Rationals Close to Irrationals and the Pell Equations Proofs

Karim El Shenawy

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Introduction

This course notebook is the collection of theorem proofs, exercises and answers from Unit 9 of the Number Theory Through Inquiry (Mathematical Association of America Textbooks).

Theorems to Mark

9.14 Theorem

If the natural number N is a perfect square, then the Pell equation

$$x^2 - Ny^2 = 1$$

has no non-trivial integer solutions.

Proof. Given natural number N is a perfect square where $N = n^2$ then the Pell equation can be expressed as,

$$\begin{aligned} x^2 - Ny^2 &= 1 && x, y, n \in \mathbf{Z} \\ x^2 - n^2y^2 &= 1 \\ x^2 - (ny)^2 &= 1 \\ (x - ny)(x + ny) &= 1 \\ \implies (x - ny) = (x + ny) &= \pm 1 && \textbf{Considering } y \neq 0 \end{aligned}$$

Then we have 2 cases, by direct proof,

- Case 1: $(x - ny) = (x + ny) = 1$ with $y \neq 0$ then

$$\begin{aligned} (x - ny) = (x + ny) &= 1 \\ 2x = 2ny &= 1 \\ x = ny &= \frac{1}{2} \end{aligned}$$

  Thus, $x = \frac{1}{2}$ which is impossible since $x \in \mathbf{Z}$.

- Case 1: $(x-ny)=(x+ny)=-1$ with $y \neq 0$ then

$$\begin{aligned} (x-ny)=(x+ny) &= -1 \\ 2x=2ny &= -1 \\ x=ny &= \frac{-1}{2} \end{aligned}$$

Thus, $x=\frac{-1}{2}$ which is impossible since $x \in \mathbf{Z}$.

Therefore, the Pell equation has no trivial solutions. $\square$

9.17 Theorem

Suppose N is a natural number and the Pell equation $x^2-Ny^2=1$ has two solutions, namely, $a^2-Nb^2=1$ and $c^2-Nd^2=1$ for some integers a, b, c and d. Then $x=ac+Nbd$ and $y=ad+bc$ is also an integer solution to the Pell equation $x^2-Ny^2=1$. That is,

$$(ac+Nbd)^2-N(ad+bc)^2=1.$$

Proof. Given $N \in \mathbf{N}$ and the Pell equation $x^2-Ny^2=1$ has two solutions, namely, $a^2-Nb^2=1$ and $c^2-Nd^2=1$ with $a,b,c,d \in \mathbf{Z}$. Suppose that $x=ac+Nbd$ and $y=ad+bc$ is also an integer solution to the Pell equation $x^2-Ny^2=1$. Then, by direct proof,

$$\begin{aligned} x^2-Ny^2 &= (ac+Nbd)^2-N(ad+bc)^2 &&= 1 \\ &= (ac+Nbd)^2-N(ad+bc)^2 &&= 1 \\ &= a^2c^2+N^2b^2d^2+2Nabcd-N(a^2d^2+b^2c^2+2abcd) &&= 1 \\ &= a^2c^2+N^2b^2d^2+2Nabcd-Na^2d^2+Nb^2c^2+2Nabcd &&= 1 \\ &= a^2c^2+N^2b^2d^2-Na^2d^2+Nb^2c^2 &&= 1 \\ &= a^2(c^2-Nd^2)-Nb^2(c^2-Nd^2) &&= 1 \\ &= (c^2-Nd^2)(a^2-Nb^2) &&= 1 \end{aligned}$$

Therefore, $(c^2-Nd^2)=(a^2-Nb^2)=1$ or $(c^2-Nd^2)=(a^2-Nb^2)=-1$. Thus, $(ac+Nbd)^2-N(ad+bc)^2=1$ holds an integer solution to the Pell equation where $x=ac+Nbd$ and $y=ad+bc$. $\square$

9.19 Theorem

Let N be a natural number and suppose that x and y are positive integers satisfying $|x-y\sqrt{N}|<\frac{1}{y}$. Then

$$|x^2-Ny^2|<3\sqrt{N}.$$

Proof. Let $N \in \mathbf{N}$ and suppose $x, y \in \mathbf{Z}_+$ satisfying $|x - y\sqrt{N}| < \frac{1}{y}$, then $|x^2 - Ny^2| < 3\sqrt{N}$ by direct proof,

$$\begin{aligned}
|x^2 - Ny^2| &= |(x - \sqrt{N}y)(x + \sqrt{N}y)| \\
&< |(x - \sqrt{N}y)|(3y\sqrt{N}) && \text{By Theorem 9.18} \\
&< \frac{1}{y} \cdot 3y\sqrt{N} && \text{Since } |x - y\sqrt{N}| < \frac{1}{y} \\
&< 3\sqrt{N}
\end{aligned}$$

Therefore, $|x^2 - Ny^2| < 3\sqrt{N}$ holds. □

9.23 Theorem

If N is a positive integer that is not a square, then the Pell equation $x^2 - Ny^2 = 1$ has a non-trivial solution in positive integers.

Proof. Given N is a positive integer that is not a square, then the Pell equation $x^2 - Ny^2 = 1$ has a non-trivial solution in positive integers. We can use 3 cases by contradiction to prove this.

