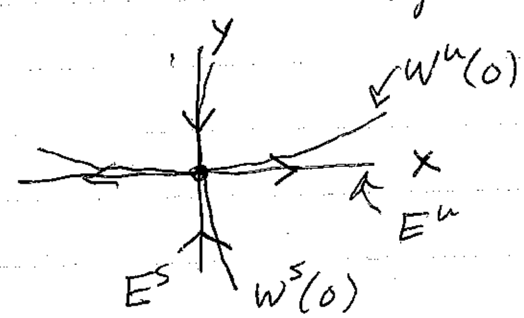
Approximating $W^s_{loc}(\cancel{x^*})$ & $W^u_{loc}(x^*)$

Ex. $\left.\begin{aligned}\dot{x} &= 2x + y^2\\ \dot{y} &= -2y + x^2 + y^2\end{aligned}\right\}$ Ex 5.12 in text

I do this ~~differently~~ from Meiss

Saddle at $(x,y)=(0,0)$ Linearization $\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$



$W^s_{loc}(0)$ is a graph over E^s

$W^s_{loc}(0) = \{(x,y) = (h_s(y), y),\ |y| < \delta\}$

$h_s(0) = h_s'(0) = 0$ since $W^s_{loc}(0)$ is tangent to E^s at $x=y=0$. $E^s = \{(0,y),\ y \in \mathbb{R}\}$

It is smooth since f is smooth & it is invariant

if $(x_0, y_0) = (h_s(y_0), y_0)$, then $\varphi_t(x_0,y_0) = (h_s(y(t)), y(t))$ $t \geq 0$

$X(t) = h_s(y(t))$

$\dot{X} = h_s'(y(t))\,\dot{y}$ ← $-2y+x^2+y^2\big|_{W^s}$

$2x+y^2\big|_{W^s}$

$$\begin{cases} 2h_s(y)+y^2 = h_s'(y)\left(-2y+h_s^2(y)+y^2\right)\\ h_s(0)=h_s'(0)=0\end{cases}$$

This is only guaranteed to be valid for y in some neighborhood of $y=0$, so try a power series soln.

$$h_s(y) = \cancelto{0}{c_0} + \cancelto{0}{c_1 y} + c_2 y^2 + c_3 y^3 + \cdots$$

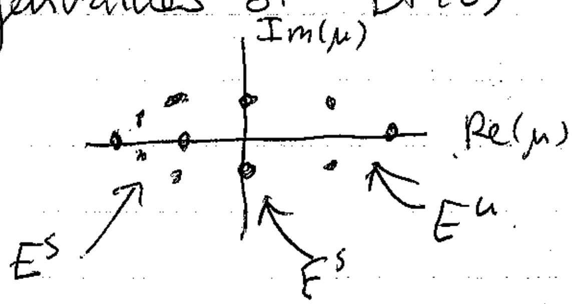
$$2c_2 y^2 + y^2 + \mathcal{O}(3) = 2c_2 y(-2y) + \mathcal{O}(3)$$

$$(6c_2+1)y^2 = \mathcal{O}(3) \qquad c_2 = -\frac{1}{6}$$

Non-hyperbolic fixed-pts. & Center Manifolds

$\dot{x} = f(x)$, $f(0)=0$, $\mu =$ eigenvalues of $Df(0)$

f is C^k, $k \geq 1$



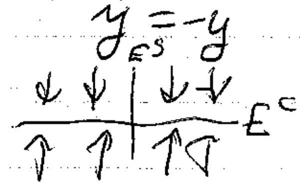
$Df(0)$ has eigenspaces $E^s \oplus E^c \oplus E^u$

Center Manifold Thm. There is a neighborhood of the origin where $\exists$ C^k locally invariant manifolds: W^s_{loc}, tangent to E^s, on which $|x(t)| \to 0$ as $t \to \infty$; W^u_{loc}, tangent to E^u, on which $|x(t)| \to 0$ as $t \to -\infty$. and a local center manifold W^s_{loc} tangent to E^c.

<u>Note:</u> uniqueness isn't mentioned for the center manifold, in contrast to stable & unstable manifolds – need not be unique. Also – cannot say anything here about asymptotic behavior of solns. in $W^c_{loc}(0)$. Nonetheless – it's useful because we can use restriction to center manifold to reduce dim. of problem. Especially powerful in bifurcation theory.

