# NOTES ON CRYPTARITHM SOLVER AND PERMUTATIONS

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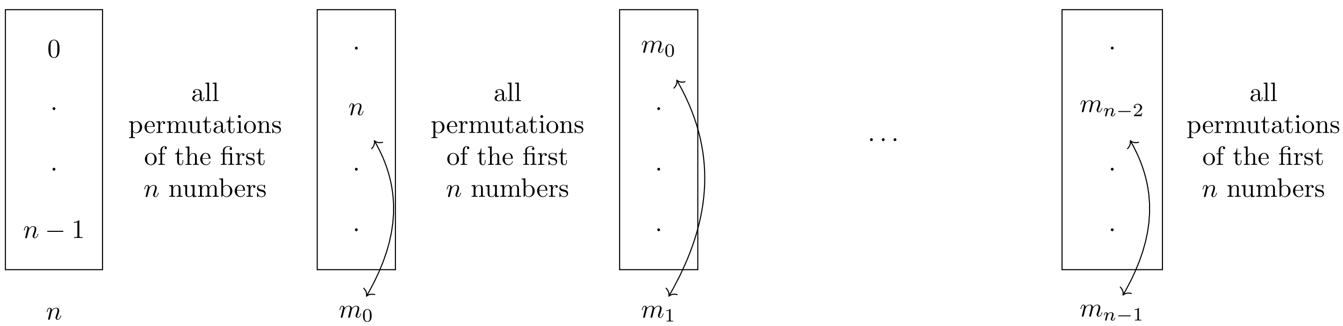
## 1. Heap's algorithm

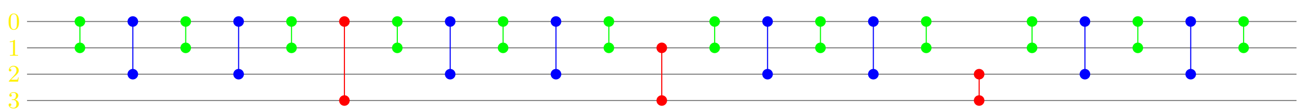
Let a nonzero natural number $n$ be given. Heap's algorithm generates all permutations of a set $S$ with $n+1$ elements, in such a way that any permutation, the first one excepted, is obtained from the previous one by exchanging two of $S$'s elements. Without loss of generality, take for $S$ the set $\{0, 1, \dots, n\}$. The recursive version of Heap's algorithm can be illustrated as follows.




So the algorithm generates all permutations of the form $L \star n$, then all permutations of the form $L \star m_0$, then all permutations of the form $L \star m_1$, ..., and eventually all permutations of the form $L \star m_{n-1}$. The scheme is correct if $\{m_0, m_1, \dots, m_{n-1}\} = \{0, \dots, n-1\}$: at every stage, the algorithm has to select a new number from the first $n$ ones (and exchange it with the current $(n+1)$st number). Heap's algorithm uses the following strategy:

- In case $n$ is odd, select the first number, then the second number, then the third number...
- In case $n$ is even, always select the first number.

The following diagram illustrates with $n = 3$.

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 0 | 1 | 2 | 3 | 1 | 0 | 3 | 1 | 0 | 0 | 2 | 3 | 0 | 2 | 3 | 3 | 2 | 1 | 3 | 2 | 1 |
| 1 | 0 | 0 | 2 | 2 | 1 | 1 | 3 | 3 | 0 | 0 | 1 | 2 | 0 | 0 | 3 | 3 | 2 | 2 | 3 | 3 | 1 | 1 | 2 |
| 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 3 | 3 | 3 | 3 | 2 | 2 | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Note that starting with $(0, 1, 2, 3)$, Heap's algorithm produces $(1, 2, 3, 0)$ as last permutation. We will see that starting with $(0, 1, 2, 3, 4, 5)$, it would produce $(3, 4, 1, 2, 5, 0)$ as last permutation; starting with $(0, 1, 2, 3, 4, 5, 6, 7)$, it would produce $(5, 6, 1, 2, 3, 4, 7, 0)$ as last permutation. More generally, starting with $(0, 1, 2, \dots, 2n+1)$, Heap's algorithm will produce as last permutation $(2n-1, 2n, 1, 2, \dots, 2n-2, 2n+1, 0)$.

Note that starting with $(0, 1, 2)$, Heap's algorithm produces $(2, 1, 0)$ as last permutation. We will see that starting with $(0, 1, 2, 3, 4)$, it would produce $(4, 1, 2, 3, 0)$ as last permutation; starting with $(0, 1, 2, 3, 4, 5, 6)$, it would produce $(6, 1, 2, 3, 4, 5, 0)$ as last permutation. More generally, starting with $(0, 1, 2, \dots, 2n)$, Heap's algorithm will produce as last permutation $(2n, 1, 2, \dots, 2n-1, 0)$.

