frac{t^2}{2} v + t w + u\right)$$

$(A-\lambda I)^2 w = 0$

Approach 2.

$$\frac{dX}{dt} = AX, \qquad X(0) = X_0$$

has soln. $\quad X(t) = e^{At} X_0$

$$\underset{\text{matrix}}{e^{At}} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \qquad (0! = 1)$$

converges for all t.

Check: $\quad X(0) = e^{A \cdot 0} X_0 = I X_0 = X_0 \quad \checkmark$

$$\frac{dX}{dt} = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}$$

$$= \lim_{h \to 0} \frac{e^{A(t+h)} X_0 - e^{At} X_0}{h}$$

$$= \lim_{h \to 0} \left(\frac{e^{At} e^{Ah} - e^{At}}{h} \right) X_0$$

$$= \underbrace{\lim_{h \to 0} \left(\frac{e^{Ah} - I}{h} \right)}_{\lim_{h \to 0} \left(\frac{Ah + \frac{1}{2} h^2 A^2 + \cdots}{h} \right) = A} \underbrace{e^{At} X_0}_{X(t)} = A X(t)$$

Important: I needed $e^{A(t+h)} = e^{At}e^{Ah} = e^{Ah}e^{At}$

~~e^{A+B}~~ $\quad e^A e^B \stackrel{?}{=} e^{(A+B)}$

not necessarily true: Baker-Campbell-Hausdorff Formula

$$e^A e^B = e^C$$

$$C = A+B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \cdots$$

where $[A,B] = AB - BA$

$e^A e^B = e^{A+B}$ only if $[A,B]=0$, ie. A & B commute

in our case $e^{A(t+h)} = e^{At}e^{Ah}$
because $[At, Ah] = th[A,A] = 0$

How do we evaluate e^{At}?

simple case of complete set of eigenvectors $(v_1, v_2, .. v_n)$ with eigenvalues $(\lambda_1, \lambda_2, .. \lambda_n)$

~~A~~ $\Rightarrow$ A can be diagonalized.

Lecture 2 P
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$$A = P\Lambda P^{-1}, \quad (P^{-1}AP = \Lambda)$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

invertible since v_j are linearly independent

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= I + P\Lambda P^{-1}t + \frac{1}{2}(P\Lambda P^{-1})(P\Lambda P^{-1})t^2 + \cdots$$

$$= P\,\underbrace{\left(I + \Lambda t + \frac{1}{2}\Lambda^2t^2 + \frac{1}{3!}\Lambda^3t^3 + \cdots\right)}_{\underbrace{\begin{pmatrix} 1+\lambda_1 t+\frac{1}{2}\lambda_1^2t^2+\cdots & & \\ & \ddots & \\ & & 1+\lambda_n t+\frac{1}{2}\lambda_n^2t^2+\cdots \end{pmatrix}}_{\begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix}}}\,P^{-1}$$

$$e^{At} = P\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}P^{-1}$$

But what if A is deficient & we can't diagonalize it?

e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only one eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

<u>Thm</u> (Proved in Chapter 2)

The matrix A on a complex vector space E has a unique decomposition, $A = S + N$, where S is <u>semi-simple</u>, N is <u>nilpotent</u> and $[S, N] = 0$

<u>semi-simple</u>: $P^{-1} S P = \Lambda$ = diagonal

<u>nilpotent</u>: $N^K = 0$, $N^{K-1} \neq 0$ nilpotency K
($K = 1$ is diagonalizable case)

$[S, N] = 0$: $SN = NS$

ex. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ✓

$e^{tA} = e^{tS} e^{tN}$ since $[S, N] = 0$

$(I + tN + \frac{1}{2} t^2 N^2 + \cdots)$ terminates at K.

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$e^{tA} = \underbrace{e^{tS}}_{\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}} \underbrace{e^{tN}}_{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = I + tN}$$

$\Rightarrow$ soln to $\dot{X} = AX$ is $X = e^{At} X_0$

$$X = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} X_0 = \begin{pmatrix} c_1 e^t + c_2 t e^t \\ c_2 e^t \end{pmatrix}$$

$X_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

soln involving eigenvector

2nd independent soln. involving generalized eigenvector

homework problem: $\dot{X} = A(t) X$
$A(t) = A(t+T)$, $X \in \mathbb{R}^2$

eigenvalues of $A(t)$ don't help in general

$[A(t), A(s)] \neq 0$ in general ...