AUTOMORPHISMS OF SPHERICAL BUILDINGS: THE QUEST TO PROVE THE GENERALISED CENTRE CONJECTURE

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Abstract. In this report we consider an unsolved conjecture of Tits, dating back to the 1950's, concerning the structure of convex subsets of spherical buildings. This conjecture is known is 'the Centre Conjecture' and one version of this conjecture was recently solved in [8]. We now look into a more general version of the Centre Conjecture, a solution to which would have applications in many areas including the structure of algebraic groups, and geometric invariant theory. One proposed strategy for proving the Generalised Centre Conjecture involves imposing additional structure onto a spherical building Δ. This will allow us to gain more information about how automorphisms of Δ affect convex subsets. In this report we look into one method of imposing additional structure onto Δ, and explicitly compute concrete examples in some simple cases. These computations reveal a new and surprising result concerning the nature of this additional structure.

Contents

1. Introduction

We start this report with a brief history of the theory of buildings before developing the formal definition and constructing a concrete example. We will then look into the statement of the Centre Conjecture and discuss the importance of this conjecture in a wider context.

1.1. Historical Background and Aims of the Project.

In 1965 Jacques Tits introduced the concept of a mathematical building [1]. Tits wanted to provide a geometric framework in which to understand certain classes of groups, and so devised the concept of a building as a simplicial complex with particular properties.

Tits was born in 1930 and showed great mathematical prowess from an early age; after his father died Tits helped to pay for his own education by tutoring students several years older than himself. Tits received his doctorate from the Free University of Brussels at the age of 20. Tits held notable academic positions such as Chair of Group Theory in the Collége de France and has received many awards including the Cantor Prize (1996) and the Abel Prize (2008) [2].

Tits himself contributed a large amount of work to the theory of buildings and eventually proved a classification theorem for certain types of buildings (proved in [3] but see Chapter 9 of [4] for an overview). Buildings themselves have been used in many areas of mathematics since their conception, including in the proof of the classification theorem for finite simple groups [5]. Other areas of application include the theory of group presentations, the study of hyperbolic groups and of groups with a BN pair (see Chapters 13 and 14 of [4] for more information).

The most comprehensive book on the theory of buildings is *Buildings: Theory and Applications* by K. S. Brown and P. Abramenko ([4]). No other text comes close to the volume of content provided in this particular text, and [4] is the recommended reference for any questions on basic building theory.

Early on in the theory of buildings, Tits conjectured that convex subcomplexes of buildings should have a natural building structure themselves or, if that is not the case, the subcomplex should have a 'centre' [6]. This is something of a generalisation of the observation that any automorphism of a sphere which stabilities the northern hemisphere fixes the north pole. Informally this is the sense in which Tits meant 'centre'. This conjecture became known as the *Centre Conjecture*.

After over 40 years the Centre Conjecture was proved for buildings of non-exceptional type by Mühlherr and Tits in 2006 [6]. Buildings of exceptional type were dealt with in [7] (2009) and [8] (2012) concluding the proof of the full result. The quest is now on to prove the general case where the subcomplex condition is relaxed to subset. This problem is known as the *Generalised Centre Conjecture* and is unsolved.

Relaxing the subcomplex condition significantly increases the complexity of the problem because we lose the inherent structure which came along with a subcomplex. An arbitrary subset of a given building need not have any inherent structure to work with when trying to look at how automorphisms of the building affect that particular subset.

In this project we will look in detail at one suggested technique to impose some additional structure on convex subsets of a given building, together with some explicit examples. Although there will not be time to pursue this research much further the hope is that this technique will allow us, in future, to look at how automorphisms of the building affect these subsets with their additional structure. This will hopefully lead to a mimicking of the proof of the Centre Conjecture given by Mühlherr and Tits for buildings of 'non-exceptional type'. The final piece of the puzzle will then be to generalise the result to all buildings covered by the original statement of the Centre Conjecture.

1.2. **The Definition of a Building.**

In order to understand the statement of the Centre Conjecture we must first understand the concept of a building. There are a few definitions we need before getting to buildings themselves.

Definition 1.1. A *simplicial complex* with vertex set V is a non-empty collection Δ of finite subsets of V (called *simplices*) such that every singleton $\{v\}$ is a simplex and every subset B of a simplex A is a simplex (which we call a *face* of A and is denoted $B \leq A$).

It is easily seen that this face relation, as the notation suggests, is a partial order because it is simply set inclusion. For more information on simplicial complexes see [9]. We have given an abstract definition of a simplicial complex; however a simplicial complex also has a geometric realisation. An example of such a geometric realisation is shown in Figure 1.

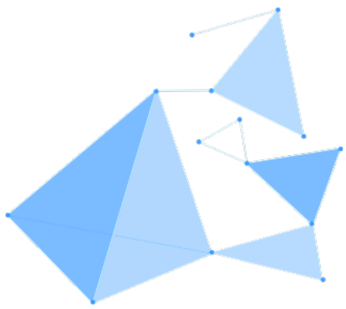


FIGURE 1. Simplicial Complex

Informally, the vertices are related to the singletons of V, vertices are then faces of lines, lines are faces of triangles, triangles are faces of tetrahedrons, etc. Each of these simplices must be joined together in a particular way; see Figure 2 for another example.

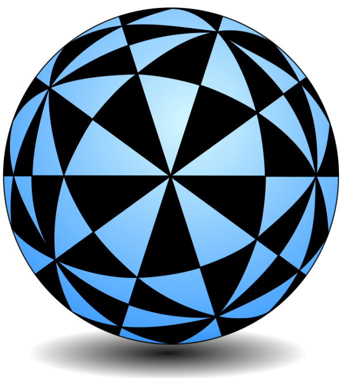


FIGURE 2. Coxeter Complex

Buildings are unions of special types of simplicial complexes. Before we get to the next definition, recall that a *Coxeter Group* is a group generated by elements of order 2.

Definition 1.2. Let W be a Coxeter group with generating set S and let $\Sigma(W, S)$ be the partially ordered set of standard cosets; ordered by reverse inclusion (thus $B \leq A$ in Σ if and only if $A \subseteq B$ as subsets of W, in which case we say that B is a *face* of A). We then call $\Sigma(W, S)$ the *Coxeter Complex* associated to W.

It is easily checked that this definition does in fact define a simplicial complex. The geometric realisation of the Coxeter complex associated with the group H_3 is shown in Figure 2. We say that a Coxeter complex is *spherical* if it is finite (or equivalently the Coxeter group itself is finite), and this will be the case which most interests us. Now we come to the formal definition of a building.

