Lecture 1 - Graph Matrices

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THE UNIVERSITY OF CHICAGO

Overview

1. Admin
2. Graphs
3. Mathematical Formulation
4. Matrices for Graphs

Admin

- **Lectures:** TR 9:30am-10:50am, TBA
- **Instructors:** Lorenzo Orecchia, Konstantinos Ameranis
- **Office Hours:**
  - Lorenzo Orecchia: Tuesday 11am-12n, JCL 315
  - Konstantinos Ameranis: Thursday 1:00-3:00pm, JCL
- **Grading:**
  - **Homework:** 35%
  - **Project:** 10%
  - **Participation:** 5%
  - **Final:** 50%
- Resources:
  - Spielman's Spectral Graph Theory Course
  - Spectral Graph Theory by Fan Chung

Graphs

Graphs are used to represent "relations" or "connections" between "things". We will call the "things" *vertices* or *nodes* and their "connections" *edges*

Examples of graphs:

Graphs

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Examples of graphs:

- Social media: Accounts are vertices, friends/messages/interactions are edges
- Networks: Devices (routers/computers/servers) are vertices, connections are edges
- Circuits: Wires are edges
- Road network: Intersections are nodes, roads are edges
- Meshes: Nodes are sampling a surface or volume, edges connect nearby nodes

Mathematical Formulation

Scalars, Vectors and Matrices

- x an unboldened literal represents a *scalar*
- $\mathbf{x}$ a boldened lower case represents a *vector*
- $\mathbf{X}$ a boldened upper case represents a *matrix*

Graphs

- $G = (V, E, w)$
- V is called the vertex set, typically $|V| = n$
- $E \subseteq V \times V$ is called the edge set $|E| = m$
- $w \in \mathbb{R}_{>0}^{|E|}$ are the weights associated with the edges

If w is omitted, then it is assumed that it is 1 for all edges

Prerequisites

This is a high level Computer Science course. At this point we expect you to have some background knowledge.

- Linear Algebra - Vectors, Matrices, Eigenvectors/Eigenvalues
- Multivariate Calculus - Partial derivatives, Hessian
- Mathematical Proof Techniques - Induction, Proof by Contradiction, etc
- Algorithms - BFS, DFS, Shortest Path, Maximum Flow

There is an **UNGRADED** quiz that will help me better understand your strengths and weaknesses

Matrices for Graphs

Theory of Algorithms → Edges described as a set
This course → Graph described with matrices
Important matrices:

Symbol	Matrix	Dimension	Definition
$\mathbf{A}$	Adjacency	$n \times n$	$\mathbf{A}_{ij} = \begin{cases} w_{ij} & \{i,j\} \in E \\ 0 & \text{o.w.} \end{cases}$
$\mathbf{W}$	Weights	$m \times m$	$\mathbf{W} = diag(\mathbf{w})$
$\mathbf{D}$	Degree	$n \times n$	$\mathbf{D} = diag(\mathbf{d}) = diag(\sum_{i \sim j} w_{ij})$
$\mathbf{B}$	Incidence	$m \times n$	$\mathbf{B}_{ij} = \mathbf{e}_j - \mathbf{e}_i$
$\mathbf{L}$	Laplacian	$n \times n$	$\mathbf{L} = \mathbf{D} - \mathbf{A} = \mathbf{B}^T\mathbf{W}\mathbf{B}$
$\mathcal{L}$	Normalized Laplacian	$n \times n$	$\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$

