Lectures 9-10: Linear Programming

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Linear Programming

Given real valued variables: $x_1, \dots, x_n$

$$\text{minimize/maximize} \sum_{i=1}^{n} c_i x_i$$

subject to a set linear constraints. Each constraint has one of the following forms.

- $\sum_{i=1}^{n} a_i x_i \geq b$
- $\sum_{i=1}^{n} a_i x_i = b$
- $\sum_{i=1}^{n} a_i x_i \leq b$

Examples

Given real valued variables: $x_1, \ldots, x_4$

minimize $x_1 + 2x_3 - 5x_4$

s.t.

- $3x_1 + 2x_4 = 1$
- $x_2 - 2x_3 \leq 0$
- $x_1 + 2x_2 \geq 2$

Examples

The max flow problem can be formulated as an LP.

Variables: $f_{\mathrm{e}} \equiv f(e)$ for all edges $e \in E$.

Objective: $\sum f_{(s,u)}$

Constraints:

- $f_e \geq 0$ for all e
- $f_e \leq c(e)$ for all e
- $\sum_v f(u,v) - \sum_w f(w,u) = 0$ for all $u \notin \{s,t\}$

Definitions

- x is feasible solution if it satisfies all the LP constraints
- x^* is an optimal solution if it is a feasible solution and
  - (maximization) $\sum c_i x_i^* \geq \sum c_i x_i'$ for every feasible solution x'
  - (minimization) $\sum c_i x_i^* \leq \sum c_i x_i'$ for every feasible solution x'

- A linear program is feasible if it has a feasible solution.
- An LP is infeasible if it doesn't have a feasible solution.
- An LP is bounded if it has an optimal solution.
- An LP is unbounded if it is feasible and doesn't have an optimal solution.

Bounded and unbounded LPs

- An LP is bounded if it has an optimal solution.
- An LP is unbounded if it is feasible and doesn't have an optimal solution.

Consider an LP. Let $f(x) = \sum c_i x_i$ be the objective function.

If the LP is bounded, then it has an optimal solution x^*. Accordingly,

$$f(x') \leq f(x^*) \text{ (if LP asks to maximize } f(x)\text{)}$$

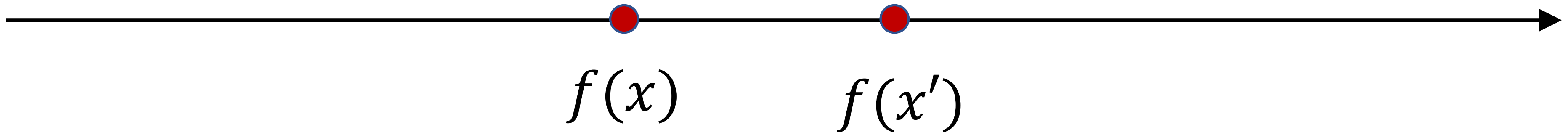
That is, f is bounded on the set of feasible solutions.

Bounded and unbounded LPs

- An LP is bounded if it has an optimal solution.
- An LP is unbounded if it is feasible and doesn't have an optimal solution.

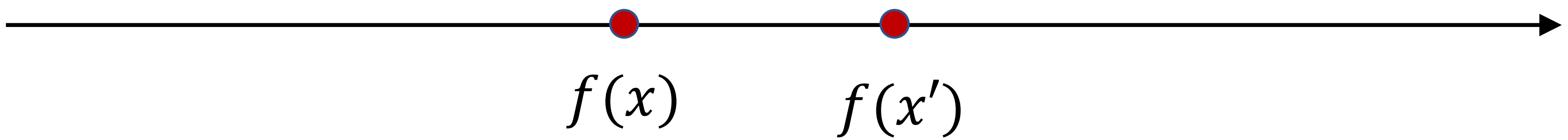
If the LP is unbounded, then for every solution x there is a better solution x'

$$f(x') \geq f(x) \text{ (if LP asks to maximize } f(x)\text{)}$$



Claim: $f(x)$ is unbounded on the set of feasible solutions.

Bounded and unbounded LPs



Claim: $f(x)$ is unbounded on the set of feasible solutions.

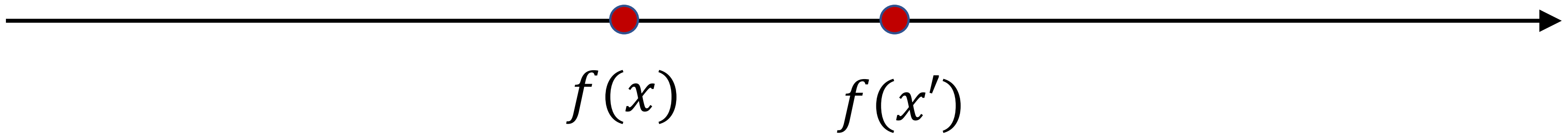
Proof: Since the set of feasible solutions is closed and f is continuous,

$$A = \{f(x): x \text{ is a feasible solution}\}$$

is a closed set. Therefore, A equals

- $[a, b]$
- $(-\infty, b]$
- $[a, \infty)$
- $(-\infty, \infty)$

Bounded and unbounded LPs



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- $[a, b]$
- $(-\infty, b]$

Then $b = f(x)$ for some x. We have, $f(x') \leq b = f(x)$. Thus, x is optimal.

Geometric interpretation

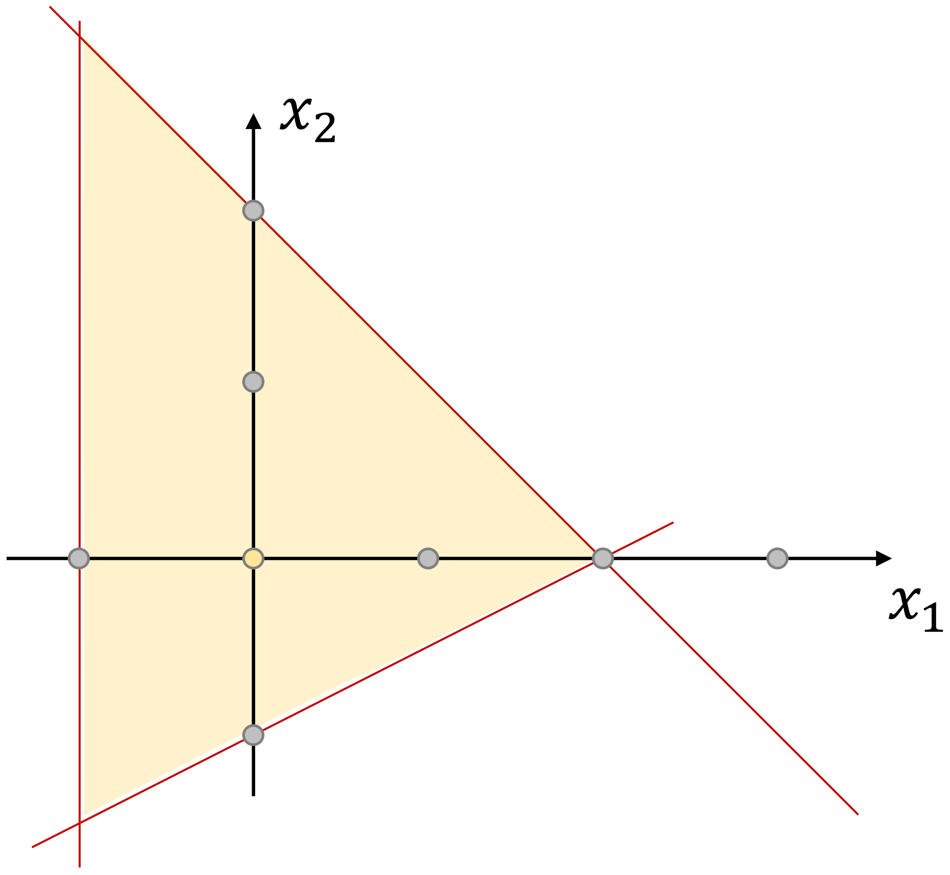
Example:

$$\max 2x_1 + x_2$$

s.t.

$$x_1 + x_2 \leq 2$$
$$x_1 \geq -1$$
$$2x_1 - x_2 \geq 1$$

Here: the set of feasible solutions is a triangle.

