Lecture Notes on

GRAPH THEORY

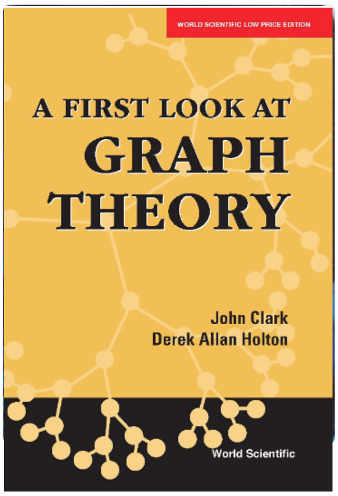
By

Dr. *Rashid A*
Assistant Lecturer
Department of Information Technology
National Institute of Technology Karnataka
Surathkal, Mangalore -575 025
E-mail: rashid.a.maths@gmail.com

Prepared for: B.Tech (in **Information Technology**) at NITK Surathkal,

Odd semester-2024 (Course: **IT203** - Discrete Mathematics)

Reference:



A FIRST LOOK
AT
GRAPH THEORY.
By
John Clark and Derek Allan Holton.

Contents

Section 1

An Introduction to Graphs

(Reference text: 'A First Look at Graph Theory' by John Clark and Derek Allan Holton)

1.1 Introduction

Graph theory is a branch of mathematics which deals the problems, with the help of diagrams. There are may applications of graph theory to a wide variety of subjects which include operations research, physics, chemistry, computer science and other branches of science. In this chapter we introduce some basic concepts of graph theory and provide variety of examples. We also obtain some elementary results.

Definition 1.1.1. *A graph $G = (V(G), E(G))$ consists of two finite sets: $V(G)$, the vertex set of the graph, often denoted by just V, which is a nonempty set of elements called vertices, and $E(G)$, the edge set of the graph, often denoted by just E, which is a possibly* *empty set of elements called edges, such that each edge e in E is assigned an unordered pair of vertices (u, v) called the end vertices of e.*

Vertices are also sometimes called points, nodes, or just dots. If e is an edge with end vertices u and v then e is said to join u and v. Note that the definition of a graph allows the possibility of the edge e having identical end vertices, i.e., it is possible to have a vertex u joined to itself by an edge - such an edge is called a loop.

Example 1.1.1. *Let $G = (V, E)$ where*

$$V = \{a, b, c, d, e\}, \quad E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$

and the ends of the edges are given by:

$$\begin{aligned} &e_1 \leftrightarrow (a, b), \quad e_2 \leftrightarrow (b, c), \quad e_3 \leftrightarrow (c, c), \quad e_4 \leftrightarrow (c, d) \\ &e_5 \leftrightarrow (b, d), \quad e_6 \leftrightarrow (d, e), \quad e_7 \leftrightarrow (b, e), \quad e_8 \leftrightarrow (b, e) \end{aligned}$$

We can then represent G diagrammatically as in Figure 1.1.

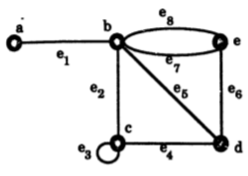


Figure 1.1: A graph with 5 vertices and 8 edges

Example 1.1.2. $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ *and* $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ *and the ends of the edges are given by:*

$$e_1 \leftrightarrow (v_1, v_2), \quad e_2 \leftrightarrow (v_2, v_3), \quad e_3 \leftrightarrow (v_2, v_4),$$
$$e_4 \leftrightarrow (v_2, v_5), \quad e_5 \leftrightarrow (v_2, v_5), \quad e_6 \leftrightarrow (v_3, v_3)$$

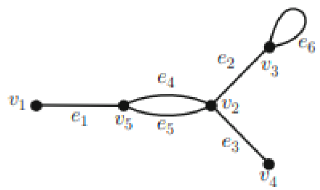


Figure 1.2: A graph with 5 vertices and 6 edges

1.2 Graph as Models

Problem 1.2.1. *Suppose that the graph of Figure* 1.3 *represents a network of telephone lines and poles. We are interested in the network's vulnerability to accidental disruption. We want to identify those lines and poles that must stay in service to avoid disconnecting the network. There is no single line whose disruption (removal)*

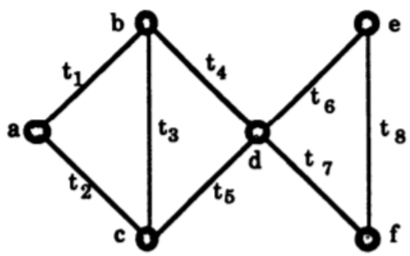


Figure 1.3: A network of telephone lines

will disconnect the graph (network), but the graph will become disconnected if we remove the two lines represented by the edges t_4 *and* t_5*, for example. When it comes to poles, the network is more vulnerable since there is a single vertex, vertex* d*, whose removal disconnects the graph.*

We may also want to find a smallest possible set of edges needed to connect the six vertices. There are several examples of such minimal sets. One is

$$\{t_1, t_3, t_5, t_6, t_7\}\,.$$

Problem 1.2.2. *Suppose that we have five people* A, B, C, D, E *and five jobs* a, b, c, d, e *and some of these people are qualified for certain jobs. Is there a feasible way of allocating one job to each person, or to show that no such matching up of jobs and people is possible. We can represent this situation by a graph having a vertex for each person and a vertex for each job, and edges joining people up to jobs for which they are qualified. Does there exist a feasible matching of people to jobs for the graph of Figure* 1.4?

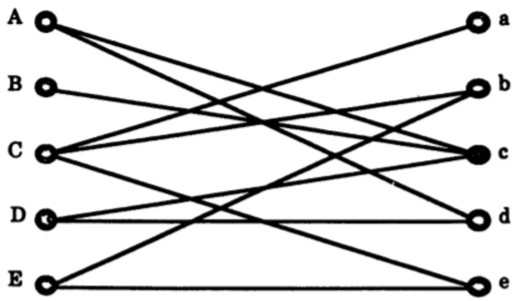


Figure 1.4: A job applications graph

The answer is no. The reason can be found by considering people A, B, and D. These three people as a set are collectively qualified for only two jobs, c and d, hence there is no feasible matching possible for these three people, much less all five people.

1.3 More definitions

Definition 1.3.1. *Let G be a graph. If two (or more) edges of G have the same end vertices then these edges are called* ***parallel****.*

For example, the edges e_4 and e_5 of the graph of Figure 1.2 are parallel.

Definition 1.3.2. *A vertex of G which is not the end of any edge is called* ***isolated****. Two vertices which are joined by an edge are said to be* ***adjacent*** *or* ***neighbours****. The set of all neighbours of a fixed vertex v of G is called the* ***neighbourhood set*** *of v and is denoted by* ***N(v)****.*

Thus, in the graph of Figure 1.2, v_1 and v_5 are adjacent but v_1 and v_2 are not. The neighbourhood set $N(v_2)$ of v_2 is $\{v_5, v_3, v_4\}$.

Definition 1.3.3. *A graph is called* ***simple*** *if it has no loops and no parallel edges.*

Example 1.3.1. *Let G be graph with 4 vertices and 3 edges given below in Figure 1.5. Which is a simple graph since it has no loops and parallel edges.*

Definition 1.3.4. *A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to the graph $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence*

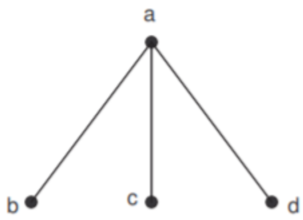


Figure 1.5: Simple graph

between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its end points the vertices u_2 and v_2 in G_2 which correspond to u_1 and v_1 respectively. Such a pair of correspondences is called a graph ***isomorphism****.*

Example 1.3.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (v_2, E_2)$ be two graphs with vertex set $V_1 = \{e_1, e_2, e_3, e_4, e_5\}$ and $V_2 = \{c_1, c_2, c_3, c_4, c_5\}$ respectively. Their Adjacency is given in the Figure 1.6. We can find and one-one correspondence between the vertices and edges of the graph G_1 and G_2 such a way that*

$e_1 \leftrightarrow c_1$ $e_2 \leftrightarrow c_3$ $e_3 \leftrightarrow c_5$ $e_4 \leftrightarrow c_2$ $e_5 \leftrightarrow c_4$

This maps preserves adjacency also.

