Euler and Milstein schemes for stochastic volatility model

Problem Statement

Consider the Heston model which is described by the bivariate stochastic process for the stock price S_t and its variance v_t:

$$dS_t = rS_t\,dt + \sqrt{v_t}S_t\,dW_t^1$$
$$dv_t = \kappa(\bar{v} - v_t)\,dt + \sigma\sqrt{v_t}\,dW_t^2$$

where $E[dW_t^1 dW_t^2] = \rho\,dt$. Derive both the Euler and Milstein schemes for this stochastic volatility model. Hint: First decouple the system of SDEs.

Euler Scheme

1. Discretize time:
$$t_n = n\Delta t$$
2. Discretize S_t equation:
$$S_{n+1} = S_n + rS_n\Delta t + \sqrt{v_n}S_n\Delta W_n^1$$
3. Discretize v_t equation:
$$v_{n+1} = v_n + \kappa(\bar{v} - v_n)\Delta t + \sigma\sqrt{v_n}\Delta W_n^2$$

Milstein Scheme

1. Discretize time:
$$t_n = n\Delta t$$
2. Discretize S_t equation:
$$S_{n+1} = S_n + rS_n\Delta t + \sqrt{v_n}S_n\Delta W_n^1$$
3. Discretize v_t equation:
$$v_{n+1} = v_n + \kappa(\bar{v} - v_n)\Delta t + \sigma\sqrt{v_n}\Delta W_n^2 + \frac{1}{4}\sigma^2\left[(\Delta W_n^2)^2 - \Delta t\right]$$

Summary of Steps and Increments

Euler Scheme

1. For S_t:

$$S_{n+1} = S_n + rS_n\Delta t + \sqrt{v_n}S_n\Delta W_n^1$$
$$\Delta S_n = rS_n\Delta t + \sqrt{v_n}S_n\Delta W_n^1$$

2. For v_t:

$$v_{n+1} = v_n + \kappa(\bar{v} - v_n)\Delta t + \sigma\sqrt{v_n}\Delta W_n^2$$
$$\Delta v_n = \kappa(\bar{v} - v_n)\Delta t + \sigma\sqrt{v_n}\Delta W_n^2$$

Milstein Scheme

1. For S_t:

$$S_{n+1} = S_n + rS_n\Delta t + \sqrt{v_n}S_n\Delta W_n^1$$
$$\Delta S_n = rS_n\Delta t + \sqrt{v_n}S_n\Delta W_n^1$$

2. For v_t:

$$v_{n+1} = v_n + \kappa(\bar{v} - v_n)\Delta t + \sigma\sqrt{v_n}\Delta W_n^2 + \frac{1}{4}\sigma^2\left[(\Delta W_n^2)^2 - \Delta t\right]$$

$$\Delta v_n = \kappa(\bar{v} - v_n)\Delta t + \sigma\sqrt{v_n}\Delta W_n^2 + \frac{1}{4}\sigma^2\left[(\Delta W_n^2)^2 - \Delta t\right]$$