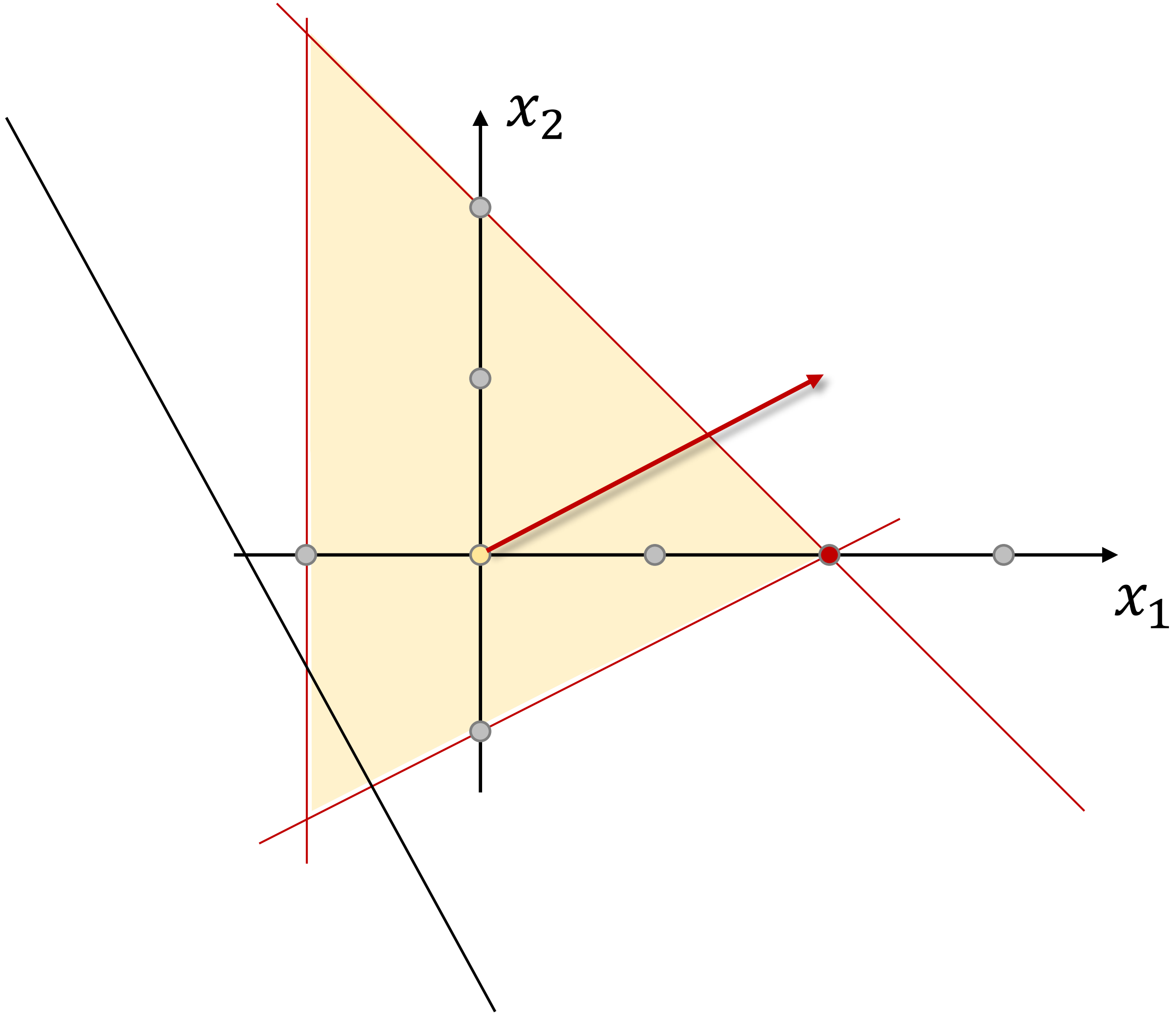


Geometric interpretation

Example:

$$\max\ 2x_1 + x_2$$

$$f(x) = 2x_1 + x_2 = (2,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



Geometric interpretation

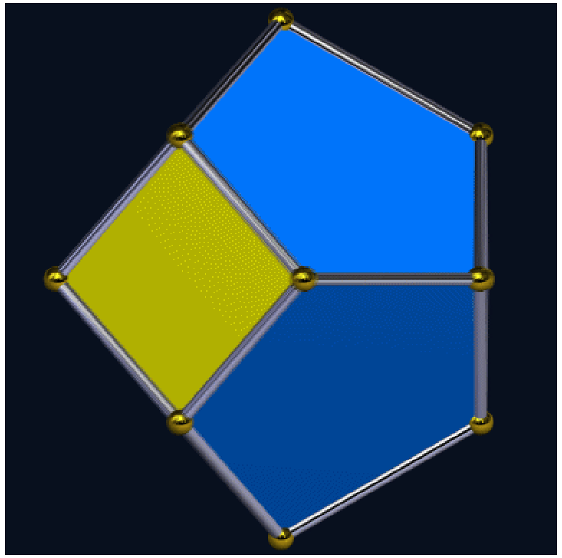
Each constraint $\sum a_i x_i \leq b$ defines a half-space in $\mathbb{R}^n$ bounded by hyperplane $\sum a_i x_i = b$.

Similarly, $\sum a_i x_i \geq b$ defines a half-space.

Constraint $\sum a_i x_i = b$ defines a hyperplane.

Their intersection is a bounded or unbounded convex polytope in $\mathbb{R}^n$.

It's called the feasible polytope.



Geometric interpretation

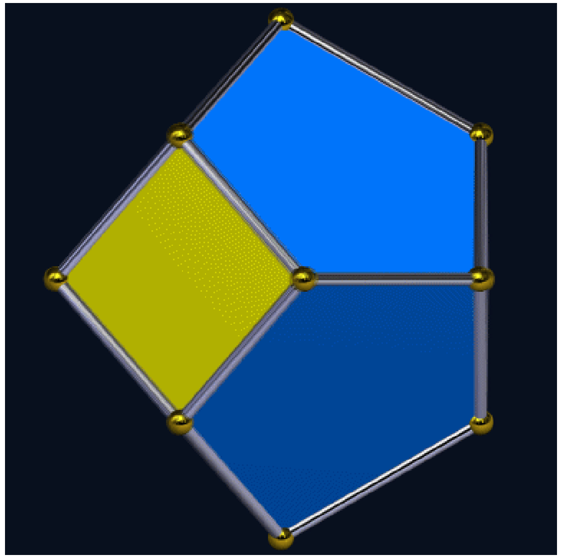
Their intersection is a bounded or unbounded convex polytope in $\mathbb{R}^n$.

It's called the feasible polytope.

x is a vertex solution or a vertex of the feasible polytope if there are no two feasible solutions x' and x'' $(x_1 \neq x_2)$ such that

$$x = \alpha x' + (1 - \alpha)x''$$

for some $\alpha \in (0,1)$



Geometric interpretation

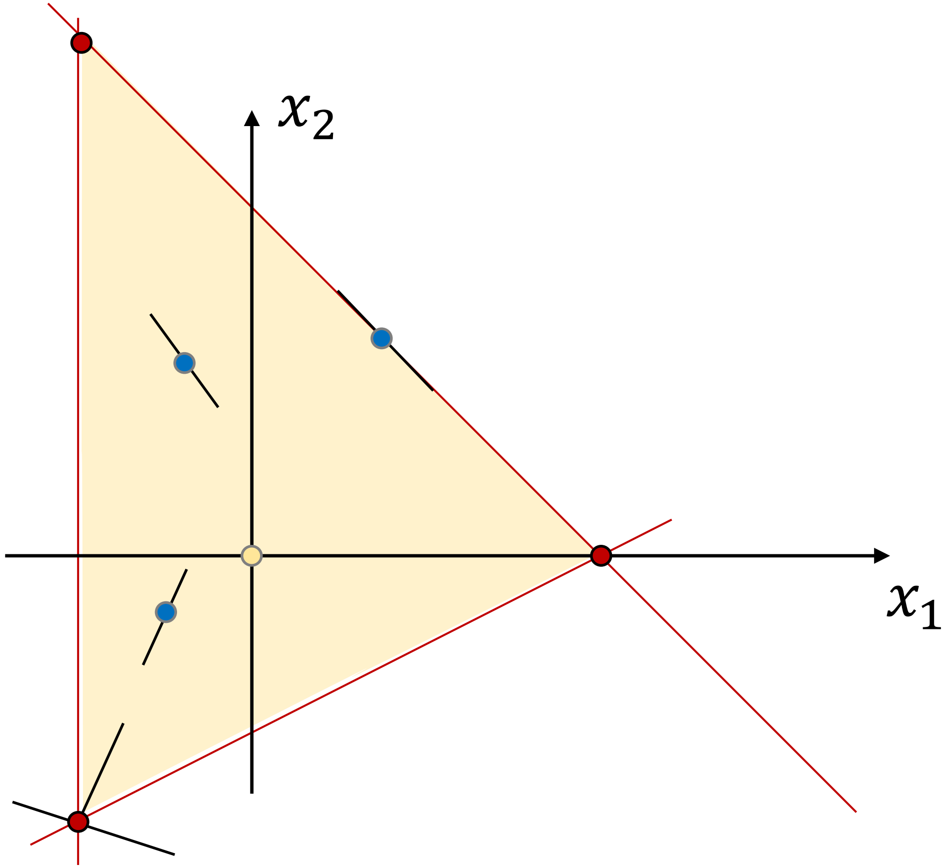
x is a vertex solution or a vertex of the feasible polytope if there are no two feasible solutions x' and x'' $(x_1 \neq x_2)$ such that

$$x = \alpha x' + (1 - \alpha)x''$$

for some $\alpha \in (0,1)$.

