

# CSCI 305, Homework # 3

Caleb Ouellette

Due date: Tuesday, May 8, midnight.

- Find  $\Theta(T(n))$  for each case.
  - For problem 1, find the solution three ways:
    - The master theorem
    - The substitution method
    - Iterate, cancel, and solve the summation to find an exact solution.
  - For all other problems, use just the master theorem.
  - When using the master theorem, be explicit about  $n^{\log_b a}$ , and about which case of the master theorem applies, about making sure the special condition applies in case 3 of the master theorem, and about proving  $f$ 's correct asymptotic relationship to  $n^{\log_b a}$ .
1. Solve this one three different ways: master theorem, substitution method, and the iterate/cancel/summation method.

$$T(n) = 2T(n/2) + n^4$$

Master theorem

$$\begin{aligned} f(n) & \text{ vs. } n^{\log_b a} \\ n^4 & \text{ vs. } n^{\log_2 2} \\ n^4 & \text{ vs. } n^1 \\ f(n) & = \Omega(n^{\log_b a + \epsilon}) \\ T(n) & = \Theta(n^4) \end{aligned}$$

$$\text{note: } af(n/b) = 2f(n/2) = 4/2^4 f(n) \leq cf(n)$$

Substitution method

Guess  $T(n) \leq c(n^4)$

$$\begin{aligned} T(n) & = 2T(n/2) + n^4 \\ T(n) & \leq 2c(n/2)^4 + n^4 \\ T(n) & \leq c/8(n^4) + n^4 \\ T(n) & \leq (c/8 + 1)n^4 \\ T(n) & \leq c(n^4) \end{aligned}$$

for  $c \leq 8/7$

We can use this same proof for  $T(n) = \Omega(n^4)$  by flipping the signs, So  $T(n) = \Theta(n^4)$ .

Iterate/cancel/summation method

$$\begin{aligned}
T(n) &= \cancel{2T(n/2)} + n^4 \\
\cancel{2T(n/2)} &= \cancel{2^2T(n/2^2)} + 2((n/2)^4) \\
\cancel{2^2T(n/2^2)} &= \cancel{2^3T(n/2^3)} + 2^2((n/2^2)^4) \\
\cancel{2^{k-1}T(n/2^{k-1})} &= \cancel{2^kT(n/2^k)} + 2^{k-1}((n/2^{k-1})^4) \\
T(n) &= 2^{\log_2 n} T(n/2^{\log_2 n}) + 2^{(\log_2 n)-1} ((n/2^{(\log_2 n)-1})^4) \\
T(n) &= nT(n/n) + \sum_{i=0}^{(\log_2 n)-1} 2^i (n/2^i)^4 \\
T(n) &= nT(1) + n^4 \sum_{i=0}^{(\log_2 n)-1} 2^i / 16^i \\
T(n) &= \Theta(n^4)
\end{aligned}$$

2. Solve this one using the master theorem.

$$T(n) = 16T(n/4) + n^2$$

$$\begin{aligned}
f(n) & \text{ vs. } n^{\log_b a} \\
n^2 & \text{ vs. } n^{\log_4 16} \\
n^2 & \text{ vs. } n^2 \\
f(n) & = \Theta(n^{\log_b a}) \\
T(n) & = \Theta(n^2 \lg n)
\end{aligned}$$

3. Solve this one using the master theorem.

$$T(n) = 2T(n/4) + \sqrt{n}$$

$$\begin{aligned}
f(n) & \text{ vs. } n^{\log_b a} \\
\sqrt{n} & \text{ vs. } n^{\log_4 2} \\
\sqrt{n} & \text{ vs. } \sqrt{n} \\
f(n) & = \Theta(n^{\log_b a}) \\
T(n) & = \Theta(\sqrt{n} \lg n)
\end{aligned}$$

4. Solve this one using the master theorem.

$$T(n) = 4T(n/3) + n \lg n$$

$$\begin{aligned}
f(n) & \text{ vs. } n^{\log_b a} \\
n \lg n & \text{ vs. } n^{\log_3 4} \\
f(n) & = O(n^{\log_b a - \epsilon}) \\
T(n) & = \Theta(n^{\log_3 4})
\end{aligned}$$

5. Solve this one using the master theorem.

$$T(n) = 4T(n/2) + n^2\sqrt{n}$$

$$\begin{array}{lll} f(n) & vs. & n^{\log_b a} \\ n^2\sqrt{n} & vs. & n^{\log_2 4} \\ n^2n^{1/2} & vs. & n^2 \\ n^{2.5} & vs. & n^2 \\ f(n) & = & \Omega(n^{\log_b a + \epsilon}) \\ T(n) & = & \Theta(n^{2.5}) \end{array}$$

$$\text{note: } af(n/b) = 4f(n/2) = 4/2^{2.5}f(n) \leq cf(n)$$