Idiot's Guide to Differential Equations

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1 Chapter 1:

1.1 Basic Definitions

Differential Equations are... simply equations containing derivatives! There are two types:

- **Ordinary Differential Equations (ODEs):** Only have one independent variable
  ODE Ex. $y' - 7y = \dfrac{d^2y}{dx} - \dfrac{dy}{dx} = e^x$
- **Partial Differential Equations (PDEs):** Have more than one independent variable
  PDE Ex. $\dfrac{d^2z}{dx^2} + \dfrac{d^2x}{dy^2} = 0$ (Notice the different variables in the denominator!)

The **Order** of a Diff Eq is the order of the highest derivative.

There are many ways that we can describe **ODEs**:

- **First Order Diff. Eq.s** can be given by the form $M(x,y)dx + N(x,y)dy = 0$
  Ex. $7x^2y' - 5xy = 0 \rightarrow 7x^2dy - 5xydx = 0$
- **Linear ODEs** have a general form of $a_n(x)\left(\dfrac{d^ny}{dx^n}\right) + ... + a_1(x)\left(\dfrac{dy}{dx}\right) + a_0(x) = g(x)$
  Ex. $y'' - 3y' - 4y = 6e^{-x}$
- In general, we can describe **nth order ODEs** using the form $F\left(x, y, \dfrac{dy}{dx}, ..., \dfrac{d^ny}{dx^n}\right)$

Furthermore, there are multiple *types* of solutions to diff. eq.s:

- **Explicit solutions:** Dependent variable can be expressed solely through the independent variable
  - This usually just means we can write $y = ...$ using only x variables.
- **Implicit solutions:** Dependent variable cannot be written directly using the independent variable
- **Singular solution:** A solution that doesn't belong to a family of solutions

We can disregard implicit solutions for now, since when we're looking or verifying a solution to a diff. eq., they're usually expressed explicitly.

Now we can move onto some *real* diff. eq.! Our first problem will ask us to *verify* a solution. That is, we're given a solution $y = ...$, and we're asked to see if it works with the given diff. eq.

Example: Verify that the indicated family of functions is a solution to the given diff. eq..

$$\frac{dy}{dx} + 4xy = 8x^3 \qquad y = 2x^2 - 1 + c_1e^{-2x^2}$$

First, don't be intimidated by the phrase "family of functions"! It just means that because there's a constant term in y (in this case, c_1), we can have multiple explicit solutions.

Our first step is to find out what order of y we need to know. In our diff. eq. on the left, we see y in two terms: $\dfrac{dy}{dx}$ and just y. We know what y is, so we need to know what $\dfrac{dy}{dx}$ is. We can easily find this by differentiating our equation for y.

$$y' = (2x^2)' + (-1)' + (c_1e^{-2x^2})' = 4x + c_1 \cdot (-4x)e^{-2x^2} = 4x - 4c_1xe^{-2x^2}$$

Now we just plug in our $\dfrac{dy}{dx}$ and y into the left side of our diff. eq.:

$$\frac{dy}{dx} + 4xy = (4x - 4c_1xe^{-2x^2}) + 4x(2x^2 - 1 + c_1e^{-2x^2}) = 4x - 4c_1xe^{-2x^2} + 8x^3 - 4x + 4c_1xe^{2x^2} = 8x^3$$

Our solution matches the right-hand side of the original diff. eq.! Thus, we can verify that this is a solution to our diff. eq.

1.2 Solutions and Intervals

We have seen in Section 1.1 how we can verify a solution for a differential equation. But what if we're given specific initial conditions? Then we have an Initial Value Problem on our hand!

- **Initial Value Problem (IVP):** An n^{th} order differential equation with n constraints, or initial conditions, on the n derivatives

This might seem scary, but really it's what we did in Section 1.1 with an added step of plugging in variables. Let's do one out!

Ex1. $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP given the initial conditions $y(-1) = 5$ and $y'(-1) = -5$.

You still with me? Good! This is basically just asking us to find the constant terms in y. So let's just try plugging $y(-1) = 5$ into $y = ...$

$$y = 5 = c_1e^{-1} + c_2e^{-(-1)} = c_1e^{-1} + c_2e$$

Hmm... we can't reduce this at all! Why don't we try differentiating y so that we can plug-and-chug y'?

$$y' = c_1e^x - c_2e^{-x}, -5 = c_1e^{-1} - c_2e^{-(-1)} = c_1e^{-1} - c_2e$$

Well that didn't help at all as well! Well wait a minute, now we have a system of equations! Why don't we try adding them together?

$$(5 = c_1e^{-1} + c_2e) + (-5 = c_1e^{-1} - c_2e) = (0 = 2c_1e^{-1})$$

Now we're getting somewhere! Obviously, $c_1 = 0$, so if we plug this into one of our conditional statements...

$$-5 = (0)e^{-1} - c_2e \to -5 = -c_2e \to c_2e = 5 \to c_2 = \frac{5}{e}$$

We get $c_1 = 0$ and $c_2 = \frac{5}{e}$, so plugging this back into y, we get our solution:

$$y = (0)e^x + \left(\frac{5}{e}\right)e^{-x} = \left(\frac{5}{e}\right)e^{-x} = 5e^{-x-1}$$

Let's look at another example, this time involving a little less thought.

Ex2. Find a solution for $y' = y^{1/3}$, $y(0) = 0$.

We can go through the rigmarole again, or we can realize... wait a minute... if we're just asked to find some y where $y(0) = 0$... isn't $y = 0$ a solution? And it is! This is a **singular solution**, and an example of work smarter, not harder.

Ex2. Determine the interval of definition of the solution for $y' + 2xy^2 = 0$ given $y(0) = -1$.

Spoiler alert, but when we solve this DE we get $y = \dfrac{1}{x^2 - 1}$. Notice that this functions is defined on the interval $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. This is the domain in which our solution to the DE is defined on. However, we're given a constraint where our solution MUST contain the point $(0, -1)$. Therefore, our solution is defined on $(-1, 1)$.

1.3 Existence and Uniqueness

Mathematicians love to get philosophical for the hell of it. Therefore, when Plato is solving an IVP DE, he always asks himself key two questions:

1. Does the DE posses any solutions? Do any of our possible solutions pass though the initial condition? Does there *exist* ...

2. When is there only one such solution that passes through our initial condition? Is our solution *unique*?

These questions are not just the incoherent rambling of a mad man, but actually a theory in Diff Eq known as... **The Existence and Uniqueness Theorem**. It is stated as follows:

Let $y' = f(x, y)$ be a 1^{st} degree ODE. There is a point $y(x_0) = y_0$, or (x_0, y_0). We want to find how many solutions for this diff. eq. pass through this point.

Let R be a rectangle with the parameters $a < x_0 < b$, $c < y_0 < d$. (This is to say, imagine a box around our point (x_0, y_0).) Don't bother picking specific boundaries for R.

- If $f(x, y)$ is continuous on R, then a solution to the IVP is guarenteed to exist.

- Furthermore, if $f_y(x, y) = \dfrac{\partial f}{\partial y}$ is continuous on R, then the solution is unique.

Not so bad, right? We just have to show that $f(x, y)$ is continuous to show that it has a solution, and that $f_y(x, y)$ is continuous to show that the solution is unique. Let's try out an example:

Ex. $\dfrac{dy}{dx} - y^{-1} = x \qquad y(2) = 3.$

We can rewrite $\dfrac{dy}{dx} - y^{-1} = x$ as $y' = y^{-1} + x$, and $y(2) = 3$ as the point $(2, 3)$.

Think of our DE as a function with two independent variables $f(x, y) = y' = y^{-1} + x$. $f(x, y)$ is continuous on $x \in (-\infty, \infty)$ and $y \in (-\infty, 0) \cup (0, \infty)$. Since our point $(2, 3)$ does not have $y = 0$, our DE is continuous at this point.

Furthermore, to show that our point is unique, we need to differentiate f in terms of y. $f_y(x, y) = \dfrac{-1}{y^2}$. Once again, because our point $(2, 3)$ is defined on $f_y(x, y) = \dfrac{-1}{y^2}$, we know that this solution is unique.

2 Chapter 2:

2.1 Autonomous Equations and Phase Lines

Before we dive into solving 1st order DEs, we're going to take a detour and talk about autonomous equations and phase lines.

- **Autonomous Equation:** An ODE with the form $y' = f(y)$ (where y' has no explicit dependence on x)

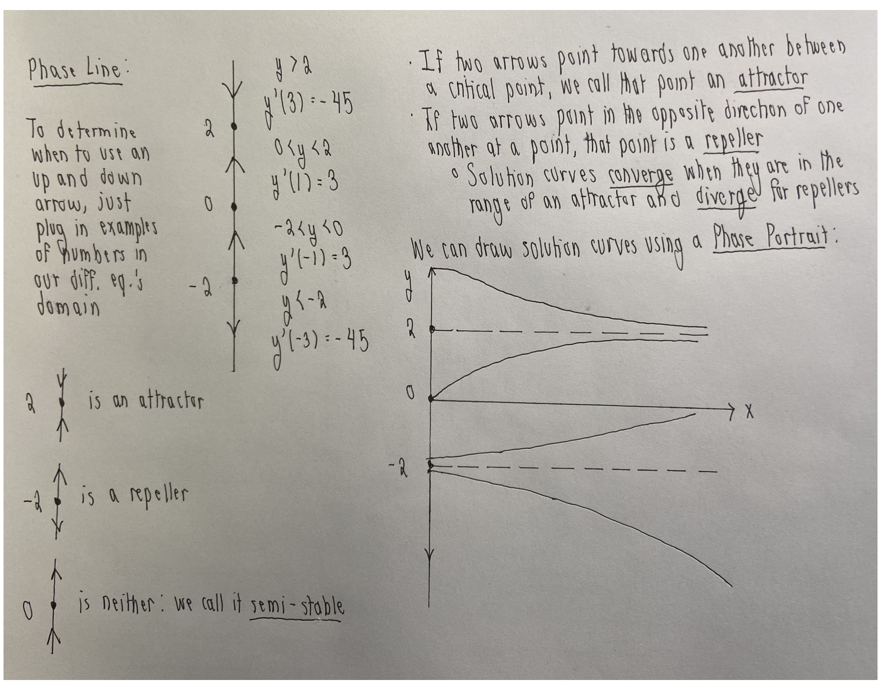
These sound TERRIBLE to solve, so we're not going to bother! Rather, we'll graph them. But first, we need to find out an Autonomous Equation's **critical points**: y values where $y' = 0$. We can do this simply by.... setting y' to 0. If c is a critical point, then we call $y = c$ an **equilibrium solution**.

- **Ex.** $\dfrac{dy}{dx} = y^2(4 - y^2) \implies y = -2, 0, 2$

This is cool and all, but the reason why Autonomous Equations are special are because of Phase Lines. **Phase Lines** allow us to determine the behavior of all solutions as depicted on a graph without having to solve an ODE. We can draw Phase Line diagram as follows:

1. Draw a vertical line representing the y-axis.
2. Add any critical points this line and determine the sign of y' in the intervals between these points.
3. Use an up arrow for when $y' > 0$ and a down arrow for when $y' < 0$.

Let's see what happens when we graph our example from above, shall we?



As depicted in the diagram above, phase lines can describe solution curves. We see that with Autonomous Equations, when we are given an initial condition, our solutions at each point are always going to be unique. Furthermore, our solution curves cannot cross our equilibrium solutions $y = c$.

2.2 Separable Equations

Now that we have a firm grasp on what a DE is and the types of DEs there are, we can jump straight into solving different types of DEs. In briefcase number 1 is the separable equation, with the form:

$$\frac{dy}{dx} = g(x)h(y)$$

We think of $g(x)$ and $h(y)$ are two equations exclusively in terms of x and y respectively.

A general rule of thumb to finding out if a 1st order DE is a separable equation or not is if we can move all y terms, including dy, to one side, and all x terms, including dx, to the other. Some examples include:

1. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2 \implies \frac{dy}{dx} = \frac{(2y+3)^2}{(4x+5)^2} \implies \frac{1}{(4x+5)^2}dx = \frac{1}{(2y+3)^2}dy$

2. $dy - (y-1)^2 dx = 0 \implies dy = (y-1)^2 dx = \frac{1}{(y-1)^2}dy = 1dx$

3. $y' = y - x$ is **NOT** separable.

In order to solve separable equations, we abide by four sacred rules, two of which we've already demonstrated in the examples above. These include:

1. Determine if the equation is separable or not.

2. Move all y terms to the left and all other terms to the right (x terms and constants).

3. Integrate both sides with respect to y or x.

4. Solve for y if possible (try to find an explicit solution). Otherwise, express C as an implicit solution.

Ex. Solve $\frac{dy}{dx} = x\sqrt{1-y^2}$.

1. If we let $g(x) = x$ and $h(y) = \sqrt{1-y^2}$, we have a $\frac{dy}{dx} = g(x)h(y)$ equation.

2. $\frac{dy}{dx} = x\sqrt{1-y^2} \implies \frac{1}{\sqrt{1-y^2}}dy = xdx$.

3. $\int \frac{1}{\sqrt{1-y^2}}dy = \arcsin(y)$, $\quad \int xdx = \frac{1}{2}x^2 + C$

4. $\arcsin(y) = \frac{1}{2}x^2 + C \implies y = \sin\left(\frac{1}{2}x^2 + C\right)$

2.3 Linear Equations

What's that behind door number two? It's the 1st order linear equation, with a form:

$$a_1(x)y' + a_0(x)y = g(x)$$

Here, $a_0(x)$, $a_1(x)$, and $g(x)$ are some functions of x.

- Linear Equations: $\dfrac{dy}{dx} + 2y = 0, \quad (x+1)\dfrac{dy}{dx} + (x+2)y = 2xe^{-x}$
- Not Linear Equations: $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$
  - Rule of Thumb (And good dating advice): If it's not linear, it's probably separable.

Solving linear equations tends to require more work than solving separable equations, and the steps are a little more convoluted. Here they are, as follows:

1. Given a linear equation $a_1(x)y' + a_0(x)y = g(x)$, if $a_1(x) \neq 1$, divide both sides of the equation to convert this into **standard linear equation form**:

$$y' + P(x)y = f(x)$$

2. Take some function $\mu(x)$ and multiply each side by $\mu(x)$: $\mu(x)y' + \mu(x)P(x)y = \mu(x)f(x)$
3. If we assume $\mu'(x) = P(x)\mu(x)$, then our second term simplifies to: $\mu(x)y' + \mu'(x)y = \mu(x)f(x)$
4. We integrate both sides. $\int \mu(x)y' + \mu'(x)y = \mu(x)y$, so we can solve $y = \dfrac{1}{\mu(x)} \int \mu(x)f(x)dx$

First, we have to address the elephant in the room: what the hell is $\mu(x)$? $\mu(x)$ is our **Integration Factor**, and we use it because as shown above, it make integrating the left-side of the equation much more easy. Generally, we define $\mu(x)$ as such:

$$y' + P(x)y = f(x) \rightarrow \mu(x) = e^{\int P(x)dx}$$

Ex. Solve $xy' + 4y = x^3 - x$.