No segment $[x', x'']$ contains x

(where x', x'' are feasible solutions not equal x)



Does every feasible LP have a vertex?

Example 1

max $x_1 + 2x_2$

s.t. no constraints

Q: Is this LP bounded?

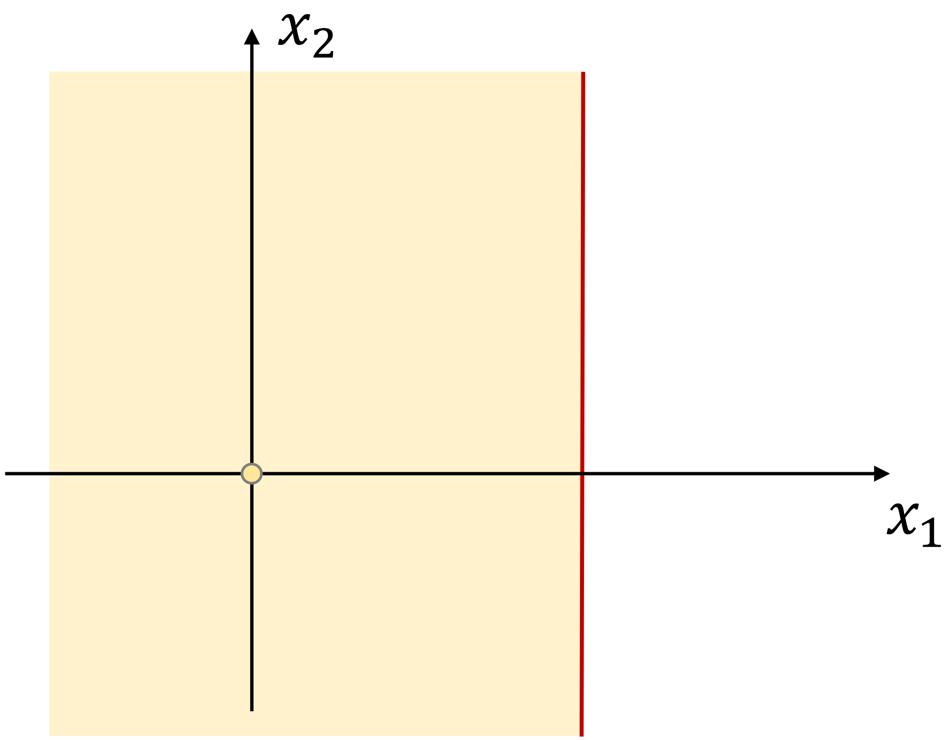
Example 2

max x_1

s.t.

$x_1 \leq 2$

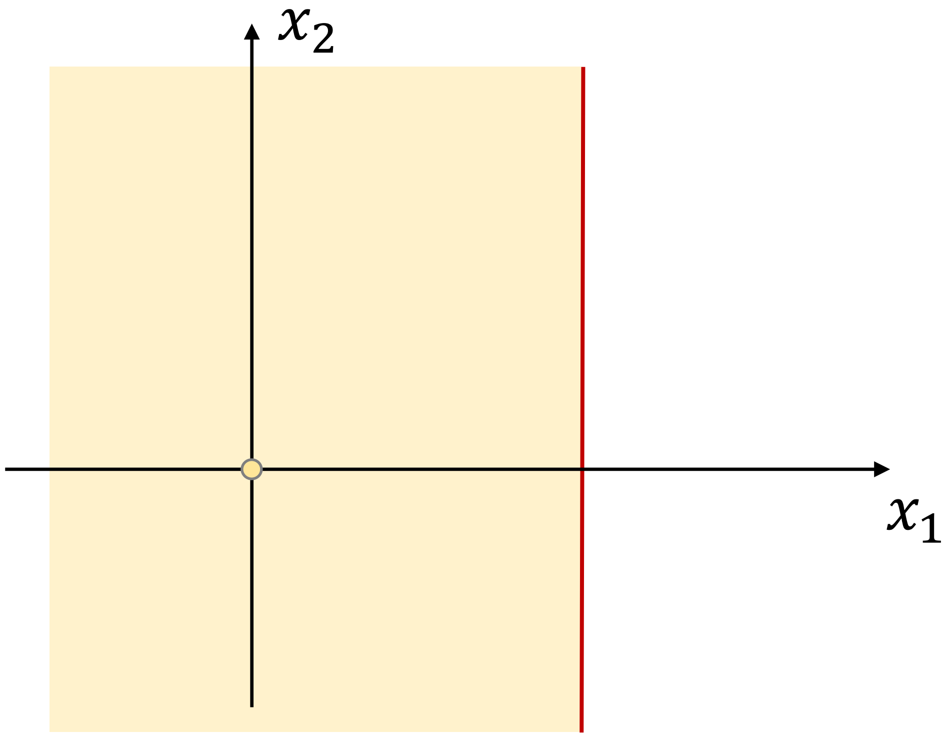
Q: Is this LP bounded?



Does every feasible LP have a vertex?

Fact:

If an LP has a vertex and bounded, then there is an optimal solution x^*,
which is a vertex.



Transforming an LP

- Transform minimization to maximization:

$$\text{maximize } c^T x$$
$$\Updownarrow$$
$$\text{minimize } (-c)^T x$$

- $a^T x \geq b$ is equivalent to $(-a^T)x \leq -b$
- $a^T x = b$ is equivalent to $a^T x \leq b$ and $(-a^T)x \leq -b$

Get constraints of the form $a^T x \leq b$

Transforming an LP

Conclusion: every LP can be transformed to the following form:

maximize $c^T x$

$a_1^T x \leq b_1$

$a_2^T x \leq b_2$

…

$a_m^T x \leq b_m$

maximize $c^T x$

s.t.

$Ax \leq b$

Canonical form

Canonical form:

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

Transforming an LP to the Canonical Form

We will see how to transform a given LP to the canonical form.

maximize $c^T x$

s.t.

$Ax \leq b$

$\Rightarrow$

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

Transforming an LP to the Canonical Form

We will see how to transform a given LP to the canonical form.

maximize $c^T x$

s.t.

$Ax \leq b$

Introduce variables x_i^+ *and* x_i^-.

Replace x_i with $x_i^+ - x_i^-$.

Add constraints: $x^+ \geq 0$ and $x^- \geq 0.$

We get an LP in the canonical form.

Introduce variables x_i^+ *and* x_i^-.

Replace x_i with $x_i^+ - x_i^-$.

Add constraints: $x^+ \geq 0$ and $x^- \geq 0$.

We get an LP in the canonical form.

If x is a feasible solution to the original LP. Then

$$x_i^+ = \max(x_i, 0) \text{ and } x_i^- = \max(-x_i, 0)$$

is a solution to the new LP, since $x_i^+ - x_i^- = x_i$.

Example: $x_i = 5$. Then $x_i^+ = 5$ and $x_i^- = 0$. We have, $x_i^+ - x_i^- = 5$.

Example: $x_i = -3$. Then $x_i^+ = 0$ and $x_i^- = 3$. We have, $x_i^+ - x_i^- = -3$.

Introduce variables x_i^+ *and* x_i^-.

Replace x_i with $x_i^+ - x_i^-$.

Add constraints: $x^+ \geq 0$ and $x^- \geq 0$.

We get an LP in the canonical form.

If x^+, x^- is a feasible solution to the new LP. Then

$$x = x^+ - x^-$$

is a solution to the original LP.

Solutions x and (x^+, x^-) have the same value.

