STAT 31210: Homework 4

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Contents

Problem 5.2

Suppose that $\{e_1, e_2, ..., e_n\}$ and $\{\bar{e}_1, \bar{e}_2, ..., \bar{e}_n\}$ are two bases of the n-dimensional linear space X, with

$$\bar{e}_i = \sum_{j=1}^{n} L_{ij} e_j, \quad e_i = \sum_{j=1}^{n} \bar{L}_{ij} \bar{e}_j$$

where L is an invertible matrix with inverse $\bar{L}$, i.e., $\sum_{j=1}^{n} L_{ij}\bar{L}_{jk} = \delta_{ik}$. Let $\{\omega_1, \omega_2, ..., \omega_n\}$ and $\{\bar{\omega}_1, \bar{\omega}_2, ..., \bar{\omega}_n\}$ be the associated dual bases of X^*.

Problem 5.2, part a

If $x = \sum x_i e_i = \sum \bar{x}_i \bar{e}_i \in X$, then prove that the components of x transform under a change of basis according to

$$\bar{x}_i = \bar{L}_{ji} x_j, \quad \forall\, i = 1, ..., n$$

Solution:

If we start with the first expansion of x in the basis $\{e_1, e_2, ..., e_n\}$ and rewrite e_i into the given form, we have

$$x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \bar{L}_{ij} \bar{e}_j.$$

After some rearranging, we have

$$x = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \bar{L}_{ij} x_i \right) \bar{e}_j.$$

Note that the summation indices can be freely interchanged, so swapping the two indices, we have,

$$x = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \bar{L}_{ji} x_j \right) \bar{e}_i.$$

By the other expansion for x in the basis $\{\bar{e}_1, \bar{e}_2, ..., \bar{e}_n\}$, we can compare the coefficients in each expansion to get

$$\bar{x}_i = \bar{L}_{ji} x_j.$$

Note that this holds for all $i = 1, ..., n$, which is what we wanted to show.

Problem 5.2, part b

If $\varphi = \sum_{i=1}^{n} \varphi_i \omega_i = \sum_{i=1}^{n} \bar{\varphi}_i \bar{\omega}_i \in X^*$, then prove that the components of φ transform under a change of basis according to

$$\bar{\varphi}_i = \sum_{j=1}^{n} L_{ij} \varphi_j, \quad \forall\, i = 1, ..., n$$

Solution:

Investigating the action of φ onto the basis vector $\bar{e}_i$, by the given mapping from $\bar{e} \mapsto e$, we have

$$\varphi(\bar{e}_i) = \varphi\left(\sum_{j=1}^{n} L_{ij} e_j\right).$$

Since φ is a linear operator, which only acts on coordinates, we have that

$$\varphi(\bar{e}_i) = \sum_{j=1}^{n} L_{ij} \varphi(e_j).$$

By our notation, we write $\varphi(\bar{e}_i) = \bar{\varphi}$ and $\varphi(e_j) = \varphi_j$, thus we can write

$$\bar{\varphi}_i = \sum_{j=1}^{n} L_{ij} \varphi_j.$$

This formula will hold for any chosen $i \leq n$, since its choice was arbitrary. Thus, we have proven the statement.

Problem 5.6

Let X be a normed linear space. Use the Hahn-Banach Theorem to prove the following statements:

a) For any $x \in X$, there is a bounded linear functional $\varphi \in X^*$ such that $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$.

b) If $x, y \in X$ and $\varphi(x) = \varphi(y)$ for any $\varphi \in X^*$, then $x = y$.

Solution:

I will include the Hahn-Banach Theorem here for completeness.

Hahn-Banach Theorem: If Y is a linear subspace of a normed linear space X and $\psi : Y \to \mathbb{R}$ is a bounded linear functional on Y with $\|\psi\| = M$, then there is a bounded linear functional $\varphi : X \to \mathbb{R}$ such that φ restricted to Y is equal to ψ and $\|\varphi\| = M$.

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a) Let $x \in X$, and define the subspace Y as any scaling of x. That is, $Y = \{\lambda x : \lambda \in \mathbb{R}\}$. Y is then a linear subspace of X, since it is closed under addition and scalar multiplication ($\mu y_1 + \gamma y_2 = \lambda(\mu + \gamma)x \in Y$). Define the functional $\psi : Y \to \mathbb{R}$ as $\psi(y \in Y) = \|y\|_X = |\lambda| \|x\|_X$. We can note that ψ is bounded, since

$$\|\psi\| = \sup \frac{|\psi(y)|}{\|y\|} = \sup \frac{|\lambda \|x\||}{\|\lambda x\|} = \sup \frac{|\lambda| \|x\|}{|\lambda| \|x\|} = 1.$$

