STAT 31210: Homework 3

Caleb Derrickson

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Collaborators: The TA's of the class, as well as Kevin Hefner, and Alexander Cram.

Contents

Problem 5.3

Let $\delta : C([0,1]) \to \mathbb{R}$ be the linear functional that evaluated an function at the origin: $\delta(f) = f(0)$. If $C([0,1])$ is equipped with the sup-norm,

$$\|f\|_1 = \int_0^1 |f(x)|\, dx,$$

show that δ is unbounded.

Solution:

Note that $\delta f \in \mathbb{R}$, so $|\delta f| = |f(0)|$. By definition, $|f(0)| \le \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty$. Then

$$\|\delta\| = \sup \frac{\|\delta f\|}{\|f\|_\infty} = \sup \frac{|f(0)|}{\|f\|_\infty} = \sup \frac{|f(0)|}{\sup |f(x)|} \le 1$$

This implies that δ is bounded. To compute the norm, we just need to find a function that achieves its max value at $x = 0$. The simplest case is to take f be a nonzero constant function, i.e. $f(x) = 2$ for all $x \in [0,1]$. Then $|f(0)| = 2$, and $\sup |f(x)| = |f(0)| = 2$. Then $\|\delta\| = 1$.

To show that when $C([0,1])$ equipped with the one-norm makes $\|\delta\|$ unbounded, we can consider the family of functions

$$\mathbb{O} = \{\{1 - nx : x \in \left[0, \frac{1}{n}\right], \text{else } 0\}, n \in \mathbb{N}\}.$$

Taking a member of that family, we can note that $\|\delta f\| = |f(0)| = 1$, and

$$\|f\|_1 = \int_0^1 |f(x)| = \int_0^{\frac{1}{n}} 1 - nx\, dx = \frac{1}{2n}$$

This implies $\|\delta\| = \sup\{2n\} = \infty$, which is unbounded.

Problem 5.7

Find the kernel and range of the linear operator $K : C([0,1]) \to C([0,1])$ defined by

$$Kf(x) = \int_0^1 \sin\left(\pi(x-y)\right) f(y)\, dy.$$

Solution: we can use an trigonometric identity to expand the operator to

$$Kf(x) = \int_0^2 \left[\sin(\pi x)\cos(\pi y) - \cos(\pi x)\sin(\pi y)\right] f(y)\, dy.$$

The kernel of K would then be functions $f(x)$ such that

$$\int_0^1 \cos(\pi y) f(y)\, dy = \int_0^1 \sin(\pi y) f(y)\, dy.$$

I have removed the x-term, since this should hold for any x. Since the sine and cosine functions are orthogonal to each-other, we need to fund functions which are orthogonal to both. As a means of testing, we can see that $\cos(2n\pi x)$ is orthogonal to $\cos(\pi x)$ and $\sin(2n\pi x)$ is orthogonal to $\sin(\pi x)$. If we multiply these two functions together, i.e., taking $f(x) = \cos(2n\pi)\sin(2m\pi)$ with $m \neq n$, we can integrate both sides and see this function is orthogonal to both $\sin(\pi x)$ and $\cos(\pi y)$. Then the kernel of the linear operator K is the family of functions of the form $\{\cos(2n\pi)\sin(2m\pi) : n \neq m\}$. The range of this linear operator are functions which are orthogonal to this family, which was found to be $\sin(\pi x)$ and $\cos(\pi x)$. Then taking functions of the form $\{\cos(2n\pi)\sin(2n\pi) : n \in \mathbb{N}\}$ gives you a value for both integrals.

Problem 5.8

Prove that equivalent norms on a normed linear space X lead to equivalent norms on the space $\mathfrak{B}(X)$ of bounded linear operators on X.