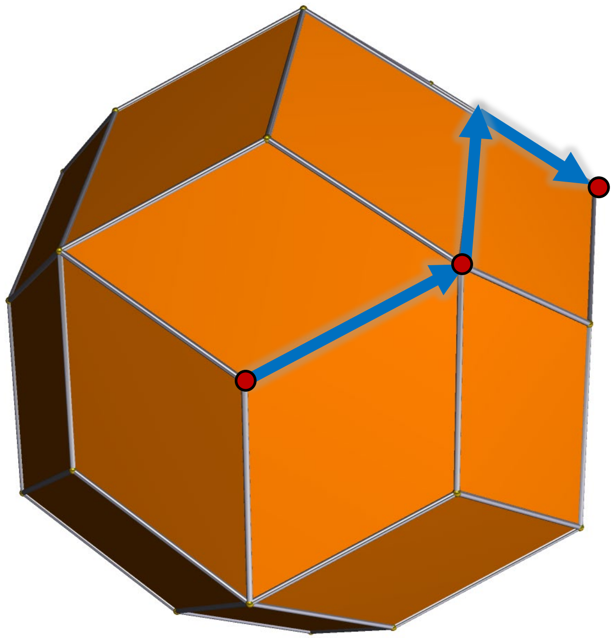
Does every feasible LP have a vertex?

Fact:

- Every LP in the canonical form has a vertex.
- If it is bounded, then there is an optimal solution x^*, which is a vertex.

Solving LPs: Simplex Method

- Find a vertex v
- Repeat
  - Consider its neighbors $u_1, \dots, u_k$
  - If there is one with $f(u_i) > f(v)$
    - choose one and let $v = u_i$ (pivot rule)
  - else: stop and return v



Works well in practice.

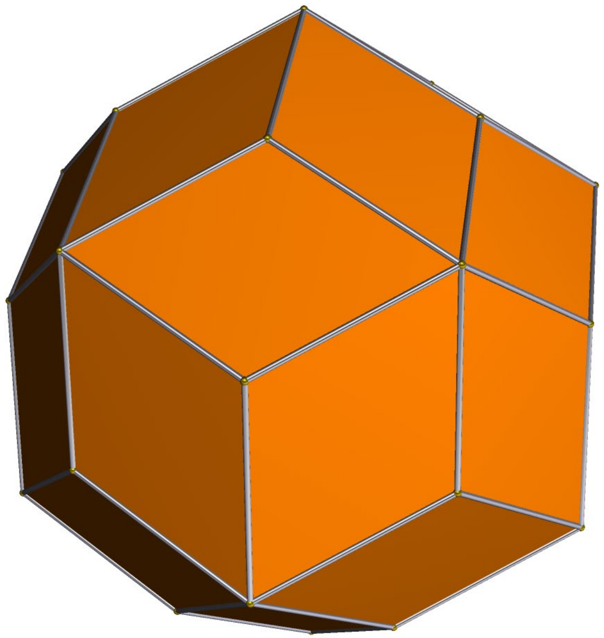
The performance depends on the pivot rule.

We don't know if there is a pivot rule that ensures polynomial running time.

Solving LPs: Simplex Method

There are polynomial-time algorithms for solving LPs:

- Ellipsoid Method
- Interior-point methods



Weighted Bipartite Matching

We are given a bipartite graph $G = (L \cup R, E)$ and edge weights w_e.

Find a matching M of maximum possible weight

$$W = \sum_{e \in M} w_e$$

We cannot reduce this problem to the Maximum Flow problem.

Another approach: use Linear Programming!

Weighted Bipartite Matching

max $\sum_e w_e x_e$

s.t.

$\sum_{e \in \partial(u)} x_e \leq 1$ for every u

$x_e \geq 0$

where $\partial(u)$ is the set of edges incident on u

Weighted Bipartite Matching

For every matching M there exists a corresponding LP solution x:

$$x_e = \begin{cases} 0, & \text{if } e \notin M \\ 1, & \text{if } e \in M \end{cases}$$

and

$$w(M) = \sum_{e \in E} x_e w_e$$

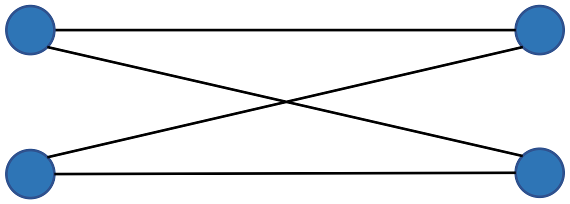
In particular,

$$LP \geq OPT$$

LP Algorithm for Matching

1. Solve the LP. Find x^*.
2. Need to construct M from the LP solution x_e^*
3. If all $x_e^* \in \{0,1\}$, then
$$M = \{e: x_e^* = 1\}$$

4. What should we do if it is not?



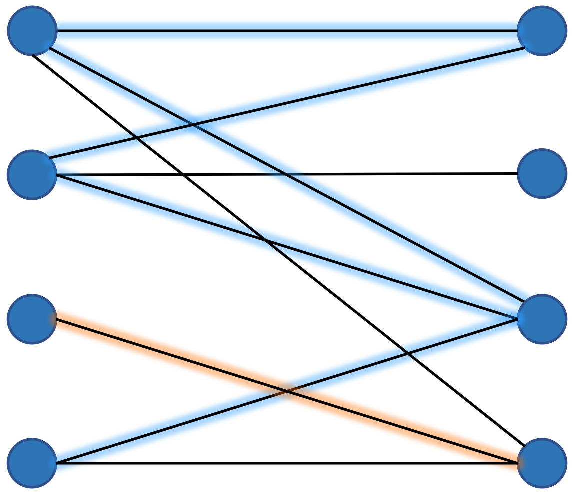
all $x_e^* = 1/2$

LP Algorithm for Matching

1. Solve the LP. Find optimal x^*, which is a vertex.
2. Prove that all $x_e^* \in \{0,1\}$.
3. Let

$$M = \{e : x_e^* = 1\}$$

Vertex Solution



Partition all edges into 3 disjoint groups:

- $E_0 = \{e : x_e^* = 0\}$
- $E_r = \{e : 0 < x_e^* < 1\}$
- $E_1 = \{e : x_e^* = 1\}$

If E_r is empty, we are done.

Focus on E_r and E_1.

Are these configurations possible?

Two edges from E_1 share a vertex.

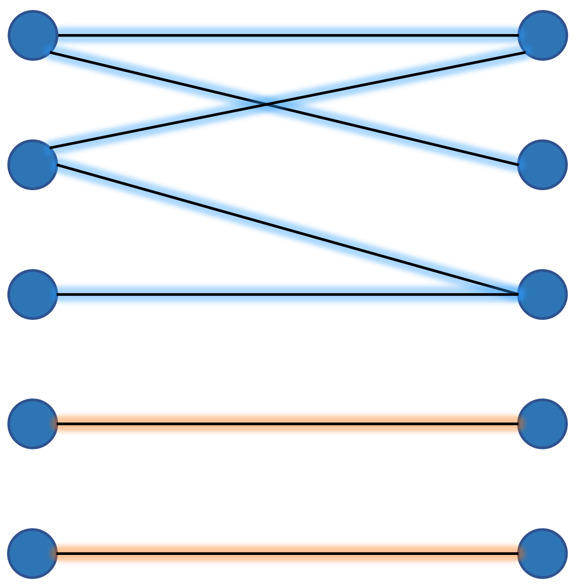
An edge from E_1 and from E_r share a vertex.

Analysis

Conclusion:

Edges in E_1 form a matching.

Edges in E_r don't share any endpoints with those in E_1.



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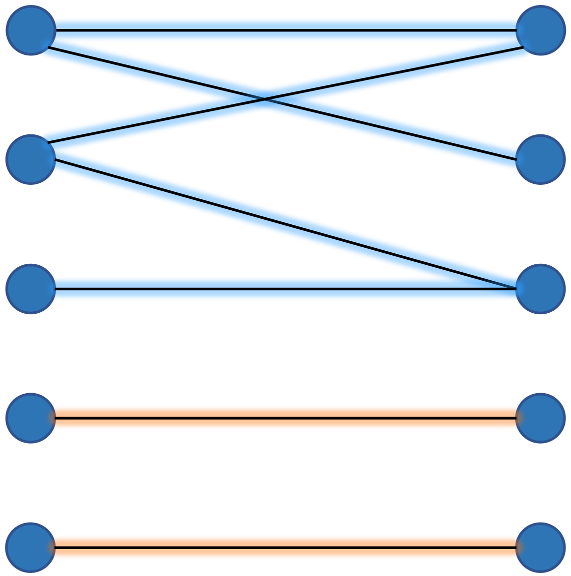

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Edges in E_1 form a matching.