Example demonstrating lack of uniqueness for W^c_{loc}:

$$\left.\begin{array}{l}\dot{x} = x^2 \\ \dot{y} = -y\end{array}\right\} \quad \text{linearize about } (0,0): \quad \begin{array}{l}\dot{x} = 0 \\ \dot{y} = -y\end{array}$$



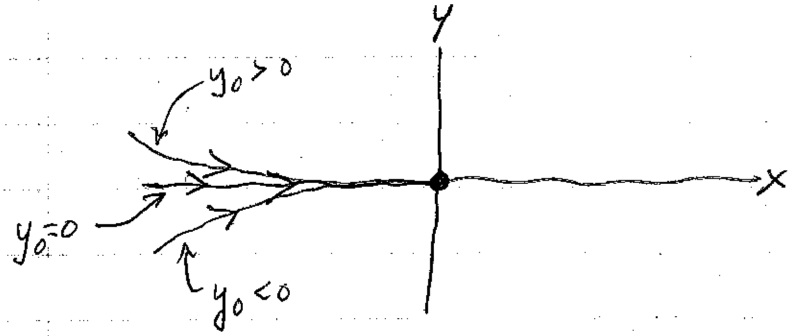
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x^2} \qquad \begin{array}{l} x(0) = x_0 < 0 \\ y(0) = y_0 \end{array}$$

(note if $y(0)=0$, then $y(t)=0$)

if $y_0 \neq 0$, then

$$\int_{y_0}^{y} \frac{dy'}{y'} = \int_{x_0}^{x} -\frac{1}{x'^2}\,dx'$$

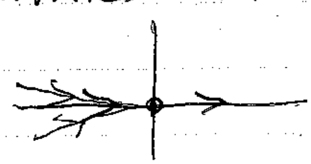
$$\ln\left|\frac{y}{y_0}\right| = \frac{1}{x} - \frac{1}{x_0} \Rightarrow y = \underbrace{\left(y_0 e^{-1/x_0}\right)}_{c,\ \text{depends on } (x_0, y_0)} e^{1/x}, \quad x<0$$



$e^{1/x}$ approaches origin for $x \to 0^-$,
very flat

Each of these is a center manifold for $y=x=0$. It is invariant (as a soln. must be) and it is tangent to the x-axis (the center eigenspace) at $x=y=0$.

What is the Taylor series for $y=ce^{1/x}$ about $x=0$ for $x<0$?
It has an essential singularity at $x=0$; not an analytic function.
Power series approximation picks off $y=0$ center manifold.
Dynamics in CM given by $\dot{x}=x^2$



Nonhyperbolic Hartman - Grobman Thm

$$(*)\begin{cases} \dot{x} = Cx + F(x,y,z) \\ \dot{y} = Sy + G(x,y,z) \\ \dot{z} = Uz + H(x,y,z) \end{cases}$$

center directions
stable directions
unstable directions

For a C^k, $k \geq 1$, vector field with fixed pt at $(0,0,0)$, F, G, H are $o(x,y,z)$.

$\exists$ neighborhood N of the origin s.t.

$$W^c_{loc} = \{(x, g(x), h(x)) : x \in E^c\} \cap N$$

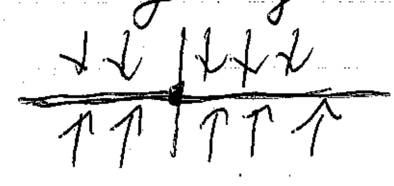
and the dynamics of $(*)$ are topologically conjugate to

$$\dot{x} = Cx + F(x, g(x), h(x))$$
$$\dot{y} = Sy$$
$$\dot{z} = Uz$$

look here for the behavior of solns. in $W^c_{loc}(0)$.

example: $\left.\begin{aligned} \dot{x} &= yx - x^3 \\ \dot{y} &= -y + ax^2 \end{aligned}\right\}$

linearized
$\dot{x} = 0$
$\dot{y} = -y$



Approximate $W^c_{loc}(0)$, which is tangent to E^c at $x=y=0$, $y=g(x)$, $g(0)=g'(0)=0$

$$y = \alpha x^2 + \cdots$$

$$\dot{y} = 2\alpha x \dot{x} + \cdots \qquad \text{For solns in } W^c_{loc}(0)$$