Solution:

Suppose we have two norms $\|\cdot\|_1,\ \|\cdot\|_2$ on X. We are given these are equivalent norms, so there exists $c_1, c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_2 \quad \forall x \in X.$$

Take $T \in \mathfrak{B}(X)$, and suppose we have two norms $\|\cdot\|_1',\ \|\cdot\|_2'$ on $\mathfrak{B}(X)$. We have that T is bounded, so $\|T\|_1'$ and $\|T\|_2'$ are defined. We can then write

$$\|Tx\|_1 \leq \|T\|_1'\|x\|_1 \leq \|T\|_1'\frac{1}{c_1}\|x\|_2 = \frac{1}{c_1}\|T\|_1' \leq \sup_{\|x\|=1}\|Tx\| = \|T\|_2'.$$

I've condensed some steps in the above lines. I've assumed that $\|x\|_2 = 1$, and from the first inequality, I substituted in $\|T\|_1'$ for any norm on T. We now have $\|T\|_1' \leq c_1\|T\|_2'$. In the other direction, we can write

$$\|Tx\|_2 \leq \|T\|_2'\|x\|_2 \leq \|T\|_2' c_2\|x\|_1 = c_2\|T\|_2' \leq \sup_{\|x\|=1}\|Tx\| = \|T\|_1'.$$

I have condensed the steps above in the same way as I did in the previous. From this, we have $c_2\|T\|_2' \leq \|T\|_2'$, therefore, we have

$$c_2\|T\|_2' \leq \|T\|_1' \leq c_1\|T\|_2',$$

this shows the equivalence of norms on $\mathfrak{B}(X)$.

Problem 5.14

Suppose that A is an $n \times n$ matrix. For $t \in \mathbb{R}$ we define $f(t) = \det e^{tA}$.

Problem 5.14, part a

Show that

$$\lim_{t \to 0} \frac{f(t) - 1}{t} = \text{tr}A$$

Solution:

Since the latter parts of this problem are true only for A being a diagonalizable matrix, I will assume it here as well. Then A can be expressed as $A = Q\Lambda Q^{-1}$, where Q is a change of basis matrix, and Λ is a diagonal matrix whose entries consists of the eigenvalues of A. If we assume this, we can rewrite $f(t)$ as

$$f(t) = \det e^{tA} = \det e^{tQ\Lambda Q^{-1}} = \det \sum_{k=0}^{\infty} \frac{(tQ\Lambda Q^{-1})^k}{k!} = \det Q \left(\sum_{k=0}^{\infty} \frac{(t\Lambda)^k}{k!} \right) Q^{-1} = \det \left(Q e^{t\Lambda} Q^{-1} \right)$$

By the properties of the determinant, this is just equal to $f(t) = \det \left(e^{t\Lambda} \right)$, since the determinant of the product is the product of the determinants. Also, the determinant of the inverse is the inverse of the determinant. The limit is then,

$$\lim_{t \to 0} \frac{\det e^{t\Lambda} - 1}{t}$$

There is some further simplification to be done to $f(t)$. Since Λ is a diagonal matrix, when we take its matrix exponential, it will just exponentiate each element of Λ . Then taking the determinant of a diagonal matrix is just the product of the elements. Therefore, $f(t) = e^{t(\lambda_1 + \lambda_2 + \cdots + \lambda_n)} = e^{t \operatorname{Tr} A}$. Rewriting 1 as e^0, and taking the series definition of e^x, we get that the limit turns into

$$\lim_{t \to 0} \frac{e^{t \operatorname{Tr} A} - e^0}{t} = \lim_{t \to 0} \frac{1}{t} \sum_{k=1}^{\infty} \frac{t^k (\text{tr}A)^k}{k!} = \lim_{t \to 0} \left[\text{tr}A + \sum_{k=2}^{\infty} \frac{t^k (\text{tr}A)^k}{k!} \right] = \text{tr}A$$

Problem 5.14, part b

Deduce that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and is a solution of the ODE $\dot{f} = (\text{tr}A)f$.