Definition 1.3. A *building* is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ (called *apartments*) satisfying the following axioms:

(B0) Each apartment Σ is a Coxeter complex.

(B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.

(B2) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing every vertex of A and every vertex of B.

An isomorphism between simplicial complexes is a bijective map of sets which preserves the simplicial structure. More information on this definition of a building can be found in [4]. We say that a building is *spherical* if every apartment is a spherical (i.e. finite) Coxeter complex.

It is interesting to note that the above definition is just one of several equivalent definitions of a building. We can also think of a building from a metric space viewpoint, and also as a purely combinatorial structure. More information on both of these approaches can be found in [4]. For a particularly accessible and informative treatment of the combinatorial approach see [10].

1.3. **Example of a Spherical Building.**

To help better understand the concept of a building, we construct the canonical example of a spherical building.

Definition 1.4. Given a vector space V over some field k, a *flag* of V is a chain of non-zero, proper subspaces of the form:

$$V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n.$$

We say that F_1 is a *subflag* of F_2 if every subspace in F_1 is contained in the flag F_2.

Now suppose V is a vector space of dimension 3 over some field k. Let X, Y and Z be the 1-dimensional subspaces generated by each element of some given basis for V, then let Σ be the collection of all flags of V generated by these subspaces.

Partially order Σ by inclusion; so for $L, M \in \Sigma$ we have $L \leq M \iff L$ is a subflag of M. Then it can easily be shown that Σ, as a partially ordered set, is a simplicial complex. Moreover Σ is the Coxeter complex of the symmetric group $\mathcal{S}_3$, which is of course finite so we have a spherical Coxeter complex.

This construction gives a Coxeter complex for each choice of basis for V. Taking the union of all such apartments gives a building Δ which is spherical; and of course this

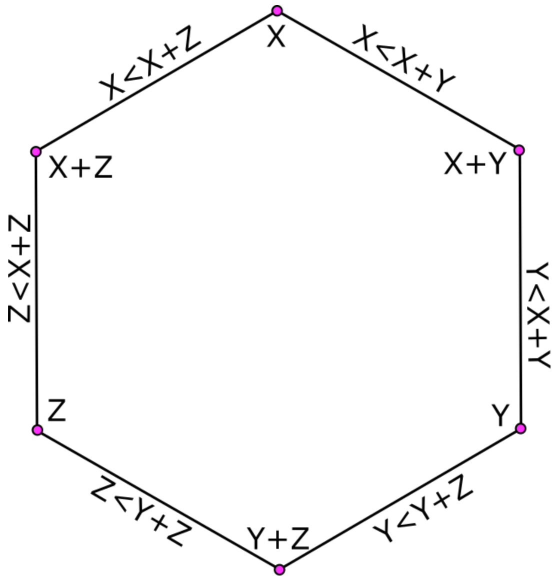


FIGURE 3. An apartment of Δ.

construction can be done for any finite dimensional vector space. For a formal treatment of this example see [4] Chapter 4.

1.4. **The Centre Conjecture.**

We can now look at the statement of the Centre Conjecture and discuss the Generalised Centre Conjecture and its applications.

It is worth noting that the notion of a convex subset of a simplicial complex is similar to that of a convex subset in Euclidean space. Formally, a subset is convex if, for any two non-opposite points of that subset, the geodesic segment joining them is also entirely within that subset. By the building axioms, any two points in a building Δ can be put into a common apartment; because each apartment is convex, the geodesic joining them will be the same no matter which apartment we choose to draw the geodesic in.

Conjecture 1.1. (The Centre Conjecture). *Let $\mathbf{\Delta} = (\Delta, \subset)$ be a spherical building and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of $\mathbf{\Delta}$. Then at least one of the following holds:*

a) *For each simplex $A \in \tilde{\Delta}$ there is a simplex $B \in \tilde{\Delta}$ which is opposite to A in $\mathbf{\Delta}$.*

b) *The group $\mathrm{Stab}_{\mathrm{Aut}(\mathbf{\Delta})}(\tilde{\Delta})$ fixes a non-trivial simplex of $\tilde{\Delta}$.*

Part (a) of the above conjecture says that $\tilde{\Delta}$ also has a building structure in its own right. In our above example, if $\tilde{\Delta}$ is a whole apartment (such as the apartment shown in Figure 3), the opposite of a given simplex A is the simplex which is directly opposite to A in the same way that X is directly opposite to $Y + Z$ in Figure 3.

An automorphism of a building is a bijective map of sets which respects the simplicial structure. However, any such map lifts to an isometry of the geometric realisation, so we can also think of automorphisms as continuous maps on a topological space. The motivation behind the original construction of buildings was that inside the automorphism group of an irreducible building lies a simple group. This provides a way of constructing all finite simple groups with a BN-pair; such as the groups of Lie type.

The fixed point of $\mathrm{Stab}_{\mathrm{Aut}(\mathbf{\Delta})}(\tilde{\Delta})$ is the 'centre' which the name of the conjecture refers to.

In 2006 Mühlherr and Tits showed that:

If $\mathbf{\Delta}$ is an irreducible spherical building which is not of type E_6, E_7, E_8 or F_4, then the Centre Conjecture holds for $\mathbf{\Delta}$ [6].

The cases E_6, E_7, E_8 and F_4 have since been tackled in [7] and [8]. Now the quest is on to solve the same problem but with the *subcomplex* condition relaxed to *subset* in the above conjecture. In this project we will investigate one proposed strategy for proving the Generalised Centre Conjecture. Informally, we will attempt to embed small buildings in larger buildings. This will add additional structure to the smaller building - while the building is embedded in the larger building. It should then be possible to look at how automorphisms of the larger building affect a given convex subset of the smaller building, with the additional structure imposed by the larger building. The hope is that this strategy will eventually lead to a proof of the Generalised Centre Conjecture.

Proving the Generalised Centre Conjecture is important for a number of reasons. As well as being an interesting unsolved problem, a solution to this conjecture has applications in the structure of algebraic groups (over arbitrary fields), representation theory, geometric invariant theory and metric geometry. In fact, the Generalised Centre Conjecture is a special case of an even more general conjecture concerning spaces with positive curvature (see [8]):

Conjecture 1.2. *Every convex subset of a* CAT(1)*-space has radius $\leq \pi/2$ or contains a sphere.*

Note that spherical buildings are an example of a CAT(1)-space. There are many analogous results to the above conjecture for spaces of non-positive curvature. This is why we believe that the Centre Conjecture can be proved with the subcomplex condition relaxed to subset, despite the lack of structure when compared to the subcomplex case. For more information on CAT(κ)-spaces, in particular spaces of non-positive curvature, see [11].