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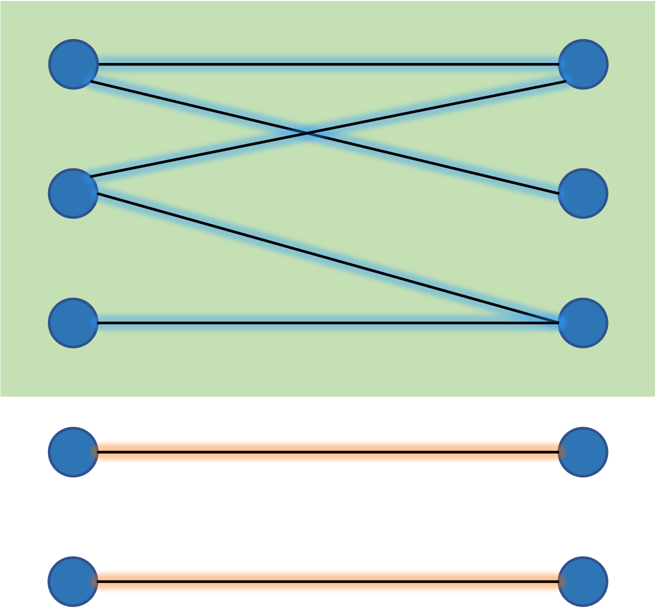


Consider subgraph H formed by edges from E_r.

Analysis

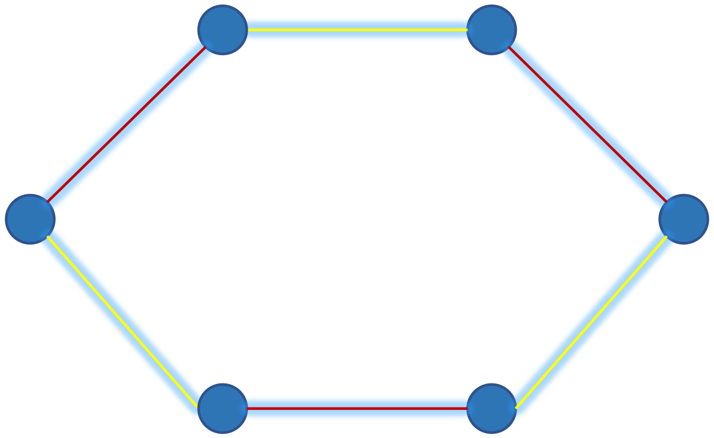
Q: Can H contain a cycle?

Assume that there a cycle C in H. Since G is a bipartite graph, C is a cycle of even length.

Divide its edges into two groups:

- put every other edge in A
- and every other in B

(so that edges of A and B alternate)



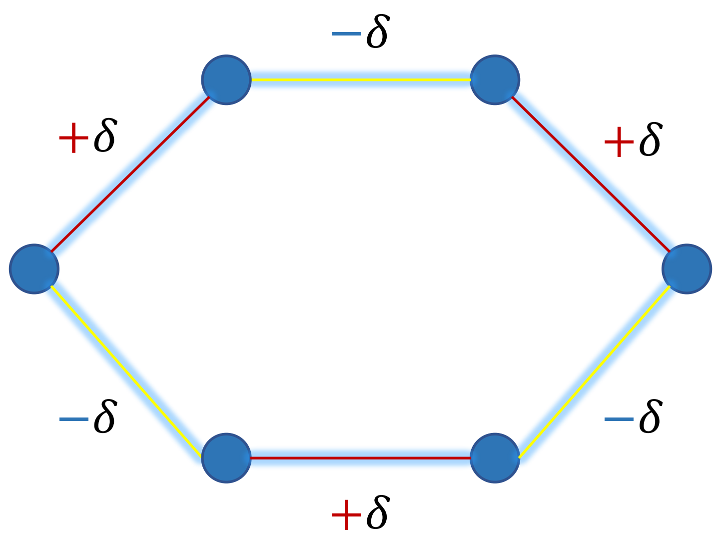
Analysis

Construct two feasible solutions x' and x'':

Let $\delta = \min_{e \in C} x_e^* > 0$.

$$x'_e = \begin{cases} x_e^* + \delta, & \text{if } e \in A \\ x_e^* - \delta, & \text{if } e \in B \\ x_e^*, & \text{otherwise} \end{cases}$$

$$x''_e = \begin{cases} x_e^* - \delta, & \text{if } e \in A \\ x_e^* + \delta, & \text{if } e \in B \\ x_e^*, & \text{otherwise} \end{cases}$$



Analysis

Note that

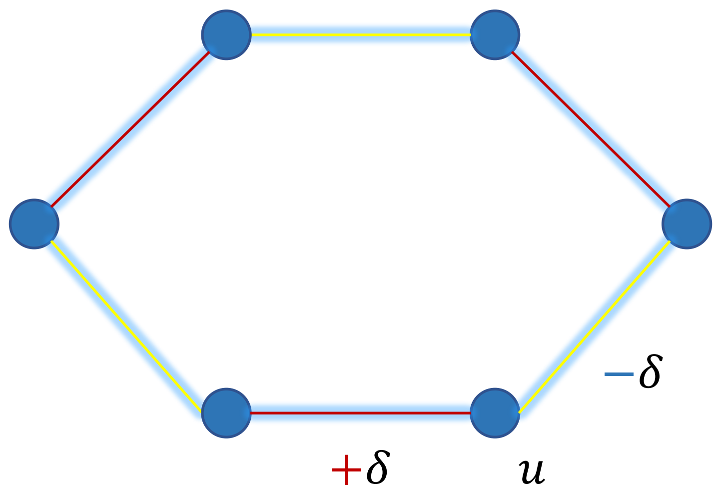
$$x^* = \frac{x' + x''}{2}$$

We show that x' and x'' are feasible solutions.

Let $\partial(u)$ be the set of edges incident on u.

$\sum_{e \in \partial(u)} x'_e = \sum_{e \in \partial(u)} x^*_e \leq 1$ if $u \notin C$.

$\sum_{e \in \partial(u)} x'_e = \sum_{e \in \partial(u)} x^*_e + \delta - \delta \leq 1$ if $u \in C$.



Analysis

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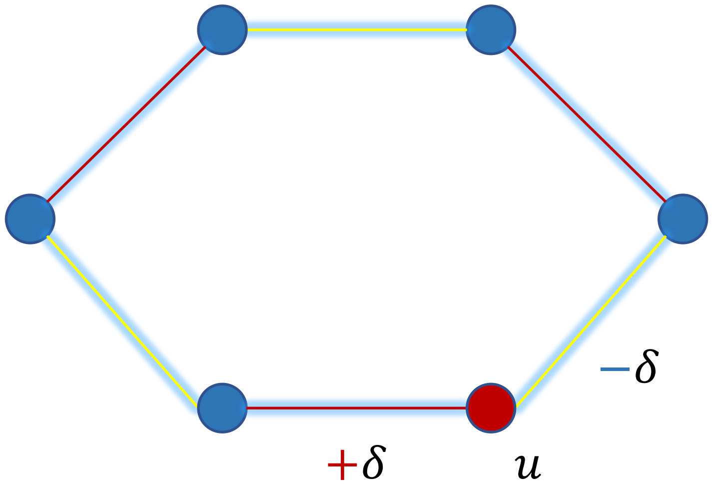


For $e \notin C$: $x'_e = x^*_e \geq 0$

For $e \in C$: $x'_e \geq x^*_e - \delta \geq 0$

We conclude that x' and (similarly) x'' are feasible solutions.

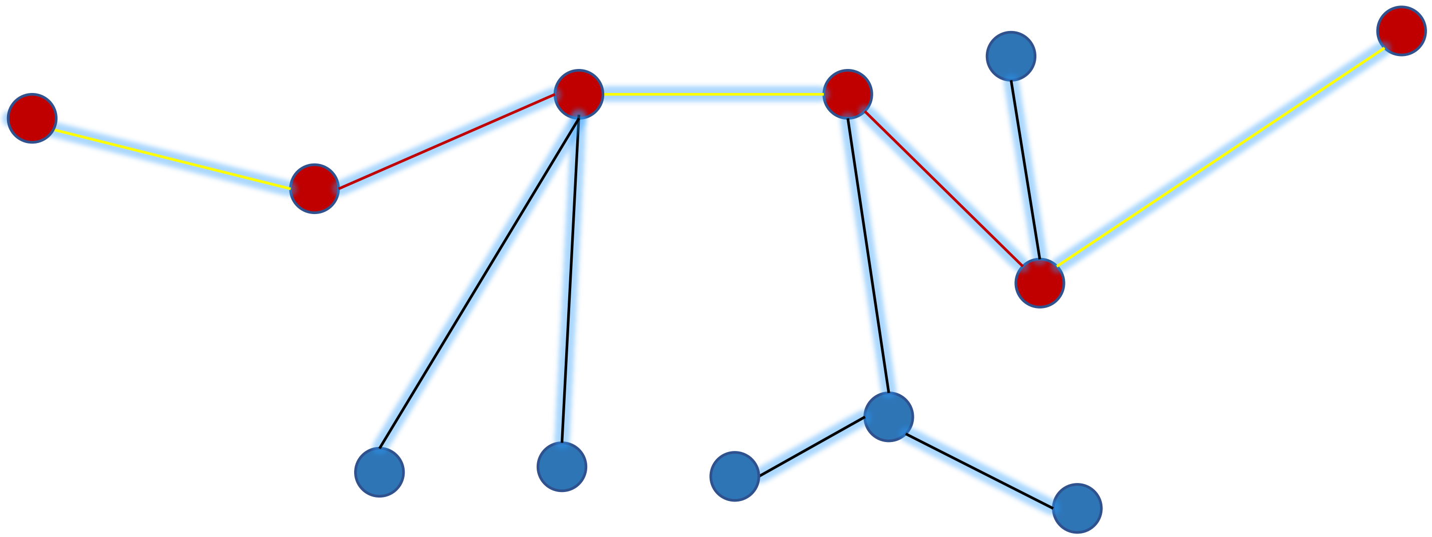
We get a contradiction: x^* is not a vertex solution.