Solution:

By the definition of the derivative, we can write the following:

$$
\begin{aligned}
\dot{f} &= \lim_{h\to 0}\left[\frac{f(t+h)-f(t)}{h}\right] && \text{(By definition.)}\\
&= \lim_{h\to 0}\left[\frac{\det e^{(t+h)A} - \det e^{tA}}{h}\right] && \text{(Substitution.)}\\
&= \lim_{h\to 0}\left[\frac{\det e^{tA+hA} - \det e^{tA}}{h}\right] && \text{(Rearranging.)}\\
&= \lim_{h\to 0}\left[\frac{\det\left(e^{tA}e^{hA}\right) - \det e^{tA}}{h}\right] && \text{(}A\text{ commutes with itself.)}\\
&= \lim_{h\to 0}\left[\frac{\det e^{tA}\det e^{hA} - \det e^{tA}}{h}\right] && \text{(Determinant property.)}\\
&= \lim_{h\to 0}\left[\frac{\det e^{tA}(\det e^{hA} - 1)}{h}\right] && \text{(Grouping.)}\\
&= \det e^{tA}\lim_{h\to 0}\left[\frac{\det e^{hA} - 1}{h}\right] && \text{(Independent of limit.)}\\
&= \det e^{tA}\text{tr}A && \text{(By part a.)}\\
\dot{f} &= (\text{tr}A)f && \text{(Substitution.)}
\end{aligned}
$$

Problem 5.14, part c

Show that

$$\det e^{A} = e^{\mathrm{tr}A}$$

Solution:

This was derived *in spirit* in the above parts, but was not explicitly shown. I will do this here. Note that this is only valid for diagonalizable matrices, since if A were not diagonalizable, then by Jordan Canonical Transformation, we would have

$$\det e^{tA} = \det\left(e^{D}\right)\det\left(e^{N}\right),$$

where D is akin to Λ in the previous parts, and N is a nilpotent matrix which contains the "non-diagonalizability" of A. This cannot be removed, but will terminate in finite iterations of the summation.

We will now prove the statement. Since A is diagonalizable, $A = Q\Lambda Q^{-1}$ for some change of basis matrix Q and Λ as described above. Note that taking any integer power of A will give back

$$A^{k} = (Q\Lambda Q^{-1})^{k} = (Q\Lambda Q^{-1})\underset{\text{k-times}}{\ldots}(Q\Lambda Q^{-1})$$

Via association, this can be written as $A^{k} = Q\Lambda^{k}Q^{-1}$. Therefore, when taking the matrix exponential,

$$e^{A} = \sum_{k=0}^{\infty}\frac{A^{k}}{k!} = \sum_{k=0}^{\infty}\frac{(Q\Lambda Q^{-1})^{k}}{k!} = Q\left(\sum_{k=0}^{\infty}\frac{\Lambda^{k}}{k!}\right)Q^{-1} = Qe^{\Lambda}Q^{-1}.$$

When taking the determinant, the determinant of the product is the product of the determinants, as well as the determinant of the inverse is the inverse of the determinants.[1] Therefore, the determinant of the matrix exponential will simplify to e^{Λ}. One property of exponentiating a diagonal matrix is that its entries are raised. We can then write

$$e^{\Lambda} = e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)}.$$

Finally, the sum of the eigenvalues of A is equal to its trace. We then end with $e^{\Lambda} = e^{\mathrm{tr}A}$, which is what we wanted to show.

[1] I like the way these sound, don't judge me.

Problem 5.15

Suppose that A and B are bounded linear operators on a Banach space.

Problem 5.15, part a

If A and B commute, then prove that $e^A e^B = e^{A+B}$.

Solution:

We will go straight into calculations.

$$
\begin{aligned}
e^A e^B &= \sum_{i=0}^{\infty} \frac{A^i}{i!} \sum_{j=0}^{\infty} \frac{B^j}{j!} && \text{(Given.)} \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{A^i B^j}{i! j!} && \text{(Limit exists for both summations.)} \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{A^i B^{(k-i)}}{i!(k-i)!} && \text{(Substituting } k = i + j\text{.)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} k! \frac{A^i B^{(k-i)}}{i!(k-i)!} && \text{(Multiplying by a 1.)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} A^i B^{(k-i)} && \text{(Rearranging.)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} (A+B)^k && \text{(Binomial Theorem, } [A, B] = 0\text{.)} \\
&= e^{(A+B)} && \text{(By definition.)}
\end{aligned}
$$