Therefore, $\|\psi\| = 1$ for any $x \in X$. By the Hahn-Banach Theorem, there exists a bounded linear functional $\varphi : X \to \mathbb{R}$ where $\varphi|_Y = \psi$ and $\|\varphi\| = 1$. Therefore, $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$ for any $x \in X$.

b) Suppose that $x \neq y \in X$, but $\varphi(y) = \varphi(x)$. Define $z = x - y \neq 0$. Note that $z \in X$, so by part a there is a linear functional φ' such that $\varphi'(z) = \|z\|$ and $\|\varphi'\| = 1$. The latter is notable, since we have that $\varphi' \neq 0$. Thus, we have that

$$\varphi'(z) = \|z\| = \|x - y\| \neq 0$$

This then implies that $\varphi'(x) \neq \varphi'(y)$. Therefore, a linear functional has been found such that for $x \neq y, \varphi(x) \neq \varphi(y)$. Note that we suppose that $\varphi(x) = \varphi(y)$ should hold for all $\varphi \in X^*$, thus $x = y$.

Problem 5.10

Suppose that $k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function. Prove that the integral operator $K : C([0,1]) \to C([0,1])$ defined by

$$Kf(x) = \int_0^1 k(x,y)f(y)\,dy \quad \text{is compact.}$$

Solution:

By Definition 5.42, we need to show that for any bounded subset in $C([0,1])$, $K(B)$ is a precompact subset of $C([0,1])$. By Theorem 2.12 (Arzelá-Ascoli), we need to show that $\overline{K(B)}$ is bounded, closed, and equicontinuous in $C([0,1])$. Define $A = [0,1]$, which is a compact subset of $\mathbb{R}$.

- Closed:

  This is by definition of the closure of a set. Thus, $\overline{K(B)}$ is closed.

- Bounded:

  Take $f \in B \subseteq C([0,1])$, then $\|f\| = \sup_{x\in A}|f(x)|$. Thus we have

  $$\|Kf\| = \left\|\int_0^1 k(x,y)f(y)\,dy\right\| \leq \sup_{x\in A}\int_0^1 |k(x,y)f(y)|\,dy \leq \sup_{x\in A}\int_0^1 |k(x,y)||f(y)|\,dy$$

  Note that by definition, $|f(y)| \leq \sup_{y\in A}|f(y)| = \|f\|$. This will add another inequality, as well as taking the norm outside of the integral (it is constant with respect to y). We then have that

  $$\|Kf\| \leq \|f\| \sup_{x\in A}\left\{\int_0^1 |k(x,y)|\,dy\right\}.$$

  We are given in Example 5.17 that the sup on the right hand side is the norm of the integral operator. Thus we have that $\|Kf\| \leq \|K\|\|f\|$, which implies that $\overline{K(B)}$ is bounded.

- Equicontinuity:

  Let $f_1, f_2 \in B$. To show equicontinuity, we need to find a δ such that, for any $\varepsilon > 0$, when $d(f_1, f_2 < \delta$, we have that $d(Kf_1, Kf_2) < \varepsilon$. Note the metric we are working under is the sup-norm, so $d(f_1, f_2) = \sup_{x\in A}|f_1(x) - f_2(x)|$, and similar for $d(Kf_1, Kf_2)$. Set $\delta = \frac{\varepsilon}{\|K\|}$. Then, we see the following:

$$\|Kf_1 - Kf_2\| = \sup_{x\in A} |Kf_1(x) - Kf_2(x)| \qquad \text{(Given.)}$$

$$= \sup_{x\in A} |K(f_1(x) - f_2(x))| \qquad \text{(Grouping.)}$$

$$\leq \sup_{x\in A} \{\|K\||(f_1(x) - f_2(x))|\} \qquad \text{(Cauchy-Schwartz.)}$$

$$= \|K\| \sup_{x\in A} \{|f_1(x) - f_2(x)|\} \qquad \text{(Rearranging.)}$$

$$< \|K\| \left(\frac{\varepsilon}{\|K\|}\right) \qquad \left(\delta = \frac{\varepsilon}{\|K\|}.\right)$$

$$= \varepsilon \qquad \text{(Simplifying.)}$$

Therefore, K is an equicontinuous linear mapping. Thus by Arzelá - Ascoli, $\overline{K(B)}$ is a compact subset of $C([0,1])$, which means that K is a compact operator.

Problem 5.11

Prove that if $T_n \to T$ uniformly, then $\|T_n\| \to \|T\|$.