Definition 1.3.5. *A simple graph G is said to be* ***complete*** *if every pair of distinct vertices of G are adjacent in G: Any two complete graphs each on a set of n vertices are isomorphic and each such graph is denoted by K_n.*

Example 1.3.3. ***Examples for complete graphs :***

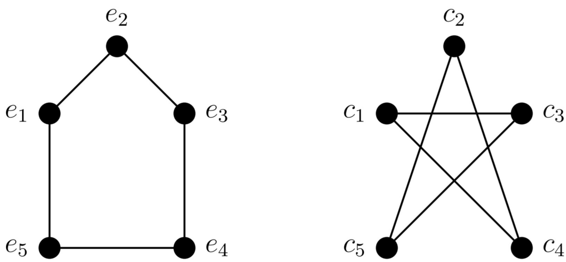


Figure 1.6: Isomorphic graphs with 5 vertices

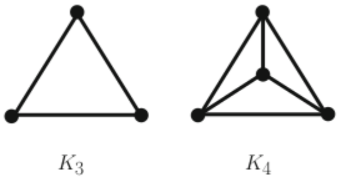
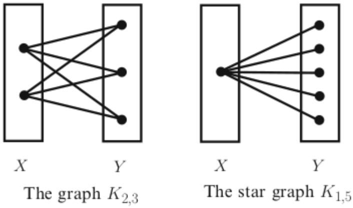


Figure 1.7: Complete Graph K_3 and K_4

Definition 1.3.6. *An **empty graph** (or trivial) is a graph with no edges. Let G be a graph. If the vertex set V of G can be partitioned into two nonempty subsets X and Y (i.e., $X \cup Y = V$ and $X \cap Y = \emptyset$) in such a way that each edge of G has one end in X and one end in Y then G is called **bipartite**. The partition $V = X \cup Y$ is called a **bipartition** of G*

*A **Complete bipartite graph** is a simple bipartite graph G, with bipartition $V = X \cup Y$, in which every vertex in X is joined to every vertex of Y. If X has m vertices and Y has n vertices, such a graph is denoted by $K_{m,n}$*

Example 1.3.4. ***Example of complete bipartite graph***



1.4 Vertex degrees

Definition 1.4.1. *An edge e of a graph G is said to be **incident** with the vertex v if v is an end vertex of e. In this case we also say that v is incident with e. Two edges e and f which are incident with a common vertex v are said to be **adjacent**.*

Let v *be a vertex of the graph G. The **degree** $d(v)$ (or $d_G(v)$ if we want to emphasize G) of v is the number of edges of G incident with v, counting each loop twice, i.e., it is the number of times v is an end vertex of an edge.*

Example 1.4.1. *Let G be a graph with 5 vertices 7 edges given below.*

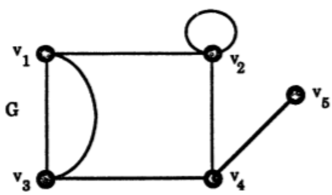


Figure 1.8: The graph G

In the graph of Figure 1.8 *we have* $d(v_1) = 3$, $d(v_2) = 4$,

$d(v_3) = 3$, $d(v_4) = 3$, $d(v_5) = 1$ *and note that in this example*

$$\begin{aligned} & d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) \\ & = 14 \\ & = 2 \times (\text{ number of edges in } G) \end{aligned}$$

Theorem 1.4.1. ***Euler :****The sum of the degrees of the vertices of a graph is equal to twice the number of its edges.*

Proof. Let the degrees of the vertices be $d_1, d_2, \ldots, d_n$. Let $n = \sum_{i=1}^{n} d_i$ and m denote the total number of edges in the graph. Notice that each edge contributes 2 to the sum of degrees of the vertices, thus $n = 2m$. □

Corollary 1.4.1.1. *In any graph G the number of vertices of odd degree is even.*

Proof. Let V_1 and V_2 be the subsets of vertices of graph G with vertices of odd and even degrees respectively. Then from the previous theorem, we have

$$\begin{aligned} 2m &= n \\ &= \textstyle\sum_{v \in V} d_G(v) \\ &= \textstyle\sum_{v \in V_1} d_G(v) + \sum_{v \in V_2} d_G(v) \end{aligned} \tag{1.1}$$

Both $2m$ and $\sum_{v \in V_2} d_G(v)$ are even, which forces $\sum_{v \in V_1} d_G(v)$ to be even.

□

Definition 1.4.2. *If for some positive integer k, $d(v) = k$ for every vertex v of the graph G then G is called* ***k-regular****. A regular graph is one that is k-regular for some k.*

1.5 Subgraphs

Definition 1.5.1. *Let H be a graph with vertex set $V(H)$ and edge set $E(H)$ and, similarly, let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then we say that H is a* ***subgraph*** *of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, we also say that G is a* ***supergraph*** *of H.*

Example 1.5.1. *For example in Figure 1.9, G_1 is a subgraph of both G_2 and G_3 but G_3 is not a subgraph of G_2.*

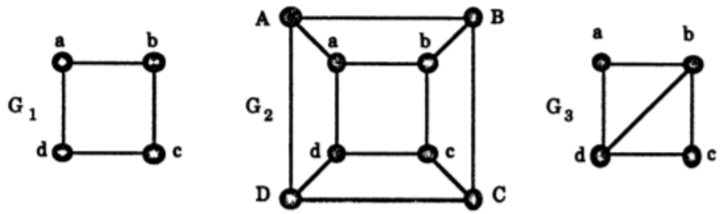


Figure 1.9: $G_1 \subseteq G_2$, $G_1 \subseteq G_3$ and G_3 not subgraph of G_2

Definition 1.5.2. *A subgraph H of a graph G is a* ***proper subgraph*** *of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A subgraph H of G is a* ***spanning subgraph*** *of G if $V(H) = V(G)$.*

From the definition we can get that ant simple graph with n vertices is a proper subgraph of complete graph K_n.

In Figure 1.7, G_1 is a proper subgraph of G_2. G_1 is a spanning subgraph of G_3, since both the graph have same vertex set.

Definition 1.5.3. *If $G = (V, E)$ and V has at least two elements (i.e., G has at least two vertices), then for any vertex v of $G, G - v$*

denotes the subgraph of G with vertex set $V - \{v\}$ and whose edges are all those of G which are not incident with v, i.e., $G - v$ is obtained from G by removing v and all the edges of G which have v as an end. $G - v$ is referred to as a ***vertex deleted subgraph****. If $G = (V, E)$ and e is an edge of G then $G - e$ denotes the subgraph of G having V as its vertex set and $E - \{e\}$ as its edge set, i.e., $G - e$ is obtained from G by removing the edge e, (but not its endpoint(s)). $G - e$ is referred to as an* ***edge deleted subgraph****.*

Example 1.5.2. ***Example of vertex deleted graph***

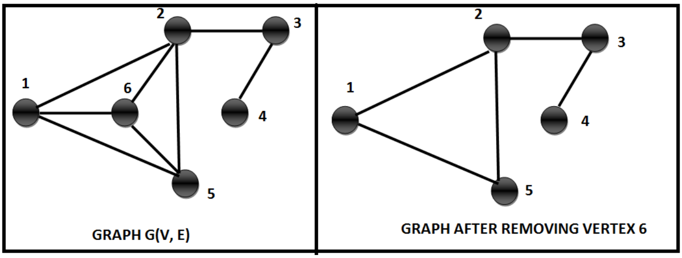


Figure 1.10: A vertex deleted graph $G - v_6$

We extend the above definition to cater for the deletion of several vertices or edges.

Definition 1.5.4. *If $G = (V, E)$ and U is a proper subset of V then $G - U$ denotes the subgraph of G with vertex set $V - U$ and whose edges are all those of G which are not incident with any vertex in U.*

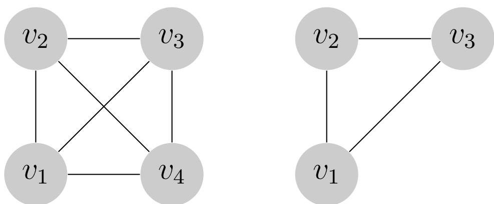
If F is a subset of the edge set E then $G - F$ denotes the subgraph of G $|$ with vertex set V and edge set $E - F$, i.e., obtained by deleting all the edges in F, but not their endpoints. $G - U$ and

*$G - F$ are also referred to as **vertex deleted subgraph** and **edge deleted subgraph**(respectively).*

Definition 1.5.5. *By deleting from a graph G all loops and in each collection of parallel edges all edges but one in the collection we obtain a simple spanning subgraph of G, called the **underlying simple graph** of G.*

Definition 1.5.6. *If U is a nonempty subset of the vertex set V of the graph G then the subgraph $G[U]$ of G induced by U is defined to be the graph having vertex set U and edge set consisting of those edges of G that have both ends in U Similarly if F is a nonempty subset of the edge set E of G then the subgraph $G[F]$ of G induced by F is the graph whose vertex set is the set of ends of edges in F and whose edge set is F.*

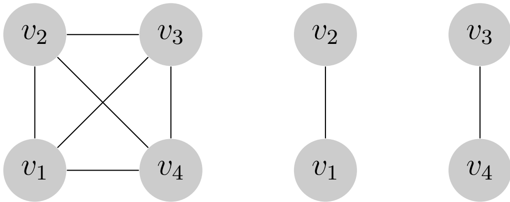
Example 1.5.3. *Let $G = K_4$ with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and let $U = \{v_1, v_2, v_3\}$. Then the induced graph of G by U, that is $G[U]$ is given in below figure.*



$G = (V, E)$ $\qquad$ $G[U]$

Definition 1.5.7. *Two subgraphs G_1 and G_2 of a graph G are said to be **disjoint** if they have no vertex in common, and **edge disjoint** if they have no edge in common.*

Example 1.5.4. *Let $G = K_4$ with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and G_1 and G_2 are subgraphs of G with vertex set $V_1(G) = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4\}$ respectively whose picture is given below.*



$G = (V, E)$ $\qquad$ G_1 $\qquad$ G_2

Here the graph G_1 and G_2 are disjoint graphs. Since they have no common vertex and also they are edge disjoint also.

