PART B.

Question 1.

a)

i. $\{-1, 0\}$

ii. $\{1, 4, 7, 10\}$

b)

i. If $|A| = |B| = n$ and $|A \cap B| = 2$ then:
$|A \cup B| = |A| + |B| - |A \cap B| =$
$= n + n - 2 = 2n - 2$

ii. $|P(A \cup B)| = 2^{|A \cup B|} = 2^{2n-2}$

c)

i. $(A-B) \cap C = (C-B) \cap A$

I will apply transformations to the left side of the equation:

$(A \cap \neg B) \cap C = (C-B) \cap A$

$A \cap \neg B \cap C = (C-B) \cap A$ - associativity

$C \cap \neg B \cap A = (C-B) \cap A$ - commutativity

$(C-B) \cap A = (C-B) \cap A$

so the equality is **True**

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ii. $(A-B) \cup C = (C-B) \cup A$
The equality is **False**, since we cannot apply transformations on either the left or the right side and get equality.
Example:
$(A \cap \neg B) \cup C = (C-B) \cup A$
$C \cup (A \cap \neg B) = (C-B) \cup A$ - commutativity
$(C \cup A) \cap (C \cup \neg B) = (C-B) \cup A$ - distributivity
There are no further steps we could apply so the equality is **False**

iii. $(A - C) \cap (C - B) =$ Ɵ
$(A \cap \neg C) \cap (C \cap \neg B) =$ Ɵ
$A \cap \neg C \cap C \cap \neg B =$ Ɵ - associativity
$\neg C \cap C =$ Ɵ
Since we have $\neg C \cap C$ which is the inverse law and results in an empty set, and aside from it all other intersections with the empty set also result in an empty set we can conclude that the equation is **True**.

d)

i.

p	q	r	¬q	p ∧ ¬q	r ∨ q	(p ∧ ¬q) -> (r ∨ q)
F	F	F	T	F	F	T
F	F	T	T	F	T	T
F	T	F	F	F	T	T
F	T	T	F	F	T	T
T	F	F	T	T	F	F
T	F	T	T	T	T	T
T	T	F	F	F	T	T
T	T	T	F	F	T	T

ii.

The compound proposition is **Not** a tautology. We can prove it by looking at the truth table below. For a proposition to be a tautology its corresponding column must only contain True values. In this case, however, there is a False value when p is True, q is False, and r is False. Therefore, it is Not a tautology.

e)

i.

The statement $\exists x \exists y (x+y=0) \vee (x*y=0)$ is **False**. We can prove it by the fact that since the universe of discourse are all positive integers, then x >= 0 and y >= 0. The addition of two positive integers may not result in 0 and the multiplication of two positive integers may not result in 0 either. Therefore there are no positive integers which would satisfy this statement.

ii.

The statement $\forall x \forall y (x*y \geq x+y)$ is **False**. We can prove it by giving a single counterexample, which is sufficient for the statement to be False. For example, when x = 1 and y = 2, then x * y = 2 and x + y = 3 which does not satisfy the relation. Therefore, the statement is **False**.

f)

i.

To negate the statement we need to add the negation sign at the beginning and then gradually move it to the right changing the quantifiers:

$\forall a \in Z, \forall b \in N, \exists c \in N \ (ac > ab)$

$\neg \forall a \in Z, \forall b \in N, \exists c \in N \ (ac > ab)$

$\exists a \in Z, \neg \forall b \in N, \exists c \in N \ (ac > ab)$

$\exists a \in Z, \exists b \in N, \neg\exists c \in N\ (ac > ab)$

$\exists a \in Z, \exists b \in N, \forall c \in N\ \neg(ac > ab)$

We can convert by removing negation to: $\exists a \in Z, \exists b \in N, \forall c \in N\ (ac <= ab)$

ii.

The original statement is **False**. It implies that for all real numbers there exists a natural number such that by multiplying the two numbers we get a number in result that is bigger than any multiplication of a real number with a natural number.

We can prove that it is **False** since if we have a natural number x then there must also exist a natural number 2x, which is greater than x. Therefore, there is no single natural number greater than all other natural numbers, hence, the statement is False.

g)

We know that $A \cap B \subseteq C$ and $x \in B$. Suppose that $x \in A - C$. This implies that $x \in A$ and $x \notin C$. If $x \in A$ and $x \in B$ then it must be that $x \in A \cap B$. Since we know that $A \cap B$ is a subset of C and $x \notin C$, we have a contradiction, which proves the original statement.

Question 3.

a)

An Euler path is a path which uses each edge exactly once.

An Euler cycle is a closed path, which means that each edge is used exactly once as well, but the path also starts and ends on the same vertex.

b)

To find the maximum number of comparisons to be made to find any record in a binary search tree with 3000 records we need to find the height of this tree. In order to do that we can use one of the two formulas. For example: ceiling(log2(N+1)). Since we have 3000 records then following the formula we have: ceiling(log2(3001)) = 12.

