THE UNIVERSITY OF CHICAGO

S310 NUM OPT

Sec 14: Interior point methods

Recall, Primal and Dual KKT conditions

$$
\begin{aligned}
A^T\lambda + s &= c, \\
Ax &= b, \\
x_i s_i &= 0, \quad i = 1, 2, \ldots, n, \\
(x, s) &\geq 0.
\end{aligned}
$$

$\longrightarrow$

$$
F(x, \lambda, s) = \begin{bmatrix} A^T\lambda + s - c \\ Ax - b \\ XSe \end{bmatrix} = 0,
$$

$$
(x, s) \geq 0,
$$

$$
X = \mathrm{diag}(x_1, x_2, \ldots, x_n),
$$

$$
S = \mathrm{diag}(s_1, s_2, \ldots, s_n)
$$

Apply Newton's Method for Nonlinear Equations

- Duality Measure: $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i s_i = \frac{x^T s}{n}$

- Residuals: $r_b = Ax - b, \qquad r_c = A^T\lambda + s - c$

- Newton's method

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_c \\ -r_b \\ -XSe \end{bmatrix}$$

- Line search to maintain positivity (note matrix is singular if x,s =0)

$$(x, \lambda, s) + \alpha(\Delta x, \Delta\lambda, \Delta s)$$

Centering

- Newton can give REALLY small steps.
- Idea: do a centering step, do not aim all the way to the solution.
- Therefore solve for Xs=\sigma \mu e, \sigma \in [0,1]

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_c \\ -r_b \\ -XSe + \sigma \mu e \end{bmatrix}$$

Conceptual Algorithm

- Compute the duality measure: $\mu_k = (x^k)^T s^k / n$;
- Choose a centering parameter: $\sigma_k \in [0, 1]$
- Solve

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} -r_c^k \\ -r_b^k \\ -X^k S^k e + \sigma_k \mu_k e \end{bmatrix},$$

- Search for nonnegative x,s

$$(x^{k+1}, \lambda^{k+1}, s^{k+1}) = (x^k, \lambda^k, s^k) + \alpha_k (\Delta x^k, \Delta \lambda^k, \Delta s^k).$$

$$(x^{k+1}, s^{k+1}) > 0$$

Why does it work? Central Path

- Feasible set:

$$\mathcal{F} = \{(x, \lambda, s) \mid Ax = b, A^T\lambda + s = c, (x, s) \geq 0\},$$

$$\mathcal{F}^o = \{(x, \lambda, s) \mid Ax = b, A^T\lambda + s = c, (x, s) > 0\},$$

- Central path :

$$(x_\tau, \lambda_\tau, s_\tau) \in \mathcal{C} \Longleftrightarrow \begin{aligned} A^T\lambda + s &= c, \\ Ax &= b, \\ x_i s_i &= \tau, \qquad i = 1, 2, \ldots, n, \\ (x, s) &> 0. \end{aligned}$$

- Minimizers of potential function:

$$\min c^T x - \tau \sum_{i=1}^{n} \ln x_i, \quad \text{subject to } Ax = b$$

Central Path …

- If problem is feasible, it exists and leads to solution.
- At each step we reduce the duality measure, because:

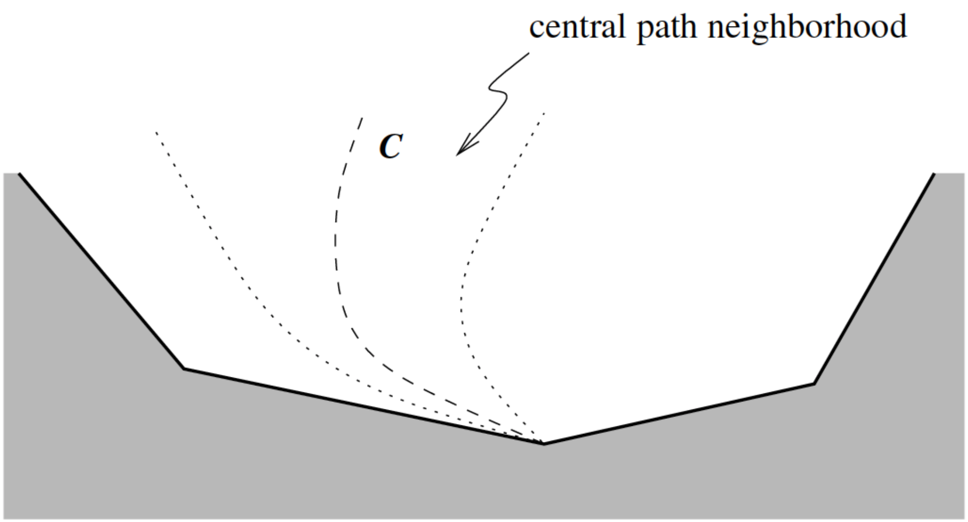
$$s^T\Delta x + x^T\Delta s = \sigma\mu e^T e - x^T s = (\sigma - 1)x^T s$$

