Term Structure Models

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Models

Table 1: Comparison of Interest Rate Models

Model	Uniqueness	Pro	Con
Vasicek	Mean-reverting with constant volatility.	Analytical tractability.	Allows negative rates.
CIR	Non-negativity via square-root volatility.	Ensures positive rates.	Difficult to calibrate.
Hull-White	Time-dependent mean reversion.	Fits initial yield curve.	Increased calibration complexity.
Black-Karasinski	Logarithmic modeling of the short rate.	Ensures positive rates.	Requires numerical solutions.
Nelson-Siegel	Empirical yield curve representation.	Intuitive interpretation of parameters.	Lacks arbitrage-free properties.

Vasicek Model

Derivation of the Vasicek Model

The Vasicek model is a one-factor short-rate model described by the following stochastic differential equation (SDE):

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t,$$

where: - r_t is the short rate at time t, - $\kappa > 0$ is the speed of mean reversion, - θ is the long-term mean level of the short rate, - $\sigma > 0$ is the volatility of the short rate, - W_t is a standard Brownian motion.

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1. Mean-Reverting Property

The term $\kappa(\theta - r_t)$ represents the mean-reverting drift, ensuring that the short rate tends to revert to the long-term mean θ over time at a speed determined by κ.

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2. Solution of the SDE

To solve the SDE, we use the method of integrating factors. Let us rewrite the SDE:

$$dr_t + \kappa r_t dt = \kappa\theta dt + \sigma dW_t.$$

Multiply through by $e^{\kappa t}$, the integrating factor:

$$e^{\kappa t}dr_t + \kappa e^{\kappa t}r_t dt = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dW_t.$$

The left-hand side becomes an exact differential:

$$d\left(e^{\kappa t}r_t\right) = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dW_t.$$

Integrating both sides from 0 to t:

$$e^{\kappa t}r_t - r_0 = \int_0^t \kappa\theta e^{\kappa s}ds + \int_0^t \sigma e^{\kappa s}dW_s.$$

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3. Simplifying the Deterministic Term

The deterministic integral is:

$$\int_0^t \kappa\theta e^{\kappa s}ds = \theta\left(e^{\kappa t} - 1\right).$$

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4. Simplifying the Stochastic Term

The stochastic integral remains as:

$$\int_0^t \sigma e^{\kappa s}dW_s.$$

For now, we leave it as is.

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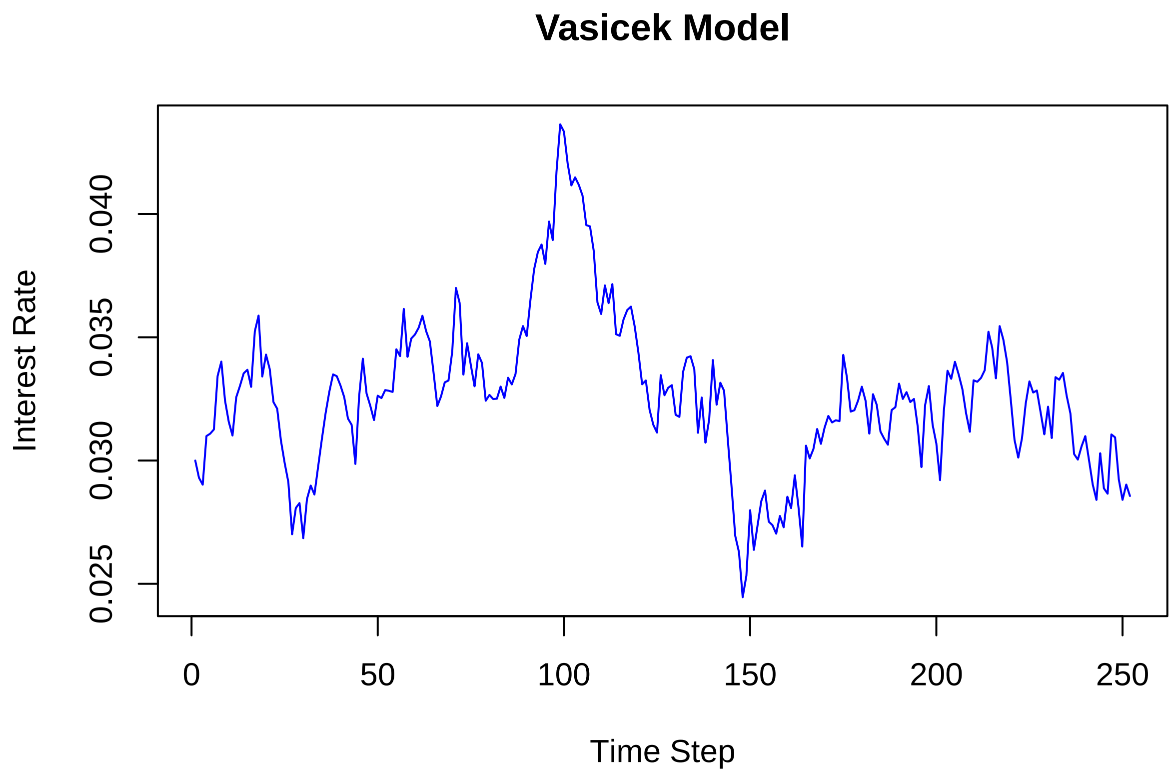
5. Final Solution