- Case 1: $N = -1$ Then the Pell equation $x^2 - Ny^2 = 1 \implies x^2 + y^2 = 1$ which leads to 4 trivial solutions: (1,0), (-1, 0), (0,1) and (0,-1).
- Case 2: $N < -1$ Then suppose $x \neq 0$ implies $x^2 - Ny^2 \geq 2$. Thus $x^2 - Ny^2 = 1$ has the trivial solutions the solutions of (1,0) and (-1,0).
- Case 3: $N = n^2$ Then $x^2 - Ny^2 = x^2 - n^2y^2 = (x - ny)(x + ny) = 1$. And thus by Theorem 9.14, the Pell equation has only trivial integer solutions.

Therefore, only if N is a positive integer that is not a square then the Pell equation $x^2 - Ny^2 = 1$ has a non-trivial solution in positive integers. □

9.27 Corollary

Let N be a natural number and r and s integers. If $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$, then so does α^k for any integer k.

Proof. Suppose $N \in \mathbf{N}$ and $r, s \in \mathbf{Z}$. Assume that $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$. Suppose integer k such that α^k holds a solution for the Pell equation. As k can be either negative or positive integer, we obtain two cases;

- Suppose $k < 0$, then by induction,
  Base case (k = -1):

$$\begin{aligned}
a^k &= a^{-1} \\
&= \frac{1}{\alpha}
\end{aligned}$$

By Theorem 9.26, α^k gives a solution when $k = -1$.
Inductive Hypothesis: Assume k = k - 1 since $k \in \mathbf{Z}, k < 0$
Inductive Step:

$$a^k = a^{k-1}$$
$$a^k = a^k a^{-1}$$
$$= a^k \frac{1}{\alpha}$$

By Theorem 9.26, we know that $\frac{1}{\alpha}$ gives a solution. Thus making a^k where $k < 0$ give a solution to $x^2 - Ny^2 = 1$.

- Suppose $k > 0$, then by induction,
  Base case (k = 1):

  $$a^k = a^1$$

  We know that α gives a solution.
  Inductive Hypothesis: Assume k = k + 1 since $k \in \mathbf{Z}, k > 0$
  Inductive Step:

  $$a^k = a^{k+1}$$
  $$a^k = a^k a$$

  We know that α gives a solution. Thus making a^k where $k > 0$ give a solution to $x^2 - Ny^2 = 1$.

Therefore, α^k is a solution for any integer k. □

9.29 Theorem

Let N be a positive integer that is not a square. Let A be the set of all real numbers of the form $r + s\sqrt{N}$, with r and s positive integers, that give solutions to $x^2 - Ny^2 = 1$. Then

1. *there is a smallest element α in A.*
2. *the real numbers α^k, $k = 1, 2, ...$ give all positive integer solutions to $x^2 - Ny^2 = 1$.*

Proof. Let N be a positive integer that is not a square. Let A be the set of all real numbers of the form $r + s\sqrt{N}$, with r and s positive integers, that give solutions to $x^2 - Ny^2 = 1$. Now,

1. By Theorem 9.23, we know that $x^2 - Ny^2 = 1$ has a non-trivial solution in positive integers. Thus, the set A cannot be empty. Moreover, since for any real number in the form of $r+s\sqrt{N}$, we can observe that $r \leq r+s\sqrt{N}$. Thus, by the Well-Ordering Axiom, the set of natural numbers has a minimal element and since it is ordered by the positive integer r, then there is a smallest element α in A.

2. Suppose that there is a smallest element α in A. Now let $\alpha = x_1 + y_1\sqrt{N}$. Since $\alpha \in A$ then $x_1^2 + y_1^2 N = 1$. By Corollary 9.27, we know that α^k, $k = 1, 2, ...$ gives all positive integer solutions to $x^2 - Ny^2 = 1$. Thus proven.

□

Practice Theorems from Rationals Close to Irrationals and the Pell Equations Proofs

9.1 Theorem

Let α be an irrational number and let b be a natural number. Then there exists an integer a such that

$$|\alpha - \tfrac{a}{b}| \leq \tfrac{1}{2b}.$$

Proof. Suppose α is an irrational number and let b be a natural number. If there exists an integer a such that

$$\begin{aligned}
|\alpha - \frac{a}{b}| &\leq \frac{1}{2b} \\
-\frac{1}{2b} \leq (\alpha - \frac{a}{b}) &\leq \frac{1}{2b} \\
-\frac{1}{2b} + \frac{a}{b} \leq (\alpha - \frac{a}{b} + \frac{a}{b}) &\leq \frac{1}{2b} + \frac{a}{b} \\
\frac{2a-1}{2b} \leq \alpha &\leq \frac{2a+1}{2b} \\
2a - 1 \leq 2b\alpha &\leq 2a + 1
\end{aligned}$$

then

$$2a - 1 \leq \ 2b\alpha \leq 2a + 1.$$

Let integer k be $2b\alpha$. Now k can either be odd or even.

- Case 1: If k is odd, then $k = 2a + 1$ for some integer a, then

  $$\begin{aligned}
  2a - 1 \leq 2b\alpha &\leq 2a + 1 \\
  2a - 1 \leq k &\leq 2a + 1 \\
  2a - 1 \leq 2a - 1 &\leq 2a + 1
  \end{aligned}$$

  Thus, it holds.

