Leonhard Euler: The Greatest Mathematician*

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1 Introduction

Among mathematicians throughout history, nobody has been as heralded and celebrated as Leonhard Euler. Euler is considered the most prolific mathematician of the 18th century, with groundbreaking work in almost all fields of mathematics including, but not limited to, number theory, complex analysis, and graph theory. Most believe that no other mathematician has published the same amount of acclaimed work as Euler - a sentiment expressed by other of the world's greatest mathematicians, such as Pierre-Simon Laplace, who was quoted saying "Read Euler, read Euler, he is the master of us all." [8]

In this article, we will explore the personal history of the man behind the math. Afterwards, we will dissect some of Euler's most infamous works: his solution to the Basel Problem and its relationship to the Riemann Zeta function.

2 Life and Major Works

Leonhard Euler was born on April 15, 1707 to Paul Euler, a protestant church pastor, and Marguerite Brucker in Basel, Switzerland. Although his family would move to Riehen, Switzerland, he continued to attend school in Basel. His primary education did not include a curriculum in mathematics; Euler would learn math from textbooks that his father provided. Furthermore, because the Euler family were friends with the Bernoulli family, Jacob Bernoulli - a prominent mathematician at the time - taught Euler more advanced topics in math during afternoon classes.

Euler originally hoped to become a protestant priest like his father. Therefore, when Euler started attending the University of Bassel, he first sought to pursue a degree in theology. When he befriended Johann Bernoulli, brother of Jacob Bernoulli, Johann noticed Euler's immense talent in mathematics and helped him foster an interest in a life of math. Eventually, Euler graduated with a master in philosophy and broke the news to his father that he was not interested in becoming a minister. Although his father was reluctant at first to accept this, he eventually allowed Euler to pursue mathematics for a living.

*Source: `https://www.overleaf.com/read/61ea1e8cbd20db779e7329bd`

After graduating, Euler applied to become a professor of physics at his alma mater. However, a lack of response drove Euler to accept a teaching position at the new Academy of Sciences in St. Petersburg, Russia. Initially, Euler taught applied mathematics in the medical department because of a vacant spot left by a physiology professor. Eventually though, he was promoted to a position in the mathematics department after Daniel Bernoulli, the head of the math department, recommended him. In 1733, political turmoil and frustration due to censorship lead Daniel Bernoulli to leave St. Petersburg, allowing Euler to become the new head of the math department. [5]

Having a secure position allowed Euler to settle down and start publishing his work. He married Katharina Gsell - another Swiss expat in Russia - and together they had thirteen children, with only five surviving. During this time, Euler wrote *Mechanica* - a textbook that combined mathematical analysis with Newtonian Dynamics (calculus-based physics that focuses on predicting motion). Furthermore, this period marked Euler's brief venture into music theory in *Attempt at a New Theory of Music*, which he would soon retract due to criticism by musicians for his work being too "technical".

Rising political turmoil in Russia forced Euler to accept a position at the University of Berlin on behalf of Frederick the Great in Prussia. During the 25 years he stayed in Berlin, Euler would produce his most acclaimed work yet. *Introduction to Infinitorium Analysis* layed the groundwork for modern mathematical analysis and functional notation. *Institutions of Differential Calculus* was the first complete text on Differential Calculus. On top of these were hundreds of papers pertaining to all different topics of mathematical sciences.

Despite Euler's immense contributions to the University of Berlin's prestige, he started to fall out of Frederick the Great's good grace. Euler was part of Frederick's court of intellectuals, and when he was unable to provide much insight on topics outside of mathematics, he found himself the subject of ridicule by philosophers such as Voltaire. When the president of the University of Berlin passed away, Euler was not given his seat, but rather the president's responsibilities. Therefore, in 1776, when Euler received an invitation to return to St. Petersburg after the political turmoil had subsided, he accepted. [8]

In his later life, Euler almost became completely blind. This is depicted in the portrait below 1. Furthermore, the same year his family moved to St. Petersburg, his wife Katharina passed away. Despite the personal setbacks, Euler remained extremely productive in his work, and he continued to publish his findings with the help of his sons. [3]

In 1783 however, Euler finally passed away from a brain hemorrage. He left a legacy of work so vast that St. Petersburg academy continued to publish his work for decades to come.