Substituting back, we get:

$$r_t = r_0 e^{-\kappa t} + \theta \left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{-\kappa(t-s)} dW_s.$$

This is the explicit solution to the Vasicek model. The first term represents the decay of the initial short rate, the second term represents the drift toward the mean level θ, and the third term represents the stochastic component driven by Brownian motion.

```r
# Vasicek

# Parameters

# Initial int rate (r0)
r0 <- 0.03

# Mean Reversion
a <- 0.1
b <- 0.05

# volatility
sigma <- 0.02

T <- 1

dt <- 1/252
n <- T/dt

# Starting Point
short_rate <- numeric(n)
short_rate[1] <- r0

#Vasicek Process

set.seed(123)


for (i in 2:n) {
  # Weiner Process description of Brownian Motion
  dW <- rnorm(1, mean = 0, sd = sqrt(dt))
  # Passing process for n periods
  short_rate[i] <- short_rate[i-1] + a * (b-short_rate[i-1]) * dt + sigma * dW
}

plot(short_rate,
     type = "l",
     col = "blue",
     xlab = "Time Step",
     ylab = "Interest Rate",
     main = "Vasicek Model")
```



Cox-Ingersol-Ross

Derivation of the CIR Model

The Vasicek model is given by the following stochastic differential equation (SDE):

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \tag{1}$$

where: - r_t is the short rate, - $\kappa > 0$ is the speed of mean reversion, - θ is the long-term mean, - σ is the volatility, - W_t is a standard Brownian motion.

While the Vasicek model allows negative interest rates, the Cox-Ingersoll-Ross (CIR) model modifies the diffusion term to avoid this.

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1. CIR Model Formulation

The CIR model introduces a square-root term in the volatility to ensure non-negativity of the short rate:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\,dW_t. \tag{2}$$

Here, the volatility term $\sigma\sqrt{r_t}$ ensures that as $r_t \to 0$, the stochastic term diminishes, preventing r_t from becoming negative.

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2. Mean-Reverting Property

Similar to the Vasicek model, the term $\kappa(\theta - r_t)$ governs the mean-reverting drift. The CIR model is mean-reverting around θ at a speed determined by κ.

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3. Solution of the CIR Model

To solve the CIR SDE, we first rewrite it in its general form:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\, dW_t.$$

Using Itô's Lemma and the substitution $r_t = x_t^2$, we rewrite the CIR SDE for x_t:

$$dx_t = \left(\frac{\kappa\theta}{2x_t} - \frac{\kappa x_t}{2}\right) dt + \frac{\sigma}{2} dW_t. \tag{3}$$

This is a transformed SDE that can be solved numerically or analytically using advanced techniques.

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4. Feller Condition

The Feller condition ensures the non-negativity of r_t:

$$2\kappa\theta \geq \sigma^2. \tag{4}$$

This condition guarantees that the diffusion term $\sigma\sqrt{r_t}$ does not dominate the drift term $\kappa(\theta - r_t)$, ensuring that $r_t \geq 0$.

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5. Final Form

The CIR process can be expressed as:

$$r_t = r_0 e^{-\kappa t} + \theta\kappa \int_0^t e^{-\kappa(t-s)} ds + \sigma \int_0^t \sqrt{r_s} e^{-\kappa(t-s)} dW_s. \tag{5}$$

This solution highlights the mean-reverting nature, stochastic component, and the non-negativity feature of the CIR model.