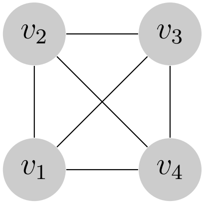
Definition 1.5.8. *Given two subgraphs G_1 and G_2 of G, the **union** $G_1 \cup G_2$ is the subgraph of G with vertex set consisting of all those vertices which are in either G_1 or G_2 (or both) and with edge set consisting of all those edges which are in either G_1 or G_2 (or both); symbolically*

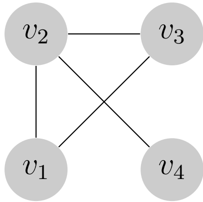
$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$
$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$

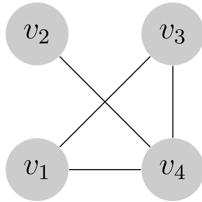
Definition 1.5.9. *If G_1 and G_2 are two subgraphs of G with at least one vertex in common then the **intersection** $G_1 \cap G_2$ is given by*

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$
$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$

Example 1.5.5. *Let $G = K_4$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and G_1 and G_2 are subgraphs of G given below.*

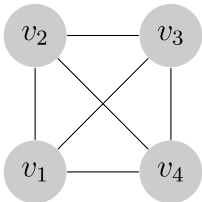


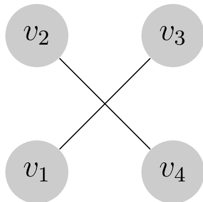




$G = (V, E)$ $\quad$ G_1 $\quad$ G_2

Then the union and intersection of graphs $G_1 \cup G_2$ and $G_1 \cap G_2$ is given below.





$G_1 \cup G_2$ $\quad$ $G_1 \cap G_2$

1.6 Paths and Cycles

Definition 1.6.1. *A **walk** in a graph G is an alternating sequence $W : v_0e_1v_1e_2v_2, \ldots, e_nv_p$ of vertices and edges, beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i. v_0 is the origin and v_p is the terminus of W and the walk W is said to join v_0 and v_p. If the graph G is simple, then a walk is determined by the sequence of its vertices. The walk is closed if $v_0 = v_p$ and otherwise it is open . A walk is called a **trail** if all the edges appearing in the walk are distinct and it is called a **path** if all the vertices are distinct. Thus, every path in G is automatically a trail in G. When writing a path, we usually omit the edges. A **cycle** is a* *closed trail in which the vertices are all distinct. A cycle is odd or even depending on whether its length is odd or even. The* ***length*** *of a walk is the number of edges in it. A walk of length 0 consists of just a single vertex.*

Clearly any two paths with the same number of vertices are isomorphic. A path with n vertices will sometimes be denoted by P_n. Note that P_n has length $n-1$.

Theorem 1.6.1. *Given any two vertices u and v of a graph G, every $u-v$ walk contains a $u-v$ path, i.e., given any walk*

$$W = ue_1v_1 \ldots v_{k-1}e_kv$$

then, after some deletion of vertices and edges if necessary, we can find a sub sequence P of W which is a $u-v$ path.

Proof. If $u = v$, i.e., if W is closed, then the trivial path $P = u$ will do. Now suppose $u \neq v$, i.e., W is open and let the vertices of W ke given, in order, by

$$u = u_0, u_1, u_2, \ldots, u_{k-1}, u_k = v$$

If none of the vertices of G occurs in W more than once then W is already a $u-v$ path and so we are finished by taking $P = W$

So now suppose that there are vertices of G that occur in W twice or more. Then there are distinct i, j, with $i < j$, say, such that $u_i = u_j$. If the terms $u_i, u_{i+1}, \ldots, u_{j-1}$ (and the preceding edges) are deleted from W then we obtain a $u-v$ walk W_1 having fewer vertices than W. If there is no repetition of vertices in W_1, then W_1 is a $u-v$ path and setting $P = W_1$ finishes the proof.

If this is not the case, then we repeat the above deletion procedure until finally arriving at a $u - v$ walk that is a path, as required. $\square$

Definition 1.6.2. *Let G be a graph. Two vertices u and v in G are said to be* ***connected*** *if there exits a $u - v$ path in G.*

Remark: The relation 'connected' forms a equivalence relation on $V(G)$.

Definition 1.6.3. *A graph G is called connected if every two of its vertices are connected. A graph that is not connected is called* ***disconnected****. Given any vertex u of a graph G, let $C(u)$ denote the set of all vertices in G that are connected to u. Then the subgraph of G induced by $C(u)$ is called the connected component containing u, or simply the* ***component containing*** *u.*

Remark: If u and v are two connected vertices in the graph G, i.e., if there is a path from u to v, then, by the remarks above, $C(u) = C(v)$ and so u and v have the same connected component. Conversely, if u and v have the same component then v is in $C(u)$ so v and u must be connected.

Example 1.6.1. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ is given below.*

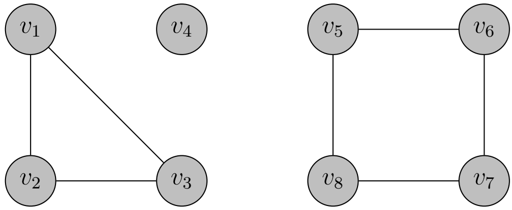


Figure 1.11: A graph with 3 connected components

The graph G has 3 components $C(v_1)$,$C(v_4)$ and $C(v_5)$.

Definition 1.6.4. *The number of components of G is called $\omega(G)$.*

Thus $\omega(G) = 3$ of a graph G in the Fig 1.11.

Definition 1.6.5. *A nontrivial closed trail in a graph G is called a **cycle** if its origin and internal vertices are distinct. In detail, the closed trail $C = v_1 v_2 \ldots v_n v_1$ is a cycle if*
(i) C has at least 1 edge and $v_1, v_2, \ldots, v_n$ are n distinct vertices.
*A cycle of length k, i.e., with k edges, is called a k-**cycle**. A k-cycle is called odd or even depending on whether k is odd or even.*
*A 3-cycle is often called a **triangle**. Clearly any two cycles of the same length are isomorphic. An n-cycle, i.e., a cycle with n vertices, will sometimes be denoted by C_n.*

Example 1.6.2. *Let $G = (V, E)$ be graph with the vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ is given in Fig 1.12.*

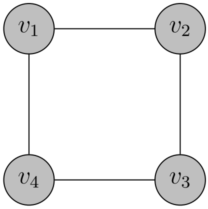


Figure 1.12: Cycle of length 4

Which is a cycle of length 4 and $C = v_1v_2v_3v_4v_1$

Theorem 1.6.2. *Let G be a nonempty graph with at least two vertices. Then G is bipartite if and only if it has no odd cycles.*

Proof. Suppose that G is bipartite with vertex set V and bipartition $V = X \cup Y$. Let $C = v_0v_1 \dots v_kv_0$ be a cycle of G. For the sake of argument, assume that v_0 is in X. Then, because G is bipartite, v_1 must be in Y. Similarly v_2 must be in X, v_3 in Y, etc. In fact, in general, the odd-indexed vertices v_{2i+1} must be in Y while the even-indexed vertices v_{2i} must be in X. Now, since v_0 is in X, we must have (at the 'other end' of the cycle) v_k in Y. Hence k must be an odd number. Thus the cycle $C = v_0v_1 \dots v_kv_0$ is even. Since C was any cycle in the graph G, G has no odd cycles.
Now, to prove the converse, we assume that G is a nonempty graph which has no odd cycles. We wish to show that G is bipartite. Now G will be bipartite if each of its nonempty connected components is bipartite, since, if these components are $C_1, C_2, C_3, \dots, C_n$ and their vertex sets $V_1, V_2, V_3, \dots, V_n$ have bipartitions $V_1 = X_1 \cup Y_1, \dots, V_n = X_n \cup Y_n$, then the vertex set V of G has bipartition

$V = X \cup Y$ where

$$X = X_0 \cup X_1 \cup X_2 \cup \ldots \cup X_n \text{ and } Y = Y_1 \cup Y_2 \cup \ldots \cup Y_n$$

where X_0 is the set of isolated vertices in G. As a result of this it is enough to show that if G is a nonempty connected graph with no odd cycles then G is bipartite.

With this assumption, let u be a fixed vertex of G. We define two subsets of the vertex set V of G as follows: X is the set of all vertices v of G with the property that any shortest $u - v$ path of G has even length, Y is the set of all vertices w of G with the property that any shortest $u - w$ path of G has odd length, i.e., X consists of those vertices of G an 'even distance' from u, while Y consists of those vertices of G an 'odd distance' from u.

