

# Chapter 8

## Normal Forms

In the rest of the course, we will allow to write formulae without explicit brackets, by following the following priority order of the logical connectives:

$$\perp, \neg, \wedge, \vee, \rightarrow, \leftrightarrow.$$

For example, by

$$\neg \neg p \vee \perp \wedge p \rightarrow \neg p \wedge q \leftrightarrow q$$

we will understand the formula

$$(((\neg \neg p \vee (\perp \wedge p)) \rightarrow (\neg p \wedge q)) \leftrightarrow q).$$

We will also write  $((\varphi_1 \vee \varphi_2) \vee \varphi_3)$  as  $\varphi_1 \vee \varphi_2 \vee \varphi_3$  (meaning that a series of  $\vee$  will be implicitly left-associated).

We will also write  $((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$  as  $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$  (meaning that a series of  $\wedge$ s will also be implicitly left-associated).

**Example 116.** *We write*

$$p \wedge q \wedge r \vee \neg p \wedge \neg q \wedge \neg r$$

*for the formula*

$$(((p \wedge q) \wedge r) \vee ((\neg p \wedge \neg q) \wedge \neg r)).$$

*We write*

$$p_1 \wedge p_2 \wedge p_3 \wedge p_4$$

*for the*

$$(((p_1 \wedge p_2) \wedge p_3) \wedge p_4).$$

We write

$$p_1 \vee p_2 \vee p_3 \vee p_4$$

for the formula

$$(((p_1 \vee p_2) \vee p_3) \vee p_4).$$

## 8.1 Replacement Theorem

We present a theorem that we have used implicitly so far:

**Theorem 117** (The Replacement Theorem). *Let  $\varphi, \varphi'$  be two formulae such that  $\varphi \equiv \varphi'$ .*

*Let  $\varphi_1$  be a formula that contains  $\varphi$  as a subformula.*

*Let  $\varphi_2$  be the formula obtained from  $\varphi_1$  by replacing one occurrence of  $\varphi$  by  $\varphi'$ .*

*Then  $\varphi_1 \equiv \varphi_2$ .*

In other words,  $\equiv$  is a *congruence*.

**Example 118.** *Let  $\varphi = p$  and  $\varphi' = \neg\neg p$ .*

*Let  $\varphi_1 = (p \vee q)$  and  $\varphi_2 = (\neg\neg p \vee q)$ .*

*By the replacement theorem, we get that  $\varphi_1 \equiv \varphi_2$ . In other words, by replacing a subformula with an equivalent one, the new formula is equivalent to the starter formula*

## 8.2 Literal

**Definition 119** (Literal). *A formula  $\varphi$  is called a literal if there exists a propositional variable  $a \in A$  such that*

$$\varphi = a \quad \text{or} \quad \varphi = \neg a.$$

**Example 120.** *The formulae  $p, q, \neg p, \neg p', q_1$  are literals, but the formulae  $(p \vee q), \neg\neg p, \neg\neg\neg q_1, (\neg p \wedge q)$  are not literals.*

## 8.3 Clauses

**Definition 121.** *A formula  $\varphi$  is called a clause if there exist  $n$  literals  $\varphi_1, \dots, \varphi_n$  such that*

$$\varphi = \varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_n.$$

In other words, a clause is a *disjunction of literals*.

**Example 122.** *The following formulae are clauses:*

1.  $p \vee q \vee r$ ;
2.  $p \vee \neg q \vee \neg r$ ;
3.  $\neg p \vee \neg q \vee \neg r$ ;
4.  $\neg p_1 \vee p_1 \vee p_2 \vee \neg q \vee \neg r$ ;
5.  $\neg p_1 \vee p_1$ ;
6.  $\neg p_1$  ( $n = 1$ ).
7.  $p$  ( $n = 1$ ).