1. First, we have to make sure that our $a_1(x) = 1$. We do this by dividing both sides by x: $\dfrac{1}{x}\left(xy' + 4y = x^3 - x\right) \rightarrow y' + \dfrac{4}{x} = x^2 - 1$
2. Now, we find our integrating function. $\mu(x) = e^{\int P(x)dx} = e^{\int \frac{4}{x}dx} = e^{4ln(x)} = x^4$.
3. We multiply our equation by $\mu(x)$: $x^4(y' + \dfrac{4}{x})y = x^4(x^2 - 1)$.
4. Integrating both sides, we get:

$$\int x^4\left(y' + \frac{4}{x}y\right)dx = x^4 \cdot y \qquad \int x^4(x^2-1)dx = \int (x^6 - x^4)dx = \frac{1}{7}x^7 - \frac{1}{5}x^5 + C$$

5. Simplifying our solution, we get: $y = \dfrac{1}{7}x^3 + \dfrac{1}{5}x + \dfrac{C}{x^4}$.

Sometimes our $g(x)$ function is a trigonometric function like $\sin(x)$ or $\cos(x)$. We're not really able to solve these using this method, but we can use the **Method of Undetermined Coefficients** which we'll discuss in a future chapter.

2.4 Homogeneous Equations

The word "homogeneous", NOT to be confused with "homogenous", is one of the more widely-used terms in math and science that has so many different meanings you loose track of what it means in the first place. Therefore, before we talk about homogeneous equations, we need to talk about what would make a function homogeneous in the first place.

- A **homogeneous function** has the same total degree for each term. You can think of this like adding up the power for every x and y variable in a term - if one term doesn't have the same total power, the function is not homogeneous.
- Ex. - $f(x,y) = x^7 + y^6x + x^4y^3$, $\quad g(x,y) = \sqrt{x^4 + y^4}$, $\quad$ NOT Homogeneous: $h(x,y) = x^6 + y^3x^2$

A **homogeneous equation** is usually given in the form:

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \implies \quad M(x,y)dx + N(x,y)dy = 0 \ ,$$

where $M(x,y)$ and $N(x,y)$ are *homogeneous* functions.

We can solve homogeneous equations using the substitution $u = \dfrac{y}{x}$. Our steps are as follows:

1. If the problem is given in $\dfrac{dy}{dx} = \dfrac{M(x,y)}{N(x,y)}$ form, convert it into $M(x,y)dx + N(x,y)dy = 0$ form.
2. Using the substitution $u = \dfrac{y}{x}$, we can substitute $y = ux$ and $dy = udx + xdu$.
3. Combine all dx and du terms into two separate terms.
4. Factor our all x terms and divide the entire equation by the larger x term. Because everything is equal to 0, we can just move both terms to the other derivative.
   - If you're doing things right your dx term should ALWAYS be $\dfrac{1}{x}dx + ...du = 0$.
5. Integrate both sides, replace yx for u then simplify. We should get $\ln|x| + ... = \ln|C|$.

Ex. Solve $-ydx + (x + \sqrt{xy})dy = 0$.

1. Our equation is already in $M(x,y)dx + N(x,y)dy = 0$ form.
2. Using our substitutions $y = ux$ and $dy = udx + xdu$, we get: $-(ux)dx + (x + \sqrt{x(ux)})(udx + xdu) = -(ux)dx + (x + \sqrt{ux^2})(udx + xdu) = -(ux)dx + (x + x\sqrt{u})(udx + xdu) = 0$.
3. Separating dx and du, we get: $-(ux)dx + (x + x\sqrt{u})(udx + xdu) = -(ux)dx + (ux + xu^{3/2})dx + (x^2 + x^2\sqrt{u})du = (xu^{3/2})dx + (x^2 + x^2\sqrt{u})du = 0$
4. Factoring out our x terms, we get: $x(u^{3/2})dx + x^2(1 + \sqrt{u})du = 0$.
5. Dividing both sides by our opposite u and x terms gives us $\dfrac{1}{x}dx + \dfrac{1+\sqrt{u}}{u^{3/2}}du = 0$.
6. When we integrate everything, we know that: $\displaystyle\int \frac{1}{x}dx = \ln|x|$ and $\displaystyle\int 0 = \ln|C|$.
   $\displaystyle\int \frac{1+\sqrt{u}}{u^{3/2}}du = \int \frac{1}{u^{3/2}}du + \int \frac{\sqrt{u}}{u^{3/2}}du = \int u^{-3/2}du + \int u^{-1}du = -2u^{-1/2} + \ln|u|$.
   Combining everything, we get $\ln|x| - 2u^{-1/2} + \ln|u| = \ln|C|$.
7. Resubstituting $y = ux$, we get $\ln|x| - 2(yx)^{-1/2} + \ln|yx| = \ln|C|$.

   Further simplification will give us $yx = \dfrac{1}{4}\left(\ln\left|\dfrac{yx^2}{C}\right|\right)^2$.

2.5 Bernoulli Equations

Unfortunately, Bernoulli isn't the name of a spaghetti sauce, but in fact another type of differential equation. Bernoulli equations have the form:

$$y' + P(x)y = f(x)y^n,$$

where $P(x)$ and $f(x)$ are any functions of x, and n is any number that's not 0 or 1.

Like Homogeneous Equations, we can use a substitution to make solving Bernoulli equations easier. Unlike Homogeneous Equations however, our substitution will get us a linear equation which we can solve like any other. Our steps are as follows:

1. Simplify the equation into a $y' + P(x)y = f(x)y^n$ form.
    - Once we know n, we know our substitution is: $u = y^{1-n}$. This gives us $\dfrac{du}{dx} = (1-n)y^{-n}y'$.
2. Multiply the entire equation by our $(1-n)y^{-n}$ term from above. This does NOT include y'!
3. Now it should be much easier to substitute our y and y' terms for u and u'.
4. Once we have all of our y terms replaced, solve like a linear equation!
    - Don't forget to resubstitute $u = yx$ at the end!

Ex. - Solve $\dfrac{dy}{dx} = y(xy^3 - 1)$.

1. $\dfrac{dy}{dx} = y(xy^3 - 1) = xy^4 - y$. This gives us $y' + y = xy^4$.
2. Given xy^4, we know that $n = 4$. Therefore, our substitutions are $u = y^{1-n} = y^{-3}$ and $u' = (1-n)y^{-n}y' = -3y^{-4}y'$.
3. If we multiply both sides of our equation by $-3y^{-4}$, we get: $-3y^{-4}y' + (-3)y^{-4}(y) = -3y^{-4}y' + (-3)y^{-3} = (xy^4)(-3y^{-4}) = -3x$.
4. Once we substitute our u terms, we get our linear equation: $-3y^{-4}y' - 3y^{-3} = -3x \implies \quad u' - 3u = -3x$.
5. We know how to solve L.E., so we will gloss over the steps for solving $u' - 3u = -3x$:
    (a) $\mu(x) = e^{\int -3dx} = e^{-3x}$.
    (b) $u' - 3u = -3x \implies \quad e^{-3x}u' - 3ue^{-3x} = -3xe^{-3x}$.
    (c) $\int e^{-3x}u' - 3ue^{-3x}dx = e^{-3x}u = \displaystyle\int -3xe^{-3x}dx = xe^{-3x} + C$.
6. Now we resubstitute: $e^{-3x}u = ye^{-3x}x = xe^{-3x} + C \to y = 1 + \dfrac{Ce^{3x}}{x}$

2.6 Numerical Approximation: Euler's Method

Solving a differential equation is hard work. Rather than figuring out what type of equation we're dealing with in the first place and doing god knows what to remove the derivatives, what if just wanted an answer that's *good enough*?

Like putting your food in the microwave for too little but not wanting to wait any longer, we have several **numerical approximation** methods with their own merits. Some of the more noteworthy ones include:

1. **Direction fields** allow us to plot the derivative of each point on a Cartesian plane. They can help us *visualize* solution curves if we have an initial value, but otherwise we're not able to *approximate* specific values.

2. **Tangent Lines** describe the instantaneous rate of change at a fixed point. That is, they tell us what the derivative is at that particular x. We can use tangent lines to approximate where our next point (x, y) might be.

3. **Euler's Method** allows us to approximate specific values based on a "step" value. The precision of our answer depends on this "step" value. As we use more steps, we're iteratively piecing together tangent lines that are more accurate to our actual curve.

Euler's Method is a **recursive** algorithm. We use the y value from a previous step to generate the value for our next step, so on and so forth. Therefore, we need an initial condition to star with.

Suppose we are given an initial condition $y(x_0) = y_0$, a function $f(x)$ and a step value h. We can then perform the following steps:

1. First, we calculate our next x value: $x_n = x_0 + n \cdot h$.
   - n refers to which step we're on.

2. Second, we calculate our next y value: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$.
   - We use y_{n+1} to signify recursion. We can calculate x_n whenever, but we specifically need the last y_n value to calculate y_{n+1}.

Ex. - Given $y' = (x - y)^2$ and $y(0) = 0.5$, approximate $y(0.5)$ using $h = 0.1$.

n	$x_0 = 0$	$y_0 = 0.5$
$n = 1$	$x_1 = 0.1$	$\mathrm{y}_1 = (0.5) + (0.1) \cdot (0 - 0.5)^2 = 0.525$
$n = 2$	$x_2 = 0.2$	$\mathrm{y}_2 = (0.525) + (0.1) \cdot (0.1 - 0.525)^2 = 0.5430625$
$n = 3$	$x_3 = 0.3$	$y_3 = (0.5430625) + (0.1) \cdot (0.2 - 0.5430625)^2 = 0.5548316879$
$n = 4$	$x_4 = 0.4$	$y_4 = 0.5613256068$
$n = 5$	$x_5 = 0.5$	$y_5 = 0.5629282019$

With $h = 0.1$, $y(0.5) \approx 0.5629282019$

2.7 Ricatti Equations

Sticking to the trend of names you'd see on Spaghetti sauce jars are Ricatti Equations. Ricatti Equations follow the general form:

$$y' = p(x) + q(x)y + r(x)y^2,$$

where $p(x)$, $q(x)$, and $r(x)$ are some functions of x.

With Bernoulli equations, you want to convert them into linear equations. With Ricatti equations, you want to convert them into Bernoulli equations, which then become linear equations. Unlike our other methods, we can only solve a Ricatti if we are already given a **particular solution**. The steps are as follows:

1. If not already in the general form, convert the equation to $y' = p(x) + q(x)y + r(x)y^2$ form.
2. Given a particular solution y_1, use the substitution $y = y_1 + u$ to reduce the Riccati's equation into a Bernoulli's equation.
   - If you do everything right then you should get a Bernoulli's equation with $n = 2$.
3. Reduce the Bernoulli equation into a linear equation using a substitution of $w = u^{-1}$.
4. After you solve the linear equation, don't forget to resubstitute!

Ex. - Solve $\dfrac{dy}{dx} = -\dfrac{4}{x^2} - \dfrac{1}{x}y + y^2$ given $y_1 = \dfrac{2}{x}$.

1. Given $y_1 = \dfrac{2}{x}$, our substitutions are $y = y_1 + u = \dfrac{2}{x} + u$ and $y' = -\dfrac{2}{x^2} + u'$.

   Plugging these into our equation, we get $\left(-\dfrac{2}{x^2} + u'\right) = -\dfrac{4}{x^2} - \dfrac{1}{x}(\dfrac{2}{x} + u) + \left(\dfrac{2}{x} + u\right)^2$.
   - Simplifying, we get $\left(-\dfrac{2}{x^2} + u'\right) = -\dfrac{4}{x^2} - \dfrac{1}{x}\left(\dfrac{2}{x} + u\right) + \left(\dfrac{2}{x} + u\right)^2$

     $= -\dfrac{4}{x^2} - \dfrac{2}{x^2} - \dfrac{u}{x} + \dfrac{4}{x^2} + \dfrac{4u}{x} + u^2 = u^2 + \dfrac{3u}{x} - \dfrac{2}{x^2}$.
2. To convert this into a Bernoulli equation, we just have to move some terms around as follows:
   $-\dfrac{2}{x^2} + u' = u^2 + \dfrac{3u}{x} - \dfrac{2}{x^2} \implies \quad u' - \dfrac{3}{x}u = u^2$.
3. Now we can solve this as a Bernoulli. Because we have $n = 2$, we can use the substitutions
   $w = u^{1-n} = u^{-1}$ and $w' = -u^{-2}u'$.

   Multiplying both sides by $-u^{-2}$ gives us $-u^{-2}\left(u' - \dfrac{3}{x}u\right) = -u^{-2}u' + \dfrac{3}{x}u^{-1} = -u^{-2} \cdot u^2 = -1$.

   Using our substitutions for w, we get $-u^{-2}u' + \dfrac{3}{x}u^{-1} = -1 \implies \quad w' + \dfrac{3}{x}w = -1$.
4. Now we have a linear equation. Our integration factor will be $\mu(x) = e^{\int 3/x dx} = e^{3\ln|x|} = x^3$.

   Multiplying, we get $x^3\left(w' + \dfrac{3}{x}w\right) = x^3w' + 3x^2w = -x^3$.

   Integrating, we get $\displaystyle\int x^3w' + 3x^2w = x^3w = \int -x^3dx = -\frac{1}{4}x^4 + C$.
5. Now we resubstitute. Since $w = u^{-1}$, we get $x^3w = -\dfrac{1}{4}x^4 + C \implies \quad x^3u^{-1} = -\dfrac{1}{4}x^4 + C$

   Since $y = y_1 + u \implies u = y - y_1 = y - \dfrac{2}{x}$, we have $x^3u^{-1} = x^3\left(y - \dfrac{2}{x}\right)^{-1} = -\dfrac{1}{4}x^4 + C$.
6. Further simplification will get us $y = \dfrac{2}{x} + \left(-\dfrac{1}{4}x + Cx^{-3}\right)^{-1}$.

3 Chapter 3:

3.1 First Order Linear Models

One of the cool things about math is that it applies to a bunch of things in real life. "What????", you, the dear reader, might be thinking at this very moment, but you know that I would never lie to you just to get this class over with. We'll go over some neat examples of how first-order differential equations are used in linear models.