Note that u itself is in X. Then, clearly, $V = X \cup Y$ and X and Y have no element in common. We show that $V = X \cup Y$ is a bipartition of G by showing that any edge of G must have one end in X and the other in Y.

Let v and w be two vertices both in X and assume they are adjacent. Let P and Q be a shortest $u - v$ path and a shortest $u - w$ path respectively, say

$$P = u_1, u_2, \ldots, u_{2n+1} \text{ and } Q = w_1, w_2, \ldots, w_{2m+1}$$

(so that $u = u_1 = w_1, v = u_{2n+1}$ and $w = w_{2m+1}$.) Suppose that w' is a vertex that the two paths have in common, and further that w' is the last such vertex. (If $v = w$ then of course $w' = v = w$. Moreover, there is always such a vertex w' since the vertex u is common to both paths.) Then that part of P from u to w' is a shortest path from u to w' and that part of Q from u to w' is a shortest path from u to w' also. In other words, we have two shortest paths from u to w'. It follows that, since these two paths have the same length, there exists an i such that $w' = u_i = w_i$. However this produces an odd cycle in G :

$$C = \underbrace{u_i u_{i+1} \cdots u_{2n+1}}_{*} \underbrace{w_{2m+1} w_{2m} \ldots w_i}_{**}$$

since if i is odd then the above parts $*$ and $**$ are both of even length while if i is even then they are both of odd length, giving the total length of C as odd $+1+$ odd or even $+1+$ even, in either case odd. Since G has no odd cycles, the assumption that v and w are adjacent is wrong.

Hence there are no edges in G joining vertices of X. A similar argument shows that there are no edges of G joining vertices of Y. Hence G is bipartite, as required. $\square$

1.7 Matrix representation of a graph

Definition 1.7.1. *Let G be a graph with n vertices, listed as v_1,v_2,v_3 …, v_n. The **adjacency matrix** of G, with respect to this particular listing of the n vertices of G, is the $n \times n$ matrix $A(G) = (a_{ij})$ where the (i, j) th entry a_{ij} is the number of edges joining the vertex v_i to the vertex v_j.*

Example 1.7.1. *Let $G = (V, G)$ be graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ is given in Figure 1.13. The Adjacency matrix $A(G)$ of graph G also listed below.*

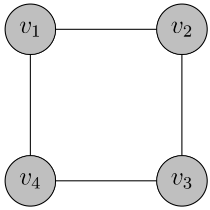


Figure 1.13: Cycle of length 4

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Figure 1.14: $A(G)$: a 4×4 matrix

Theorem 1.7.1. *Let G be a graph with n vertices $v_1, \ldots, v_n$ and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let k be any positive integer and let A^k denote the matrix multiplication of k copies of A. Then the (i, j) th entry of A^k is the number of different $v_i - v_j$ walks in G of length k.*

Theorem 1.7.2. *Let G be a graph with n vertices $v_1, \ldots, v_n$ and let A denote the adjacency matrix of G with respect to this listing of the vertices. Let $B = (b_{ij})$ be the matrix*

$$B = A + A^2 + \cdots + A^{n-1}$$

Then G is a connected graph if and only if for every pair of distinct indices i, j we have $b_{ij} \neq 0$, i.e., if and only if B has no zero entries *off the main diagonal.*

Proof. Let $a_{ij}^{(k)}$ denote the (i, j) th entry of the matrix A^k for each $k = 1, \ldots, n-1$ Then

$$b_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)} + \cdots + a_{ij}^{(n-1)}$$

However, by Theorem 1.5, $a_{ij}^{(k)}$ denotes the number of distinct walks of length k from v_i to v_j and so i.e., b_{ij} is the number of different $v_i - v_j$ walks of length less than n. Now suppose that G is connected. Then for every pair of distinct indices i, j there is a path from v_i to v_j. Since G has only n vertices this path goes through at most n vertices and so it has length less than n, i.e., there is at least 1 path from v_i to v_j of length less than n. Hence $b_{ij} \neq 0$ for each i, j with $i \neq j$, as required.

Conversely, suppose that for each distinct pair i, j we have $b_{ij} \neq 0$. Then, from above, there is at least 1 walk (of length less than n) from v_i to v_j. In particular, v_i is connected to v_j. Thus G is a connected graph, as required, since i and j were an arbitrary pair of distinct vertices. □

Definition 1.7.2. *Suppose that G has n vertices, listed as $v_1, \ldots, v_n$, and t edges, listed as $e_1, \ldots, e_t$. The **incidence matrix** of G, with respect to these particular listings of the vertices and edges of G, is the $n \times t$ matrix $M(G) = (m_{ij})$ where m_{ij} is the number of times that the vertex v_i is incident with the edge e_j, i.e.,*

$$m_{ij} = \begin{cases} 0 & \textit{if } v_i \textit{ is not an end of } e_j \\ 1 & \textit{if } v_i \textit{ is an end of the non-loop } e_j \\ 2 & \textit{if } v_i \textit{ is an end of the loop } e_j \end{cases}$$

Example 1.7.2. *Let $G = (v, G)$ be graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ is given in Figure 1.15. The incidence matrix $M(G)$ of graph G also listed below.*

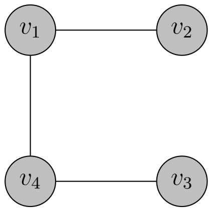


Figure 1.15: A graph with 4 vertices and 3 edges

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Figure 1.16: $M(G)$: a 4×3 matrix

Section 2

Trees and Connectivity

2.1 Definitions and Simple Properties

Definition 2.1.1. *A Graph G is called* ***acyclic*** *if it contains no cycles.*

Definition 2.1.2. *A Graph G is called a* ***Tree*** *if it is a connected acyclic graph.*

Example 2.1.1. *Let $G = (V, E)$ is a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ given in Figure 2.1.*

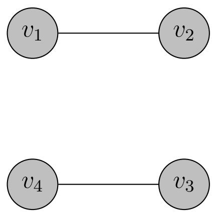


Figure 2.1: Acyclic graph

The graph G do not contain any cycles. Hence G is a acyclic graph.

Example 2.1.2. *Let $G = (V, E)$ is a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ given in Figure 2.2.*

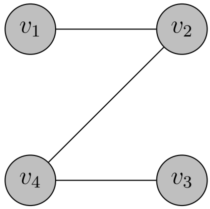


Figure 2.2: A Tree with 4 vertices

The graph G is a connected graph also it do not contain a cycle. Hence it is a tree.

Theorem 2.1.1.

1. *Let u and v be distinct vertices of a tree T. Then there is precisely one path from u to v.*

2. *Let G be a graph without any loops. If for every pair of distinct vertices u and v of G there is precisely one path from u to v, then G is a tree.*

Proof.

1. Suppose that the result is false. Then there are two different paths from u to v, say $P = uu_1u_2 \dots u_mv$ and $P' = uv_1v_2 \dots v_nv$. Let w be the first vertex after u which belongs to both P and P'. (The vertex w might well be v, but at least there is such a vertex) Then $w = u_i = v_j$ for some indices i and j. (See Figure 2.3.)

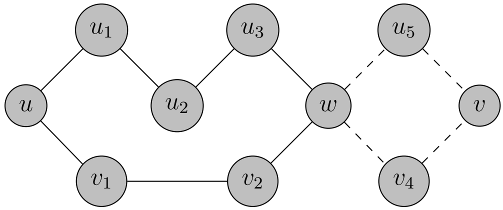


Figure 2.3: A Tree with 4 vertices

This produces a cycle $C = uu_1 \ldots u_i v_{j-1} \ldots v_1 u$, since no two of the $'u'$ terms are repeated (since P is a path), no two of the $'v'$ terms are repeated (since P' is a path), and, by the definition of w, no" u " term in C is the same as a vv'' term. Since T is a tree, T bas no cycles. This contradiction means that our initial assumption (the first sentence of the proof) must be false. Thus there is precisely one path from u to v.

2. Since, by assumption, there is a path between each pair of vertices u and v, G must be connected. Thus, to show that G is a tree it remains to show that G has no cycles. Firstly, since G has no loops it has no cycles of length one. Now suppose that G has a cycle of length greater than one, say $C = v_1 v_2 \ldots v_n v_1$ where $n \geq 2$. Since, by definition, a cycle is a trail, the edge $v_n v_1$ does not appear in the path $v_1 v_2 \ldots v_n$. Thus $P = v_1 v_n$ and $P' = v_1 v_2 \ldots v_n$ are two different paths from v_1 to v_n. This contradicts our assumptions. Hence G has no cycles and so is a tree.