*The following formulae are not clauses:*

1.  $p \wedge r$ ; (conjunction instead of disjunction)
2.  $p \vee \neg \neg q \vee \neg r$ ; (not a disjunction of literals)
3.  $\neg \neg p \vee p \wedge \neg p_1$ . (we have  $\neg \neg$ , so no literals; we also have  $\wedge$ )

**Remark.** For  $n = 0$ , we get the empty clause, denoted by  $\square$ . We consider that  $\square$  represents an unsatisfiable formula (for example,  $\perp$ ), without being a formula itself.

## 8.4 Conjunctive Normal Form

**Definition 123** (CNF). A formula  $\varphi$  is in CNF if there exist  $n$  clauses  $\varphi_1, \dots, \varphi_n$  such that

$$\varphi = \varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n.$$

In other words, a formula in CNF is a conjunction of disjunctions of literals. Or, a formula in CNF is a conjunction of clauses. CNF is conveniently also an abbreviation of clausal normal form, which means the same thing as conjunctive normal form.

**Example 124.** *The following formulae are in CNF:*

1.  $(\neg p \vee q) \wedge (r \vee \neg p \vee r') \wedge (\neg p \vee \neg r)$ ;
2.  $\neg p \wedge (r \vee \neg p \vee r') \wedge \neg r$ ;
3.  $\neg p \wedge r \wedge \neg r$ ;
4.  $\neg p$ ;

5.  $p$ .

The following formulae are not in CNF:

1.  $\neg(\neg p \vee q) \wedge (r \vee \neg p \vee r') \wedge (\neg p \vee \neg r)$ ; (the first clause is negated)
2.  $\neg p \wedge \neg(r \vee \neg p \vee r')$ ; (second clause is negated)
3.  $\neg p \vee (r \wedge \neg p \wedge r')$ ; (the main connective is the disjunction, instead of a conjunction)
4.  $\neg\neg p$ ; (we have  $\neg\neg$ , so no literals)
5.  $p \vee (q \wedge r)$ . (disjunction of conjunctions, not a conjunction of disjunctions (of literals))

## 8.5 Bringing a Formula into CNF

**Theorem 125** (Theorem for Bringing a Formula into CNF). *For any formula  $\varphi \in \mathbb{PL}$ , there exists a formula  $\varphi' \in \mathbb{PL}$  that is in CNF such that  $\varphi \equiv \varphi'$ .*

**Proof:** [Proof Sketch]

By repeatedly applying the replacement theorem using the following equivalences:

1.  $(\varphi_1 \leftrightarrow \varphi_2) \equiv ((\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1))$ ;
2.  $(\varphi_1 \rightarrow \varphi_2) \equiv (\neg\varphi_1 \vee \varphi_2)$ ;
3.  $(\varphi_1 \vee (\varphi_2 \wedge \varphi_3)) \equiv ((\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3))$ ;
4.  $((\varphi_2 \wedge \varphi_3) \vee \varphi_1) \equiv ((\varphi_2 \vee \varphi_1) \wedge (\varphi_3 \vee \varphi_1))$ ;
5.  $(\varphi_1 \vee (\varphi_2 \vee \varphi_3)) \equiv ((\varphi_1 \vee \varphi_2) \vee \varphi_3)$ ;
6.  $(\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)) \equiv ((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$ ;
7.  $\neg(\varphi_1 \vee \varphi_2) \equiv (\neg\varphi_1 \wedge \neg\varphi_2)$ ;
8.  $\neg(\varphi_1 \wedge \varphi_2) \equiv (\neg\varphi_1 \vee \neg\varphi_2)$ ;
9.  $\neg\neg\varphi \equiv \varphi$ .

The first two equivalences ensure the fact that all double implications and all implications disappear from the formula.

The third and the fourth equivalence ensure that in the tree of the formula, all disjunctions “go under” the conjunctions.

Equivalences 5 and 6 ensure the associativity of the chains of disjunctions and conjunctions, respectively (it allows us to write  $\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \dots \vee \varphi_n$  instead of  $((((\varphi_1 \vee \varphi_2) \vee \varphi_3) \vee \dots) \vee \varphi_n)$ ).

Equivalences 7 and 8 ensure that the negations end up “under” the conjunctions and disjunctions in the tree of the formula.

