$\dot{X} = F(X;\Lambda)$, $X \in \mathbb{R}^n$, $\Lambda \in \mathbb{R}^k$, F smooth

Defining conditions for a steady state bifurcation: $F(X_0;\Lambda_0) = 0$

$\text{Det}[D_X F(X_0;\Lambda_0)] = 0 \Leftarrow \mu = 0$ is spectrum of $D_X F$

(implicit function theorem cannot be applied to $F(X_0;\Lambda_0) = 0$)

After (extended) center manifold reduction, assuming $\mu = 0$ is simple eigenvalue, Let $\lambda \in \mathbb{R}$ be bifurcation parameter of interest, and assume that the bifurcation happens at $\lambda = 0$, for equilibrium at $x = 0$.

$\dot{x} = f(x;\lambda)$ $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$

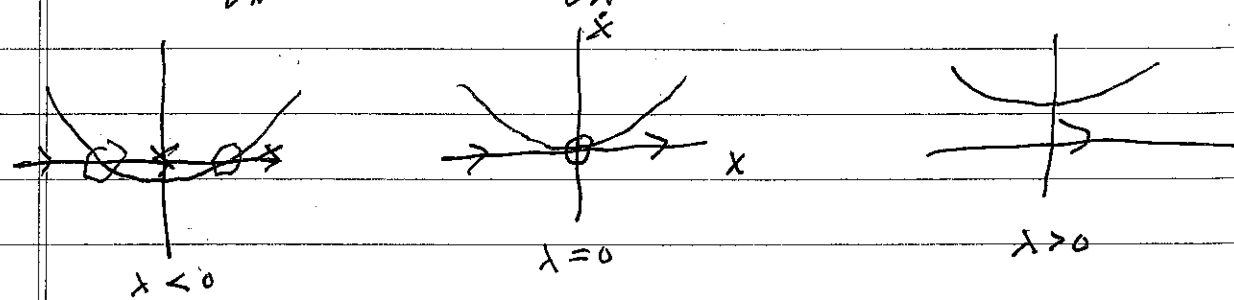
defining conditions for bifurcation are now $f(0;0) = \frac{\partial f}{\partial x}(0;0) = 0$, Taylor expand f in x & λ about $x = \lambda = 0$

$$f(x;\lambda) = \cancelto{0}{f(0;0)} + \cancelto{0}{\frac{\partial f}{\partial x}(0;0)}\, x + \frac{\partial f}{\partial \lambda}(0;0)\lambda$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0;0)x^2 + \frac{\partial^2 f}{\partial x \partial \lambda}(0;0)\, x\lambda + \frac{1}{2}\frac{\partial^2 f}{\partial \lambda^2}(0;0)\lambda^2 + \cdots$$

Saddle-node or fold: $\frac{\partial f}{\partial \lambda}(0,0) \neq 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$

transversality condition non-degeneracy condition if $\frac{\partial^2 f}{\partial x^2}(0,0) > 0$, $\frac{\partial^2 f}{\partial \lambda}(0,0) > 0$

