STAT 31210: Homework 5

Caleb Derrickson

February 8, 2024

Collaborators: The TA's of the class, as well as Kevin Hefner, and Alexander Cram.

Contents

Problem 12.6

Use the Dominated Convergence Theorem to prove Corollary 12.36 for differentiation under an integral sign.

Solution:

I will include the Dominated Convergence Theorem and Corollary 12.36 for reference.

Dominated Convergence Theorem: Suppose that (f_n) is a sequence of integrable functions, $f_n : X \to \overline{\mathbb{R}}$, on a measure space (X, A, μ) that converges pointwise to a limiting function $f : X \to \overline{\mathbb{R}}$. If there is an integrable function $g : X \to [0, \infty]$ such that

$$|f_n(x)| \leq g(x) \quad \text{for all } x \in X \text{ and } n \in \mathbb{N},$$

then f is integrable and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Corollary 12.36: Suppose that (X, A, μ) is a complete measure space, $I \subset \mathbb{R}$ is an open interval, and $f : X \times I \to \overline{\mathbb{R}}$ is a measurable function such that:

- $f(\cdot, t)$ is integrable on X for each $t \in I$;
- $f(x, \cdot)$ is differentiable in I for each $x \in X \setminus N$, where $\mu(N) = 0$;
- there is an integrable function $g : X \to [0, \infty]$ such that

$$\left|\frac{\partial f}{\partial t}(x, t)\right| \leq g(x) \quad \text{a.e. in } X \text{ for every } t \in I.$$

Then

$$\varphi(t) = \int_X f(x, t) \, d\mu(x)$$

is a differentiable function of t in I, and

$$\frac{d\varphi}{dt}(t) = \int_X \frac{\partial f}{\partial t}(x, t) \, d\mu(x).$$

Suppose we have a sequence of functions $d_n : X \times I \to \overline{\mathbb{R}}$ defined as

$$d_n(x,t) = \frac{f(x, t + \frac{1}{n}) - f(x,t)}{1/n}$$

Note that this is the n-th approximation to $\frac{\partial f}{\partial t}$. Since for some n sufficiently large, there exists some $\varepsilon > 0$ for which $|d_n - \frac{\partial f}{\partial t}| < \varepsilon$. By the reverse triangle inequality, we see that

$$|d_n - \frac{\partial f}{\partial t}| < \varepsilon \implies |d_n| < |\frac{\partial f}{\partial t}| + \varepsilon.$$

Since the partial derivative is bounded by some $g : X \to [0, \infty]$, we have that $|d_n| \leq g(x)$, since ε is arbitrary. Therefore, we have

$$\lim_{n\to\infty} \int_X d_n \; d\mu(x) = \int_X \lim_{x\to\infty} d_n \; d\mu(x) = \int_X \frac{\partial f}{\partial t}(x,t) \; d\mu(x).$$

Note that d_n is differentiable in I for all n, thus the derivative is defined. This then proves Corollary 12.36.

Problem 12.8

Let $f_n : X \to \mathbb{C}$. be a sequence of measurable functions converging to f pointwise almost everywhere. Suppose there exists $g \in L^p(X)$ such that $|f_n| \leq g$ almost everywhere. Then $f_n \to f$ in the L^p-norm.

Solution:

We can first note that $\lim_{n\to\infty} |f_n(x)| = |f(x)| \leq |g(x)|$. This implies that $|f_n(x)| \leq |g(x)|$, so we can rewrite the convergence of f_n as

$$|f_n - f|^p \leq (|f_n| + |f|)^p \leq (|g| + |g|)^p = 2^p|g|^p.$$

Since $g \in L^p$, we have that $\int |g|^p < \infty$, therefore, Theorem 12.35 applies. Therefore, we can write

$$\lim_{n\to\infty} \|f_n - f\|_p^p = \lim_{n\to\infty} \int |f_n - f|^p \, d\mu = \int \lim_{n\to\infty} |f_n - f|^p = 0.$$

Therefore, $f_n \to f$ in the L^p norm.