□

Theorem 2.1.2. *Let T be a tree with at least two vertices and let $P = u_0 u_1 \ldots u_n$ be a longest path in T (so that there is no path in*

T of length greater than n). Then both u_0 and u_n have degree 1, i.e., $d(u_0) = 1 = d(u_n)$

Proof. Suppose that $d(u_0) > 1$. The edge $f = u_0 u_1$ contributes 1 to the degree of u_0 and so there must be another edge e from u_0 to a vertex v of T (which is different: from f). If this vertex v is one of the vertices of the path P then we may set $v = u_i$for some $i = 0, 1, \ldots, n$ and this produces a cycle $C = u_0 u_1 \ldots u_i u_0$ (the last edge being e). Since T is a tree it has no cycles and so this is a contradiction. Thus the remaining possibility for v is that it is not one of the vertices of the path P. But then $P_1 = v u_0 u_1 \ldots u_n$, where the first edge is e_2 gives a path of length $n+1$ in T, in contradiction to our assumption that P is a longest path (and of length n). This final contradiction shows that there is no such edge e and so $d(u_0) = 1$, as required. Similarly $d(u_n) = 1$, as required. □

Corollary 2.1.2.1. *Any tree T with at least two vertices has more than one vertex of degree 1.*

Proof. In such a tree T there is a longest path P (of length greater than 0) and so the Theorem produces at least two vertices of degree 1. □

Theorem 2.1.3. *If T is a tree with n vertices then it has precisely $n-1$ edges.*

Proof. We use induction on n. When $n = 1$, i.e., T has only 1 vertex, then, since it has no loops, T can not have any edges, i.e., it has $n - 1 = 0$ edges. This establishes that the result is true for $n = 1$

Now suppose that the result is true for $n = k$ where k is some positive integer. We use this to show that the result is true for $n = k + 1$. Let T be a tree with $k + 1$ vertices and let u be a vertex of degree 1 in T. (Note that such a vertex exists by Corollary 2.3.) Let $e = uv$ denote the unique edge of T which has u as an end. Then if x and y are vertices in T both different from u, any path P joining x to y does not go through the vertex u since if it did it would involve the edge e twice. Thus the subgraph $T - u$, obtained from T by deleting the vertex u (and the edge e), is connected. Moreover if C is a cycle in $T - u$ then C would be a cycle in T - impossible, since T is a tree. Thus the subgraph $T - u$ is also acyclic. Hence $T - u$ is a tree. However $T - u$ has k vertices (since T has $k + 1 vertices$) and so, by our induction assumption, $T - u$ has $k - 1$ edges. Since $T - u$ has exactly 1 edge less than T (the edge e), it follows that T has k edges, as required. In other words, assuming the result is true for k, we have shown that it is true for $k + 1$. Thus, by the principle of mathematical induction, it is true for all positive integers k. $\square$

Note: An acyclic graph is also called **Forest**.

Theorem 2.1.4. *Let G be an acyclic graph with n vertices and k connected components, i.e., $\omega(G) = k$. Then G has $n - k$ edges.*

Proof. Denote the k components of G by $C_1, C_2, \ldots, C_k$ and suppose that for each $i, 1 \leq i \leq k$, the i th component C_i has n_i vertices. Then $n = n_1 + n_2 + \cdots + n_k$ Also, since each C_i is a tree, by Theorem 2.1.3 it has $n_i - 1$ edges and so since each edge of G belongs to precisely one component of G the total number of edges in G is $(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)$. Thus G has $(n_1 + n_2 + \cdots + n_k) - k$ edges, i.e., $n - k$ edges, as required. $\square$

2.2 Bridges

Theorem 2.2.1. *Let e be an edge of the graph G and, as usual, let $G - e$ be the subgraph obtained by deleting e. Then $\omega(G) \leq \omega(G - e) \leq \omega(G) + 1$.*

Definition 2.2.1. *An edge e of a graph G is called a* ***bridge*** *(or a cut edge or an isthmus) if the subgraph $G - e$ has more connected components than G has.*

Example 2.2.1. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is given in figure 2.4*

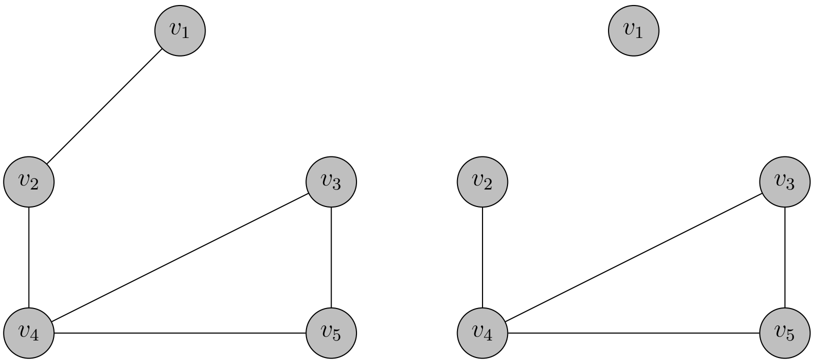


Figure 2.4: A Graph G and $G - e_1$

Here the edge $e_1 = v_1v_2$ is a bridge, since the graph G with removal of the edge e_1 has more components than G. That is, $G - e_1$ has 2 components but G has only one component.

Theorem 2.2.2. *An edge e of a graph G is a bridge if and only if e is not part of any cycle in G.*

Proof. Let e have end vertices u and v. If e is not a bridge then, by the above remarks, it is either a loop or there is a path $P = uu_1 \dots u_n v$ from u to v, different from the edge e. If it is a loop then it forms a cycle (by itself). If there is such a path P then $C = uu_1 \dots u_n vu$, the concatenation of P with e, is a cycle in G. This shows that if e is not a bridge then it is part of a cycle. This is equivalent to saying that if e is not part of any cycle then e must be a bridge.

Conversely, suppose that e is part of some cycle $C = u_0 u_1 \dots u_m$ in G. Let $e = u_i u_{i+1}$. In the case where $m = 1, C = u_0 u_1$ and so C is just the edge e and e is a loop. On the other hand, if $m > 1$ then $P = u_i u_{i-1} \dots u_0 u_{m-1} \dots u_{i+1}$ is a path from u to v different from e. (See Figure 2.5.) Thus, by the remarks preceding the proof, e is not a bridge. This shows that if e is a bridge then it is not part of any cycle in G, completing the proof. □

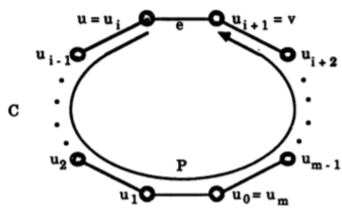


Figure 2.5:

Theorem 2.2.3. *Let G be a connected graph. Then G is a tree if and only if every edge of G is a bridge, i.e., if and only if for every edge e of G the subgraph $G - e$ has two components.*

Proof. Suppose that G is a tree. Then G is acyclic, i.e., it has no cycles, and so no edge of G belongs to a cycle. In other words, if e is any edge of G then, by Theorem 2.2.2, it is a bridge, as required.

Conversely suppose that G is connected and that every edge e of G is a bridge. Then G can have no cycles since any edge belonging to a cycle is not a bridge, by Theorem 2.2.2. Hence G is acyclic and so is a tree as required. □

Corollary 2.2.3.1. *A connected graph G with n vertices has at least $n-1$ edges.*

Theorem 2.2.4. *Let G be a graph with n vertices. Then the following three statements are equivalent:*
(i) G is a tree,
(ii) G is an acyclic graph with $n-1$ edges,
(iii) G is a connected graph with $n-1$ edges.

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii): Suppose that G is a tree. Then by definition G is an acyclic graph and by Theorem 2.1.3 it has $n-1$ edges. Thus (ii) holds.
(ii) $\Rightarrow$ (iii): Assume that G is an acyclic graph with $n-1$ edges and, as usual, let $\omega(G)$ denote the number of connected components of G. Then, by Theorem 2.1.4, G has $n-\omega(G)$ edges. Thus $\omega(G)=1$, in other words, G is connected. This establishes (iii).
(iii) $\Rightarrow$ (i): Assume that G is a connected graph with $n-1$ edges. To prove (i), i.e., that G is a tree, we must show that G is acyclic. We do this by contradiction. Thus, assume that G is not acyclic. Then G contains a cycle and every edge of this cycle can not be a bridge, by Theorem 2.2.2. Choose such an edge e. Then, since e is

not a bridge, $G - e$ is still connected. However $G - e$ has $n - 2$ edges and n vertices, which is impossible by the above corollary. This contradiction has arisen from our assumption that G is not acyclic. Hence G is acyclic and so a tree as required. $\square$

2.3 Spanning Trees

Definition 2.3.1. *A* ***spanning tree*** *of a graph G is a spanning subgraph of G that is a tree.*

Theorem 2.3.1. *A graph G is connected if and only if it has a spanning tree.*

Proof. Suppose that G is connected with n vertices and q edges. Then, by Corollary 2.2.3.1, we have $q \geq n - 1$. If $q = n - 1$ then, by (iii) $\Rightarrow$ (i) of Theorem 2.2.4, G is a tree and so we can take $T = G$ as a spanning tree of G.

If $q > n - 1$ then, by Theorem 2.1.3 (or by (i) $\Rightarrow$ (iii) of Theorem 2.2.4), G is not a tree and so G must contain a cycle. Let e_1 be an edge of such a cycle. Then the subgraph $G - e_1$ is connected (since e_1 is not a bridge), has n vertices, and has $q - 1$ edges. If $q - 1 = -1$ then, repeating the above argument gives $T = G - e_1$ as a spanning tree of G.

