STAT 31210: Homework 8

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Contents

Exercise 9.1

Prove that $\rho(A^*) = \overline{\rho(A)}$, where $\overline{\rho(A)}$ is the set $\{\lambda \in \mathbb{C} : \overline{\lambda} \in \rho(A)\}$.

Solution:

We should first show that $((A - \lambda\mathbb{I})^{-1})^* = (A^* - \overline{\lambda}\mathbb{I})^{-1}$. Let $x, y \in \mathcal{H}$, then, for $\overline{\lambda} \in \rho(A^*)$.

$$(A^* - \overline{\lambda}\mathbb{I})^{-1}y = x$$

$$y = (A^* - \overline{\lambda}\mathbb{I})x$$

$$y^* = ((A^* - \overline{\lambda}\mathbb{I})x)^*$$

$$y^* = (A^*x - \overline{\lambda}x)^*$$

$$y^* = (A^*x)^* - (\overline{\lambda}x)^*$$

$$y^* = x^*A - \lambda x^*$$

$$y^* = x^*(A - \lambda\mathbb{I})$$

$$y^* = x^*(A - \lambda\mathbb{I})$$

$$y^*(A - \lambda\mathbb{I})^{-1} = x^*$$

$$((A - \lambda\mathbb{I})^{-1})^*y = x$$

For the proof above to hold, we need to show that if $\overline{\lambda} \in \rho(A^*)$, then $\lambda \in \rho(A)$. For this, we need to show that $(A - \lambda\mathbb{I})$ is bijective. Let $x, y \in \mathcal{H}$, then

$$(A - \lambda\mathbb{I})x = \left(x^*(A - \overline{\lambda}\mathbb{I})^*\right)^* = (y^*)^* = y.$$

This holds by standard adjoint properties, as well as $(A - \overline{\lambda}\mathbb{I})$ being surjective. We next need to show that $(A - \lambda\mathbb{I})$ is injective. Take $x_1, x_2 \in \mathcal{H}$ which under the given operator maps to the same $y \in \mathcal{H}$. Then,

$$(A - \lambda\mathbb{I})x_1 - (A - \lambda\mathbb{I})x_1 = (A - \lambda\mathbb{I})(x_1 - x_2) = \left(x_1^*(A^* - \overline{\lambda}\mathbb{I})\right)^* - \left(x_2^*(A^* - \overline{\lambda}\mathbb{I})\right)^* = (y^* - y^*)^* = 0.$$

This impies that $x_1^* - x_2^* \in \ker(A^* - \lambda\mathbb{I})$, which is equal to zero since $(A^* - \lambda\mathbb{I})$ is injective. Therefore, $x_1^* = x_2^*$, so $x_1 = x_2$, implying that $(A - \lambda\mathbb{I})$ is injective, hence bijective. Then $\lambda \in \rho(A)$.

Finally, we note that, by above, we have that $(A^* - \overline{\lambda}\mathbb{I})^{-1} = ((A - \lambda\mathbb{I})^{-1})^*$. This implies that if $\overline{\lambda} \in \rho(A^*)$, then $\lambda \in \rho(A)$, which implies that $\overline{\lambda} \in \overline{\rho(A)}$. This satisfies one direction of the equality. Next we take a $\lambda \in \rho(A)$. Then by the above calculations, we have that $\lambda \in \overline{\rho(A^*)}$, which implies that $\overline{\lambda} \in \rho(A^*)$. This shows the equality to hold.

Exercise 9.3

Suppose that A is a bounded linear operator of a Hilbert space and $\mu, \lambda \in \rho(A)$. Prove that the resolvent set R_λ of A satisfies the *resolvent equation*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu.$$

Solution:

We will go straight into calculations.