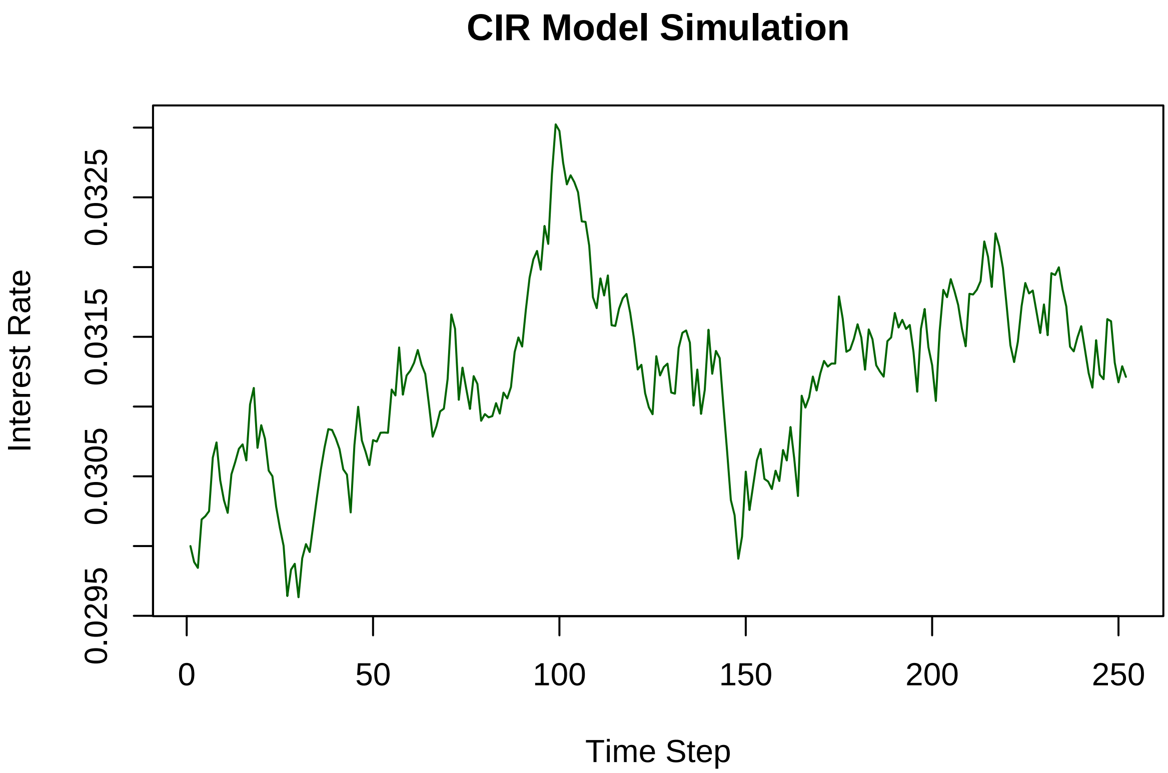
```r
# Cox-Ingersol-Ross (CIR Model)
# Similar to Vasicek, bound above/at 0
# Implements additional process onto variance

# Define parameters for the CIR model
a <- 0.1     # Speed of mean reversion
b <- 0.05    # Long-term mean
sigma <- 0.02 # Volatility
r0 <- 0.03   # Initial interest rate
T <- 1       # Time horizon in years
dt <- 1/252  # Daily time step
n <- T/dt    # Number of steps

# Initialize short rate vector
short_rate <- numeric(n)
short_rate[1] <- r0

# Simulate CIR process
set.seed(123)
for (i in 2:n) {
  dW <- rnorm(1, mean = 0, sd = sqrt(dt))
  short_rate[i] <- short_rate[i-1] + a * (b - short_rate[i-1]) * dt + sigma * sqrt(short_rate[i-1]) * dW
  short_rate[i] <- max(short_rate[i], 0)  # Ensure non-negativity
}

# Plot the simulated short rate path
plot(short_rate, type = "l", col = "darkgreen", xlab = "Time Step", ylab = "Interest Rate", main = "CIR
```



Hull-White Model

Derivation of the Hull-White Model

The Vasicek model is given by the following stochastic differential equation (SDE):

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \tag{6}$$

where: - r_t is the short rate, - $\kappa > 0$ is the speed of mean reversion, - θ is the long-term mean, - σ is the volatility, - W_t is a standard Brownian motion.