Problem 5.15, part b

If $[A,[A,B]] = [B,[A,B]] = 0$, then prove that

$$e^A e^B = e^{A+B+[A,B]/2}$$

Solution:

Note that since A and B commute with $[A,B]$, $e^{A+B+[A,B]/2} = e^{A+B}e^{[A,B]/2}$. Then showing $e^A e^B e^{-[A,B]/2}$ equals e^{A+B} is equivalent to showing the given statement. Define

$$X(t) = e^{tA}e^{tB}e^{-t^2[A,B]/2}$$

$$Y(t) = e^{t(A+B)}$$

Note that $X(t=0) = \mathbb{I} = Y(t=0)$, thus if we show that $X(t)$ and $Y(t)$ solve the same differential equation, then by uniqueness of solutions of ODE's with defined initial conditions, $X(t) = Y(t)$. The right side is simple to differentiate:

$$\frac{dY}{dt} = \frac{d}{dt}e^{t(A+B)} = e^{t(A+B)}(A+B) = Y(t)(A+B).$$

The right hand side however is more involved. We can first apply the product rule to get

$$\frac{dX}{dt} = \frac{d}{dt}\left[e^{tA}e^{tB}e^{-t^2[A,B]/2}\right] = \left[\frac{d}{dt}e^{tA}\right]e^{tB}e^{-t^2[A,B]/2} + e^{tA}\left[\frac{d}{dt}e^{tB}\right]e^{-t^2[A,B]/2} + e^{tA}e^{tB}\left[\frac{d}{dt}e^{-t^2[A,B]/2}\right]$$

Since any matrix C commutes with its matrix exponential, differentiating the first two terms are an equivalent process. We will then handle the third term separately.

$$\begin{aligned}
\frac{d}{dt}e^{-t^2[A,B]/2} &= \frac{d}{dt}\sum_{k=0}^{\infty}\left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k && \text{(Given.)}\\
&= \sum_{k=0}^{\infty}\frac{d}{dt}\left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k && \text{(Sum independent of differentiation.)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(2k)(t^{2k-1})}{2^k k!}([A,B])^k && \text{(Differentiating.)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(t^{2k-1})}{2^{(k-1)}(k-1)!}([A,B])^k && \text{(Simplifying.)}\\
&= (-t[A,B])\sum_{k=0}^{\infty}\frac{(-1)^{(k-1)}t^{2(k-1)}}{2^{(k-1)}(k-1)!}([A,B])^{(k-1)} && \text{(Pulling out extra terms.)}
\end{aligned}$$

$$= (-t[A,B]) \sum_{k=0}^{\infty} \left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k \qquad (k-1 \to k.)$$

$$= (-t[A,B]) e^{-t^2[A,B]/2} \qquad \text{(By definition.)}$$

Therefore, we can write

$$\frac{dX}{dt} = e^{tA} A e^{tB} e^{-t^2[A,B]/2} + e^{tA} e^{tB} B e^{-t^2[A,B]/2} + e^{tA} e^{tB} e^{-t^2[A,B]/2} (-t[A,B])$$

I claim that if B commutes with $[A,B]$, then B commutes with the matrix exponential of $[A,B]$. I will show this below. Note that

$$Be^{(-t^2[A,B]/2)} = B\left(\sum_{k=0}^{\infty} \left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k\right) = \lim_{n\to\infty}\left(B\sum_{k=0}^{n} \left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k\right)$$

I will then show that, by induction on n, that the partial sums and B commute.

- Base case: $n = 0$

  Then the partial summation is just the term evaluated at $k = 0$, so

$$B\sum_{k=0}^{n} \left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k = B\left(\frac{-t^2}{2}\right)^0 \frac{1}{0!}([A,B])^0 = B\mathbb{I} = \mathbb{I}B = \sum_{k=0}^{n} \left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k B$$

- Induction Step:

  We will next suppose this property holds up to some $j < n$ case. We will then show that the $j+1$ case follows. Then

$$B\sum_{k=0}^{j+1} \left(\frac{-t^2}{2}\right)^k \frac{1}{k!}([A,B])^k = B\left(\text{J - case} + \frac{-t^2}{2}([A,B])\right) = \left(\text{J - case} + \frac{-t^2}{2}([A,B])\right)B$$

  Note that the last equality is allowed since B commutes with both terms - the first term from the induction hypothesis, and the second term is by assumption of the problem. Therefore, the claim has been shown by induction.