Growth and Decay: Initial-value problems given by $\dfrac{dx}{dt} = kx$, $x(t_0) = x_0$ give you a constant rate of proportionality k and usually ask you to find the proportion of x if we know x at a given time t_0. You've most definitely done problems such as these in other classes, some examples being:

- In your biology class, you're asked to find the population of bacteria if you know the rate at which they grow as well as the population at a specific time.
- In your chemistry class, you're asked to find the half-life at which an isotope disintegrate, or perhaps the age of a fossil using carbon dating.

Ex1. - The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

For any half-life problem, we have to solve an IVP given by $\dfrac{dA}{dt} = kA$, $A(0) = A_0$. I'll spare you the work of solving this linear equation and tell you directly that the general solution is $A(t) = A_0 e^{kt}$. First, we need to find our decay constant k. After 3.3 hours, we're left with half of our original amount. We can conceptualize this in our equation as $0.5 \cdot A_0 = A_0 e^{(3.3)k}$. Solving this would give us $k = \dfrac{\ln(0.5)}{3.3}\left(\dfrac{\text{grams}}{\text{hour}}\right)$. We want to find out how long will it take for 90% of our lead to decay, so we'd be left with 10% of our 1 gram, or $0.1A_0$. This would give us $0.1 = e^{(\ln(0.5)/3.3)t} \implies \quad t = \dfrac{3.3\ln(0.1)}{\ln(0.5)} = 10.96$ hours.

Newton's Law of Cooling/Warming: Given a constant of proportionality k, the temperature of the object $T(t)$, and a constant ambient temperature T_m, the rate at which an object warms/cools is given by:

$$\frac{dT}{dt} = k(T - T_m)$$

Ex2. - A small metal bar, whose initial temperature was $20°C$ is dropped into a container of boiling water. How long will it take the bar to reach $90°C$ if it is known that its temperature increases $2°$ in 1 second?

First, we need to solve our general differential equation. Given $\dfrac{dT}{dt} = k(T - T_m)$, if we separate the variables we get $\dfrac{dT}{T - T_m} = kdt$. We can solve this like a linear or separable equation, but regardless we get $\ln|T - T_m| = kt + c \implies \quad T = T_m + ce^{kt}$. Remember that the boiling point of water is $100°C$, so our ambient temperature $T_m = 100°C$. We have to find our k and c first for $T = 100 + ce^{kt}$.

1. To find c first, remember that we are given an initial condition $T(0) = 20$. Plugging this into our equation, we get $20 = 100 + ce^{k(0)} = 100 + c$. This gives us $c = -80$, so we have $T = 100 - 80e^{kt}$.
2. To find k, remember that we are given the fact that after 1 second, our temperature increases by $2°C$. This means that after 1 second, our initial temperature can be given by $20 + 2 = 100 - 80e^{k(1)} \to -78 = -80e^{k}$. Solving for k, we get $k = \ln(\frac{78}{80}) = -0.025$.

Overall, we have our solution equation $T = 100 - 80e^{(-0.025)t}$. If we solve for $T = 90°C$, we get $90 = 100 - 80e^{(-0.025)t} \to \dfrac{1}{8} = e^{(-0.025)t} \implies t = \dfrac{\ln(\frac{1}{8})}{(-0.025)} = 83$. It will take 83 seconds to reach $90°C$.

3.2 First Order Nonlinear Models

In this chapter we'll continue discussing some applications of first-order differential equations, this time with a focus on nonlinear models. I have nothing witty to say here.

Logistic Equation: Suppose want to model an environment of individuals with a *carrying capacity* - that is, there is a specific number such that when the number of individuals is greater than this carrying capacity, they can't sustain themselves and they start to die off until they reach their limit. Nature is grim stuff, but math can be even grimmer. We can model this with a nonlinear equation $\dfrac{dP}{dt} = P(a - bP)$.

Ex1 - A model for the population $P(t)$ in a suburb of a large city is given by the initial value problem $\dfrac{dP}{dt} = P(10^{-1} - 10^{-7}P)$, $P(0) = 5000$, where t is measured in months. What is the limiting value of the population? At what time will the population be equal to one-half of this limiting value?

We can make our lives easier by solving the logistic equation. We will use separation of equations. Separating our variables to each side, we get $\dfrac{dp}{P(a-bP)} = dt$. If we try to break our fraction apart, we get $\dfrac{dp}{P(a-bP)} = \left(\dfrac{1/a}{P} + \dfrac{b/a}{a-bP}\right) dP$. Don't try to think about this too hard unless you want a brain hemorrhage. Integrating both sides gives us $\dfrac{1}{a}\ln|P| - \dfrac{1}{a}\ln|a-bP| = t + c \ln\left|\dfrac{P}{a-bP}\right| = a(t+c) \implies \dfrac{P}{a-bP} = ce^{at}$. It follows that $P(t) = \dfrac{ac}{bc + e^{-at}}$. If $P(0) = P_0$, we find $c = \dfrac{P_0}{a - bP_0}$, so our solution equation becomes $P(t) = \dfrac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$.

To calculate the carrying capacity, we just need to find $\dfrac{a}{b}$. Easy peasy: $\dfrac{a}{b} = \dfrac{(10^{-1})}{(10^{-7})} = 10^6$. To find the time when our population will reach half the limiting-value, we just have to solve $P(t) = \dfrac{aP_0}{bP_0 + (a - bP_0)e^{-at}} \to 5^6 = \dfrac{(10^{-1})\cdot(5000)}{(10^{-1})(5000) - (10^{-7})(5000)e^{(10^{-1})t}}$. How fun! Let's just skim through this, shall we? $5^6 = \dfrac{(10^{-1})\cdot(5000)}{(10^{-1})(5000) - (10^{-7})(5000)e^{(10^{-1})t}} \to (10^{-1})(5000) - (10^{-7})(5000)e^{(10^{-1})t} = \dfrac{(10^{-1})}{5^6} = \dfrac{1}{2\cdot 5^7} \to (10^{-7})(5000)e^{(10^{-1})t} = \dfrac{1}{2\cdot 5^7} - (10^{-1})(5000) \implies t = 10 \cdot \ln(\dfrac{\frac{1}{2\cdot 5^7} - (10^{-1})(5000)}{(10^{-7})(5000)} = 52.93$ months. Isn't applied math just magical?

Chemical Reactions: Suppose we combined a grams of chemical A with b grams of chemical B. Suppose there are M parts of A and N parts of B that form a new compound according to $X(t)$, where t is time. The rate at which our new chemical is created is:

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \text{ where } \quad \alpha = a\left(\frac{M+N}{M}\right) \quad \text{and} \quad \beta = b\left(\frac{M+N}{N}\right)$$

.

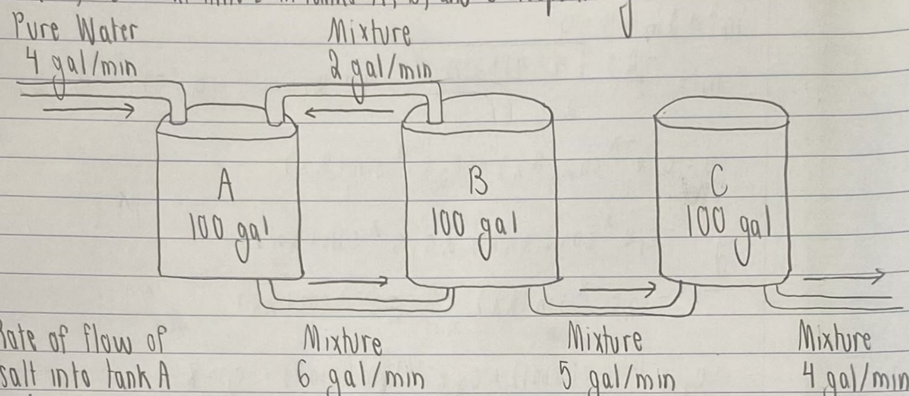
3.3 First Order Systems of Equations Models

You know what our previous examples were missing? More differential equations! This time however, we'll focus more on creating a **system of linear equations** using a model rather than solving them.

Mixtures: Suppose we have a couple of tanks containing salt and brine. Each of the tanks are connected to one another so that they might pump or receive mixture at a constant rate. We want to construct a mathematical model that describes the number of pounds $x_1(t)$, $x_2(t)$, etc. of salt in tanks A, B, etc. respectively at time t.

- We can find the net change of how much salt is in each tank, or $\frac{dx_i}{dt}$, by subtracting the *input rate* of salt by the *output rate* of salt.

Ex. - Construct a mathematical model for the number of pounds of salt $x_1(t)$, $x_2(t)$, $x_3(t)$ at time t in tanks A, B, and C respectively.



Rate of flow of salt into tank A

$$\frac{dx_1}{dt} = \underbrace{\underbrace{\left(4\,\frac{\text{gal}}{\text{min}}\right)\cdot\left(0\,\frac{\text{lb}}{\text{gal}}\right)}_{\text{No salt from pure water}} + \underbrace{\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right)}_{\text{Flowing from tank B into tank A}}}_{\text{Input rate of salt}} - \underbrace{\underbrace{\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)\left(6\,\frac{\text{gal}}{\text{min}}\right)}_{\text{Flowing out of tank A into tank B}}}_{\text{Output rate of salt}}$$

Our system of net rate of salt into each tank:

$$\begin{cases} dx_1/dt = (1/50)\,x_2 - (3/50)\,x_1 \\ dx_2/dt = (3/50)\,x_1 - (1/50)\,x_2 - (1/20)\,x_2 \\ dx_3/dt = (1/20)\,x_2 - (1/25)\,x_3 \end{cases}$$

Lotka-Volterra Predator-Prey Model: Suppose $x(t)$ denotes a predator population and $y(t)$ denotes a prey population. The population of predator will decrease if there are not enough prey and will grow if there is an amble population of prey proportional to the population of predator. The inverse is true for prey. Regardless, the rate at which their populations grow is jointly proportional to one another. Given constants a, b, c, d, we can model this as follows:

$$\frac{dx}{dt} = -ax + bxy \qquad \frac{dy}{dt} = dy - cxy$$

4 Chapter 4:

4.1 Linear Operators and Linear Independence

Now we can start dealing with higher-order differential equations! To kick things off, let's look at some boring notation.

Operators are functions that take in a function and output a function. Sounds confusing? Well, turns out you've been using operators the *entire time*, just not explicitly. Let's look at some examples:

- D is the derivative operator - that is, it takes a function and returns its derivative. We can specify how many derivatives we want using D^n, where n is how many times we want to differentiate.
  - Ex. - $D^2[\sin(x)] = D[\cos(x)] = -\sin(x)$
- L is the linear operator - it allows us to isolate the functions in an equation like such:
  - $L[c_1y_1 + ... + c_ky_k] = c_1L[y_1] + ... + c_kL[y_k]$.

  This might seem a little weird at first - but just you wait.

Let's look at our linear equation again. Recall that an n^{th} order linear equation has the form:

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + ... + a_1(x)y' + a_0y = g(x)$$

A **homogeneous** linear equation is set equal to 0, that is, $g(x) = 0$.

Imagine we have a homogeneous linear equation given by $a_n(x)y^n + a_{n-1}(x)y^{n-1} + ... + a_1(x)y' + a_0y = 0$. There are two possible solutions to this:

1. All of our constants $a_n, ..., a_0$ are 0.
2. Some of our constants $a_n, ..., a_0$ are nonzero.

If we can solve our homogeneous linear equation using the second approach, then we have a **linearly independent** equation. In terms of linear operators, this means if we have $L[y] = 0$, then we know there exists terms such that $L[y] = L[c_1y_1 + ...c_ny_n] = c_1L[y_1] + ... + c_kL[y_n]$, where $c_i \neq 0$.

We can verify linear independence using the **Wronskian**, which is a hellish-looking determinant matrix consisting of all of our functions and their possible determinants. Our set of solutions $y_1, ...y_n$ is linearly independent if and only if $W(y_1, ..., y_n) \neq 0$.

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} \mathrm{f}_1 & \mathrm{f}_2 & \cdots & \mathrm{f}_n \\ \mathrm{f}_1' & \mathrm{f}_2' & \cdots & \mathrm{f}_n' \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{f}_1^{n-1} & \mathrm{f}_2^{n-1} & \cdots & \mathrm{f}_n^{n-1} \end{vmatrix}$$

$$\text{For two functions } y_1 \text{ and } y_2,\ W(y_1, y_2) = \begin{vmatrix} \mathrm{y}_1 & \mathrm{y}_2 \\ \mathrm{y}_1' & \mathrm{y}_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

Example - Verify if the given set of functions is linearly independent: $f_1(x) = x$, $f_2(x) = 4x - 3x^2$

$$W(f_1, f_2) = \begin{vmatrix} \mathrm{x} & \mathrm{4x - 3x^2} \\ \mathrm{(x)'} & \mathrm{(4x - 3x^2)'} \end{vmatrix} = \begin{vmatrix} \mathrm{x} & \mathrm{4x - 3x^2} \\ 1 & \mathrm{4 - 6x} \end{vmatrix} = (x)(4-6x) - (4x - 3x^2)(1) = -3x^2 \neq 0$$

4.2 Homogeneous Second Order Equations with Constant Coefficients

What a mouthful for a section title! Thankfully, none of the math we will be as convoluted as saying that title. Suppose we are given a 2^{nd} order linear equation with the form:

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants.

We want to find a solution in the form $y = e^{mx}$. If we plug this into our L.E., we get $am^2e^{mx} + bme^{mx} + ce^{mx} = 0$. If we extract e^{mx} out of this, we get $am^2 + bm + c = 0$. This is our **auxiliary equation**, which is just a simple quadratic equation. If we try to find the root of this quadratic, we have three possible cases:

1. **Two distinct roots** $m_1 \neq m_2$: We end up with $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. Because this is linearly independent, our general solution is the sum $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

2. **Repeated root** m: $y_1 = e^{mx}$ is a solution, and using **Reduction of Order** we get $y_2 = xe^{mx}$. This gives us the general solution $y = c_1 e^{mx} + c_2 x e^{mx}$.

3. **Complex root**: Given the complex roots $m = \alpha \pm \beta\iota$, our general solution is $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$.

That's it! Once we've solved our quadratic we're as good as done. Let's do an example of each.

Ex1. - $y'' + 8y' + 16y = 0$.

We know $a = 1$, $b = 8$ and $c = 16$. This gives us our auxiliary equation $m^2 + 8m + 16m = 0$. Factoring this gives us $m^2 + 8m + 16m = (m+4)^2 = 0 \implies m = -4$. This is a *repeated root*. Therefore, our general solution is $y = c_1 e^{-4x} + c_2 x e^{-4x}$.

Ex2. - $y'' - 4y' + 5y = 0$.

