Optimal Hedging with Margin Constraints and Default Aversion and its Application to Bitcoin Perpetual Futures

Carol Alexander[a], Jun Deng[b], and Bin Zou[c]

[a]University of Sussex Business School and Peking University HSBC Business School. Jubilee Building, University of Sussex, Falmer, Brighton, BN1 9RH, UK. Email: c.alexander@sussex.ac.uk.

[b]School of Banking and Finance, University of International Business and Economics. No.10, Huixin Dongjie, Chaoyang District, Beijing, 100029, China. Email: jundeng@uibe.edu.cn.

[c]Department of Mathematics, University of Connecticut. 341 Mansfield Road U1009, Storrs, Connecticut 06269-1009, USA. Email: bin.zou@uconn.edu.

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Abstract

We consider a futures hedging problem subject to a budget constraint that limits the ability of a hedger with default aversion to meet margin requirements. We derive a semi-closed form for an optimal hedging strategy with dual objectives – to minimize both the variance of the hedged portfolio and the probability of forced liquidations due to margin calls. An empirical analysis of bitcoin shows that the optimal strategy not only achieves superior hedge effectiveness, but also reduces the probability of forced liquidations to an acceptable level. We also compare how the hedger's default aversion impacts the performance of optimal hedging based on minute-level data across major bitcoin spot and perpetual futures markets.

Keywords: Cryptocurrency; Futures Hedging; Leverage; Perpetual Swap; Extreme Value Theorem

JEL Classification: G32; G11

1 Introduction

The cryptocurrency bitcoin is the most volatile of all financial assets held by institutional investors. From 7 to 12 March 2020 its spot price fell by 60%, its 30-day implied volatility index almost touched 200% and by 31 March 2020 its 30-day statistical volatility had risen to about 170%.[1] This extremely high price uncertainty imposes tremendous risk to various participants in the cryptocurrency markets and generates a strong hedging demand by investors with long positions in bitcoin. These include, but are certainly not limited to: bitcoin miners such as AntPool;[2] centralised and decentralised cryptocurrency exchanges like Coinbase and Uniswap; businesses and retailers that accept bitcoin as payment, e.g. Microsoft and Wikipedia; and individuals and institutions who hold bitcoin in their asset portfolios.[3]

It is common to hedge long (short) spot price risk by buying (selling) futures contracts on the same asset, or one that is very highly correlated; see Lien and Tse (2002). There is by now a huge literature on hedging with futures, examining the use of commodities, currency, interest rates, stocks, and indexes futures as effective hedges of spot price risk. We refer to Ederington (1979) and Figlewski (1984a) for classical contributions, and Lien and Yang (2008), Mattos et al. (2008), Acharya et al. (2013), Cifarelli and Paladino (2015), Wang et al. (2015), Billio et al. (2018), and Xu and Lien (2020) for more recent works.

Bitcoin futures were introduced in December 2017 and by November 2020 their monthly trading volume had reached $1.32 trillion notional.[4] Nevertheless, at the time of writing there is little in-depth research into the role that bitcoin futures can play to hedge bitcoin spot risk. The two papers that study similar contracts to us agree that they are excellent hedging instruments to segmented bitcoin spot markets: Alexander et al. (2020) show that BitMEX perpetual contracts achieve outstanding hedge effectiveness against the spot risk on Bitstamp, Coinbase, and Kraken. Deng et al. (2020) show that OKEx inverse quarterly contracts are excellent hedging tools for the spot risk on Bitfinex and OKEx, and are superior to CME standard futures in terms of hedge effectiveness. However, neither of these studies accounts for the impact of margin constraints and default aversion on hedging performance.[5]

There are several reasons why the hedging problem is more challenging for bitcoin than it is for other assets: first, many different types of bitcoin futures are traded on multiple exchanges, and the best choice of product and exchange is a problem not often considered for traditional assets;

[1] Details of the bitcoin implied volatility index can be accessed here and the statistical volatility figure is from the Forbes report which can be accessed here.

[2] Antpool is currently one of the largest of several very large mining cooperatives that aim to optimize bitcoin revenue for their members. See Alsabah and Capponi (2020) and Cong et al. (2020a) for further analysis on the centralisation effects of these large pools.

[3] For example, Satoshi Nakamoto, Roger Ver, Charlie Shrem, and the institutions listed in this Forbes report.

[4] See the CryptoCompare December 2020 Exchange Review here.

[5] Deng et al. (2020) claim that margins have little effect on the trading profit and loss and they implicitly assume that the hedger has an infinite supply of bitcoin to meet margin calls.

secondly, a key feature of bitcoin futures is the way that different exchanges apply margin calls to different products, so we need to model margin constraints when making our choice;[6] and thirdly, most exchanges are at most lightly regulated and consequently the risk of default is far higher than it is on standard derivatives exchanges – so the hedger's aversion to default needs to be part of the mathematical problem.

The methodological contribution of this paper is to provide a semi-closed solution to a hedging problem that is entirely new to the finance literature, and which incorporates two key features of bitcoin markets, i.e. margin constraints and default aversion. The margin constraint imposes a positive probability that the hedger's futures positions will be forced to liquidate during the hedge horizon, and such forced liquidations are seen as default events. The hedger seeks an optimal hedging strategy with dual objectives to minimize the risk of the hedged portfolio and the default probability. We apply the extreme value theorem to obtain the optimal strategy and show that both margin constraint and default aversion play a key role in determining its characteristics.

We consider a hedger who has limited resources to meet the margin requirement while trading bitcoin futures, and hence is subject to an upper margin constraint. A second key novelty in our model is to characterize the hedger as a default averse agent. We define a default event for futures traders as the circumstance that they cannot raise enough bitcoin (or fiat currency) to meet the margin calls, i.e. a default event happens upon a forced liquidation. A hedger is called default averse if she views default as an undesirable event. There are at least two theoretical reasons to consider a default averse hedger. First, a default (forced liquidation) is an unexpected and adverse event to hedgers, causing negative impact on their financial goals. This is certainly the case for those who use futures as a hedging instrument. An early liquidation means a failure to hedge the spot risk for the chosen horizon (ending with a naked spot position). Second, when the above defined default happens, hedgers fail to pay liabilities in full amount to their counterpart in futures trading. This hurts their creditworthiness immediately and may further cause exchanges to require higher collateral or enforce more strict policies in their future trading. Including default aversion is also a natural choice in our model; because of the already imposed margin constraint, the default probability is strictly positive as long as hedgers trade futures.

An empirical analysis considers different bitcoin spot exchanges and perpetual futures contracts. This way, we investigate three important topics in risk management: hedge effectiveness, default probability, and leverage. Since bitcoin markets are highly segmented, we compare results for the three most liquid bitcoin perpetual futures, BitMEX, Deribit, and OKEx, and three choices for bitcoin spot prices, i.e. the .BXBT index on BitMEX, and the two largest spot markets, Bitstamp and Coinbase. Our main empirical findings are threefold: (1) A tighter margin constraint or a

[6] Margin mechanisms are essential for ensuring the stability and integrity of futures markets, and there is abundant research on this topic for traditional futures like commodities, precious metals, stock indexes, and currencies; see Figlewski (1984b), Longin (1999), Cotter (2001), Daskalaki and Skiadopoulos (2016), Alexander et al. (2019) and many others.

higher degree of default aversion decreases the hedger's positions in bitcoin futures, justifying the incorporation of these two features in the analysis; (2) The optimal strategy leads to a highly efficient and robust hedging performance across the considered spot and futures exchanges for most hedgers, with exception only for those with extremely tight constraint and high default aversion; (3) By following the optimal strategy, the hedger is able to control the default probability at a reasonable level, close to or less than 1% in all scenarios examined, which is also robust to the choice of bitcoin futures and spot markets.

In the following: Section 2 provides the background for this study, describing the trading characteristics of different types of bitcoin futures on various exchanges; Section 3 formulates the optimal hedging problem under both margin constraint and default aversion, and derives the optimal hedging strategy; Section 4 summarizes the bitcoin spot and futures data used in the empirical analysis, and explains the estimation procedure for model parameters; Section 5 presents empirical results – on sensitivity analysis, hedge effectiveness, and implied leverage under the optimal strategy; and Section 6 concludes. Technical proofs are deferred to Appendix A. Additional computation results are provided in an online supplementary appendix. Results are derived using Matlab code which is available from the authors on reasonable request.

2 Features of Bitcoin Futures Contracts

There are two distinctive types of bitcoin futures contracts: (1) standard futures that have a fixed expiry date and a regular schedule for new issues; and (2) 'perpetual futures' or just *perpetuals*, because they have no expiry date. Perpetual futures are an innovative financial product, so far unique to the cryptocurrency markets.[7] At the time of writing, standard bitcoin futures are traded on the Chicago Mercantile Exchange (CME), Bakkt and numerous other less-regulated exchanges which also trade a variety of direct and inverse perpetual futures. For instance, quarterly contracts are traded on BitMEX and Deribit, and quarterly/weekly are contracts traded on Huobi, Kraken and OKEx.

Apart from their term, a key difference between the two types of contracts is that bitcoin standard futures are denominated and settled in US dollars (domestic currency) with bitcoin as the underlying asset (foreign currency), while the majority of bitcoin perpetual futures contracts are *inverse* futures because they take the opposite design, i.e. they set bitcoin as the denomination unit (domestic currency) and the US dollar as the underlying asset (foreign currency). For instance, one CME bitcoin standard futures contract has a notional value of 5 bitcoin, while one BitMEX bitcoin inverse futures contract has a notional value of 1 US dollar. Please see Alexander et al. (2020) for further details of the different contract specifications.

[7]The official name is [exchange] *perpetual contract*, e.g. BitMEX perpetual contract. Some exchanges call them perpetual *swaps* to highlight that they also have features in common with currency swaps, except there is no exchange of notional. For an example of their terms and conditions, see Deribit.

Should a hedger use standard or inverse futures to hedge the spot price risk of bitcoin? Considering liquidity, trading volumes are much lower on the CME contracts at the time of writing, but on the less-regulated derivatives exchanges like BitMEX, OKEx, Binance, Huobi or Deribit there is not a huge difference between trading volumes on perpetuals compared with standard futures.[8] In terms of availability, bitcoin spot is traded continuously – the markets do not even close on religious holidays – but the CME closes at weekends and holidays. Therefore, we only consider the less-regulated exchanges. Then, comparing hedging costs for standard futures versus perpetuals, the latter are hardly influenced by the swings between backwardation and contango that can induce a high degree of roll-cost uncertainty into hedging with standard futures contracts. This is because the price of the perpetual futures is kept very close to the underlying price by changing the funding rate to be positive (negative) when the perpetual price is greater (less) than the underlying price. Because of this mechanism the basis risk is very small, the perpetual contract price is continually re-aligned with the spot price, and these contracts are highly effective hedges.

To summarise, we have both theoretical and practical reasons to assume hedgers use bitcoin perpetual futures. First, hedgers can use such contracts to match their preferred horizon, of any length *exactly*. Second, an immediate benefit without an expiry is that there is no need to consider rolling the hedge, which can be a highly technical issue in hedging.[9] Third, unregulated exchanges only offer one inverse perpetual futures contract, but several active regular futures contracts. If we otherwise chose regular futures, we would need to debate on which regular contract(s) to use or even on how to construct a synthetic contract with a constant maturity using several contracts. Lastly, a perpetual contract is likely more cost efficient than a regular contract, since no rolling means there is only one entry and one exit in transactions, and the funding payments are usually very small and balance out over the hedge horizon as they regularly fluctuate between positive and negative cash flows.[10] Finally, the open interest and trading volumes of bitcoin futures varies considerably over time, so it may be difficult to arrive at a commonly-accepted preferment of one contract over another. Therefore, this study considers three of the main perpetuals that are settled in bitcoin (XBT), i.e. BitMEX, Deribit, and OKEx.[11] For the spot price we consider the BitMEX underlying, i.e. the .BXBT index, and the bitcoin prices on the two largest spot exchanges, Bitstamp and Coinbase.[12]

[8] See the CryptoCompare Monthly Exchange Review, available here.

[9] When using standard futures in hedging, one often closes the current contract *one week* prior to its expiry and rolls to the next closest contract. For example, Deng et al. (2020) use OKEx quarterly futures and indeed follow such an ad-hoc one-week rule.

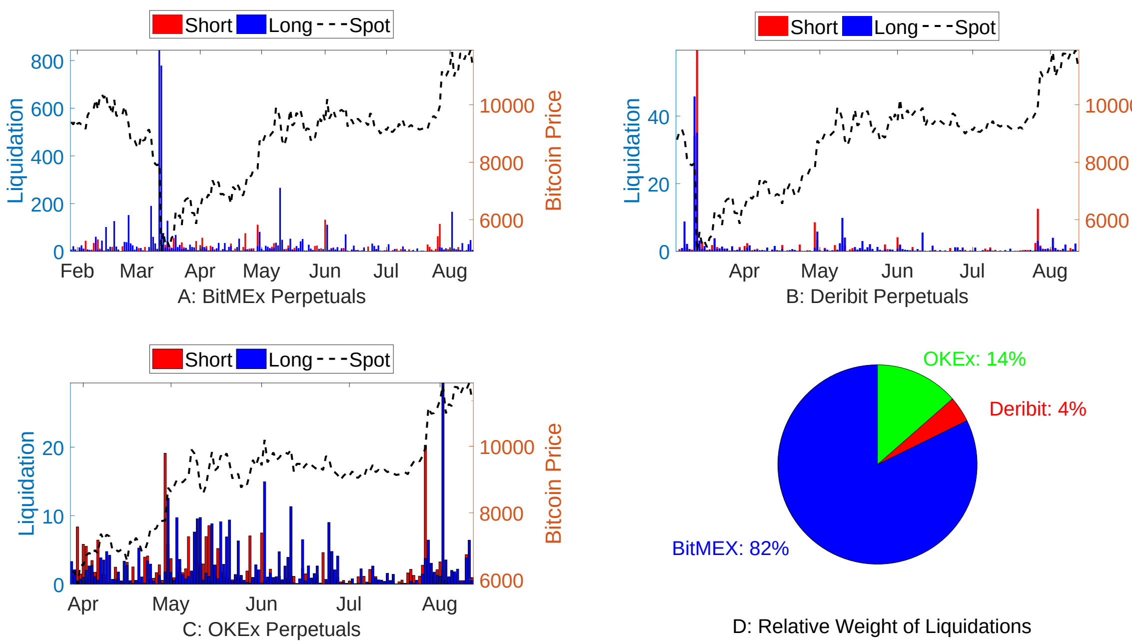
[10] See, for example, Deribit perpetual funding rate.