2. Preliminaries

In this chapter we will develop the notation and ideas required to proceed to the next, more technical, chapter. We start by introducing the special linear group $\mathrm{SL}_n(k)$ and some key ideas surrounding this group. We will then develop the concept of a vector building before aligning this viewpoint with the traditional definition of a building given in the introduction. Much of the content in this chapter is treated, in a slightly more general setting, in [12].

2.1. The Algebraic Group $\mathrm{SL}_n(k)$.

We wish to prove the Centre Conjecture for all buildings of algebraic groups. This is a large task as there many algebraic groups. Fortunately there is a result stating that if the Centre Conjecture is true for the algebraic group $\mathrm{SL}_n(k)$ then the conjecture is true for all algebraic groups, see [13].

Definition 2.1. An *algebraic group* is an affine algebraic variety with a group structure such that the multiplication and inverse maps:

$$m : G \times G \to G,\ \iota : G \to G$$

are morphisms of affine varieties.

Given this definition, the special linear group is algebraic as the vanishing set, in the variety of $n \times n$ matrices with entries in some field k, of the polynomial $\det(A) = 1$ for all $A \in G$. For more information on algebraic groups see [14].

2.2. Symmetric Powers and Cocharacters.

From this point onwards, our field k is always algebraically closed. We will also denote $\mathrm{SL}_{n+1}(k)$ by G for simplicity. G has a natural module, which we shall refer to as E, spanned by the basis vectors $e_1, \ldots, e_{n+1}$, where $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the ith position. G then acts on E by matrix multiplication (thinking of each e_i is a column vector, i.e. a $(n+1) \times 1$ matrix).

Definition 2.2. The rth *symmetric power* of E, denoted $S^r(E)$, is the module spanned by the basis vectors $e_1^{p_1} \cdot \ldots \cdot e_{n+1}^{p_{n+1}}$ where $\sum_{i=1}^{n+1} p_i = r$; i.e. monomials of degree r.

The action of G on $S^r(E)$ is given by the extension of the action of G on E. So if $g \in G$ and f is a monomial of degree r then:

$$g \cdot f(e_1, \ldots, e_{n+1}) = f(g \cdot e_1, \ldots, g \cdot e_{n+1}),$$

where $\cdot$ represents matrix multiplication. The following example illustrates how we should interpret this definition.

Example 2.1. Let $G = \mathrm{SL}_2(k)$ then E is the module spanned by $e_1 = x = (1,0)$ and $e_2 = y = (0,1)$. The 2nd symmetric power $S^2(E)$ is then spanned by the elements:

$$x^2,\, xy,\, y^2.$$

So $S^2(E)$ is spanned by 3 vectors whereas E was only spanned by 2. In general the size of the spanning set of $S^r(E)$ is $\binom{r+n}{n}$ (the number of ways of placing r unlabelled balls into $n+1$ labeled boxes as we have $\dim(E) = n+1$). The fact that our spanning set grows so quickly will prove to be extremely useful later on. Consider the matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acting on the vector x^2:

$$\begin{aligned} g \cdot x^2 &= (g \cdot x)^2 \\ &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)^2 \\ &= \begin{pmatrix} a \\ c \end{pmatrix}^2, \end{aligned}$$

where we interpret this expression as:

$$\begin{pmatrix} a \\ c \end{pmatrix}^2 = (ax + cy)^2 = a^2x^2 + 2acxy + c^2y^2,$$

which is a linear combination of our basis polynomials so is an element of $S^2(E)$.

Definition 2.3. The *standard diagonal maximal torus*, T, of G is the subgroup of G which is made up of diagonal matrices.

The reason for introducing the notion of the symmetric power of a module is because the given basis vectors $e_1^{p_1} \cdot \ldots \cdot e_{n+1}^{p_{n+1}}$ are weight vectors for the action of the standard diagonal maximal torus of G. The diagonal matrix $\mathrm{diag}(t_1, \ldots, t_{n+1})$ is acting with weight $\prod_{i=1}^{n+1} t_i^{p_i}$ on the vector $\prod_{i=1}^{n+1} e_i^{p_i}$. Hence if $t = \mathrm{diag}(t_1, \ldots, t_{n+1})$ then $t \cdot \prod_{i=1}^{n+1} e_i^{p_i} = (\prod_{i=1}^{n+1} t_i^{p_i})(\prod_{i=1}^{n+1} e_i^{p_i})$.

Definition 2.4. A *cocharacter* of G is a homomorphism of algebraic groups from the multiplicative group of the field k to G.

From the theory of algebraic groups, it turns out that every cocharacter of the standard diagonal maximal torus, T, has the form:

$$t \mapsto \mathrm{diag}(t^{a_1}, \ldots, t^{a_{n+1}}),$$

where each $a_i \in \mathbb{Z}$ and $\sum_{i=1}^{n+1} a_i = 0$. Hence the set of cocharacters $Y(T)$ is isomorphic as an abelian group to the group $\mathbb{Z}^n$.

We shall parameterise a cocharacter of T by specifying a tuple

$$\lambda = (a_1, \ldots, a_n) \in \mathbb{Z}^n;$$

then $\lambda(t) = \mathrm{diag}(t^{a_1}, \ldots, t^{a_n}, t^{a_{n+1}})$ with $a_{n+1} = -\sum_{i=1}^{n} a_i$. Therefore, we use the convention that the group operation on $Y(T)$ is written additively: given $\lambda, \mu \in Y(T)$, we can define $\lambda + \mu \in Y(T)$ by $(\lambda + \mu)(t) := \lambda(t)\mu(t)$ for all $t \in k \setminus \{0\}$.

We have that $Y(T)$ is a $\mathbb{Z}$-module, but for our purposes we will need to extend this definition to a $\mathbb{Q}$-vector space. This construction is similar to how $\mathbb{Q}$ is obtained from $\mathbb{Z}$. Firstly define $\sim$ on $Y(T) \times \mathbb{Z} \setminus \{0\}$ by:

$$(\lambda, l) \sim (\mu, m) \iff m\lambda = l\mu.$$

It is left to the reader to check that this is in fact an equivalence relation and we denote the equivalence class of (λ, l) by λ/l.