Problem 12.12

Prove the following generalization of Hölder's inequality: if $1 \leq p_i \leq \infty$, where $i = 1, ..., n$ satisfy

$$\sum_{i=1}^{n} \frac{1}{p_i} = 1$$

and $f_i \in L^{p_i}(X, \mu)$, then $f_1 \cdots f_n \in L^1(X, \mu)$ and

$$\left| \int f_1 \cdots f_n \, d\mu \right| \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

Solution:

This claim will be proven via induction on n.

- Base Case: $n = 1$

  Then the summation runs only over one p_i, in particular p_1. Since $\frac{1}{p_1} = 1$, this implies that $p_1 = 1$. Then

  $$\left| \int f_1 \, d\mu \right| = \|f_1\|_1 \leq \|f_1\|_1$$

  Since equality is a subcase of "$\leq$".

- Induction Hypothesis:

  Next we assume the inequality above holds for some cases up to and including the case k, $k > 1$. This implies that

  $$\sum_{i=1}^{k} \frac{1}{p_i} = 1,$$

  and

  $$\left| \int f_1 \cdots f_k \, d\mu \right| \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

  We will now show that the subsequent case, $k + 1$, holds.

- Induction Case: $n = k + 1$.

  Since each $1 \leq p_i \leq \infty$, we need to consider the case when $p_{k+1} = \infty$. If this is true, then

  $$\sum_{i=1}^{k+1} \frac{1}{p_i} = \sum_{i=1}^{k} \frac{1}{p_i} = 1.$$

  Thus, Hölder's inequality can be applied, where we separate the function f_{k+1} from the first k functions. Then,

  $$\left| \int f_1 \cdots f_{k+1} \, d\mu \right| \leq \|f_1 \cdots f_k\|_1 \|f_{k+1}\|_\infty.$$

Then, we can apply the induction hypothesis to get

$$\left| \int f_1 \cdots f_{k+1} \, d\mu \right| \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k} \|f_{k+1}\|_\infty$$

Which is what we wanted to show.

We next assume that $p_{k+1} \neq \infty$. Define the following Hölder conjugates:

$$p := \frac{p_{k+1}}{p_{k+1} - 1}, \quad q := p_{k+1}$$

Applying Hölder's Inequality in a similar fashion to the previous case gives,

$$\|f_1 \cdots f_k f_{k+1}\|_1 \leq \|f_1 \cdots f_k\|_p \|f_{k+1}\|_q$$

Which by the Induction Hypothesis, satisfies the given equality. Next, we need to show that the summation is satisfied. From the definition of p, q, we have

$$\sum_{i=1}^{k+1} \frac{1}{p_i} = \frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{p_{k+1}} + \frac{1}{p_{k+1}} = 1$$

Problem 12.15

If $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, where $p < q$, prove that $f \in L^r(\mathbb{R}^n)$ for any $p < r < q$, and show that

$$\|f\|_r \leq (\|f\|_p)^{\frac{1/r-1/q}{1/p-1/q}} (\|f\|_q)^{\frac{1/p-1/r}{1/p-1/q}},$$

This result is one of the simplest examples of an *interpolation inequality*.

Solution:

Suppose there are constants $\alpha \in (0,1)$, and $m, n > 1$ be conjugates. By Hölder's Inequality, we then have

$$\|f\|_r^r = \int |f|^r = \int |f|^{\alpha r} |f|^{(1-\alpha)r} \leq \left(\int |f|^{\alpha m r} \right)^{1/m} \left(\int |f|^{(1-\alpha)nr} \right)^{1/n}$$

We wish to have the right hand side to have the form given, that is, we need to choose α, m, n such that

$$\alpha m r = p \tag{1}$$

$$(1-\alpha) n r = q \tag{2}$$

$$\frac{1}{m} + \frac{1}{n} = 1 \tag{3}$$

We will first solve for α. Note that (1) implies that $\alpha = \frac{p}{mr}$. (2) implies $\frac{1}{n} = \frac{(1-\alpha)r}{q}$, and (3) implies $\frac{1}{m} = 1 - \frac{1}{n}$. Plugging this all in, we have that

$$\alpha = \frac{p}{r}\left(1 - \frac{(1-\alpha)r}{q}\right).$$

We can then solve for α.