The previous formulas can be used to generalise Heap's algorithm and generate all sequences of $k$ numbers chosen from $\{0, 1, \dots, n\}$: when the last number has been selected, and the penultimate number has been selected, ..., and the

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$(n-k+1)$st number has been selected, it suffices to stop the recursion and "simulate" all permutations of the remaining $n+1-k$ numbers by applying those formulas.

The following is a possible implementation of Heap's algorithm.

```
def permute(L):
    for L in heap_permute(L, len(L)):
        yield L

def heap_permute(L, length):
    if length == 1:
        yield L
    else:
        length -= 1
        for i in range(length):
            for L in heap_permute(L, length):
                yield L
            if length % 2:
                L[i], L[length] = L[length], L[i]
            else:
                L[0], L[length] = L[length], L[0]
        for L in heap_permute(L, length):
            yield L
```

## 2. Proof of correctness

We prove that Heap's algorithm is correct and that moreover, the following holds for all $n \geq 1$:

(1) starting with $(0, 1, 2, \ldots, 2n)$, all permutations of $(0, 1, 2, \ldots, 2n)$ are generated, ending in
$$(2n, 1, 2, \ldots, 2n-1, 0)$$
(2) starting with $(0, 1, 2, \ldots, 2n+1)$, all permutations of $(0, 1, 2, \ldots, 2n+1)$ are generated, ending in
$$(2n-1, 2n, 1, 2, \ldots, 2n-2, 2n+1, 0)$$

Proof is by induction. The base case $n = 1$ is straightforward, so let $n \geq 1$ be given, and assume that (1) holds. We show that (2) holds too.

- Starting from $(0, 1, 2, \ldots, 2n-1, 2n) \star 2n+1$, Heap's algorithm generates all permutations of the form $L \star 2n+1$, ending with $(2n, 1, 2, \ldots, 2n-1, 0) \star 2n+1$.
- Permuting first and last elements, $(2n, 1, 2, \ldots, 2n-1, 0) \star 2n+1$ is changed to $(2n+1, 1, 2, \ldots, 2n-1, 0) \star 2n$. Starting from $(2n+1, 1, 2, \ldots, 2n-1, 0) \star 2n$, the algorithm then generates all permutations of the form $L \star 2n$, ending with $(0, 1, 2, \ldots, 2n-1, 2n+1) \star 2n$.
- Permuting second and last elements, $(0, 1, 2, \ldots, 2n-1, 2n+1) \star 2n$ is changed to $(0, 2n, 2, \ldots, 2n-1, 2n+1) \star 1$. Starting from $(0, 2n, 2, \ldots, 2n-1, 2n+1) \star 1$, the algorithm then generates all permutations of the form $L \star 1$, ending with $(2n+1, 2n, 2, \ldots, 2n-1, 0) \star 1$.
- Permuting third and last elements, $(2n+1, 2n, 2, \ldots, 2n-1, 0) \star 1$ is changed to $(2n+1, 2n, 1, 3 \ldots, 2n-1, 0) \star 2$. Starting from $(2n+1, 2n, 1, 3 \ldots, 2n-1, 0) \star 2$, the algorithm then generates all permutations of the form $L \star 2$, ending with $(0, 2n, 1, 3 \ldots, 2n-1, 2n+1) \star 2$.
- Permuting fourth and last elements, $(0, 2n, 1, 3 \ldots, 2n-1, 2n+1) \star 2$ is changed to $(2n+1, 2n, 1, 2 \ldots, 2n-1, 0) \star 3$. Starting from $(2n+1, 2n, 1, 2 \ldots, 2n-1, 0) \star 3$, the algorithm then generates all permutations of the form $L \star 3$... till all permutations of the form $L \star 2n-1$, ending in $(2n+1, 2n, 1, 2, \ldots, 2n-2, 0) \star 2n-1$.
- Permuting last two elements, $(2n+1, 2n, 1, 2, \ldots, 2n-2, 0) \star 2n-1$ is changed to $(2n+1, 2n, 1, 2, \ldots, 2n-2, 2n-1) \star 0$. Starting from $(2n+1, 2n, 1, 2, \ldots, 2n-2, 2n-1) \star 0$, the algorithm then generates all permutations of the form $L \star 0$, ending with $(2n-1, 2n, 1, 2, \ldots, 2n-2, 2n+1) \star 0$.