Definition 2.5. Following the above construction we say that λ/l is a *rational cocharacter* of T. Repeating the above process for each maximal torus of G and making the correct identifications gives the set of rational cocharacters of G.

Note that the set of rational cocharacters of T is now a $\mathbb{Q}$-vector space, so we are identifying elements of $Y(T)$ as integer points inside $\mathbb{Q}^n$. This could also have been achieved using the tensor product.

2.3. **Parabolic Subgroups.**

We want to make a correspondence between cocharacters and certain types of subgroups and this correspondence is the key to constructing a vector building.

Definition 2.6. A *parabolic subgroup* of $G = \mathrm{SL}_n(k) = \mathrm{SL}(E)$ is the stabiliser of a flag in E.

To elaborate; if the flag is $\mathcal{F} : U_1 \subset U_2 \subset \cdots \subset U_m$, then the corresponding parabolic $P = \mathrm{Stab}_G(\mathcal{F})$ is the set of elements of $\mathrm{SL}(E)$ which map each subspace U_i to U_i for each i. If we wish to see these subgroups in matrix form we can choose a basis in a particular way which is compatible with the flag. First choose a basis for U_1, extend this to U_2, then extend to U_3, etc. then P consists of matrices of 'upper staircase form'. We illustrate this with the following example.

Example 2.2. Let G be $\mathrm{SL}_4(k)$ and consider the flag:

$$\{0\} \subset < e_1, e_2 > \subset < e_1, e_2, e_3 > \subset E$$

where $< e_i >$ is the subspace spanned by the standard basis vector e_i. Note that the flag itself does not include the zero subspace or the whole space E, by definition, but we show them here for clarity. With respect to the basis $e_1, \ldots, e_4$ any matrix of the form:

$$\begin{pmatrix} * & & & \\ & * & & \\ 0 & 0 & * & \\ 0 & 0 & 0 & * \end{pmatrix}$$

with any entries from k (such that the product of the diagonal entries - shown by *'s - is 1) above the 'staircase' stabilises the flag. This is because the upper 2×2 block stabilises the first subspace in the flag which has dimension 2. The next 1×1 block corresponds to an extension of this basis to the 3-dimensional subspace which appears next in the flag.

All that is left to do at this point is to extend by one more basis element to fill the whole space, so we get a final 1×1 block.

It is left to the reader to check that matrices of this general shape form subgroups of $\mathrm{SL}_n(k)$.

These subgroups correspond to cocharacters in the following way: given a diagonal cocharacter λ with $\lambda(t) = \mathrm{diag}(t^{a_1}, \ldots, t^{a_{n+1}})$ for $t \in k^*$, take a general matrix $g = (g_{ij})$, then conjugating g by $\lambda(t)$ gives the matrix;

$$\lambda(t) g \lambda(t)^{-1} = (t^{a_i - a_j} g_{ij}). \tag{2.1}$$

This can be seen to be true by observing that when multiplying a general matrix g by a diagonal matrix $\mathrm{diag}(b_1, \ldots, b_{n+1})$ on the left this gives a matrix where each row is multiplied by the diagonal element corresponding to that row, i.e. $\mathrm{diag}(b_1, \ldots, b_{n+1})(g_{ij}) = (b_i g_{ij})$, with the same being true but for columns instead of rows when multiplying on the right. Of course the inverse of $\mathrm{diag}(b_1, \ldots, b_{n+1})$ is $\mathrm{diag}(b_1^{-1}, \ldots, b_{n+1}^{-1})$; giving the above formula.

We can formally think of the limit of (2.1) as $t \to 0$, and we see that this can only exist if $g_{ij} = 0$ whenever $a_j > a_i$. Then the set of g for which the limit exists gives a parabolic subgroup, and in fact all parabolics arise in this way. It is in this sense that cocharacters correspond to parabolics.

Example 2.3. If we have the flag $\mathcal{F}$ given above, and we have chosen a basis compatible with that flag, then we can choose a cocharacter λ by specifying the values a_i for $1 \leq i \leq n$.

Suppose the dimension of U_j is d_j for each $1 \leq j \leq m$. Then set $a_1 = a_2 = \cdots = a_{d_1} = m$, $a_{d_1+1} = \cdots = a_{d_1+d_2} = m - 1$, and so on, and then fill in the last $n - \sum_{j=1}^{m} d_j$ diagonal entries with the same value so as to make the determinant 1.

In the case of Example 2.2 $\lambda(t) = \mathrm{diag}(t^2, t^2, t^1, t^{-5})$ will do, as we have $n + 1 = 4$, $m = 2$, $\dim(U_1) = 2$, $\dim(U_2) = 3$.

Doing this ensures that we have a non-increasing sequence of values a_i, with the changes happening at the 'transitions' in the flag $\mathcal{F}$. It is now the case that the parabolic corresponding to λ is none other than the parabolic corresponding to $\mathcal{F}$. Note that there is a lot of choice here, which explains why in general a parabolic corresponds to (the interior of) a simplex in the building.

Given a cocharacter λ, we denote the corresponding parabolic by P_λ.

2.4. **Vector Buildings.**

We now have all the tools in place to discuss the idea of a *vector building* using the terminology developed in the previous sections. It may be useful at this stage to recall the notion of a *rational cocharacter* (definition 2.5).

So far the definition of a rational cocharacter has very much depended upon the choice of basis we have used for a given subspace in the flag which that cocharcter is acting upon. We want to declare two cocharacters to be the equivalent if they act on the same flag with the same weight on each subspace in that flag: we illustrate this with an example.

Example 2.4. Let G be $\mathrm{SL}_4(k)$ again and consider the flag:

$$\{0\} \subset U \subset E$$

where $\dim U = 2$. Let λ be the cocharacter which acts with weight a on U and let $\{x, y\}$ be a basis for U. With respect to this basis we have:

$$\lambda(t) = \begin{pmatrix} t^a & 0 & 0 & 0 \\ 0 & t^a & 0 & 0 \\ 0 & 0 & t^{-a} & 0 \\ 0 & 0 & 0 & t^{-a} \end{pmatrix}$$

Now suppose that μ is another cocharacter which acts with weight a on U and let $\{x', y'\}$ be a different basis for U. such that $\mu(t)$ has the same form as $\lambda(t)$ with respect to the basis $\{x', y'\}$. Hence there exists a change of basis matrix A such that $\lambda(t) = A\mu(t)A^{-1}$.