- Case 2: If k is even, then $k = 2a$ for some integer a, then

  $$\begin{aligned}
  2a - 1 \leq 2b\alpha &\leq 2a + 1 \\
  2a - 1 \leq k &\leq 2a + 1 \\
  2a - 1 \leq 2a &\leq 2a + 1
  \end{aligned}$$

  Thus, it holds as well.

Therefore, there exists an integer a such that

$$|\alpha - \tfrac{a}{b}| \leq \tfrac{1}{2b}.$$

$\square$

9.2 Exercise

Among the first eleven multiples of $\sqrt{2}$,

$$0\sqrt{2}, \sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, ..., 10\sqrt{2},$$

find the two whose difference is closest to a positive integer. Feel free to use a calculator. Use those two multiples to find a good rational approximation for $\sqrt{2}$. By good, we mean that you find integers a and b such that

$$|\tfrac{a}{b} - \sqrt{2}| \leq \tfrac{1}{b^2}.$$

Solution. We can notice using a calculator that $7\sqrt{2}$ and $2\sqrt{2}$ are the two whose difference is closest to a positive integer. Since $7\sqrt{2} - 2\sqrt{2} = 5\sqrt{2} \approx 7.07 \approx 7 \in \mathbf{Z}$.
Now, we can observe that $\sqrt{2} = \frac{7}{5} \approx 1.4 \Longrightarrow |\frac{7}{5} - \sqrt{2}| \approx 0.014213$ and that $\frac{1}{5^2} = 0.04$. Thus,

$$|\tfrac{7}{5} - \sqrt{2}| \leq \tfrac{1}{5^2}.$$

9.3 Exercise

Repeat the previous exercise for $\sqrt{7}$ using the first 13 multiples of $\sqrt{7}$.

Solution. We can notice using a calculator that $10\sqrt{7}$ and $7\sqrt{7}$ are the two whose difference is closest to a positive integer. Since $10\sqrt{7} - 7\sqrt{7} = 3\sqrt{7} \approx 7.937 \approx 8 \in \mathbf{Z}$.
Now, we can observe that $\sqrt{7} = \frac{8}{3} \approx 2.6667 \Longrightarrow |\frac{8}{3} - \sqrt{7}| \approx 0.0209$ and that $\frac{1}{3^2} = 0.111$. Thus,

$$|\tfrac{8}{3} - \sqrt{7}| \leq \tfrac{1}{3^2}.$$

9.4 Exercise

Repeat the previous exercise for π, using the first 15 multiples of π.

Solution. We can notice using a calculator that 14π and 7π are the two whose difference is closest to a positive integer. Since $14\pi - 7\pi = 7\pi \approx 21.99 \approx 22 \in \mathbf{Z}$.
Now, we can observe that $\pi = \frac{22}{7} \approx 3.1428 \Longrightarrow |\frac{22}{7} - \pi| \approx 0.0012644$ and that $\frac{1}{7^2} = 0.0204$. Thus,

$$|\tfrac{22}{7} - \pi| \leq \tfrac{1}{7^2}.$$

9.5 Exercise

Let α be an irrational number.

1. *Imagines making a list of the first 11 multiples of α. Can you predict how close to an integer the nearest difference between two of those numbers must be?* Yes, when the difference is under $\frac{1}{11}$.

2. *Now imagine making a list of 11 multiples of α, but not the first 11. Can you still predict how close to an integer the nearest difference between two of those numbers must be?* Yes, when the difference is under $\frac{1}{11}$ which is the number of multiples.

3. *Now imagine making a list of 50 multiples of α, rather than just 11. Can you predict how close to an integer the nearest difference between two of those numbers must be?* Yes, when the difference is under $\frac{1}{50}$.

4. *What is the general relationship between how many multiples of α we consider and how well we can rationally approximate α using our multiples?* When the difference is under $\frac{1}{\text{number of multiples of } \alpha}$.

9.6 Theorem

Let K be a positive integer. Then, among any K real numbers, there is a pair of them whose difference is within $\frac{1}{K}$ of being an integer.

Proof. Suppose K is a positive integer and that there exists the set of real numbers $A = \{a_1, a_2, a_3, ..., a_K\}$. No assume that each element a_i from the set A is ordered by the value b_i where b_i is the difference between a_i and it's approximation to an integer, say, c_i. Thus, $b_i = |a_i - c_i|$. Therefore, each b_i would also be ordered as $b_1, b_2, b_3, ..., b_K$ where

$$0 \leq b_1 \leq b_2 \leq b_3 \leq ... \leq b_K < 1.$$

We can notice that the sum of differences between consecutive b_i will sum to 1,

$$(b_2 - b_1) + (b_3 - b_2) + ... + (b_K - b_{K-1}) + (1 + b_1 - b_K) = 1$$

All these differences are also greater or equal to zero and at least one of them must be less than or equal to $\frac{1}{K}$ since

$$\begin{aligned} 0 < \tfrac{1}{K} - 1 < b_1 - b_K < 1 \\ 0 < 1 - \tfrac{1}{K} < b_K - b_1 < 1 \end{aligned}$$