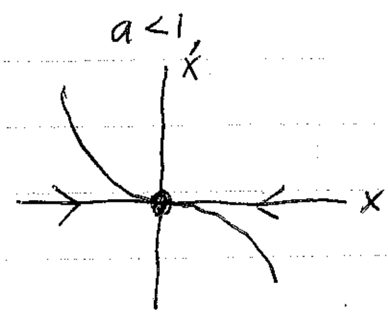
$$-y + a x^2 \Big|_{y=\alpha x^2+\cdots} \qquad\qquad yx - x^3 \Big|_{y=\alpha x^2+\cdots}$$

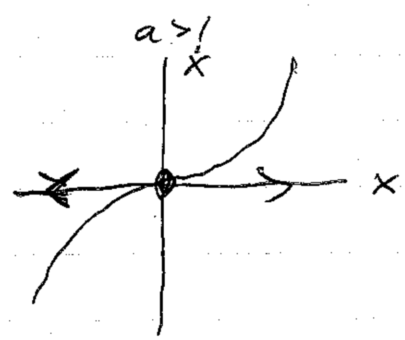
$$-\alpha x^2 + a x^2 + \Theta(3) = \Theta(4)$$

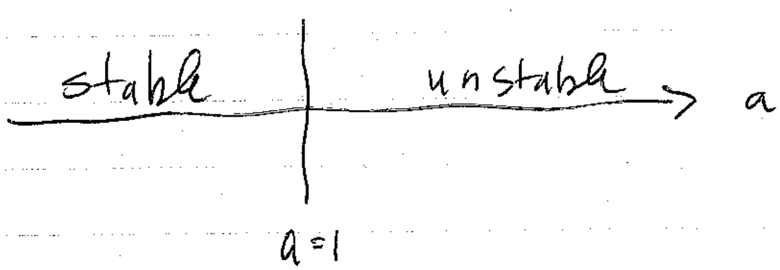
$$\Rightarrow \quad \alpha = a \qquad W^c_{loc}:\ y = a x^2 + \Theta(3)$$

Approximate dynamics in W^c_{loc}

$$\dot{x} = yx - x^3 \Big|_{W^c_{loc}} = (a-1)x^3 + \Theta(x^4)$$







What about $a=1$?

$$\left.\begin{aligned}\dot{x} &= yx - x^3 = x(y-x^2)\\ \dot{y} &= -y + x^2 = -(y-x^2)\end{aligned}\right\} \quad \begin{aligned}&\dot{x}=\dot{y}=0\\ &\text{on } y=x^2\end{aligned}$$

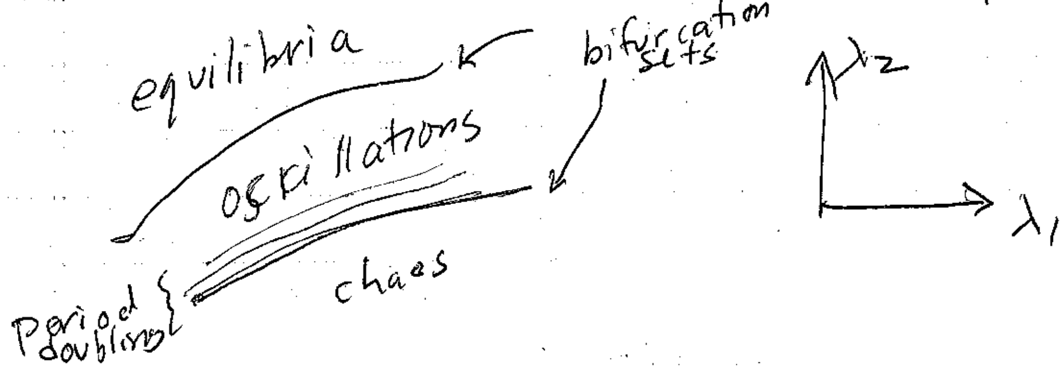
$a<1$ $\qquad$ $a=1$ $\qquad$ $a>1$

Center manifold plays a fundamental role in local bifurcation theory – next topic for next week

~~Note: we already~~

Goal of bifurcation theory is to determine parameter sets where there is a qualitative change in behavior for parameterized families of dynamical systems

$$\dot{x} = f(x;\lambda), \quad x \in \mathbb{R}^n, \quad \underbrace{\lambda \in \mathbb{R}^k}_{\text{parameters}}$$



Rössler period-doubling bifurcations, associated with μ crossing $|\mu|=1$ at $\mu=-1$