Therefore, the equivalence relation, $\circledast$, we want is:

$$\begin{aligned} \lambda \circledast \mu \Longleftrightarrow\ & \lambda,\ \mu \text{ act with the same weight on each subspace in the flag and} \\ & \lambda(t) = A\mu(t)A^{-1} \text{ for some change of basis matrix } A. \end{aligned}$$

Definition 2.7. The *vector building* of G can be realised geometrically as the set of rational cocharacters under the above equivalence relation $\circledast$. The *spherical building* Δ is the the unit sphere inside the vector building. Since every rational cocharacter can be multiplied by a suitably large positive number to give an integer cocharacter, the spherical building can be realised simply as the projection of all integer cocharacters onto the unit sphere.

For our fixed maximal torus T, each equivalence class of $\circledast$ has only one representative in $Y(T)$, so we can still safely think of the points of $Y(T)$ as points of the vector building. We write λ, μ, etc. for such points, and $\overline{\lambda}$ for the associated point of Δ.

3. The Density of r-Points as $r \to \infty$

The central idea behind this project is that by taking larger and larger symmetric powers of E, we can lay more and more structure on the spherical building Δ. We need a terminology to distinguish between the original points of Δ and the new 'points' coming from this extra structure. This terminology will be developed in this chapter before we state and prove our main result concerning the nature of our new 'points'.

3.1. **The r-Points of Δ.**

Let r be some positive integer and let $\lambda = (a_1, \ldots, a_n) \in Y(T)$. We are interested in situations where the weight of λ is the same for two different basis vectors of $S^r(E)$. Suppose we have two basis vectors $\prod_{i=1}^{n+1} x_i^{p_i}$ and $\prod_{i=1}^{n+1} x_i^{q_i}$. The weight of λ on the first is

$$a_1p_1 + \cdots + a_np_n - (a_1 + \cdots + a_n)p_{n+1} = a_1(p_1 - p_{n+1}) + \cdots + a_n(p_n - p_{n+1}),$$

and the second gives the same formula with every p replaced with a q. So the difference in the two weights for λ is

$$\sum_{i=1}^{n} a_i(p_i - q_i - p_{n+1} + q_{n+1}).$$

Now, recalling that $\sum_{i=1}^{n+1} p_i = r = \sum_{i=1}^{n+1} q_i$, we can rewrite this as

$$\sum_{i=1}^{n} a_i \left(2(p_i - q_i) + \sum_{1 \le j \le n, j \ne i} (p_j - q_j) \right). \tag{3.1}$$

Bearing this in mind, for each pair of basis vectors, we set $X_i := 2(p_i - q_i) + \sum_{1 \le j \le n, j \ne i}(p_j - q_j)$ for each $1 \le i \le n$. In this way each distinct pair of basis vectors gives us a nonzero vector $\mathbf{X} = (X_1, \ldots, X_n) \in \mathbb{Z}^n$. Note that (3.1) is simply the dot product: $\lambda \bullet \mathbf{X}$.

Given a pair of basis vectors of $S^r(E)$, we call the pair *interesting for* λ if the corresponding vector $\mathbf{X} = (X_1, \ldots, X_n)$ lies in the hyperplane orthogonal to $\lambda = (a_1, \ldots, a_n)$; for then the sum in (3.1) (i.e. $\lambda \bullet \mathbf{X}$) is zero, and we have two basis vectors with the same λ-weight. Now we can make our main definition of this chapter.

Definition 3.1. we call $\overline{\lambda}$ an *r-point* of the building Δ if there are enough pairs of basis vectors which are interesting for λ so that the corresponding vectors $(X_1, \ldots, X_n)$ (for each pair) span the entire hyperplane orthogonal to λ.

This means that the one-dimensional subspace of $\mathbb{Q}^n$ spanned by λ can be uniquely determined by looking at pairs of basis vectors with the same weight for λ. Uniqueness is due to the fact that the hyperplane orthogonal to λ is uniquely defined. Therefore, if we can span this hyperplane with such pairs, then the only vectors in $\mathbb{Q}^n$ which correspond to those pairs are the scalar multiples of λ.

Example 3.1. Let E be the natural module of $\mathrm{SL}_3(k)$ (so $n+1=3$) and let $r=2$ so $S^2(E) = \mathrm{span}\{x^2, y^2, z^2, xy, xz, yz\}$.

Let $\lambda = (1,1,1)$ and consider the pair of basis vectors x^2, y^2. Note that $x^2 = x^2 \cdot y^0 \cdot z^0$ and $y^2 = x^0 \cdot y^2 \cdot z^0$. So we have:

$$\begin{aligned} a_1 &= 1, & a_2 &= 1, & a_3 &= 1 \\ p_1 &= 2, & p_2 &= 0, & p_3 &= 0 \\ q_1 &= 0, & q_2 &= 2, & q_3 &= 0 \end{aligned}$$

Substituting these values into (3.1) gives:

$$\begin{aligned} &1(2(2-0)+(0-2)+(0-0)) \\ &\quad + 1(2(0-2)+(2-0)+(0-0)) \\ &\quad + 1(2(0-0)+(2-0)+(0-2)) \\ &\quad = 1(2)+1(-2)+1(0) \\ &\quad = 0. \end{aligned}$$

So in this case $(X_1, X_2, X_3) = (2, -2, 0)$. We now need one more pair of basis vectors which are interesting for λ to show that $\overline{\lambda}$ is a 2-point of Δ, as the (hyper)plane orthogonal to λ is only 2-dimensional.

To find a second spanning vector, choose the pair xz, z^2. It is left to the reader to check that this pair is interesting for λ and gives $(X_1, X_2, X_3) = (1, 0, -1)$ using the above procedure. It is clear that $(2, -2, 0)$ and $(1, 0, -1)$ are linearly independent, so they span a 2-dimensional plane which is orthogonal to $\lambda = (1,1,1)$. Therefore, $\overline{\lambda}$ is a 2-point of Δ.

Conversely, if $\lambda = (1,1,2)$ then it is possible to show (either by hand or by writing a short computer program) that $(X_1, X_2, X_3) = (1, -1, 0)$ and $(2, -2, 0)$ are the only distinct tuples corresponding to pairs of basis vectors for $S^2(E)$. Hence because $(1, -1, 0)$ and $(2, -2, 0)$ are not linearly independent they do not span a 2-dimensional plane, so $\lambda = (1,1,2)$ is not a 2-point of Δ.