Solution:

We have that $T_n \to T$ uniformly, that is, $\lim_{n\to 0} \|T_n - T\| = 0$. Let $\varepsilon > 0$. Since $T_n \to T$, there exists some $N \in \mathbb{N}$ such that when $n \geq N \implies \|T_n - T\| < \varepsilon$. A quick shortcut can be made via the reverse triangle inequality, but I will explain it slightly more. Consider $\|T_n\|$. This is equivalent to $\|T_n - T + T\|$, that is, adding and subtracting the same number such that we maintain equality. By the triangle inequality, we have that

$$\|T_n - T + T\| \leq \|T_n - T\| + \|T\|.$$

Therefore, we have that $\|T_n\| - \|T\| \leq \|T_n - T\|$. By symmetry of the argument, we have that

$$\Big|\|T_n\| - \|T\|\Big| \leq \|T_n - T\|.$$

Since $\|T_n - T\| < \varepsilon$ for sufficiently large n, then by the reverse triangle inequality explained above, we have that

$$\Big|\|T_n\| - \|T\|\Big| < \varepsilon, \quad \text{For sufficiently large } n.$$

Therefore, $\lim_{n\to\infty} \Big|\|T_n\| - \|T\|\Big| = 0$, or equivalently, $\|T_n\| \to \|T\|$.

Problem 5.17

Suppose that $K : X \to X$ is a bounded linear operator on a Banach space X with $\|K\| < 1$. Prove that $\mathbb{I} - K$ is invertible and

$$(\mathbb{I} - K)^{-1} = \mathbb{I} + K + K^2 + K^3 + ...,$$

where the series on the right hand side converges uniformly in $\mathfrak{B}(X)$.

Solution:

We will first show that $\mathbb{I} - K$ is invertible. Note that this is equivalent to showing $\|\mathbb{I} - K\| > 0$, since if it were, then $\mathbb{I} - K$ would have a nonzero kernel. By the reverse triangle inequality, we can write

$$\Big|\|\mathbb{I}\| - \|K\|\Big| \leq \|\mathbb{I} - K\|$$

We have that $\|K\| < 1$, and $\|\mathbb{I}\| = 1$. Therefore, $\|\mathbb{I}\| - \|K\| > 0$, implying that $\|\mathbb{I} - K\| > 0$, thus $\mathbb{I} - K$ is invertible. Next, we need to show that its inverse is of the given form. Define A_n as the n-th iterate of the sequence on the right hand side, so we can write

$$A_n = \sum_{j=0}^{n} K^j$$

Multiplying on the left by $(\mathbb{I} - K)$ gives

$$(\mathbb{I} - K)A_n = (\mathbb{I} - K)\sum_{j=0}^{n} K^j = \sum_{j=0}^{n} K^j - K^{j+1}$$

We note that $(\mathbb{I} - K)$ is a telescoping series, so we will only be left with K^0 and $-K^{n+1}$, giving

$$(\mathbb{I} - K)A_n = \mathbb{I} - K^{n+1}$$

Next we need to show that $K^{n+1} \to 0$ as $n \to \infty$ in order to justify the equality. Note that $\|K\| < 1$, so $\|K\|^2 \leq \|K\|$. Thus for some index m, we note that $\|K\|^m \leq \|K\|^{m-1} < ... < 1$, thus the sequence $\|K\|^m$ is a monotonically decreasing sequence in $\mathbb{R}$. By the properties of the norm, this sequence is bounded below by zero, and is bounded above by the previous iterate of the sequence. Thus, there is a converging subsequence $\|K\|^{\varphi(n)} \to 0$ as $\varphi(n) \to \infty$, which implies $\|K\|^n \to 0$, so $K^n \to 0$ as $n \to \infty$. This implies that $\lim_{n\to\infty} A_n = (\mathbb{I} - K)^{-1}$. We can recover the bound of the norm on the right hand side by the following:

$$
\begin{aligned}
&\left\|\lim_{n\to\infty} A_n\right\| && \text{(Given.)}\\
&= \lim_{n\to\infty} \|A_n\| && \text{(Limits exist.)}\\
&= \lim_{n\to\infty} \left\|\sum_{j=0}^{n} K^j\right\| && \text{(Definition.)}\\
&\leq \lim_{n\to\infty} \sum_{j=0}^{n} \left\|K^j\right\| && \text{(Triangle inequality.)}\\
&\leq \lim_{n\to\infty} \sum_{j=0}^{n} \|K\|^j && \text{(Cauchy-Schwartz.)}\\
&= \frac{1}{1-\|K\|} && \text{(Geometric series.)}
\end{aligned}
$$

Therefore, the norm on the right hand side is bounded by $(1-\|K\|)^{-1}$. This is well defined, since $\|K\| < 1$. Therefore, $\lim_{n\to\infty} A_n \in \mathfrak{B}(X)$, which completes the proof.