$$\begin{aligned}
R_\lambda - R_\mu &= (\lambda \mathbb{I} - A)^{-1} - (\mu \mathbb{I} - A)^{-1} && \text{(Given.)} \\
&= (\lambda \mathbb{I} - A)^{-1} \left[\mathbb{I} - (\lambda \mathbb{I} - A)(\mu \mathbb{I} - A)^{-1} \right] && \text{(Factoring.)} \\
&= (\lambda \mathbb{I} - A)^{-1} (\mu \mathbb{I} - A)^{-1} \left[(\mu \mathbb{I} - A) - (\lambda \mathbb{I} - A) \right] && \text{(Factoring.)} \\
&= (\lambda \mathbb{I} - A)^{-1} (\mu \mathbb{I} - A)^{-1} \left[(\mu - \lambda) \mathbb{I} \right] && \text{(Simplifying.)} \\
&= (\mu - \lambda)(\lambda \mathbb{I} - A)^{-1} (\mu \mathbb{I} - A)^{-1} && \text{(Rearranging.)} \\
&= (\mu - \lambda) R_\lambda R_\mu && \text{(Definition.)}
\end{aligned}$$

Exercise 9.6

Let G be a multiplication operator on $L^2(\mathbb{R})$ defined by

$$Gf(x) = g(x)f(x)$$

where g is continuous and bounded. Prove that G is a bounded linear operator on $L^2(\mathbb{R})$ given by

$$\sigma(G) = \overline{\{g(x) : x \in \mathbb{R}\}}$$

Can an operator of this form have eigenvalues?

Solution:

Let us first show that G is linear. Take $h, f \in L^2(\mathbb{R})$, and $\mu, \lambda \in \mathbb{R}$. Then

$$G(\lambda h + \mu f)(x) = g(x)(\lambda h(x) + \mu f(x)) = \lambda g(x)h(x) + \mu g(x)f(x) = \lambda Gh(x) + \mu Gf(x).$$

Next, let us show that G is bounded. Suppose that $|g(x)|$ is bounded by M. Then

$$\|Gf\|^2 = \int_{\mathbb{R}} |g(x)f(x)|^2 \, dx \leq \int_{\mathbb{R}} |g(x)|^2 |f(x)|^2 \, dx \leq \sup_{x \in \mathbb{R}} |g(x)|^2 \int_{\mathbb{R}} |f(x)|^2 \, dx = M^2 \|f\|^2$$

This implies that $\|Gf\|^2 \leq M^2 \|f\|^2$, which means $\|Gf\| \leq M\|f\|$. Therefore, G is bounded.

Finally, we need to show that the spectrum of G is given by the above set. We will show this via inclusions on both sides. For the sake of simplicity, denote the set $\overline{\{g(x) : x \in \mathbb{R}\}}$ by A.

$\underline{\lambda \in \sigma(G) \implies \lambda \in A}$:

Since $\lambda \in \sigma(G)$, then $\sigma \notin \rho(G)$. Therefore, $(\lambda \mathbb{I} - G)$ is not bijective. Therefore, we should break this into cases based on whether the operator is not injective or surjective. We will take the two cases based on these. Note that a λ could satisfy both; in this case, we take either branch.

Case 1: $\lambda \mathbb{I} - G$ is not injective.

This implies that $\ker(\lambda \mathbb{I} - G) \neq \{0\}$. This implies there exists $f \neq 0$ for which $f \in \ker(\lambda \mathbb{I} - G)$. Then $(\lambda \mathbb{I} - G)f = 0$, so $(g(x) - \lambda)f(x) = 0$ (a.e.). this implies then that $\lambda = g(x)$ for some $x \in \mathcal{M}$, where $\mathcal{M}$ is a subset of measure nonzero on the real line. This implies $\lambda \in A$.

Case 2: range$(\lambda \mathbb{I} - G) \neq \mathcal{H}$.

Suppose false. That is, range$(\lambda \mathbb{I} - G) \neq \mathcal{H}$, yet $\lambda \notin A$. The first property implies there exists $z \in \mathcal{H}$ such that $(\lambda \mathbb{I} - G)y \neq z$ for any $y \in \mathcal{H}$. The second property means that $\lambda \neq g(x)$ for any $x \in \mathbb{R}$. Therefore, $\lambda - g(x) \neq 0$ for any x. Then, we have that

$$(\lambda \mathbb{I} - G)y \neq z \implies \lambda y - Gy \neq z \implies (\lambda - g(x))y \neq z \implies y \neq \frac{z}{\lambda - g(x)}$$

Note that $\lambda - g(x) \in \mathbb{R}$ for any $x \in \mathbb{R}$, so we can essentially treat it as a nonzero scalar quantity. We have that $\frac{z}{\lambda - g(x)} \in \mathcal{H}$, by linearity. But such a y cannot exist, by assumption. Therefore, we have a contradiction. Which implies that $\lambda \in A$.