3.2. **The Density of r-Points.**

We are now in a position to look at the main result of this chapter.

Proposition 3.1. *Every point of Δ is an r-point for some r. If $\overline{\lambda}$ is an r-point of Δ, then $\overline{\lambda}$ is an s-point for all $s \geq r$.*

Proof. The hyperplane orthogonal to λ is $n-1$ dimensional, so is spanned by $n-1$ rational vectors. By multiplying up with suitable integers, we can actually assume that these are integer vectors. We are done if we can find r such that each of these basis vectors for the hyperplane is equal to a vector $(X_1, \ldots, X_n)$ attached to a pair of basis vectors for $S^r(E)$.

Turning the argument in the previous paragraph around, we are done if for any given tuple $(X_1, \ldots, X_n) \in \mathbb{Z}^n$ we can find a suitable r and a suitable pair of basis vectors of

$S^r(E)$ such that the corresponding vector is $(X_1, \dots, X_n)$.

Now, given r, the numbers p_i and q_i can be freely chosen in the range $0 \le p_i, q_i \le r$, so the differences $p_i - q_i$ can be freely chosen in the range $[-r, r]$. Let us denote each difference by $p_i - q_i = d_i$. Then given $(X_1, \dots, X_n)$ we are searching for solutions to the linear system of equations

$$X_i = 2d_i + \sum_{1 \le j \le n, j \ne i} d_i, \qquad 1 \le i \le n.$$

This system has a solution as long as the matrix of coefficients is invertible. The matrix is

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}$$

This matrix is invertible, so there is a suitable choice of d_is. However, these $d_i s$ might not be integers: we can fix this by multiplying the X_is up to clear out denominators; this will only scale the vector $(X_1, \dots, X_n)$, so we remain in the same hyperplane.

All we need to do then, is to pick r large enough so that it is possible to choose suitable d_is for each basis vector for the hyperplane orthogonal to λ; since the only constraints on the d_is is the range $[-r, r]$, this is possible for suitably large r; moreover, it will still be possible for all $s \ge r$. □

Example 3.2. Let $n + 1 = 5$, $\mathbf{X} = (28, -33, 50, 18)$. How large does r need to be so that we can find a pair of basis vectors for $S^r(E)$ which correspond to $\mathbf{X}$? Following the guidance of the proof of Proposition 3.1 we must solve the following system of linear equations:

$$\begin{aligned} 28 &= 2d_1 + d_2 + d_3 + d_4 \\ -33 &= d_1 + 2d_2 + d_3 + d_4 \\ 50 &= d_1 + d_2 + 2d_3 + d_4 \\ 18 &= d_1 + d_2 + d_3 + 2d_4 \end{aligned}$$

It is left to the reader to check that $d_1 = 77/5$, $d_2 = -228/5$, $d_3 = 187/5$, $d_4 = 27/5$ is a solution, so to make each d_i an integer we must multiply $\mathbf{X}$ by 5. After solving this system again with $5\mathbf{X}$ we get:

$$p_1 - q_1 = 77,\ p_2 - q_2 = -228,\ p_3 - q_3 = 187,\ p_4 - q_4 = 27,\ p_5 - q_5 = -63,$$

where the relationship between p_5 and q_5 comes from the fact that $n+1 = 5$ and $d_{n+1} = -\sum_{i=1}^{n} d_i = -63$ in this case. Hence, if $\{x,\ y,\ z,\ w,\ v\}$ is a basis for E, the pair

$$x^{77} z^{187},\ y^{228} v^{63}$$

of basis vectors for $S^{291}(E)$ will do, so we need $r \ge 291$.

It is now clear that as $r \to \infty$ the number of r-points increases and eventually fills up Δ. This is how we obtain the structure required to see how automorphisms of the spherical building Δ affect subsets which previously had no intrinsic structure.

4. Computing the r-Points for Small n

The observant reader may have realised that everything done so far is easily computable for $G = \mathrm{SL}_3(k)$ or $\mathrm{SL}_4(k)$. This chapter documents those computations and the results they produce. We end the chapter with a somewhat surprising example which illustrates how our intuition can be misleading in this area.

4.1. The Structure of Δ for $\mathrm{SL}_3(k)$.

Firstly we look at $\mathrm{SL}_3(k)$. In this case the natural module E is spanned by x, y, z. So any cocharacter has the form (a, b) and acts with weight a on x, b on y and $-a-b$ on z. As in the previous section we are interested in situations when the weight of this cocharacter is the same for two different basis vectors of $S^r(E)$. We show the calculation for $r = 2$.

Example 4.1. We have $\{x^2, y^2, z^2, xy, xz, yz\}$ as a basis for $S^2(E)$. Hence the cocharacter (a, b) acts on this basis with corresponding weights:

$$\begin{aligned} x^2 &\longleftrightarrow 2a & \qquad xy &\longleftrightarrow a+b \\ y^2 &\longleftrightarrow 2b & xz &\longleftrightarrow -b \\ z^2 &\longleftrightarrow -2a-2b & yz &\longleftrightarrow -a \end{aligned}$$

The distinct points and lines in Δ correspond to distinct flags of E, or equivalently; distinct sequences of inequalities between the above expressions.

For example the inequality $2b > a + b > 2a > -a > -b > -2a - 2b$ corresponds to the flag $\{0\} \subset \{y^2\} \subset \{y^2, xy\} \subset \{y^2, xy, x^2\} \subset \{y^2, xy, x^2, yz\} \subset \{y^2, xy, x^2, yz, xz\} \subset \{y^2, xy, x^2, yz, xz, z^2\}$ which we get by starting with the subspace corresponding to $2a$, as this has the most weight, and then successively adding in each subspace according to its weight as we go along the inequality.

We investigate the possibilities by fixing one of a or b and varying the other. 2-Points in Δ occur when two or more of the above weights are equal (e.g. $2a = 2b$). Regions in between (such as $a > b > 0$) correspond to lines in Δ.

Start by fixing $b = 0$ and vary a from large positive to zero to large negative. For large positive a we get:

$$2a > a > 0 = 0 > -a > -2a$$

where we list 0 twice (as both $2b$ and $-b$ are zero) and it is convention to list the value of every weight to show that two weights are equal and we have a 2-point. This sequence of inequalities holds for all $a > 0$ so now if $a = 0$ we get:

$$0 = 0 = 0 = 0 = 0 = 0$$

however, the zero point is not on the boundary of the circle so we are not interested in this case. Now if $a < 0$ we get:

$$-2a > -a > 0 = 0 > a > 2a$$

which is yet another distinct point.