□

9.7 Theorem

Let α be a positive irrational number and K be a positive integer. Then there exist positive integers a, b and c with $0 \leq a < b \leq K$ and $0 \leq c \leq K\alpha$ such that

$$|\tfrac{c}{b-a} - \alpha| \leq \tfrac{1}{(b-a)^2}.$$

Proof. Let α be a positive irrational number and K be a positive integer. Now suppose there exist positive integers a, b and c with $0 \leq a < b \leq K$ and $0 \leq c \leq K\alpha$. Using Theorem 9.6, we can find a and b since $0 \leq a < b \leq K$ and that $b - a = K$. Now again using Theorem 9.6 and by direct proof, we can say that

$$\begin{aligned}
|c - \alpha K| &\leq \frac{1}{K} \\
|c - \alpha(b-a)| &\leq \frac{1}{K} && K = (b-a) \\
|(c - \alpha(b-a)) \times \frac{1}{b-a}| &\leq \frac{1}{K(b-a)} && \text{dividing } (b-a) \\
|\frac{c}{b-a} - \alpha| &\leq \frac{1}{K(b-a)} \\
|\frac{c}{b-a} - \alpha| &\leq \frac{1}{(b-a)(b-a)} \\
|\frac{c}{b-a} - \alpha| &\leq \frac{1}{(b-a)^2} && K = (b-a)
\end{aligned}$$

Thus, we can express α, K, a, b, and c as

$$|\tfrac{c}{b-a} - \alpha| \leq \tfrac{1}{(b-a)^2}.$$

□

9.8 Theorem (Dirichlet's Rational Approximation Theorem, Version I)

Let α be a real number. Then there exist infinitely many rational numbers $\frac{a}{b}$ satisfying

$$|\tfrac{a}{b} - \alpha| \leq \tfrac{1}{b^2}.$$

Proof. Let α be a real number. Suppose, that there exist rational number $\frac{a}{b}$ satisfying

$$|\tfrac{a}{b} - \alpha| \leq \tfrac{1}{b^2}.$$

Since $|a - b\alpha|$ is always greater than 0 when α is irrational. We can assume that these exist integers $K_i, i \in \mathbf{Z}$ such that, by Theorem 9.6,

$$|\tfrac{a_i}{b_i} - \alpha| > \tfrac{1}{K_i}.$$

Now by Theorem 9.7, we know that $|\frac{c}{b-a} - \alpha| \leq \frac{1}{(b-a)^2}$ which only holds if $0 \leq a < b \leq K$ and $0 \leq c \leq K\alpha$. Suppose let $K_1 = 2$ then we can express Theorem 9.7 as,

$$\frac{1}{K_{i+1}} < |\frac{c_i}{b_i - a_i} - \alpha| \leq \frac{1}{K_i}.$$

Which we know becomes

$$|\frac{c_i}{b_i - a_i} - \alpha| \leq \frac{1}{(b_i - a_i)^2}.$$

This implies that we can find a_{i+1} and b_{i+1} such that the difference of $a_{i+1}\alpha$, $b_{i+1}\alpha$ from c_{i+1} is less than $\frac{1}{K_{i+1}}$. However, by

$$\frac{1}{K_{i+1}} < |\frac{c_i}{b_i - a_i} - \alpha| \leq \frac{1}{K_i}$$

a_{i+1} and b_{i+1} must be distinct from each of the previous a_i and b_i. Therefore, we have found another rationale number thus proving that there exist infinitely many rational numbers $\frac{a}{b}$. □

9.9 Theorem

Show that Versions I and II of Dirichlet's Rational Approximation Theorem can be deduced from one another.

Proof. By direct proof,

$$\begin{aligned}
|\frac{a}{b} - \alpha| &\leq \frac{1}{b^2} \\
|a - b\alpha| &\leq \frac{1b}{b^2} && \text{multiplying by b} \\
|a - b\alpha| &\leq \frac{1}{b}
\end{aligned}$$

□

9.10 Exercise

Show that if N is a natural number which is not a square and $x = a$ and $y = b$ is a positive integer solution to the Pell equation $x^2 - Ny^2 = 1$, then $\frac{a}{b}$ gives a good rational approximation to $\sqrt{N}$.

Proof. By direct proof,

$$\begin{aligned}
x^2 - Ny^2 &= 1 \\
\frac{x^2}{y^2} - N &= \frac{1}{y^2} \\
\frac{a}{b}^2 - N &= \frac{1}{b^2} \\
(\frac{a}{b} - \sqrt{N})(\frac{a}{b} + \sqrt{N}) &= \frac{1}{b^2}
\end{aligned}$$

For this statement to be true, both $(\frac{a}{b} - \sqrt{N})$ and $(\frac{a}{b} + \sqrt{N})$ must be equal or less than $\frac{1}{b^2}$. Thus, by Dirichlet's Rationale approximation this holds. $\square$

9.11 Theorem

Let N be a positive integer that is not a square. If $x = a$ and $y = b$ is a solution in positive integers to $x^2 - Ny^2 = 1$, then

$$|\tfrac{a}{b} - \sqrt{N}| \leq \tfrac{1}{b^2}.$$

Proof. Let N be a positive integer that is not a square. Suppose $x = a$ and $y = b$ is a solution in positive integers to $x^2 - Ny^2 = 1$, then

$$\begin{aligned} x^2 - Ny^2 &= 1 \\ a^2 - Nb^2 &= 1 \\ (a - b\sqrt{N})(a + b\sqrt{N}) &= 1 \end{aligned}$$