If $q - 1 > n - 1$ then $G - e_1$ is not a tree so, as before, there is a cycle in $G - e_1$. Removing an edge e_2 from such a cycle gives a subgraph $G - \{e_1, e_2\} = (G - e_1) - e_2$ which is connected, has n vertices and $q - 2$ edges. We keep on repeating this process, deleting $q-n+1$ edges altogether, to eventually produce a subgraph T which is connected, has n vertices and $q - (q - n + 1) = n - 1$

edges. Thus by Theorem 2.2,4, T is a tree and since it has the same vertex set as G it is a spanning tree of G.

Conversely, if G has a spanning subtree T, then given any two vertices u and v of G then u and v are also vertices of the connected subgraph T. Thus u and v are connected by a path in T and so by a path in G. This shows that G is connected. $\square$

Example 2.3.1. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is given in Figure 2.6*

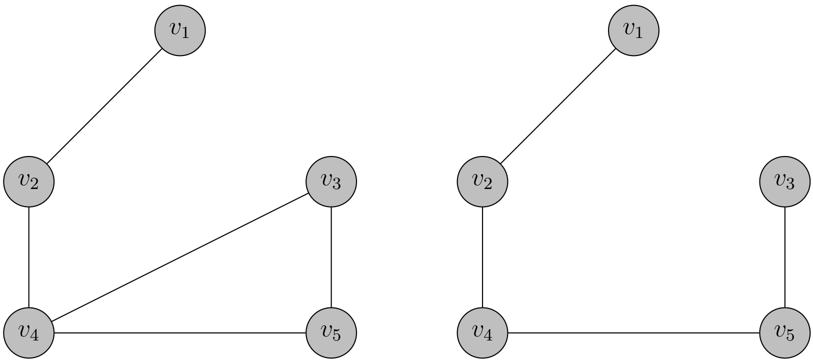


Figure 2.6: A Graph G and a spanning subgraph H

The graph H is a spanning subgraph of G and also tree. Hence it's a spanning tree.

Example 2.3.2. *Figure 2.7 illustrates a graph K_4 and all it's different spanning subgraphs.*

Theorem 2.3.2. *(Cayley, 1889) The complete graph K_n has n^{n-2} different spanning trees.*

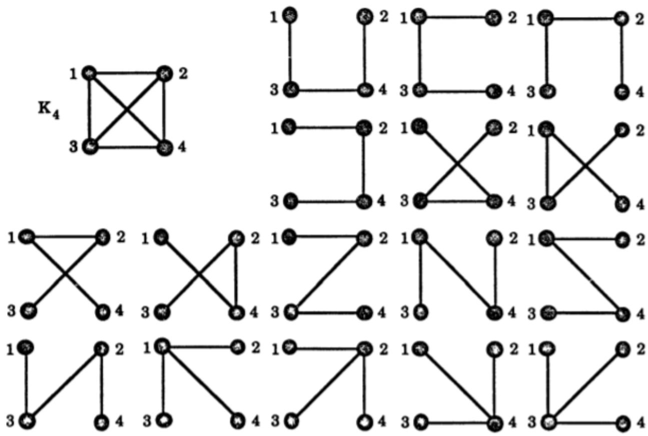


Figure 2.7: K_4 and it's 16 different spanning graphs

2.4 Cut Vertices and Connectivity

Definition 2.4.1. *A vertex v of a graph G is called a **cut vertex** (or articulation point) of G if $\omega(G - v) > \omega(G)$.*

Example 2.4.1. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is given in figure 2.8.*

Here the vertex v_4 is a cut vertex. Since $\omega(G - v_4) = 2$ and $\omega(G) = 1$. That is, $G - v_4$ has more components than G.

Theorem 2.4.1. *Let v be a vertex of the connected graph G. Then v is a cut vertex of G if and only if there are two vertices u and w of G, both different from v, such that v is on every $u - w$ path in G.*

Proof. First let v be a cut vertex of G. Then $G - v$ is disconnected and so there are vertices u and w of G which lie in different components of $G - v$. Thus, although there is a path in G from u to w,

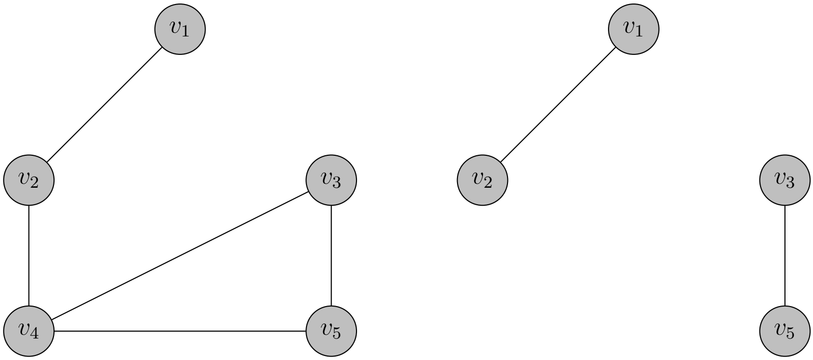


Figure 2.8: A Graph G and $G - v_4$

there is no such path in $G - v$. This implies that every path in G from u to w contains the vertex v, as required.

Conversely, suppose that u and w are two vertices of G, different from v, such that every path in G from u to w contains v. Then there can be no path from u to w in $G - v$. Thus $G - v$ is disconnected (with u and w lying in different components). Hence v is a cut vertex, as required. $\square$

Theorem 2.4.2. *Let G be a graph with n vertices, where $n \geq 2$. Then G has at least two vertices which are not cut vertices.*

Proof. Clearly we may suppose that G is a connected graph. We proceed by assuming the result is false for our G and so the proof will be complete if we derive a contradiction from this.

Thus we are assuming that there is at most one vertex in G which is not a cut vertex. Now let u, v be vertices in G such that the distance $d(u, v)$ between them is the greatest of distances between pairs of vertices in G, i.e., $d(u, v) = \mathrm{diam}(G)$. Since G

is connected and has at least two vertices, $u \neq v$. Thus, by our assumption, one of these two vertices must be a cut vertex, say v. Then $G - v$ is disconnected and so there is a vertex w in G which does not belong to the same component as u does in $G - v$. This implies that every uw path in G contains the vertex v.

It follows from this that the shortest path in G from u to w contains the shortest path from u to v and this contradiction completes our proof. $\square$

Definition 2.4.2. *Let G be a simple graph. The **vertex connectivity** of G, denoted by $\kappa(G)$, is the smallest number of vertices in G whose deletion from G leaves either a disconnected graph or K_1.*

Example 2.4.2. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is given in Figure 2.9.*

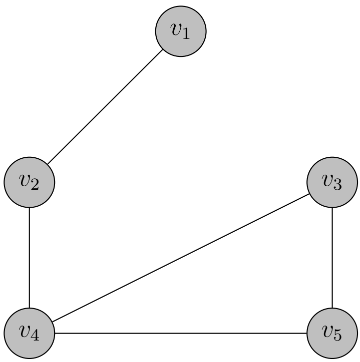


Figure 2.9: A Graph G with 5 vertices

Here vertex connectivity of G is one. Since the vertex v_4 is a cut vertex of the graph G deletion of that vertex results in a disconnected graph. Hence $\kappa(G) = 1$.

Example 2.4.3. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is given in figure 2.8.*

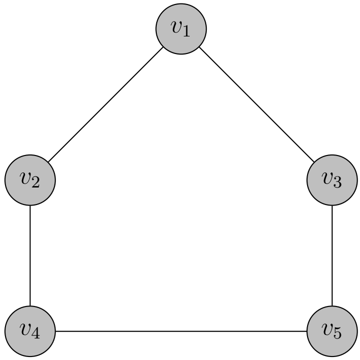


Figure 2.10: $G = C_5$

Here vertex connectivity of C_5 is two. Since it has no cut vertex therefore $\kappa(G) > 1$. Also removal of the vertices v_2 and v_3 result's in a disconnected graph. Hence $\kappa(G) = 2$.

Definition 2.4.3. *A simple graph G is called n-**connected** (where $n \geq 1$) if $\kappa(G) \geq n$.*

Note: It follows that G is 1 -connected if and only if G is connected and has at least two vertices. Moreover G is 2 -connected if and only if G is connected with at least three vertices but no cut vertices.