[11] We use XBT as the code of bitcoin currency, which is used by the International Standards Organization (ISO), see Kraken.com. Although popular, we do not consider the Binance perpetuals because they are quoted and settled in the stable coin tether instead of XBT.

[12] We selected these exchanges because they had the largest trading volumes at the time of writing. However, it should be noted that the unregulated derivatives exchanges may engage in wash trading practices that artificially inflate volumes – see Cong et al. (2020b) – and that new exchanges can quickly rise in volume-based rankings. For instance, Binance's monthly trading volume in November 2020 was up 132% at $405bn since October, making it the

Turning now to margining, in the previous literature on bitcoin futures margins are either set to zero or not considered at all – see Baur and Dimpfl (2019), Alexander et al. (2020) and others. However, we argue that the role of margin requirements is particularly important when trading bitcoin, because leverage is exceptionally high on the less-regulated exchanges, despite the very high volatility of bitcoin. Furthermore, most online cryptocurrency exchanges employ a platform that automatically liquidates the futures positions once traders cannot meet the margin requirement. Put differently, automatic liquidations happen when the losses from futures trading exceed the traders' margin constraint.

Figure 1: Short and Long Forced Liquidations of bitcoin Perpetual Futures



Note. We plot both the short (in red) and long (in blue) forced liquidations of bitcoin perpetual futures on BitMEX in Panel A (from February to August 2020), Deribit in Panel B (from March to August 2020), and OKEx in Panel C (from April to August 2020). In Panels A-C, the left y-axis is the size (volume) of the forced liquidations in million USD, while the right y-axis is the bitcoin spot price in USD, with the black dashed line denoting the spot price. The pie chart in Panel D plots the relative weight of forced liquidations among the three perpetual futures during the overlapping period (from April to August 2020). We obtain data from coinalyze.net.

Figure 1 depicts daily time series of the USD amounts of long and short forced liquidations of bitcoin perpetual futures on BitMEX, Deribit, and OKEx during recent months. Upon closer inspection of these data, we find that the average daily volume of short liquidations is around 11 million USD, while that of long liquidations is about triple the size, at around 32 million USD. On 'Black Thursday' (12 March 2020) a total of 800 million USD worth of long positions in BitMEX perpetual futures were forced into liquidation. This is the main reason why 82% of the liquidation amounts are on BitMEX. Also, apart from Black Thursday, the relative amount of forced liquidation

largest derivatives exchange by monthly trading volume; see CryptoCompare Exchange Review November 2020.

is much lower on Deribit than the other two exchanges. These data testify to the utmost importance of considering margin and liquidation mechanisms when deciding which bitcoin futures to trade.

Because the size of forced liquidations on bitcoin futures can be extremely high, the probability of margin calls will also be high. To verify this empirically, we use data from BitMEX bitcoin perpetual futures to compute the probability that a hedger receives a margin call during an interval of five minutes. We do this for different leverages from 5X to 100X, corresponding to 20% to 1% margins,[13] and hedging horizons between 1 and 30 days. Table 1 presents the results. For positions with the highest leverage (100X), more than 60% of all 1-day hedges end up with facing margin calls; that is a huge risk to hedgers for such a short hedge horizon. The probability quickly gains as the hedge horizon increases (in particular from 1day to 15 days). For the 30-day horizon, even 5X leveraged positions receive margin calls around 40% of the time. The important message from Table 1 is that failing to account for default aversion likely leads to failing to achieve the hedging purpose.

Table 1: Historical Probability of Margin Call for BitMEX Perpetual Futures

	Leverage	5X	15X	20X	50X	100X
Short	1d	0.30%	8.97%	16.05%	44.33%	63.19%
	5d	3.42%	37.26%	47.15%	75.67%	84.79%
	15d	19.54%	65.52%	68.97%	85.06%	91.95%
	30d	41.86%	67.44%	72.09%	83.72%	93.02%
Long	1d	1.44%	12.70%	18.40%	43.73%	61.22%
	5d	9.51%	40.68%	49.81%	67.68%	80.23%
	15d	24.14%	56.32%	66.67%	80.46%	89.66%
	30d	37.21%	76.74%	86.05%	93.02%	95.35%

Note. Here we define a margin call as an event when losses from trading futures exceed the margin deposit over the hedge horizon. We use the 5min historical data of bitcoin perpetual futures on BitMEX from January 1, 2017 to August 12, 2020 to compute the probability of margin calls. We perform computations under different leverage levels, ranging from 5X to 100X, and different hedge horizons, ranging from 1d (day) to 30d.

3 Problem Formulation and Main Results

We consider a discrete-time economy indexed by $\mathcal{T} := \{n\,\Delta t\}_{n=0,1,2,\cdots}$, where time steps Δt are equal. The time-interval Δt can be interpreted as the frequency that the hedged position is monitored.[14] We denote the bitcoin spot price by $S = (S_t)_{t\in\mathcal{T}}$ and the perpetual futures price by

[13]The reciprocal of the leverage is the initial margin percentage, e.g. 100X leverage corresponds to 1% initial margin.

[14]Since bitcoin price is highly volatile, in the empirical analysis we choose monitoring frequencies $\Delta t = 15$ min, 30 min and 1hr.

$F = (F_t)_{t\in\mathcal{T}}$. Both S and F are denominated in fiat currency USD. Define the n-period differences:

$$\Delta S_t(n) := S_{t+n\Delta t} - S_t \qquad \text{and} \qquad \Delta F_t(n) := F_{t+n\Delta t} - F_t, \tag{1}$$

where $t \in \mathcal{T}$ and $n = 1, 2, \cdots$. If $n = 1$ in (1), we simply write ΔS_t and ΔF_t and then – when it is possible without ambiguity – we may also suppress the time subscript t: e.g. ΔS denotes the change in the bitcoin spot price over the time period Δt. Now the nominal value $\widehat{F}$ of a bitcoin inverse perpetual futures contract, and its difference operators are denoted:[15]

$$\widehat{F}_t := \frac{1}{F_t}, \qquad \Delta\widehat{F}_t(n) := \widehat{F}_t - \widehat{F}_{t+n\Delta t}, \qquad \Delta\widehat{F}_t := \widehat{F}_t - \widehat{F}_{t+\Delta t}, \tag{2}$$

where $t \in \mathcal{T}$ and $n = 1, 2, \cdots$. The n-period difference $\Delta\widehat{F}_t(n)$ is the profit and loss (P&L) of a unit long position in bitcoin inverse futures that is opened at t and closed at $t + n\Delta t$. We comment that $\Delta\widehat{F}_t(n)$ in (2) is defined as 'opening price − closing price' which is different from the definition of $\Delta F_t(n)$ in (1). Hence, an increase in the inverse futures price leads to a positive P&L for a long position in inverse futures.

For convenience, and without loss of generality, we consider a representative hedger who holds 1 bitcoin at time $t \in \mathcal{T}$ and who seeks to use a position on an inverse perpetual contract to hedge the spot price volatility from t to $t + N\Delta t$, for a positive integer $N \geq 1$. Because this is an inverse contract, the hedger of a long position *shorts* θ units of bitcoin inverse futures, where the hedging position θ is established at time t and carried over for N time periods. In other words, the hedger follows a static hedge strategy to protect against the volatility of the bitcoin spot price. The (realized) P&L on the hedge is $-\theta\,\Delta\widehat{F}_t(N)$, denominated in XBT. Now, the hedged portfolio consists of one bitcoin spot (long position) and θ bitcoin inverse futures (short position) from t to $t + N\Delta t$. The P&L of such a portfolio is given by $\Delta S_t(N) - \theta\,\Delta\widehat{F}_t(N) \cdot S_{t+N\Delta t}$, where we use the spot price at time $t + N\Delta t$ to convert the trading P&L of inverse futures, $-\theta\,\Delta\widehat{F}_t(N)$, from XBT to USD. This P&L ignores value changes in the margin account, which are minimal because funding rate variations are specifically designed to balance any P&L in the margin account arising from changes in the price of bitcoin. This result naturally leads to the first optimization objective:

- Under the minimum-variance framework, the hedger wants to choose θ to minimize the variance of the P&L of the hedged portfolio, given by:

$$\sigma^2_{\Delta h}(\theta) := \mathbb{V}\mathrm{ar}\big(\Delta S_t(N) - \theta\,\Delta\widehat{F}_t(N)\,S_{t+N\Delta t}\big), \tag{3}$$

[15] Note that $\widehat{F}$ in (2) is defined in terms of one unit of fiat currency (1 USD), which corresponds to the numerator 1 in the definition. If a bitcoin inverse futures contract has a notional value different from 1 (say 100), we simply multiply $\widehat{F}$ by a factor (100).

where $\mathbb{V}\text{ar}(\cdot)$ denotes the variance of a random variable.[16]

As readily seen from (3), the minimum-variance hedging of inverse futures is already more complex than that of standard futures – recall that in the classical minimum-variance framework of Ederington (1979) and Figlewski (1984a) the corresponding variance to be minimized is simply $\mathbb{V}\text{ar}\big(\Delta S_t(N) - \theta\,\Delta F_t(N)\big)$.[17]

The margin mechanism is key to the integrity and stability of futures markets. In Deng et al. (2020), the authors study hedging of bitcoin futures without imposing any margin constraint, i.e. they implicitly assume the hedger has an *infinite* supply of bitcoin to meet margin requirement. However, this is contradicted by the empirical findings presented in Figure 1 and Table 1. To trade bitcoin inverse futures, the hedger needs to deposit a certain amount of bitcoin to meet the initial margin requirement and maintain the amount at a specified level, both of which are articulated using futures contracts. Futures are marked 'period to period' and a positive $\Delta\widehat{F}_t$ means a marked loss to the hedger from t to $t+\Delta t$ is $\theta \cdot \Delta\widehat{F}_t$ which may result in a margin call or even a forced liquidation. This call will be triggered if $\theta \cdot \Delta\widehat{F}_t$ exceeds the amount of bitcoin in the hedger's margin account at time $t + \Delta t$.

We model the impact of margin constraint on the hedger's decisions using the basic approach to margining proposed by Longin (1999). Let m be a positive number, which can be interpreted as an upper constraint on the hedger's ability to meet margin requirement in trading bitcoin futures. That is, if there is an extreme price movement, such that $\theta \cdot \Delta\widehat{F}_{t+n\Delta t} > m$, i.e. the total marked loss is greater than the upper constraint m, the hedger will be forced to liquidate the short position in bitcoin inverse futures, being unable to acquire enough bitcoin to pay the long counterparty.[18] We call such an extreme scenario a *default* event, since the hedger indeed fails to cover the full losses of the short position. This argument motivates us to incorporate the second optimization objective:

- Under the margin constraint m, the hedger wants to minimize the default probability given by:

$$P(m,\theta) = \mathbb{P}\text{rob}(\text{default events}) := \mathbb{P}\text{rob}\left(\theta \cdot \max_{0\le n\le N-1} \Delta\widehat{F}_{t+n\Delta t} > m\right), \tag{4}$$

where $\mathbb{P}\text{rob}(\cdot)$ denotes the probability of an event.

By definition (4), the default probability $P(m,\theta)$ is a decreasing function of the constraint m and

[16] Regarding the subscript Δh of notation $\sigma^2_{\Delta h}(\theta)$ in (3), Δ means the difference in portfolio values (i.e. P&L) and h denotes the **h**edged portfolio.

[17] Minimizing the portfolio variance (risk) is an important criterion in portfolio management, dating back to Markowitz's seminar mean-variance portfolio theory, and is recently adopted in robo-advising; see Capponi et al. (2020) and Dai et al. (2020a,b).

[18] Recall that $\Delta\widehat{F}_{t+n\Delta t}$ is the P&L of trading one unit of bitcoin inverse futures for one period from $t+n\Delta t$ to $t+(n+1)\Delta t$, and a positive $\Delta\widehat{F}_{t+n\Delta t}$ means losses to short hedgers.

an increasing function of the position θ. The economic meaning is clear: more financial reserves (larger m) or less risk taking activities (smaller θ) both reduce the chance of default.

From the arguments leading to (3) and (4) the hedger has dual objectives to minimize, and a natural way is to aggregate them into one objective. To balance the magnitude of two objectives we multiply the default probability $P(m,\theta)$ by a factor $\sigma^2_{\Delta S_t(N)}$, the variance of the N-period spot price changes and to aggregate them we introduce a parameter γ. This way, we consider the following optimal hedging problem for the hedger:

Problem 3.1. *The hedger, possessing one bitcoin at time t, seeks an optimal static hedging strategy θ^* for N periods that solves the following problem:*

$$\min_{\theta>0} \quad \left\{\sigma^2_{\Delta h}(\theta) + \gamma\, \sigma^2_{\Delta S_t(N)}\, P(m,\theta)\right\}, \tag{5}$$

where $\sigma^2_{\Delta h}(\theta)$ is given by (3), $\gamma > 0$ is the aggregation factor, $\sigma^2_{\Delta S_t(N)}$ is the variance of the N-period spot price changes, $m > 0$ represents the margin constraint, and $P(m,\theta)$ is given by (4).

One may also interpret γ as a *default aversion* parameter that captures the extent of the hedger's dislike of the default events defined in (4). As γ increases, the hedger fears default events more and will therefore trade in a more conservative way. The limiting case of $m = +\infty$ (or $\gamma = 0$) is studied in Deng et al. (2020) and corresponds to the scenario where default events have no impact on the hedging decisions. Note that another extreme case is when $\theta = 0$, in which case default will not occur because the hedger has no position in futures.