Then we know that $\implies a - b\sqrt{N} = \frac{1}{a - b\sqrt{N}} > 0$ thus $a > b\sqrt{N})$. Therefore, by direct proof,

$$\begin{aligned} |\frac{a}{b} - \sqrt{N}| &= \frac{a - b\sqrt{N}}{b} \\ &= \frac{1}{b(a + b\sqrt{N})} \\ &< \frac{1}{b(b\sqrt{N} + b\sqrt{N})} \\ &< \frac{1}{2b^2\sqrt{N}} \\ &= \frac{1}{2b^2\sqrt{N}} \\ &< \frac{1}{2b^2} \\ &< \frac{1}{b^2} \end{aligned}$$

Thus,

$$|\tfrac{a}{b} - \sqrt{N}| \leq \tfrac{1}{b^2}.$$

$\square$

9.12 Question

For every natural number N, there are some trivial values of x and y that satisfy the Pell equation $x^2 - Ny^2 = 1$. What are those trivial solutions?

Solution. Let us express the Pell equation $x^2 - Ny^2 = 1$ as $x \cdot x - Ny \cdot y = 1$. Then there exist integers a and b where $a = x$ and $b = Ny$ with $gcd(a, b) = 1$. Therefore, the trivial solutions can be

- $x = 1$ and $y = 0$
- $x = -1$ and $y = 0$

9.13 Question

For what values of natural number N can you easily show that there are non-trivial solutions to the Pell equation $x^2 - Ny^2 = 1$?

Solution. For values of 9, 16, 25, 36 and 49. Thus perfect squares.

9.14 Theorem

If the natural number N is a perfect square, then the Pell equation

$$x^2 - Ny^2 = 1$$

has no non-trivial integer solutions.

Proof. Given natural number N is a perfect square where $N = n^2$ then the Pell equation can be expressed as,

$$\begin{aligned} x^2 - Ny^2 &= 1 && x, y, n \in \mathbf{Z} \\ x^2 - n^2y^2 &= 1 \\ x^2 - (ny)^2 &= 1 \\ (x - ny)(x + ny) &= 1 \\ \implies (x - ny) = (x + ny) &= \pm 1 && \textbf{Considering } y \neq 0 \end{aligned}$$

Then we have 2 cases, by direct proof,

- Case 1: $(x - ny) = (x + ny) = 1$ with $y \neq 0$ then

  $$\begin{aligned} (x - ny) = (x + ny) &= 1 \\ 2x = 2ny &= 1 \\ x = ny &= \frac{1}{2} \end{aligned}$$

  Thus, $x = \frac{1}{2}$ which is impossible since $x \in \mathbf{Z}$.

- Case 1: $(x - ny) = (x + ny) = -1$ with $y \neq 0$ then

  $$\begin{aligned} (x - ny) = (x + ny) &= -1 \\ 2x = 2ny &= -1 \\ x = ny &= \frac{-1}{2} \end{aligned}$$

  Thus, $x = \frac{-1}{2}$ which is impossible since $x \in \mathbf{Z}$.

Therefore, the Pell equation has no trivial solutions. □

9.15 Exercise

Find, by trial and error, at least two non-trivial solutions to each of the Pell equations $x^2 - 2y^2 = 1$ and $x^2 - 3y^2 = 1$.

Solution.

- $x^2 - 2y^2 = 1$
  - with $x = 3, y = 2$ then $x^2 - 2y^2 = 3^2 - 2(2^2) = 9 - 8 = 1$.
  - with $x = 17, y = 12$ then $x^2 - 2y^2 = 17^2 - 2(12^2) = 289 - 288 = 1$.
- $x^2 - 3y^2 = 1$
  - with $x = 2, y = 1$ then $x^2 - 3y^2 = 2^2 - 3(1^2) = 4 - 3 = 1$.
  - with $x = 7, y = 4$ then $x^2 - 3y^2 = 7^2 - 3(4^2) = 49 - 48 = 1$.

9.16 Question

To know all the integer solutions to a Pell equation, why does it suffice to know just the positive integer solutions?

Solution. Suppose that all integers solutions (positive or negative) are known, then the Pell equation, $x^2 - Ny^2 = 1$ where $\exists x, y \in \mathbf{Z}$, can be,

- $x^2 - N(-y)^2 = x^2 - Ny^2 = 1$ where all solutions are in form $(x, -y)$;
- $(-x)^2 - Ny^2 = x^2 - Ny^2 = 1$ where all solutions are in form $(-x, y)$;
- $(-x)^2 - N(-y)^2 = x^2 - Ny^2 = 1$ where all solutions are in form $(-x, -y)$;

9.17 Theorem

Suppose N is a natural number and the Pell equation $x^2 - Ny^2 = 1$ has two solutions, namely, $a^2 - Nb^2 = 1$ and $c^2 - Nd^2 = 1$ for some integers a, b, c and d. Then $x = ac + Nbd$ and $y = ad + bc$ is also an integer solution to the Pell equation $x^2 - Ny^2 = 1$. That is,

$$(ac + Nbd)^2 - N(ad + bc)^2 = 1.$$

Proof. Given $N \in \mathbf{N}$ and the Pell equation $x^2 - Ny^2 = 1$ has two solutions, namely, $a^2 - Nb^2 = 1$ and $c^2 - Nd^2 = 1$ with $a, b, c, d \in \mathbf{Z}$. Suppose that $x = ac + Nbd$ and $y = ad + bc$ is also an integer solution to the Pell equation

