Group theory for Maths, Physics and Chemistry students

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Chapter 1

Introduction

1.1 Symmetry

Group theory is an abstraction of symmetry

Symmetry is the notion that an object of study may look the same from different points of view. For instance, the chair in Figure 1.1 looks the same as its reflection in a mirror that would be placed in front of it, and our view on the wheel depicted next to the chair doesn't change if we rotate our point of view over $\pi/6$ around the shaft.

But rather than changing viewpoint ourselves, we think of an object's symmetry as transformations of space that map the object 'into itself'.

What do we mean when we say that an object is *symmetric*? To answer this question, consider once more the chair in Figure 1.1. In this picture we see a plane V cutting the chair into two parts. Consider the *transformation* r of three-dimensional space that maps each point p to the point p' constructed as follows:

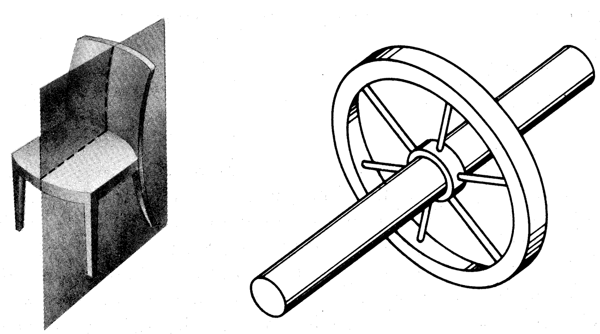

Figure 1.1: Bilateral and rotational symmetry. From: [7].

let l be the line through p and perpendicular to V. Denoting the distance of a point x to V by $d(x, V)$, p' is the unique point on l with $d(p', V) = d(p, V)$ and $p \neq p'$. The map r is called the *reflection in* V. Now apply r to all points in the space occupied by the chair. The result is a new set of points, but because of the choice of V it equals the old space occupied by the chair. For this reason, we call the chair *invariant* under r, or say that r is a *symmetry (transformation)* of the chair.

Transforming any point in space to its reflection in the mirror W and next rotating it over an angle of π around the axis $V \cap W$, gives the same result as the reflection r in V. This is an illustration of what we observed about symmetry: the change of view on the chair from the original chair to its mirror image, was 'neutralized' to the original look, but then on a transformed chair (transformed by means of r). Symmetry then is the phenomenon that the image chair is indistinguishable from the original chair.

Next, consider the wheel depicted in Figure 1.1. It also has a symmetry, but of a different type: the rotation s around the wheel's axis m over $2\pi/6$ moves all points occupied by the wheel to other points occupied by it. For this reason we call s a symmetry of the wheel. This type of symmetry is called *rotational symmetry.* Note that the rotations around m over the angles $2k\pi/6$ are symmetries of the wheel as well, for $k = 1, 2, 3, 4, 5$. The wheel also has reflection symmetry: the reflection t in the plane W perpendicular to m and cutting the wheel into two thinner wheels of the same size also maps the wheel to itself.

Symmetry registers regularity, and thus records beauty. But it does more than that, observing symmetry is useful. Symmetry considerations lead to efficiency in the study of symmetric objects. For instance,

- when designing a chair, the requirement that the object be symmetric may reduce drawings to half the chair;

- when studying functions on the plane that respect rotational symmetry around a 'centre' p in the plane, we can reduce the number of arguments from the (usual) two for an arbitrary function on the plane to one, the distance to p.

Symmetry conditions

A few more observations can be made regarding the previous cases.

1. A symmetry of an object or figure in space is a transformation of space that maps the object to itself. We only considered the space occupied by the chair and the wheel, respectively, and not the materials involved. For example, if the left legs of the chair are made of wood and the right ones of iron, one could argue that the chair is *not* symmetric under reflection in V. Usually, we shall speak about symmetries of sets of points, but this subtlety should be remembered.

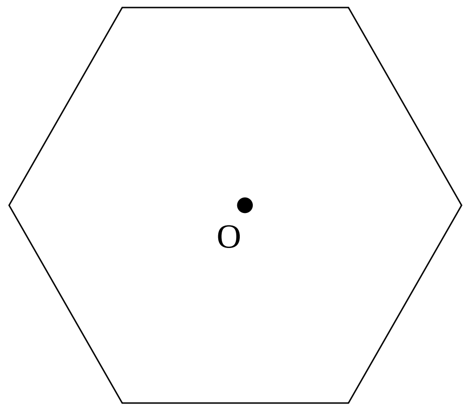

Figure 1.2: The Hexagon

2. Symmetry transformations are bijective, i.e., they have an inverse transformation. In the case of r, it is r itself. In the case of s, it is rotation about m over $-2\pi/6$, or, which amounts to the same, $10\pi/6$.

3. Two transformations can be composed to obtain another transformation; if both translations are symmetries of some object, then so is their composition. In the case of s, we see that $s \circ s$ (s applied twice) is a symmetry of the wheel as well; it is rotation over the angle $2 \cdot (2\pi/6)$. Also, $t \circ s$ (first apply s, then t) is a symmetry of the wheel. Usually, we will leave out the composition sign $\circ$ and write ts instead of $t \circ s$, and s^2 for ss.

4. There is one trivial transformation which is a symmetry of any object, namely the identity, sending each point to itself. It is usually denoted by e (but also by 0 or 1, see below).

Groups

The transformations under which a given object is invariant, form a *group*. Group theory was inspired by these types of group. However, as we shall see, 'group' is a more general concept. To get a feeling for groups, let us consider some more examples.