We know $a = 1$, $b = -4$ and $c = 5$. Our auxiliary equation is $m^2 - 4m + 5 = 0$. Factoring gives us $m^2 - 4m + 5 = (m-5)(m+1) = 0$, which gives us *two distinct roots* $m_1 = -1$ and $m_2 = 5$. Thus, our general solution is $y = c_1 e^{-x} + c_2 e^{5x}$.

Ex3. - $3y'' + 2y' + y = 0$.

We know $a = 3$, $b = 2$ and $c = 1$, so our auxiliary equation is $3m^2 + 2m + 1 = 0$. This is not a quadratic that we can normally solve by factoring, so we be using the **quadratic formula**. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-(2) \pm \sqrt{(2)^2 - 4(3)(1)}}{2(3)} = \dfrac{-2 \pm \sqrt{-8}}{6}$. Our **discriminant** is negative, so we have a *complex root* on our hands. Using i, we get $x = \dfrac{-2 \pm \sqrt{-8}}{6} = \dfrac{-2 \pm 2\sqrt{2}i}{6} = \dfrac{-1 \pm i\sqrt{2}}{3}$. In total, this gives us a general solution of $y = c_1 e^{-x/3}\cos\left(\dfrac{\sqrt{2}}{3}x\right) + c_2 e^{-x/3}\sin\left(\dfrac{\sqrt{2}}{3}x\right)$.

4.3 Method of Undetermined Coefficients

Previously mentioned in Section 2.3, here it is! The **Method of Undetermined Coefficients** allows us to solve non-homogeneous equations, particularly involving trigonometric functions. Here, we will be focusing on non-homogeneous, 2^{nd}-order constant linear equations that have the form

$$ay'' + by' + cy = g(x).$$

Here, a,b and c are constants and $g(x) \neq 0$. When we use this method, our general solution can be given by the form of $y = y_c + y_p$.

- y_c is the **complementary** solution associated with the **homogeneous** equation. We find y_c by solving our equation where $g(x) = 0$.
- y_p is the **particular** solution. It is given in the same form of $g(x)$, initially with constants that we denote by using capital-letters.
  - If $g(x) = x^n$, y_p would be a polynomial up to x^n so $y_p = Ax^n + Bx^{n-1} + ... + Z$.
  - For exponential functions $g(x) = e^{nx}$, $y_p = Ae^{nx}$.
  - For trigonometric functions $g(x) = \sin(nx)$ or $g(x) = \cos(nx)$, $y_p = A\sin(nx) + B\cos(nx)$.
  - Ex. - $g(x) = (x^2+1)\sin(3x) + 5xe^{6x} \implies$
    $y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x) + (Gx + H)e^{6x}$

Here are the steps to performing the Method of Undetermined Coefficients:

1. To find y_c, solve the *associated* homogeneous equation.
2. To find y_p, we isolate each term in $g(x)$ and compare it to y_c
   (a) If any term in y_c has the same general form, we need to multiply ONLY that portion of y_p by x.
   (b) To determine the coefficients of y_p, we substitute y_p and its derivatives into our original DE.
3. We repeat our last step for EVERY term of $g(x)$ so we get $y_p = y_{p_1} + y_{p_2} + ...$
4. Last, we combine our solutions in the form $y = y_c + y_p$.

Example - $y'' + 2y' + y = \sin(x) + 3\cos(2x)$.

1. First, we need to find y_c. If we set $g(x) = 0$, we get $y'' + 2y' + y = 0$. This gives us the auxiliary equation $m^2 + 2m + 1 = 0$. This gives us a repeated root $m = -1$. Therefore, $y_c = c_1e^{-x} + c_2xe^{-x}$.
2. Now we need to find the general form for y_p. Because $g(x) = sin(x) + 3cos(2x)$, we have two particular solutions $y_{p_1} = A\sin(x) + B\cos(x)$ and $y_{p_2} = C\sin(2x) + D\cos(2x)$. We consider each case separately.
   - Our y_{p_1} does not share similar terms with our y_c so we don't have to change anything. For $y_{p_1} = A\sin(x) + B\cos(x)$, differentiating we get $y'_{p_1} = A\cos(x) - B\sin(x)$ and $y''_{p_1} = -A\sin(x) - B\cos(x)$. When we plug this in, we get $y'' + 2y' + y = (-A\sin(x) - B\cos(x)) + 2(A\cos(x) - B\sin(x)) + (A\sin(x) + B\cos(x)) = 2A\cos(x) - 2B\sin(x) = \sin(x) + 3\cos(2x)$. This would give us $A = 0$ and $B = -\frac{1}{2}$. Therefore, $y_{p_1} = -\frac{1}{2}\cos(x)$.
   - When we perform the same operations above for our y_{p_2}, we get $C = 0$ and $D = -1$. Therefore, $y_{p_2} = -\cos(2x)$.
3. We are given $y_c = c_1e^{-x} + c_2xe^{-x}$, $y_{p_1} = -\frac{1}{2}\cos(x)$ and $y_{p_2} = -\cos(2x)$. Thus, our final solution is $y = y_c + y_p = y_c + y_{p_1} + y_{p_2} = c_1e^{-x} + c_2xe^{-x} - \frac{1}{2}\cos(x) - \cos(2x)$.

4.4 Variation of Parameters

What's wrong? You don't like the Method of Undetermined Coefficients? You want to solve more tedious integrals? You want to use the Wronskian? Well look no further, because our next method can solve ANY 2^{nd} order linear equation.

The **Variation of Parameters** method is a lot more straightforward than the Method of Undetermined Coefficients. Assume we are given a function $y'' + P(x)y' + Q(x) = g(x)$. The steps are as follows:

1. Find the complementary solution $y_c = c_1y_1 + c_2y_2$.
2. Determine the Wronskian of y_1 and y_2, given by $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_2 \cdot y_1'$.
3. Find $u_1 = -\displaystyle\int \frac{y_2(x)g(x)}{W(x)}dx$ and $u_2 = \displaystyle\int \frac{y_1(x)g(x)}{W(x)}dx$.
   - If we include constants c_1 and c_2 as part of our evaluated integrals for u_1 and u_2, then we don't have to add the complementary solution to our general solution.
4. Our general solution will be $y = u_1y_1 + u_2y_2$.

Example - $y'' + 3y' + 2y = \sin(e^x)$

1. First, we need to find the complementary solution. Converting $y'' + 3y' + 2y = 0$ into an auxiliary equation, we get $m^2 + 3m + 2 = 0$. This gives us two real roots $m_1 = -1$ and $m_2 = -2$. Therefore, we have $y_c = c_1e^{-x} + c_2e^{-2x}$, so $y_1 = e^{-x}$ and $y_2 = e^{-2x}$.
2. Next, we calculate the Wronskian.

   $$W(e^{-x}, e^{-2x}) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = (-2e^{-2x})(e^{-x}) - (-e^{-x})(e^{-2x}) = -2e^{-3x} + e^{-3x} = -e^{-3x}.$$

3. Now, we calculate our u_1 and u_2.

   $$u_1 = -\int \frac{y_2(x)g(x)}{W(x)}dx = -\int \frac{(e^{-2x})(\sin(e^x))}{-e^{-3x}}dx = \int e^x \sin(e^x)dx = -\cos(e^x) + C_1.$$

   $$u_2 = \int \frac{y_1(x)g(x)}{W(x)}dx = \int \frac{(e^{-x})(\sin(e^x))}{(-e^{-3x})} = -\int e^{2x}\sin(e^x)dx$$

   We need to use u substitution. Let $u = e^x$, so $du = e^x dx$. Then $-\int e^{2x}\sin(e^x)dx = -\int u\sin(u)du$.

   Now we use integration by parts, but with v and t as our variables. Let $v = u$ and $t = \sin(u)$. Then $-\int u\sin(u)du = -\int vt'du = -(vt - \int tv'du) = -(u\cdot(-\cos(u)) - \int \cos(u)du) = u\cos(u) + \sin(u) + C_2$.
4. We have our y_1, y_2, u_1, and u_2.

   Our general solution is $y = u_1 \cdot y_1 + u_2 \cdot y_2 = (C_1 - \cos(e^x))e^{-x} + (u\cos(u) + \sin(u) + C_2)e^{-2x}$.

4.5 Cauchy-Euler Equations

We're able to solve 2^{nd} order linear equations such as $ay'' + by' + cy = g(x)$, homogeneous or not, pretty easily when all of the coefficients for our y terms are constants. Once we start dealing outside the realm of constants, things get hard quick.

There is, however, one type of 2^{nd} order equation we can consistently solve. Second-order **Cauchy-Euler Equations** are given by the general form (where a, b and c are constants and $g(x)$ is any function of x):

$$ax^2y'' + bxy' + cy = g(x).$$

We want our complementary solution to have the form $y_c = x^m = e^{m\ln|x|}$. This is because when we substitute $y = x^m$ into our Cauchy-Euler equation, we get $ax^2(m(m-1)x^{m-2}) + bx(mx^{m-1}) + c(x^m) = x^m[am(m-1)+bm+c] = 0$, which will give us an auxiliary equation we can work with: $am^2+(b-a)m+c = 0$. As a quadratic, we have three possible solution cases:

1. **Distinct Real Roots**: If $m_1 \neq m_2$, then $y_c = c_1x^{m_1} + c_2x^{m_2}$.
2. **Repeated Root**: Given one root m, $y_c = c_1x^m + c_2x^m \ln|x|$.
3. **Complex Case:** Given $m = \alpha \pm \beta i$, $y_c = c_1x^\alpha \cos(\beta \ln|x|) + c_2x^\alpha \sin(\beta \ln|x|)$.

Once we get our complimentary solution y_c, if our Cauchy-Euler is not homogeneous we can use **Variations of Parameters** to get the particular solution y_p. Before we can calculate u_1 and u_2 however, remember that our y'' term cannot have any additional terms. Thus, we first need to divide our equation by x^2, so we're solving $y'' + \frac{b}{x}y' + \frac{c}{x^2}y = \frac{g(x)}{x^2}$.

Example - $x^2y'' + xy' - y = \ln(x)$.

1. First we need to find the complementary solution. We have $a = 1$, $b = 1$ and $c = -1$ so our auxiliary equation is $(1)m^2 + ((1)-(1))m + (-1) = m^2 - 1 = (m+1)(m-1) = 0$. This gives us two distinct roots $m = \pm 1$. Therefore, $y_c = c_1x^{-1} + c_2x$.

2. Because this equation is non homogeneous, we proceed with Variation by Parameters. Given $y_1 = x^{-1}$ and $y_2 = x$, we take the Wronskian:

   $$W(y_1, y_2) = \begin{vmatrix} \text{x}^{-1} & \text{x} \\ \text{-x}^{-2} & 1 \end{vmatrix} = (x^{-1}1) - (x \cdot (-x^{-2})) = 2x^{-1}.$$

3. Before we can calculate u_1 and u_2, we have to isolate our y'' term. We will divide our equation by x^2:
   $\frac{1}{x^2}(x^2y'' + xy' - y = \ln(x)) \implies \quad y'' + \frac{1}{x}y' - \frac{1}{x^2} = \frac{\ln(x)}{x^2}$, meaning $g(x) = \frac{\ln(x)}{x^2}$.
   $u_1 = -\int \frac{y_2(x) \cdot g(x)}{w(x)}dx = -\int \frac{(x)(\frac{\ln(x)}{x^2})}{(2x^{-1})}dx = -\int \frac{x^{-1}\ln(x)}{2x^{-1}}dx = -\frac{1}{2}\int \ln(x)dx$
   We can solve this integral using integration by parts. Let $u = ln(x)$, $du = \frac{1}{x}dx$, $dv = dx$, and $v = x$. Then $-\frac{1}{2}\int \ln(x) = -\frac{1}{2}\left(x\ln(x) - \int x \cdot \frac{1}{x}dx\right) = \frac{1}{2}x - \frac{1}{2}x\ln(x) + C_1$
   $u_2 = \int \frac{y_1(x) \cdot g(x)}{w(x)} = \frac{1}{2}\int \frac{\ln(x)}{x^2}dx$. If $u = \ln(x)$, $du = \frac{1}{x}dx$, $dv = \left(\frac{1}{x^2}\right)dx$, $v = \left(\frac{-1}{x}\right)$, then $\frac{1}{2}\int \frac{\ln(x)}{x^2} = \frac{1}{2}\left((\ln(x))\left(\frac{-1}{x}\right) - \int \left(\frac{-1}{x}\right)\left(\frac{1}{x}\right)dx\right) = \frac{1}{2}\left(\frac{-\ln(x)}{x} + \int \frac{1}{x^2}dx\right) = \frac{-1}{2x}(\ln(x)+1) + C_2$.

4. Our general solution is $y = u_1y_1 + u_2y_2 = \left(\frac{1}{2}x - \frac{1}{2}x\ln(x) + C_1\right) \cdot x^{-1} + \left(\frac{-1}{2x}(\ln(x)+1) + C_2\right) \cdot x$.

4.6 Extension of Ideas to Higher-Order Equations

Scientists have shown that there is a clear correlation between how many higher-order differential equations you have to solve and how fast you loose your mind. Therefore, today we'll be looking at how to solve systems of second-order differential equations. Our method in question is similar to dealing with systems like in Algebra, called **systematic elimination**. The steps are as follows:

1. Rewrite each differential equation using the differential operator notation.
   - **Ex. -** $x'' = D^2x, \quad y' = Dy, \quad x'' - 3x' + 5x + y' + 5y = (D^2 - D + 3)x + (D + 5)y.$
2. Add and subtract each diff. eq. until you get two equations with only a power of x or y multiplied by a term consisting of linear differential operators and constants. We treat our differential operator D as a constant, so you can multiply each equation using a term of D.
3. Our auxiliary equation for $x(t)$ and $y(t)$ is our term of linear differential operators and constants, where each D is an m. From here, we solve for $x(t)$ and $y(t)$.
4. To eliminate constants beyond c_1 and c_2, substitute $x(t)$ and $y(t)$ into one of our original DEs

Ex. - $x'' - 2y'' - 2y' = \sin(t)$, $x + y' = 0$.