$x^2 - Ny^2 = 1$. Then, by direct proof,

$$\begin{aligned}
x^2 - Ny^2 &= (ac + Nbd)^2 - N(ad + bc)^2 &= 1\\
&= (ac + Nbd)^2 - N(ad + bc)^2 &= 1\\
&= a^2c^2 + N^2b^2d^2 + 2Nabcd - N(a^2d^2 + b^2c^2 + 2abcd) &= 1\\
&= a^2c^2 + N^2b^2d^2 + 2Nabcd - Na^2d^2 + Nb^2c^2 + 2Nabcd &= 1\\
&= a^2c^2 + N^2b^2d^2 - Na^2d^2 + Nb^2c^2 &= 1\\
&= a^2(c^2 - Nd^2) - Nb^2(c^2 - Nd^2) &= 1\\
&= (c^2 - Nd^2)(a^2 - Nb^2) &= 1
\end{aligned}$$

Therefore, $(c^2 - Nd^2) = (a^2 - Nb^2) = 1$ or $(c^2 - Nd^2) = (a^2 - Nb^2) = -1$. Thus, $(ac + Nbd)^2 - N(ad + bc)^2 = 1$ holds an integer solution to the Pell equation where $x = ac + Nbd$ and $y = ad + bc$. □

9.18 Theorem

Let N be a natural number and suppose that x and y are positive integers satisfying $|x - y\sqrt{N}| < \frac{1}{y}$. Then

$$x + y\sqrt{N} < 3y\sqrt{N}.$$

Proof. Let $N \in \mathbf{N}$ and suppose $x, y \in \mathbf{Z}_+$ satisfying $|x - y\sqrt{N}| < \frac{1}{y}$, then by direct proof,

$$\begin{aligned}
|x - y\sqrt{N}| &< \frac{1}{y}\\
\frac{x}{y} - \sqrt{N} &< \frac{1}{y^2}\\
\text{Since } y \geq 1 \implies \frac{1}{y^2} &\leq \frac{1}{y} \leq 1 :\\
\frac{x}{y} < \frac{1}{y^2} + \sqrt{N} &< 1 + \sqrt{N}\\
\text{As } 1 \leq \sqrt{N}, N \in \mathbf{N} :\\
\frac{x}{y} &< \sqrt{N} + \sqrt{N}\\
x &< 2y\sqrt{N}\\
x + y\sqrt{N} &< 3y\sqrt{N}
\end{aligned}$$

□

9.19 Theorem

Let N be a natural number and suppose that x and y are positive integers satisfying $|x - y\sqrt{N}| < \frac{1}{y}$. Then

$$|x^2 - Ny^2| < 3\sqrt{N}.$$

Proof. Let $N \in \mathbf{N}$ and suppose $x, y \in \mathbf{Z}_+$ satisfying $|x - y\sqrt{N}| < \frac{1}{y}$, then $|x^2 - Ny^2| < 3\sqrt{N}$ by direct proof,

$$\begin{aligned} |x^2 - Ny^2| &= |(x - \sqrt{N}y)(x + \sqrt{N}y)| \\ &< |(x - \sqrt{N}y)|(3y\sqrt{N}) && \text{By Theorem 9.18} \\ &< \frac{1}{y} \cdot 3y\sqrt{N} && \text{Since } |x - y\sqrt{N}| < \frac{1}{y} \\ &< 3\sqrt{N} \end{aligned}$$

Therefore, $|x^2 - Ny^2| < 3\sqrt{N}$ holds. ☐

9.20 Theorem

There exists a non-zero integer K such that the equation

$$x^2 - Ny^2 = K$$

has infinitely many solutions in positive integers.

Proof. Let there exist a non-zero integer K such that the equation $x^2 - Ny^2 = K$. Now we can express it as $x^2 = K + -Ny^2$. Now this is a linear Diophantine equation. Since $gcd(1, N) = 1$, this implies N divides K. Thus there are infinitely many solutions to x^2 and y^2 since there are infinitely many perfect squares among them. This is guaranteed if N divides K, thus a suitable K must be chosen. ☐

9.21 Lemma

Let n be a natural number and suppose that $(x_i, y_i), i = 1, 2, 3, ...$ are infinitely many ordered pairs of integers. Then there exist distinct natural numbers j and k such that

$$x_j \equiv x_k \pmod{n} \text{ and } y_j \equiv y_k \pmod{n}.$$

Proof. Let $n, j, k \in \mathbf{N}$ and suppose that $(x_i, y_i), i = 1, 2, 3, ...$ are infinitely many ordered pairs of integers. Suppose we choose (x_j, y_j) and (x_k, y_k) where $j \neq k$ and where $x_k = x_j + n$ and $y_k = y_j + n$. This implies $(x_k, y_k) = (x_j + n, y_j + n)$ which is not equal to (x_j, y_j). However, $x_j + n \equiv x_j \pmod{n}$ and $y_j + n \equiv y_j \pmod{n}$. Therefore, $x_j \equiv x_k \pmod{n}$ and $y_j \equiv y_k \pmod{n}$. ☐

9.22 Lemma

Let N be a natural number and K be a non-zero integer and let (x_i, y_j) and (x_k, y_k) be distinct integer solutions to $x^2 - Ny^2 = K$ satisfying

$$x_j \equiv x_k \ (\text{mod } |K|) \text{ and } y_j \equiv y_k \ (\text{mod } |K|).$$

Then

$$x = \frac{x_j x_k - y_j y_k N}{K} \text{ and } y = \frac{x_j y_k - x_k y_i}{K}$$

are integers satisfying $x^2 - Ny^2 = 1$.