The last equivalence ensures that there are no double negations “one under the other” in the abstract syntax tree.

Applying the equivalences above eventually stops (they cannot be applied ad infinitum – why?).

The result will be a formula in which all conjunctions are “above” all disjunctions, which are “above” all possible negations, meaning a formula in CNF. q.e.d.

**Example 126.** *Let us bring the formula  $((\neg p \rightarrow \neg q) \leftrightarrow (q \rightarrow p))$  into CNF*

$$\begin{aligned}
 & ((\neg p \rightarrow \neg q) \leftrightarrow (q \rightarrow p)) \\
 \stackrel{1}{=} & ((\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)) \wedge ((q \rightarrow p) \rightarrow (\neg p \rightarrow \neg q)) \\
 \stackrel{2}{=} & ((\neg \neg p \vee \neg q) \rightarrow (\neg q \vee p)) \wedge ((\neg q \vee p) \rightarrow (\neg \neg p \vee \neg q)) \\
 \stackrel{2}{=} & (\neg(\neg \neg p \vee \neg q) \vee (\neg q \vee p)) \wedge (\neg(\neg q \vee p) \vee (\neg \neg p \vee \neg q)) \\
 \stackrel{7}{=} & ((\neg \neg \neg p \wedge \neg \neg q) \vee (\neg q \vee p)) \wedge ((\neg \neg q \wedge \neg p) \vee (\neg \neg p \vee \neg q)) \\
 \stackrel{9}{=} & ((\neg p \wedge q) \vee (\neg q \vee p)) \wedge ((q \wedge \neg p) \vee (p \vee \neg q)) \\
 \stackrel{4}{=} & ((\neg p \vee (\neg q \vee p)) \wedge (q \vee (\neg q \vee p))) \wedge ((q \vee (p \vee \neg q)) \wedge (\neg p \vee (p \vee \neg q))) \\
 \stackrel{5}{=} & (((\neg p \vee \neg q) \vee p) \wedge ((q \vee \neg q) \vee p)) \wedge (((q \vee p) \vee \neg q) \wedge ((\neg p \vee p) \vee \neg q)) \\
 = & ((\neg p \vee \neg q \vee p) \wedge (q \vee \neg q \vee p)) \wedge ((q \vee p \vee \neg q) \wedge (\neg p \vee p \vee \neg q)) \\
 \stackrel{6}{=} & (((\neg p \vee \neg q \vee p) \wedge (q \vee \neg q \vee p)) \wedge (q \vee p \vee \neg q)) \wedge (\neg p \vee p \vee \neg q) \\
 = & (\neg p \vee \neg q \vee p) \wedge (q \vee \neg q \vee p) \wedge (q \vee p \vee \neg q) \wedge (\neg p \vee p \vee \neg q).
 \end{aligned}$$

## 8.6 Disjunctive Normal Form

**Definition 127** (DNF). *A formula is in DNF if it is a disjunction of conjunctions of literals.*

**Example 128.** *The following formulae are in DNF:*

$$1. (\neg p \wedge q) \vee (r \wedge \neg p \wedge r') \vee (\neg p \wedge \neg r);$$

2.  $\neg p \vee (r \wedge \neg p \wedge r') \vee \neg r$ ;
3.  $\neg p \vee r \vee \neg r$ ;
4.  $\neg p$ ;
5.  $p$ .

The following formulae are not in DNF:

1.  $\neg(\neg p \wedge q) \vee (r \wedge \neg p \wedge r') \vee (\neg p \wedge \neg r)$ ;
2.  $\neg p \vee \neg(r \wedge \neg p \wedge r')$ ;
3.  $\neg p \wedge (r \vee \neg p \vee r')$ ;
4.  $\neg\neg p$ ;
5.  $p \wedge (q \vee r)$ .