Analysis

Thus, H does not contain any clycles.

$\Rightarrow$ H is a forest.

Consider a tree in H and a path P between two leaves.



Analysis

Construct two feasible solutions x' and x'':

Let $\delta = \min\ \{x_e^*, 1 - x_e^*: e \in P\}\ > 0.$

$$x'_e = \begin{cases} x_e^* + \delta, & \text{if } e \in A \\ x_e^* - \delta, & \text{if } e \in B \\ x_e^*, & \text{otherwise} \end{cases}$$

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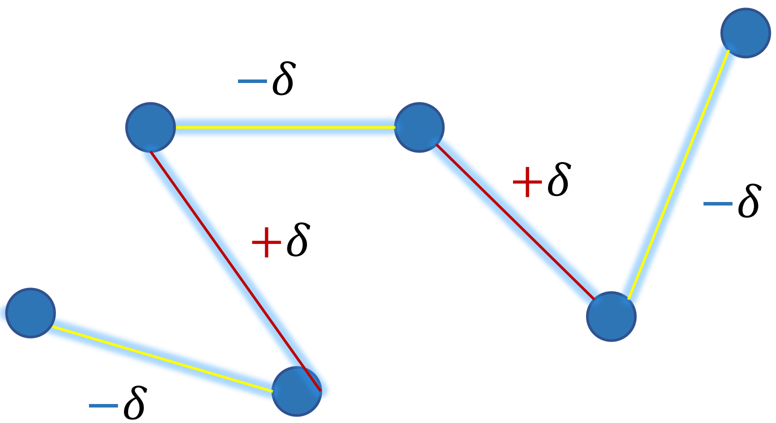


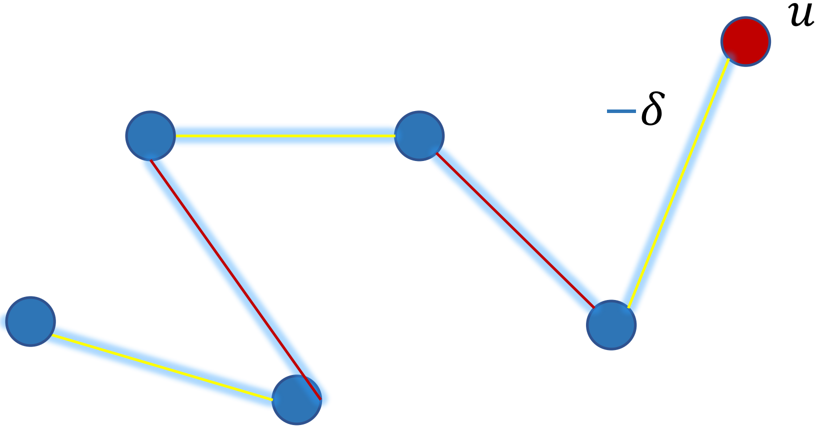
Analysis

x' and x'' are feasible solutions.

Then

why?

$$\sum_{e \in \partial(u)} x'_e = x^*_e \pm \delta \le 1$$



Conclusion

We proved that $x_e^* \in \{0,1\}$ for every edge e in G.

In fact, we showed that every vertex of the feasible polytope is integral.

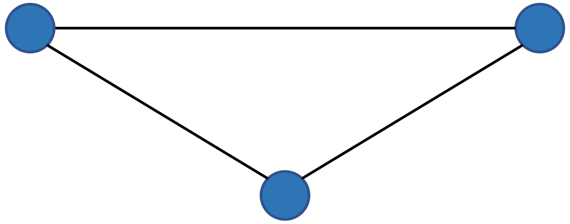
We say that the LP is integral.

Algorithm

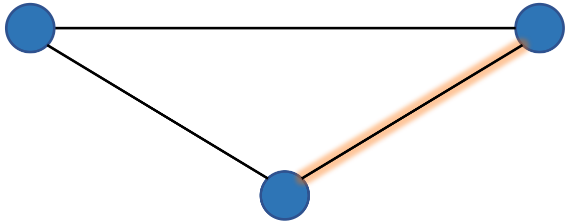
- Find an optimal vertex solution x_e^*
- Let $M = \{e : x_e^* = 1\}$

Non-bipartite graph

If G is non-bipartite, a vertex solution might not be integral.



all $x_e^* = 1/2$



OPT = 1

LP = $3/2$

The integrality gap is the ratio between the best LP and optimal solution.

In this example, the gap is $3/2$.

LP Duality

LP Duality

We define a dual for a linear program. Assume A is an $m \times n$ matrix.

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$x \in \mathbb{R}^n$

primal

minimize $b^T y$

s.t.

$A^T y \geq c$

$y \geq 0$

$y \in \mathbb{R}^m$

dual

LP Duality

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$x \in \mathbb{R}^n$

primal

minimize $b^T y$

s.t.

$A^T y \geq c$

$y \geq 0$

$y \in \mathbb{R}^m$

dual

Primal	**Dual**
n variables	n constraints
m constraints	m variables

Weak Duality

maximize $c^T x$
s.t.
$Ax \leq b$
$x \geq 0$
$x \in \mathbb{R}^n$
primal

minimize $b^T y$
s.t.
$A^T y \geq c$
$y \geq 0$
$y \in \mathbb{R}^m$
dual

Consider feasible solutions:

$x \in \mathbb{R}^n$ for the primal and
$y \in \mathbb{R}^m$ for the dual.

$$c^T x \leq \left(A^T y\right)^T x = y^T A x \leq y^T b$$

value of solution x

since $A^T y \geq c$ and $x \geq 0$

since $Ax \leq c$ and $y \geq 0$

value of solution y

Weak Duality

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$x \in \mathbb{R}^n$

primal

minimize $b^T y$

s.t.

$A^T y \geq c$

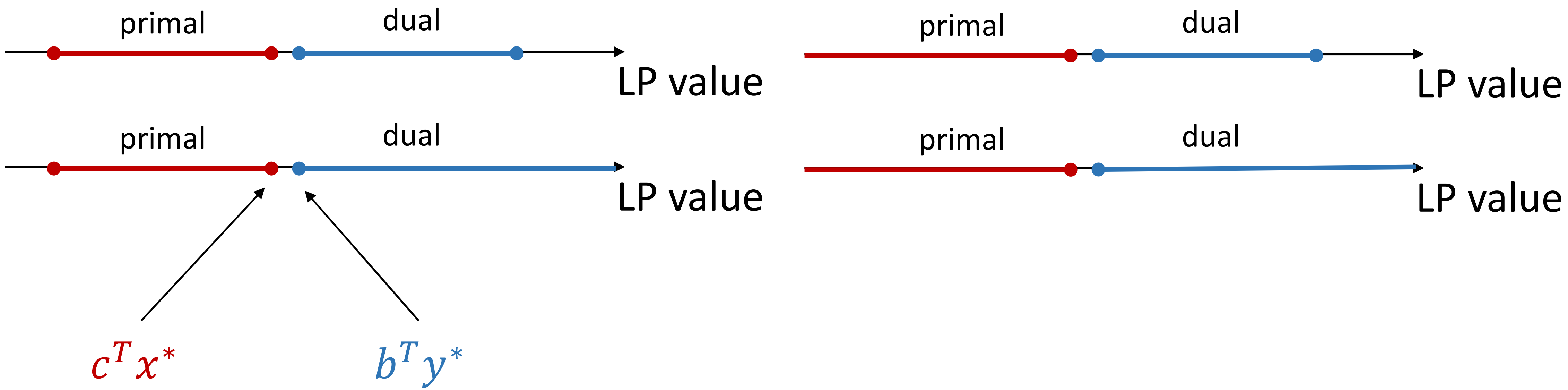
$y \geq 0$

$y \in \mathbb{R}^m$

dual

Assume that both the primal and duals LPs are feasible.