1. Remember that $' = D$, so $\begin{cases} \text{x'' - 2y'' -2y'} = \sin(t) \\ \text{x + y'} = 0 \end{cases} \implies \begin{cases} D^2x - 2D(D-1)y = \sin(t) \\ x + Dy = 0 \end{cases}$.
2. Now we begin the process of elimination. First, we create equations with like terms.
   - $2(D-1)\cdot(x + Dy) = 2(D-1)x + 2D(D-1)y = 0.$
   - $D^2 \cdot (x + Dy) = D^2x + D^3y = 0.$

   Adding and subtracting our terms above to $D^2x - 2D(D-1)y = \sin(t)$, we get:

   $$\begin{array}{lll} 2(D-1)x + 2D(D-1)y & = 0 & D^2x + D^3y = 0 \\ +[D^2x - 2(D-1)y & = \sin(t)] & -[D^2x - 2D(D-1)y = \sin(t)] \\ \hline D^2x + 2(D-1)x & = -\sin(t) & D^3y + 2D(D-1)y = \sin(t) \end{array}$$

   Further simplification gives us our equations:

   - $D^2x + 2(D-1)x = -\sin(t) \implies \quad (D^2 + 2D - 2)x = -\sin(t).$
   - $D^3y + 2D(D-1)y = \sin(t) \implies \quad D(D^2 + 2D - 2)y = \sin(t).$
     - We only know how to work with 2nd order diff. equations, so we can get rid of our extra D terms as follows: $D(D^2 + 2D - 2)y = \sin(t) \to (D^2 + 2D - 2)y = D^{-1}(\sin(t)) = -\cos(t)$
3. Now we will try to find a solution for $y(t)$ and $x(t)$. Both of our equations above have the same differential operator term $(D^2 + 2D - 2)$. This can be transformed into an auxiliary equation $m^2 + 2m - 2 = 0$. Solving this will result in two real roots $m = -1 \pm \sqrt{3}$. Therefore, our complimentary solutions are the same: $x_c(t) = y_c(t) = c_1e^{(\sqrt{3}-1)t} + c_2e^{-(\sqrt{3}+1)t}$.

   Using the method of undetermined coefficients, we can find the particular solutions for $y(t)$ and $x(t)$. We get: $y_p(t) = \frac{2}{5}\sin(t) + \frac{3}{5}\cos(t)$ and $x_p(t) = \frac{-3}{13}\sin(t) + \frac{2}{13}\cos(t)$. In total, we have:

   - $x(t) = c_1e^{(\sqrt{3}-1)t} + c_2e^{-(\sqrt{3}+1)t} - \frac{3}{13}\sin(t) + \frac{2}{13}\cos(t).$
   - $y(t) = c_3e^{(\sqrt{3}-1)t} + c_4e^{-(\sqrt{3}+1)t} + \frac{2}{5}\sin(t) + \frac{3}{5}\cos(t).$
4. We want to express c_3 and c_2 in terms of c_1 and c_2. We can plug our $x(t)$ and $y(t)$ into our initial diff. eq. $x + Dy = 0$. Doing so gives us:

   $$(c_1 + c_3(\sqrt{3}-1))e^{(\sqrt{3}-1)t} + (c_2 - (\sqrt{3}+1)c_4)e^{-(\sqrt{3}+1)t} + \left(\frac{2}{5} - \frac{3}{13}\right)\sin(t) + \left(\frac{3}{5} + \frac{2}{13}\right)\cos(t) = 0$$

   Our coefficients needs to be equal to 0, so $c_1 + c_3(\sqrt{3}-1) = 0$ and $c_2 - (\sqrt{3}+1)c_4 = 0$. Solving these gives us $c_3 = \frac{c_1}{1-\sqrt{3}}$ and $c_4 = \frac{c_2}{\sqrt{3}+1}$. Thus, we can re-express our solution $y(t)$ as $y(t) = (\frac{c_1}{1-\sqrt{3}})e^{(\sqrt{3}-1)t} + (\frac{c_2}{\sqrt{3}+1})e^{-(\sqrt{3}+1)t} + \frac{2}{5}\sin(t) + \frac{3}{5}\cos(t)$.

4.7 Green's Function

Remember Green's Theorem from Multivariable Calculus? No? Well, buckle up, because it's back, baby! (In a sense, no - not really.)

On the topic of ways we can solve a second order differential equation, **Green's Theorem** provides a very convoluted approach, but is the best method for approaching **Initial Value Problems**. Recall that when we use Variation of Parameters, we find our y_c first, then using our y_1 and y_2 to determine u_1 and u_2. Green's Theorem takes a different approach:

1. Find the complementary solution $y_c = c_1y_1 + c_2y_2$.
2. Find the Wronskian $W(x) = (y_1, y_2) = \begin{vmatrix} \mathrm{y}_1 & \mathrm{y}_2 \\ \mathrm{y}_1' & \mathrm{y}_2' \end{vmatrix} = y_1 \cdot y_2' - y_2 \cdot y_1'$.
3. Find $G(x,t) = \dfrac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$.
   - BE CAREFUL!!! We are dealing with TWO independent variables x and t.
4. After finding $G(x,t)$, our particular solution is given by $y_p(x) = \int_{x_0}^{x} G(x,t) \cdot f(t)dt$. Here, $f(t)$ is just our $g(x)$ from our general form $a(x)y'' + b(x)y' + c(x)y = g(x)$ but with t instead of x. We can choose to evaluate this integral or leave it as is.
   - Notice that our integral starts at x_0. When we are given an initial value $y(x_0) = y_0$, we can just plug x_0 into our integral.
   - It's a lot harder to work coefficients when evaluating this integral, so we will just add the complementary solution to get our particular solution. That is, $y = y_c + y_p$, and we do NOT have to include coefficients for y_p.

Example - $y'' + 9y = x + \sin(x)$, $y(0) = 1$, $y'(0) = -1$.

1. First, we find the complementary solution. $y'' + 9y = 0$ will give us an auxiliary equation of $m^2 + 9 = 0$. We can find m using the quadratic formula: $m = \dfrac{-(0) \pm \sqrt{(0) - 4(9)(1)}}{2(1)} = \dfrac{\pm\sqrt{-36}}{2} = \pm 3i$. Since m is a complex root, we have $y_c = c_1 \cos(3x) + c_2 \sin(3x)$.
2. We have $y_1 = \cos(3x)$ and $y_2 = \sin(3x)$.

   $$W(x) = W(y_1, y_2) = \begin{vmatrix} \cos(3x) & \sin(3x) \\ \text{-3}\sin(3x) & 3\cos(3x) \end{vmatrix} = 3\cos^2(3x) + 3\sin^2(3x) = 3.$$
3. $G(x,t) = \dfrac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} = \dfrac{\cos(3t) \cdot \sin(3x) - \cos(3x) \cdot \sin(3t)}{3} = \dfrac{1}{3}\sin 3(x-t)$.
   - This is just another one of those weird trigonometric properties.
4. We have our $G(x,t)$. Thus, $y_p = \displaystyle\int_{x_0}^{x} G(x,t) \cdot f(t)dt = \int_{x_0}^{x} \frac{1}{3}\sin 3(x-t)(t + \sin(t))dt$.

   Since we are given initial conditions at $x_0 = 0$, $y_p = \displaystyle\int_0^x \frac{1}{3}\sin 3(x-t)(t + \sin(t))dt$.

   We won't bother solving this integral because I'm too lazy.
5. Our general solution is: $y = y_c + y_p = c_1\cos(3x) + c_2\sin(3x) + \displaystyle\int_0^x \frac{1}{3}\sin 3(x-t)(t + \sin(t))dt$

5 Chapter 5:

5.1 Second-Order Linear Models

Who do you think I am, some hippy-dippy engineer? No! This is a pure math household! No modelling or applications allowed (except for that whole chapter we did on first-order models)!

In other words, these are not the droids you're looking for. Move along!

6 Chapter 6:

6.1 Series Solutions at Ordinary Points

You ever look at a differential equation and tell yourself "This is great, but it needs more series..."? No? That's the absolute last thing you'd think about? Well buckle up - you're in for a bumpy ride these next two chapters.

Remember that we can convert a *homogeneous* 2^{nd} order equations like $a(x)y'' + b(x)y' + c(x)y = 0$ into **standard form** by dividing both sides by $a(x)$: $y'' + P(x)y' + Q(x)y = 0$. Because $P(x) = \frac{b(x)}{a(x)}$ and $Q(x) = \frac{c(x)}{a(x)}$, we say that P and Q are **analytic** at points where $a(x) \neq 0$. $x = x_0$ is an **ordinary point** if P and Q are analytic at x_0. Otherwise, if either P or Q is not analytic at x_0 (doesn't require both), x_0 is a **singular point**.

If $x = x_0$ is an ordinary point, then we have a solution to the equation in the form $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$. If there are no singular points, then we can guarantee that we will have two series solutions centered at $x_0 = 0$. The steps to finding series solutions at ordinary points include:

1. Given $a(x)y'' + b(x)y' + c(x)y = 0$, find any x for which $a(x) = 0$. It could be real or complex! Given two roots x_1 and x_2, our solutions will converge for *at least* $|x| < x_1 \cdot x_2$.
2. Substitute $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$ into our standard equation $y'' + P(x)y' + Q(x)y = 0$. We want to combine all series into one.
3. Try substituting values in for n in c_n until you can define a clear relationship between each c_n value. Once we can define each c_n in terms of c_0, substitute $c_0 = 1$ and redefine the series exclusively in terms of x and n.

Example - $(x-1)y'' + y' = 0$.

1. If we set $x - 1 = 0$, we get $x = 1$. Therefore, our solution converges when $|x| < 1$.
2. We know $y = \sum_{n=0}^{\infty} c_n x^n$. Differentiating, we get $y' = \sum_{n=1}^{\infty} n \cdot c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-2}$.

   Plugging this in, we get $(x-1)y'' + y' = (x-1)(\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-2}) - (\sum_{n=1}^{\infty} n \cdot c_n x^{n-1})$.

   If we distribute, we get $(x-1)(\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-2}) + (\sum_{n=1}^{\infty} n \cdot c_n x^{n-1}) =$
   $x(\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-2}) - \sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-2}) - (\sum_{n=1}^{\infty} n \cdot c_n x^{n-1}) =$
   $(\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-1}) - (\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-2}) - (\sum_{n=1}^{\infty} n \cdot c_n x^{n-1})$.

   In order to combine these sums, we need our indexes to start at the same value and our x terms to have the same power. We do this by factoring out terms from our sums and substituting n for some k.

   $(\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-1}) - (\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-2}) - (\sum_{n=1}^{\infty} n \cdot c_n x^{n-1}) =$
   $(\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-1}) - 2c_2 - (\sum_{n=3}^{\infty} n(n-1) \cdot c_n x^{n-2}) - c_1 - (\sum_{n=2}^{\infty} n \cdot c_n x^{n-1})$

   (a) Let $k = n - 1$. Then $\sum_{n=2}^{\infty} n(n-1) \cdot c_n x^{n-1} = \sum_{k=1}^{\infty} (k+1)k \cdot c_{k+1} x^k$.
   (b) Let $k = n - 2$. Then $\sum_{n=3}^{\infty} n(n-1) \cdot c_n x^{n-2} = \sum_{k=1}^{\infty} (k+1)(k+2.c_{k+2} x^k$.
   (c) Let $k = n - 1$. Then $\sum_{n=2}^{\infty} n \cdot c_n x^{n-1} = \sum_{k=1}^{\infty} (k+1) \cdot c_{k+1} x^k$.

   $\sum_{k=1}^{\infty} (k+1)k \cdot c_{k+1} x^k - \sum_{k=1}^{\infty} (k+1)(k+2.c_{k+2} x^k - \sum_{k=1}^{\infty} (k+1) \cdot c_{k+1} x^k - (2c_1 + c_1) =$
   $\sum_{k=1}^{\infty} [(k^2-1)c_{k+1} - (k+2)(k+1)c_{k+2}]x^k - 2c_2 - c_1 = 0$.

3. Now we need to find some pattern. For $(k^2-1)c_{k+1} - (k+2)(k+1)c_{k+2} = 0$, $c_{k+1} = \frac{(k-1)}{(k+2)} c_{k+1}$. $2c_2 + c_1 = 0$ gives us $c_2 = \frac{-1}{2} c_1$. Every sum will contain c_1 as the coefficient of our series. Furthermore, since c_0 is an arbitrary term, we can just tack it onto the end. Therefore, $y = c_0 + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$.

6.2 Series Solutions at Singular Points

Recall that for a standard homogeneous linear $y'' + P(x)y' + Q(x) = 0$, we say that P and Q are *analytic* at *ordinary points* - that is, they are defined on points $x = x_0$. **Singular points** are points where either P or Q is NOT analytic on.

- We say x_0 is a **regular singular point** if the functions $p(x) = x \cdot P(x)$ and $q(x) = x^2 \cdot Q(x)$ were analytic at a point $x = x_0$.
  - Any Cauchy-Euler equation (Given by $ax^2y'' + bxy' + cy = 0$) has a singular point at $x_0 = 0$.

Frobenius' Theorem states "If x_0 is a regular singular point, then there exists at least one solution of the form $y = (x - x_0)^r \cdot \sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r}$.

- This means Cauchy-Euler equations have a general series solution of $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.
- Other than this, solving at singular points involves almost the exact same process as solving at ordinary points. Keep in mind that with each r, we have a different solution.

Example - $xy'' - xy' + y = 0$ at $x_0 = 0$.

1. We know that $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Differentiating, we get $y' = \sum_{n=0}^{\infty}(n + r) \cdot c_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(r + n)(r + n - 1) \cdot c_n x^{n+r-2}$. If we plug this into our equation, we get $xy'' - xy' + y = 0 = x(\sum_{n=0}^{\infty}(r + n)(r + n - 1) \cdot c_n x^{n+r-2}) - x(\sum_{n=0}^{\infty}(n + r) \cdot c_n x^{n+r-1}) + (\sum_{n=0}^{\infty} c_n x^{n+r})$.

   Distributing each term, we get $x(\sum_{n=0}^{\infty}(r + n)(r + n - 1) \cdot c_n x^{n+r-2}) - x(\sum_{n=0}^{\infty}(n + r) \cdot c_n x^{n+r-1}) + (\sum_{n=0}^{\infty} c_n x^{n+r}) = \sum_{n=0}^{\infty}(r + n)(r + n - 1) \cdot c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n + r) \cdot c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r}$.

2. Our first goal is to combine these sums by changing them to like powers. Although this seems like much more of a pain at first since we're dealing with two variables, we are actually allowed to factor out the x^r term in all of our sums.

   This leaves us with $\sum_{n=0}^{\infty}(r + n)(r + n - 1) \cdot c_n x^{n-1} - \sum_{n=0}^{\infty}(n + r) \cdot c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$. Now we need to make some substitutions.

   (a) For our first sum, set $k = n - 1$, so $n = k + 1$. This gives us $\sum_{n=0}^{\infty}(r + n)(r + n - 1) \cdot c_n x^{n-1} = \sum_{k=-1}^{\infty}(r + k + 1)(r + k) \cdot c_{k+1} x^k$.

   (b) For our second and third sum, we can simply let $n = k$. Nothing to show here!

