But, by definition, $height\Big((\varphi_1 \vee \varphi_2)\Big) = 1 + max\Big(height\big(\varphi_1\big), height\big(\varphi_2\big)\Big)$ and, as $max(a, b) \leq a + b$ (for any naturals $a, b \in \mathbb{N}$), we have that $height\Big((\varphi_1 \vee \varphi_2)\Big) \leq 1 + height\big(\varphi_1\big) + height\big(\varphi_2\big)$. But $height\big(\varphi_i\big) \leq size\big(\varphi_i\big)$ $(1 \leq i \leq 2)$ by our hypothesis, and therefore $height\Big((\varphi_1 \vee \varphi_2)\Big) \leq 1 + size\big(\varphi_1\big) + size\big(\varphi_2\big)$. But, by definition, $size\Big((\varphi_1 \vee \varphi_2)\Big) = 1 + size\big(\varphi_1\big) + size\big(\varphi_2\big)$, and therefore $height\Big((\varphi_1 \vee \varphi_2)\Big) \leq size\Big((\varphi_1 \vee \varphi_2)\Big)$, what we had to prove.

4. (Inductive Case iii) Similar to Inductive Case ii.

q.e.d.

4.4 Exercise Sheet

Exercise 19. *Compute, using the function subf, the set of subformulae of the following formulae:*

1. $((\mathtt{p} \wedge \neg\mathtt{q}) \wedge \mathtt{r})$; *2.* $((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$; *3.* $\neg((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$.

Exercise 20. *Compute the abstract syntax trees of the following formulae:*

1. $((\mathtt{p} \wedge \neg\mathtt{q}) \wedge \mathtt{r})$;
2. $((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$;
3. $\neg((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$;
4. $(\neg(\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$.

Exercise 21. *Recall the recursive definition of the function height* $: \mathbb{PL} \to \mathbb{N}$*, which computes, given a formula, the height of its abstract syntax tree. Compute the height of the formulae shown in Exercise 20.*

Exercise 22. *Recall the recursive definition of the function size* $: \mathbb{PL} \to \mathbb{N}$*, which computes the number of nodes in abstract syntax tree of a formula. Compute the size of the formulae shown in Exercise 20.*

Exercise 23. *Recall the recursive definition of the function prop* $: \mathbb{PL} \to 2^A$*, which computes, given a formula, the set of propositional variables occuring in the formula. Compute the set of propositional variables occuring in the formulae shown in Exercise 20.*

Exercise 24. *Show by structural induction that* $height(\varphi) < size(\varphi) + 1$ *for any formula* $\varphi \in \mathbb{PL}$.