**Exercise 129.** State and sketch the proof of a constructive theorem to bring a formula into DNF.

## 8.7 The Link between DNF and CNF

**Definition 130.** The complement of a formula  $\varphi \in \mathbb{PL}_{\neg, \vee, \wedge}$  is denoted by  $\varphi^c$  and is defined as follows:

1.  $a^c = \neg a$ , for any  $a \in A$ ;
2.  $(\neg\varphi)^c = \varphi$ , for any  $\varphi \in \mathbb{PL}$ ;
3.  $((\varphi_1 \vee \varphi_2))^c = (\varphi_1^c \wedge \varphi_2^c)$ , for any  $\varphi_1, \varphi_2 \in \mathbb{PL}$ ;
4.  $((\varphi_1 \wedge \varphi_2))^c = (\varphi_1^c \vee \varphi_2^c)$ , for any  $\varphi_1, \varphi_2 \in \mathbb{PL}$ .

**Example 131.** 1.  $(\neg p)^c = p$  (the complement of  $\neg p$  is not  $\neg\neg p$ );

2.  $(\neg(\neg p \wedge q) \vee (r \wedge \neg p \wedge r') \vee (\neg p \wedge \neg r))^c =$   
 $(\neg p \wedge q) \wedge (\neg r \vee p \vee \neg r') \wedge (p \vee r)$ ;
3.  $((\neg p \vee q) \wedge (r \vee \neg p \vee r') \wedge (\neg p \vee \neg r))^c =$   
 $(p \wedge \neg q) \vee (\neg r \wedge p \wedge \neg r') \vee (p \wedge r)$ .

**Theorem 132** (The Complement is Equivalent to the Negation). For any formula  $\varphi \in \mathbb{PL}_{\neg, \wedge, \vee}$ , we have that

$$\varphi^c \equiv \neg\varphi.$$

Exercise: prove the theorem above by structural induction.

**Example 133.**  $(p \wedge (q \vee \neg r))^c = (\neg p \vee (\neg q \wedge r)) \equiv \neg(p \wedge (q \vee \neg r)).$

**Theorem 134** (The Link between CNFs and DNFs). *Let  $\varphi_1$  be a formula in CNF and  $\varphi_2$  a formula in DNF.*

*Then  $\varphi_1^c$  is in DNF and  $\varphi_2^c$  is in CNF.*

## 8.8 Exercise Sheet

**Exercise 135.** Write the following formulae with as few brackets as possible:

1.  $((p \wedge \neg q) \wedge r);$
2.  $((p \vee \neg q) \wedge r);$
3.  $\neg((p \vee \neg q) \wedge r).$

**Exercise 136.** Compute the abstract syntax tree for the following formulae (pay attention to the priority of the logical connectives):

1.  $p \wedge q \vee r;$
2.  $\neg p \wedge \neg q \vee \neg r;$
3.  $p \wedge \neg q \vee r;$
4.  $\neg p \wedge \neg(q \vee r);$
5.  $p \vee \neg p \wedge \neg(q \vee r).$

**Exercise 137.** Bring the following formulae into CNF:

1.  $((p \wedge q) \vee r);$
2.  $((p \vee q) \wedge r);$
3.  $\neg((p \vee q) \wedge r);$
4.  $\neg((p \wedge q) \vee r);$
5.  $((p \wedge q) \vee (\neg p \wedge \neg q));$
6.  $((p \wedge (q \wedge r)) \vee \neg p);$
7.  $\neg(\neg(p \wedge q) \vee (p \vee q));$
8.  $(\neg(p \wedge q) \rightarrow (\neg p \wedge \neg q));$
9.  $(p \leftrightarrow (q \rightarrow (\neg p \wedge \neg q)));$
10.  $((p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p));$
11.  $(p_1 \wedge q_1) \vee (p_2 \wedge q_2) \vee \dots \vee (p_n \wedge q_n)$  (first solve the cases  $n = 2$  and  $n = 3$ , then generalize);

**Exercise 138.** Bring into CNF the formula  $p \vee q \rightarrow p \wedge \neg r$  (mind the priority of the logical connectives).

**Exercise 139.** Compute the complement of the formulae in CNF computed above.

