CSCI 305, Homework # 3

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Due date: Tuesday, May 8, midnight.

- Find $\Theta(T(n))$ for each case.
- For problem 1, find the solution three ways:
  - The master theorem
  - The substitution method
  - Iterate, cancel, and solve the summation to find an exact solution.
- For all other problems, use just the master theorem.
- When using the master theorem, be explicit about $n^{\log_b a}$, and about which case of the master theorem applies, about making sure the special condition applies in case 3 of the master theorem, and about proving f's correct asymptotic relationship to $n^{\log_b a}$.

1. Solve this one three different ways: master theorem, substitution method, and the iterate/cancel/summation method.

$$T(n) = 2T(n/2) + n^4$$

Master theorem

$$\begin{array}{rcl} f(n) & vs. & n^{log_b a} \\ n^4 & vs. & n^{log_2 2} \\ n^4 & vs. & n^1 \\ f(n) & = & \Omega(n^{log_b a + \epsilon}) \\ T(n) & = & \Theta(n^4) \end{array}$$

note: $af(n/b) = 2f(n/2) = 4/2^4 f(n) \leq cf(n)$

Substitution method
Guess $T(n) \leq c(n^4)$

$$\begin{array}{rcl} T(n) & = & 2T(n/2) + n^4 \\ T(n) & \leq & 2c(n/2)^4 + n^4 \\ T(n) & \leq & c/8(n^4) + n^4 \\ T(n) & \leq & (c/8 + 1)n^4 \\ T(n) & \leq & c(n^4) \end{array}$$

for $c \leq 8/7$
We can use this same proof for $T(n) = \Omega(n^4)$ by flipping the signs, So $T(n) = \Theta(n^4)$.

Iterate/cancel/summation method

$$
\begin{aligned}
T(n) &= \cancel{2T(n/2)} + n^4 \\
\cancel{2T(n/2)} &= \cancel{2^2T(n/2^2)} + 2((n/2)^4) \\
\cancel{2^2T(n/2^2)} &= \cancel{2^3T(n/2^3)} + 2^2((n/2^2)^4) \\
\cancel{2^{k-1}T(n/2^{k-1})} &= \cancel{2^kT(n/2^k)} + 2^{k-1}((n/2^{k-1})^4) \\
T(n) &= 2^{log_2 n}T(n/2^{log_2 n}) + 2^{(log_2 n)-1}((n/2^{(log_2 n)-1})^4) \\
T(n) &= nT(n/n) + \sum_{i=0}^{(log_2 n)-1} 2^i(n/2^i)^4 \\
T(n) &= nT(1) + n^4 \sum_{i=0}^{(log_2 n)-1} 2^i/16^i \\
T(n) &= \Theta(n^4)
\end{aligned}
$$

2. Solve this one using the master theorem.

$$T(n) = 16T(n/4) + n^2$$

$$
\begin{array}{rcl}
f(n) & vs. & n^{log_b a} \\
n^2 & vs. & n^{log_4 16} \\
n^2 & vs. & n^2 \\
f(n) & = & \Theta(n^{\log_b a}) \\
T(n) & = & \Theta(n^2 \lg n)
\end{array}
$$

3. Solve this one using the master theorem.

$$T(n) = 2T(n/4) + \sqrt{n}$$

$$
\begin{array}{rcl}
f(n) & vs. & n^{log_b a} \\
\sqrt{n} & vs. & n^{log_4 2} \\
\sqrt{n} & vs. & \sqrt{n} \\
f(n) & = & \Theta(n^{\log_b a}) \\
T(n) & = & \Theta(\sqrt{n} \lg n)
\end{array}
$$

4. Solve this one using the master theorem.

$$T(n) = 4T(n/3) + n \lg n$$

$$
\begin{array}{rcl}
f(n) & vs. & n^{log_b a} \\
n \lg n & vs. & n^{log_3 4} \\
f(n) & = & O(n^{\log_b a - \epsilon}) \\
T(n) & = & \Theta(n^{\log_3 4})
\end{array}
$$

5. Solve this one using the master theorem.

$$T(n) = 4T(n/2) + n^2\sqrt{n}$$

$$\begin{aligned}
f(n) &\quad vs. \quad n^{log_b a} \\
n^2\sqrt{n} &\quad vs. \quad n^{log_2 4} \\
n^2 n^{1/2} &\quad vs. \quad n^2 \\
n^{2.5} &\quad vs. \quad n^2 \\
f(n) &\quad = \quad \Omega(n^{log_b a+\epsilon}) \\
T(n) &\quad = \quad \Theta(n^{2.5})
\end{aligned}$$

note: $af(n/b) = 4f(n/2) = 4/2^{2.5}f(n) \leq cf(n)$