   In total, we are left with $\sum_{k=-1}^{\infty}(r + k + 1)(r + k) \cdot c_{k+1} x^k + \sum_{k=0}^{\infty}(k + r) \cdot c_k x^k + \sum_{k=0}^{\infty} c_k x^k$. Because the index of our first sum is different however, we have to factor out a $k = -1$ term. This leaves us with $r(r-1)c_0 x^{-1} + \sum_{k=0}^{\infty}(r+k+1)(r+k) \cdot c_{k+1} x^k + \sum_{k=0}^{\infty}(k+r) \cdot c_k x^k + \sum_{k=0}^{\infty} c_k x^k = r(r-1)c_0 x^{-1} + \sum_{k=0}^{\infty}[(r + k + 1)(r + k)c_{k+1} - (r + k - 1)c_k]x^k = 0$.

3. Now we get into the weird part. Remember that our equation is set equal to 0, so our leftover term and our sum must both be equal to 0. We can set our term inside the sum equal to 0 and solve for c_{k+1} as follows: $(r + k + 1)(r + k)c_{k+1} - (r + k - 1)c_k = 0 \to c_{k+1} = \frac{(r+k-1)c_k}{(r+k+1)(r+k)}$. If we set $r(r - 1)c_0 x^{-1} = 0$, solving for r we get $r = 0, 1$. Therefore, we have two possible solutions for each r. We will try to find a pattern for each.

   (a) If $r = 0$, then $c_{k+1} = \frac{(r+k-1)c_k}{(r+k+1)(r+k)} = \frac{(k-1)c_k}{(k+1)(k)} = \frac{c_k}{k+1} - \frac{c_k}{k(k+1)}$. Our second term does not exist when $k = 0$, so we disregard it. When $k = 0$, $c_1 = c_0$. When $k = 1$, $c_2 = \frac{1}{2}c_1$. When $k = 2$, $c_3 = \frac{1}{3}c_2$. This would give us a series that looks like $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$. We get our particular solution by multiplying this series by x^r, so $y_1 = x^0(1+x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots) = 1+x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$.

   (b) If $r = 1$, then $c_{k+1} = \frac{(r+k-1)c_k}{(r+k+1)(r+k)} = \frac{(k)c_k}{(k+1)(k)} = \frac{c_k}{k+1}$. This would give the same series as our term above. Therefore, our particular solution would be $y_2 = x^1(1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \cdots$.

   In conclusion, our general solution would be $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are the particular solutions discovered above.

7 Chapter 7:

7.1 Review of Initial-Value Problems

Alright - now that you had a brain aneurysm from working with series, let's take a deep breath and rewind back a bit to the beginning from where it all began... back when you thought it would be a good idea to start reading this "guide".

When we find the general solution to a differential equation, each of our terms usually have a constant. These constants allow us to describe a family of possible solutions. Therefore, if we wanted to find a specific **solution curve**, we'll have to specify some **initial conditions**.

- If we want to find the solution curve for a second order ODE, we'll usually be given initial conditions in the form $y(x_0) = y_0$ and $y'(x_1) = y_1$.
- **Ex. -** If our general solution is $y = c_1 e^{6x} + c_2 e^{-4x}$, we can find c_1 and c_2 using $y(6) = 5$ and $y'(7) = 3$.

Initial-value problem just differential equations we need to solve but with initial conditions tacked onto the end. Thankfully, they don't change our method for finding the general solution - it just means we have to do some more work at the end. Our steps to solving an initial-value problem are as follows:

1. Find the general solution to the differential equation.
2. Adjust our general solution so that we can plug in our initial-conditions. This just means differentiating our general solution.
3. After plugging in our initial-values, we are given a system of equations. Solve the system of equations to find each constant (c_1 and c_2 for 2nd order ODEs).

Ex. - $y'' + 2y' + 5y = 0$, $y'(0) = 5$, $y(0) = 7$

1. This is a standard homogeneous 2nd order equation with constant coefficients.
   (a) If we convert this into an auxiliary equation, we get $m^2 + 2m + 5 = 0$.
   (b) Using the quadratic formula, we get $m = \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2} = -1 \pm 2i$.
   (c) We have a complex solution, where $\alpha = -1$ and $\beta = 2$.
   Therefore, our general solution is $y = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x)$.
2. We need to differentiate our general solution so that we can fit $y'(0) = 5$.
   Doing so will give us $y' = -c_1 e^{-x}\cos(2x) - 2c_1 e^{-x}\sin(2x) - c_2 e^{-x}\sin(2x) + 2c_2 e^{-x}\cos(2x)$.
   (a) $y(0) = c_1 e^{-(0)}\cos(2(0)) + c_2 e^{-(0)}\sin(2(0)) = 7 \implies c_1 = 7$.
   (b) $y'(0) = -c_1 e^{-(0)}\cos(2(0)) - 2c_1 e^{-(0)}\sin(2(0)) - c_2 e^{-(0)}\sin(2(0)) + 2c_2 e^{-(0)}\cos(2(0)) = 5$
   $\implies -c_2 + 2c_2 = 5 \implies \quad 2c_2 = 5 + c_1 = 5 + 7 = 12 \implies \quad c_2 = 6$.
3. Our constants just kinda came around from our step above, so we're done!
   Our specific solution is $y = 7e^{-x}\cos(2x) + 6e^{-x}\sin(2x)$.

That was a pretty straight-forward example since it technically just solves itself. However, it doesn't take a genius to realize that if you have multiple terms - perhaps a Cauchy-Euler equation - things can start getting out of control really fast. "If only we'd spend the next couple of sections discussing some sort of function just for initial-value problems!", I hear you saying. Well my friend, you're in for a pleasant surprise.

7.2 Definition of the Laplace Transform

Time for something new! Recall that in general calculus, differentiation and integration take one function and give you another function. We call these types of techniques *transforms*. Our next transform will be particularly helpful for solving initial-value problems.

Before we show you what's behind the curtain however, I'm going to cram another definition down your throat. Given any function $f(t)$, we say that f is of **exponential order** if these exists some $c, M > 0$ and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$. In English, this means that if we always have some exponential function *outgrow* some function after a certain point ($t > T$), then that function is of exponential order. (Ex. - $e^{45t} < e^{46t}$, $t^{18} < t^{19}$, etc.)

Laplace Transformations: Let f be a function of exponential order that is defined for $t \geq 0$. Then the Laplace Transformation of f is $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$.

- This transform a function of t into a function of s, so $L\{f(t)\} = F(s)$.
- We have some important properties to note about the Laplace Transformation:
  1. $L\{0\} = 0$.
  2. L is a linear operator, so $L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\} = aF(s) + bG(s)$.
  3. L is defined for complex values of s, but we usually assume that s is real.
     - If we're given $f(t) + i \cdot g(t)$, then $L\{f(t) + i \cdot g(t)\} = F(s) + i \cdot G(s)$.

That's it! Doing a Laplace Transforms just involves solving a weird integral. Even better is the fact that we don't have to bother solving most Laplace transformations if we can break up our function into a couple of other basic functions! In the page after this, I've generously included a table of the most common Laplace transformations (that I definitely did not steal from my professor), so make sure you remember all of those. No whining. But regardless, let's try out some examples.

Ex1. - $f(t) = te^{4t}$.

$L\{f(t)\} = \displaystyle\int_0^\infty e^{-st} te^{4t} dt = \int_0^\infty te^{t(4-s)} dt$. From this point, we can use integration by parts.
Let $u = t$, $du = dt$, $dv = e^{t(4-s)}dt$, and $v = \dfrac{1}{4-s}e^{t(4-s)}$.

$$\int_0^\infty te^{t(4-s)}dt = \left[t \cdot \left(\frac{1}{4-s}e^{t(4-s)}\right)\right]_0^\infty - \int_0^\infty \left(\frac{1}{4-s}e^{t(4-s)}\right) dt = -\int_0^\infty \left(\frac{1}{4-s}e^{t(4-s)}\right) dt = \frac{-1}{(4-s)^2}.$$

Ex2. - $f(t) = \begin{cases} \sin(t) & 0 \leq t < \pi \\ 0 & t \geq \pi \end{cases}$

Given a piecewise function, we can divide the Laplace integral according to the interval as follows:
$L\{f(t)\} = \displaystyle\int_0^\pi \sin(t)e^{-st}dt + \int_\pi^\infty (0)e^{-st}dt = \int_0^\pi \sin(t)e^{-st}dt$. Voila! We're left with a definite integral. But be warned, because this is an absolute HELLISH integral to solve. I'll spare you the torture and tell you straight-up that our answer is $\displaystyle\int_0^\pi \sin(t)e^{-st}dt = \frac{1 + e^{-\pi s}}{s^2 + 1}$.

7.3 Laplace Transform Table

Common Laplace transforms:

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{n+1}}$ $(s > 0)$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{s-a}$ $(s > a)$
$f(t) = \sin(kt)$	$F(s) = \dfrac{k}{s^2+k^2}$
$f(t) = \cos(kt)$	$F(s) = \dfrac{s}{s^2+k^2}$
$f(t) = e^{at}\sin(kt)$	$F(s) = \dfrac{k}{(s-a)^2+k^2}$
$f(t) = e^{at}\cos(kt)$	$F(s) = \dfrac{s-a}{(s-a)^2+k^2}$
$f(t) = \sinh(kt)$	$F(s) = \dfrac{k}{s^2-k^2}$
$f(t) = \cosh(kt)$	$F(s) = \dfrac{s}{s^2-k^2}$
$f(t) = U(t-a)$	$F(s) = \dfrac{e^{-as}}{s}$ $(s > 0)$

Properties of Laplace transforms:

Rule for L	Rule for L^{-1}
$L\{y'\} = sY(s) - y(0),\ L\{y''\} = s^2Y(s) - sy(0) - y'(0)$	
$L\{y^{(n)}\} = s^nY(s) - s^{n-1}y(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0)$	
$L\{e^{at}f(t)\} = F(s-a)$	$L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\} = e^{at}f(t)$
$L\{f(t-a)U(t-a)\} = e^{-as}L\{f(t)\} = e^{-as}F(s)$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)U(t-a)$
$L\{t^nf(t)\} = (-1)^n\dfrac{d^n}{ds^n}F(s)$	
$L\{(f*g)(t)\} = F(s)G(s)$	$L^{-1}\{F(s)G(s)\} = \displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau$
f periodic, period $T \Rightarrow L\{f(t)\} = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}f(t)\,dt$	

7.4 The Inverse Laplace Transform

You memorized all of the common Laplace Transformations on the table, right? Good, because I wasn't kidding when I said that you'd need to know all of them going further. I told you that I would never lie to you, but you didn't listen, did you?

There's nothing much to say here: an **Inverse Laplace Transform** can be defined as $L^{-1}\{F(s)\} = f(t)$. It's literally just working backwards! But that's why it's important to know your basic Laplace transformations - because you're going to have to decide how you want to break a function $F(s)$ apart. Let's dive into a big example.

Ex. - $F(s) = \dfrac{s}{(s-2)(s-3)(s-6)}$.

"The hell is that thing?", you might be asking. "No function could produce THAT!" Well hold your horses bud, because using a technique you learned all the way back in calculus, we can break this thing down. The technique in question is **Partial Fraction Decomposition**. Let's go through this step-by-step:

Given $\dfrac{s}{(s-2)(s-3)(s-6)}$, we want to find A, B, C where $\dfrac{s}{(s-2)(s-3)(s-6)} = \dfrac{A}{s-2} + \dfrac{B}{s-3} + \dfrac{C}{s-6}$.

If we multiply both sides by $(s-2)(s-3)(s-6)$, we get $s = A(s-3)(s-6) + B(s-2)(s-6) + C(s-2)(s-3)$.

We can find each constant A, B, C by plugging in values of s that will cancel out other terms:

- Let $s = 2$. This gives us $2 = A(2-3)(4-6) + B(2-2)(2-6) + C(2-2)(2-3) = A(-1)(-2) = 2A$. If $2A = 2$, then $A = 1$.
- Let $s = 3$. This gives us $3 = A(3-3)(3-6) + B(3-2)(3-6) + C(3-2)(3-3) = B(1)(-3) = -3B$. Thus, $B = -1$.
- Let $s = 6$. This gives us $6 = C(6-2)(6-3) = 12C$, so $C = \dfrac{1}{2}$.

Overall, we have $\dfrac{s}{(s-2)(s-3)(s-6)} = \dfrac{1}{s-2} - \dfrac{1}{s-3} + \dfrac{1}{2}\left(\dfrac{1}{s-6}\right)$. But we're not done yet!

Our goal is to find $L^{-1}\left\{\dfrac{s}{(s-2)(s-3)(s-6)}\right\}$. Recall that L is a linear operator, so this would mean that $L^{-1}\left\{\dfrac{s}{(s-2)(s-3)(s-6)}\right\} = L^{-1}\left\{\dfrac{1}{s-2}\right\} - L^{-1}\left\{\dfrac{1}{s-3}\right\} + \dfrac{1}{2}L^{-1}\left\{\dfrac{1}{s-6}\right\}$.

Recall that for some exponential function $f(t) = e^{at}$, we have $L\{f(t)\} = \dfrac{1}{s-a}$.

This would mean that $L^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$.

Therefore, $L^{-1}\left\{\dfrac{1}{s-2}\right\} - L^{-1}\left\{\dfrac{1}{s-3}\right\} + \dfrac{1}{2}L^{-1}\left\{\dfrac{1}{s-6}\right\} = e^{2t} - e^{3t} + \dfrac{1}{2}e^{6t}$. That's it!

We can conclude that $L^{-1}\{F(s)\} = e^{2t} - e^{3t} + \dfrac{1}{2}e^{6t}$.

7.5 Solving a First-Order IVP with the Laplace Transform

Laplace Transformations are okay and all, but what's the point? Well, as everything in a class called "Differential Equations", it's about solving differential equations. Consider a first-order linear ODE $f'(t)$. If we take the Laplace we get the following:

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt = -f(0) + sL\{f(t)\}.$$

This is *really* nice if we are given an initial-value of $f(0)$ since we can just plug it in. Generally, our steps to solving a first-order IVP are as follows:

1. Given $ay' + by + c = g(t)$, take the Laplace transform of every member: $aL\{y'\} + bL\{y\} = L\{g(t) - c\}$.
   - We "absorb" c into $g(t)$ so it doesn't needs its own separate transform.
   - Taking the Laplace is really easy for our left side, because we can just substitute $L\{y'\} = sY(s) - y(0)$ and $L\{y\} = Y(s)$.
   - However, you're tough out of luck with $g(t)$ - you just have to face it like a champ.

   You should be left with $a(sY(s) - y(0)) + bY(s) = G(s)$.

2. Separate all variables with $Y(s)$ on one side and all others on the other side so that you are left with $Y(s) = \cdots$. For $a(sY(s) - y(0)) + bY(s) = G(s)$, we'd have $asY(s) - ay(0) + bY(s) = G(s)$, so $a \cdot s \cdot Y(s) + b \cdot Y(s) = G(s) + a \cdot y(0)$ gives us a general formula $Y(s) = \dfrac{G(s) + ay(0)}{a \cdot s + b}$.