Definition 2.4.4. *Let u and v be two vertices of a graph G. A collection $\{P_{(1)}, \ldots, P_{(n)}\}$ of $u - v$ paths is said to be **internally disjoint** if, given any distinct pair $P_{(i)}$ and $P_{(j)}$ in the collection, u and v are the only vertices $P_{(i)}$ and $P_{(j)}$ have in common.*

Theorem 2.4.3. ***(Whitney, 1932)*** *Let G be a simple graph with at least three vertices. Then G is 2-connected if and only if for each pair of distinct vertices u and v of G there are two internally disjoint $u - v$ paths in G.*

Section 3

Euler Tours and Hamiltonian Cycles

3.1 Euler Tours

Definition 3.1.1. *A trail in G is called an* ***Euler trail*** *if it includes every edge of G.*

Definition 3.1.2. *A tour of G is a closed walk of G which includes every edge of G at least once.*

An ***Euler tour*** *of G is a tour which includes each edge of G exactly once.*

Definition 3.1.3. *A graph G is called* ***Eulerian*** *or Euler if it has an Euler tour.*

Example 3.1.1. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ is given in Figure 3.1.*

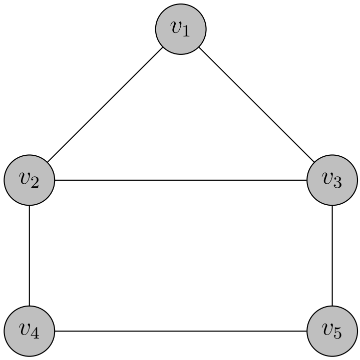


Figure 3.1: Eulerian graph G

The trail $v_2 - v_1 - v_3 - v_2 - v_4 - v_5 - v_3$ is a euler trail since it's contain all the edges of G exactly once. $v_2 - v_1 - v_3 - v_2 - v_4 - v_5 - v_3$ *is a euler tour in G. Hence G is a eulerian graph.*

Theorem 3.1.1. *Let G be a graph in which the degree of every vertex is at least two. Then G contains a cycle.*

Proof. If G is not simple then it contains a cycle since any loop is a cycle of length 1 while a pair of parallel edges gives a cycle of length 2 . We now suppose that G is simple. Let v_0 be any vertex of G. Since $d\left(v_0\right) \geq 2$, we can choose an edge e_1 with one end v_0 and the other v_1, say. Since $d\left(v_1\right) \geq 2$ we can choose an edge e_2 with one end v_1 and the other v_2, say, different from v_0. We repeat this process so that (see Figure 3.2) at the $(i+1)$ th stage we have an edge e_i incident with v_i and v_{i+1} and $v_{1+1} \neq v_{i-1}$.

Since G has only finitely many vertices, we must eventually choose a vertex which has been chosen before. If v_k is the first such vertex then the walk between the first two occurrences of v_k is

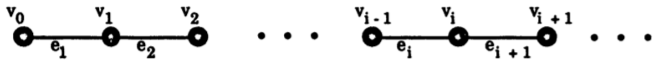


Figure 3.2:

a cycle (since the internal vertices of this walk are distinct and also different from v_k as v_k is the first vertex to be repeated).(See Figure 3.3.)$\mathbf{v_{k+2}}$ $\square$

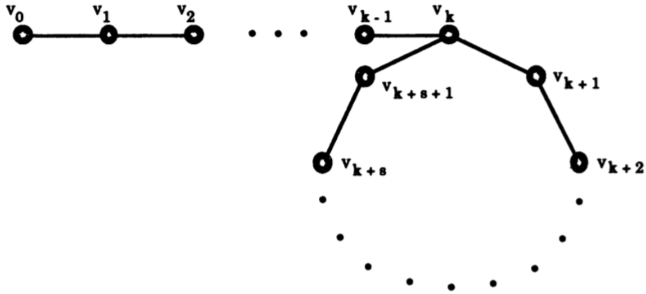


Figure 3.3:

Theorem 3.1.2. *A connected graph G is Euler if and only if the degree of every vertex is even.*

3.2 Hamiltonian Graphs

Definition 3.2.1. *A **Hamiltonian path** in a graph G is a path which contains every vertex of G*

Definition 3.2.2. *A **Hamiltonian cycle** (or Hamiltonian circuit) in a graph G is a cycle which contains every vertex of G.*

Note: A hamiltonian cycle in G with initial vertex v contains every other vertex of G precisely once and then ends back at v.

Definition 3.2.3. *A graph G is called* ***Hamiltonian*** *if it has a Hamiltonian cycle.*

Example 3.2.1. *Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ is given below. Consider the path $v_1 -$*

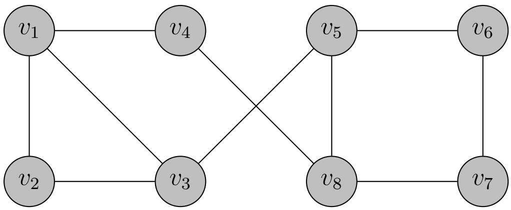


Figure 3.4: Hamiltonian graph G

$v_2 - v_3 - v_5 - v_6 - v_7 - v_8 - v_4$ which contain all the vertices of G. Hence it's a hamiltonian path. $v_1 - v_2 - v_3 - v_5 - v_6 - v_7 - v_8 - v_4 - v_1$ which is a cycle contain all the vertices of G. Hence G is a Hamiltonian graph.

Definition 3.2.4. *A simple graph G is called* ***maximal non-Hamiltonian*** *if it is not Hamiltonian but the addition to it of any edge connecting two non-adjacent vertices forms a Hamiltonian graph.*

Example 3.2.2. *Let $G = (V, E)$ is a graph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ given in figure 3.5.*

The graph G is maximal non-hamiltonian graph since addition of any edge become a hamiltonian graph.

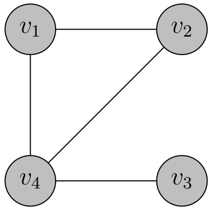


Figure 3.5: A maximal non-hamiltonian graph G

Theorem 3.2.1. ***(Dirac, 1952)*** *If G is a simple graph with n vertices, where $n \geq 3$, and the degree $d(v) \geq n/2$ for every vertex v of G, then G is Hamiltonian.*

Theorem 3.2.2. *Let G be a simple graph with n vertices and let u and v be non-adjacent vertices in G such that*

$$d(u) + d(v) \geq n$$

Let $G + uv$ denote the supergraph of G obtained by joining u and v by an edge. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian

Proof. Suppose that G is Hamiltonian. Then, as noted earlier, the supergraph $G + uv$ must also be Hamiltonian.

Conversely, suppose that $G + uv$ is Hamiltonian. Then, if G is not Hamiltonian, just as in the proof of Theorem 3.6 we obtain the inequality $d(u)+d(v) < n$. However, by hypothesis, $d(u)+d(v) \geq n$. Hence G must be Hamiltonian also, as required. $\square$

Definition 3.2.5. *Let G be a simple graph. If there are two nonadjacent vertices u_1 and v_1 in G such that $d\left(u_1\right) + d\left(v_1\right) \geq n$ in G,* *join u_1 and v_1 by an edge to form the supergraph G_1. Then, if there are two nonadjacent vertices u_2 and v_2 such that $d\,(u_2)+d\,(v_2) \geq n$ in G_1, join u_2 and v_2 by an edge to form the supergraph G_2. Continue in this way, recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains. The final supergraph thus obtained is called the* ***closure*** *of G and is denoted by $c(G)$.*

Example 3.2.3. *We give an example of a closure operation in Figure 3.6. For this example $c(G) = K_7$*

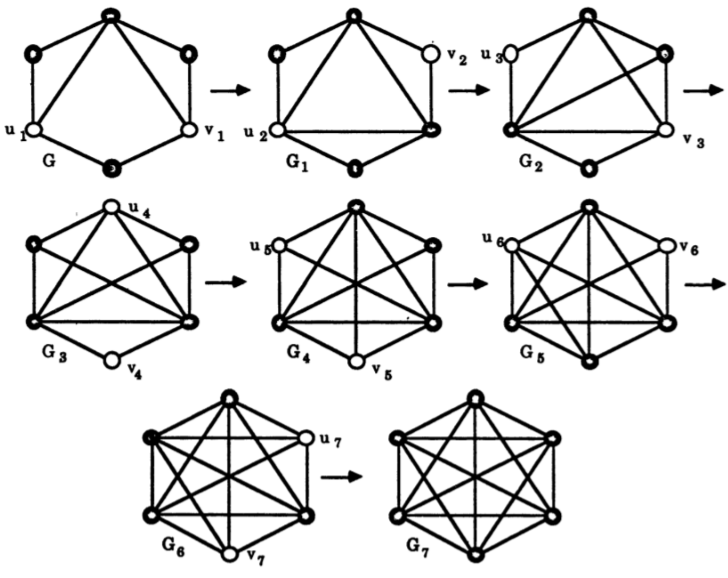


Figure 3.6: Closure operation of graph G

Note: For a graph G on n vertices of $d(u) + d(v) < n$ for any pair u, v of nonadjacent vertices in G and so the closure operation does not get off the ground, i.e., $c(G) = G$.