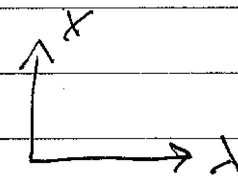


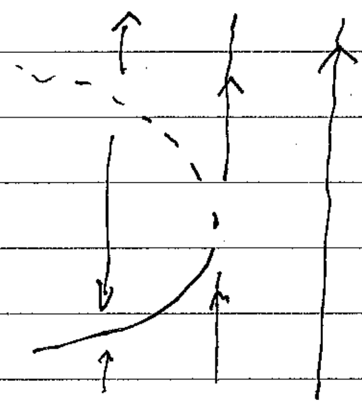
2 equilibria
1 - stable &
1 - unstable

1 unstable
equilibrium

no equilibria

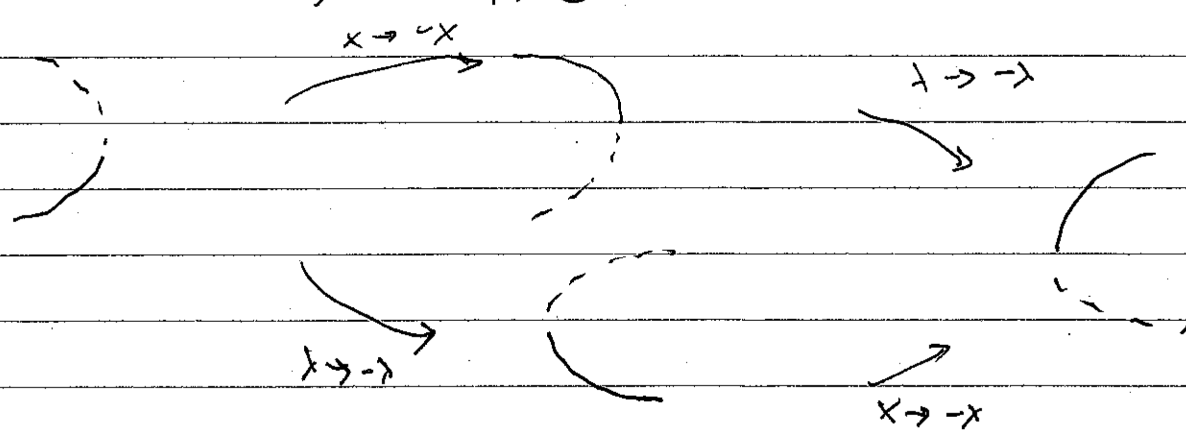
Bifurcation diagram:



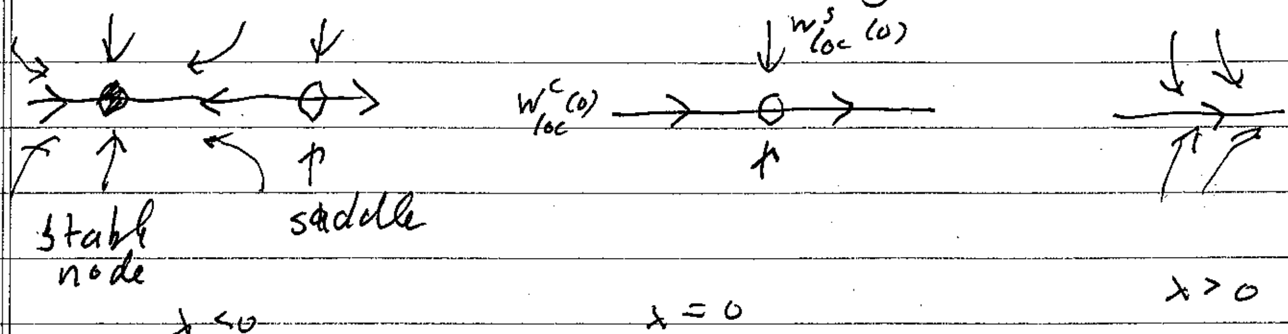


—— stable (on CM)
- - - - unstable

other sign choices lead to diagrams obtained by Flipping about $\dot{x}=0$, $\lambda=0$ or both



Saddle-node terminology comes from higher dimensional cases of this, e.g in $\mathbb{R}^2$



<u>Transcritical bifurcation</u> happens naturally

in problems where there is an equilibrium that exists for all values of the parameter λ, e.g. a zero population state in population dynamics, or a no infection state in epidemiology. These may be some kind of trivial equilibrium

If $x=0$ is always an equilibrium

$$\Rightarrow \quad \dot{x} = f(x,\lambda) = x\, g(x;\lambda)$$

$$\Rightarrow \quad \frac{\partial f}{\partial \lambda}(0;0)\Big|_{x=\lambda=0} = x\,\frac{\partial g}{\partial \lambda}(0,0)\Big|_{x=\lambda=0} = 0$$

, so our transversality condition is violated automatically

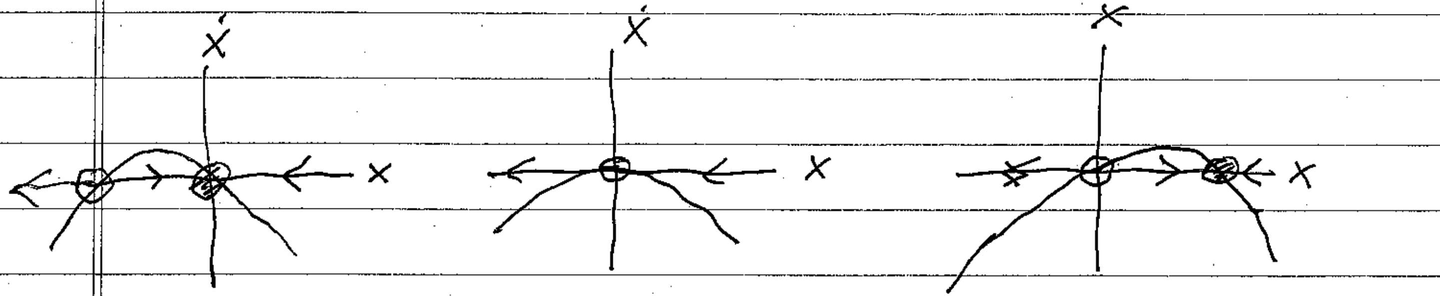
it's replaced by $\dfrac{\partial^2 f}{\partial x \partial \lambda}(0,0) \neq 0$

