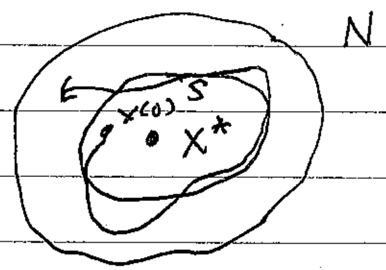
Stability of an equilibrium x^* of $\dot{x} = f(x)$, equilibrium satisfies $f(x^*) = 0$

An equilibrium is <u>Lyapunov stable</u>

if for <u>every neigbborhood</u> N of x^*

there exists a neighborhood $S \subset N$

s.t. for all $x(0) \in S$, $x(t) \in N$ for all $t \geq 0$

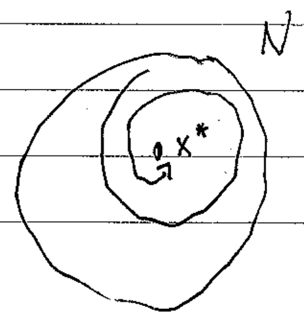


<u>unstable</u>: not Lyapunov stable
(no such neighborhood S exists)

<u>asymptotically stable</u>: Lyapunov stable

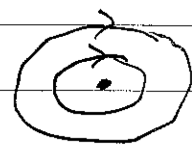
<u>and</u> there exists a neighborhood N s.t.

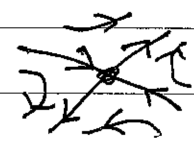
$\lim_{t \to \infty} x(t) = x^*$ for all $x(t) \in N$



example: pendulum $\dot{\Theta} = \Omega$
$\dot{\Omega} = -\sin\Theta$

2 equilibria $x_1^* = (\Theta^*, \Omega^*) = (0,0)$ 
$x_2^* = (\Theta^*, \Omega^*) = (\pi, 0)$ 

x_1^*  Lyapunov stable, but not asymptotically stable

x_2^*  unstable

Another stability concept "linearly stable"

$\dot{x} = f(x)$, $f(x^*) = 0$, f is continuously differentiable (C^1)

let $x = x^* + y$

$$\dot{x} = \dot{y}$$
$$f(x) = f(x^* + y) = f(x^*) + D_x f(x^*) y + \cancel{}$$

linear stability; neglect error term

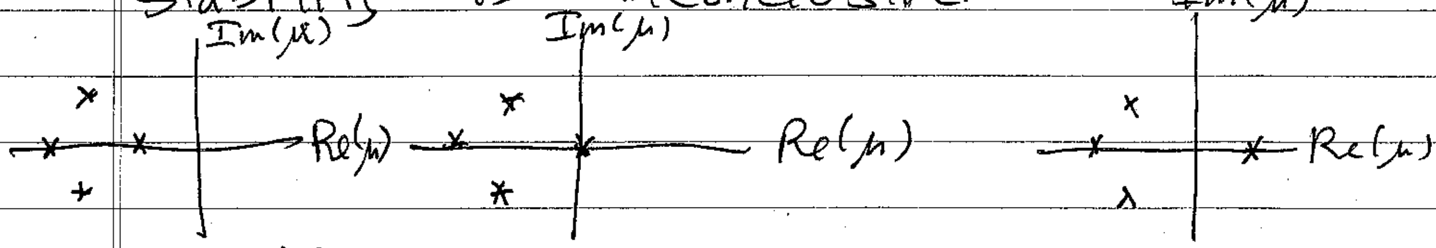
$$\dot{y} = \underbrace{D_x f(x^*)}_{n \times n \text{ constant matrix}} y$$

$$D_x f(x^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdot & — & - \\ & \ddots & & \\ & & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=x^*}$$

← Jacobian matrix

if all eigenvalues of $D_x f(x^*)$ satisfy $Re(\mu) < 0$, then x^* is linearly asymptotically stable.
if <u>any</u> eigenvalue of $D_x f(x^*)$ has $Re(\mu) > 0$ then x^* is linearly unstable

We will show (later) that <u>linear asymptotic stability</u> <u>implies</u> asymptotic stability. Likewise linear instability implies instability.
However, if we find that x^* is neither linearly asymptotically stable, nor unstable, then linearization to determine stability is inconclusive.



asym. stable (?) unstable

homework problem: parametrically ~~forced~~ pendulum

$$\ddot{\Theta} = -\frac{g}{l}\left(1 - \epsilon\cos(\omega t)\right)\sin\Theta$$

forcing frequency

$\frac{g}{l} = \omega_0^2$, where

ω_0 = natural frequency of small amplitude motion $\ddot{\Theta} \approx -\omega_0^2\Theta \Rightarrow \cos(\omega_0 t)$, $\sin(\omega_0 t)$

Question: For what (ϵ, ω)-pairs is the equilibrium $(\Theta, \Omega) = (0,0)$ destabilized? Exploring the impact of resonant forcing...

Linearized problem takes the form

$$\ddot{\Theta} = -\omega_0^2\left(1 - \epsilon\cos(\omega t)\right)\Theta$$

or
$$\begin{pmatrix}\dot{\Theta}\\ \dot{\Omega}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\omega_0^2(1-\epsilon\cos(\omega t)) & 0\end{pmatrix}\begin{pmatrix}\Theta\\ \Omega\end{pmatrix}$$

Detour:

$$\dot{X} = A(t)X, \qquad X \in \mathbb{R}^2$$

$A = 2\times 2$ T-periodic matrix $\left(T = \frac{2\pi}{\omega}\right)$

Question: when is $X=0$ stable/unstable in the parameter space of (ω_0^2/ω^2), ϵ?

First: How not to solve $\dot{X} = A(t)X$

(a) Mistake 1: compute eigenvalues & eigenvectors of $A(t)$, which would be time-dependent and write soln. that way e.g. $X = c_1 v_1(t) e^{\mu_1(t)t} + c_2 v_2(t) e^{\mu_2(t)t}$

You can check - this isn't a soln. ...

(b) Mistake 2: if $\dot{x} = a(t)x$, $x \in \mathbb{R}$, then $x(t) = e^{\int_0^t a(\xi)d\xi} x_0$, so try

$$X = e^{\int_0^t A(s)ds} X_0$$

You can check - this isn't a soln...

$$\dot{X} = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}$$

$$= \lim_{h\to 0} \left[\frac{e^{\int_0^{t+h} A(s)ds} - e^{\int_0^t A(s)ds}}{h} \right] X_0 \stackrel{?}{=} AX$$

$$e^{\int_0^{t+h} A(s)ds} = e^{\int_0^t A(s)ds + \int_t^{t+h} A(s)ds}$$

$$\neq e^{\int_0^t A(s)ds} e^{\int_t^{t+h} A(s)ds}$$

since $\int_0^t A(s)ds$ ~~may~~ may not commute with $\int_t^{t+h} A(s)ds$..

What do we know?

$$\left.\begin{aligned} &\dot{X} = A(t)X \\ &X(0) = X_0 \\ &A(t) = A(t+T) \end{aligned}\right\}$$

linear, homogeneous; 2 linearly independent solns. $x_1(t)$ & $x_2(t)$ needed for general soln.

A Fundamental Matrix Soln. $\Phi(t)$

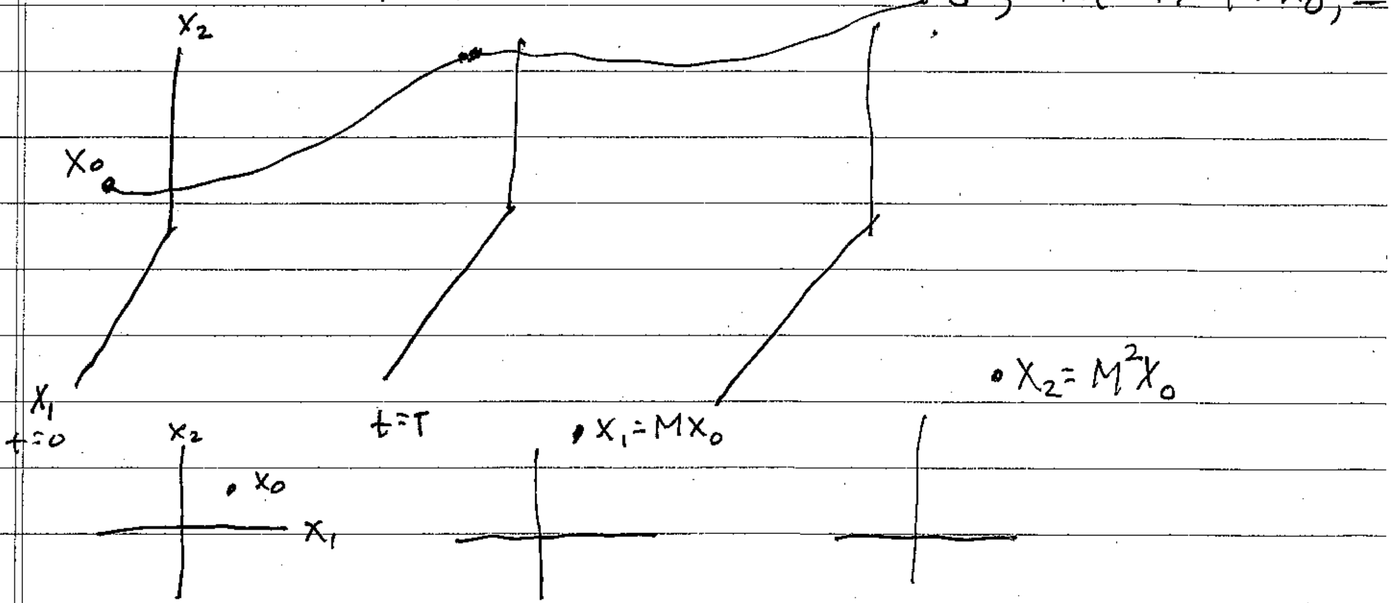
$$\dot{\Phi} = A(t)\Phi, \quad \Phi(0) = Id$$

$$\Rightarrow X(t) = \Phi(t) X_0$$

Useful to examine the Monodromy matrix M to determine is $X=0$ is stable or not.

$$M = \Phi(T)$$

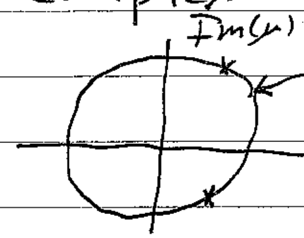
if $X(0) = X_0$, then $X(T) = M X_0$, $X(2T) = M^2 X_0$, —



eigenvalues μ_j of M, called <u>Floquet multipliers</u> determine stability of $X=0$ for $\dot{X}=A(t)X$

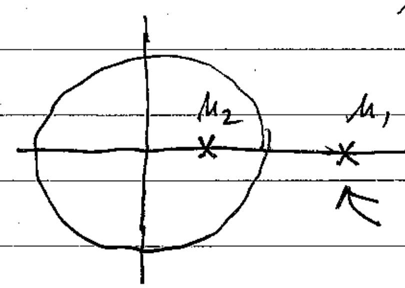
You should prove that either

① eigenvalues ~~[illegible]~~ are complex $\mu_2=\bar{\mu}_1$ and $|\mu|=1$



$|X|$ neither grows <u>nor</u> decays in this case...

or ② eigenvalues are real and $\mu_2 = 1/\mu_1$



instability $|X|$ is growing (exponentially) at rate determined by μ_1

For our problem, we have no way to compute, analytically $\Phi(t)$, but we can at least numerically estimate M.

Fix (α, β)

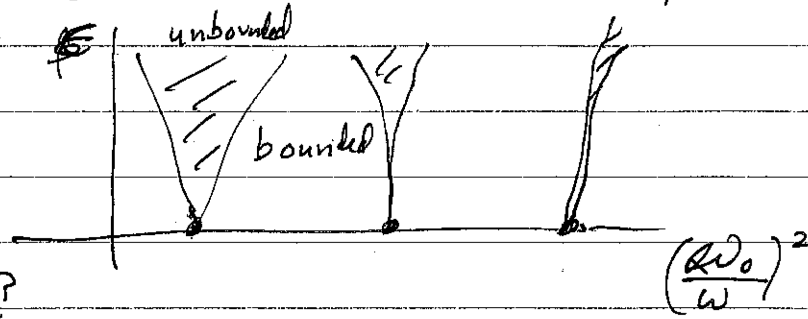
let $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, solve $\dot{X} = A(t) X$ to time T to obtain $M X_0 = \begin{pmatrix} a \\ b \end{pmatrix}$

& also let $\tilde{X}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to get $M\tilde{X}_0 = \begin{pmatrix} c \\ d \end{pmatrix}$

$\Rightarrow \quad M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

explore on a grid in the (α, β)-plane

Should find:



"resonance tongues"

What is associated $\left(\frac{\omega_0}{\omega}\right)$?

<u>Next topic</u>: Existence & Uniqueness of soln. to initial value problem

$$(*) \begin{cases} \dot{X} = f(X) & f: \mathbb{R}^n \to \mathbb{R}^n \\ X(0) = X_0 & X_0 \in \mathbb{R}^n \end{cases}$$

A soln. of $(*)$ exists, on some time interval containing $t=0$ provided f is continuous.