Then $c^T x \leq b^T y$ for every feasible solutions x of P and y of D.



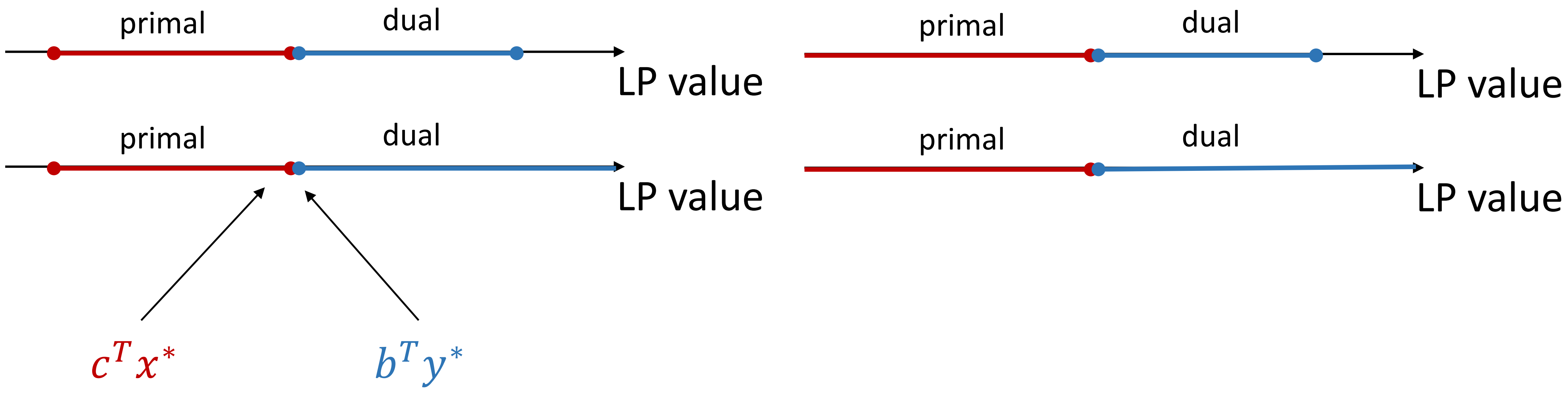
Is there a gap between them?

Strong Duality

Assume that both the primal and duals LPs are feasible.

Then $c^T x^* = b^T y^*$.

primal	dual
maximize $c^T x$	minimize $b^T y$
s.t.	s.t.
$Ax \leq b$	$A^T y \geq c$
$x \geq 0$	$y \geq 0$
$x \in \mathbb{R}^n$	$y \in \mathbb{R}^m$



There is no gap!

Strong Duality

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$x \in \mathbb{R}^n$

primal

minimize $b^T y$

s.t.

$A^T y \geq c$

$y \geq 0$

$y \in \mathbb{R}^m$

dual

We have the following possibilities:

- P and D are feasible. Then both are bounded and $c^T x^* = b^T y$.
- P is feasible and unbounded and D is infeasible.
- P is infeasible and D is feasible and unbounded.
- P and D are infeasible.

Complementary Slackness

maximize $c^T x$
s.t.
$Ax \leq b$
$x \geq 0$
$x \in \mathbb{R}^n$
primal

minimize $b^T y$
s.t.
$A^T y \geq c$
$y \geq 0$
$y \in \mathbb{R}^m$
dual

Assume that both P and D are feasible.

$$c^T x^* \leq \left(A^T y^*\right)^T x^* = y^{*T} A x^* \leq b^T y^* = c^T x^*$$

strong duality

Thus, both $\leq$ are equalities:

$$c^T x^* = \left(A^T y^*\right)^T x^*$$
$$(Ax^*)^T y^* = b^T y^*$$

Complementary Slackness

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$x \in \mathbb{R}^n$

primal

minimize $b^T y$

s.t.

$A^T y \geq c$

$y \geq 0$

$y \in \mathbb{R}^m$

dual

We have,

$$(Ax^*)^T y^* = b^T y^*$$

or

$$(b - Ax^*)^T y^* = 0$$

Now, for every i

$$(b - Ax^*)_i \geq 0$$

$$y_i \geq 0$$

Q: What does it mean that $\sum_i (b - Ax^*)_i y_i = 0$?

Complementary Slackness

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$x \in \mathbb{R}^n$

primal

minimize $b^T y$

s.t.

$A^T y \geq c$

$y \geq 0$

$y \in \mathbb{R}^m$

dual

Q: What does it mean that $\sum_i (b - Ax^*)_i y_i = 0$?

A: It must be the case that either $(b - Ax^*)_i = 0$ or $y_i = 0$ for every i.

Complementary Slackness

- If i-th primal constraint is not tight, then $y_i = 0$.
- If $y_i > 0$, then i-th primal constraint is tight.

Complementary Slackness

maximize $c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$x \in \mathbb{R}^n$

primal

minimize $b^T y$

s.t.

$A^T y \geq c$

$y \geq 0$

$y \in \mathbb{R}^m$

dual

Complementary Slackness

- If i-th primal constraint is not tight, then $y_i = 0$.
- If $y_i > 0$, then i-th primal constraint is tight.

- If i-th dual constraint is not tight, then $x_i = 0$.
- If $x_i > 0$, then i-th dual constraint is tight.

Dual of Dual?

We defined the dual only for LPs in
the canonical form, but it can be written for any LP.

Q: What is the dual of the dual?

$$\begin{array}{l} \text{maximize } c^T x \\ \text{s.t.} \\ Ax \le b \\ x \ge 0 \\ x \in \mathbb{R}^n \end{array}$$

primal

$$\begin{array}{l} \text{minimize } b^T y \\ \text{s.t.} \\ A^T y \ge c \\ y \ge 0 \\ y \in \mathbb{R}^m \end{array}$$

dual

$$\begin{array}{l} \min b^T y \\ \text{s.t.} \\ A^T y \ge c \\ y \ge 0 \end{array} \xrightarrow{\text{canonical form}} \begin{array}{l} \max\ -b^T y \\ \text{s.t.} \\ -A^T y \le -c \\ y \ge 0 \end{array} \xrightarrow{\text{dual}} \begin{array}{l} \min\ -c^T x \\ \text{s.t.} \\ -Ax \ge -b \\ x \ge 0 \end{array}$$

Dual for the Maximum Flow LP

max $\sum_{e \in in(t)} f_e$

s.t.

$f_e \leq c(e)$ for all e

$\sum_{e \in out(u)} f_e - \sum_{e \in in(u)} f_e \leq 0$ for all $u \notin \{s, t\}$

$\sum_{e \in in(u)} f_e - \sum_{e \in out(u)} f_e \leq 0$ for all $u \notin \{s, t\}$

$f_e \geq 0$ for all e

dual variables

y_e

d_u^+

d_u^-

Dual Objective

max $\sum_{e \in in(t)} f_e$

s.t.

	dual variables
$f_e \leq c(e)$	y_e
$\sum_{e \in out(u)} f_e - \sum_{e \in in(u)} f_e \leq 0$	d_u^+
$\sum_{e \in in(u)} f_e - \sum_{e \in out(u)} f_e \leq 0$	d_u^-
$f_e \geq 0$ for all e	

Dual objective: min $\sum_e c(e) y_e$

Dual Constraint for Edge (a, b)

max $\sum_{e \in in(t)} f_e$

s.t.

	dual variables
$f_e \leq c(e)$	y_e
$\sum_{e \in out(u)} f_e - \sum_{e \in in(u)} f_e \leq 0$	d_u^+
$\sum_{e \in in(u)} f_e - \sum_{e \in out(u)} f_e \leq 0$	d_u^-
$f_e \geq 0$ for all e	

Find all occurrences of $f_{(a,b)}$ in the primal LP.

Dual Constraint for Edge (a, b)

max $\sum_{e \in in(t)} f_e$

dual variables

$f_{(a,b)} \leq c(a,b)$	y_e
$\sum_{e \in out(a)} f_e - \sum_{e \in in(a)} f_e \leq 0$	d_a^+
$\sum_{e \in out(b)} f_e - \sum_{e \in in(b)} f_e \leq 0$	d_b^+
$\sum_{e \in in(b)} f_e - \sum_{e \in out(a)} f_e \leq 0$	d_b^-
$\sum_{e \in in(u)} f_e - \sum_{e \in out(a)} f_e \leq 0$	d_a^-

$a, b \notin \{s, t\}$: $y_{(a,b)} + d_a^+ - d_b^+ + d_b^- - d_a^- \geq 0$

Dual for the Maximum Flow LP

min $\sum_e c(e) y_e$

s.t.