While the Vasicek model assumes constant parameters, the **Hull-White model** introduces time-dependent drift to better fit the initial term structure of interest rates.

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1. Hull-White Model Formulation

The Hull-White model modifies the drift term in the Vasicek SDE to include a time-dependent mean $\theta(t)$:

$$dr_t = \kappa\big(\theta(t) - r_t\big)dt + \sigma dW_t. \tag{7}$$

Here: - $\theta(t)$ is a deterministic function of time, often chosen to fit the initial term structure of interest rates.

2. Mean-Reverting Property

As with the Vasicek and CIR models, the term $\kappa(\theta(t) - r_t)$ ensures mean reversion, but now the mean $\theta(t)$ varies with time, allowing the model to adapt to the initial term structure.

3. Solution of the Hull-White SDE

To solve the SDE, we rewrite it:

$$dr_t + \kappa r_t dt = \kappa\theta(t)dt + \sigma dW_t.$$

Multiply through by the integrating factor $e^{\kappa t}$:

$$e^{\kappa t}dr_t + \kappa e^{\kappa t}r_t dt = \kappa\theta(t)e^{\kappa t}dt + \sigma e^{\kappa t}dW_t.$$

The left-hand side simplifies to an exact differential:

$$d\big(e^{\kappa t}r_t\big) = \kappa\theta(t)e^{\kappa t}dt + \sigma e^{\kappa t}dW_t.$$

Integrate both sides from 0 to t:

$$e^{\kappa t}r_t - r_0 = \int_0^t \kappa\theta(s)e^{\kappa s}ds + \int_0^t \sigma e^{\kappa s}dW_s. \tag{8}$$

4. Simplifying the Deterministic Term

The deterministic integral is:

$$\int_0^t \kappa\theta(s)e^{\kappa s}ds. \tag{9}$$

This integral depends on the functional form of $\theta(t)$. For specific choices of $\theta(t)$, such as a constant or linear function, it can be solved explicitly.

5. Stochastic Term

The stochastic integral remains as:

$$\int_0^t \sigma e^{\kappa s}dW_s. \tag{10}$$

The stochastic term contributes to the volatility of the short rate and can be expressed as a Wiener process with modified variance.

6. Final Solution

Combining the deterministic and stochastic components, the solution to the Hull-White model is:

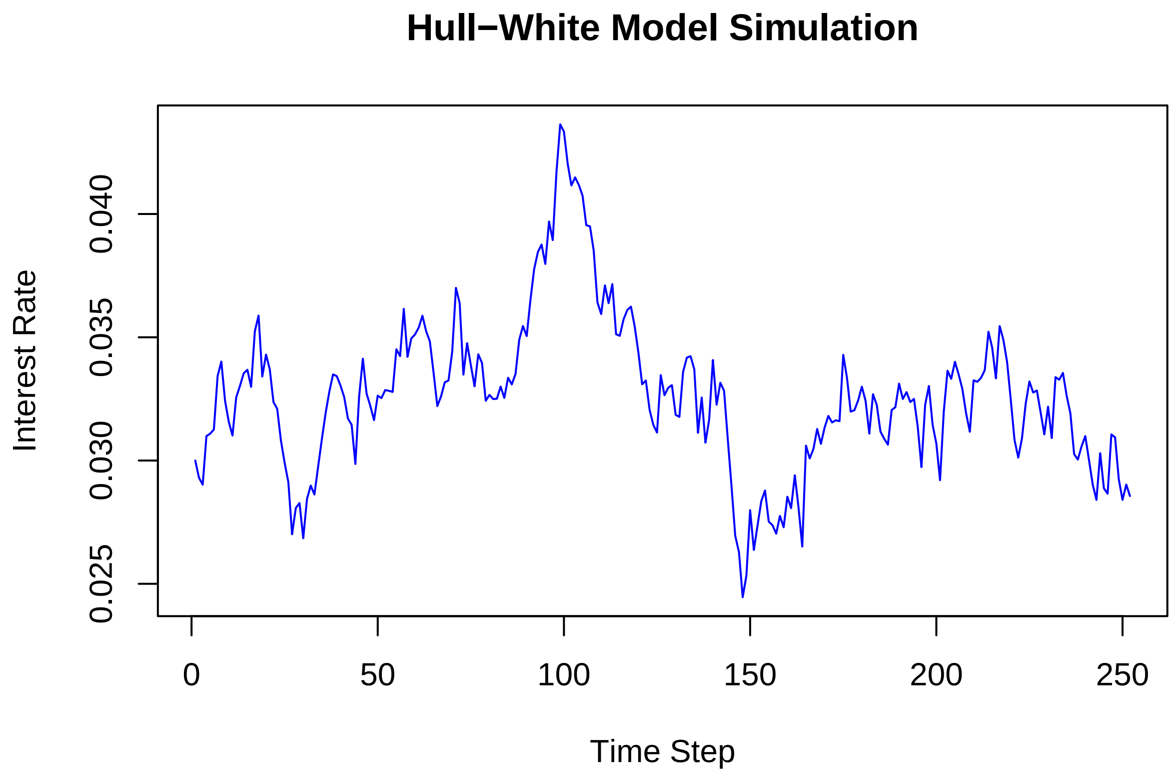
$$r_t = r_0 e^{-\kappa t} + e^{-\kappa t} \int_0^t \kappa \theta(s) e^{\kappa s} ds + \sigma \int_0^t e^{-\kappa(t-s)} dW_s. \tag{11}$$

```r
# Hull-White Model


#parameters
a <- 0.1
sigma <- 0.02
r0 <- 0.03
T <- 1
dt <- 1/252
n <- T/dt
short_rate <- numeric(n)
short_rate[1] <- r0


set.seed(123)
for (i in 2:n) {
  dW <- rnorm(1, mean = 0, sd = sqrt(dt))
  theta <- 0.05
  short_rate[i] <- short_rate[i-1] + a * (theta - short_rate[i-1]) * dt + sigma * dW
}

# Plot the simulated short rate path
plot(short_rate, type = "l", col = "blue", xlab = "Time Step", ylab = "Interest Rate", main = "Hull-Whi
```