The main theoretical results of the paper is a semi-closed solution to Problem (5) which is given in the following theorem and the proof is given in the Appendix A.

Theorem 3.2. *The optimal hedging strategy θ^* to Problem (5) is given by*

$$\theta^* = \frac{F_t^2}{S_t}\,\theta_0, \tag{6}$$

where θ_0 is a positive root to the equation $f(x) = 0$, where:

$$\begin{aligned} f(x) := {} & \frac{\gamma m W_t}{2\alpha}\,\frac{\sigma^2_{\Delta S_t(N)}}{\sigma^2_{\Delta F_t(N)}}\,\exp\left[-\left(1+\tau\frac{\frac{mW_t}{x}-\beta}{\alpha}\right)^{-\frac{1}{\tau}}\right]\left(1+\tau\frac{\frac{mW_t}{x}-\beta}{\alpha}\right)^{-\frac{1}{\tau}-1}\cdot\frac{1}{x^2} \\ & + x - \frac{\sigma^2_{\Delta S_t(N),\Delta F_t(N)}}{\sigma^2_{\Delta F_t(N)}}. \end{aligned} \tag{7}$$

In (7), $W_t := S_t/F_t^2$, constants α, β and τ are estimation parameters of the right tail of $\Delta\widehat{F}$ based on the extreme value theorem (see (A.2) in Appendix), $\sigma^2_{\Delta S_t(N)}$ (resp. $\sigma^2_{\Delta F_t(N)}$) denotes the variance of the N-period spot (resp. futures) price changes, and $\sigma^2_{\Delta S_t(N),\Delta F_t(N)}$ denotes the covariance between the two random variables.

To provide more insight to the rather complex expression in (6), let us take a closer look at two extreme cases when $\gamma = 0$ or $m = +\infty$. In these two cases, the first term of $f(x)$ in (7) becomes zero and the optimal strategy $\widetilde{\theta}^*$ ($=\theta^*|_{\gamma=0} = \theta^*|_{m=\infty}$) is reduced to

$$\widetilde{\theta}^* = \frac{F_t^2}{S_t} \underbrace{\left(\rho_{\Delta S_t(N), \Delta F_t(N)} \frac{\sigma_{\Delta S_t(N)}}{\sigma_{\Delta F_t(N)}} \right)}_{:=\widetilde{\theta}_0} = \frac{F_t^2}{S_t} \widetilde{\theta}_0, \tag{8}$$

where $\rho_{\Delta S_t(N), \Delta F_t(N)}$ is the correlation between $\Delta S_t(N)$ and $\Delta F_t(N)$. Note that $\widetilde{\theta}_0$ is the optimal hedging strategy when a standard futures contract is used as the hedging instrument. Therefore, even in such simplified cases, the optimal strategy of inverse futures $\widetilde{\theta}^*$ still differs from that of standard futures by a factor F_t^2/S_t. Since $S_t \approx F_t$ and the current bitcoin price is above 5 digits USD, missing this factor could lead to catastrophic consequences when using inverse futures to hedge spot risk. Incorporating a margin constraint m and default aversion γ significantly complicates the analysis and introduces a non-linear adjustment to correct $\widetilde{\theta}_0$ into θ_0.

4 Data Summary and Parameters Estimation

4.1 Data Description

At the time of writing, the top three exchanges trading bitcoin perpetual futures by average daily volumes are BitMEX, OKEx, and Deribit, with relative market shares given respectively by 77%, 17%, and 6% (see the right panel of Figure 2). All three exchanges are online trading platforms that operate continuously and we refer readers to their websites for full contract specifications.[19] Since bitcoin spot markets are severely segmented, we also consider three different possibilities for the bitcoin spot price in the empirical analysis: the .BXBT index on BitMEX and the two largest bitcoin spot markets, Bitstamp and Coinbase.[20] We retrieve data at the 5-minute frequency on these prices using the API (application programming interface) provided by CoinAPI covering the periods shown in Table 2.[21] Please refer to Table 2 for the information of the bitcoin spot and futures price data used in the empirical analysis.

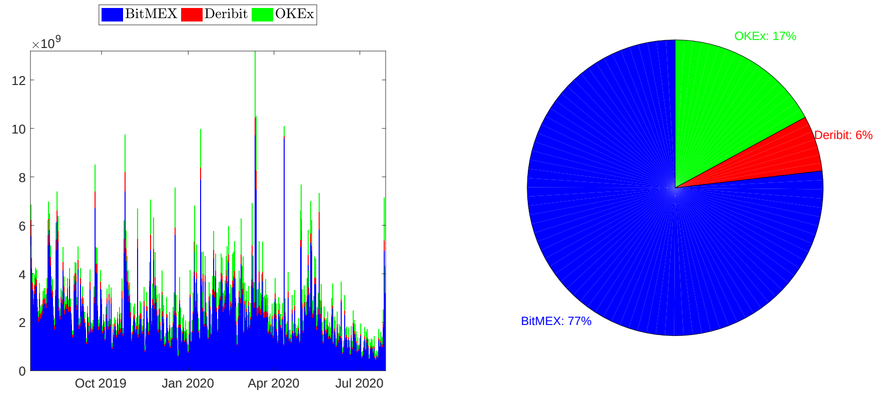
To calculate 1-period changes in the spot price ΔS, the futures price ΔF, and the 1-period change in the inverse futures price, $\Delta \widehat{F}$, we consider the time step at three different values $\Delta t =$ 15 minutes, 30 minutes and 1 hour. Recall that $\Delta \widehat{F}_s = \widehat{F}_s - \widehat{F}_{s+\Delta t} = 1/F_s - 1/F_{s+\Delta t}$. Table 3

[19] For instance, the full specifications of BitMEX perpetual futures (XBTUSD) can be accessed from link.

[20] Bitstamp and Coinbase are two dominant exchanges for trading bitcoin with the latter having the biggest trading volumes; see Figure 2 and Table 3 in Alexander and Heck (2020). The .BXBT index is the reference price for the BitMEX perpetual futures (XBTUSD), which tracks the bitcoin price every minute and is calculated as the weighted average of the last price of 5 currently active constituent exchanges: Bitstamp (25.73%), Bittrex (2.63%), Coinbase (46.98%), Gemini (6.69%), and Kraken (17.97%); see details on BitMEX link.

[21] CoinAPI is a platform providing fast, reliable, and unified data APIs for cryptocurrency markets.

Figure 2: Trading Volumes of bitcoin Perpetual Futures on BitMEX, Deribit, and OKEx



Note. The left panel plots daily trading volumes of bitcoin perpetual futures on BitMEX, Deribit, and OKEx, while the right panel plots their relative weight of aggregate volumes. The unit of y-axis in the left panel is USD. For both panels, data spans from July 2019 to July 2020.

Table 2: Data Information of bitcoin Spot and Perpetual Futures

	Exchange	Start Date	End Date	# of Obs.
	Bitstamp	2017/1/1	2020/8/12	380,161
bitcoin Spot	Coinbase	2017/1/1	2020/8/12	380,161
	BitMEX	2018/4/26	2020/8/12	241,921
	BitMEX	2017/1/1	2020/8/12	380,161
Perpetual Futures	Deribit	2018/8/15	2020/7/27	205,345
	OKEx	2019/7/18	2020/7/31	112,897

Note. This table reports the start/end date and the number of observations for each of the three spot and three futures price data sources used in the empirical analysis.

reports the summary statistics of the price changes ΔS and ΔF at different time steps Δt. The mean is very close to zero, but the standard deviation is very large. Expressed in daily terms, it ranges from 37 to 167. There is a strong negative skewness especially in the BitMEX index and OKEx perpetual futures and the kurtosis is extremely high. Using data at the base frequency of 5 minutes data the correlations between ΔS and ΔF, for all possible combinations of bitcoin spot and perpetual futures, are displayed in Table 4. As expected, there exists a strong positive correlation. The highest correlation between ΔS and ΔF is 0.96 (Coinbase spot and OKEx futures) while the lowest is 0.79 in the cases of BitMEX spot and BitMEX/OKEx futures.

4.2 Parameter Estimation

To calculate the optimal hedging strategy θ^* in (6), we need to estimate the variances $\sigma^2_{\Delta S(N)}$ and $\sigma^2_{\Delta F(N)}$, the covariance $\sigma^2_{\Delta S(N),\Delta F(N)}$ and the tail distribution parameters α, β and τ. Here we demonstrate how these quantities are estimated as time series, using a fixed-size window that rolls over the entire sample. For illustration purpose, we fix the monitoring frequency Δt to be 30 minutes and the hedge horizon $N\Delta t =$ to be 5 days. We also fix the length of the rolling window

Table 3: Summary Statistics of the bitcoin Spot and Futures Price Changes

	BitMEX Index ΔS			Bitstamp ΔS			Coinbase ΔS		
Variables	15min	30min	1h	15min	30min	1h	15min	30min	1h
Min	-1274.79	-1006.46	-1184.61	-1011.89	-1328.27	-1295.01	-1261.21	-1517.11	-1605.00
P25	-8.38	-11.90	-16.32	-8.58	-11.73	-16.15	-8.00	-11.17	-15.81
Median	0.15	0.19	0.33	0.16	0.29	0.59	0.06	0.26	0.56
P75	8.66	12.36	16.92	9.14	12.62	17.82	8.59	12.04	17.33
Max	710.71	645.02	949.39	1045.00	1392.98	1218.00	1116.64	1241.36	1198.12
Mean	0.03	0.07	0.14	0.08	0.17	0.34	0.08	0.17	0.33
Skewness	-2.72	-1.52	-1.14	-0.82	-0.75	-0.74	-0.85	-0.94	-0.67
Kurtosis	120.78	59.61	49.75	64.78	53.15	37.47	92.57	61.55	40.65
Daily S.D.	30.79	43.33	62.19	41.04	57.08	79.80	41.04	56.66	79.22
# Obs.	80,640	40,320	20,160	126,720	63,360	31,680	126,720	63,360	31,680
	BitMEX ΔF			Deribit ΔF			OKEx ΔF		
	15min	30min	1h	15min	30min	1h	15min	30min	1h
Min	-1011.00	-1285.00	-1389.00	-1405.50	-1037.50	-1291.00	-1452.00	-1154.60	-1420.80
P25	-8.00	-11.00	-15.50	-8.00	-11.50	-15.50	-11.20	-15.20	-21.60
Median	0.00	0.29	0.50	0.00	0.00	0.50	0.10	0.00	0.60
P75	8.50	12.00	17.00	8.50	12.00	17.00	11.70	15.90	21.90
Max	1204.50	1048.00	1059.50	608.00	661.00	942.50	723.80	676.00	945.10
Mean	0.08	0.17	0.34	0.07	0.15	0.29	0.05	0.10	0.20
Skewness	-0.74	-0.74	-0.77	-2.81	-1.75	-1.39	-3.56	-2.24	-1.52
Kurtosis	71.52	50.89	40.15	131.20	61.18	54.44	167.44	74.55	59.49
Daily S.D.	42.05	58.21	81.63	31.30	44.13	64.03	34.02	46.45	66.22
# Obs.	126720	63360	31680	68448	34224	17112	37632	18816	9408

Note. This table reports the summary statistics of spot price changes ΔS and futures price changes ΔF across different spot and futures markets. P25 and P75 refer to 25% and 75% quantiles, S.D. denotes standard deviation, and # Obs. means the number of observations.

Table 4: Correlations of bitcoin Spot and Perpetual Futures Price Changes

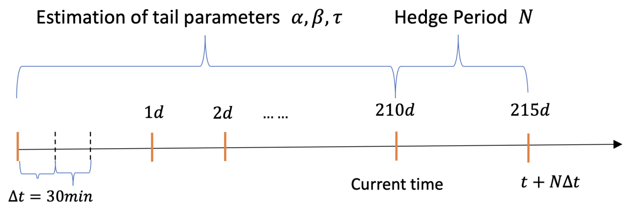
		Spot			Futures		
	variables	ΔBitMEX	ΔBitstamp	ΔCoinbase	ΔBitMEX	ΔDeribit	ΔOKEx
Spot	ΔBitMEX	1.00	0.80	0.81	0.79	0.80	0.79
	ΔBitstamp	0.80	1.00	0.82	0.87	0.89	0.93
	ΔCoinbase	0.81	0.82	1.00	0.88	0.92	0.96
Futures	ΔBitMEX	0.79	0.87	0.88	1.00	0.92	0.95
	ΔDeribit	0.80	0.89	0.92	0.92	1.00	0.88
	ΔOKEx	0.79	0.93	0.96	0.95	0.88	1.00

Note. This table reports the correlation coefficient between ΔS and ΔF for different choices of bitcoin spot S and perpetual futures F. All the results are computed using the 5-min data over the full available sample period.

to be 210 days and use the observations on ΔS, ΔF and $\Delta\widehat{F}$ at $\Delta t = 30$-min frequency.

To estimate the desired variances and covariance we derive a sub-sample from our 210-day window of 30-minute returns, which is a sub-sample of 42 observations on each 5-day price difference $\Delta S(N)$ and $\Delta F(N)$, and apply the standard operators to this sub-sample. Then we compute another sub-sample which consists of the maximum 30-minute price change $\Delta\widehat{F}$ on each day – this is taken as the 'extreme value' for that day. That is, we take the maximum of $\left(\Delta\widehat{F}_{t+\Delta t}, \Delta\widehat{F}_{t+2\Delta t}, \cdots, \Delta\widehat{F}_{t+48\Delta t}\right)$ as the extreme value on day t. This way, we construct a sub-sample of 210 extreme values which we use to estimate the right tail distribution parameters α, β and τ for that window (see (A.2)). Please refer to Figure 3 for graphic illustrations of the procedure. Please see Online Supplementary Appendix I for further details.

Figure 3: Parameters Estimation Procedure



The parameter τ measures the heaviness of the right tail, i.e. the losses to hedgers with short futures positions, and this parameter has a major impact on the default probability. On the other hand, the scale parameter α and the location parameter β represent the dispersion and the average of the extreme value observations, which are less important because we can always change the scale and location. So let us now discuss our findings on τ, since it is the most important parameter for hedging. Table 5 reports summary statistics of the τ parameter estimates, just for the case $\Delta t =$ 30min and $N\Delta t = 5$ days, and Figure 4 depicts how this estimate evolves as the 210-day window rolls over the sample.

