# Chapter 5

# Semantics of Propositional Logic

The set $B = \{0, 1\}$ is called the set of boolean values (also called truth values). The value 0 denotes falsehood and the value 1 denotes truth.

The function $\overline{\phantom{x}} : B \to B$ is called *logical negation* and is defined as follows: $\overline{0} = 1$ and $\overline{1} = 0$.

The function $+ : B \times B \to B$ is called *logical disjunction* and is defined as follows: $0 + 0 = 0, 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 1$.

The function $\cdot : B \times B \to B$ is called *logical conjunction* and is defined as follows: $0 \cdot 0 = 0, 0 \cdot 1 = 0, 1 \cdot 0 = 1, 1 \cdot 1 = 1$.

The tuple $(B, +, \cdot, \overline{\phantom{x}})$ is a *boolean algebra*.

## 5.1 Assignments

A *truth assignment* (or simply *assignment* from hereon) is any function $\tau : A \to B$. In other words, an assignment is a function that associates to any propositional variable a truth value.

**Example 25.** *Let $\tau_1 : A \to B$ be a function defined as follows:*

1. $\tau_1(\mathtt{p}) = 1$;
2. $\tau_1(\mathtt{q}) = 0$;
3. $\tau_1(\mathtt{r}) = 1$;
4. $\tau_1(a) = 0$ *for all* $a \in A \setminus \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$.

*As it is a function from $A$ to $B$, $\tau_1$ is a truth assignment.*

**Example 26.** *Let $\tau_2 : A \to B$ be a function defined as follows:*

1. $\tau_2(\mathtt{p}) = 0$;
2. $\tau_2(\mathtt{q}) = 0$;
3. $\tau_2(\mathtt{r}) = 1$;
4. $\tau_2(a) = 1$ *for all* $a \in A \setminus \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$.

*As it is a function from $A$ to $B$, $\tau_2$ is also a truth assignment.*

**Example 27.** *Let $\tau_3 : A \to B$ be a function defined as follows:*

1. $\tau_3(a) = 0$ *for all* $a \in A$.

*As it is a function from $A$ to $B$, $\tau_3$ is also a truth assignment.*

## 5.2 Truth Value of A Formula in An Assignment

The truth value of a formula $\varphi$ in an assignment $\tau$ is denoted by $\hat{\tau}\Big(\varphi\Big)$ and is defined recursively as follows:

$$\hat{\tau}\Big(\varphi\Big) = \begin{cases} \tau(\varphi), & \text{if } \varphi \in A; \\ \overline{\hat{\tau}\Big(\varphi'\Big)}, & \text{if } \varphi = \neg\varphi' \text{ and } \varphi' \in \mathbb{PL}; \\ \hat{\tau}\Big(\varphi_1\Big) \cdot \hat{\tau}\Big(\varphi_2\Big), & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \text{ and } \varphi_1, \varphi_2 \in \mathbb{PL}; \\ \hat{\tau}\Big(\varphi_1\Big) + \hat{\tau}\Big(\varphi_2\Big), & \text{if } \varphi = (\varphi_1 \vee \varphi_2) \text{ and } \varphi_1, \varphi_2 \in \mathbb{PL}. \end{cases}$$

In fact $\hat{\tau} : \mathbb{PL} \to B$ is a function called *the homomorphic extension* of the assignment $\tau : A \to B$ to the entire set of formulae $\mathbb{PL}$, but do not feel obligated to recall this name.

Here is an example of how to compute the truth value of the formula $(\mathtt{p} \vee \mathtt{q})$ in the truth assignment $\tau_1$:

$\hat{\tau}_1\Big((\mathtt{p} \vee \mathtt{q})\Big) = \hat{\tau}_1\Big(\mathtt{p}\Big) + \hat{\tau}_1\Big(\mathtt{q}\Big) = \tau_1(\mathtt{p}) + \tau_1(\mathtt{q}) = 1 + 0 = 1.$

We conclude that the truth value of $(\mathtt{p} \vee \mathtt{q})$ in $\hat{\tau}_1$ is 1.

Here is another example, where we compute the truth value of the formula $\neg(\mathtt{p} \wedge \mathtt{q})$ in the truth assignment $\tau_1$:

$\hat{\tau}_1\Big(\neg(\mathrm{p} \wedge \mathrm{q})\Big) = \overline{\hat{\tau}_1\Big((\mathrm{p} \wedge \mathrm{q})\Big)} = \overline{\hat{\tau}_1\Big(\mathrm{p}\Big) \cdot \hat{\tau}_1\Big(\mathrm{q}\Big)} = \overline{\tau_1(\mathrm{p}) \cdot \tau_1(\mathrm{q})} = \overline{1 \cdot 0} = \overline{0} = 1.$

We conclude that the truth value of the formula $\neg(\mathrm{p} \wedge \mathrm{q})$ in $\tau_1$ is 1.

Here is yet another example, where we compute the truth value of the formula $\neg\neg\mathrm{q}$ in the truth assignment $\tau_2$:

$\hat{\tau}_2\Big(\neg\neg\mathrm{q}\Big) = \overline{\hat{\tau}_2\Big(\neg\mathrm{q}\Big)} = \overline{\overline{\hat{\tau}_2\Big(\mathrm{q}\Big)}} = \overline{\overline{\tau_2(\mathrm{q})}} = \overline{\overline{0}} = \overline{1} = 0.$

So the truth value of $\neg\neg\mathrm{q}$ in $\tau_2$ is 0.

**Remark.** *Important!*

*It does not make sense to say "the truth value of a formula". It makes sense to say "the truth value of a formula in an assignment".*

*Also, it does not make sense to say "the formula is true" or "the formula is false". Instead, it makes sense to say "this formula is true/false in this assignment".*

*This is because a formula could be true in an assignment but false in another. For example, the formula $\neg\neg\mathrm{p}$ is true in $\tau_1$, but false in $\tau_2$.*

**Definition 28** (Satisfaction)**.** *An* assignment $\tau$ satisfies $\varphi$ *if* $\hat{\tau}\Big(\varphi\Big) = 1$.

Instead of $\tau$ *satisfies* $\varphi$, we may equivalently say any of the following:

1. $\tau$ is a model of $\varphi$;

2. $\tau$ is true of $\varphi$;

3. $\varphi$ holds in/at $\tau$;

4. $\tau$ makes $\varphi$ true;

We write $\tau \models \varphi$ (and we read: $\tau$ is a model of the formula $\varphi$; or: $\tau$ satisfies $\varphi$, etc.) iff $\hat{\tau}\Big(\varphi\Big) = 1$. We write $\tau \not\models \varphi$ (and we read: $\tau$ is not a model of the formula $\varphi$; or: $\tau$ does not satisfy $\varphi$) iff $\hat{\tau}\Big(\varphi\Big) = 0$.

**Definition 29** (The satisfaction relation (written $\models$))**.** *The relation $\models$, between assignments and formulae, is called the* satisfaction relation*. It is defined as follows: $\tau \models \varphi$ iff $\hat{\tau}\Big(\varphi\Big) = 1$.*

**Example 30.** *The assignment $\tau_1$ defined above is a model for $\neg(\mathrm{p} \wedge \mathrm{q})$. The assignment $\tau_1$ is not a model of the formula $(\neg\mathrm{p} \wedge \mathrm{q})$.*

## 5.3 Satisfiability

**Definition 31.** *A formula $\varphi$ is* satisfiable *if, by definition, there exists at least an assignment $\tau$ such that $\tau \models \varphi$ (i.e., if there exists at least a model of $\varphi$).*

**Example 32.** *The formula $(p \vee q)$ is satisfiable, since it has a model (for example $\tau_1$ above).*

**Example 33.** *The formula $\neg p$ is also satisfiable: for example, the assignment $\tau_3$ above makes the formula true.*

**Example 34.** *The formula $(p \wedge \neg p)$ is not satisfiable, since it is false in any assignment.*

**Proof:** *Let us consider an arbitrary assignment $\tau : A \to B$.*

*We have that $\hat{\tau}\Big((p \wedge \neg p)\Big) = \hat{\tau}\Big(p\Big) \cdot \hat{\tau}\Big(\neg p\Big) = \tau(p) \cdot \overline{\hat{\tau}\Big(p\Big)} = \tau(p) \cdot \overline{\tau(p)}$.*
*But $\tau(p)$ can be either $0$ or $1$:*

1. *in the first case ($\tau(p) = 0$), we have that $\hat{\tau}\Big((p \wedge \neg p)\Big) = \ldots = \tau(p) \cdot \overline{\tau(p)} = 0 \cdot \overline{0} = 0 \cdot 1 = 0$;*
2. *in the second case ($\tau(p) = 1$) , we have that $\hat{\tau}\Big((p \wedge \neg p)\Big) = \ldots = \tau(p) \cdot \overline{\tau(p)} = 1 \cdot \overline{1} = 1 \cdot 0 = 0$.*

*Therefore, in any case, we have that $\hat{\tau}\Big((p \wedge \neg p)\Big) = 0$. But $\tau$ was chosen arbitrarily, and therefore the result must hold for any assignment $\tau$: $(p \wedge \neg p)$ is false in any assignment, which means that it is not satisfiable.*

*q.e.d.*

**Definition 35** (Contradiction)**.** *A formula that is not satisfiable is called a* contradiction.

**Example 36.** *As we have seen above, $(p \wedge \neg p)$ is a contradiction.*

## 5.4 Valid Formulae

**Definition 37.** *A formula $\varphi$ is* valid *if, by definition, any assignment $\tau$ has the property that $\hat{\tau}\Big(\varphi\Big) = 1$ (any assignment is a model of the formula).*

We sometimes write $\models \varphi$ instead of $\varphi$ is valid.

**Definition 38** (Tautology)**.** *A valid formula is also called a* tautology*.*

**Example 39.** *The formula* $(\mathrm{p} \vee \neg\mathrm{p})$ *is valid, because it is true in any assignment: let* $\tau$ *be an arbitrary assignment; we have that* $\hat{\tau}\Big((\mathrm{p} \vee \neg\mathrm{p})\Big) = \tau(\mathrm{p}) + \overline{\tau(\neg\mathrm{p})}$*, which is either* $0+1$ *or* $1+0$*, which is* $1$ *in any case.*

**Example 40.** *The formula* $\mathrm{p}$ *is not valid (because there is an assignment (for example* $\tau_3$*) that makes it false).*

## 5.5 Contingent Formulae

**Definition 41.** *A formula that is neither a contradiction nor a tautology is called* contingent*.*

**Example 42.** *Examples of formulas of each type are shown here:*

1. $(\mathrm{p} \wedge \neg\mathrm{p})$ *is a contradiction.*
2. $\mathrm{p}$ *is contingent.*
3. $(\mathrm{p} \vee \neg\mathrm{p})$ *is a tautology.*

## 5.6 Equivalence

**Definition 43.** *We say that two formulae* $\varphi_1, \varphi_2 \in \mathbb{PL}$ *are* equivalent *and we write* $\varphi_1 \equiv \varphi_2$ *if, for any assignment* $\tau : A \to B$*,* $\hat{\tau}\Big(\varphi_1\Big) = \hat{\tau}\Big(\varphi_2\Big)$*.*

Intuitively, formulae that are equivalent have the same meaning (express the same thing).

**Example 44.** *At the start of the course, someone asked whether* $\mathrm{p}$ *and* $\neg\neg\mathrm{p}$ *are equal. Of course not, I said, since one has* $1$ *symbol and the other* $3$ *symbols from the alphabet.*

*However, we are now ready to understand the relation between them:* $\mathrm{p} \equiv \neg\neg\mathrm{p}$*. In other words, even if they are not equal, they are equivalent: they express the same thing.*

*To prove* $\mathrm{p} \equiv \neg\neg\mathrm{p}$*, we have to show they have the same truth value in any assignment. Let* $\tau$ *be an arbitrary assignment. We have that* $\hat{\tau}\Big(\neg\neg\mathrm{p}\Big) = \overline{\overline{\tau(\mathrm{p})}} = \tau(\mathrm{p}) = \hat{\tau}\Big(\mathrm{p}\Big)$*. In summary,* $\hat{\tau}\Big(\neg\neg\mathrm{p}\Big) = \hat{\tau}\Big(\mathrm{p}\Big)$*. As* $\tau$ *was chosen arbitrarily, it follows that* $\hat{\tau}\Big(\neg\neg\mathrm{p}\Big) = \hat{\tau}\Big(\mathrm{p}\Big)$ *for any assignment* $\tau$ *and therefore* $\mathrm{p}$ *is equivalent to* $\neg\neg\mathrm{p}$*.*

**Example 45.** *The following equivalence holds:* $(p \vee q) \equiv \neg(\neg p \wedge \neg q)$ *(check it).*

Here are two more equivalences, known as De Morgan's laws:

**Theorem 46.** *For any formulae* $\varphi_1, \varphi_2 \in \mathbb{PL}$*, we have that:*

1. $\neg(\varphi_1 \vee \varphi_2) \equiv (\neg\varphi_1 \wedge \neg\varphi_2)$;

2. $\neg(\varphi_1 \wedge \varphi_2) \equiv (\neg\varphi_1 \vee \neg\varphi_2)$.

## 5.7 Semantical Consequence

**Definition 47.** *Let* $\Gamma = \{\varphi_1, \ldots, \varphi_n, \ldots\}$ *be a set of formulae. We say that* $\varphi$ *is a* semantical consequence *of* $\Gamma$ *and we write* $\Gamma \models \varphi$*, if is any model of all formulae in* $\Gamma$ *is a model of* $\varphi$ *as well.*

We also say that $\varphi$ is a logical consequence of $\Gamma$ or that $\varphi$ is a tautological consequence of $\Gamma$ instead of $\varphi$ is a semantical consequence of $\Gamma$.

**Example 48.** *Let* $\Gamma = \{p, (\neg p \vee q)\}$*. We have that* $\Gamma \models q$*.*

*Indeed, let* $\tau$ *be a model of* $p$ *and of* $(\neg p \vee q)$*. As* $\tau$ *is a model of* $p$*, by definition, we have that* $\tau(p) = 1$*.*

*As* $\tau$ *is a model of* $(\neg p \vee q)$*, it follows that* $\hat{\tau}\Big((\neg p \vee q)\Big) = 1$*. But* $\hat{\tau}\Big((\neg p \vee q)\Big)\overline{\tau(p)} + \tau(q)$*. But* $\tau(p) = 1$*, and therefore* $\hat{\tau}\Big((\neg p \vee q)\Big) = 0 + \tau(q) = \tau(q)$*. This means that* $\tau(q) = 1$*.*

*This means that* $\tau$ *is a model of* $q$*. We assumed that* $\tau$ *is a model of* $p$ *and of* $(\neg p \vee q)$ *and we show that necessarily* $\tau$ *is a model of* $q$*. But this is exactly the definition of* $\{p, (\neg p \vee q)\} \models q$*, what we had to show.*

**Example 49.** *We have that* $p, (p \vee q) \not\models \neg q$*, that is* $\neg q$ *is not a logical consequence of* $p, (p \vee q)$*. To show this "unconsequence", it is sufficient to find a model of* $p$ *and* $(p \vee q)$ *that is not a model of* $q$*. Any assignment* $\tau$ *with* $\tau(p) = 1$ *and* $\tau(q) = 0$ *will do.*

**Notation.** *We sometimes write* $\varphi_1, \ldots, \varphi_n \models \varphi$ *instead of* $\{\varphi_1, \ldots, \varphi_n\} \models \varphi$*.*

**Remark.** *When* $n = 0$*, the notation above allows us to we write* $\models \varphi$ *instead of* $\{\} \models \varphi$*. This is consistent will the notation for validity, since a formula that is a logical consequence of the empty set is valid (and vice-versa).*

## 5.8 Consistent set of formulae

Another semantical notion is that occurs relatively often in practice is the *(inconsistent) set of formulae.*

**Definition 50** (Consistent set of formulae). *A set $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of formulae is* consistent *if there is an assignment $\tau$ that satisfies all $\varphi_i$ ($1 \leq i \leq n$).*

**Remark.** *Pay attention! The assignment $\tau$ must be the* same *for each formula in the set (not a different assignment $\tau$ for each formula $\varphi_i$, where $1 \leq i \leq n$)!*

A set of formulae is inconsistent if it is not consistent.

**Lemma 51** (The link between consistent sets and the logical connector $\wedge$). *A set of formulae $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is consistent if and only if the formula*

$$(((\varphi_1 \wedge \varphi_2) \wedge \ldots) \wedge \varphi_n)$$

*is satisfiable.*

**Lemma 52** (The link between inconsistent sets and the logical connector $\wedge$). *A set of formulae $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is inconsistent if and only if the formula*

$$(((\varphi_1 \wedge \varphi_2) \wedge \ldots) \wedge \varphi_n)$$

*is not satisfiable.*

**Exercise 53.** *Show that, for any formula $\varphi$, if $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is inconsistent then*

$$\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \models \varphi.$$

**Exercise 54.** *Show that if*

$$\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \models (\mathrm{p} \wedge \neg\mathrm{p}),$$

*then $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is inconsistent.*

## 5.9 Application 1

John writes the following code:

```
if (((y % 4 == 0) && (y % 100!= 0)) || (y%400 == 0))
   printf("%d is a leap y", y);
else
   printf("%d is not a leap y", y);
```

Jill simplifies the code:

```
if (((y % 4 != 0) || (y % 100 == 0)) && (y%400 != 0))
   printf("%d is not a leap y", y);
else
   printf("%d is a leap y", y);
```

Is Jill right? It is difficult to tell just by looking at the code, but we can use our logic-fu to model the problem above and to determine whether the two programs behave in the same manner.

First of all, we will "translate" the conditions in the if-else statements into propositional logic. Supposed the value of `y` is fixed. We identify the "atomic propositions" and we replace them with propositional variables as follows:

1. the propositional variable $\mathtt{p}$ will stand for `(y % 4 == 0)`;
2. the propositional variable $\mathtt{q}$ will stand for `(y % 100 = 0)`;
3. the propositional variable $\mathtt{r}$ will stand for `(y % 400 == 0)`.

Taking into account the translation key above, we can see that John's condition is, in propositional logic speak, $((\mathtt{p} \wedge \neg\mathtt{q}) \vee \mathtt{r})$.

Jill's formula is, in propositional logic speak,$((\neg\mathtt{p} \vee \mathtt{q}) \wedge \neg\mathtt{r})$.

Also notice that the branches in the two programs are reversed (the if branch of John's program is the else branch of Jill's program and vice-versa). In order for the two programs to have the same behaviour, it is sufficient for the negation of John's formula to be equivalent to Jill's formula. Is this the case? I.e., is it the case that

$$\neg((\mathtt{p} \wedge \neg\mathtt{q}) \vee \mathtt{r}) \equiv ((\neg\mathtt{p} \vee \mathtt{q}) \wedge \neg\mathtt{r})?$$

By applying De Morgan's laws, we can see that the equivalence does hold and therefore the two programs have the same behaviour. So Jill's changes do not break the program – they are correct.

## 5.10 Exercise Sheet

**Exercise 55.** *Let $\tau : A \to B$ be the truth assignment defined as follows: $\tau(\mathtt{p}) = 1$, $\tau(\mathtt{q}) = 0$, $\tau(\mathtt{r}) = 0$, $\tau(a) = 0$ for any other propositional variable $a \in A \setminus \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$.*

*Find the truth value of the following formulae in the assignment $\tau$:*

1. $(\mathtt{p} \land \mathtt{q})$;
2. $(\mathtt{q} \land \mathtt{p})$;
3. $\neg\mathtt{q}$;
4. $(\neg\mathtt{q} \land \mathtt{r})$;
5. $((\neg\mathtt{q} \land \mathtt{r}) \lor \neg\mathtt{p})$.

**Exercise 56.** *Find an assignment $\tau$ that is a model for the following formulae (one assignment for each formula):*

1. $(\mathtt{p} \land \mathtt{q})$;
2. $(\mathtt{p} \land \neg\mathtt{q})$;
3. $((\mathtt{p} \land \neg\mathtt{q}) \lor \mathtt{q})$.

*Is there a single assignment that makes all these formulas true?*

**Exercise 57.** *Find an assignment $\tau$ in which the following formulae are false (one assignment per formula):*

1. $(\mathtt{p} \lor \mathtt{q})$;
2. $(\mathtt{q} \land (\mathtt{p} \lor \neg\mathtt{q}))$;
3. $((\mathtt{p} \land \neg\mathtt{q}) \lor \mathtt{q})$.

**Exercise 58.** *Which of the following formulae are satisfiable?*

1. $(\mathtt{p} \land \neg\mathtt{p})$;
2. $(\mathtt{p} \lor \neg\mathtt{p})$;
3. $((\mathtt{p} \lor \neg\mathtt{p}) \land \neg\mathtt{q})$;
4. $((\mathtt{p} \lor \neg\mathtt{p}) \land (\neg\mathtt{p} \land \mathtt{q}))$;
5. $((\mathtt{p} \lor \neg\mathtt{q}) \land (\neg\mathtt{p} \lor \mathtt{r}))$.

**Exercise 59.** *Which of the following formulae are valid?*

1. $(p \wedge \neg p)$;
2. $(p \vee \neg p)$;
3. $p$;
4. $((p \vee \neg p) \wedge \neg q)$;
5. $(p \rightarrow \neg p)$;
6. $((p \wedge q) \vee (\neg p \wedge r))$.

**Exercise 60.** *Give 5 examples of contradictions.*

**Exercise 61.** *Give 5 examples of tautologies.*

**Exercise 62.** *Prove that, for any formulae $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{PL}$, the following equivalences hold:*

1. $(\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)) \equiv ((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$;
2. $(\varphi_1 \wedge \varphi_2) \equiv (\varphi_2 \wedge \varphi_1)$;
3. $(\varphi_1 \vee (\varphi_2 \vee \varphi_3)) \equiv ((\varphi_1 \vee \varphi_2) \vee \varphi_3)$;
4. $(\varphi_1 \vee \varphi_2) \equiv (\varphi_2 \vee \varphi_1)$;
5. $(\neg(\neg\varphi_1)) \equiv \varphi_1$;
6. $(\neg(\varphi_1 \wedge \varphi_2)) \equiv ((\neg\varphi_1) \vee (\neg\varphi_2))$;
7. $(\neg(\varphi_1 \vee \varphi_2)) \equiv ((\neg\varphi_1) \wedge (\neg\varphi_2))$.

**Exercise 63.** *Can you prove that, for any formulae $\varphi_1, \varphi_2 \in \mathbb{PL}$, $(\varphi_1 \vee \varphi_2) \equiv \varphi_1$ if and only if $\varphi_1 \in \mathbb{PL}$ is a tautology?*

**Exercise 64.** *Can you prove that, for any formulae $\varphi_1, \varphi_2 \in \mathbb{PL}$, $(\varphi_1 \wedge \varphi_2) \equiv \varphi_2$ if and only if $\varphi_1 \in \mathbb{PL}$ is a tautology?*

**Exercise 65.** *Can you prove that, for any formulae $\varphi_1, \varphi_2 \in \mathbb{PL}$, $(\varphi_1 \wedge \varphi_2) \equiv \varphi_1$ if and only if $\varphi_1 \in \mathbb{PL}$ is a contradiction?*

**Exercise 66.** *Can you prove that, for any formulae $\varphi_1, \varphi_2 \in \mathbb{PL}$, $(\varphi_1 \vee \varphi_2) \equiv \varphi_2$ if and only if $\varphi_1 \in \mathbb{PL}$ is a contradiction?*

**Exercise 67.** *Show that $\neg p$ is a semantical consequence of $(\neg p \vee \neg p)$.*

**Exercise 68.** *Show that $p$ is not a semantical consequence of $(\neg q \vee p)$.*

**Exercise 69.** *Show that $p$ is a semantical consequence of $(\neg q \vee p)$ and $q$.*

**Exercise 70.** *Show that $p_3$ is a logical consequence of the formulae $(\neg p_1 \vee (p_2 \vee p_3))$, $((\neg\neg p_2 \vee \neg p_4) \wedge (\neg\neg p_4 \vee \neg p_2))$ and $(p_1 \wedge \neg p_4)$.*