Black - Karasinski Model

Derivation of the Black-Karasinski Model

The Black-Karasinski model assumes that the logarithm of the short rate follows an Ornstein-Uhlenbeck process. Starting with the Vasicek model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t,$$

where: - r_t is the short rate, - $\kappa > 0$ is the mean reversion speed, - θ is the long-term mean, - σ is the volatility, - W_t is a standard Brownian motion.

The Black-Karasinski model modifies this by defining the logarithm of the short rate, $x_t = \log(r_t)$, as following an Ornstein-Uhlenbeck process:

$$dx_t = \kappa(\mu(t) - x_t)dt + \sigma dW_t,$$

where: - $x_t = \log(r_t)$, - $\mu(t)$ is a time-dependent mean function.

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1. Model Formulation in Terms of r_t

Rewriting the SDE in terms of the short rate $r_t = e^{x_t}$, we have:

$$dr_t = r_t\big[\kappa(\mu(t) - \log(r_t))dt + \sigma dW_t\big].$$

This SDE ensures that $r_t > 0$, since the logarithmic transformation guarantees positivity of r_t throughout.

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2. Mean-Reverting Property

The drift term $r_t\kappa(\mu(t) - \log(r_t))dt$ ensures that the short rate reverts to a time-dependent mean $e^{\mu(t)}$. This mean-reverting property allows the model to flexibly adapt to changes in the term structure over time.

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3. Solution of the Black-Karasinski Model

To solve the SDE, we first solve for x_t, which satisfies:

$$dx_t = \kappa(\mu(t) - x_t)dt + \sigma dW_t.$$

Multiply through by $e^{\kappa t}$, the integrating factor:

$$e^{\kappa t}dx_t + \kappa e^{\kappa t}x_t dt = \kappa\mu(t)e^{\kappa t}dt + \sigma e^{\kappa t}dW_t.$$

The left-hand side becomes:

$$d\big(e^{\kappa t}x_t\big) = \kappa\mu(t)e^{\kappa t}dt + \sigma e^{\kappa t}dW_t.$$

Integrating both sides from 0 to t, we get:

$$x_t = x_0 e^{-\kappa t} + \int_0^t \kappa\mu(s)e^{-\kappa(t-s)}ds + \sigma\int_0^t e^{-\kappa(t-s)}dW_s.$$

---

4. Back Transformation to r_t

Returning to the original short rate $r_t = e^{x_t}$, the solution becomes:

$$r_t = r_0^{e^{-\kappa t}} \exp\left(\int_0^t \kappa\mu(s)e^{-\kappa(t-s)}ds + \sigma\int_0^t e^{-\kappa(t-s)}dW_s\right).$$

This form highlights the log-normal distribution of r_t under the Black-Karasinski model.