$$\alpha = \frac{p}{r}\left(1 - \frac{(1-\alpha)r}{q}\right) \qquad \text{(Given.)}$$

$$\alpha = \frac{p}{r}\left(1 - \frac{r}{q}\right) + \frac{\alpha p}{q} \qquad \text{(Distributing.)}$$

$$\alpha\left(1 - \frac{p}{q}\right) = \frac{p}{r}\left(1 - \frac{r}{q}\right) \qquad \text{(Rearranging.)}$$

$$\alpha = \left(\frac{pq - rp}{rq}\right)\left(\frac{q-p}{q}\right)^{-1} \qquad \text{(Rearranging.)}$$

$$= \left(\frac{pq - rp}{rq}\right)\left(\frac{q}{q-p}\right) \qquad \text{(Taking inverse.)}$$

$$= \frac{pq - rp}{rq - rp} \qquad \text{(Simplifying.)}$$

$$= \frac{1/r - 1/q}{1/p - 1/q} \qquad \text{(Dividing by } rpq\text{.)}$$

From the original equation, we now have that

$$\|f\|_r \leq \left((\|f\|_p^p)^{1/m} (\|f\|_q^q)^{1/n} \right)^{1/r} = \|f\|_p^{p/mr} \|f\|_q^{q/nr}.$$

From (1) and (2), we can substitute the powers by α and $1 - \alpha$, respectively. Note that

$$1 - \alpha = 1 - \frac{1/r - 1/q}{1/p - 1/q} = \frac{1/p - 1/q}{1/p - 1/q} - \frac{1/r - 1/q}{1/p - 1/q} = \frac{1/p - 1/r}{1/p - 1/q}.$$

We therefore have the desired inequality.

Problem 12.17

Prove that the unit ball in $L^p([0,1])$, where $1 \leq p \leq \infty$, is not strongly compact.

Solution:

As a counterexample, we let $n \in \mathbb{N}$, so that $I_n = (2^{-n}, 2^{-(n-1)})$, and $f_n = 2^{n/p}\chi_{I_n}$. Note that

$$\|f_n\|_p = \int_0^1 |2^{n/p}\chi_{I_n}(x)|\, d\mu(x) = \int_{2^{-n}}^{2^{-n+1}} 2^n\, d\mu(x) = 2^n(2^{-n+1} - 2^{-n}) = 2 - 1 = 1.$$

Similarly, we have that $\|f_n - f_m\|\infty = 1$. For $m \neq n$. For $p \in [1, \infty)$, we have that

$$\|f_n - f_m\|_p = (2^n\chi_{I_n}(x) - 2^m\chi_{I_m}(x)\, d\mu(x))^{1/p} = (1+1)^{1/p} = 2^{1/p}.$$

Note that the integral above can be broken into two integrals, which are both of the previous integral. Since we have the difference under the sup-norm is 1, we cannot take infinitely large p to maintain the p-norm less than ε. Therefore, no subsequence of (f_n) can be Cauchy, so none can converge.

Problem 12.18

Give an example of a bounded sequence in $L^1([0,1])$ that does not have a weakly convergent subsequence. Why does this not contract the Banach-Analoglu Theorem?

Solution:

The choice of f_n is similar to Problem 2.17, with $f_n = 2^n \chi_{I_n}$. Suppose that (f_{n_j}) is a subsequence of (f_n), and define a function $g \in L^\infty$ by

$$g = \sum_{j=1}^{\infty} (-1)^j \chi_{I_{n_j}}.$$

Take $\varphi \in (L^1)^*$ as $\varphi(f) = \int fg$. Then,

$$\int_0^1 2^{n_j} \chi_{I_{n_j}}(x) \sum_{k=1}^{\infty} (-1)^k \chi_{I_{n_k}}(x)\, d\mu(x) = \int_{2^{-n_j}}^{2^{-n_j+1}} 2^{n_j} (-1)^j \, d\mu(x) = 2^{n_j}(2^{-n_j} - 2^{-n_j+1})(-1)^j = (-1)^j.$$

Note that when taking the sum, the only term that will survive with respect to the outside indicator function is the one related to n_j. Therefore, (f_{n_j}) cannot converge weakly. Note that this does not contradict the Banach-Alaoglu Theorem since L^1 is not reflexive.