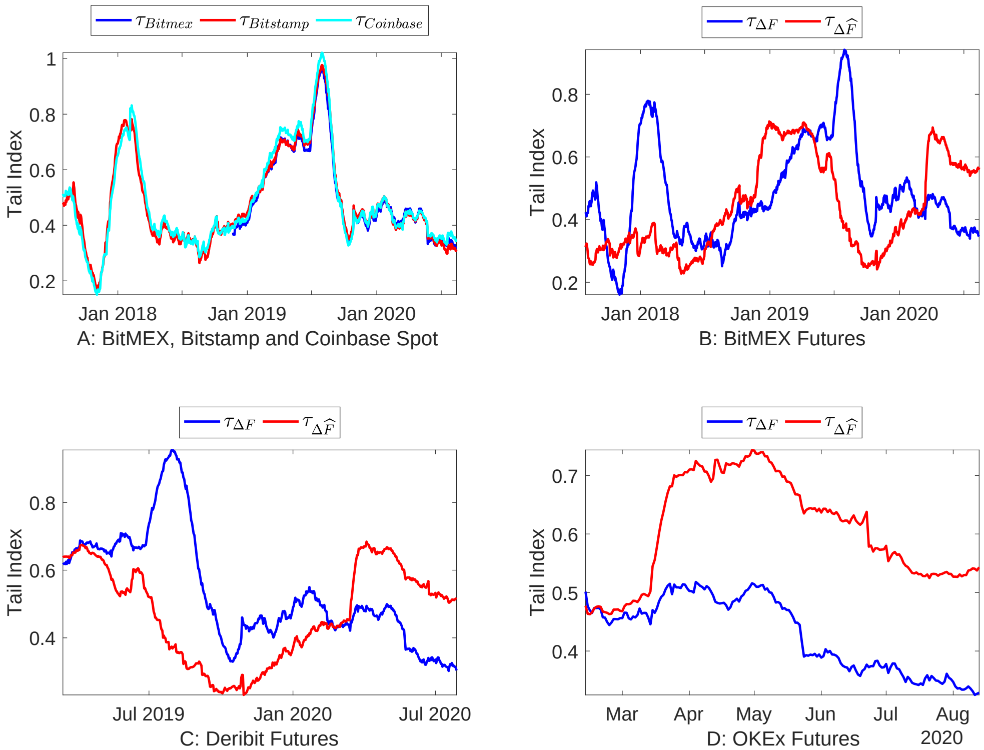
Table 5: Summary Statistics of Estimated Tail Index τ

	ΔS Spot Price Difference			ΔF Futures Price Difference					
Variables	$\tau_{\Delta S}$ BitMEX	$\tau_{\Delta S}$ Bitstamp	$\tau_{\Delta S}$ Coinbase	$\tau_{\Delta F}$ BitMEX	$\tau_{\Delta \widehat{F}}$ BitMEX	$\tau_{\Delta F}$ Deribit	$\tau_{\Delta \widehat{F}}$ Deribit	$\tau_{\Delta F}$ OKEx	$\tau_{\Delta \widehat{F}}$ OKEx
Min	0.32	0.17	0.15	0.16	0.23	0.30	0.23	0.32	0.46
Mean	0.52	0.48	0.49	0.48	0.44	0.54	0.47	0.43	0.61
Median	0.45	0.43	0.43	0.45	0.39	0.49	0.49	0.45	0.62
Max	0.97	0.98	1.02	0.94	0.71	0.95	0.68	0.52	0.74
Daily S.D.	0.16	0.17	0.17	0.16	0.15	0.16	0.14	0.06	0.09
Count	631	1111	1111	1111	1111	504	504	183	183

Note. This table reports summary statistics for the estimated tail index parameter τ for ΔS, ΔF and $\Delta \widehat{F}$ using a rolling 210-day sample from all available data sources. We set $\Delta t = 30$ min and $N\Delta t = 5$ days.

The most important finding is that τ remains positive over the entire sample period, for all variables, which immediately implies the limiting distributions are Fréchet type – see also Figure A.1 in the appendix. Moreover, they have very heavy right tails. This indicates that taking short positions can lead to substantial losses. From Table 5, we observe that the mean and median of the estimated τ's are almost the same except for $\Delta \widehat{F}$ computed from the OKEx perpetual futures and possibly the Deribit perpetual futures as well. Panel A of Figure 4 shows that the estimated τ of the spot price changes ΔS are very close to each other for the three spot exchanges BitMEX, Bitstamp and Coinbase. In comparison, regarding panels B – D, the estimated τ's of ΔF and $\Delta \widehat{F}$

Figure 4: Rolling Estimation of τ for ΔS, ΔF, and $\Delta \widehat{F}$



Note. This figure plots the rolling estimation of the tail index parameter τ for ΔS, ΔF, and $\Delta \widehat{F}$ using a rolling window of 210 days. The full sample data are specified in Table 2. For illustration we set the monitoring frequency of $\Delta t = 30$ min and a hedge horizon $N\Delta t = 5$ days. Panel A plots the estimated τ of ΔS for BitMEX, Bitstamp and Coinbase. This shows that the different τ estimates are very close to each other for the three exchanges. Panels B to D plot the estimated τ of ΔF and $\Delta \widehat{F}$ for the bitcoin perpetual futures traded on BitMEX, Deribit and OKEx, respectively. These exhibit very different features across the three exchanges. All we can say is that, in most cases, $\tau_{\Delta \widehat{F}}$ is larger than $\tau_{\Delta F}$.

vary more significantly from exchange to exchange, and for a fixed exchange, $\Delta \widehat{F}$ produces a higher τ than ΔF in most cases.

Now at each time t we use the rolling window of the preceding 210 days to estimate all the parameters and we apply (6) to compute the current optimal strategy θ_t^*, which will be followed from t to $t + N\Delta t$. We denote by ΔV^* the P&L of the hedged portfolio for N periods under the optimal strategy θ^*. For instance, the P&L of strategy θ_t^* from t to $t + N\Delta t$ is given by $\Delta V_t^* = \Delta S_t(N) - \theta_t^* \cdot \Delta \widehat{F}_t(N)\, S_{t+N\Delta t}$. Once we have the realized price data at time $t + N\Delta t$, we can compute ΔV_t^* and store it. Then, we roll the 210-day sample of 30-minute data forward by 30-minutes and repeat all parameter estimates, compute the optimal strategy and calculate ΔV_t^*. We continue this process until we arrive at the last available time point, i.e. exactly $N\Delta t$ periods prior to the end of the full sample. This constitutes our realized time series of ΔV^*, the P&L of the optimally hedged portfolio, that will be used to investigate hedge effectiveness in the next section.

5 Empirical Analysis

In this section, we conduct empirical analysis to investigate the economic consequences of the optimal hedging strategy θ^*, derived in (6), for the representative hedger. We begin with a sensitivity analysis for θ^* in Section 5.1. Next, we study two important topics related the hedger's dual objectives: hedge effectiveness in Section 5.2 and default probability in Section 5.3. We close the section with investigations on the implied leverage under θ^* in Section 5.4.

5.1 Sensitivity Analysis

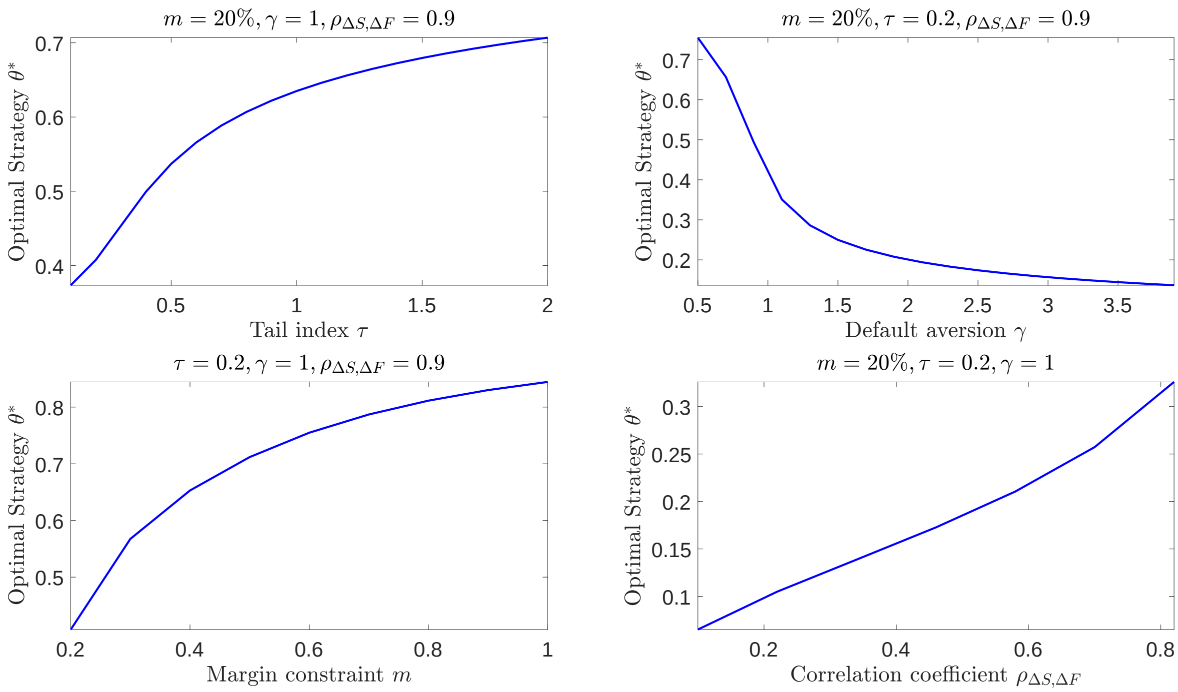
To understand the results obtained in this sub-section, recall that the representative hedger seeks the optimal strategy θ^* with dual objectives to minimize the portfolio variance (risk) $\sigma^2_{\Delta h}$ in (3) and the default probability $P(m, \theta)$ in (4) and that the hedger's optimal strategy θ^* can be computed efficiently once all the parameters in (6) have been estimated or assigned. Since we are mostly interested in the *qualitative* behavior of the optimal strategy θ^* with respect to various parameters, we assign reasonable base values and conduct sensitivity analysis focusing on four parameters: the default aversion γ and margin constraint m, which are specific to the hedger's own profile, and the tail index τ and correlation coefficient $\rho_{\Delta S,\Delta F}$ which are purely market-specific. The results are presented in Figure 5.

We now discuss the findings from Figure 5. The first impression is that both margin constraint and default aversion have a major effect on the optimal strategy θ^*, by carefully examining the relative scales along the y-axis. This provides further justifications for incorporating them in the problem setup (5). The impact of the margin constraint m and the default aversion γ on the optimal strategy θ^* is only through their influence on the default probability. When default aversion γ increases, minimizing the default probability becomes a more important task to the hedger and a natural solution is to reduce the size of the futures position – because, by definition (4), a decrease of θ leads to a smaller $P(m, \theta)$. This observation amounts to the optimal strategy θ^* being a decreasing function of γ, as shown in the upper right panel. When m increases, the hedger's financial reserves improve, naturally reducing the likelihood of default and thus alleviating the constraint on trading. In consequence, we expect the hedger to increase their futures position, and this is confirmed by the lower left panel in Figure 5.

Next we analyse the effect of the other two parameters, the tail index τ and the correlation coefficient $\rho_{\Delta S,\Delta F}$, which are market-related and independent of the hedger's particular profile. From the standard hedging theory and also the result in (8), the correlation coefficient $\rho_{\Delta S,\Delta F}$ appears as a positive multiplier in the optimal strategy and a near-linear positive relation between $\rho_{\Delta S,\Delta F}$ and θ^* is anticipated, which is verified by the lower right panel in Figure 5.[22] According

[22]Recall from (8) that, without margin constraint and default aversion, the optimal strategy $\widetilde{\theta}^*$ is indeed a linear function of $\rho_{\Delta S,\Delta F}$. Once both features are included, a non-linear adjustment is needed; however, the leading term

Figure 5: Sensitivity Analysis of the Optimal Strategy θ^*



Note. We conduct sensitivity analysis of the optimal strategy θ^* on four parameters: tail index τ, default aversion γ, margin constraint m, and correlation coefficient $\rho_{\Delta S,\Delta F}$. Here, we express m in percentage since the hedger's initial holding in bitcoin is normalized to one. In each of the four panels, we study the impact of one target parameter (shown as the x-axis label) on the optimal strategy, while fixing other three parameters as shown on top. We set the remaining parameters as: $\sigma^2_{\Delta S} = \sigma^2_{\Delta F} = 1$, $S_t = F_t = 1$, $\alpha = \beta = 0.2$.

to the extreme value theorem, tail index τ measures the heaviness of the tails of $\Delta\widehat{F}$; a larger τ means heavier tails or more extreme price changes (see Figure A.1). The impact of τ on the optimal strategy is rather complex and highly non-linear, as seen from the definition of f in (7). The upper left panel in Figure 5 indicates a positive relationship between τ and θ^*.[23]

5.2 Hedge Effectiveness

In this sub-section, we study the hedge effectiveness of bitcoin perpetual futures in hedging bitcoin spot risk. To that end, we consider two strategies (portfolios) for the hedger: the first one is an *unhedged* portfolio holding one bitcoin, and the second one is an optimally *hedged* portfolio that consists of one bitcoin and a short position of θ^* perpetual futures contracts, where the optimal strategy θ^* is given by (6). Since the tail index parameter τ is rather stable with respect to the choice of time step Δt (see Table I.1), we fix the monitoring frequency $\Delta t = 30$ minutes in the following analysis.

First, we focus on the P&L of these two (unhedged and hedged) portfolios. We apply the rolling

is still determined by $\widetilde{\theta}^*$.