$f(0;0) = 0, \quad \frac{\partial f}{\partial x}(0;0) = 0; \quad \frac{\partial f}{\partial \lambda}(0;0) = 0$

$\frac{\partial^2 f}{\partial x^2}(0;0) \neq 0, \quad \frac{\partial^2 f}{\partial x \partial \lambda}(0;0) \neq 0 \quad \Rightarrow$ transcritical

example: $\dot{x} = x(\lambda - x) = \lambda x - x^2$

$\left(\frac{\partial^2 f}{\partial x \partial \lambda} > 0, \quad \frac{\partial^2 f}{\partial x^2} < 0 \right)$

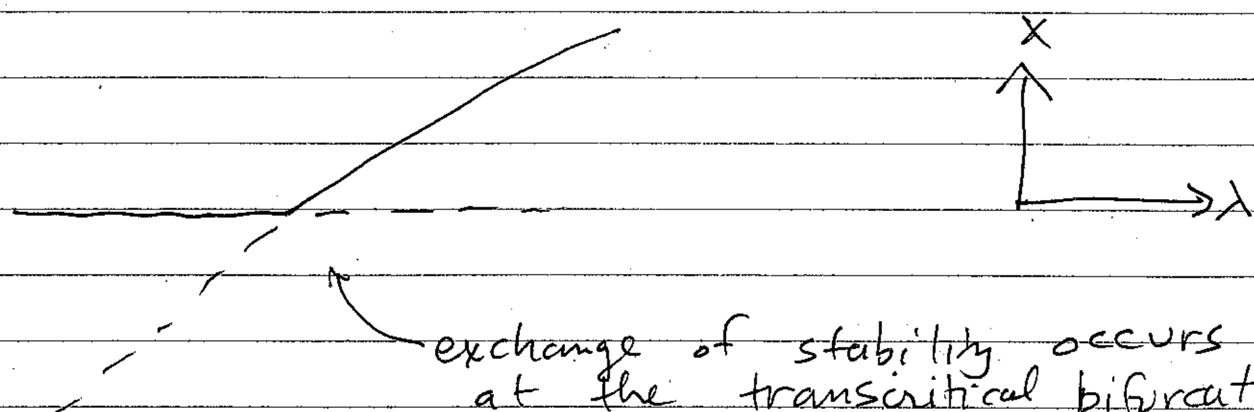


$\lambda < 0$
1 unstable
1 stable at $x=0$

$\lambda = 0$
1 unstable equilibrium

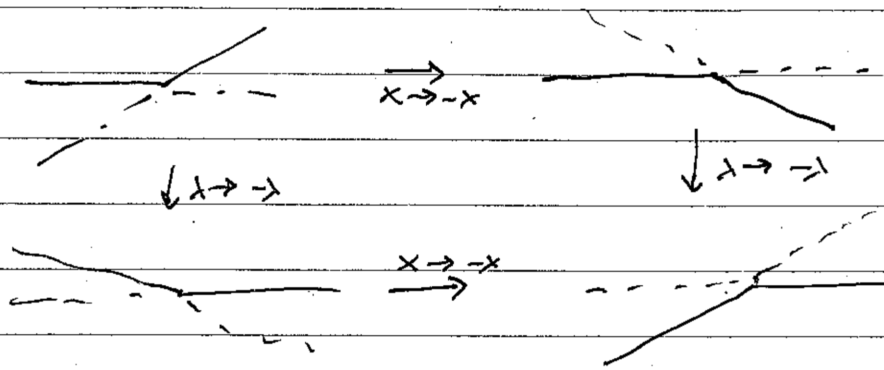
$\lambda > 0$
1 unstable
1 (positive) stable

Bifurcation Diagram



exchange of stability occurs at the transcritical bifurcation from the "trivial equilibrium" ($x=0$) to the nontrivial one ($x \neq 0$)

Again, we can find all versions via reflections in x & λ:



Final example: pitchfork bifurcation occurs as the generic bifurcation in systems with a reflection symmetry $x \to -x$.