Figure 1: Portrait of Euler by Swiss Painter Jakob Handmann. Notice the poor shape of his right eye compared to his left. (source: https://en.wikipedia.org/wiki/Leonhard_Euler#/media/File:Leonhard_Euler_-_edit1.jpg)

3 Basel Problem and Riemann Zeta Function

The Basel Problem was first proposed by Italian mathematician Pietro Mengoli in 1650. The question asks if a precise sum exists for the sum of the reciprocals of the squares of numbers from 1 to n. [6]

$$\sum_{i=0}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \tag{1}$$

The problem was eventually named after the town of Basel, Switzerland, after the Bernoulli family had tried, yet failed to solve the problem, and after Euler proposed a solution. In 1735, Euler found that a precise sum exists and in 1740 he later published his finding in an article titled "On the sums of series of reciprocals". He was 28 years old when he solved this infamous problem, bringing him immediate fame. A general outline of his proof is as follows:

Proof. We want to develop two infinite series for $\sin(x)$ - an infinite sum $\sum$, and an infinite product $\prod$. [2]

1. Set $\sin(x)$ to a power series $\sin(x) = a_0 + a_1x + a_2x^2 + ...$ We want to determine the coefficients a_0, a_1, etc.

   If we set $x = 0$, then all of our x terms dissapear, so we are left with $\sin(0) = a_0$. This gives us $a_0 = 0$. If we take the derivative of everything, we are left with $\frac{d}{dx}\sin(x) = \cos(x) = a_1 + 2a_2x + ...$ If we set $x = 0$, then we can once again isolate our next coefficient, giving us $a_1 = cos(0) = 1$.

If we continue to take derivatives, evaluating at 0, then isolating the constant, we end up with the Taylor Series expansion of $\sin(x)$ given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \tag{2}$$

2. To find the infinite product, recall that $\sin(x) = 0$ when x is a multiple of π, or $x = d \cdot \pi$ where $d \in \mathbb{Z}$. This means each $d\pi$, $(x - d\pi)$ is a part of our infinite product of $\sin(x)$. This gives us the following:

$$\sin(x) = a \cdot (x) \cdot (x - \pi) \cdot (x + \pi) \cdot (x - 2\pi) \cdot (x + 2\pi) \cdots \tag{3}$$

Here, the constant a is our *scaling factor* that we wish to determine. Notice that we can combine terms with similar coefficients for π as follows:

$$\sin(x) = a \cdot (x) \cdot [(x)^2 - (\pi)^2] \cdot [(x)^2 - (2\pi)^2] \cdots$$

We can isolate a from the rest of our equation as follows:

$$\begin{aligned} a &= \frac{\sin(x)}{(x) \cdot [(x)^2 - (\pi)^2] \cdot [(x)^2 - (2\pi)^2] \cdots} \\ &= \frac{\sin(x)}{x} \cdot \frac{1}{[(x)^2 - (\pi)^2] \cdot [(x)^2 - (2\pi)^2] \cdots} \end{aligned} \tag{4}$$

Recall that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, so when $x = 0$, we are left with:

$$a = \frac{1}{[-(\pi)^2] \cdot [-(2\pi)^2] \cdot [-(3\pi)^2] \cdots}$$

[Quick Sidenote: This is technically a convergent series, meaning as x approaches 0, a should also approach 0. However, for the sake of keeping this proof as general as possible, we digress.]

If we plug our a back into our equation 3, we get:

$$\sin(x) = x \cdot \frac{[(x)^2 - (\pi)^2] \cdot [(x)^2 - (2\pi)^2] \cdot [(x)^2 - (3\pi)^2] \cdots}{[-(\pi)^2] \cdot [-(2\pi)^2] \cdot [-(3\pi)^2] \cdots}$$

We can simplify this to get our final infinite product of $\sin(x)$:

$$\sin(x) = x \cdot \left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{2^2\pi^2}\right) \cdot \left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots \tag{5}$$

Now we have our infinite sum 2 and our infinite product 5. We can set them equal to one another after dividing $\sin(x)$ by x as follows:

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ... = \left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{2^2\pi^2}\right) \cdot \left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots$$

