NOTES ON NASH EQUILIBRIUM

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1. 2×2 games, pure strategies, and Pareto optimality

We consider two players, Ruth and Charlie, who both can take one of two possible decisions. Each of the four pairs (d_R, d_C) of possible decisions taken by both players is associated with a pair (p_R, p_C) of expected *payoffs* (the indexes R and C refer to Ruth and Charlie, respectively).

For a first example, consider the *prisoner's dilemma*. Ruth and Charlie have been arrested and charged with robbery. It is believed that they were actually carrying guns, making them liable for the more severe charge of armed robbery. Ruth and Charlie, who are held in separate cells and cannot communicate, are each offered the following deal, knowing that the other is offered the same deal:

- if they testify that their partner was armed and their partner does not testify against them, then their sentence will be suspended while their partner will spend 15 years in jail;
- if they testify against each other, then they will both spend 10 years in jail;
- if neither testifies against the other, then they will both spend 5 years in jail.

	Not testify	Testify	
Testify	(-15, 0)	(-10, -10)	Charlie
Not testify	(-5, -5)	(0, -15)	
	Ruth		

Both Ruth and Charlie deciding to testify is a Nash equilibrium:

- Charlie testifying, Ruth finds herself better off by testifying (-10) than by not testifying (-15);
- Ruth testifying, Charlie finds himself better off by testifying (-10) than by not testifying (-15).

Note that Ruth and Charlie would in fact be better off if they did not testify (they would spend 5 years rather than 10 years in jail). But that is not what Game theory recommends, and that is not what is observed in practice either... A way to express that Ruth and Charlie would be better off if they did not testify is to say that this pair of decision is not *Pareto optimal*: there exists another pair of decision (namely, both of them not testifying) associated with an outcome which is at least as good for both players, and better for at least one of them (in this case, better for both of them).

For a second example, consider the *game of chicken*. Ruth and Charlie drive towards each other at high speed following a white line drawn on the middle of the road. If both swerve (chicken out), then it is a draw (0 to both). If one swerves and the other does not, then the one who did not chicken out wins (1) while the other one loses (-1). If neither chickens out, then they won't have a chance to play again, and their common payoff could be reasonably set to $-\infty$, but we set it arbitrarily to -10.

	Not swerve	Swerve	
Swerve	(1, -1)	(0, 0)	Charlie
Not swerve	(-10, -10)	(-1, 1)	
	Ruth		

Ruth not chickening out and Charlie chickening out is a Nash equilibrium:

- Charlie swerving, Ruth finds herself better off by not swerving (1) than by swerving (0);
- Ruth not swerving, Charlie finds himself better off by swerving (-1) than by not swerving (-10).

By symmetry, Ruth chickening out and Charlie not chickening out is also a Nash equilibrium.

2. Mixed strategies

Testifying or not testifying, swerving or not swerving, are *pure strategies*. More generally, Ruth can testify or swerve with probability p, and Charlie can testify or swerve with probability q. Ruth opts for a pure strategy iff p is 0 or 1, and similarly Charlie opts for a pure strategy iff q is 0 or 1; otherwise, they opt for a *mixed strategy*.

Let us consider the game of chicken with both Ruth and Charlie swerving with probability 0.9. Then Ruth's expectation is $0.1(0.1 \times -10 + 0.9 \times 1) + 0.9(0.1 \times -1 + 0.9 \times 0) = -0.1$; by symmetry, Charlie's expectation is also -0.1. It turns out that this strategy is also a Nash equilibrium, as we now show.

If Ruth swerves with probability p and Charlie swerves with probability q, then Ruth's expectation is equal to

$$(1-p)[(1-q) \times -10 + q \times 1] + p[(1-q) \times -1 + q \times 0]$$

which simplifies to

$$(-10q+9)p + 11q - 10$$

Ruth's aim is to *maximise* her expectation, that is, maximise the value of the above expression, which is achieved by:

- setting p to 0 if $q > 0.9$;
- setting p to 1 if $q < 0.9$;
- taking for p an arbitrary value if $q = 0.9$.

By symmetry, Charlie maximises his expectation by:

- setting q to 0 if $p > 0.9$;
- setting q to 1 if $p < 0.9$;
- taking for q an arbitrary value if $p = 0.9$.

This means that both players accepting the opponent's strategy as it is (they can only decide for themselves), Ruth and Charlie will both not regret their strategy in three cases, the three Nash equilibria:

- $p = 0$ and $q = 1$
- $p = 1$ and $q = 0$
- $p = 0.9$ and $q = 0.9$
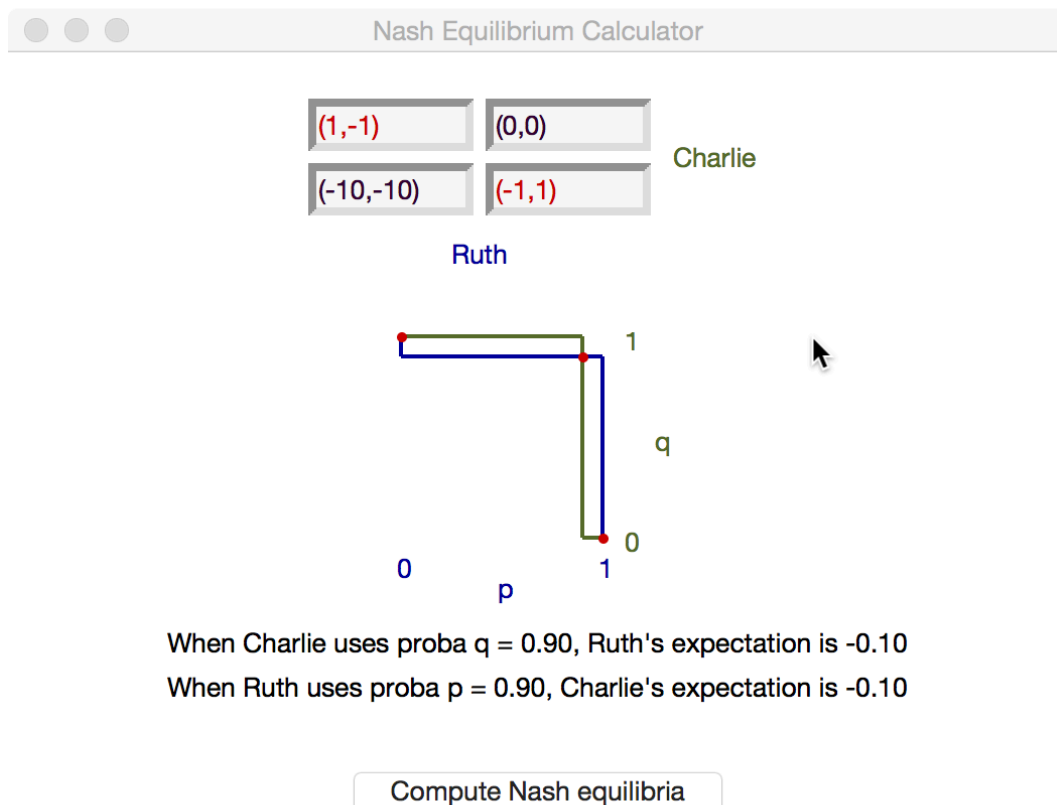
3. No regret graphs

Use the following notation for Ruth's and Charlie's payoffs:

$(a_2,\ b_2)$	$(a_4,\ b_4)$
$(a_1,\ b_1)$	$(a_3,\ b_3)$

Charlie

Ruth

Then Ruth's expectation is

$$(1-p)[(1-q)a_1 + qa_2] + p[(1-q)a_3 + qa_4]$$

which can be written as

$$(a_1 - a_2 - a_3 + a_4)pq + (a_3 - a_1)p + (a_2 - a_1)q + a_1$$

whereas Charlie's expectation is

$$(1-p)[(1-q)b_1 + qb_2] + p[(1-q)b_3 + qb_4]$$

which can be written as

$$(b_1 - b_2 - b_3 + b_4)pq + (b_3 - b_1)p + (b_2 - b_1)q + b_1$$

Set:

- $D_R = a_1 - a_2 - a_3 + a_4$ and $D_C = b_1 - b_2 - b_3 + b_4$;
- $E_R = a_3 - a_1$ and $E_C = b_2 - b_1$;
- $F_R = a_2 - a_1$ and $F_C = b_3 - b_1$.

So Ruth's expectation is $(D_R q + E_R)p + F_R q + a_1$ and Charlie's expectation is $(D_C p + E_C)q + F_C p + b_1$. Both players aim at maximising their expectation, which determines:

- Ruth's No regret graph, consisting of all pairs of numbers of the form:
  - $(0, q)$ with $D_R q + E_R < 0$,
  - (p, q) with $0 \le p \le 1$ and $D_R q + E_C = 0$,
  - $(1, q)$ with $D_R q + E_R > 0$;
- Charlie's No regret graph, consisting of all pairs of numbers of the form:
  - $(p, 0)$ with $D_C p + E_C < 0$,
  - (p, q) with $0 \le q \le 1$ and $D_C p + E_C = 0$,
  - $(q, 1)$ with $D_C q + E_C > 0$.

The possible No regret graphs for Ruth are:
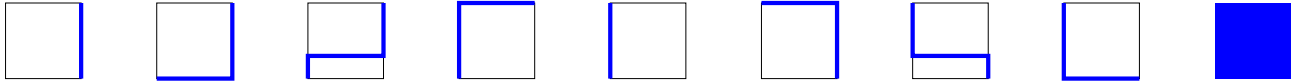
The possible No regret graphs for Charlie are:

Any possible No regret graph for Ruth intersects any No regret graph for Charlie, which shows the existence of a Nash equilibrium; the Nash equilibria are all intersection points.

Note from the graphs that if Ruth achieves a Nash equilibrium using a pure strategy, then Charles can also also use a pure strategy; similarly, if Charlie achieves a Nash equilibrium using a pure strategy then Ruth can also also use a pure strategy.

Note from the equations that when Ruth achieves a Nash equilibrium using a mixed strategy, then $D_R q + E_R = 0$ and her expectation does not depend on her own probability of choosing one decision over the alternative; similarly, when Charlie achieves a Nash equilibrium using a mixed strategy, then $D_C p + E_C = 0$ and his expectation does not depend on his own probability of choosing one decision over the alternative.