Proof. Incomplete. ☐

9.23 Theorem

If N is a positive integer that is not a square, then the Pell equation $x^2 - Ny^2 = 1$ *has a non-trivial solution in positive integers.*

Proof. Given N is a positive integer that is not a square, then the Pell equation $x^2 - Ny^2 = 1$ has a non-trivial solution in positive integers. We can use 3 cases by contradiction to prove this.

- Case 1: $N = -1$ Then the Pell equation $x^2 - Ny^2 = 1 \Longrightarrow x^2 + y^2 = 1$ which leads to 4 trivial solutions: (1,0), (-1, 0), (0,1) and (0,-1).
- Case 2: $N < -1$ Then suppose $x \neq 0$ implies $x^2 - Ny^2 \geq 2$. Thus $x^2 - Ny^2 = 1$ has the trivial solutions the solutions of (1,0) and (-1,0).
- Case 3: $N = n^2$ Then $x^2 - Ny^2 = x^2 - n^2y^2 = (x - ny)(x + ny) = 1$. And thus by Theorem 9.14, the Pell equation has only trivial integer solutions.

Therefore, only if N is a positive integer that is not a square then the Pell equation $x^2 - Ny^2 = 1$ has a non-trivial solution in positive integers. ☐

9.24 Exercise

Follow the steps of the preceding theorems to find several solutions to the Pell equations $x^2 - 5y^2 = 1$ *and* $x^2 - 6y^2 = 1$ *and then give some good rational approximations to* $\sqrt{5}$ *and* $\sqrt{6}$.

Solutions.

- $x^2 - 5y^2 = 1 \Longrightarrow y^2 = \frac{x^2-1}{5}$
  - x must be a positive integer
  - $\frac{x^2-1}{5}$ must be an integer square
  - $x^2 - 1$ is even, then x must be odd

  After trial and error we can conclude to $x = 9$ and $y = \sqrt{\frac{9^2-1}{5}} = 4$. Thus (9,4) is a non-trivial solution.
- $x^2 - 6y^2 = 1 \Longrightarrow y^2 = \frac{x^2-1}{6}$
  - x must be a positive integer
  - $\frac{x^2-1}{6}$ must be an integer square

- $x^2 - 1$ is even, then x must be odd

After trial and error we can conclude to $x = 5$ and $y = \sqrt{\frac{5^2-1}{6}} = 2$. Thus (5,2) is a non-trivial solution.

9.25 Theorem

Let N be a natural number and $r_1, r_2, s_1,$ and s_2 be integers. If $\alpha = r_1 + s_1\sqrt{N}$ and $\beta = r_2 + s_2\sqrt{N}$ both give solutions to the Pell equation $x^2 - Ny^2 = 1$, then so does $\alpha\beta$.

Proof. Let $N \in \mathbf{N}$ and $r_1, r_2, s_1, s_2 \in \mathbf{Z}$. Now suppose that $\alpha = r_1 + s_1\sqrt{N}$ and $\beta = r_2 + s_2\sqrt{N}$ both give solutions to the Pell equation $x^2 - Ny^2 = 1$. Now let's compute the $\alpha\beta$,

$$\begin{aligned}\alpha\beta &= (r_1 + s_1\sqrt{N})(r_2 + s_2\sqrt{N}) \\ &= r_1r_2 + s_1s_2N + \sqrt{N}(r_2s_1 + r_1s_2)\end{aligned}$$

Suppose that $\alpha\beta$ holds the solution to the Pell equation such that $x = r_1r_2 + s_1s_2N$ and $y = r_2s_1 + r_1s_2$. Then the solution to $x^2 - Ny^2 = 1$ would be

$$\begin{aligned}x^2 - Ny^2 &= 1 \\ (r_1r_2 + s_1s_2N)^2 - N(r_2s_1 + r_1s_2)^2 &= 1 \\ r_1^2r_2^2 + s_1^2s_2^2N^2 + 2r_1r_2s_1s_2N - Nr_2^2s_1^2 - Nr_1^2s_2^2 - 2r_1r_2s_1s_2N &= 1 \\ r_1^2r_2^2 + s_1^2s_2^2N^2 - Nr_2^2s_1^2 - Nr_1^2s_2^2 &= 1 \\ r_1^2(r_2^2 - Ns_2^2) - Ns_1^2(r_2^2 - Ns_2^2) &= 1 \\ (r_1^2 - Ns_1^2)(r_2^2 - Ns_2^2) &= 1\end{aligned}$$

Thus by definition, $\alpha\beta$ give solutions to the Pell equation $x^2 - Ny^2 = 1$. □

9.26 Theorem

Let N be a natural number and r and s integers. If $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$, then so does $\frac{1}{\alpha}$.