We now have that

$$\frac{dX}{dt} = e^{tA} A e^{tB} e^{-t^2[A,B]/2} + e^{tA} e^{tB} e^{-t^2[A,B]/2} B + e^{tA} e^{tB} e^{-t^2[A,B]/2} (-t[A,B])$$

We ideally want to rewrite the first term in the same form as the last two. Investigating the first few terms of Ae^{tB}, we can see the following:

$$
\begin{aligned}
Ae^{tB} &= e^{tB}e^{-tB}Ae^{tB} && \text{(Multiplying by } \mathbb{I}.\text{)} \\
&= e^{tB}\left[\left(\sum_{k=0}^{\infty}\frac{(-tB)^k}{k!}\right)A\left(\sum_{j=0}^{\infty}\frac{(tB)^j}{j!}\right)\right] && \text{(Expanding.)} \\
&= e^{tB}\left[(\mathbb{I} - tB + \frac{t^2}{2}b^2 - \dots)A(\mathbb{I} + tB + \frac{t^2}{2}B^2 + \dots)\right] && \text{(First terms.)} \\
&= e^{tB}\left[(\mathbb{I} - tB + \frac{t^2}{2}b^2 - \dots)(A + tAB + \frac{t^2}{2}AB^2 + \dots)\right] && \text{(Multiplying.)} \\
&= e^{tB}\left[A + tAB + \frac{t^2}{2}AB^2 - tBA - t^2BAB - \frac{t^3}{2}BAB^2 + \frac{t^2}{2}B^2A + \dots\right] && \text{(Expanding.)} \\
&= e^{tB}\left[A + t(AB - BA) + \frac{t^2}{2}(AB^2 - 2BAB + B^2A) + \dots\right] && \text{(Grouping.)}
\end{aligned}
$$

Expanding $[B, [A, B]]$, we see that

$$[B, [A, B]] = B(AB - BA) - (AB - BA)B = -(AB^2 - 2BAB + B^2A)$$

This is equivalent to the third term in the expansion. Since we are assuming this term equals zero, the expansion equals zero after the second term[2]. We see that

$$Ae^{tB} = e^{tB}\left[A + t(AB - BA)\right],$$

so we can finally rewrite the first term as

$$e^{tA}Ae^{tB}e^{-t^2[A,B]/2} = e^{tA}e^{tB}\left[A + t[A, B]\right]e^{-t^2[A,B]/2} = e^{tA}e^{tB}e^{-t^2[A,B]/2}\left[A + t[A, B]\right]$$

The last equality holds via my argument above. We have that

$$
\begin{aligned}
\frac{dX}{dt} &= e^{tA}e^{tB}e^{-t^2[A,B]/2}\left[A + t[A, B]\right] + e^{tA}e^{tB}e^{-t^2[A,B]/2}B + e^{tA}e^{tB}e^{-t^2[A,B]/2}(-t[A, B]) \\
&= X(t)\left[A + t[A, B] + B - t[A, B]\right] \\
&= X(t)\left[A + B\right]
\end{aligned}
$$

[2]This doesn't *immediately* prove that, but terms further in the expansion has this term nested in it, so all higher terms equal zero.

We can now rejoice, since $X(t)$ and $Y(t)$ solve the same differential equation with the same initial condition. Thus by uniqueness of the solution, $X(t) = Y(t)$, which, when setting $t = 1$, we have that

$$e^A e^B e^{-[A,B]/2} = e^{A+B} \implies e^A e^B = e^{A+B} e^{[A,B]/2} = e^{A+B+[A,B]/2}$$

Which is what we wanted to show.