$\underline{\lambda \in A \implies \lambda \in \sigma(G)}$:

Let $\lambda \in A$. Since A is closed, there exists some sequence λ_n for which $\lambda_n \to \lambda$ as $n \to \infty$. Since g is continuous, we have there existing some $x_n \subset \mathbb{R}$ for which $g(x_n) = \lambda_n$. Then, $Gf(x_n) = g(x_n)g(x_n)$. We have then that $(\lambda\mathbb{I} - G)f(x_n) = \lambda f(x_n) - g(x_n)f(x_n) = (\lambda - g(x_n))f(x_n)$. Note that $\|\lambda\mathbb{I} - G\| = \|\lambda - g(x_n)\| = \|\lambda - \lambda_n\|$. Furthermore,

$$\left\|(\lambda\mathbb{I} - G)^{-1}\right\| = \frac{1}{\|\lambda_n - g(x)\|}$$

The inverse of $(\lambda\mathbb{I} - G)$ can be taken, since we can arbitrarily pick some sequence with this restriction. Note however that the norm of $(\lambda\mathbb{I} - G) \to 0$ as $n \to \infty$, which means that the inverse goes to infinity. This implies that G is not bounded, which violates our assumption. Therefore, $\lambda \notin \rho(G)$ so $\lambda \in \sigma(G)$.

Exercise 9.7

Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\, dy.$$

Exercise 9.7, part a

Find the adjoint operator K^*.

Solution:

The Adjoint of K will be the operator K^* such that

$$\langle Kf|g\rangle = \langle f|K^* g\rangle$$

for $f, g \in L^2([0,1])$. Taking the inner product, we have that

$$\langle Kf|g\rangle = \int_0^1 (Kf)(x)g(x)\, dx = \int_0^1 g(x) \int_0^x f(y)\, dy\, dx = \int_0^1 f(y) \int_y^1 g(x)\, dx\, dy = \langle f|K^* g\rangle\,.$$

The second to last equality is given to us by Fubini's theorem, where the two sets

$$\{[x,y] : x \in [0,1] \text{ and } y \in [0,x]\} \quad \text{and} \quad \{[x,y] : x \in [y,1] \text{ and } y \in [0,1]\}$$

characterize the same regions in $\mathbb{R}^2$. Here, I propose that

$$K^* f(x) = \int_x^1 f(x)\, dx.$$

Exercise 9.7, part b

Show that $\|K\| = 2/\pi$.

Solution:

This part requires a few steps before getting to the result. We should first show that $\|K\|^2 = \|K^*K\|$. For the purposes of this analysis, we will assume $f \leq 1$, by the properties of the norm.

$$\|Kf\|^2 = \langle Kf|Kf\rangle = \langle f|K^*Kf\rangle \leq \|f\|\|K^*Kf\| \leq \|f\|^2\|K^*K\| = \|K^*K\|$$

Similarly, we can write,

$$\|Kf\|^2 = \langle Kf|Kf\rangle = \langle KK^*f|f\rangle \leq \|KK^*f\|\|f\| \leq \|f\|^2\|KK^*\| = \|K^*K\|.$$

Therefore, we have that $\|K\|^2 = \|K^*K\|$. Note that K^*K is self adjoint. From the result of Theorem 9.16, its norm is equal to its largest eigenvalue. Suppose then that f is the corresponding eigenfunction. Then,

$$K^*Kf = \lambda f$$

Assume that f has integral F, which in turn has integral E. Differentiating both sides twice gives,

$$\lambda\frac{\partial^2}{\partial x^2}f = \frac{\partial^2}{\partial x^2}\int_x^1\int_0^y f(u)\,du dy = \frac{\partial^2}{\partial x^2}\int_x^1 F(y) - F(0)dy = \frac{\partial^2}{\partial x^2}\left[E(1) - E(x) - F(0) + xF(0)\right] = -f(x).$$