[23]This positive relationship can also be verified mathematically by showing that the first non-linear term in (7) increases as τ increases.

window estimation procedure described in Section 4.2 under three possible hedge horizons $N\Delta t =$ 5 days, 15 days, and 1 month. The margin constraint m is taken to be 10% for the moment. This is equivalent to assuming that the hedger who is long 1 bitcoin is able to acquire an additional 0.1 bitcoin to top up the margin requirement when trading bitcoin futures. Figures 6 and 7 draw the histograms of the value changes of these two portfolios, along with the optimal strategy θ^*, in the cases that $\gamma = 20$ and $\gamma = 100$ respectively. For illustration we choose the spot price to be the BitMEX .BXBT index and the futures price to be that of BitMEX perpetuals in Figures 6 and 7.

Recall that γ is the default aversion parameter (or aggregation factor) in the main problem (5). A larger γ means the hedger is more risk averse to 'default events' and concerns more about forced liquidations. When the hedger's default aversion is low (e.g. $\gamma = 20$ in Figure 6), the volatility of the hedged portfolio is much smaller than that of the unhedged portfolio, indicating the optimal hedging strategy works as hoped in reducing risk. However, if the hedger has a very high default aversion (e.g. $\gamma = 100$ as in Figure 7), the degree of volatility reduction from hedging is limited.

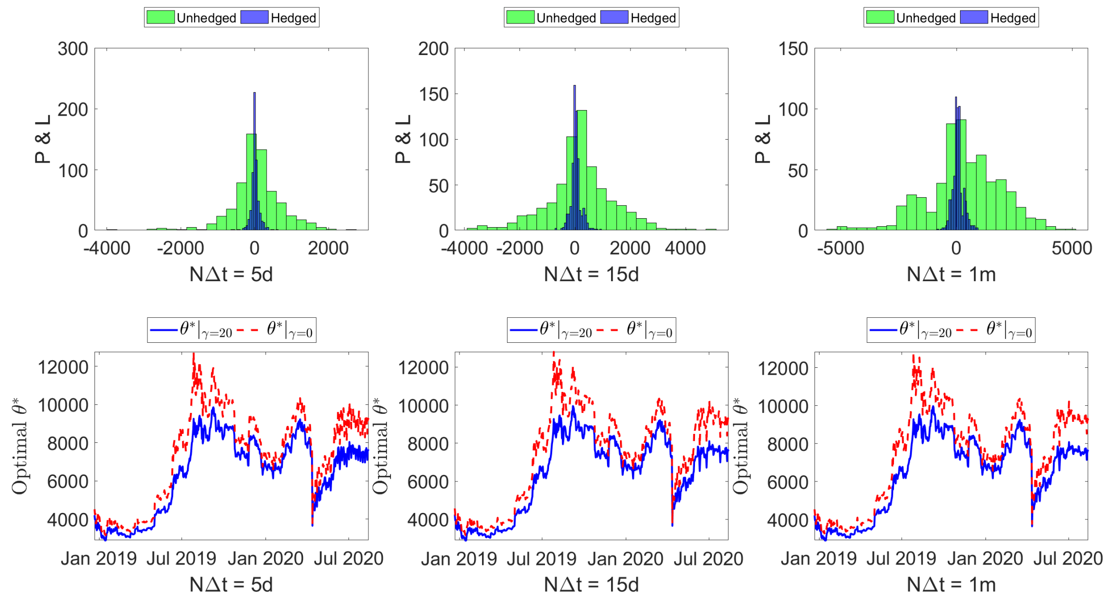
The first economic explanation is that the non-linear payoff of bitcoin inverse futures induces an *asymmetry* effect, whereby a decline in the bitcoin spot price imposes a stronger effect on the P&L of inverse futures than an increase in spot price of the same size. This effect makes inverse futures considerably more risky than bitcoin spot, as already noted by Deng et al. (2020). Secondly, a more risk averse hedger acts more conservatively by shorting less futures contracts than the 'global' optimal level (i.e. θ^* under $\gamma = 0$, shown in red in the low panels of Figures 6 and 7), which naturally leads to a diminished effect on volatility reduction. In addition, we observe from both figures that the volatility of both portfolios increases with the hedge horizon $N\Delta t$, although the optimal strategy θ^* remains relatively stable.

To investigate the hedging performance of the optimal strategy θ^*, we define hedge effectiveness (HE) as the percentage reduction in portfolio variance. Let $\Delta V(\theta)$ denote the P&L of a portfolio with one bitcoin and θ short positions in bitcoin futures. It is clear that $\theta = 0$ corresponds to the unhedged portfolio. Mathematically, we define HE by

$$\mathrm{HE}(\theta) = 1 - \frac{\mathbb{V}\mathrm{ar}(\Delta V(\theta))}{\mathbb{V}\mathrm{ar}(\Delta V(0))}, \qquad \theta > 0, \tag{9}$$

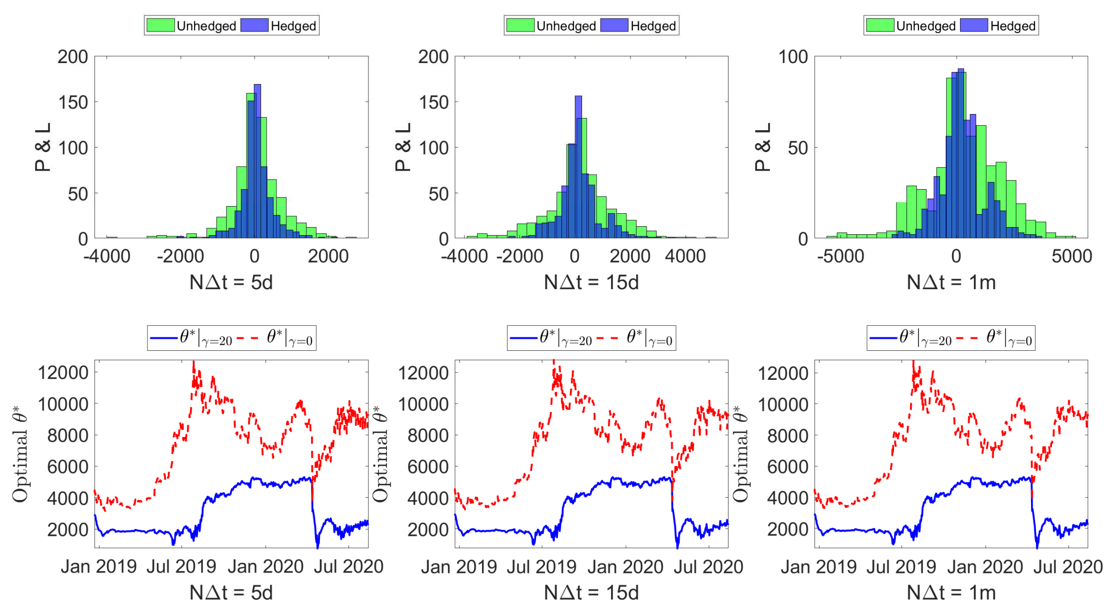
where $\mathbb{V}\mathrm{ar}(\cdot)$ denotes the variance of a random variable. We calculate the hedge effectiveness of the optimal portfolio (i.e. when $\theta = \theta^*$ given by (6)) for different pairs of bitcoin spot and perpetual futures under different margin constraints m, default aversion γ, and hedge horizon $N\Delta t$. We report the results in Table 6 under three margin constraint levels ($m = 10\%, 20\%, 50\%$), two default aversion levels ($\gamma = 20, 100$), and two hedge horizon levels ($N\Delta t =$ 5d, 30d). In each of Tables II.1-II.9 in Online Supplementary Appendix II, we calculate HE defined in (9) for one pair of bitcoin spot (from BitMEX, Bitstamp, or Coinbase) and perpetual futures (from BitMEX, Deribit, or OKEx) under four margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$), four default aversion

Figure 6: P&L of Unhedged and Hedged Portfolios, and Optimal Strategy θ^* under $\gamma = 20$



Note. The top three panels plot the histogram of the P&L of the unhedged and optimally hedged portfolios under three different hedge horizons 5d, 15d and 1m. The lower three panels plot the optimal hedging strategy θ^* under $\gamma = 20$ for the three hedge horizons, along with θ^* under $\gamma = 0$ for comparison purpose. We observe that the hedger trades less bitcoin futures when she is default averse, captured by a strictly positive γ. We set monitoring frequency $\Delta t = 30\text{min}$, $m = 10\%$ and $\gamma = 20$. For illustration purpose, we only focus on BitMEX index and BitMEX perpetual futures.

Figure 7: P&L of Unhedged and Hedged Portfolios, and Optimal Strategy θ^* under $\gamma = 100$



Note. All the explanations in note of Figure 6 apply here as well, except now we set the default aversion parameter $\gamma = 100$.

levels ($\gamma = 20, 50, 100, 200$), and four hedge horizon levels ($N\Delta t$ = 5d, 15d, 30d, 60d).

Table 6: Hedge Effectiveness of bitcoin Perpetual Futures

			Spot Markets					
Futures	m	γ	BitMEX 5d	Bitstamp 5d	Coinbase 5d	BitMEX 30d	Bitstamp 30d	Coinbase 30d
BitMEX	$m = 10\%$	$\gamma = 20$	93.61%	74.98%	75.08%	94.45%	78.53%	76.82%
		$\gamma = 100$	68.92%	51.38%	55.65%	67.31%	52.28%	51.78%
	$m = 20\%$	$\gamma = 20$	98.75%	92.31%	92.28%	99.14%	94.68%	94.35%
		$\gamma = 100$	89.89%	76.00%	75.50%	90.07%	79.38%	78.67%
	$m = 50\%$	$\gamma = 20$	99.54%	99.20%	99.39%	99.87%	99.68%	99.71%
		$\gamma = 100$	98.69%	96.12%	96.29%	99.00%	97.69%	97.57%
Deribit	$m = 10\%$	$\gamma = 20$	93.99%	93.96%	93.92%	94.74%	94.72%	94.71%
		$\gamma = 100$	70.18%	70.01%	69.93%	69.11%	69.06%	69.05%
	$m = 20\%$	$\gamma = 20$	99.03%	99.08%	99.06%	99.26%	99.27%	99.26%
		$\gamma = 100$	90.72%	90.65%	90.59%	91.11%	91.08%	91.07%
	$m = 50\%$	$\gamma = 20$	99.77%	99.85%	99.84%	99.92%	99.94%	99.94%
		$\gamma = 100$	99.03%	99.08%	99.06%	99.22%	99.23%	99.22%
OKEx	$m = 10\%$	$\gamma = 20$	97.68%	97.77%	97.79%	97.83%	97.84%	97.86%
		$\gamma = 100$	76.71%	76.74%	76.77%	76.12%	76.14%	76.17%
	$m = 20\%$	$\gamma = 20$	99.60%	99.71%	99.71%	99.74%	99.75%	99.75%
		$\gamma = 100$	94.67%	94.80%	94.81%	95.35%	95.37%	95.39%
	$m = 50\%$	$\gamma = 20$	99.86%	99.96%	99.96%	99.97%	99.98%	99.98%
		$\gamma = 100$	99.42%	99.55%	99.55%	99.64%	99.66%	99.66%

Note. This table reports the hedge effectiveness, defined in (9), of the optimal strategy θ^* at different hedge horizon $N\Delta t$, margin constraint m, and default aversion γ. We consider three bitcoin spot markets BitMEX (.BXBT index), Bitstamp and Coinbase, and three bitcoin perpetual futures on BitMEX, Deribit and OKEx. We set Δt = 30min.

We summarize the key findings on hedge effectiveness as follows:

- The margin constraint has a major impact on both the hedge effectiveness and the optimal strategy θ^* in bitcoin futures markets. In all cases, once the hedger has sufficient capital to meet margin requirement (e.g. $m = 50\%$), the optimal hedging strategy achieves superior hedge effectiveness (with HE close to 100%). This finding offers strong support to the inclusion of margin constraint in the hedging analysis of bitcoin futures. The hedging performance of the optimal strategy may be over-exaggerated if the analysis does not consider the possibility of margin constraint (such as Alexander et al. (2020) and Deng et al. (2020)).

- The effect of margin constraint is more profound for hedgers with high default aversion (large γ). To see this, we consider the case of using BitMEX perpetual futures to hedge the spot risk on Bitstamp given $\gamma = 100$. By comparing the hedge effectiveness under $m = 10\%$ and

$m = 50\%$, we observe a very significant improvement: HE increases sharply from around 40% to 95% for both 5d and 30d hedge horizon. This extreme example also serves as another evidence to the above conclusion that it is critical to include margin constraint in the hedging study of bitcoin inverse futures. In addition, for a fixed margin constraint level m, hedgers with lower default aversion short more futures and are able to reduce the portfolio risk in a more significant amount than those with higher default aversion.

- As margin constraint parameter m increases (i.e. the hedger has a 'deeper' pocket), the optimal strategy's hedge effectiveness improves to almost 100% rather quickly across all pairs of spot and futures markets. This result testifies that bitcoin perpetual futures (offered by BitMEX, Deribit, and OKEx) provide an effective instrument to hedge against the price volatility of bitcoin spot price, which is consistent with the findings in Alexander et al. (2020). On the other hand, there is no clear winner among the three perpetual futures considered in the analysis, as they deliver similar hedging performance across different spot markets. However, upon close examination, we find that (1) OKEx perpetual futures is marginally better than the other two perpetual futures in terms of HE, and (2) BitMEX perpetual futures is not a good choice for high default averse hedgers who want to hedge the spot risk on Bitstamp and Coinbase.

- As the hedge horizon ($N\Delta t$) increases, the hedge effectiveness improves, although at a tiny scale. To see this, we refer to Tables II.1-II.9 in Online Supplementary Appendix II where we consider four different choices for the hedge horizon at $N\Delta t$ = 5d, 15d, 30d, and 60d. This result is due to the fact that a longer horizon naturally smooths the price time series.