Graphs as Operators

Linear operator $G(\mathbf{x}) = \mathbf{Ax}$

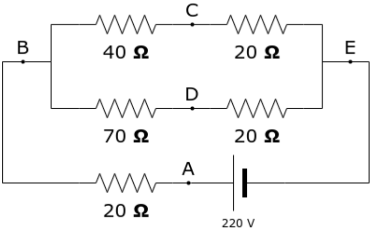

Figure: Electrical Network

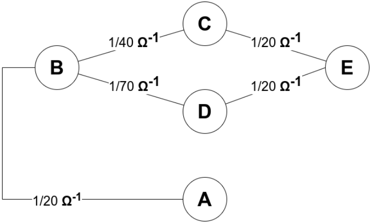

Figure: Network as a graph

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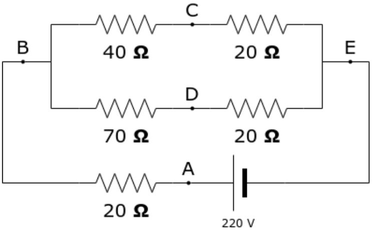

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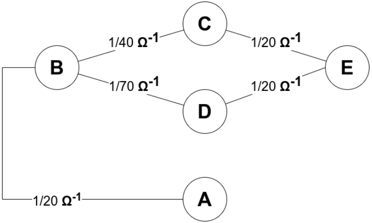
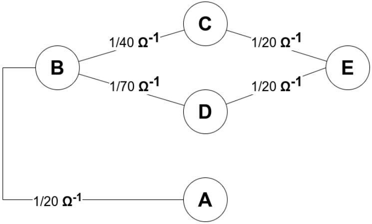


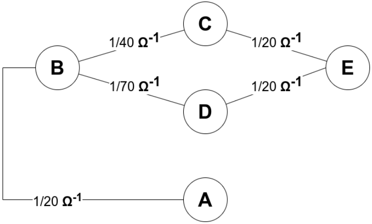
Figure: Network as a graph

Ohm's Law: $i_{uv} = \frac{v_v - v_u}{R_{uv}}$
$\mathbf{w} = 1/R$ & "stacking" Ohm's law in a vector we get $\mathbf{i} = \mathbf{W}\mathbf{B}\mathbf{v}$
Kirchhof's Current Law: $\mathbf{i}_{ext} = \mathbf{B}^T\mathbf{i} \Rightarrow \mathbf{i}_{ext} = \mathbf{B}^T\mathbf{W}\mathbf{B}\mathbf{v} = \mathbf{L}\mathbf{v}$

Graphs as Operators



Graphs as Operators



$$\begin{bmatrix} i_{AB} \\ i_{BC} \\ i_{BD} \\ i_{CE} \\ i_{DE} \end{bmatrix} = \begin{bmatrix} -1/20 & 1/20 & 0 & 0 & 0 \\ 0 & -1/40 & 1/40 & 0 & 0 \\ 0 & -1/70 & 0 & 1/70 & 0 \\ 0 & 0 & -1/20 & 0 & 1/20 \\ 0 & 0 & 0 & -1/20 & 1/20 \end{bmatrix} \begin{bmatrix} v_A \\ v_B \\ v_C \\ v_D \\ v_E \end{bmatrix} = \begin{bmatrix} \frac{v_B - v_A}{20} \\ \frac{v_C - v_B}{40} \\ \frac{v_D - v_B}{70} \\ \frac{v_E - v_C}{20} \\ \frac{v_E - v_D}{20} \end{bmatrix}$$

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Quadratic form $G(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$
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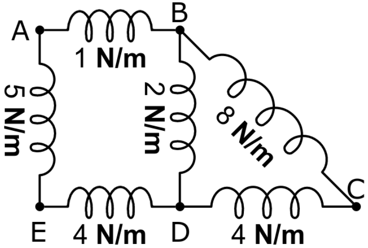

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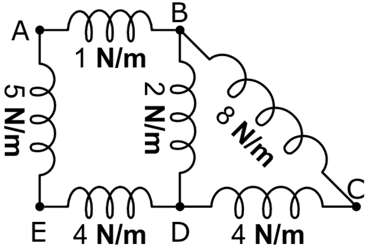

$E = \sum_{uv} k_{uv}(x_v - x_u)^2 = \mathbf{x}^T\mathbf{L}\mathbf{x}$