3. Take the inverse Laplace transformation of $Y(s)$ to find our solution $y(t)$.

Ex. - $y' + 6y = e^{4t}, \quad y(0) = 2.$

1. Taking the Laplace of both sides, we get $L\{y' + 6y\} = L\{e^{4t}\}$. We consider each side separately.
   - $L\{y' + 6y\} = L\{y'\} + 6L\{y\} = (sY(s) - y(0)) + 6Y(s) = (6 + s)Y(s) - 2.$
   - Recall that $L\{e^{at}\} = \dfrac{1}{s-a}$. Therefore, $L\{e^{4t}\} = \dfrac{1}{s-4}$.

2. We have $(6 + s)Y(s) - 2 = \dfrac{1}{s-4}$.

   Adding a two to both sides gives us $(6 + s)Y(s) = \dfrac{1}{s-4} + 2 = \dfrac{1}{s-4} + \dfrac{2s-8}{s-4} = \dfrac{2s-7}{s-4}$.

   Dividing both sides by $(6 + s)$ gives us $Y(s) = \dfrac{1}{6+s} \cdot \dfrac{2s-7}{s-4} = \dfrac{2s-7}{(s+6)(s-4)}$.

3. Now we take the inverse Laplace of both sides. $L^{-1}\{Y(s)\} = y(t) = L^{-1}\left\{\dfrac{2s-7}{(s+6)(s-4)}\right\}$.

   As discussed in the previous chapter, we can solve this using Partial Fraction Decomposition:

   (a) Set $\dfrac{2s-7}{(s+6)(s-4)} = \dfrac{A}{s+6} + \dfrac{B}{s-4}$. Simplifying this, we get $2s - 7 = A(s - 4) + B(s + 6)$.

   (b) Let $s = 4$. Then $2(4) - 7 = 1 = A(4 - 4) + B(4 + 6)$, so $10B = 1 \implies B = \frac{1}{10}$.

   (c) Let $s = -6$. Then $2(-6) - 7 = -20 = A(-6 - 4) + B(-6 + 6)$ so $-10A = -20 \implies A = 2$.

   (d) Overall, we know that $\dfrac{2s-7}{(s+6)(s-4)} = \dfrac{2}{s+6} + \dfrac{1}{10}\left(\dfrac{1}{s-4}\right)$.

   We have $L^{-1}\left\{\dfrac{2s-7}{(s+6)(s-4)}\right\} = L^{-1}\left\{\dfrac{2}{s+6} + \dfrac{1}{10}\left(\dfrac{1}{s-4}\right)\right\} = 2L^{-1}\left\{\dfrac{1}{s+6}\right\} + \dfrac{1}{10}L^{-1}\left\{\dfrac{1}{s-4}\right\}$.

   Therefore, $y(t) = 2L^{-1}\left\{\dfrac{1}{s+6}\right\} + \dfrac{1}{10}L^{-1}\left\{\dfrac{1}{s-4}\right\} = 2e^{-6t} + \dfrac{1}{10}e^{4t}$.

7.6 Solving a Second-Order IVP with the Laplace Transform

If we take the Laplace of a second-order linear ODE, say $f''(t)$, we get the following:
$L\{f''(t)\} = \int_0^\infty e^{-st}f''(t)dt = \left[e^{-st}f'(t)\right]_0^\infty + s\int_0^\infty e^{-st}f'(t)dt = -f'(0) + sL\{f'(t)\} = s[sF(s) - f(0)] - f'(0)$

If you look at $L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$, $L\{f'(t)\} = sF(s) - f(0)$, and $L\{f(t)\} = F(s)$, you'll notice that a pattern starts to emerge. Given any $f^n(t)$, the Laplace can be given by:

$$L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

Ex. - $y'' + y = \sqrt{2}\sin(\sqrt{2}t)$, $\quad y(0) = 10$, $\quad y'(0) = 0$

1. We take the Laplace of both sides: $L\{y'' + y\} = L\{\sqrt{2}\sin(\sqrt{2}t)\}$. We consider each side separately.
   - $L\{y'' + y\} = L\{y''\} + L\{y\} = s^2Y(s) - sy(0) - y'(0) + Y(s) = (s^2+1)Y(s) - sy(0) - y'(0)$. When we plug in our initial values, we get $(s^2+1)Y(s) - sy(0) - y'(0) = (s^2+1)Y(s) - 10s$.
   - For $L\{\sqrt{2}\sin(\sqrt{2}t)\} = \sqrt{2}L\{\sin(\sqrt{2}t)\}$, recall that for $f(t) = \cos(kt)$ we have $L\{f(t)\} = \frac{k}{s^2+k^2}$.
     Therefore, $\sqrt{2}L\{\sin(\sqrt{2}t)\} = \sqrt{2}\left(\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right) = \frac{2}{s^2+2}$.

   Overall, we have $(s^2+1)Y(s) - 10s = \frac{2}{s^2+2}$.
2. Given $(s^2+1)Y(s) - 10s = \frac{2}{s^2+2}$, adding $10s$ to both sides gives us
   $(s^2+1)Y(s) = \frac{2}{s^2+2} + 10s = \frac{2}{s^2+2} + \frac{10s^3+20s}{s^2+2} = \frac{10s^3+20s+2}{s^2+2}$.
   If we divide by (s^2+1), we get $Y(s) = \frac{10s^3+20s+2}{(s^2+1)(s^2+2)}$.
3. Now we gotta find the inverse Laplace of $Y(s) = \frac{10s^3+20s+2}{(s^2+1)(s^2+2)}$. And you guessed it, we're gonna have to break apart THAT fraction! Oh boy!
   (a) Because we cannot factor (s^2+1) and (s^2+2) into simpler terms, we can partially decompose this as follows: $\frac{10s^3+20s+2}{(s^2+1)(s^2+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2}$. If we get rid of our denominators, we end up with $10s^3+20s+2 = (As+B)(s^2+2) + (Cs+D)(s^2+1)$.
   (b) Our method for dealing with these types of partial fraction problems involves a little more than just plugging in numbers for s. But for now, it'd be best to expand our right-hand expression, so $10s^3+20s+2 = (As+B)(s^2+2) + (Cs+D)(s^2+1) =$
   $As^3 + 2As + Bs^2 + 2B + Cs^3 + Cs + Ds^2 + D = (A+C)s^3 + (B+D)s^2 + (2A+C)s + (2B+D)$.
   (c) Now that we have $10s^3+20s+2 = (A+C)s^3 + (B+D)s^2 + (2A+C)s + (2B+D)$, notice that for each s term the coefficients on each side need to be equal to one another. This gives us a system of equations: $A+C = 10$, $B+D = 0$, $2A+C = 20$, $2B+D = 2$. To spoil the surprise for you though, if we solve for each constant we get $A = 10$, $B = 2$, $C = 0$, and $D = -2$.
   (d) In conclusion, $\frac{10s^3+20s+2}{(s^2+1)(s^2+2)} = \frac{10s+2}{s^2+1} - \frac{2}{s^2+2}$.

   We have our decomposed fractions. But how in god's name are we gonna find $L\left\{\frac{10s+2}{s^2+1} - \frac{2}{s^2+2}\right\}$? Using our handy-dandy table, of course! Let's take each term into consideration:
   - For our first fraction, $L\left\{\frac{10s+2}{s^2+1}\right\} = 10L\left\{\frac{s}{s^2+1}\right\} + 2L\left\{\frac{1}{s^2+1}\right\} = 10\cos(t) + 2\sin(t)$.
   - $L\left\{\frac{2}{s^2+2}\right\} = \sqrt{2}L\left\{\frac{\sqrt{2}}{s^2+2}\right\} = \sqrt{2}\sin(\sqrt{(2)}t)$.

   In conclusion, we have $y(t) = 10\cos(t) + 2\sin(t) - \sqrt{2}\sin(\sqrt{(2)}t)$. I'm going to bed.

7.7 Derivatives of Transforms

We've basically explored the nitty-gritty of what the Laplace transformation has to offer, so now we'll dive into another cool property. Suppose we were to take the derivative of $F(s) = L\{f(t)\}$. This would give us the following:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \left(\frac{d}{ds}e^{-st}\right)f(t)dt = -\int_0^\infty t\left[e^{-st}f(t)\right]dt = -L\{tf(t)\}$$

This would mean that $L\{tf(t)\} = -\frac{d}{ds}L\{f(t)\}$. If we try this for any t^n, where n is any natural number, we find the general formula for the **derivative of transforms**:

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

This is really useful when we want to find the Laplace of a function with a t^n term. Some examples include:

Ex1 - $f(t) = te^{2t}\sin(6t)$.
Using our theorem above, $L\{te^{2t}\sin(6t)\} = -\frac{d}{ds}L\{e^{2t}\sin(6t)\}$. According to our table, $L\{e^{at}\sin(kt)\} = \frac{k}{(s-a)^2-k^2}$, so $-\frac{d}{ds}L\{e^{2t}\sin(6t)\} = -\frac{d}{ds}\left(\frac{6}{(s-2)^2-36}\right) = -\frac{d}{ds}(6(s^2-4s-32)^{-1}) = \frac{6(2s-4)}{s^2-4s-32}$.

Ex2 - $f(t) = t\cos(kt)$.
We have to use the quotient rule for this one: $L\{t\cos(kt)\} = -\frac{d}{ds}L\{\cos(kt)\} = -\frac{d}{ds}\left(\frac{s}{s^2+k^2}\right) = -\left(\frac{(\frac{d}{ds}s)(s^2+k^2)-(\frac{d}{ds}(s^2+k^2))(s)}{(s^2+k^2)^2}\right) = \frac{2s^2-(s^2-k^2)}{(s^2+k^2)^2} = \frac{s^2+k^2}{(s^2+k^2)^2} = \frac{1}{s^2+k^2}$

Ex3 - Solve $y'' + 9y = \cos(3t)$, $y(0) = 2$, $y'(0) = 5$.

1. We already know the rigmarole, so we'll skim over a good amount of parts.
   $L\{y''+9y\} = (s^2Y(s) - sy(0) - y'(0)) + 9Y(s) = (s^2+9)Y(s) - 2s - 5 = L\{\cos(3t)\} = \frac{s}{s^2+9}$.
2. Eventually, we get $Y(s) = \frac{s}{(s^2+9)^2} + \frac{(2s+5)}{(s^2+9)}$.
3. I'll toss you a bone on this one: we know that $L\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}$.

   Therefore, for our first integral $\frac{s}{(s^2+9)^2}$ if we let $k = 3$ then $\frac{s}{(s^2+9)^2} = \frac{1}{6}(\frac{2\cdot 3\cdot s}{(s^2+(3)^2)^2})$.

   This gives us $L^{-1}\left\{\frac{1}{6}\left(\frac{2\cdot 3\cdot s}{(s^2+(3)^2)^2}\right)\right\} = \frac{1}{6}(t\sin(3t))$.

   For our other integral, $L^{-1}\left\{\frac{(2s+5)}{(s^2+9)}\right\} = 2L^{-1}\left\{\frac{s}{(s^2+9)}\right\} + 5L^{-1}\left\{\frac{1}{s^2+9}\right\} = 2\cos(3t) + \frac{5}{9}\sin(3t)$.

   In conclusion, $y(t) = \frac{1}{6}(t\sin(3t)) + 2\cos(3t) + \frac{5}{9}\sin(3t)$.

8 Chapter 8:

8.1 Eigenvalues and Eigenvectors

You've done it - you've reached the final chapter of this guide. We've saved the best for last, so get ready for the apex of Differential Equations and everything about equations that are differentiable... *get ready for Linear Algebra!* Let's get straight to business.

Consider a *system* of linear first-order equations. We can break out linear equation down into a vector X of all our possible $x_i(t)$ in each equation, a matrix A for all coefficients $a_{ij}(t)$ of each $x_i(t)$, and another vector $F(t)$ of all additional $f_i(t)$:

$$\begin{cases} \frac{dx_1}{dt} = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \frac{dx_1}{dt} = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \vdots \\ \frac{dx_1}{dt} = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \end{cases}, \vec{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \dots & a_{2n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}, \vec{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

We can rewrite our system as $\vec{X}' = A\vec{X} + \vec{F}$. If all of our equations are homogeneous (meaning no $f_i(t)$ for all $\frac{dx_i}{dt}$), then our homogeneous system can be given by $X' = A\vec{X}$. We solve this system by finding the **solution vector** X that satisfies this system.

Recall that when we have a first-order equation like $y' = ky$, where k is a constant, we have a solution in the form $y = Ce^{rt}$. We can expand this idea to our homogeneous system $\vec{X}' = A\vec{X}$ by assuming that our solution vector has a general solution $\vec{X} = \vec{K}e^{\lambda t}$. In $\vec{X} = \vec{K}e^{\lambda t}$, $\vec{K}$ is our **eigenvector** corresponding to A and λ is our **eigenvalue**. Here are some other properties that you should know about eigenvectors and eigenvalues:

- Each eigenvector corresponds to an eigenvalue. The number of eigenvalues is the same as the number of columns in the matrix A. Therefore, for $A_{2\times 2}$, we have λ_1 and λ_2.
- Eigenvectors are a *special* kind of vector because taking the product of an eigenvector $\vec{K}$ of a matrix A is the same as multiplying $\vec{K}$ by its eigenvalue λ, so $A\vec{K} = \lambda\vec{K}$.
  - If we take $A\vec{K} = \lambda\vec{K}$, then $A\vec{K} - \lambda\vec{K} = 0$. Since $\vec{K} = I\vec{K}$ (I is our identity matrix which just has 1s on the diagonal), we have $(A - \lambda I)\vec{K} = 0$. We see this below.

$$(A - \lambda I)\vec{K} = \begin{pmatrix} (a_{11} - \lambda)k_1 & a_{12}k_2 & \dots & a_{1n}k_n \\ a_{21}k_1 & (a_{22} - \lambda)k_2 & \dots & a_{2n}k_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}k_1 & a_{n2}k_2 & \dots & (a_{nn} - \lambda)k_n \end{pmatrix} = 0$$

- We find eigenvalues by solving the **characteristic polynomial**. In general, our polynomial has the form $\det(A - \lambda I) = 0$. For $A_{2\times 2}$, we can simplify it as $p(\lambda) = \lambda^2 - Tr(A)\lambda + \det(A)$
  - $Tr(A)$: Trace of a matrix, for $A_{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad Tr(A) = a + d$

I've just thrown a LOT of information at you, so don't feel too bad right now if you don't understand this. Save feeling bad for when you take Analysis, or if you're an engineer, the fact that you'll be doing modelling problems until you retire. We'll cover examples in the next chapters.