If $\tilde{x}(t)$ solves $\dot{x} = f(x)$, then so does $-\tilde{x}(t)$. Implications on form of $f(x)$:

$$\dot{\tilde{x}} = f(\tilde{x})$$
$$\Rightarrow (-\tilde{x})^{\cdot} = f(-\tilde{x})$$
$$= -f(\tilde{x})$$

f is Z_2-equivariant ~~there~~

$$-f(x) = f(-x) \qquad \Rightarrow f(x) \text{ is odd in } x$$

$$\Rightarrow \dot{x} = x\, g(x^2; \lambda)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = 0, \quad \frac{\partial f}{\partial \lambda}(0,0) = 0,$$

~~non-genericity~~ by symmetry

Two distinct cases to consider here

$\frac{\partial^3 f}{\partial x^3}(0,0) > 0$ & $\frac{\partial^3 f}{\partial x^3}(0,0) < 0$

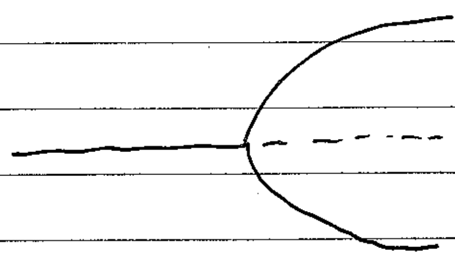
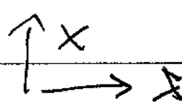
examples:

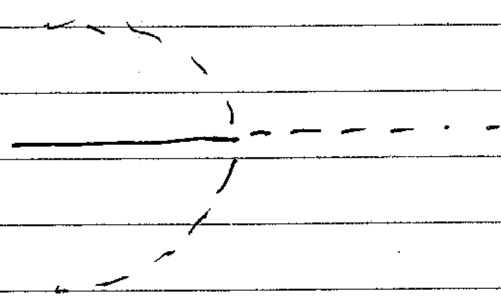
(1) $\dot{x} = \lambda x - x^3$

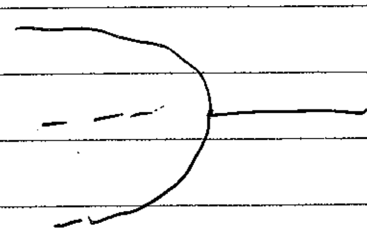
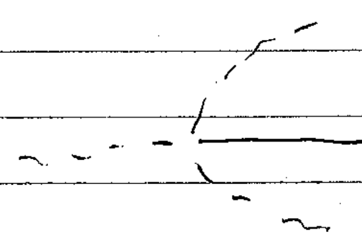
(2) $\dot{x} = \lambda x + x^3$

we can't get (2) from (1) by letting $x \to -x$ since this ~~has that~~ ~~so has this symmetry~~ is symmetric under $x \to -x$.

Bifurcation Diagrams:

(1)  "supercritical pitchfork" 

(2)  "subcritical pitchfork"

$\lambda \to -\lambda$:  

supercritical subcritical

<u>Example</u> : Lorenz equations, supercritical pitchfork at $r=1$.

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - xz - y \\ \dot{z} &= xy - bz \end{aligned}\right\} \text{ consider } r \text{ as bifurcation parameter}$$

reflection symmetry $(x,y,z) \to (-x,-y,z)$

$$\gamma = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \qquad \gamma^2 = I$$

$$\dot{X} = F(X) \quad \& \quad \gamma F(X) = F(\gamma X) \qquad Z_2\text{-equivariant}$$

~~trivial equilibrium exists because of this symmetry~~ $\cancel{F(0) = F(\gamma 0) = \gamma F(0)}$