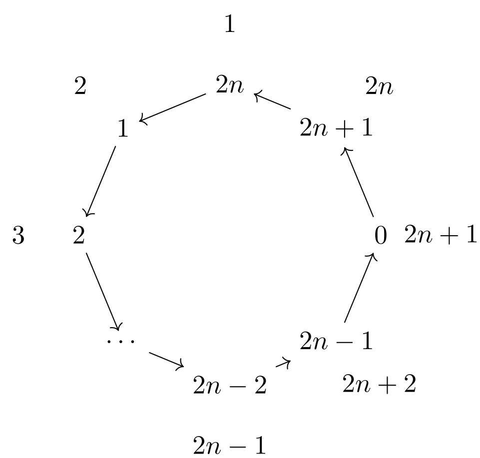
So we have established that (2) holds.

Now assume that (2) holds. We show that (1) with $n$ replaced by $n+1$ holds too. The inner circle of the following diagram shows how elements move from one position to another one after all permutations of a list consisting of the $2n+2$

numbers 0, …, $2n+1$ have been performed by Heap's algorithm. For instance, the first element ends up as the last element, moving from position (index) 0 and eventually ending up at position (index) $2n+1$. After all permutations of $(0, 1, 2, \ldots, 2n, 2n+1)$ have been generated, ending in $(2n-1, 2n, 1, 2, \ldots, 2n-2, 2n+1, 0)$, so after all permutations of $(0, 1, 2, \ldots, 2n, 2n+1) \star 2n+2$ have been generated, ending in $(2n-1, 2n, 1, 2, \ldots, 2n-2, 2n+1, 0) \star 2n+2$, Heap's algorithm replaces the element now at position 0, that is, $2n-1$ (originally at position $2n-1$), with $2n+2$. This is depicted in the following diagram with $2n+2$ on the outer circle facing $2n-1$ on the inner circle. At the end of each of the following stages, the algorithm permutes the element currently at position 0 with the element currently at position $2n+2$, that is, the element at position 0 at the end of previous stage. Hence as illustrated in the diagram, move to position $2n+2$: first $2n-1$ replaced by $2n+2$, then $2n-2$ replaced by $2n-1$, …, then 2 replaced by 3, then 1 replaced by 2, then $2n$ replaced by 1, then $2n+1$ replaced by $2n$, and eventually 0 replaced by $2n+1$. Finally, all permutations of the numbers then at position 0, …, $2n+1$ (those numbers being 1, 2, …, $2n+2$) are generated, corresponding to a last, $(2n+2)$nd rotation in the following diagram, hence a rotation following a "full circle". This means that:

- ends up at position 0 the element which at the beginning of this last round of permutations, is a position $2n-1$, that is, $2n+2$,
- ends up at position 1 the element which at the beginning of this last round of permutations, is a position $2n$, that is, 1,
- …

resulting in the final list $(2n+2, 1, 2, 3, \ldots, 2n-1, 2n, 2n+1) \star 0$. So we have established that (1) with $n$ replaced by $n+1$ holds.




## 3. Notes on the implementation of the cryptarithm solver

The first version is a minor adaptation of code written by Raymond Hettinger as part of ActiveState Code Recipes. It uses the `permutations` function, imported from `itertools` (and also the `findall` function, imported from `re`). The second version does not import anything.

The first version filters out the permutations that assign 0 to one of the letters that start a word; the second version does not produce those permutations. The first version creates a string where the letters starting a word come first, followed by the letters not starting a word; it is the other way around for the second version. For instance, with the cryptarithm `SEND + MORE == MONEY`, the first version creates a string which could be `SMENDORY`, whereas the second version creates a string which could be `ENDORYSM`. Let us still use that example to explain how we proceed in the second version.

- Starting with the list $L = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$, we use the generalisation of Heap's algorithm to generate lists of the form $L_1L_2$, one for each possible list $L_2$ of two nonzero digits (so we ignore 0, as if $L$ started with the element of index 1). This determines a possible assignment to `SM`.
- For each list of the form $L_1L_2$ generated as described, we make a copy of $L_1L_2$ and we use again the generalisation of Heap's algorithm to generate from the copy lists of the form $L_{11}L_{12}L_2$, one for each possible list $L_{12}$ of six digits, amongst those in $L_1$ (so 0 is now allowed but we ignore $L_2$, as if we were working with a list that ended at index 7). We return the last 8 digits of $L_{11}L_{12}L_2$, that is, $L_{12}L_2$, allocating the digits in $L_{12}$ to `ENDORY` and the digits in $L_2$ to `SM`.