$y_{(a,b)} + d_a^+ - d_b^+ + d_b^- - d_a^- \geq 0$ for $(a, b) \in E$ s.t. $a \neq s$ and $b \neq t$

$y_{(s,b)} - d_b^+ + d_b^- \geq 0$ for $(s, b) \in E$ s.t. $b \neq t$

$y_{(a,t)} + d_a^+ - d_a^- \geq 1$ for $(a, t) \in E$ s.t. $a, b \notin \{s, t\}$

$y_{(s,t)} \geq 1$ if edge (s, t) is present

$y_e \geq 0, d_u^+ \geq 0, d_u^- \geq 0$

To simplify this LP, let $d_u = d_u^+ - d_u^-$, $d_s = 0$, $d_t = 1$

Dual for the Maximum Flow LP

min $\sum_e c(e) y_e$

s.t.

$y_{(a,b)} + d_a - d_b \geq 0$ for $(a, b) \in E$ s.t. $a \neq s$ and $b \neq t$

$y_{(s,b)} + d_s - d_b \geq 0$ for $(s, b) \in E$ s.t. $b \neq t$

$y_{(a,t)} + d_a - d_t \geq 0$ for $(a, t) \in E$ s.t. $a, b \notin \{s, t\}$

$y_{(s,t)} + d_s - d_t \geq 0$ if edge (s, t) is present

$y_e \geq 0, d_s = 0, d_t = 1$

To simplify this LP, let $d_u = d_u^+ - d_u^-$, $d_s = 0$, $d_t = 1$

Dual for the Maximum Flow LP

min $\sum_e c(e)y_e$

s.t.

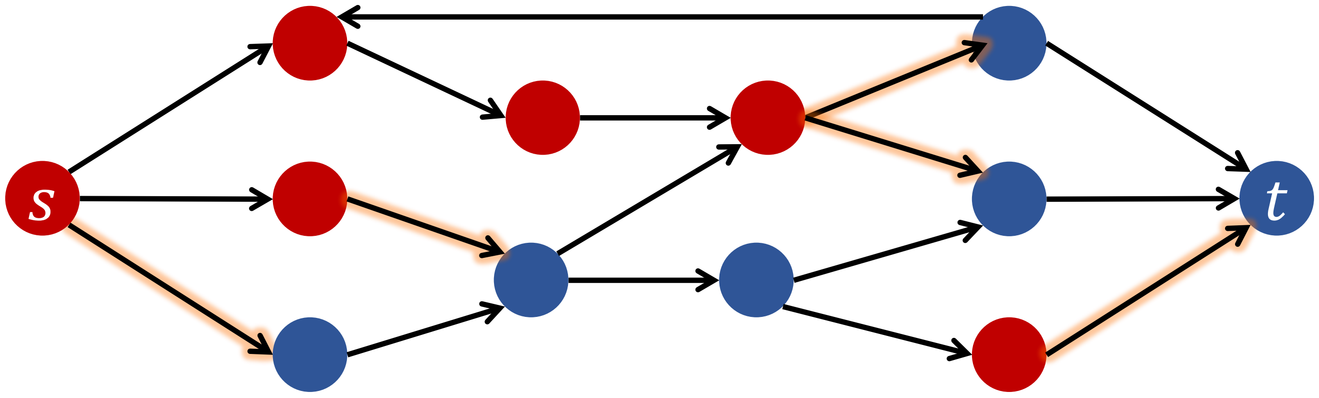
$y_{(a,b)} \geq d_b - d_a$

$d_s = 0$

$d_t = 1$

$y_e \geq 0$

This is an LP for Minimum Cut!



Intended solution

$d_u = 0$ for $u \in A$

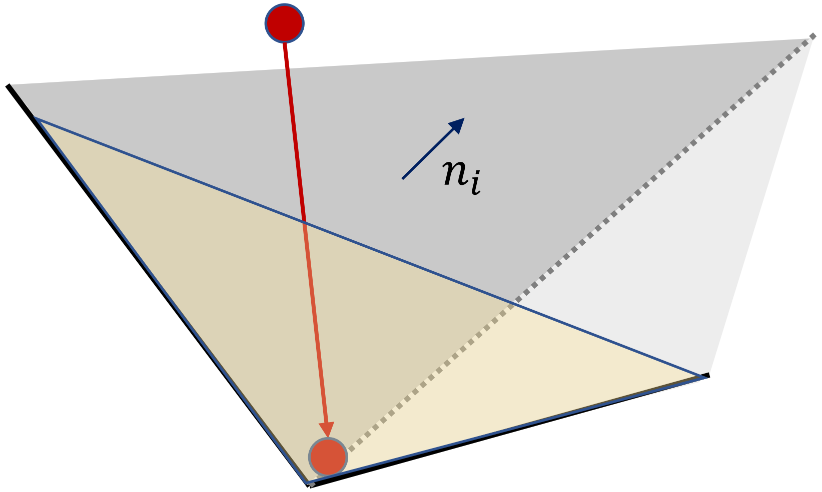
$d_u = 1$ for $u \in B$

$y_e = 1$ if e is cut by (A, B)

$y_e = 0$ if e is not cut

Physical Interpretation of Duality

We have a vessel in the shape of a convex polyhedron. We drop a small ball (test particle) inside of it. It will fall down and may move for a while.



Q: Where will it eventually stop?
A: Let $u = (x, y, z)$ be its location. Then

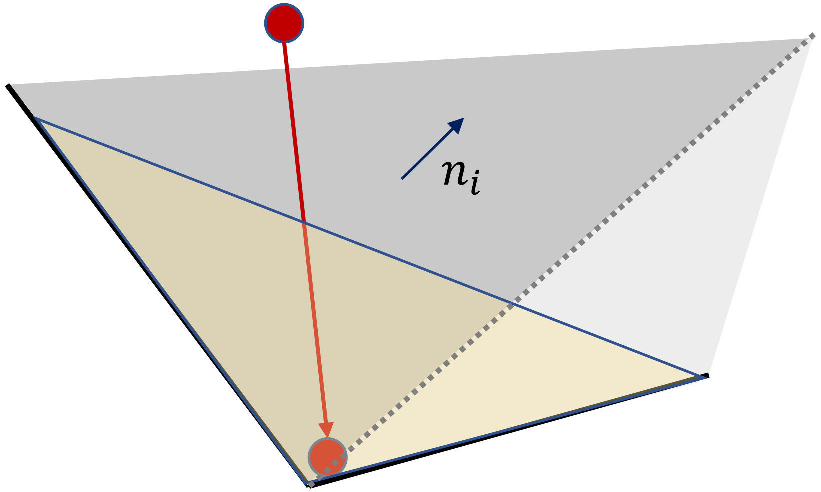
$$\boxed{\begin{array}{l}\min z \\ \text{s.t.} \\ n_i^T u \geq b_i\end{array}}$$

where equations $n_i^T u \geq b_i$ define the faces of the polyhedron.

Physical Interpretation of Duality

A: Let $u = (x, y, z)$ be its location. Then

$$\min z$$
s.t.
$$n_i^T u \geq b_i$$



Dual:

$$\max \textstyle\sum_i \alpha_i b_i$$
s.t.
$$\sum_i \alpha_i n_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$\alpha_i \geq 0$$

surface contact forces

The sum of all forces acting on the body — the contact forces and the force of gravity is 0.

Contact forces are pointed "upward".

Physical Interpretation of Duality

A: Let $u = (x, y, z)$ be its location. Then

$$\min z$$

s.t.

$$n_i^T u \geq b_i$$

Dual:

max $\sum_i \alpha_i b_i$

s.t.

$$\sum_i \alpha_i n_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\alpha_i \geq 0$$

Complementary slackness:
If wall i acts on the ball $(\alpha_i > 0)$,
then the ball touches the wall

$$n_i^T u = b_i$$

Proof System Interpretation of Duality

Assume that

(i) $Ax \leq b$

(ii) $x \geq 0$

This system of inequalities implies another inequality $c^T x \leq M$ if and only if this inequality can be proved from (i) and (ii) by

- adding up inequalities of type (i) with some non-negative coefficients y_i
- using that $c^T x \leq \tilde{c}^T x$ if $c \leq \tilde{c}$.