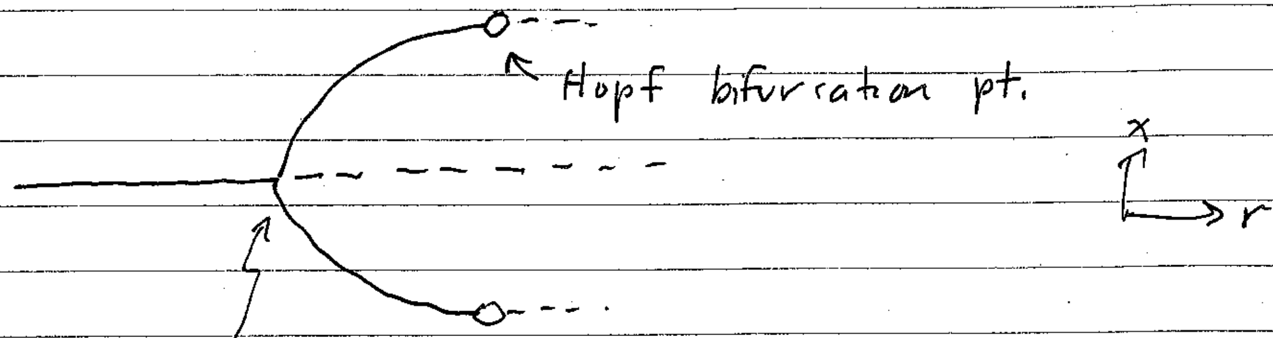
trivial equilibrium $X_0 = 0$ exists for all r
& it is symmetric : $\gamma X_0 = X_0$.

other equilibria?

$$\begin{aligned} x &= y \\ z &= x^2/b \\ rx - x - x^3/b &= 0 \\ b(r-1) &= x^2 \end{aligned}$$

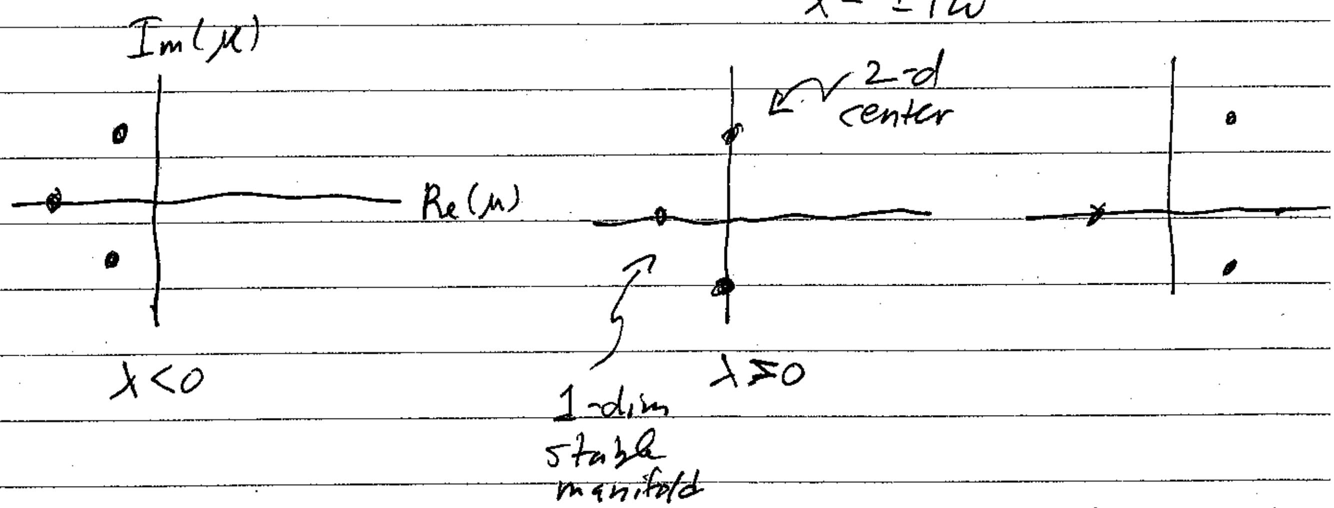
$$X_\pm := (x,y,z)_\pm = \left(\pm\sqrt{b(r-1)},\ \pm\sqrt{b(r-1)},\ r-1\right)$$

$$\gamma X_+ = X_- \quad , \text{ related by symmetry}$$



supercritical pitchfork at $r=1$

<u>Hopf bifurcation</u>: $D_x F(X_0; \lambda_0)$ has a purely imaginary pair of eigenvalues $\lambda = \pm i\omega$



note: $\pm i\omega$ good estimate for frequency of limit cycle produced there

~~the center eigenspace~~

linear problem: