Chapter 1

Motivation and introduction

First-order logic, what we will be studying next, is an extension of propositional logic, extension that brings more expressivity. The additional expressivity is necessary in order to model certain statements that cannot be expressed in propositional logic.

In propositional logic, we cannot express naturally the following statement: *All men are mortal.*

To model a statement in propositional logic, we identify the atomic propositions. Then we associate to each atomic proposition a propositional variable. The atomic propositions are the propositions that cannot be split into one or more smaller propositions, linked among them by the logical connectives of propositional logic: $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

We notice that the statement *All men are mortal* cannot be decomposed into smaller statements linked among them by the logical connectives of propositional logic, as is described above. Therefore, in propositional logic, the statement is atomic. So we associate to the entire statement a propositional variable $\mathtt{p} \in A$.

Let us now model the statement *Socrates is a man.* Obviously, to this second statement we must associate another propositional variable $\mathtt{q} \in A$. Let us assume that $\mathtt{p}$ and $\mathtt{q}$ are true. Formally, we work in a truth assignment $\tau : A \to B$ where $\tau(\mathtt{p}) = 1$ and $\tau(\mathtt{q}) = 1$. Can we draw the conclusion that *Socrates is mortal* in the truth assignment τ?

No, because to the statement *Socrates is mortal* we should associate a third propositional variable $\mathtt{r} \in A$. We cannot draw any conclusion on $\tau(\mathtt{r})$ from $\tau(\mathtt{p}) = 1$ and $\tau(\mathtt{q}) = 1$. So, from the semantics of propositional logic, we cannot draw the conclusion that $\mathtt{r}$ is true in any truth assignment that makes both $\mathtt{p}$ and $\mathtt{q}$ true. This is despite the fact that, in any world where *All men are mortal* and *Socrates is a man*, we can draw the conclusion that *Socrates* is mortal without failure. This difference between reality and our modelling indicates that our modelling is not sufficient for our purposes.

First-order logic includes, in addition to propositional logic, the notion of *quantifier* and the notion of *predicate*. The universal quantifier is denoted by $\forall$ and the existential quantifier is denoted by $\exists$.

A predicate is a statement whose truth value depends on zero or more parameters. For example, for the statements above, we will be using two predicates: $\mathtt{Man}$ and $\mathtt{Mortal}$. The predicate $\mathtt{Man}$ is the predicate that denotes the quality of being a man: $\mathtt{Man}(\mathsf{x})$ is true iff x is a man. The predicate $\mathtt{Mortal}$ is true when its argument is mortal. As the predicates above have only one argument/parameter, they are called *unary* predicates. Predicates generalize propositional variables by the fact that they can take arguments. Actually, propositional variable can be regarded as predicates with no arguments.

In this way, the statement *All men are mortal* will be modelled by the formula

$$\big(\forall \mathsf{x}.(\mathtt{Man}(\mathsf{x}) \rightarrow \mathtt{Mortal}(\mathsf{x}))\big),$$

which is read as follows: *for any* x*, if* $\mathtt{Man}$ *of* x*, then* $\mathtt{Mortal}$ *of* x. The statement *Socrate is a men* shall be modelled by the formula $\mathtt{Man}(s)$, where s is a *constant* that denotes Socrates, just like 0 denotes the natural number *zero*. For example, $\mathtt{Man}(\mathsf{s})$ is true (as s stands for a particular man – Socrates), but $\mathtt{Man}(\mathsf{l})$ is false if l is a constant standing for the dog *Lassie*.

The statement *Socrates is mortal* shall be represented by $\mathtt{Mortal}(\mathsf{s})$ (recall that the constant s stands for Socrates). The statement $\mathtt{Mortal}(\mathsf{s})$ is true, as Socrates is mortal; likewise, the statement $\mathtt{Mortal}(\mathsf{l})$ is also true.

We shall see that in first-order logic, the formula $\mathtt{Mortal}(\mathsf{s})$ is a logical consequence of the formulae $\big(\forall \mathsf{x}.(\mathtt{Man}(\mathsf{x}) \rightarrow \mathtt{Mortal}(\mathsf{x}))\big)$ and respectively $\mathtt{Man}(\mathsf{s})$. Therefore, first-order logic is sufficiently expressive to explain theoretically the argument by which we deduce that *Socrates is mortal* from the facts that *All men are mortal* and *Socrates is a man*.