Theorem 3.2.3. ***(Bondy and Chvatal, 1976)*** *A simple graph G is Hamiltonian if and only if its closure $c(G)$ is Hamiltonian.*

Proof. Since $c(G)$ is a supergraph of G, if G is Hamiltonian then $c(G)$ must be Hamiltonian. Conversely, suppose that $c(G)$ is Hamiltonian. Let $G, G_1, G_2, \ldots, G_{k-1}, G_k = c(G)$ be the sequence of graphs obtained by performing the closure procedure on G. Since $c(G) = G_k$ is obtained from G_{k-1} by setting $G_k = G_{k-1} + uv$, where u, v is a pair of nonadjacent vertices in G_{k-1} with $d(u) + d(v) \geq n$, it follows by Theorem 3.2.2 that G_{k-1} is Hamiltonian. Similarly G_{k-2}, so $G_{k-3}, \ldots$, so G_1, and so G must be Hamiltonian, as required. □

Corollary 3.2.3.1. *Let G be a simple graph on n vertices, with $n \geq 3$. If $c(G)$ is complete, i.e., if $c(G) = K_n$, then G is Hamiltonian.*

Proof. This is immediate from the Theorem since any complete graph is Hamiltonian. □

3.3 Plane and Planar graphs

Definition 3.3.1. *A **plane graph** is a graph drawn in the plane (of the paper, blackboard, etc.) in such a way that any pair of edges meet only at their end vertices (if they meet at all).*

*A **planar graph** is a graph which is isomorphic to a plane graph, i.e., it can be (re)drawn as a plane graph.*

Example 3.3.1. *We give a example of a planar graphs in Figure 3.7*

All 5 graphs are planar but G_1 and G_4 are not plane graphs.

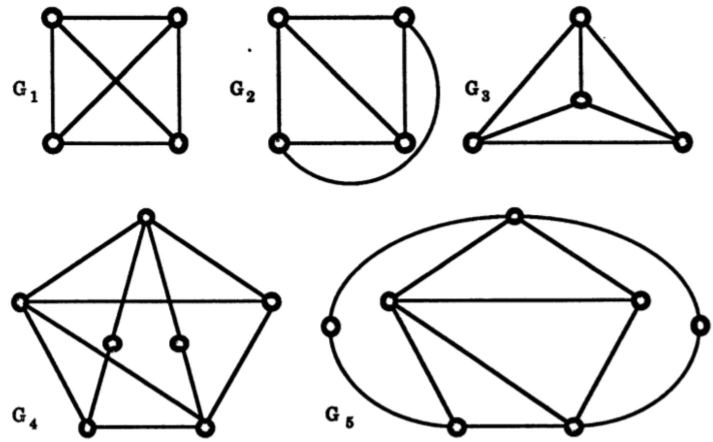


Figure 3.7: 5 planar graphs

Definition 3.3.2. *A **Jordan curve** in the plane is a continuous non-self-intersecting curve whose origin and terminus coincide.*

Example 3.3.2. *For example, in Figure* 3.8 *the curve* C_1 *is not a Jordan curve because it intersects itself,* C_2 *is not a Jordan curve since its origin and terminus do not coincide, i.e., its two end points do not meet up, but* C_3 *is a Jordan curve.*

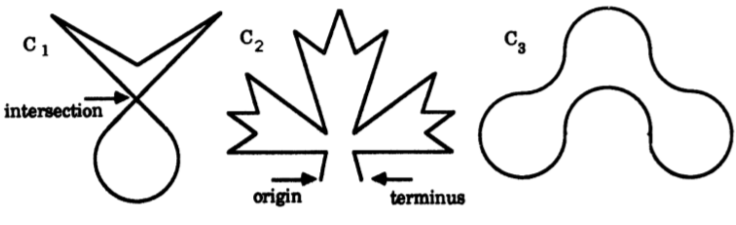


Figure 3.8: C_1 and C_2 are not Jordan curves but C_3 is

Definition 3.3.3. *If* J *is a Jordan curve in the plane then the part of the plane enclosed by* J *is called the **interior** of* J *and denoted* *by int J. We exclude from int J the points actually lying on J. Similarly the part of the plane lying outside J is called the* ***exterior*** *of J and denoted by ext J.*

Theorem 3.3.1. *The Jordan Curve Theorem states that if J is a Jordan curve, if x is a point in int J and y is a point in ext J then any (straight or curved) line joining x to y must meet J at some point, i.e., must cross J.*

Theorem 3.3.2. *K_5, the complete graph on five vertices, is nonplanar.*

Theorem 3.3.3. *The complete bipartite graph $K_{3,3}$ is not planar*

3.4 Euler's Formula

Definition 3.4.1. *A plane graph G partitions the plane into a number of regions called the* ***faces*** *of G. More precisely, if x is a point on the plane which is not in G, i.e., is not a vertex of G or a point on any edge of G, then we define the* ***face of*** G ***containing*** x *to be the set of all points on the plane which can be reached from x by a (straight or curved) line which does not cross any edge of G or go through any vertex of G.*

Example 3.4.1. *For example, for the point x in the graph G_1 of Figure 3.9, the face containing x is shown as the interior of the cycle $v_2 - v_4 - v_3 - v_6 - v_5 - v_2$. In this example obviously the face of G_1 containing the point y is the same face as that containing x. It is bounded by the cycle $v_2v_4v_3v_6v_5v_4$. The face of G_1 containing the point z is not bounded by any cycle. It is called the exterior face. of G_1.*

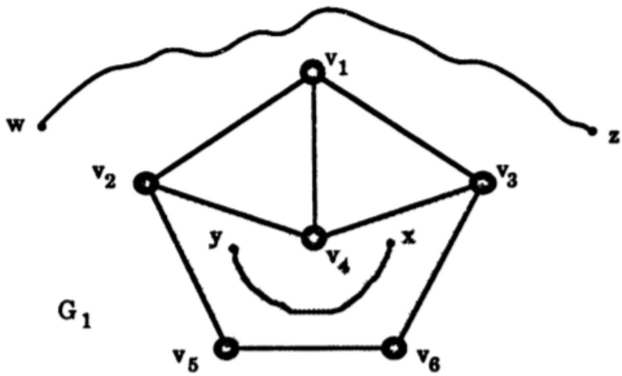


Figure 3.9:

Note: Any plane graph has exactly one exterior face. Any other face is bounded by a closed walk in the graph and is called an interior face.

Theorem 3.4.1. *(Euler's Formula) Let* G *be a connected plane graph, and let* $n, e,$ *and* f *denote the number of vertices, edges and faces of G, respectively. Then*

$$n - e + f = 2$$

Corollary 3.4.1.1. *Let* G *be a plane graph with* n *vertices,*e *edges,* f *faces and* k *connected components. Then*

$$n - e + f = k + 1$$

Corollary 3.4.1.2. *Let* G_1 *and* G_2 *be two plane graphs which are both redrawings of the same planar graph* G*. Then* $f\,(G_1) = f\,(G_2)$*, i.e.,* G_1 *and* G_2 *have the same number of faces.*

Proof. Let $n(G_1), n(G_2)$ denote the number of vertices and $e(G_1)$, $e(G_2)$ the number of edges in G_1, G_2 respectively. Then, since G_1 and G_2 are both isomorphic to G we have $n(G_1) = n(G_2)$ and $e(G_1) = e(G_2)$. Using Euler's Formula we get

$$f(G_1) = e(G_1) - n(G_1) + 2 = e(G_2) - n(G_2) + 2 = f(G_2)$$

as required. $\square$

Definition 3.4.2. *Let φ be a face of a plane graph G. We define the* ***degree*** *of φ, denoted by $d(\varphi)$, to be the number of edges on the boundary of φ.*

Theorem 3.4.2. *Let G be a simple planar graph with n vertices and e edges, where $n \geq 3$. Then*

$$e \leq 3n - 6$$

Corollary 3.4.2.1. *If G is a simple planar graph then G has a vertex v of degree less than 6 , i.e., there is a v in $V(G)$ with $d(v) \leq 5$*

Proof. If G has only one vertex this vertex must have degree 0. If G has only two vertices then both must have degree at most 1. Thus we can suppose that $n \geq 3$, i.e., that G has at least three vertices. Now if the degree of every vertex of G is at least six we have

$$\sum_{v \in V(G)} d(v) \geq 6n$$

However, by Theorem 1.4.1, $\sum_{v \in V(G)} d(v) = 2e$. Thus $2e \geq 6n$ and so $e \geq 3n$. This is impossible since, by the above theorem,

$e \leq 3n - 6$. This contradiction shows that G must have at least one vertex of degree less than 6, as required. $\square$

Corollary 3.4.2.2. *K_5 is nonplanar.*

Proof. Here $n = 5$ and $e = (5 \times 4)/2 = 10$ so that $3n - 6 = 9$. Thus $e > 3n - 6$ and so, by the theorem, $G = K_5$ can not be planar. $\square$

Corollary 3.4.2.3. *$K_{3,3}$ is nonplanar.*

Proof. Since $K_{3,3}$ is bipartite it contains no odd cycles (by Theorem 1.6.1) and so in particular no cycle of length three. It follows that every face of a plane drawing of $K_{3,3}$, if such exists, must have at least four boundary edges. Thus, using the argument of the proof of Theorem 3.4.2, we get $b \geq 4f$ and then $4f \leq 2e$, i.e., $2f \leq e = 9$. This gives $f \leq 9/2$. However, by Euler's Formula, $f = 2 - n + e = 2 - 6 + 9 = 5$, a contradiction. $\square$