We then have the differential equation $\lambda\frac{\partial^2}{\partial x^2}f = -f(x)$, which, when denoting $\omega^2 = \frac{1}{\lambda}$, has solution

$$f(x) = c_1e^{i\omega x} + c_2e^{-i\omega x}$$

To get the value for λ, we need to plug this back into the equation $K^*Kf = \lambda f$ to get the following:

$$K^*Kf = \int_x^1\int_0^y c_1e^{i\omega u} + c_2e^{-i\omega u}\,du\,dy \qquad \text{(Given.)}$$

$$= \int_x^1\left[\frac{c_1}{i\omega}e^{i\omega u} - \frac{c_2}{i\omega}e^{-i\omega u}\right]_0^y\,dy \qquad \text{(Integrating.)}$$

$$= \int_x^1\left[\frac{c_1}{i\omega}e^{i\omega y} - \frac{c_2}{i\omega}e^{-i\omega y} - \frac{c_1}{i\omega} + \frac{c_2}{i\omega}\right]\,dy \qquad \text{(Taking limits.)}$$

$$= \left[-\frac{c_1}{\omega^2}e^{i\omega y} - \frac{c_2}{\omega^2}e^{-\omega y} - \frac{c_1}{i\omega}y + \frac{c_2}{i\omega}y\right]_x^1 \qquad \text{(integrating.)}$$

$$= -\frac{1}{\omega^2}(c_1e^{i\omega} + c_2e^{-i\omega}) + \frac{1}{i\omega}(c_2 - c_1) + \frac{1}{\omega^2}f(x) + \frac{1}{i\omega}(c_1 - c_2)x \qquad \text{(Taking bounds.)}$$

Since we have that $K^*Kf = \lambda f(x)$, we require $c_1 = c_2$ and the first term equal zero. This then implies

$$c_1 e^{i\omega} + c_2 e^{-i\omega} = 0 \iff \cos(\omega) = 0$$

We then get that $\omega = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$. Therefore,

$$\lambda = \frac{1}{\omega^2} = \frac{4}{(2n+1)^2\pi^2}$$

We want the largest value for λ to relate it to the norm of K^*K. Therefore,

$$\|K\|^2 = \frac{4}{\pi^2} \implies \|K\| = \frac{2}{\pi},$$

which is what we wanted.

Exercise 9.7, part c

Show that the spectral radius of K is equal to zero.

Solution:

The easiest way to show this is to first find the resolvent set $\rho(K)$, then taking its complement. If $\lambda \in \rho(K)$, then $(K - \lambda\mathbb{I})$ is bijective. Let $g, f \in L^2([0,1])$. By bijectivity, $f = (K - \lambda\mathbb{I})g$. The following thus holds:

$$f = (K - \lambda\mathbb{I})g \qquad \text{(Given.)}$$

$$f = \int_0^x g(t)\,dt - \lambda g(x) \qquad \text{(Given.)}$$

$$f = \int_0^x g(t)\,dt - \lambda \frac{d}{dx}\int_0^x g(t)\,dt \qquad \text{(Fundamental Theorem.)}$$

$$-\frac{1}{\lambda} f(x) = \frac{d}{dx}\int_0^x g(t)\,dt - \frac{1}{\lambda}\int_0^x g(t)\,dt \qquad \text{(Rearranging.)}$$

$$-\frac{1}{\lambda} e^{-x/\lambda} f(x) = e^{-x/\lambda}\frac{d}{dx}\int_0^x g(t)\,dt - \frac{1}{\lambda}e^{-x/\lambda}\int_0^x g(t)\,dt \qquad \text{(Multiplying both sides.)}$$

$$\frac{d}{dx}\left(e^{-x/\lambda}\int_0^x g(t)\,dt\right) = -\frac{1}{\lambda}e^{-x\lambda} f(x) \qquad \text{(Product rule.)}$$