5.3 Default Probability

We regard a default event for futures traders as the circumstance that they cannot meet a margin calls – see (4) – and is forced to liquidate their futures positions. Clearly, as readily seen from Figure 1 and Table 1, both the historical forced liquidation volumes and the simulated margin call probabilities (using BitMEX perpetual futures data) are very high, imposing enormous risk to futures traders. Our approach to alleviating this risk is to take account of the impact of the default probability $P(m, \theta)$, defined in (4), on the hedger's hedging strategy θ. To be precise, we assume the representative hedger is default averse and aims to minimize the default probability $P(m, \theta)$, as formulated in Problem (5). With that in mind, we naturally hypothesize that following the optimal strategy θ^* should reduce the default probability to a reasonable level. But how much of a reduction is possible? So in this sub-section we ask, to what degree does the optimal hedging strategy reduce the default probability?

To investigate this, we obtain the default probability $P(m, \theta^*)$ in (10) under the optimal strategy θ^*, which we call the 'optimal default probability' for short. We derive this using the extreme value theorem (A.2) in Appendix A as:

$$P(m,\theta^*) \simeq 1 - \exp\left[-\left(1+\tau\left(\frac{\frac{m}{\theta^*}-\beta}{\alpha}\right)\right)^{-1/\tau}\right], \tag{10}$$

where α and β are the scale and location parameters respectively, and τ is the tail index parameter. All the parameters α, β, and τ have been estimated and summarized in Table 5 and the optimal strategy θ^* has been obtained as well – see, e.g. the lower panels in Figures 6 and 7. Hence, we can calculate the right hand side of (10), and use it as an approximation to the optimal default probability $P(m,\theta^*)$. We emphasis that the default probabilities reported in Table 7 are all at the monitoring frequency of Δt =30min.

Table 7: Default Probability Under the Optimal Strategy θ^*

			Spot Markets					
			BitMEX	Bitstamp	Coinbase	BitMEX	Bitstamp	Coinbase
Futures	m	γ		$N\Delta t = 5$d			$N\Delta t = 30$d	
BitMEX	$m = 10\%$	$\gamma = 20$	0.55%	0.52%	0.52%	0.55%	0.52%	0.52%
		$\gamma = 100$	0.23%	0.19%	0.19%	0.23%	0.19%	0.19%
	$m = 20\%$	$\gamma = 20$	0.20%	0.19%	0.20%	0.20%	0.20%	0.20%
		$\gamma = 100$	0.14%	0.12%	0.12%	0.14%	0.12%	0.12%
	$m = 50\%$	$\gamma = 20$	0.05%	0.04%	0.04%	0.05%	0.04%	0.04%
		$\gamma = 100$	0.04%	0.03%	0.03%	0.04%	0.03%	0.03%
Deribit	$m = 10\%$	$\gamma = 20$	0.53%	0.54%	0.54%	0.54%	0.54%	0.54%
		$\gamma = 100$	0.22%	0.22%	0.22%	0.22%	0.22%	0.22%
	$m = 20\%$	$\gamma = 20$	0.19%	0.19%	0.19%	0.19%	0.19%	0.19%
		$\gamma = 100$	0.13%	0.13%	0.13%	0.13%	0.13%	0.13%
	$m = 50\%$	$\gamma = 20$	0.04%	0.04%	0.04%	0.04%	0.04%	0.04%
		$\gamma = 100$	0.04%	0.04%	0.04%	0.04%	0.04%	0.04%
OKEx	$m = 10\%$	$\gamma = 20$	0.57%	0.58%	0.57%	0.57%	0.57%	0.57%
		$\gamma = 100$	0.26%	0.26%	0.26%	0.26%	0.26%	0.26%
	$m = 20\%$	$\gamma = 20$	0.20%	0.21%	0.21%	0.21%	0.21%	0.21%
		$\gamma = 100$	0.15%	0.15%	0.15%	0.16%	0.16%	0.16%
	$m = 50\%$	$\gamma = 20$	0.05%	0.05%	0.05%	0.05%	0.05%	0.05%
		$\gamma = 100$	0.04%	0.04%	0.04%	0.04%	0.04%	0.04%

Note. This table reports the default probability $P(m,\theta^*)$ under the optimal strategy θ^*, which is computed using (10). We perform calculations for all pairs of bitcoin spot (BitMEX, Bitstamp, and Coinbase) and bitcoin perpetual futures (BitMEX, Deribit, and OKEx) under different margin constraint m ($m = 10\%, 20\%, 50\%$), different default aversion parameter γ ($\gamma = 20, 100$), and different hedge horizon $N\Delta t$ ($N\Delta t = 5$d, 30d). Default probabilities reported here are all at the monitoring frequency of Δt =30min.

Table 7 reports the results and we outline the main findings in the following:

- The optimal default probability $P(m,\theta^*)$ is very small, close to or less than 1%, in all scenarios. Therefore, this result strongly supports our hypothesis that the hedger is able to reduce the default probability to a desirable level by following the optimal strategy θ^*. We also emphasize that such a finding is robust, in the sense that it holds in regardless of the margin constraint level m, default aversion γ, hedge horizon $N\Delta t$, and the choice of bitcoin spot and perpetual futures.

- The margin constraint m has a major impact on the optimal default probability. A larger m leads to a smaller optimal default probability. For a hedger with relatively low default aversion ($\gamma = 20$), increasing the margin constraint from 10% to 20% reduces the optimal default probability by about 50%. To achieve the similar 50% of reduction, a hedger with relatively high default aversion ($\gamma = 100$) needs to increase m from 10% to 50% or more. With sufficient resource to meet margin requirement (say $m \geq 50\%$), the hedger is able to control the optimal default probability below 0.05%. Based on these findings, we conclude that it is critical to consider the margin constraint in the analysis of optimal futures hedging for a default averse hedger.

- As expected, the optimal default probability decreases as the default aversion γ increases. Hedgers with higher γ are more default averse and hence act more conservatively by taking less shorts positions in bitcoin perpetual futures when hedging spot risk (see the lower panels in Figures 6 and 7 for evidence).

- In the computations leading to Table 7, we consider three markets for bitcoin spot (BitMEX, Bitstamp, and Coinbase) and three bitcoin perpetual futures (BitMEX, Deribit, and OKEx). We do not see any significant difference on the optimal default probability among different pairs of bitcoin spot and perpetual futures, although Deribit perpetual futures is the most preferred choice. This provides a positive signal to apply our approach to the cases when a different bitcoin spot market or/and a different perpetual futures contract is taken.

- Hedge horizon only has a marginal effect on the optimal default probability, with a weak positive correlation. That is, as the hedge horizon $N\Delta t$ increases, the optimal default probability $P(m, \theta^*)$ increases, at a negligible scale though.

5.4 Implied Leverage

A distinguish feature of various bitcoin derivatives is high leverage, e.g. up to 100X for BitMEX perpetual futures. In the last part of empirical analysis, we study the *implied leverage* under the optimal strategy θ^* taken by the representative hedger.

Recall that in our setup, the margin constraint m is applied to the entire short position of θ^* futures contracts. Hence, the margin constraint per contract is m/θ^* in XBT. Since the nominal value per futures contract is equal to $\widehat{F} = 1/F$ also in XBT, the implied leverage in futures under the optimal strategy θ^* is given by $\theta^*/(F \cdot m)$. We then compute the implied leverage under the optimal strategy θ^* and present the results in Table 8. We summarize our observations from Table 8 as follows:

Table 8: Implied Leverage Under the Optimal Strategy θ^*

			Spot Markets					
			BitMEX	Bitstamp	Coinbase	BitMEX	Bitstamp	Coinbase
Futures	m	γ		$N\Delta t = 5$d			$N\Delta t = 30$d	
BitMEX	$m = 10\%$	$\gamma = 20$	8.75	8.32	8.32	8.81	8.34	8.36
		$\gamma = 100$	6.02	5.86	5.86	6.06	5.88	5.88
	$m = 20\%$	$\gamma = 20$	4.76	4.70	4.71	4.80	4.72	4.74
		$\gamma = 100$	4.17	4.11	4.12	4.19	4.13	4.14
	$m = 50\%$	$\gamma = 20$	1.96	1.96	1.97	1.98	1.97	1.98
		$\gamma = 100$	1.90	1.91	1.91	1.92	1.92	1.93
Deribit	$m = 10\%$	$\gamma = 20$	8.73	8.75	8.76	8.77	8.77	8.77
		$\gamma = 100$	6.03	6.04	6.04	6.03	6.03	6.03
	$m = 20\%$	$\gamma = 20$	4.77	4.79	4.79	4.80	4.80	4.80
		$\gamma = 100$	4.19	4.20	4.20	4.20	4.20	4.20
	$m = 50\%$	$\gamma = 20$	1.97	1.97	1.97	1.98	1.98	1.98
		$\gamma = 100$	1.91	1.92	1.92	1.92	1.92	1.92
OKEx	$m = 10\%$	$\gamma = 20$	8.81	8.83	8.83	8.87	8.87	8.87
		$\gamma = 100$	5.89	5.89	5.89	5.94	5.94	5.94
	$m = 20\%$	$\gamma = 20$	4.79	4.80	4.80	4.81	4.81	4.81
		$\gamma = 100$	4.16	4.17	4.17	4.18	4.18	4.18
	$m = 50\%$	$\gamma = 20$	1.97	1.98	1.98	1.98	1.98	1.98
		$\gamma = 100$	1.91	1.92	1.92	1.92	1.92	1.92

Note. This table reports the implied leverage under the optimal strategy θ^*, which is defined as $\theta^*/(F \cdot m)$. We perform calculations for all pairs of bitcoin spot (BitMEX, Bitstamp, and Coinbase) and bitcoin perpetual futures (BitMEX, Deribit, and OKEx) under different margin constraint m ($m = 10\%, 20\%, 50\%$), different default aversion parameter γ ($\gamma = 20, 100$), and different hedge horizon $N\Delta t$ ($N\Delta t$ = 5d, 30d). All Leverages reported here are at the monitoring frequency of Δt =30min.

- The implied leverage the optimal strategy θ^* varies from 1X to 9X, which is at a reasonable level for bitcoin futures. For instance, BitMEX perpetual futures allow up to 100X leverage, while the implied leverage in the case of BitMEX perpetual futures is below 5X for a hedger with a moderate margin constraint (say $m \geq 20\%$). The maximum implied leverage is 8.68 when Deribit perpetual futures are used as the hedging instrument, $m = 10\%$, and $\gamma = 20$. We thus conclude that the hedger reduces leverage considerably in trading perpetual futures when she follows the optimal strategy θ^* derived under the margin constraint.

- Similar to the analysis of hedge effectiveness and default probability, both the margin constraint and default aversion play a key role in determining the implied leverage. Given a sufficiently large margin constraint (say $m = 50\%$), the implied leverage is less than 2X in all cases, and remains stable for different default aversion γ and pairs of bitcoin spot and perpetual futures. On the other hand, the impact of hedge horizon $N\Delta t$ on the implied leverage is negligible.

6 Conclusion

Motivated by both empirical findings and practical needs, we study how margin constraints and default aversion affect the optimal hedging of bitcoin using futures. A representative investor is assumed to hold one bitcoin and hedges against its price volatility by trading bitcoin perpetual inverse futures, with a limited ability to acquire bitcoin to meet margin calls. The representative investor is averse to both market volatility and default risks, where default is defined as the circumstance when marked trading losses exceeding the margin constraint. The main theoretical contribution of this paper is to formulate the optimal hedging problem under dual objectives: minimizing both the volatility of the hedged portfolio and the probability of default. An application of the extreme value theorem yields the optimal hedging strategy in semi-closed form.

In the empirical part of this paper we investigate three important topics in risk management: hedge effectiveness, default probability, and implied leverage. Based on three popular bitcoin perpetual futures (BitMEX, Deribit, and OKEx) and three bitcoin spot markets (BitMEX, Bitstamp, and Coinbase) we conduct a sensitivity analysis of the optimal strategy, to examine how it changes with various parameters controlling the characteristics of both the contract and the hedger.

The primary goal of using futures in hedging is to reduce the risk exposure to the spot markets; see for instance Ederington (1979) and Figlewski (1984a). As such, we analyze the hedge effectiveness of the optimal hedging strategy, i.e. whether and to what degree the optimal strategy helps reduce the variance. However, of particular interest here, given the novelty of our theoretical results, is the effect of margin constraint and default aversion on the hedger's decisions. We find that the optimal hedging strategy achieves superior hedge effectiveness, in terms of reducing portfolio volatility, as long as the hedger is not overly default averse or the margin constraint is not too tight.

As is well known, bitcoin futures are among the most leveraged financial products in the markets (e.g. BitMEX perpetual futures allow up 100X leverage). But the high leverage feature of bitcoin futures is a double-edged sword: on the positive side, this feature helps attract a large amount of noise traders and speculators, and thus improves the overall market depth and liquidity; while on the negative side, it could easily lead to forced automatic liquidations (default events). Therefore, in the last part, we investigate the implied leverage under the optimal strategy. By following our strategy the hedger is able to reduce the default probability significantly (less than 1% in most cases) and control the leverage in futures trading to a reasonable level (mostly below 5X).

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Appendix

A Theoretical Appendix

In this section, we provide detailed derivations leading to our main results in Theorem 3.2.

Step 1. Approximation of the variance term $\sigma^2_{\Delta h}(\theta)$ defined in (3).
Recall that $\sigma^2_{\Delta h}(\theta)$ is the variance of the P&L of the hedged portfolio consisting of one long position in bitcoin spot and θ short positions in bitcoin inverse futures. By following the same arguments in Deng et al. (2020) (see Eq.(3.1) therein), we obtain

$$\sigma^2_{\Delta h}(\theta) \simeq \sigma^2_{\Delta S_t(N)} + \left(\frac{S_t\theta}{F_t^2}\right)^2 \sigma^2_{\Delta F_t(N)} - 2\frac{S_t\theta}{F_t^2}\,\sigma^2_{\Delta S_t(N),\Delta F_t(N)}, \tag{A.1}$$

where $\sigma^2_{\Delta S_t(N)}$ (resp. $\sigma^2_{\Delta F_t(N)}$) denotes the variance of the N-period spot (resp. futures) price changes, and $\sigma^2_{\Delta S_t(N),\Delta F_t(N)}$ denotes the covariance between the two random variables. The approximation result in (A.1) depends on one condition, $S_t/F_t \simeq S_{t+N\Delta t}/F_{t+N\Delta t}$. Since this ratio is almost unchanging over time, being very close to one for all the exchanges considered, we conclude that the approximation to $\sigma^2_{\Delta h}(\theta)$ in (A.1) is very accurate.