For $a = 0$ we get:

$$(b > 0) \qquad 2b > b > 0 = 0 > -b > -2b$$

$$(b < 0) \qquad -2b > -b > 0 = 0 > b > 2b$$

as expected. Now if we focus our attention on the case where $a, b > 0$ we see that putting the weights in order is not as straightforward. Suppose $a > b > 0$ then we get:

$$2a > a + b > 2b > -b > -a > -2a - 2b.$$

Continuing, $a = b > 0$ gives:

$$2a = 2b = a + b > -a = -b > -2a - 2b$$

and finally if $b > a > 0$:

$$2b > a + b > 2a > -a > -b > -2a - 2b.$$

A similar situation occurs when $a, b < 0$:

$$(0 > a > b) \qquad -2a - 2b > -b > -a > 2a > a + b > 2b$$

$$(0 > a = b) \qquad -2a - 2b > -a = -b > a + b = 2a = 2b$$

$$(0 > b > a) \qquad -2a - 2b > -a > -b > 2b > a + b > 2a$$

We are now have two quadrants left to check; $a > 0 > b$ and $b > 0 > a$. We shall start with $a > 0 > b$. To begin with, let a be very small and $-b$ large before increasing the size of a until $a = -b$:

$$(2a < -b) \qquad -2a - 2b > -b > 2a > -a > a + b > 2b$$

$$(2a = -b) \qquad -2a - 2b = -b > 2a > -a > a + b > 2b$$

$$(2a > -b) \qquad -b > -2a - 2b > 2a > -a > a + b > 2b$$

Now if $a = -b$ we get:

$$2a > -b > 0 = 0 > -a > 2b.$$

When $a > -b$ we need to specify the relationship between a and $-2b$, giving:

$$(a < -2b) \qquad 2a > -b > a + b > -2a - 2b > -a > 2b$$

$$(a = -2b) \qquad 2a > -b = a + b > -a = -2a - 2b = 2b$$

$$(a > -2b) \qquad 2a > a + b > -b > 2b > -a > -2a - 2b$$

So we have three distinct points in this region corresponding to $2a = -b$, $a = -b$ and

$a = -2b$. At this point we leave it as an exercise to the reader to compute the sequences of inequalities in the region where $b > 0 > a$, again we should find points when $2a = -b$, $a = -b$ and $a = -2b$.

Importantly, we see that there are six new points which we did not see in $S^1(E)$.

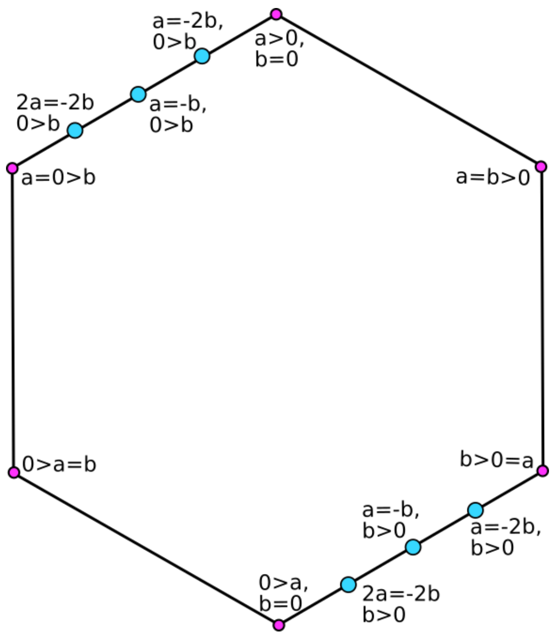


FIGURE 4. The new points of Δ laid on top of $S^1(E)$.

As r increases, we see more and more r-points appearing. The above calculation was somewhat long and tedious and this was only for the second symmetric power. Therefore, to calculate the r-points for $r > 2$ we resorted to programming a computer to find the r-points for each r.

Instead of picking a region to look at, e.g. when $a, b > 0$, we instead program the computer to move around the unit circle in the (a, b)-plane substituting the values of a, b into the expressions for the weights of each basis vector for $S^r(E)$ and sorting the list to obtain to the sequence of inequalities (remember, we are only concerned with the direction each point is in, because a point in Δ is the projection of a λ onto the unit sphere, so we do not lose anything by simply looking at directions).

So, for example, the basis for $S^3(E)$ is:

$$\{x^3,\ y^3,\ z^3,\ x^2y,\ x^2z,\ xy^2,\ xz^2,\ xyz,\ y^2z,\ z^2y\}$$

which means we have to order 10 expressions for each choice of a, b - something a computer can do very easily. As we move around the unit circle, we then keep track of the angles when the sequence of inequalities changes, just as we did in the worked example above. The code, written in Python, can be found here: `https://github.com/CallMeLuke94/final-year-project`.

Of course this process is only an approximation with accuracy depending on the step size we choose to travel around the circle. To visualise the results we created a Processing sketch which can be found here: `http://www.openprocessing.org/sketch/226756`.

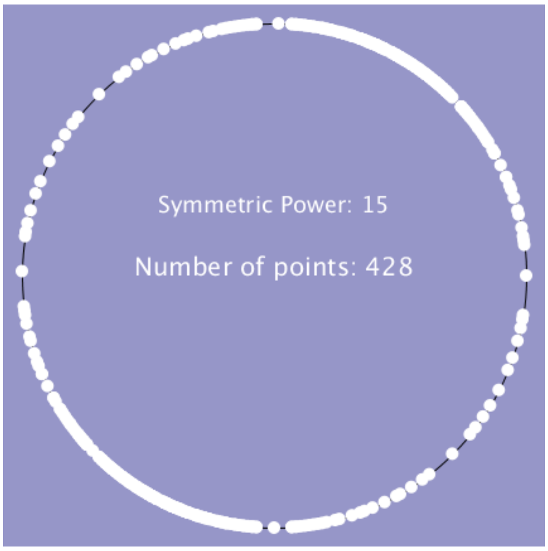


FIGURE 5. Δ for $S^{15}(E)$.

As expected, for each symmetric power we get the triangulated boundary of the unit circle, with more and more points appearing at each stage. Therefore we always have a simplicial complex at every symmetric power.

