Chapter 6

Logical Connectives

There are two more important connectives in propositional logic: the conditional (implication) and the equivalence (double implication).

We use the syntax $(\varphi_1 \to \varphi_2)$ for an implication and $(\varphi_1 \leftrightarrow \varphi_2)$ for a double implication.

The semantics of an implication $(\varphi_1 \to \varphi_2)$ is given by

$$\hat{\tau}\Big((\varphi_1 \to \varphi_2)\Big) = \overline{\hat{\tau}\Big(\varphi_1\Big)} + \hat{\tau}\Big(\varphi_2\Big),$$

for any truth assignment τ.

It is easy to see that $(\varphi_1 \to \varphi_2) \equiv (\neg\varphi_1 \vee \varphi_2)$.

Similarly, the semantics of a double implication $(\varphi_1 \leftrightarrow \varphi_2)$ is defined such that

$$(\varphi_1 \leftrightarrow \varphi_2) \equiv ((\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)).$$

Example 71. *We have that* $(\mathtt{p} \to \mathtt{p})$ *is a valid formula. Why? The formula* $(\mathtt{p} \to \mathtt{p})$ *is equivalent to* $(\neg\mathtt{p} \vee \mathtt{p})$*, and this formulae is obviously valid.*

6.1 Several Propositional Logics

Up to this point, we have studied the propositional logic of the connectives $\neg, \wedge, \vee$, and we have denoted its set of formulae by $\mathbb{PL}$. In fact, depending on the set of logical connectives that we need, there are several propositional logics. The logic that we have studied up to this point is $\mathbb{PL}_{\neg,\wedge,\vee} = \mathbb{PL}$.

Depending on the logical connectives that are allowed, we can obtain other interesting propositional logics:

1. $\mathbb{PL}_{\neg,\vee}$ is the propositional logic that allows as connectives $\neg$ and $\vee$.

2. $\mathbb{PL}_{\bot,\rightarrow}$ is the propositional logic in which that only connectives are $\bot$ (an arity 0 connective) and $\rightarrow$. The formula $\bot$ (read as: *bottom*) is false in any truth assignment.

3. $\mathbb{PL}_{\vee,\wedge}$ is a logic in which the only connectives are $\vee$ and $\wedge$.

Exercise 72. *Write the definition of the formal syntax for each of the logics above.*

What do $\mathbb{PL}, \mathbb{PL}_{\neg,\vee}, \mathbb{PL}_{\bot,\rightarrow}$ have in common? They are *equiexpressive* (equally expressive). This means that for any formula $\varphi \in \mathbb{PL}$ there is a formula $\varphi' \in \mathbb{PL}_{\neg,\vee}$ so that $\varphi \equiv \varphi'$ (and vice-versa, for any formula $\varphi' \in \mathbb{PL}_{\neg,\vee}$ there is an equivalent formula in $\mathbb{PL}$).

How can we show that $\mathbb{PL}$ and $\mathbb{PL}_{\neg,\vee}$ are equally expressive? For one of the directions, it is sufficient to translate all conjunctions in $\mathbb{PL}$ as follows:

$$(\varphi \wedge \varphi') \equiv \neg(\neg\varphi \vee \neg\varphi').$$

At the end we will obtain an equivalent formula that does not contain conjunctions (and is therefore in $\mathbb{PL}_{\neg,\vee}$). Vice-versa, any formula in $\mathbb{PL}_{\neg,\vee}$ is already a formula in $\mathbb{PL}$.

How can we show that $\mathbb{PL}_{\neg,\vee}$ and $\mathbb{PL}_{\bot,\rightarrow}$ are equally expressive?

We translate all disjunctions and negations using the following equivalences:

1. $(\varphi \vee \varphi') \equiv (\neg\varphi \rightarrow \varphi')$;

2. $\neg\varphi \equiv (\varphi \rightarrow \bot)$.

The translation operation stops after a finite number of steps and the result is a formula equivalent to the starter formula, but which uses only the connectives $\bot$ and $\rightarrow$.

The logic $\mathbb{PL}_{\vee,\wedge}$ is strictly less expressive. For example, in this logic there are no unsatisfiable formulae.

Exercise 73. *Explain why in $\mathbb{PL}_{\vee,\wedge}$ all formulae are satisfiable.*

Remark. *By* propositional logic *we understand any logic that is as expressive as $\mathbb{PL}$. For example, $\mathbb{PL}_{\neg,\wedge}$, $\mathbb{PL}_{\neg,\vee}$, $\mathbb{PL}_{\neg,\rightarrow}$ are all propositional logics, but $\mathbb{PL}_{\neg}$ and $\mathbb{PL}_{\wedge,\vee}$ are not propositional logics (they are less expressive).*

6.2 The relation between implications and semantical consequence

There is a strong link between implications and the concept of semantical consequence, link which is formalized in the following theorem.

Theorem 74 (The relation between implications and logical consequences)**.** *For any two formulae $\varphi_1, \varphi_2 \in \mathbb{PL}$, we have that $\varphi_1 \models \varphi_2$ if an only if the formula $(\varphi_1 \rightarrow \varphi_2)$ is valid.*

The following more general theorem also holds:

Theorem 75 (The generalized relation between implications and logical consequence)**.** *For any formulae $\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi \in \mathbb{PL}$, we have that $\varphi_1, \varphi_2, \ldots, \varphi_n \models \varphi$ if and only if the formula $(((\varphi_1 \land \varphi_2) \land \ldots) \land \varphi_n) \rightarrow \varphi$ is valid.*

A similar relation exists between the double implication logical connective and the concept of semantical equivalence:

Theorem 76 (The relation between double implication and semantic equivalence)**.** *For any two formulae $\varphi_1, \varphi_2 \in \mathbb{PL}$, we have that $\varphi_1 \equiv \varphi_2$ if and only if the formula $(\varphi_1 \leftrightarrow \varphi_2)$ is valid.*

6.3 *Translating* propositions from English into $\mathbb{PL}$

By *translation* we understand modeling an English proposition as a formula of propositional logic. The purpose of this modeling could be: clarifying the meaning of a proposition by eliminating possible syntactical ambiguities, checking whether the proposition is valid, etc.