$$e^{-x/\lambda}\int_0^x g(t)\,dt = -\frac{1}{\lambda}\int_0^x e^{-u/\lambda} f(u)\,du \qquad \text{(Integrating both sides.)}$$

$$\int_0^x g(t)\,dt = -\frac{1}{\lambda}\int_0^x e^{(x-u)/\lambda} f(u)\,du \qquad \text{(Rearranging.)}$$

$$f = -\frac{1}{\lambda}\int_0^x e^{(x-u)/\lambda} f(u)\,du - \lambda g(x) \qquad \text{(Plugging back into 3.)}$$

$$g(x) = -\frac{1}{\lambda} f(x) - \frac{1}{\lambda^2}\int_0^x e^{(x-u)/\lambda} f(u)\,du \qquad \text{(Rearranging.)}$$

$$\implies g(x) = (K - \lambda\mathbb{I})^{-1} f = -\frac{1}{\lambda} f(x) - \frac{1}{\lambda^2}\int_0^x e^{(x-u)/\lambda} f(u)\,du$$

Therefore, an explicit formula for the inverse has been found. We can see that this formula will not hold only for $\lambda = 0$, which implies that $0 \notin \rho(K)$. Then $0 \in \sigma(K)$. This is the only value inside the spectrum of K, since if there were any other nonzero values in the spectrum, then its inverse would not be defined, which is only true for the zero value. Therefore, $\sigma(K) = \{0\}$.

Exercise 9.8

We define the right shift operator S on $\ell^2(\mathbb{Z})$ by

$$S(x)_k = x_{k-1} \quad \text{for all } k \in \mathbb{Z},$$

where $x = (x_k)_{k=-\infty}^{\infty}$ is in $\ell^2(\mathbb{Z})$. Prove the following facts.

a) The point spectrum of S is empty.

b) $\text{range}(\lambda\mathbb{I} - S) = \ell^2(\mathbb{Z})$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$.

c) $\text{range}(\lambda\mathbb{I} - S) = \ell^2(\mathbb{Z})$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.

d) The spectrum of S consists of the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and is purely continuous.

Solution:

a) The point spectrum of S is empty.

Suppose false, that is there exists a $\lambda \in \sigma(S)$ for which $(S - \lambda\mathbb{I})$ is not injective. This would imply that for $x^1, x^1 \in \ell^2(\mathbb{Z})$, $x^1 \neq x^2$, we have that

$$(S - \lambda\mathbb{I})x^1 = (S - \lambda\mathbb{I})x^2.$$

When rearranging, we have that

$$S(x^1 - x^2) = \lambda(x^1 - x^2)$$

This implies that the action that S does to the vector $x^1 - x^2$ simply multiplies it by some λ. Since S is the shift operator, we then have that

$$(x^1 - x^2)_{k-1} = \lambda(x^1 - x^2)_k, \quad \forall k \in \mathbb{Z}$$

Note that $x^1 - x^2 \in \ell^2(\mathbb{Z})$, this means that

$$\sum_{k\in\mathbb{Z}} (x^1 - x^2)_k < \infty.$$

Therefore, its series is bounded. This means, by the relation we found between successive terms, we have

$$\sum_{k\in\mathbb{Z}} \left\|(x^1 - x^2)_k - \lambda(x^1 - x^2)_{k-1}\right\| = 0 \implies \left\|x^1 - x^2 - \lambda x^1 + \lambda x^2\right\| = 0 \implies \left\|(1 - \lambda)(x^1 - x^2)\right\| = 0$$

Since $x^1 \neq x^2$, we have that $|\lambda| = 1$. Plugging $\lambda = 1$, (as an example) back into the equation expressing non-injectivity, we have

$$(S - \mathbb{I})x^1 = (S - \mathbb{I})x^2 = y$$

When rearranging, we have that

$$0 = S(x^1 - x^2) + (x^1 - x^2) = y - x^1 - Sx^1$$

Note that $S(x^1 - x^2) = x^1 - x^2$, implying $S(x^2 - x^1) = x^2 - x^1$ when multiplying both sides by -1. We then get that

$$2(x^1 - x^2) = y - x^1 - Sx^1 = 0 \implies x^1 - x^2 = 0 \implies x^1 = x^2.$$

This violates the non-injectivity of $(S - \mathbb{I})$, implying that λ is not in the point spectrum. Therefore, the point spectrum is empty.

b) range$(\lambda\mathbb{I} - S) = \ell^2(\mathbb{Z})$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$.