Notice that every term has an x term with a power that is a multiple of 2, so for each term of $\frac{\sin(x)}{x}$, we have x^{2m} for some $m \in \mathbb{Z}$, $m \geq 0$. In particular, we can re-express our infinite product as follows:

$$\frac{\sin(x)}{x} = 1 - x^2 \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots \right) + x^4(\cdots) + \cdots$$

Let's take a closer look at the coefficient for our x^2 term in our infinite sum and product. In our infinite sum, the coefficient for x^2 is $\frac{1}{3!}$. We have just shown above that for our infinite product, the coefficient for x^2 is $\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)$. From this we can derive the following property: [1]

$$\frac{1}{3!} = \frac{1}{6} = \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots \right) = \frac{1}{\pi^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

Eureka! Our expression on the right-hand side is just the Basel Problem 1 multiplied by $\frac{1}{\pi^2}$. From here, we can finally conclude what we wanted to show: that there is a precise solution to Basel Problem.

$$\frac{1}{6} = \frac{1}{\pi^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) \implies \sum_{i=0}^{n} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\square$

This is a problem that has stumped mathematicians for close to a hundred years. Therefore, when this proof was published, it sparked greater interests in using power series to analyze problems in number theory. More importantly however would be Euler using Basel's problem to create the **Riemann Zeta function** shown below: [9]

$$\zeta(s) = \sum_{i=0}^{\infty} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{6}$$

Although today the function is named after Bernhard Riemann, an 19th century mathematician who studied the function extensively, it was first introduced by Euler. Putting this problem into perspective, when $s = 1$, we have the **Harmonic Series**. When $s = 2$, we have the Basel problem.

The Riemann Zeta function is one of the most important complex variables because of its connections to prime numbers. A couple of years after solving Basel's problem, Euler proved the identity below:

$$\sum_{i=0}^{n} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \tag{7}$$

This identity 7 is used to calculate the probability that s random integers are **co-prime** - that is, they have no common integers other than 1. The zeta function has also been used to calculate the distribution of prime numbers given a specific domain. [4]

When we have $s > 0$, we have a convergent series. When we have $s < 0$, we have a divergent series. We had seen from the proof of the Basel Problem that we get a definite number $(\frac{\pi^2}{6})$ when $s = 2$. Riemann's work on the zeta function showed that no matter what s was, even if it was a complex number you would always get a definite number. When $s = -1$, one of the most famous properties in mathematics arises:

$$s = -1 \rightarrow \zeta(-1) = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \cdots = 1 + 2 + 3 + \cdots = -\frac{1}{12} \tag{8}$$

There is only one exception to trend this however: there is no well-defined point for $s = 1$, which is our Harmonic Series. It is from this that we get one of the most famous conjectures in mathematics that remains not proven.

We wish to find a point where $\zeta(s) = 0$. Coincidentally, all the negative even numbers will produce $\zeta(s) = 0$, so we call these *trivial zeros*. Many have suggested that there exist non-trivial zeros between 0 and 1. **Riemann's Hypothesis** suggests that the only non-trivial zeros exist on a line $s = \frac{1}{2} + y\iota$, where y is any number on our imaginary axis. [4] This is pictured in the diagram 2 below.

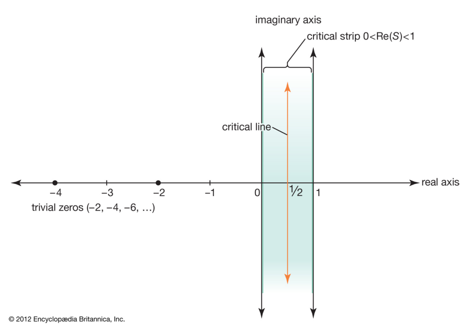

Figure 2: Diagram of Riemann's Hypothesis: Non-trivial zeros are proposed to only exist on the Critical Line (source: `https://www.britannica.com/science/Riemann-hypothesis`)

There are other applications of the zeta functions outside pure analytical mathematics. In probability, the Zipf Law is used to study the relationship between arbitrary rankings and the frequency of data in social systems, such as how frequently words are used in the English language. [10] In quantum field theory, the Cashmir Effect analyzes the relationship between materials and electromagnetic forces, an example being how metal conducts heat based on its shape and position. [7]

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