Step 2. Approximation of the default probability $P(m,\theta)$ defined in (4).
Due to the margin constraint m, the representative hedger may not be able to meet margin calls if the futures price encounters a sharp movement, which is referred to as a default event. The probability of default is then given by (4). Since the hedger holds short positions in the futures, the right tail of the value changes $\Delta\widehat{F}$ (defined in (2)) is the risk part to the hedger. This argument naturally inspires us to apply the extreme value theorem to study the (right) tail risk of $\Delta\widehat{F}$. We refer to Embrechts et al. (1997) for general theory and Longin (1999) for applications in optimal margin problem in futures. By applying the extreme value theorem, we obtain the following convergence result[24]

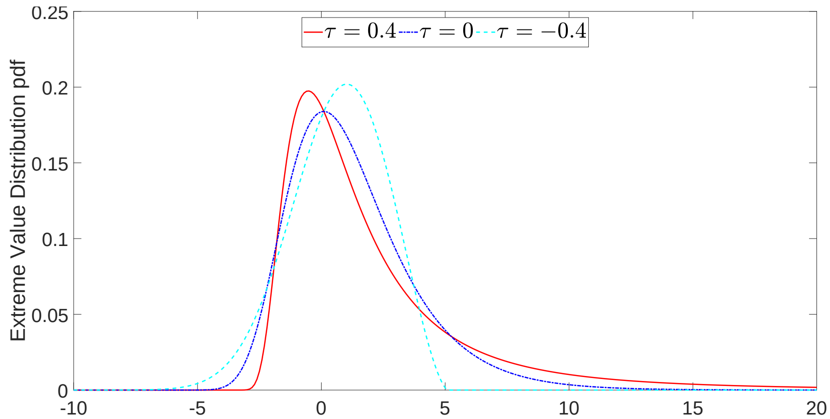
$$\mathbb{P}\left(\max_{0\le n\le N-1} \Delta\widehat{F}_{t+n\Delta t} \le x\right) \quad\longrightarrow\quad \exp\left[-\left(1+\tau\left(\frac{x-\beta}{\alpha}\right)\right)^{-1/\tau}\right], \tag{A.2}$$

where α, β, and τ are respectively the scale parameter, location parameter, and right tail index of the value changes of the inverse futures $\Delta\widehat{F}$, and all of them can be easily estimated from bitcoin futures prices. The most important parameter among the three is the right tail index τ, which measures (right) tail fatness of the data. To be precise, the sign of τ determines the type of the extreme value distribution as follows: the limiting case $\tau = 0$ corresponds to the double exponential

[24] Note that the limiting result in (A.2) is *not* achieved as $N \to \infty$, but under that the number of observations over N periods goes to infinity. For instance, let us take $\Delta t = 1$h (hour) and $N = 8$, meaning the hedger wants to hedge the position for 8 hours. Then the result in (A.2) holds as long as we have enough observations of the maximum price changes over a 8h ($N\Delta t$) window.

Gumbel distribution, $\tau > 0$ the Fréchet distribution, and $\tau < 0$ the Weibull distribution. Figure A.1 illustrates an example of these three types of distributions. We observe from Figure A.1 that the bigger the τ, the fatter the right tail. The limiting result in (A.2) is *model-free*, i.e. it holds for any distribution of $\Delta\widehat{F}$ under mild conditions.

Figure A.1: Generalized Extreme Value Distribution



Note. We set the scale and location parameters $\alpha = 2$ and $\beta = 0.1$ and three levels for the tail index parameter τ at $\tau = 0.4, 0, -0.4$ that represents Fréchet (red), Gumbel (blue) and Weibull (cyan) distributions, respectively.

Using (A.2), we approximate the default probability $P(m, \theta)$ by

$$P(m,\theta) \simeq 1 - \exp\left[-\left(1+\tau\left(\frac{\frac{m}{\theta}-\beta}{\alpha}\right)\right)^{-1/\tau}\right]. \tag{A.3}$$

We comment that the right hand side in (A.3) provides an accurate approximation to $P(m, \theta)$, which is empirically verified in the Online Supplementary Appendix I.

Step 3. Approximation of the main problem defined in (5).
Using (A.1) and (A.3), the hedger's optimization problem in (5) can be approximated by

$$\begin{aligned}\min_{\theta>0} \quad & \sigma^2_{\Delta S_t(N)} + \left(\frac{S_t\theta}{F_t^2}\right)^2 \sigma^2_{\Delta F_t(N)} - 2\frac{S_t\theta}{F_t^2}\,\sigma^2_{\Delta S_t(N),\Delta F_t(N)} \\ & + \gamma\,\sigma^2_{\Delta S_t(N)}\left[1 - \exp\left[-\left(1+\tau\left(\frac{\frac{m}{\theta}-\beta}{\alpha}\right)\right)^{-1/\tau}\right]\right].\end{aligned} \tag{A.4}$$

Since the approximations in Steps 1 and 2 are accurate enough, Problem (A.4) serves as a good approximation to Problem (5), and from now on we focus on Problem (A.4). To solve this, we apply a change of variable $\widetilde{\theta} = S_t\theta/F_t^2$. We then obtain the first-order condition of Problem (A.4) by $2\,\sigma^2_{\Delta F_t(N)}\, f(\widetilde{\theta}) = 0$, where f is defined in (7). Hence, θ^* given in (6) is a necessary solution to Problem (A.4).

Step 4. Verification of the optimal solution θ^* given by (6).

Since $S_t \approx F_t$ and the bitcoin price S_t is in the scale of thousands (even tens of thousands), the linear term in function f has the dominating role in its second derivative. As a result, the second derivative of the objective functional in Problem (A.4) is positive, and thus the necessary condition of optimality is also sufficient. Therefore, θ^* in (6) is indeed an optimal strategy to Problem (A.4).

Step 5. Showing that the equation $f(x) = 0$ has a positive solution if $\tau > 0$.
The only part of the proof remaining now is to find a positive solution to $f(x) = 0$. As the time series of $\Delta\widehat{F}$ has very fat tails, we expect $\tau > 0$ (verified in the empirical analysis, see Table 5 and Figure 4). Now under the condition $\tau > 0$, we have

$$\lim_{x \to 0} f(x) < 0 \qquad \text{and} \qquad \lim_{x \to \infty} f(x) > 0,$$

where we have used the fact that S and F are positively correlated (which is always the case for spot and futures, but see also Table 4). Together with the continuity property of function f, we apply the mean value theorem to conclude that there always exists a positive root to the equation $f(x) = 0$.

Since the function f is highly non-linear, the uniqueness result regarding its root is not available in general. However, in all scenarios considered during our empirical analysis, we always find a unique solution to $f(x) = 0$. $\square$

Online Supplementary Appendix

I Approximation Accuracy of Tail Distribution

In this section, we investigate the approximation accuracy to the tail distribution based on the extreme value theorem, which is derived in (A.2). The ultimate purpose here is to verify that the right hand side of (A.3) provides an accurate approximation to the default probability $P(m, \theta)$.

In the empirical investigation, we calculate the left hand side of (A.2) directly from the historical data. To calculate the right hand side, we estimate the parameters α, β and τ from data. We refer to Hosking et al. (1985) and Longin (1999) for parameter estimation details. Once we have successfully obtained both sides of (A.2), we can easily examine the approximation accuracy. In the analysis, we consider three variables (time series): 1-period spot price difference ΔS, 1-period futures price difference ΔF, and 1-period inverse futures' value changes $\Delta \widehat{F}$, although (A.2) only concerns $\Delta \widehat{F}$. Please refer to (1) and (2) for the definition of ΔS, ΔF, and $\Delta \widehat{F}$. We include the results of ΔS and ΔF to showcase the model-free feature and powerful applications of the extreme value theorem.

We take the BitMEX .BXBT Index to be the spot price S and the BitMEX perpetual XBTUSD futures price to be the futures price F, sampled at 5min frequency from Jan. 1, 2017 to Aug. 12, 2020. We consider three possible time steps Δt = 15min, 30min, and 1h, and obtain the *full sample* at the frequency of Δt from raw data. Since the hedger in our analysis shorts bitcoin inverse futures, the right tail of the futures price changes ΔF (or $\Delta \widehat{F}$) is the risk (losses) to the hedger. As such, we use the maximum price change of each day sampled at monitoring frequency Δt = 15min, 30min, and 1h to form a sample of 'right tail', which we use to estimate the parameters (α, β, and τ).

Table I.1: Estimated Parameters α, β, and τ Using BitMEX Full Sample

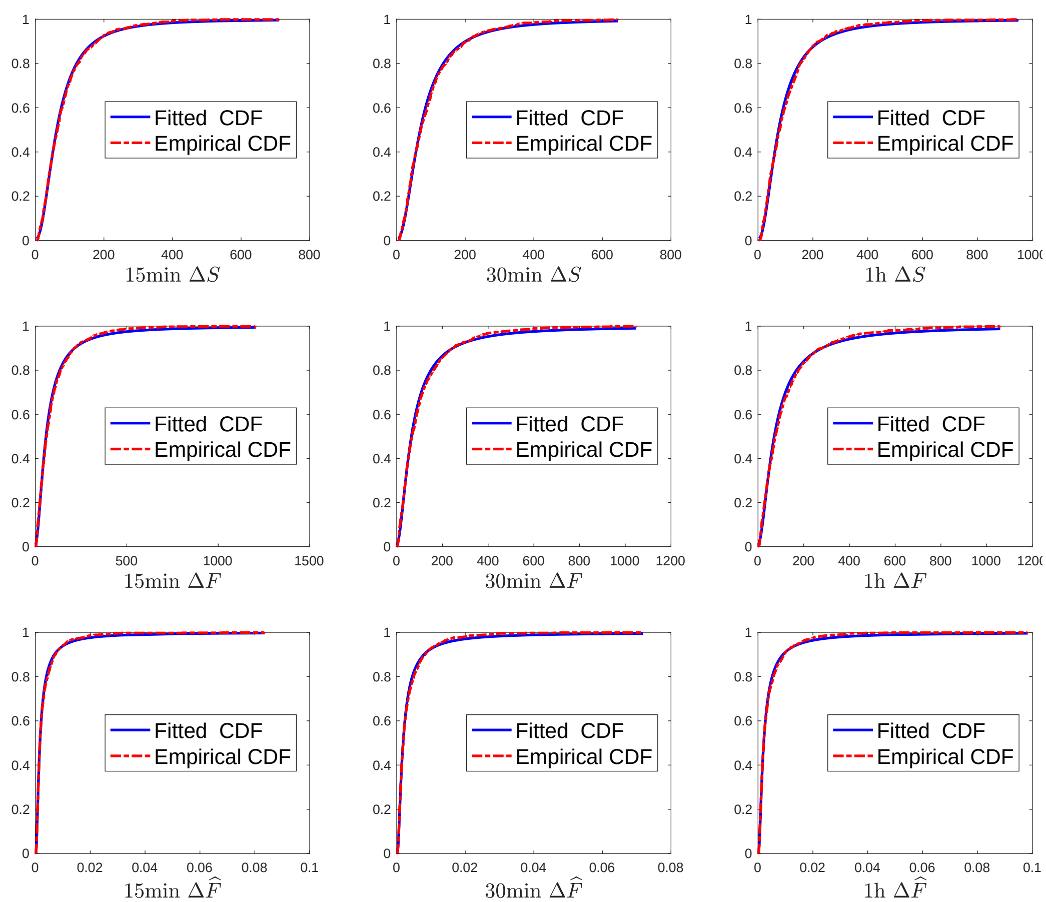
Variable	Δt	τ	α	β
ΔS	15min	0.37	36.19	47.83
	30min	0.40	40.54	52.55
	1h	0.42	45.19	56.98
ΔF	15min	0.54	39.05	42.29
	30min	0.57	43.50	46.59
	1h	0.59	48.41	51.18
$\Delta \widehat{F}$	15min	0.70	1.04E-06	1.24E-06
	30min	0.72	1.18E-06	1.37E-06
	1h	0.73	1.34E-06	1.52E-06

Note. This table reports estimated parameters α, β, and τ for ΔS, ΔF, and $\Delta \widehat{F}$ using the full sample of BitMEX data. We set time step Δt = 15min, 30min and 1h. Since the values of $\Delta \widehat{F}$ are quite small, we multiply $\Delta \widehat{F}$ by 1000 to alleviate numerical errors. This does not affect the tail index τ of $\Delta \widehat{F}$, even through it does change α and β.

Recall from (A.2) that, the scale parameter α and location parameter β represent the dispersion and the average of the extremes, while the right tail index τ determines the type of the extreme value distribution. We report the estimation results in Table I.1. We find that τ is positive in all scenarios, and as a result, the limiting distributions are all Fréchet type (see Figure A.1). Although the scale and location parameters α and β depart from each other, the estimated values of the tail index τ are rather stable for all variables (ΔS, ΔF, and $\Delta\widehat{F}$) and for all choices of Δt (15min, 30min, and 1h). The right tail indices τ of inverse futures $\Delta\widehat{F}$ are larger than those of ΔS and ΔF. That means inverse futures has more extreme values and fatter right tail.

Next, we use the estimated parameters in Table I.1 to examine the accuracy of the limiting result in (A.2). We plot the empirical and fitted cumulative distribution function (CDF) for ΔS, ΔF, and $\Delta\widehat{F}$ in Figure I.1. The fitted CDF's show great accuracy to their empirical counterpart for all three variables ΔS, ΔF, and $\Delta\widehat{F}$ under all three choices of $\Delta t =$ 15min, 30min and 1h. That gives us confidence to use the simplified problem in (A.4) to approximate the original problem in (5), and claim that the optimal strategy θ^* to Problem (A.4) is near optimal to Problem (5).