Problem 6.2

Consider $C([0,1])$ with the sup-norm. Let

$$N = \left\{ f \in C([0,1]) : \int_0^1 f(x)\,dx = 0 \right\}$$

be the closed linear subspace of $C([0,1])$ of functions with zero mean. Let

$$X = \{f \in C([0,1]) : f(0) = 0\}$$

and define $M = N \cap X$.

Problem 6.2, part a

If $u \in C([0,1])$, prove that

$$d(u,N) = \inf_{n \in N} \|u - n\| = |\overline{u}|$$

where $|\overline{u}| = \int_0^1 u(x)\,dx$ is the mean of u, so the infimum is attained when $n = u - \overline{u} \in N$.

Solution:

We first consider two cases, where $u \in C([0,1]) \setminus N, u \in N$. If $u \in N$, then $d(u,N) = \inf_{n\in N} \|u - n\| = 0$, since the norm is positive function, and equals zero only when the term inside is equal zero. This implies $n = u$ is the unique element. If $u \in C([0,1]) \setminus N$, we then have by Theorem 6.13 that there us a unique closest element for u in N, denoted y, such that

$$\|u - y\| = \min_{z \in N} \|u - z\|.$$

Furthermore, the element y is the unique element of N with the property that $(u - y) \perp N$. Denote $y = u - \overline{u}$. Note that since $u \notin N$, $\int_0^1 u(x)\,dx \neq 0$, thus $\overline{u} \notin N$. We next need to show that $y \in N$. This is shown by the following:

$$\int_0^1 u(x) - \overline{u}\,dx = \int_0^1 u(x)\,dx - \int_0^1 \overline{u}\,dx = \overline{u} - \overline{u} = 0.$$

Next we need to show that $(u - y) \perp N$. Take $n \in N$, then

$$\int_0^1 (u - y)(x) n(x)\,dx = \overline{u} \int_0^1 n(x) = 0.$$

Therefore, $d(u,N) = |\overline{u}|$.

Problem 6.2, part b

If $u(x) = x \in X$, show that

$$d(x, M) = \inf_{m \in M} \|u - m\| = 1/2,$$

but that the infimum is not attained for any $m \in M$.

Solution:

From part a, we see that

$$d(x, M) = \left| \int_0^1 x \, dx \right| = \frac{1}{2}.$$

Therefore, we choose $y = x - \frac{1}{2}$. Note that $y \notin M$, however, since setting $x = 0$, then $y = -\frac{1}{2}$. This violates the property any element has in M. Therefore, the claim has been shown.

Problem 6.5

Suppose that $\{H_n : n \in \mathbb{N}\}$ is a set of orthogonal closed subspaces of a Hilbert space H. We define the infinite direct sum

$$\bigoplus_{n=1}^{\infty} H_n = \left\{ x_n : x_n \in H_n \text{ and } \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \right\}.$$

Prove that $\bigoplus_{n=1}^{\infty} H_n$ is a closed linear subspace of H.

Solution:

For the sake of simplicity, let $H = \bigoplus_{n=1}^{\infty}$. To show that H closed, take a sequence x_n in H. We want to show that $x_n \to x \in H$. Note that, since $x_n \in H$, then x_n can be written as a summation of sequences y_n^k, for which each $y_n^k \in H_k$. Then $x_n = \sum_k y_n^k$. Then,

$$\|x_n - x_m\|^2 = \left\| \sum_k y_n^k - y_m^k \right\|^2 = \sum_k \left\| y_n^k - y_m^k \right\|^2$$

Note that since each H_k is closed, then $\sum_k \left\| y_n^k - y_m^k \right\|^2 \to 0$. Note that summing over all k implies

$$\left\| y_n^k - y_m^k \right\|^2 \leq \sum_k \left\| y_m^k - y_n^k \right\|^2 \to 0.$$

This tells us $\left\| y_n^k - y_m^k \right\| \to 0$ for all k. Since each H_k is closed, we have that $y_n^k \to y^k$ for some $y^k \in H_k$. We can then show that $x_n \to \sum_k y^k$.