4.2. The Structure of Δ for $\mathrm{SL}_4(k)$.

The situation is rather more complex when $n+1=4$. Even the second symmetric power is extremely complicated, beyond the scope of computation by hand. Fortunately a computer is not fazed by this increased complexity and will still happily substitute values for a, b, and c this time into the many expressions and put them in order.

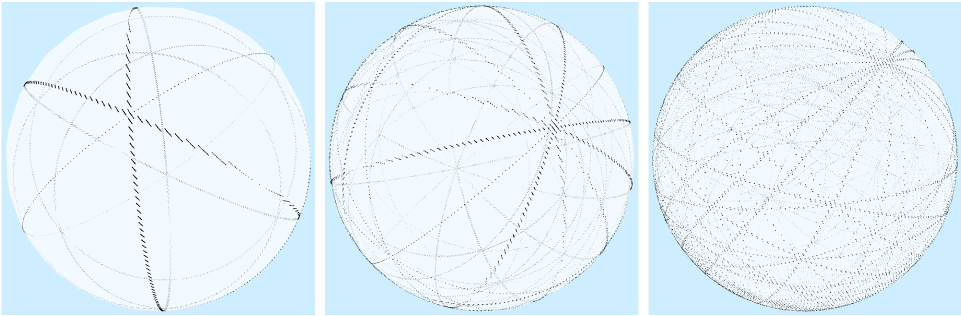
The Python and Processing code for this case can again be found here: `https://github.com/CallMeLuke94/final-year-project`.

Images of this animation are shown in figure 6. We see that, instead of single points, whole great circles are added for each symmetric power. Furthermore, for the third symmetric power the structure is no longer simplicial. This result was somewhat surprising.

As a concrete example in figure 7 we see a quadrilateral region in the centre of the image surrounded by the four purple vectors. These vectors are:

$$(-9,-1,-13), \quad (-5,-1,-9), \quad (-1,0,-2), \quad (-1,0,-3/2).$$

To give a sense of location we also plot the blue vector (towards the left of the image) which is the south pole $(0,0,-1)$, the red vector (towards the right) which is the negative x-axis $(-1,0,0)$, and the green vector (towards the top which) is $(-1,-1,-1)$.



(A) $S^1(E)$ (B) $S^2(E)$ (C) $S^3(E)$

FIGURE 6. The first, second and third symmetric powers when $G = \mathrm{SL}_4(k)$.

The flag corresponding to the interior of this region is given by:

$$3c < a + 2c < 2a + c < b + 2c < 3a < a + b + c < 2a + b < 2b + c$$
$$< a + 2b < c - a - b < 3b < -b < a - b - c < -a < -c < b - a - c$$
$$< -2a - 2b - c < -a - 2b - 2c < -2a - b - 2c < -3a - 3b - 3c.$$

It is somewhat disappointing to see that the simplicial structure is no longer present once $r \geq 3$; however, maybe we should not be too surprised to discover that this is the case. In the $\mathrm{SL}_3(k)$ case there was no room on the 1-dimensional boundary of a circle to disrupt the simplicial structure (all we can do is further triangulate the line). In the $\mathrm{SL}_4(k)$ case we are working on the 2-dimensional surface of a sphere and we are adding great circles at each symmetric power. Every time we add another great circle, it becomes harder and harder to maintain the simplicial structure, especially given how the laying of addition great circles is not done in a perfectly symmetric way.

It is worth noting, also, that we should not expect to see a simplicial structure when $n + 1 \geq 4$ and r is sufficiently large (most likely $r \geq 3$ again). Even so, we do still see a general complex structure allowing us to look at subcomplexes in this new structure.

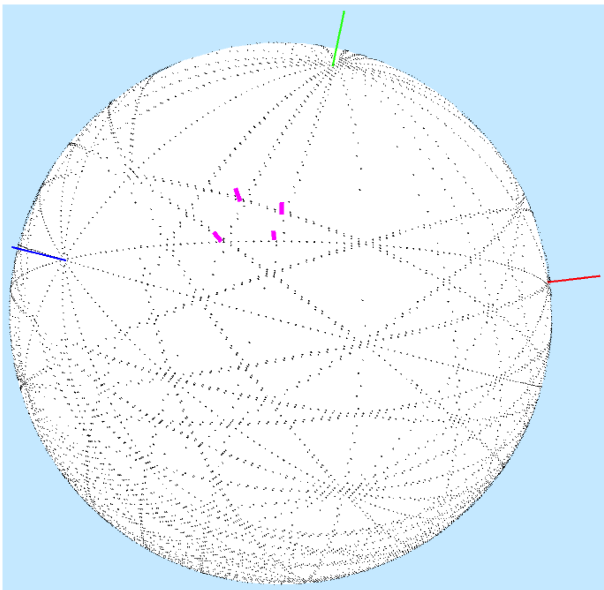


FIGURE 7. The purple vectors surround a quadrilateral region.

5. Conclusion

We have come a long way since the start of this report: from the basic definition of a building, to seeing new and surprising results about the structure of buildings raised to higher symmetric powers. We now take the time to reflect on this journey and comment upon where the future of this work lies.

Firstly, the work in this report is far from finished. The Generalised Centre Conjecture is still just a conjecture and we are still a long way from formulating a proof. However, we have made a significant step along the path to a proof.

We have seen some new results; in particular proposition 3.1 tells us that we can lay as much additional structure onto Δ as we wish. Furthermore we have seen what this structure looks like for $n+1 = 3$, 4. We also now know that we should not expect to see a simplicial structure for $n+1 \geq 4$ and large enough r, in particular we have a concrete example of quadrilateral region when $n+1 = 4$ and $r = 3$.

Fortunately, the case when $n+1 = 3$ is easily understood, and a proof of the Generalised Centre Conjecture seems within reach for $G = \mathrm{SL}_3(k)$.

The future of this research is clear. We have seen that the additional structure from raising E to higher and higher symmetric powers gives us a (*not necessarily simplicial*) complex. The next step towards a proof of the Generalised Centre Conjecture is to prove the conjecture for subcomplexes in this new structure for a given r.

The final challenge will then be to prove that, given any convex subset Σ of a spherical building Δ, that we can approximate Σ by a subcomplex of Δ in a high enough symmetric power. This will prove the Generalised Centre Conjecture as we will know that the conjecture holds for subcomplexes when r is large.

These final two steps could take some considerable time, but we are at the very least in a position to pursue such a course of research.

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