Quinn Nolan

Homework 1

1 i) $\langle \psi | \phi \rangle = \langle |\psi\rangle, |\phi\rangle \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\rangle = \bar{1} \cdot \frac{1}{\sqrt{2}} + \bar{0} \cdot \frac{1}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}}$

$\langle \phi | \psi \rangle = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \frac{\bar{1}}{\sqrt{2}} \cdot 1 + \frac{\bar{1}}{\sqrt{2}} \cdot 0 = \boxed{\frac{1}{\sqrt{2}}}$

ii) $\langle \psi | \phi \rangle = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\rangle = 0 \cdot \frac{1}{\sqrt{2}} + \bar{1} \cdot \frac{1}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}}$

$\langle \phi | \psi \rangle = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \frac{\bar{1}}{\sqrt{2}} \cdot 0 + \frac{\bar{1}}{\sqrt{2}} \cdot 1 = \boxed{\frac{1}{\sqrt{2}}}$

iii) $\langle \psi | \phi \rangle = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\rangle = \frac{\bar{1}}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{\bar{1}}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) = \boxed{0}$

$\langle \phi | \psi \rangle = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\rangle = \frac{\bar{1}}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{-\bar{1}}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}}\right) = \boxed{0}$

iv) $\langle \psi | \phi \rangle = \left\langle \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \bar{i} \cdot 1 + \bar{0} \cdot 0 = \boxed{-i}$

$\langle \phi | \psi \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix} \right\rangle = \bar{1} \cdot i + \bar{0} \cdot 0 = \boxed{i}$

Assume U is square

v) $\langle \psi | \phi \rangle = \left\langle U \begin{pmatrix} 1 \\ 0 \end{pmatrix}, U \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \bar{1} \cdot 0 + \bar{0} \cdot 1 = \boxed{0}$

$\langle \phi | \psi \rangle = \left\langle U \begin{pmatrix} 0 \\ 1 \end{pmatrix}, U \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \bar{0} \cdot 1 + \bar{1} \cdot 0 = \boxed{0}$

2 i) $\boxed{\text{NO}}$ ii) $\boxed{\text{NO}}$ iii) $\boxed{\text{YES}}$ iv) $\boxed{\text{NO}}$ v) $\boxed{\text{YES}}$

3 i) No, needs to be 2 quantum states

ii) No, $|1\rangle$ and $|+\rangle$ are not orthogonal since $\langle |1\rangle, |+\rangle \rangle = \frac{1}{\sqrt{2}} \neq 0$

iii) No, $|\phi^+\rangle$ and $|00\rangle$ are not orthogonal since $\langle |\phi^+\rangle, |00\rangle \rangle = \frac{1}{\sqrt{2}} \neq 0$

iv) Yes

v) Yes

v) = Yes

4 i) $M^* = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $MM^* = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot1+1\cdot1 & 1\cdot0+1\cdot1 \\ 0\cdot1+1\cdot1 & 0\cdot0+1\cdot1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Since $MM^* \neq I$, M is not unitary

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4 ii) $M^* = \begin{bmatrix} \bar{i} & 0 \\ 0 & \bar{i} \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$ $MM^* = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$= I$ Since $MM^* = I$, M is unitary

iii) $M^* = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $MM^* = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$
Since $MM^* \neq I$, M is not unitary

iv) $M^* = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $MM^* = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sin^2\theta + \cos^2\theta & -2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$
$= \begin{bmatrix} 1 & -\sin(2\theta) \\ -\sin(2\theta) & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ Since $MM^* \neq I$, M is not unitary

v) Since M is not square, M is not unitary

5 i) $|\psi\rangle = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ i/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ i^2/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ i^3/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ i/2 \\ i^2/2 \\ i^3/2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$?
$x_0x_1 = 1/2$ $x_0y_1 = i/2$ $y_0x_1 = i^2/2$ $y_0y_1 = i^3/2$
$|\phi_1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i^2/\sqrt{2} \end{pmatrix}$ $|\phi_2\rangle = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$ $|\phi_1\rangle \otimes |\phi_2\rangle = |\psi\rangle$

ii) $|\psi\rangle = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -i/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -i/2 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$? $x_0x_1 = \frac{1}{2}$ $x_0y_1 = 0$
$y_0x_1 = -i/2$ $y_0y_1 = 1/\sqrt{2}$
Not possible since $x_0x_1 = \frac{1}{2}$ so $x_0 \neq 0$, $x_1 \neq 0$ and $x_0y_1 = 0$ so y_1 must be 0 but $y_0y_1 = \frac{1}{\sqrt{2}} \neq 0$. This is a contradiction.

iii) $|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$? $x_0x_1 = 0$ $x_0y_1 = 1$ $x_0 = 1$
$y_0x_1 = 0$ $y_0y_1 = 0$ $y_1 = 1$
$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\otimes$ $|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $|\phi_1\rangle \otimes |\phi_2\rangle = |\psi\rangle$

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6. The unitary we will apply is a 22.5° rotation counterclockwise: $U = \begin{pmatrix} \cos\frac{\pi}{8} & -\sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{pmatrix}$

$U|0\rangle = \begin{pmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} \end{pmatrix}$ Collapses to $|0\rangle$ with probability $\frac{\sqrt{2}+1}{2\sqrt{2}} \approx 0.8536$

Collapses to $|1\rangle$ with probability $\frac{\sqrt{2}-1}{2\sqrt{2}} \approx 0.1464$

$U|+\rangle = \begin{pmatrix} \frac{\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}}}{2\sqrt{2}} \\ \frac{\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}}{2\sqrt{2}} \end{pmatrix}$ Collapses to $|0\rangle$ with probability $\frac{2-\sqrt{2}}{4} \approx 0.1464$

Collapses to $|1\rangle$ with probability $\frac{2+\sqrt{2}}{4} \approx 0.8536$

<u>Strategy</u>: If you see "0", guess $|0\rangle$; If you see "1", guess $|+\rangle$

This has the worst case incorrect guessing probability of ≈ 0.1464