9.4 Soundness

Like natural deduction, resolution is sound. This section shows that this is indeed the case.

Lemma 148. *Let* $\varphi \in \{\varphi_1, \ldots, \varphi_n\}$. *We have that*

$$\varphi_1, \ldots \varphi_n \models \varphi.$$

Exercise 149. *Prove Lemma 148.*

Lemma 150. *Let C, D be two clauses and let $a \in A$ be a propositional variable. We have that*

$$C \cup \{a\}, D \cup \{\neg a\} \models C \vee D.$$

Exercise 151. *Prove Lemma 150.*

Theorem 152 (Soundness of Resolution). *If there is a proof by resolution of*

Theorem 152 (Soundness of Resolution)**.** *If there is a proof by resolution of φ from $\varphi_1, \ldots, \varphi_n$, then*

$$\varphi_1, \ldots, \varphi_n \models \varphi.$$

Proof: Let $\psi_1, \ldots, \psi_m$ be a proof by resolution of φ from $\varphi_1, \ldots, \varphi_n$.

We will prove by induction on $i \in \{1, 2, \ldots, m\}$ that

$$\varphi_1, \ldots, \varphi_n \models \psi_i.$$

Let $i \in \{1, 2, \ldots, m\}$ be an integer. We assume by the induction hypothesis that

$$\varphi_1, \ldots, \varphi_n \models \psi_l \text{ for any } l \in \{1, 2, \ldots, i-1\}$$

and we prove that

$$\varphi_1, \ldots, \varphi_n \models \psi_i.$$

By the definition of a formal proof by resolution, we must be in one of the following two cases:

1. $\psi_i \in \{\varphi_1, \ldots, \varphi_n\}$. In this case we have

   $$\varphi_1, \ldots, \varphi_n \models \psi_i$$

   by Lemma 148, which is what we had to show.

2. ψ_i was obtained by resolution from ψ_j, ψ_k with $1 \leq j, k < i$. In this case, ψ_j must be of the form $\psi_j = C \cup a$, ψ_k must be of the form $\psi_k = D \cup \{\neg a\}$ and $\psi_i = C \cup D$, where C, D are clauses and $a \in A$ is a propositional variable.

   By the induction hypotheses that $\varphi_1, \ldots, \varphi_n \models \psi_j$ and that $\varphi_1, \ldots, \varphi_n \models \psi_k$. Replacing ψ_j and ψ_k as detailed above, we have that

   $$\varphi_1, \ldots, \varphi_n \models C \cup \{a\}$$

   and that

   $$\varphi_1, \ldots, \varphi_n \models D \cup \{\neg a\}.$$

   We prove that

   $$\varphi_1, \ldots, \varphi_n \models C \cup D.$$

   Let τ be a model of φ_1, ..., and φ_n. We have that τ is a model of $C \cup \{a\}$ and of $D \cup \{\neg a\}$ by the semantical consequences above. By Lemma 150, it follows that τ is a model of $C \cup D$. But $\psi_i = C \cup D$ and therefore τ is a model ψ_i. As τ was any model of all of φ_1, ..., and φ_n, it follows that

   $$\varphi_1, \ldots, \varphi_n \models \psi_i,$$

   which is what we had to prove.

In both cases, we have established

$$\varphi_1, \dots, \varphi_n \models \psi_i$$

for all $1 \leq i \leq m$. As $\psi_1, \dots, \psi_m$ is a proof of φ, it follows that $\psi_m = \varphi$ and therefore, for $i = m$, we have

$$\varphi_1, \dots, \varphi_n \models \varphi,$$

which is what we had to prove.

q.e.d.