$$\left\| x_n - \sum_k y^k \right\|^2 = \left\| \sum_k (y_n^k - y_n) \right\|^2 = \sum \left\| y_n^k - y_n \right\|^2 \leq \liminf_{m \to \infty} \sum_k \left\| y_n^k - y_m^k \right\|^2.$$

Since each $\left\| y_n^k - y_m^k \right\|$ can be made less than ε for sufficiently large m, n, we have that $\left\| x_n - \sum_k y^k \right\|^2 < \varepsilon$. Therefore, $x = \sum_k y^k$. We now need to show that $y_k \in H$. Note that

$$\sum_k \left\| y^k \right\|^2 \leq \liminf_{m \to \infty} \sum_k \left\| y_m^k \right\|^2 = \liminf_{m \to \infty} \|x_m\|^2 = \|x\|^2 < \infty.$$

Therefore, $y_k \in H$, so H is closed.

Problem 6.11

Prove that if M is a dense linear subspace of a Hilbert space H, then H has an orthonormal basis consisting of elements in M. Does the same result hold for arbitrary dense subsets of H?

Solution:

Since we are not given information on the dimensionality of H, we need to consider two separate cases of when H is finite or infinite dimensional. We will first consider the case when H is finite dimensional. Since any subspace of a linear space is closed in finite dimensions, the only dense linear subspace of H is the space H itself. The claim that H then has an orthonormal basis from M holds since H has an orthonormal basis.

Next consider the case when H is infinite dimesnsional. Since we suppose that M is a dense linear subspace of H, we can take H to be separable. Because of this, there is a countable dense subset $\{x_n : n \in \mathbb{N}\}$ of H. Note that this subset is agnostic to M. Since M itself is dense in H, we can take a sequence $x_{m,n} \in M$ such that $x_{m,n} \to x_n$ as $m \to \infty$. Then the set $\{x_{m,n} : m, n \in \mathbb{N}\}$ is a countable subset of M, which is dense in H.

To connect this new set to H, suppose that we have an arbitrary subset P of M that is dense in H. Let $P_B = \{x_n : n \in \mathbb{N}\}$ then be the largest linearly independent subset of P. Then since P_B spans every element of B, $B \subseteq P_B$, thus P_B is dense in H. Therefore, the closure of P_B is then equal to H. Via Gram-Schmidt, we can take the basis P_B and transform it into an orthonormal set of vectors, P_N whose closed span is equal to the span of P_B. This implies that P_N is an orthonormal basis of H. Since we have that elements of P_N are linear combinations of elements of M, $P_N \subseteq M$, thus P_N is an orthonormal basis of H whose elements belong to the dense linear subspace M.

To give a counterexample for any arbitrary dense subset, take $H = \mathbb{R}^2$, and consider P_B to be any rotation of the basis vectors, e_1, e_2 by an irrational angle, say $n\pi$ degrees for any $n \in \mathbb{N}$. Then, P_B is dense in H, however, P_B does not have a finite set of linearly independent vectors. Since π is irrational, rotations of the basis vectors will never overlap since, if they did, then that would imply that π was irrational, which is not the case. Therefore, the result cannot hold for arbitrary dense subsets of a Hilbert space.

Problem 6.14

Define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right).$$

Problem 6.14, part a

Show that

$$\varphi_n(x) = e^{-x^2/2} H_n(x)$$

is an orthogonal set in $L^2(\mathbb{R})$.

Solution:

First, since each φ_n is orthogonal to every function of the form $e^{(-x^2/2)} p_m$, where p_m is a polynomial of lower degree of n, we just need to show that $e^{-x^2/2} x^m$ is orthogonal to φ_n. Integrating by parts m times, and taking each residual from integration by parts to go to zero, we can see that

$$\int_{\mathbb{R}} e^{-x^2/2} \varphi_n(x)\, dx = (-1)^n \int_{\mathbb{R}} x^m \frac{d^n}{dx^n} \left(e^{-x^2}\right)\, dx = (-1)^{m+n} m! \int_{\mathbb{R}} \frac{d^{n-m}}{dx^{n-m}} \left(e^{-x^2}\right)\, dx = 0.$$

The last integral is equal zero since the differentiated function vanishes as $|x| \to \infty$.