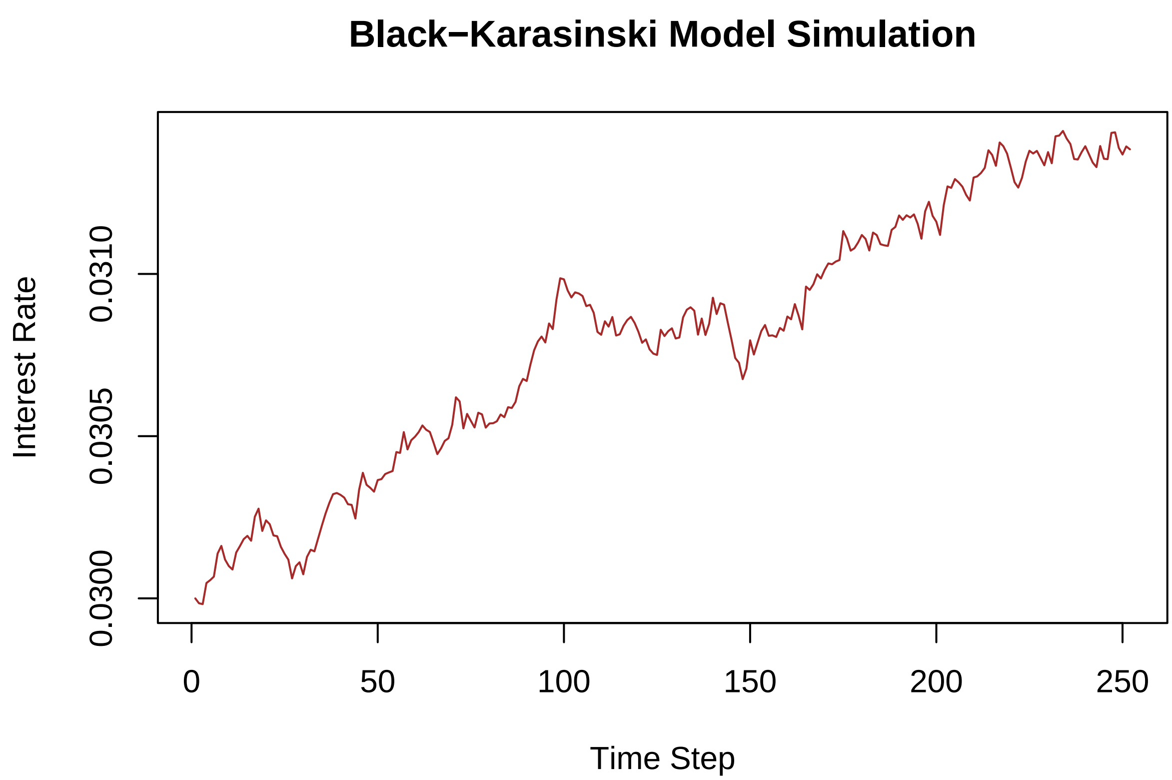
```r
# Define parameters for Black-Karasinski model
a <- 0.1      # Speed of mean reversion
sigma <- 0.02 # Volatility
r0 <- 0.03    # Initial interest rate
T <- 1        # Time horizon in years
dt <- 1/252   # Daily time step
n <- T/dt     # Number of steps

# Initialize short rate vector
short_rate <- numeric(n)
short_rate[1] <- log(r0)

# Simulate Black-Karasinski process
set.seed(123)
for (i in 2:n) {
  dW <- rnorm(1, mean = 0, sd = sqrt(dt))
  short_rate[i] <- short_rate[i-1] + a * (log(0.05) - short_rate[i-1]) * dt + sigma * dW
}

# Convert back to interest rates
short_rate <- exp(short_rate)

# Plot the simulated short rate path
plot(short_rate, type = "l", col = "brown", xlab = "Time Step", ylab = "Interest Rate", main = "Black-Ka
```



Nelson - Siegel

Derivation of the Nelson-Siegel Model

The **Nelson-Siegel model** is a parsimonious representation of the term structure of interest rates. It provides a functional form for the yield curve, characterized by level, slope, and curvature factors.