Figure I.1: Empirical and Fitted Tail CDF of ΔS, ΔF, and $\Delta\widehat{F}$ Using BitMEX Full Sample



Note. All the empirical CDF's are obtained using the maximum price change of each day sampled at monitoring frequency $\Delta t =$ 15min, 30min, and 1h to form a sample of 'right tail'. All the fitted tail CDF's are computed by the right hand side of (A.2) under the estimated parameters in Table I.1. We use the full sample data from BitMEX.

II Tables of Hedge Effectiveness

In this section, we report full details on the hedge effectiveness (HE) of different pairs of bitcoin spot and perpetual futures. Each of the tables below considers a pair of bitcoin spot (from BitMEX .BXBT index, Bitstamp, or Coinbase) and perpetual futures (from BitMEX, Deribit, or OKEx) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60$d), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$).

Table II.1: Hedge Effectiveness for BitMEX Spot and BitMEX Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	93.61%	93.97%	94.45%	95.79%
	$\gamma = 50$	82.05%	81.63%	82.32%	83.61%
	$\gamma = 100$	68.92%	67.18%	67.31%	64.51%
	$\gamma = 200$	55.70%	51.99%	51.40%	42.31%
$m = 20\%$	$\gamma = 20$	98.75%	99.03%	99.14%	99.45%
	$\gamma = 50$	95.77%	95.98%	96.24%	97.24%
	$\gamma = 100$	89.89%	89.68%	90.07%	91.53%
	$\gamma = 200$	79.96%	78.69%	78.87%	78.37%
$m = 50\%$	$\gamma = 20$	99.54%	99.82%	99.87%	99.93%
	$\gamma = 50$	99.34%	99.60%	99.66%	99.78%
	$\gamma = 100$	98.69%	98.92%	99.00%	99.30%
	$\gamma = 200$	96.73%	96.81%	96.93%	97.64%
$m = 80\%$	$\gamma = 20$	99.57%	99.86%	99.90%	99.95%
	$\gamma = 50$	99.53%	99.81%	99.85%	99.92%
	$\gamma = 100$	99.38%	99.64%	99.69%	99.80%
	$\gamma = 200$	98.84%	99.05%	99.12%	99.35%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60$d), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the BitMEX perpetual futures to hedge the spot risk on BitMEX (.BXBT index).

Table II.2: Hedge Effectiveness for BitMEX Spot and Deribit Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	93.99%	94.12%	94.74%	96.28%
	$\gamma = 50$	82.83%	82.19%	83.33%	85.76%
	$\gamma = 100$	70.18%	68.31%	69.11%	69.17%
	$\gamma = 200$	56.46%	53.66%	53.56%	48.54%
$m = 20\%$	$\gamma = 20$	99.03%	99.13%	99.26%	99.56%
	$\gamma = 50$	96.24%	96.24%	96.65%	97.81%
	$\gamma = 100$	90.72%	90.29%	91.11%	93.29%
	$\gamma = 200$	81.36%	79.91%	80.88%	82.75%
$m = 50\%$	$\gamma = 20$	99.77%	99.89%	99.92%	99.96%
	$\gamma = 50$	99.59%	99.69%	99.76%	99.85%
	$\gamma = 100$	99.03%	99.08%	99.22%	99.52%
	$\gamma = 200$	97.29%	97.19%	97.52%	98.35%
$m = 80\%$	$\gamma = 20$	99.79%	99.92%	99.95%	99.97%
	$\gamma = 50$	99.76%	99.88%	99.92%	99.95%
	$\gamma = 100$	99.63%	99.74%	99.79%	99.87%
	$\gamma = 200$	99.18%	99.23%	99.35%	99.58%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60\text{d}$), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the Deribit perpetual futures to hedge the spot risk on BitMEX (.BXBT index).

Table II.3: Hedge Effectiveness for BitMEX Spot and OKEx Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	97.68%	97.91%	97.83%	97.34%
	$\gamma = 50$	90.08%	90.76%	90.30%	87.80%
	$\gamma = 100$	76.71%	77.48%	76.12%	69.17%
	$\gamma = 200$	58.80%	58.75%	56.05%	42.73%
$m = 20\%$	$\gamma = 20$	99.60%	99.72%	99.74%	99.61%
	$\gamma = 50$	98.28%	98.60%	98.60%	97.94%
	$\gamma = 100$	94.67%	95.46%	95.35%	93.14%
	$\gamma = 200$	86.00%	87.42%	86.87%	80.42%
$m = 50\%$	$\gamma = 20$	99.86%	99.95%	99.97%	99.95%
	$\gamma = 50$	99.76%	99.87%	99.90%	99.82%
	$\gamma = 100$	99.42%	99.61%	99.64%	99.40%
	$\gamma = 200$	98.27%	98.71%	98.74%	97.86%
$m = 80\%$	$\gamma = 20$	99.88%	99.96%	99.99%	99.96%
	$\gamma = 50$	99.85%	99.94%	99.97%	99.93%
	$\gamma = 100$	99.75%	99.87%	99.90%	99.82%
	$\gamma = 200$	99.41%	99.62%	99.65%	99.39%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60$d), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the OKEx perpetual futures to hedge the spot risk on BitMEX (.BXBT index).

Table II.4: Hedge Effectiveness for Bitstamp Spot and BitMEX Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	74.98%	76.33%	78.53%	86.28%
	$\gamma = 50$	64.77%	63.61%	63.29%	73.30%
	$\gamma = 100$	51.38%	52.22%	52.28%	60.29%
	$\gamma = 200$	43.56%	41.35%	39.25%	45.51%
$m = 20\%$	$\gamma = 20$	92.31%	94.11%	94.68%	97.16%
	$\gamma = 50$	84.35%	86.31%	87.65%	92.85%
	$\gamma = 100$	76.00%	77.54%	79.38%	86.91%
	$\gamma = 200$	65.97%	66.72%	68.71%	77.38%
$m = 50\%$	$\gamma = 20$	99.20%	99.66%	99.68%	99.72%
	$\gamma = 50$	98.12%	98.97%	99.03%	99.43%
	$\gamma = 100$	96.12%	97.46%	97.69%	98.67%
	$\gamma = 200$	92.41%	94.26%	94.90%	96.85%
$m = 80\%$	$\gamma = 20$	99.54%	99.81%	99.86%	99.76%
	$\gamma = 50$	99.34%	99.73%	99.76%	99.74%
	$\gamma = 100$	98.87%	99.45%	99.49%	99.62%
	$\gamma = 200$	97.70%	98.65%	98.74%	99.20%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60\text{d}$), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the BitMEX perpetual futures to hedge the spot risk on Bitstamp.

Table II.5: Hedge Effectiveness for Bitstamp Spot and Deribit Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	93.96%	94.14%	94.72%	96.28%
	$\gamma = 50$	82.70%	82.19%	83.29%	85.75%
	$\gamma = 100$	70.01%	68.28%	69.06%	69.16%
	$\gamma = 200$	56.29%	53.62%	53.51%	48.47%
$m = 20\%$	$\gamma = 20$	99.08%	99.16%	99.27%	99.57%
	$\gamma = 50$	96.23%	96.26%	96.64%	97.81%
	$\gamma = 100$	90.65%	90.30%	91.08%	93.29%
	$\gamma = 200$	81.24%	79.89%	80.84%	82.74%
$m = 50\%$	$\gamma = 20$	99.85%	99.92%	99.94%	99.96%
	$\gamma = 50$	99.66%	99.72%	99.77%	99.86%
	$\gamma = 100$	99.08%	99.11%	99.23%	99.52%
	$\gamma = 200$	97.30%	97.21%	97.52%	98.36%
$m = 80\%$	$\gamma = 20$	99.88%	99.95%	99.97%	99.98%
	$\gamma = 50$	99.84%	99.91%	99.93%	99.96%
	$\gamma = 100$	99.71%	99.76%	99.81%	99.88%
	$\gamma = 200$	99.23%	99.25%	99.36%	99.58%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60\text{d}$), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the Deribit perpetual futures to hedge the spot risk on Bitstamp.

Table II.6: Hedge Effectiveness for Bitstamp Spot and OKEx Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	97.77%	97.95%	97.84%	97.33%
	$\gamma = 50$	90.15%	90.82%	90.32%	87.78%
	$\gamma = 100$	76.74%	77.54%	76.14%	69.14%
	$\gamma = 200$	58.79%	58.81%	56.08%	42.67%
$m = 20\%$	$\gamma = 20$	99.71%	99.75%	99.75%	99.62%
	$\gamma = 50$	98.40%	98.65%	98.61%	97.94%
	$\gamma = 100$	94.80%	95.52%	95.37%	93.12%
	$\gamma = 200$	86.11%	87.49%	86.90%	80.39%
$m = 50\%$	$\gamma = 20$	99.96%	99.98%	99.98%	99.96%
	$\gamma = 50$	99.87%	99.91%	99.91%	99.83%
	$\gamma = 100$	99.55%	99.66%	99.66%	99.40%
	$\gamma = 200$	98.42%	98.76%	98.76%	97.86%
$m = 80\%$	$\gamma = 20$	99.97%	99.99%	99.99%	99.98%
	$\gamma = 50$	99.95%	99.97%	99.97%	99.95%
	$\gamma = 100$	99.87%	99.91%	99.91%	99.83%
	$\gamma = 200$	99.55%	99.66%	99.67%	99.40%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60$d), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the OKEx perpetual futures to hedge the spot risk on Bitstamp.

Table II.7: Hedge Effectiveness for Coinbase Spot and BitMEX Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	75.08%	75.06%	76.82%	86.16%
	$\gamma = 50$	62.33%	62.19%	63.68%	72.68%
	$\gamma = 100$	55.65%	50.14%	51.78%	60.43%
	$\gamma = 200$	41.25%	39.77%	38.79%	45.62%
$m = 20\%$	$\gamma = 20$	92.28%	93.43%	94.35%	96.89%
	$\gamma = 50$	84.02%	85.20%	87.07%	92.51%
	$\gamma = 100$	75.50%	76.22%	78.67%	86.57%
	$\gamma = 200$	65.99%	65.49%	68.13%	77.08%
$m = 50\%$	$\gamma = 20$	99.39%	99.62%	99.71%	99.72%
	$\gamma = 50$	98.35%	98.79%	99.00%	99.34%
	$\gamma = 100$	96.29%	97.11%	97.57%	98.50%
	$\gamma = 200$	92.44%	93.67%	94.66%	96.59%
$m = 80\%$	$\gamma = 20$	99.63%	99.85%	99.91%	99.81%
	$\gamma = 50$	99.51%	99.72%	99.80%	99.76%
	$\gamma = 100$	99.09%	99.37%	99.50%	99.59%
	$\gamma = 200$	97.93%	98.45%	98.71%	99.11%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60\text{d}$), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the BitMEX perpetual futures to hedge the spot risk on Coinbase.

Table II.8: Hedge Effectiveness for Coinbase Spot and Deribit Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	93.92%	94.12%	94.71%	96.28%
	$\gamma = 50$	82.63%	82.15%	83.28%	85.76%
	$\gamma = 100$	69.93%	68.24%	69.05%	69.17%
	$\gamma = 200$	56.22%	53.64%	53.51%	48.54%
$m = 20\%$	$\gamma = 20$	99.06%	99.15%	99.26%	99.57%
	$\gamma = 50$	96.20%	96.24%	96.63%	97.82%
	$\gamma = 100$	90.59%	90.27%	91.07%	93.30%
	$\gamma = 200$	81.16%	79.85%	80.83%	82.75%
$m = 50\%$	$\gamma = 20$	99.84%	99.91%	99.94%	99.96%
	$\gamma = 50$	99.65%	99.71%	99.76%	99.86%
	$\gamma = 100$	99.06%	99.10%	99.22%	99.52%
	$\gamma = 200$	97.26%	97.20%	97.51%	98.36%
$m = 80\%$	$\gamma = 20$	99.87%	99.95%	99.96%	99.98%
	$\gamma = 50$	99.83%	99.90%	99.93%	99.96%
	$\gamma = 100$	99.69%	99.76%	99.80%	99.88%
	$\gamma = 200$	99.21%	99.24%	99.35%	99.59%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60\text{d}$), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the Deribit perpetual futures to hedge the spot risk on Coinbase.

Table II.9: Hedge Effectiveness for Coinbase Spot and OKEx Futures

m	γ	5d	15d	30d	60d
$m = 10\%$	$\gamma = 20$	97.79%	97.96%	97.86%	97.34%
	$\gamma = 50$	90.18%	90.84%	90.34%	87.79%
	$\gamma = 100$	76.77%	77.58%	76.17%	69.16%
	$\gamma = 200$	58.83%	58.85%	56.10%	42.70%
$m = 20\%$	$\gamma = 20$	99.71%	99.76%	99.75%	99.62%
	$\gamma = 50$	98.41%	98.65%	98.62%	97.94%
	$\gamma = 100$	94.81%	95.53%	95.39%	93.13%
	$\gamma = 200$	86.13%	87.51%	86.93%	80.41%
$m = 50\%$	$\gamma = 20$	99.96%	99.98%	99.98%	99.96%
	$\gamma = 50$	99.87%	99.90%	99.91%	99.83%
	$\gamma = 100$	99.55%	99.66%	99.66%	99.40%
	$\gamma = 200$	98.42%	98.76%	98.77%	97.86%
$m = 80\%$	$\gamma = 20$	99.97%	99.99%	99.99%	99.98%
	$\gamma = 50$	99.95%	99.97%	99.98%	99.95%
	$\gamma = 100$	99.86%	99.91%	99.91%	99.83%
	$\gamma = 200$	99.54%	99.66%	99.67%	99.40%

Note. This table reports the hedge effectiveness defined in (9) for the optimal hedging strategy θ^* given by (6) under four different default aversion levels ($\gamma = 20, 50, 100, 200$), four different hedge horizon levels ($N\Delta t = 5, 15, 30, 60\text{d}$), and four different margin constraint levels ($m = 10\%, 20\%, 50\%, 80\%$). The hedger shorts the OKEx perpetual futures to hedge the spot risk on Coinbase.