Problem 6.14, part b

show that the n-th Hermite function φ_n is an eigenfunction of the linear operator

$$H = -\frac{d^2}{dx^2} + x^2$$

with eigenvalue $\lambda_n = 2n + 1$.

Solution:

First, let

$$A = \frac{d}{dx} + x, \quad A^* = -\frac{d}{dx} + x.$$

We will first show that $AA^* - 1 = H$. Taking a test function ψ, we see that,

---

$$(AA^*)\psi = A(A^*\psi)$$

$$= A\left(-\frac{d\psi}{dx} + x\psi\right)$$

$$= -A\left(\frac{d\psi}{dx}\right) + A\left(x\psi\right)$$

$$= -\left(\frac{d^2\psi}{dx^2} + x\frac{d\psi}{dx}\right) + \frac{d}{dx}\left(x\psi\right) + x^2\psi$$

$$= -\left(\frac{d^2\psi}{dx^2} + x\frac{d\psi}{dx}\right) + x\frac{d\psi}{dx} + \psi + x^2\psi$$

$$= -\frac{d^2\psi}{dx^2} + \psi + x^2\psi$$

$$= \left(-\frac{d^2}{dx^2} + 1 + x^2\right)\psi$$

Therefore, the action that AA^* preforms on ψ is equivalent to the form above, which is equivalent to $H + 1$. This implies that $AA^* - 1 = H$, which is what we wanted to show.

Next, we need to prove the following recurrence relation between Hermite Polynomials:

$$\frac{dH_n}{dx} = 2nH_{n-1} = -H_{n+1} + 2xH_n$$

We can relate the first and the third relations together via:

$$\frac{dH_n}{dx} = (-1)^n \frac{d}{dx}\left[e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right]$$

$$= (-1)^n e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) + (-1)^n 2xe^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$= -H_{n+1} + 2xH_n$$

Next, we can observe the differentiation term in the $n+1$ Hermite polynomial.

$$\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) = \frac{d^n}{dx^n}\left(-2xe^{-x^2}\right)$$

$$= -2x\frac{d^n}{dx^n}\left(e^{-x^2}\right) - 2n\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)$$

We can then multiply both sides by $(-1)^{n+1}e^{x^2}$ to get

$$H_{n+1} = 2xH_n - 2nH_{n-1}$$

Using the found equation above, we can remove the H_n term from both sides to get

$$\frac{dH_n}{dx} = 2nH_{n-1}$$

Next, we need to investigate the actions A and A^* have on on φ_n. From the found relations above, we have

$$A\varphi_n = \left(\frac{d}{dx} + x\right)\left(e^{-x^2/2}H_n\right)$$

$$= \frac{d}{dx}\left[e^{-x^2/2}H_n\right] + xe^{-x^2/2}H_n$$

$$= -xe^{-x^2/2}H_n + e^{-x^2/2}\frac{dH_n}{dx} + xe^{-x^2/2}H_n$$

$$= e^{-x^2/2}\frac{dH_n}{dx}$$

$$= 2ne^{-x^2/2}H_{n-1}$$

$$= 2n\varphi_{n-1}$$

Similarly,

$$
\begin{aligned}
A^*\varphi_n &= \left(-\frac{d}{dx}+x\right)\left(e^{-x^2/2}H_n\right) \\
&= -\frac{d}{dx}\left[e^{-x^2/2}H_n\right] + xe^{-x^2/2}H_n \\
&= xe^{-x^2/2}H_n - e^{-x^2/2}\frac{dH_n}{dx} + xe^{-x^2/2}H_n \\
&= e^{-x^2/2}\left(-\frac{dH_n}{dx}+2xH_n\right) \\
&= e^{-x^2/2}H_{n+1} \\
&= \varphi_{n+1}
\end{aligned}
$$

We can now finally see that action H has on φ_n. Via everything we have shown above, we can write

$$
\begin{aligned}
H\varphi_n &= (AA^*-1)\varphi_n \\
&= AA^*\varphi_n - \varphi_n \\
&= A(\varphi_{n+1}) - \varphi_n \\
&= 2(n+1)\varphi_n - \varphi_n \\
&= (2n+1)\varphi_n
\end{aligned}
$$

Therefore, φ_n is an eigenfunction of H with eigenvalue $2n+1$.