To *translate* propositions from English into propositional logic, we should perform the following steps:

1. Identifying the atomic propositions and associating propositional variables to them;

2. Identifying the logical connectives and their relative order.

For example, let us *translate* the proposition *I want to learn Logic and pass the examination if the subject is interesting* into propositional logic.

The first step is to identify the atomic propositions. In our case, we find three atomic propositions:

1. *I want to learn Logic*;

2. *I want to pass the examination*; (note that the words *I want* do not appear explicitly in the proposition, but they are *implied*)

3. *the subject is interesting.*

Pay attention! The connectives themselves are not part of the atomic propositions. For example, the third atomic proposition is not *if the subject is interesting.*

We associate a propositional variable to each atomic proposition:

propositional variable	atomic proposition
p	*I want to learn Logic*
q	*I want to pass the examination*
r	*the subject is interesting.*

Pay attention! For an accurate *translation*, if a proposition occurs several times (even if it does not use exactly the same words), we should associate to all of its occurences the same propositional variable.

The next step is to identify the logical connectives. In the example above, there are two logical connectives:

1. the connective *and* in the context *[...] Logic and pass [...]*, which indicates a conjunction;

2. the connective *if-then* (even if the word *then* does not appear explicitly) in the context *[...] the examination if the subject [...]*, which indicates an implication.

What is the order of the two connectives? In other words, it the entire proposition a conjunction or an implication? What is the main connective? Our English-language intuition tells us that it is more likely an implication (for example, because the verb *want* is implied (not explicit) in the proposition *I want to pass the examination*). The *translation* is therefore

$$(\mathtt{r} \rightarrow (\mathtt{p} \wedge \mathtt{q})),$$

because the proposition associated to the propositional variable r is the antecedent of the implication, even if syntactically it appears at the end of the sentence.

> Pay attention! In spite of the name, *propositional variables* are not variables in a mathematical sense. A frequent mistake is to write
>
> $$\mathtt{p} = \textit{I want to learn Logic},$$
>
> which is not correct, because $\mathtt{p}$ is only equal to $\mathtt{p}$ and nothing else. Any variable (in a mathematical sense) in the lecture is written with a regular black font. For example, we often use the mathematical variable φ for propositional formulae.

Exercise 77. *Translate the following proposition into propositional logic:* Either I pass Logic, or I don't.

Pay attention to the words that do not explictly appear in the atomic propositions. Pay attention to the meaning of the connective either-or *(hint: it is not among the connectives that we discussed so far, but it can be* emulated/simulated*).*

6.4 Application 2

The following puzzle is from the book *Peter Smith. An Introduction to Formal Logic.* Either the butler or the cook committed the murder. The victim died from poison if the cook did the murder. The butler did the murder only if the victim was stabbed. The victim didn't die from poison.

Does it follow that the victim was stabbed?

We associate to every atomic proposition a propositional variable as follows:

1. For the proposition "the butler commited the murder" the propositional variable $\mathtt{p}$;
2. For the proposition "the cook commited the murder" the propositional variable $\mathtt{q}$;
3. For the proposition "the victim died of posion" the propositional variable $\mathtt{r}_1$;
4. For the proposition "the victim was stabbed" the propositional variable $\mathtt{r}_2$.

The hypotheses of the puzzle are modeled as propositional formulae in propositional logic as follows:

1. $((\mathtt{p} \vee \mathtt{q}) \wedge \neg(\mathtt{p} \wedge \mathtt{q}))$ (exclusive disjunction between $\mathtt{p}$ and $\mathtt{q}$);
2. $(\mathtt{q} \rightarrow \mathtt{r}_1)$;
3. $(\mathtt{p} \rightarrow \mathtt{r}_2)$ (mind the direction of the implication);
4. $\neg\mathtt{r}_1$.

The question of the puzzle is simply the formula $\mathtt{r}_2$.
To answer the puzzle with yes or no, it is sufficient to check whether

$$\Big\{((\mathtt{p} \vee \mathtt{q}) \wedge \neg(\mathtt{p} \wedge \mathtt{q})), (\mathtt{q} \rightarrow \mathtt{r}_1), (\mathtt{p} \rightarrow \mathtt{r}_2), \neg\mathtt{r}_1\Big\} \models \mathtt{r}_2.$$

The logical consequences does hold (left as an exercise to the reader).

6.5 Exercise sheet

Exercise 78. *Associate to each of the following statements a formula in $\mathbb{PL}$ that models its meaning in English.*

1. *If it rains outside, I stay inside or go to the mall. I don't stay inside unless I'm bored. It rains and I'm not bored.*
2. *I study logic only if it is not possible to go outside. It is possible to go outside if it is not raining and if it is hot. As I am not studying logic and outside is hot, it means it is raining.*
3. *Thing go well in the country if the leaders of the country are not thiefs and if the economy is healthy. People go abroad if and only if things do not go well in the country. The economy is healthy, but people leave abroad.*