Proof. Suppose $N \in \mathbf{N}$ and $r, s \in \mathbf{Z}$. Assume that $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$. Then $\frac{1}{\alpha}$ can be expressed as

$$\begin{aligned}
\frac{1}{\alpha} &= \frac{1}{r + s\sqrt{N}} \\
&= \frac{1}{r + s\sqrt{N}} \cdot \frac{r - s\sqrt{N}}{r - s\sqrt{N}} \\
&= \frac{r - s\sqrt{N}}{r^2 - s^2 N} \\
&= \frac{r - s\sqrt{N}}{1} && \text{By definition } r^2 - s^2 N = 1 \\
&= r - s\sqrt{N}
\end{aligned}$$

By direct proof, we can demonstrate that $\frac{1}{\alpha} = r - s\sqrt{N}$ is a solution;

$$\begin{aligned}
\frac{1}{\alpha} &= r - s\sqrt{N} \\
1 &= \alpha \cdot r - s\sqrt{N} \\
1 &= (r + s\sqrt{N}) \cdot (r - s\sqrt{N}) \\
1 &= r^2 - s^2 N
\end{aligned}$$

Thus, $\frac{1}{\alpha}$ is a solution. $\square$

9.27 Corollary

Let N be a natural number and r and s integers. If $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$, then so does α^k for any integer k.

Proof. Suppose $N \in \mathbf{N}$ and $r, s \in \mathbf{Z}$. Assume that $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$. Suppose integer k such that α^k holds a solution for the Pell equation. As k can be either negative or positive integer, we obtain two cases;

- Suppose $k < 0$, then by induction,
  Base case (k = -1):

  $$\begin{aligned}
  a^k &= a^{-1} \\
  &= \frac{1}{\alpha}
  \end{aligned}$$

  By Theorem 9.26, α^k gives a solution when $k = -1$.
  Inductive Hypothesis: Assume k = k - 1 since $k \in \mathbf{Z}, k < 0$
  Inductive Step:

  $$\begin{aligned}
  a^k &= a^{k-1} \\
  a^k &= a^k a^{-1} \\
  &= a^k \frac{1}{\alpha}
  \end{aligned}$$

By Theorem 9.26, we know that $\frac{1}{\alpha}$ gives a solution. Thus making a^k where $k < 0$ give a solution to $x^2 - Ny^2 = 1$.

- Suppose $k > 0$, then by induction,
  Base case (k = 1):

$$a^k = a^1$$

  We know that α gives a solution.
  Inductive Hypothesis: Assume k = k + 1 since $k \in \mathbf{Z}, k > 0$
  Inductive Step:

$$a^k = a^{k+1}$$
$$a^k = a^k a$$

  We know that α gives a solution. Thus making a^k where $k > 0$ give a solution to $x^2 - Ny^2 = 1$.

Therefore, α^k is a solution for any integer k. □

9.28 Exercise

Let N be a natural number and r and s integers. Show that if $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$, then so does each of

$$r - s\sqrt{N}, -r + s\sqrt{N}, \text{ and } -r - s\sqrt{N}.$$

Solution. If $\alpha = r + s\sqrt{N}$ gives a solution to $x^2 - Ny^2 = 1$ then

- With $r - s\sqrt{N} = r + (-s)\sqrt{N}$:

$$x^2 - Ny^2 = 1$$
$$r^2 - Ns^2 = 1$$
$$r^2 - N(-s)^2 = 1$$

  Thus, $r - s\sqrt{N}$ is a solution.

- With $-r + s\sqrt{N}$:

$$x^2 - Ny^2 = 1$$
$$r^2 - Ns^2 = 1$$
$$(-r)^2 - Ns^2 = 1$$

  Thus, $-r + s\sqrt{N}$ is a solution.

- With $-r - s\sqrt{N} = -r + (-s)\sqrt{N}$:

$$x^2 - Ny^2 = 1$$
$$r^2 - Ns^2 = 1$$
$$(-r)^2 - N(-s)^2 = 1$$

  Thus, $-r - s\sqrt{N}$ is a solution.

9.29 Theorem

Let N be a positive integer that is not a square. Let A be the set of all real numbers of the form $r + s\sqrt{N}$, with r and s positive integers, that give solutions to $x^2 - Ny^2 = 1$. Then

1. *there is a smallest element α in A.*
2. *the real numbers α^k, $k = 1, 2, ...$ give all positive integer solutions to $x^2 - Ny^2 = 1$.*

Proof. Let N be a positive integer that is not a square. Let A be the set of all real numbers of the form $r + s\sqrt{N}$, with r and s positive integers, that give solutions to $x^2 - Ny^2 = 1$. Now,

1. By Theorem 9.23, we know that $x^2 - Ny^2 = 1$ has a non-trivial solution in positive integers. Thus, the set A cannot be empty. Moreover, since for any real number in the form of $r+s\sqrt{N}$, we can observe that $r \leq r+s\sqrt{N}$. Thus, by the Well-Ordering Axiom, the set of natural numbers has a minimal element and since it is ordered by the positive integer r, then there is a smallest element α in A.
2. Suppose that there is a smallest element α in A. Now let $\alpha = x_1 + y_1\sqrt{N}$. Since $\alpha \in A$ then $x_1^2 + y_1^2 N = 1$. By Corollary 9.27, we know that α^k, $k = 1, 2, ...$ gives all positive integer solutions to $x^2 - Ny^2 = 1$. Thus proven.

$\square$

9.30 Blank Paper Exercise

- Diophantine Approximation
- Rational Approximation
- Dirichlet's Rational Approximation
- Pell Equation solutions (Trivial and non-Trivial)
- Bovine Math