8.2 Real, Distinct Eigenvalues

Suppose we are given an $n \times n$ matrix A. We have previously mentioned that there's an eigenvalue for every column, so we'll have $\lambda_1, \lambda_2, \cdots, \lambda_n$. This would give us eigenvectors $\vec{K}_1, \vec{K}_2, \cdots, \vec{K}_n$. For each eigenvalue λ_i and eigenvector $\vec{K}_i$, we have a specific solution $X_i = \vec{K}_i e^{\lambda_1 t}$. Therefore, our general solution for $\vec{X}$ can be given by:

$$X = c_1\vec{K}_1 e^{\lambda_1 t} + c_2\vec{K}_2 e^{\lambda_2 t} + \cdots c_n\vec{K}_n e^{\lambda_n t}$$

Our steps to solving a first-order, homogeneous, linear system are as follows:

1. If we are given equations in linear form, convert the coefficients into a matrix.
2. Using the characteristic equation $\det(A - \lambda I)$, find the n eigenvalues of A.
   - $A - \lambda I$ just means add a $-\lambda$ to all of our coefficients on our main diagonal.
   - For 3×3 matrices, if your entries above or below your diagonal aren't all 0, then use the cofactors $(-1)^{i+j}A_{ij}$ of a row or column.
   - For 2×2 matrices, we can use $det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c$.
3. For each eigenvalue λ_i, find the corresponding eigenvector K_i by row-reducing $[(A - \lambda_k I) \mid 0]$.
   - $[(A - \lambda_k I) \mid 0]$ is just our $(A - \lambda_k I)$ matrix but with a column of 0's tacked onto the end.
   - If you run into any free variables, just use whatever you want. I'm not particularly picky.
4. After you've found all of the eigenvectors, plug them into the general solution equation above.

Ex. - Solve $\begin{cases} \frac{dx}{dt} = 2x - 7y \\ \frac{dy}{dt} = 5x + 10y + 4z \\ \frac{dz}{dt} = 5y + 2z \end{cases}$

1. Ignoring our variables for now, $\begin{cases} \frac{dx}{dt} = 2x - 7y \\ \frac{dy}{dt} = 5x + 10y + 4z \\ \frac{dz}{dt} = 5y + 2z \end{cases} \implies A = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$

2. $\det(A - I\lambda) = \begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 5 \cdot (-1)^{2+3} \begin{vmatrix} 2-\lambda & 0 \\ 5 & 4 \end{vmatrix} + (2-\lambda) \cdot (-1)^{3+3} \begin{vmatrix} 2-\lambda & -7 \\ 5 & 10-\lambda \end{vmatrix}$
   $= (-5)[(2-\lambda)(4) - (0)(5)] + (2-\lambda)[(2-\lambda)(10-\lambda) - (5)(-7)] = (2-\lambda)((2-\lambda)(10-\lambda) + 15)$
   $= -(\lambda - 2) \cdot (\lambda^2 - 12\lambda + 35) = -(\lambda-2)(\lambda-7)(\lambda-5) = 0 \implies \lambda_1 = 2,\ \lambda_2 = 7,\ \lambda_3 = 5$

3. $\lambda_1 = 2 : [(A - 2I) \mid 0] = \begin{pmatrix} (2-2) & -7 & 0 & 0 \\ 5 & (10-2) & 4 & 0 \\ 0 & 5 & (2-2) & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{4}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x + \frac{4}{5}z = 0 \\ y = 0 \\ z \text{ is free} \end{cases}$

   $\lambda_1 = 2 : [(A - 7I) \mid 0] = \begin{pmatrix} (2-7) & -7 & 0 & 0 \\ 5 & (10-7) & 4 & 0 \\ 0 & 5 & (2-7) & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{7}{5} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x + \frac{7}{5}z = 0 \\ y - z = 0 \\ z \text{ is free} \end{cases}$

   $\lambda_1 = 5 : [(A - 5I) \mid 0] = \begin{pmatrix} (2-5) & -7 & 0 & 0 \\ 5 & (10-5) & 4 & 0 \\ 0 & 5 & (2-5) & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{7}{5} & 0 \\ 0 & 1 & \frac{-3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x + \frac{7}{5}z = 0 \\ y - \frac{3}{5}z = 0 \\ z \text{ is free} \end{cases}$

   If $z = 1$, then $K_1 = \begin{pmatrix} -4/5 \\ 0 \\ 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} -7/5 \\ 1 \\ 1 \end{pmatrix}$, and $K_3 = \begin{pmatrix} -7/5 \\ 3/5 \\ 1 \end{pmatrix}$.

4. $\vec{X}(t) = c_1 \begin{pmatrix} -4/5 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -7/5 \\ 1 \\ 1 \end{pmatrix} e^{7t} + c_3 \begin{pmatrix} -7/5 \\ 3/5 \\ 1 \end{pmatrix} e^{5t}$.

8.3 Repeated Eigenvalues

Each eigenvalue is special in its own way. But what if they're not? What if we run into a repeated eigenvalue? There are two cases to consider. We'll consider an example for each.

Case 1: λ is repeated, but we have two linearly independent eigenvectors.

- Sometimes we run into this situation when we have more than one free variable. When this occurs, we can divide up our eigenvector for our repeated eigenvalue into two separate eigenvectors.

Ex1. - Solve $X' = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} X$.

1. $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 = 0 \to \lambda = 5.$

2. $[(A - 5I) \mid 0] = \begin{pmatrix} (5-5) & 0 & 0 \\ 0 & (5-5) & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x \text{ is free} \\ y \text{ is free} \end{cases}$

   $\vec{K} = \begin{pmatrix} x \\ y \end{pmatrix}$ for any $x, y \to \ \vec{K} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \ \vec{K}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{K}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

3. $\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{5t} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{5t}.$

Case 2: λ is repeated but we only have one eigenvector $\vec{K}$

- For a 2×2 matrix, if we only have $\vec{K}$, we need to find another vector $\vec{P}$ by solving $(A - \lambda I)\vec{P} = \vec{K}$. This will give us our second solution $X_2 = \vec{K}te^{\lambda t} + \vec{P}e^{\lambda t}$. Remember that our general solution is $\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2$, where $\vec{X}_1 = \vec{K}e^{\lambda t}$.

- If we have only one eigenvalue λ for a 3×3 matrix (eigenvalue with a multiplicity of 3), then after finding $\vec{P}$ we need to find a third vector $\vec{Q}$ using $(A - \lambda I)\vec{Q} = \vec{P}$. Our third solution will have the form $\vec{X}_3 = \vec{K}\frac{t^2}{2}e^{\lambda t} + \vec{P}te^{\lambda t} + \vec{Q}e^{\lambda t}$. Our general solution is $\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 + c_3\vec{X}_3$.

Ex2. - Solve $X' = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} X$.

1. $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 3 \\ -3 & 5-\lambda \end{vmatrix}$

   $= (-1-\lambda)(5-\lambda) - (3)(-3) = (\lambda+1)(\lambda-5) + 9 = (\lambda-2)^2 = 0 \to \lambda = 2$

2. $[(A - 2I) \mid 0] = \begin{pmatrix} (-1-2) & 3 & 0 \\ -3 & (5-2) & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x = y \\ y \text{ is free} \end{cases} \to$ Let $y = 3$, so $\vec{K} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

   $[(A - 2I) \mid \vec{K}] = \begin{pmatrix} -3 & 3 & 3 \\ -3 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x = y + 1 \\ y \text{ is free} \end{cases} \to$ We chose $y = 3$, so $\vec{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

3. $\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2 = c_1 \cdot \vec{K}e^{\lambda t} + c_2[\vec{K}te^{\lambda t} + \vec{P}e^{\lambda t}] = c_1 \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 3 \\ 3 \end{pmatrix} te^{2t} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{2t} \right].$

8.4 Complex Eigenvalues

You knew as soon as you saw the characteristic equation that we were gonna have to do some freaky stuff with imaginary numbers. And you'd be right! The world isn't all sunshine and rainbows - there exist quadratics with complex solutions. Luckily, for eigenvalues and eigenvectors, it's not too bad.

If solving our characteristic equation results in complex eigenvalues given by $\lambda = \alpha \pm i\beta$, then our eigenvectors $\vec{K}_1$ for $\lambda_1 = \alpha + i\beta$ and $\vec{K}_2$ for $\lambda_2 = \alpha - i\beta$ will have both real and complex parts. Conveniently, we can just focus on our first eigenvalue $\lambda_1 = \alpha + i\beta$ which will give us $\vec{K}_1$. We should be able to divide our eigenvector into $\vec{K}_1 = \vec{B}_1 + i\vec{B}_2$. This will give us a general solution of:

$$\vec{X}(t) = c_1 e^{\alpha t}[\vec{B}_1 \cos(\beta t) - \vec{B}_2 \sin(\beta t)] + c_2 e^{\alpha t}[\vec{B}_2 \cos(\beta t) + \vec{B}_1 \sin(\beta t)].$$

Example - $X' = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} X$

1. $\det(A - \lambda I) = \begin{pmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix} = (5-\lambda)(3-\lambda) - (1)(-2) = 15 - 8\lambda + \lambda^2 + 2 = \lambda^2 - 8\lambda + 17.$

2. We'll use everyone's favorite *quadratic formula* to find λ:

   $\lambda = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(17)}}{2(1)} = \dfrac{8 \pm \sqrt{64-68}}{2} = 4 \pm \dfrac{\sqrt{-4}}{2} = 4 \pm i.$

   Wow! Complex eigenvalues in a chapter about complex eigenvalues! Life is full of surprises, isn't it?

3. We only have to consider one λ, so consider $\lambda_1 = 4 + i$.

   $[A - (4+i) \mid 0] = \begin{pmatrix} 5-(4+i) & 1 & 0 \\ -2 & 3-(4+i) & 0 \end{pmatrix} \to \begin{cases} (1-i)x + y = 0 \\ -2x - (1+i)y = 0 \end{cases}$

   Our first equation gives us $y = -(1-i)x = (i-1)x$. We'll ignore our second equation because we can. Let $x = 1$. This will give us a solution vector of $\vec{K}_1 = \begin{pmatrix} 1 \\ i-1 \end{pmatrix}$.

4. Our eigenvalue $\lambda_1 = 4 + i$ gives us $\alpha = 4$ and $\beta = 1$.

   Splitting our eigenvector gives us $\vec{K}_1 = \begin{pmatrix} 1 \\ i-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   This means that $\vec{B}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{B}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   Thus, our general solution is $\vec{X}(t) = c_1 e^{4t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)\right] + c_2 e^{4t}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(t)\right]$.

   Further simplification leads to $\vec{X}(t) = c_1 e^{4t}\begin{pmatrix} \cos(t) \\ -[\cos(t) + \sin(t)] \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} \sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}$.

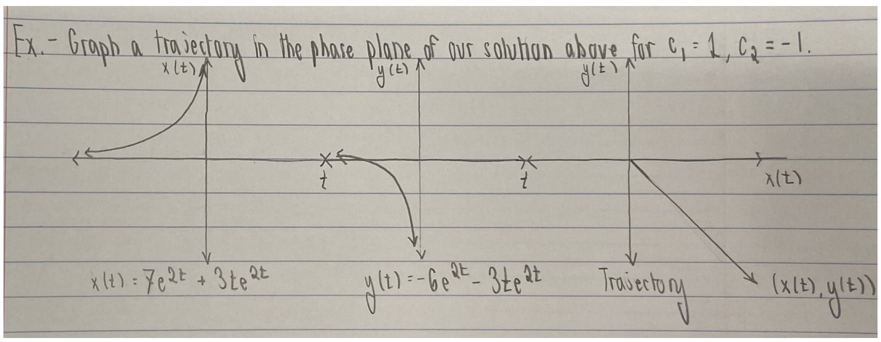
8.5 Phase Portraits

Remember that one chapter on Phase Lines, which we could use to solve autonomous equations like $y' = f(y)$? Well get ready to meet its underachieving little brother, Phase Portraits!

Recall that a matrix is just a fancy way of representing a system of first-order linear equations. When we solve a system $X' = AX$ where A is a 2×2 matrix, our solution has a general form $\vec{X} = c_1\vec{X}_1 + c_2\vec{X}_2$. We can split our solution back into a system of equations using the entries at each corresponding row.

Ex. - $\vec{X} = c_1 \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 3 \\ 3 \end{pmatrix} te^{2t} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{2t} \right] \rightarrow \begin{cases} x = 3c_1e^{2t} + (3te^{2t} + 4e^{2t})c_2 \\ y = 3c_1e^{2t} + (3te^{2t} + 3e^{2t})c_2 \end{cases}$

Keep in mind that here, our x and y are just variables that will help denote each equation, so we could have whatever variable name we want. We can create a **phase plane** by drawing a curve of points $(x(t), y(t))$. Each curve will have a specific c_1 and c_2, so we call these curves **trajectories**.



A **phase portrait** is just a diagram consisting of all the possible trajectories of a system of first-order linear ODEs. We can usually generalize the direction of our trajectories using **half-lines** which correspond to an eigenvalue. Half-lines act line asymptotes, and are linear like the trajectory in our example above.

Like when working with critical points in phase lines, we can describe the origin in a phase portrait as either a **repeller**, **attractor**, or a **saddle point** depending on the direction of the trajectories. We will consider the general scenario for each possible eigenvalue:

- **Distinct Real Eigenvalues:** Given λ_1 and λ_2, our general solution is $X = c_1K_1e^{\lambda_1 t} + c_2K_2e^{\lambda_2 t}$. As $t \to \infty$, $e^{\lambda t}$ either approaches 0 or ∞ depending on the sign of λ. This will determine the direction of our trajectories.
  1. **Opposite Signs:** Our trajectories will approach the half-lines for our positive eigenvalue, so they point away from the origin, and diverge from our negative eigenvalue half-lines, so they point towards our origin. This makes our origin a *saddle point*
  2. **Positive Signs:** All of our trajectories point away from the origin, making it a *repeller.*
  3. **Negative Signs:** All of our trajectories point towards the origin, making it an *attractor.*
- **Repeated Eigenvalue:** Given one λ, our general solution is $X = c_1Ke^{\lambda t} + c_2(Kte^{\lambda t} + Pe^{\lambda t}) = e^{\lambda t}(c_2Kt + (c_1K + c_2P))$. Thus, if $\lambda < 0$, our origin is an *attractor*, and if $\lambda > 0$, our origin is a *repeller.*
- **Complex Eigenvalues:** We know a phase portrait is depicting a system with complex eigenvalues $\lambda = \alpha \pm \beta i$ if our trajectories form a spiral graph.
  - If $\alpha > 0$, our spirals move away from our origin, creating a **spiral source**.
  - If $\alpha < 0$, our spirals move towards the origin, creating a **spiral sink**.