Suppose false, that is, there exists some $z \in \ell^2(\mathbb{Z})$ for which $(\lambda\mathbb{I} - S)x \neq z$ for any $x \in \ell^2(\mathbb{Z})$. Taking the inner product of these two values gives us

$$\langle z|(\lambda\mathbb{I} - S)x\rangle \neq \|z\|^2 \iff \langle z|\lambda x\rangle - \langle z|Sx\rangle \neq \|z\|^2.$$

Take $x = z$, then

$$\lambda\|z\|^2 - \langle z|Sz\rangle \neq \|z\|^2 \iff (\lambda - 1)\|z\|^2 \neq \langle z|Sz\rangle$$

Rewriting the norm as an inner product of z with itself, we can rearrange to get that

$$\lambda \langle z|z\rangle \neq \langle z|Sz + z\rangle$$

This implies that $z \neq Sz + z$, so $Sz \neq 0$. Then $z \notin \ker(S)$. Therefore, $z \in$ range(S), so there exists some $y \in \ell^2(\mathbb{Z})$ for which $Sz = y$. Then,

$$(S - \lambda\mathbb{I})z = Sz - \lambda z \neq 0 - 0 \implies z \notin \ker(\lambda\mathbb{I} - S), \implies z \in \text{range}(\lambda\mathbb{I} - S).$$

Therefore, we have found a contradiction, implying that range$(\lambda\mathbb{I} - S) = \ell^2(\mathbb{Z})$.

c) range$(\lambda\mathbb{I} - S) = \ell^2(\mathbb{Z})$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$:

Let us first consider the case when $\lambda = 0$. In this case, we wish to show that range$(\mathbb{I} - S) = \ell^2(\mathbb{Z})$. Suppose this is false, that is, there exists some $z \in \ell^2(\mathbb{Z})$ for which $Sx \neq z$ for any $x \in \ell^2(\mathbb{Z})$. Then, taking the inner product implies

$$\langle z|Sx\rangle \neq \langle z|z\rangle \implies \langle z|Sx - z\rangle \neq 0$$

Let x be defined as the element for which, when S is applied to it, equals z. That is, we take $x = S^*z$, where S^* is the left shift operator. Clearly, the inverse of the right shift operator is the left shift operator (over $\mathbb{Z}$, this is not the case over $\mathbb{N}$). Then,

$$\langle z|SS^*z - z\rangle \neq 0 \implies \langle z|z - z\rangle \neq 0.$$

This is a contradiction. Therefore, range$(S) = \ell^2(\mathbb{Z})$. Note that if $\lambda \neq 0$, the same proof from part b applies, since I did not use $|\lambda| > 1$; only $\lambda \neq 0$.

d) <u>The spectrum of S consists of the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and is purely continuous.</u>

From part a, we found that $\lambda \in \sigma(S)$ only when $|\lambda| = 1$. Thus, the first clause of the statement is true. We just need to show that the spectrum is purely continuous. Since $|\lambda| = 1$ is not in the point spectrum, then necessarily, $(S - \lambda\mathbb{I})$ is surjective (if it wasn't then $\lambda \notin \sigma(S)$. Thus, we need to show that range$(S - \lambda\mathbb{I})$ is dense in $\ell^2(\mathbb{Z})$ when $|\lambda| = 1$. By Theorem 8.17, for a bounded linear operator A defined on Hilbert space $\mathcal{H}$, then $\overline{\text{range}(A)} \oplus \ker(A^*) = \mathcal{H}$. For this theorem to apply, we need to show that $S - \mathbb{I}$ is bounded (it is clearly linear), and that $\ker(A^*) = 0$. Let $x \in \ell^2(\mathbb{Z})$. Then, by Parseval's identity, for and orthonormal basis $\{e_k\}$ for $\ell^2(\mathbb{Z})$, we have