$$\begin{aligned}\bar{x}^T\bar{s} &= (x+\alpha\Delta x)^T(s+\alpha\Delta s)\\ &= x^T s + \alpha(s^T\Delta x + x^T\Delta s) + \alpha^2\Delta x^T\Delta s\\ &= (1-\alpha(1-\sigma))x^T s + \alpha^2\Delta x^T\Delta s\end{aligned}$$

- So we are following this central path with \mu->0

Some Geometry

- We cannot go all the way to the boundary, Newton is singular.
- So need to stay in "good" neighborhoods.



$$\mathcal{N}_2(\theta) = \{(x, \lambda, s) \in \mathcal{F}^o \mid \|XSe - \mu e\|_2 \leq \theta \mu\}$$

$$\mathcal{N}_{-\infty}(\gamma) = \{(x, \lambda, s) \in \mathcal{F}^o \mid x_i s_i \geq \gamma \mu \quad \text{all } i = 1, 2, \ldots, n\}$$

A long-step path following algorithm

- Inner loop:

Choose $\sigma_k \in [\sigma_{\min}, \sigma_{\max}]$;
Solve (14.10) to obtain $(\Delta x^k, \Delta\lambda^k, \Delta s^k)$;
Choose α_k as the largest value of α in $[0, 1]$ such that

$$(x^k(\alpha), \lambda^k(\alpha), s^k(\alpha)) \in \mathcal{N}_{-\infty}(\gamma)$$

$$\text{Set } (x^{k+1}, \lambda^{k+1}, s^{k+1}) = (x^k(\alpha_k), \lambda^k(\alpha_k), s^k(\alpha_k))$$

- Can prove I can take large enough steps and (delta independent of n)

$$\mu_{k+1} \leq \left(1 - \frac{\delta}{n}\right) \mu_k,$$

- Then we obtain that: $\mu_k \leq \epsilon \mu_0, \qquad \text{for all } k \geq K \qquad K = O(n \log 1/\epsilon)$
- It is a polynomial algorithm! (sort of)

Infeasible starting points

- There is no need for points to be feasible!
- At each iteration we would obtain residuals decrease by 1-\alpha.
- The analysis becomes different, and the central path may not be defined.
- Idea get infeasible, but positive starting points, by solving

$$\min_x \tfrac{1}{2}x^T x \ \text{ subject to } Ax = b,$$

$$\min_{(\lambda,s)} \tfrac{1}{2}s^T s \ \text{ subject to } A^T\lambda + s = c.$$

- Then perturbing

$$\hat{x} = \tilde{x} + \delta_x e, \quad \hat{s} = \tilde{s} + \delta_s e$$

Linear Algebra

- We solve the system: $$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_c \\ -r_b \\ -r_{xs} \end{bmatrix}$$
- Eliminate last row:

$$\begin{bmatrix} -D^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -r_c + X^{-1} r_{xs} \\ -r_b \end{bmatrix},$$

$$D = S^{-1/2} X^{1/2}. \qquad \Delta s = -X^{-1} r_{xs} - X^{-1} S \Delta x,$$

- Can use Bunch Kaufman now.

Linear Algebra, one step further

- Or we can eliminate the first row as well.

$$AD^2A^T\Delta\lambda = -r_b - AXS^{-1}r_c + AS^{-1}r_{xs}$$

$$\Delta s = -r_c - A^T\Delta\lambda,$$

$$\Delta x = -S^{-1}r_{xs} - XS^{-1}\Delta s,$$

- And use Cholesky, or sparse Cholesky.
- Note: entries in D can go to infinity ….

Summary

- All IP methods for LP are variations on this theme.
- Iterative methods for interior point have not so far been succesful (good research topic ...)
- Interior point methods have polynomial complexity (compare to simplex which does not).
- In practice, simplex is competitive for *most* problems.
- And it is irreplaceable in integer programming.