The yield at time t for a bond maturing at time T is expressed as:

$$y(t,\tau) = \beta_0 + \beta_1 \frac{1-e^{-\lambda\tau}}{\lambda\tau} + \beta_2 \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right),$$

where: - $y(t,\tau)$ is the yield at time t for maturity $\tau = T - t$, - $\beta_0, \beta_1, \beta_2$ are parameters that represent level, slope, and curvature, - λ is a decay parameter that governs the exponential decay of the slope and curvature factors, - τ is the time to maturity.

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1. Model Intuition

The Nelson-Siegel model decomposes the yield curve into three components: 1. **Level (β_0)**: Represents the long-term average yield and shifts the entire curve up or down. 2. **Slope (β_1)**: Represents the short-term deviations from the level and controls the steepness of the curve. 3. **Curvature (β_2)**: Captures the hump-shaped behavior of the curve, typically around medium maturities.

The decay parameter λ determines the speed at which the slope and curvature factors decay to zero as maturity increases.

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2. Functional Form

2.1. Level Component (β_0)

The level factor is independent of τ, making it constant across all maturities:

$$\text{Level: } \beta_0.$$

2.2. Slope Component (β_1)

The slope factor decreases as τ increases, with the decay controlled by λ:

$$\text{Slope: } \beta_1 \frac{1-e^{-\lambda\tau}}{\lambda\tau}.$$

2.3. Curvature Component (β_2)

The curvature factor increases initially, reaches a maximum, and then decays back to 0 for large τ:

$$\text{Curvature: } \beta_2 \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right).$$

---

3. Special Cases

3.1. When $\lambda \to 0$

The slope and curvature factors degenerate into linear terms, and the model reduces to a simple linear function of maturity.

3.2. When $\lambda \to \infty$

The slope factor decays instantaneously, leaving only the level factor (β_0) and curvature factor (β_2) contributing to the yield.

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4. Estimation

The parameters $\beta_0, \beta_1, \beta_2$, and λ are typically estimated using historical yield curve data by minimizing the squared error between the observed and modeled yields:

$$\min_{\beta_0,\beta_1,\beta_2,\lambda} \sum_{\tau} \left[y_{\text{obs}}(t,\tau) - y(t,\tau)\right]^2 .$$

This estimation provides the best fit for the yield curve at a given time t.

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5. Final Form

The Nelson-Siegel model can be summarized as:

$$y(t,\tau) = \beta_0 + \beta_1 \frac{1-e^{-\lambda\tau}}{\lambda\tau} + \beta_2 \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right).$$

```r
#Nelson-Siegel Model
library(minpack.lm)

# Define a function for Nelson-Siegel yield curve
nelson_siegel <- function(t, beta0, beta1, beta2, lambda) {
  return(beta0 + beta1 * (1 - exp(-lambda * t)) / (lambda * t) + beta2 * ((1 - exp(-lambda * t)) / (lam
}

# Example maturity and yield data
maturity <- c(1, 2, 3, 5, 7, 10)
yield <- c(0.02, 0.025, 0.027, 0.03, 0.032, 0.035)

# Fit the Nelson-Siegel model to the data
ns_model <- nlsLM(yield ~ nelson_siegel(maturity, beta0, beta1, beta2, lambda),
                  start = list(beta0 = 0.03, beta1 = -0.01, beta2 = 0.01, lambda = 0.5))

# Display model summary
summary(ns_model)
```

```
##
## Formula: yield ~ nelson_siegel(maturity, beta0, beta1, beta2, lambda)
##
## Parameters:
##          Estimate Std. Error t value Pr(>|t|)
## beta0   3.901e-02  2.384e-03   16.36  0.00371 **
## beta1  -2.373e-02  1.908e-03  -12.44  0.00640 **
## beta2   6.005e-07  2.291e+00    0.00  1.00000
## lambda  5.133e-01  4.957e+01    0.01  0.99268
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0009498 on 2 degrees of freedom
##
## Number of iterations to convergence: 14
## Achieved convergence tolerance: 1.49e-08
```

```r
# Plot the fitted yield curve
fitted_yield <- predict(ns_model)
plot(maturity, yield, pch = 19, col = "red", xlab = "Maturity (Years)", ylab = "Yield", main = "Nelson-S
lines(maturity, fitted_yield, col = "blue", lwd = 2)
```