$$\|(S - \lambda\mathbb{I})x\|^2 = \sum_k |\langle e_k|(S - \lambda\mathbb{I})x\rangle|^2 \leq \sum_k |\langle e_k|Sx\rangle|^2 + |\lambda|^2 \sum_k |\langle e_k|x\rangle|^2$$

$$= \|Sx\|^2 + |\lambda|^2\|x\|^2 \leq (\|S\|^2 + |\lambda|^2)\|x\|^2$$

Since the right shift operator is bounded, then $S - \lambda\mathbb{I}$ is bounded. Next, we need to show that $\ker(S^* - \overline{\lambda}\mathbb{I}) = \{0\}$. If $z \in \ker(S^* - \overline{\lambda}\mathbb{I})$, then $S^*z - \overline{\lambda}z = 0$, so $S^*z = \overline{\lambda}z$. Taking the adjoint of both sizes implies that

$$z^*S = \lambda z^* \implies z^*(S - \lambda\mathbb{I}) = 0 \implies z^* \in \ker(S - \lambda\mathbb{I})$$

Note that I am borrowing notation from linear algebra when taking the adjoint. Since we have that $S - \lambda\mathbb{I}$ is injective, then its kernel is equal to zero. Therefore, $z^* = 0$, so $z = 0$. Therefore, $\ker(A^*) = \{0\}$, so $\ell^2(\mathbb{Z}) = \overline{\text{range}(S - \lambda\mathbb{I})}$. Therefore the range of $S - \lambda\mathbb{I}$ is dense in $\ell^2(\mathbb{Z})$, implying that all λ's in the spectrum of S are in the continuous spectrum.

Exercise 9.12

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}$, and $A \in \mathcal{B}(\mathcal{H})$ such that

$$\sum_n \|Ae_n\|^2 < \infty.$$

Exercise 9.12, part a

Prove that the Hilbert-Schmidt norm defined in (9.18) is independent of the basis. That is, show that for any other orthonormal basis $\{f_n\}$ one has

$$\sum_n \|Af_n\|^2 = \sum_n \|Ae_n\|^2.$$

Solution:

By above, and Parseval's identity (both are orthonormal bases of $\mathcal{H}$), we can write,

$$\|A\|_{HS}^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle e_k|Ae_n\rangle|^2 \quad (\text{ AND }) \quad \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle f_k|Ae_n\rangle|^2$$

$$\sum_{n=1}^{\infty} \|Af_n\|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle f_k|Af_n\rangle|^2 \quad (\text{ AND }) \quad \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle e_k|Af_n\rangle|^2$$

These two can be related to each-other by the following: we can rewrite the second line's second implication as

$$\sum_{n=1}^{\infty} \|Af_n\|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\overline{\langle Af_n|e_k\rangle}|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle Af_n|e_k\rangle|^2$$

Since $\|A\| = \|A^*\|$, we can freely interchange the place of A inside the above inner product. Then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle Af_n|e_k\rangle|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle f_n|Ae_k\rangle|^2$$

Note that the two summations, that of the first line and the one directly above, are equivalent (up to summation indices). Therefore,

$$\sum_n \|Af_n\|^2 = \sum_n \|Ae_n\|^2.$$

Exercise 9.12, part b

Prove that

$$\|A\|_{HS} = \|A^*\|_{HS}.$$

Solution:

Without loss of generality, take the orthonormal basis $\{f_k\}$ from above[1]. Then,

$$\|A\|_{HS}^2 = \sum_n \|Af_n\|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle e_k|Af_n\rangle|^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle A^*e_k|f_n\rangle|^2 = \sum_{k=1}^{\infty} \|A^*e_k\|^2 = \|A^*\|_{HS}^2$$

[1]This is without loss of generality since we showed in the previous part that the Hilbert-Schmidt norm is independent of basis.