Therefore, we need **at most 12 comparisons** to find any record in a binary search tree with 3000 records.

c)

i.

A path in a graph is a walk where no vertices or edges are repeated.

ii.

Step 1

Unvisited vertices: {A, B, C, D, E, F, G, H, I}

Vertex	**Minimum distance**	**Last visited**
A	inf	None
B	Inf	None
C	Inf	None
D	Inf	None
E	Inf	None
F	Inf	None
G	Inf	None
H	Inf	None
I	Inf	None

Step 2

Unvisited vertices: {B, C, D, E, F, G, H, I}

Vertex	**Minimum distance**	**Last visited**
A	0	
B	23	A
C	39	A
D	56	A
E	Inf	None
F	Inf	None
G	Inf	None
H	Inf	None
I	Inf	None

Step 3

Unvisited vertices: {C, D, E, F, G, H, I}

Vertex	**Minimum distance**	**Last visited**
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	Inf	None
G	Inf	None
H	Inf	None
I	Inf	None

Step 4

Unvisited vertices: {D, E, F, G, H, I}

Vertex	Minimum distance	Last visited
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	83	C
G	Inf	None
H	122	C
I	Inf	None

Step 5

Unvisited vertices: {E, F, G, H, I}

Vertex	Minimum distance	Last visited
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	~~83~~, 72	~~C~~, D
G	88	D
H	122	C
I	Inf	None

Step 6

Unvisited vertices: {F, G, H, I}

Vertex	Minimum distance	Last visited
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	~~83~~, 72	~~C~~, D
G	88	D
H	122	C
I	Inf	None

Step 7

Unvisited vertices: {G, H, I}

Vertex	Minimum distance	Last visited
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	~~83~~, 72	~~C~~, D
G	~~88~~, 87	~~D~~, F
H	~~122~~, 99	~~C~~, F
I	116	F

Step 8

Unvisited vertices: {H, I}

Vertex	Minimum distance	Last visited
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	~~83~~, 72	~~C~~, D
G	~~88~~, 87	~~D~~, F
H	~~122~~, 99	~~C~~, F
I	~~116~~, 108	~~F~~, G

Step 9

Unvisited vertices: {I}

Vertex	Minimum distance	Last visited
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	~~83~~, 72	~~C~~, D
G	~~88~~, 87	~~D~~, F
H	~~122~~, 99	~~C~~, F
I	~~116~~, 108	~~F~~, G

Step 10

Unvisited vertices: {}

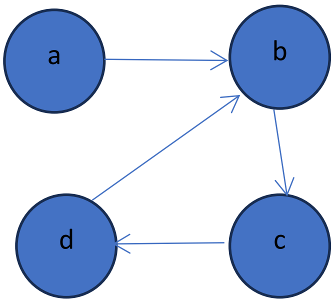
Vertex	**Minimum distance**	**Last visited**
A	0	
B	23	A
C	39	A
D	~~56,~~ 51	~~A~~, B
E	50	B
F	~~83~~, 72	~~C~~, D
G	~~88~~, 87	~~D~~, F
H	~~122~~, 99	~~C~~, F
I	~~116~~, 108	~~F~~, G

Answer: From the table we can deduce that the shortest path from A to I is:

A -> B -> D -> F -> G -> I with the associated cost of 108

d)

i.



ii.

Reflexive: R is not reflexive since not all elements are related to themselves (none of them in fact)

Symmetric: R is not symmetric since there are elements when xRy but not yRx. For example aRb but not bRa.

Anti-symmetric: R is anti-symmetric since for every connection xRy there is no corresponding connection yRx. Also there are no parallel edges in the graph above.

Transitive: R is not transitive since there exists a relation aRb and bRc but there is no aRc. Hence, the relation is not transitive iii.

In order to get the transitive closure we need to add the following relations: {(a,c), (a,d), (b,d)}

iv.

	A	B	C	d
A	0	1	0	0
B	0	0	1	0
C	0	0	0	1
D	0	1	0	0

e)

i.

$S(2) = 2^{1-1} + 2^{2-1} = 1 + 2 = 3$

ii.

1) Basis step S(1): $2^1 - 1 = 2^1 - 1$ => $1 = 1$ which confirms the basis step

2) Induction step: We assume that the hypothesis $S(n) = 2^n - 1$ is true.

Then: $(2^{1-1}) + (2^{2-1}) + \ldots + (2^{n-1}) = 2^n - 1$

Then: $(2^{1-1}) + (2^{2-1}) + \ldots + (2^{n-1}) + (2^{(n+1)-1}) = 2^{n+1} - 1$

Since $(2^{1-1}) + (2^{2-1}) + \ldots + (2^{n-1}) = 2^n - 1$

Then: $2^n - 1 + (2^{(n+1)-1}) = 2^{n+1} - 1$

Then: $2^n - 1 + 2^n = 2^{n+1} - 1$

Since $2^n + 2^n = 2^{n+1}$ we have

$2^{n+1} - 1 = 2^{n+1} - 1$ which concludes the proof of the hypothesis.