The clauses $\varphi_1, \ldots, \varphi_n$ are called the premisses of the formal proof and φ is the conclusion.

Here is an example of a proof of $\{\mathtt{p}, \neg\mathtt{q}\}$ from $\{\neg\mathtt{r}, \mathtt{p}, \neg\mathtt{r}'\}$, $\{\mathtt{r}, \mathtt{p}\}$ and $\{\mathtt{r}', \neg\mathtt{q}\}$:

1. $\{\neg\mathtt{r}, \mathtt{p}, \neg\mathtt{r}'\}$; (premiss)
2. $\{\mathtt{r}, \mathtt{p}\}$; (premiss)
3. $\{\mathtt{r}', \neg\mathtt{q}\}$; (premiss)
4. $\{\mathtt{p}, \neg\mathtt{r}'\}$; (B.R., 2, 1, $a = \mathtt{r}$)
5. $\{\mathtt{p}, \neg\mathtt{q}\}$. (resolution, 3, 4, $a = \mathtt{r}'$)

We sometimes also say *derivation (by resolution) of* φ instead of *formal proof of* φ. Here is a derivation of the empty clause from $\{\mathtt{p}, \neg\mathtt{q}\}, \mathtt{q}, \neg\mathtt{p}$:

1. $\{\mathtt{p}, \neg\mathtt{q}\}$; (premiss)
2. $\{\mathtt{q}\}$; (premiss)
3. $\{\neg\mathtt{p}\}$; (premiss)
4. $\{\mathtt{p}\}$; (B.R., 2, 1, $a = \mathtt{q}$)
5. $\square$. (B.R., 4, 3, $a = \mathtt{p}$)

A derivation of $\square$ from $\varphi_1, \ldots, \varphi_n$ is sometimes also called a *refutation of* $\varphi_1, \ldots, \varphi_n$.

Exercise 147. *Starting with the same premisses, find a different proof by resolution of* $\square$.

9.4 Soundness

Like natural deduction, resolution is sound. This section shows that this is indeed the case.

Lemma 148. *Let* $\varphi \in \{\varphi_1, \ldots, \varphi_n\}$. *We have that*

$$\varphi_1, \ldots \varphi_n \models \varphi.$$

Exercise 149. *Prove Lemma 148.*

Lemma 150. *Let* C, D *be two clauses and let* $a \in A$ *be a propositional variable. We have that*

$$C \cup \{a\}, D \cup \{\neg a\} \models C \vee D.$$

Exercise 151. *Prove Lemma 150.*

Theorem 152 (Soundness of Resolution)**.** *If there is a proof by resolution of φ from $\varphi_1, \ldots, \varphi_n$, then*

$$\varphi_1, \ldots, \varphi_n \models \varphi.$$

Proof: Let $\psi_1, \ldots, \psi_m$ be a proof by resolution of φ from $\varphi_1, \ldots, \varphi_n$.

We will prove by induction on $i \in \{1, 2, \ldots, m\}$ that

$$\varphi_1, \ldots, \varphi_n \models \psi_i.$$

Let $i \in \{1, 2, \ldots, m\}$ be an integer. We assume by the induction hypothesis that

$$\varphi_1, \ldots, \varphi_n \models \psi_l \text{ for any } l \in \{1, 2, \ldots, i-1\}$$

and we prove that

$$\varphi_1, \ldots, \varphi_n \models \psi_i.$$

By the definition of a formal proof by resolution, we must be in one of the following two cases:

1. $\psi_i \in \{\varphi_1, \ldots, \varphi_n\}$. In this case we have

   $$\varphi_1, \ldots, \varphi_n \models \psi_i$$

   by Lemma 148, which is what we had to show.

2. ψ_i was obtained by resolution from ψ_j, ψ_k with $1 \leq j, k < i$. In this case, ψ_j must be of the form $\psi_j = C \cup a$, ψ_k must be of the form $\psi_k = D \cup \{\neg a\}$ and $\psi_i = C \cup D$, where C, D are clauses and $a \in A$ is a propositional variable.

   By the induction hypotheses that $\varphi_1, \ldots, \varphi_n \models \psi_j$ and that $\varphi_1, \ldots, \varphi_n \models \psi_k$. Replacing ψ_j and ψ_k as detailed above, we have that

   $$\varphi_1, \ldots, \varphi_n \models C \cup \{a\}$$

   and that

   $$\varphi_1, \ldots, \varphi_n \models D \cup \{\neg a\}.$$

   We prove that

   $$\varphi_1, \ldots, \varphi_n \models C \cup D.$$

   Let τ be a model of φ_1, …, and φ_n. We have that τ is a model of $C \cup \{a\}$ and of $D \cup \{\neg a\}$ by the semantical consequences above. By Lemma 150, it follows that τ is a model of $C \cup D$. But $\psi_i = C \cup D$ and therefore τ is a model ψ_i. As τ was any model of all of φ_1, …, and φ_n, it follows that

   $$\varphi_1, \ldots, \varphi_n \models \psi_i,$$

   which is what we had to prove.

In both cases, we have established

$$\varphi_1, \ldots, \varphi_n \models \psi_i$$

for all $1 \leq i \leq m$. As $\psi_1, \ldots, \psi_m$ is a proof of φ, it follows that $\psi_m = \varphi$ and therefore, for $i = m$, we have

$$\varphi_1, \ldots, \varphi_n \models \varphi,$$

which is what we had to prove.

q.e.d.

9.5 Completeness

A proof system must be sound in order to be of any use (otherwise, we could use it to prove something false).

However, it is also nice when a proof system is complete, in the sense of allowing to prove any true statement.

Unfortunately, resolution is not complete, as shown by the following example:

Theorem 153 (Incompleteness of Resolution)**.** *There exist clauses $\varphi_1, \ldots, \varphi_n, \varphi$ such that*

$$\varphi_1, \ldots, \varphi_n \models \varphi,$$

but there is no resolution proof of φ from $\varphi_1, \ldots, \varphi_n$.

Proof: Let $n = 2$, $\varphi_1 = \mathtt{p}$, $\varphi_2 = \mathtt{q}$ and $\varphi = \{\mathtt{p}, \mathtt{q}\}$. We clearly have that $\mathtt{p}, \mathtt{q} \models (\mathtt{p} \vee \mathtt{q})$, but there is no way to continue the following resolution proof:

1. $\{\mathtt{p}\}$; (premiss)
2. $\{\mathtt{q}\}$; (premiss)
3. $\cdots$

because there is no negative literal anywhere. Therefore, only $\mathtt{p}$ and $\mathtt{q}$ can be derived by resolution from $\mathtt{p}, \mathtt{q}$. q.e.d.

However, resolution still has a weaker form of completeness called refutational completeness:

Theorem 154 (Refutational Completness of Resolution)**.** *If the CNF formula $\varphi_1 \wedge \ldots \wedge \varphi_n$ is unsatisfiable, then there is a derivation by resolution of $\square$ starting from the clauses $\varphi_1, \varphi_2, \ldots, \varphi_n$.*

The proof is beyond the scope of the course, but it is not too complicated in case you want to prove it yourself and I recommend this exercise for the more curious minds among you.