Planar groups

The hexagon, as depicted in Figure 1.2, is a two-dimensional object, lying in the plane. There are lots of transformations of the plane leaving it invariant. For example, rotation r around 0 over $2\pi/6$ is one of them, as is reflection s in the vertical line l through the barycentre. Therefore, the hexagon has at least

Figure 1.3: A frieze pattern. From: [10].

the following symmetries:

$$e, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5.$$

In fact, these are all symmetries of the hexagon. Thus, each is a (multiple) product of r and s. These two elements are said to *generate* the group of symmetries.

Exercise 1.1.1. Verify that the elements listed are indeed distinct and that

$$r^6 = s^2 = e, \text{ and } srs = r^5.$$

Exercise 1.1.2. Show that sr is a reflection. In which line?

Figure 1.3 shows the palmette motif, which is very frequently used as a decoration on the upper border of wallpaper. Imagine this pattern to be repeated infinitely to the left and right. Then the translation t over the vector pointing from the top of one palmette to the top of the second palmette to its right is a symmetry of the pattern, and so is the reflection r in the vertical line l through the heart of a given palmette. With these two, all symmetries of the frieze pattern can be listed. They are:

$$e, t, t^2, t^3, \ldots \text{ and } r, rt, rt^2, rt^3, \ldots.$$

We find that the symmetry group of the pattern is infinite. However, when ignoring the translational part of the symmetries, we are left with essentially only two symmetries: e and r.

Exercise 1.1.3. Find frieze patterns that have essentially different symmetry groups from the one of Figure 1.3.

A pattern with a more complicated symmetry group is the wallpaper pattern of Figure 1.4, which Escher designed for 'Ontmoeting' (Dutch for 'Encounter'). It should be thought of as infinitely stretched out vertically, as well as horizontally. The translations s and t over the vectors a (vertical from one nose

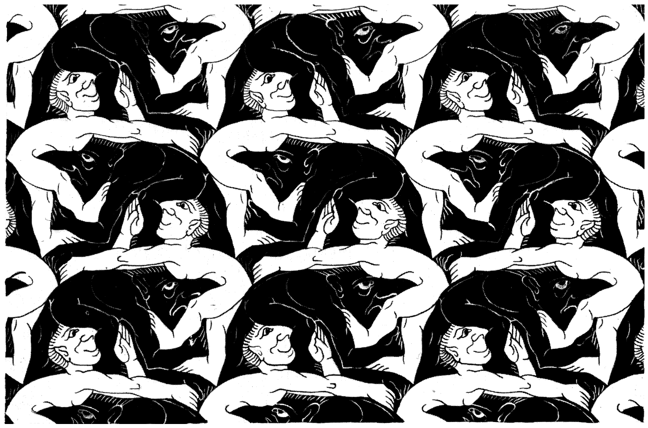

Figure 1.4: Wallpaper pattern for Ontmoeting. From: M.C.Escher, 1944.

to the next one straight above it) and b (horizontal, from one nose to the next one to its right) leave the pattern invariant, but this is not what makes the picture so special. It has another symmetry, namely a *glide-reflection* g, which is described as: first apply translation over $\frac{1}{2}a$, and then reflect in the vertical line l equidistant to a left oriented and a neighbouring right oriented nose. Note that $g^2 = s$.

Exercise 1.1.4. List all symmetries of the pattern in Figure 1.4.

Space groups

Having seen symmetries of some 2-dimensional figures, we go over to 3-dimensional ones.

The cube in Figure 1.5 has lots of symmetries. Its symmetry group is generated by the rotations r_3, r_4, r_4' whose axes are the coordinate axes, and the reflection t through the horizontal plane dividing the cube in two equal parts.

Exercise 1.1.5. What is the size of the symmetry group of the cube?

Closely related is the methane molecule, depicted in Figure 1.6. Not all of the symmetries of the cube are symmetries of the molecule; in fact, only those that map the hydrogen atoms to other such atoms are; these atoms form a regular *tetrahedron.*

Exercise 1.1.6. Which of the symmetries of the cube are symmetries of the methane molecule?

Finally consider the cubic grid in Figure 1.7. Considering the black balls to be natrium atoms, and the white ones to be chlorine atoms, it can be seen as the structure of a salt-crystal. It has translational symmetry in three perpendicular

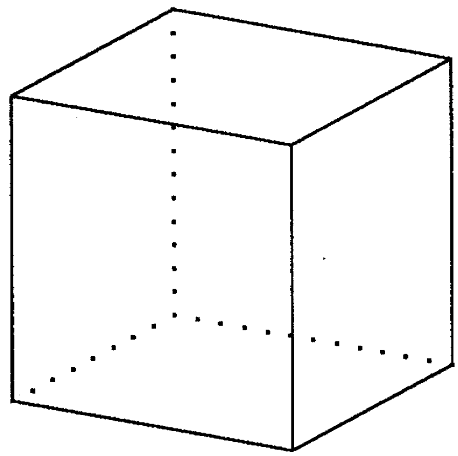

Figure 1.5: The cube.

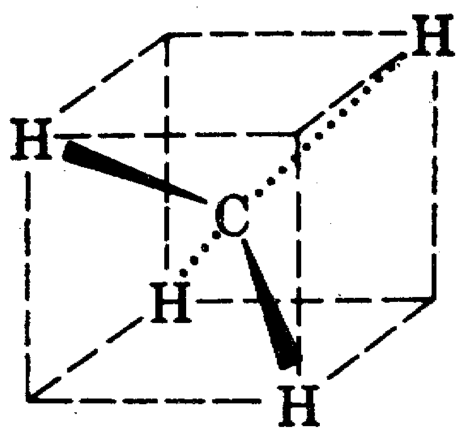

Figure 1.6: The methane molecule. From: [6].

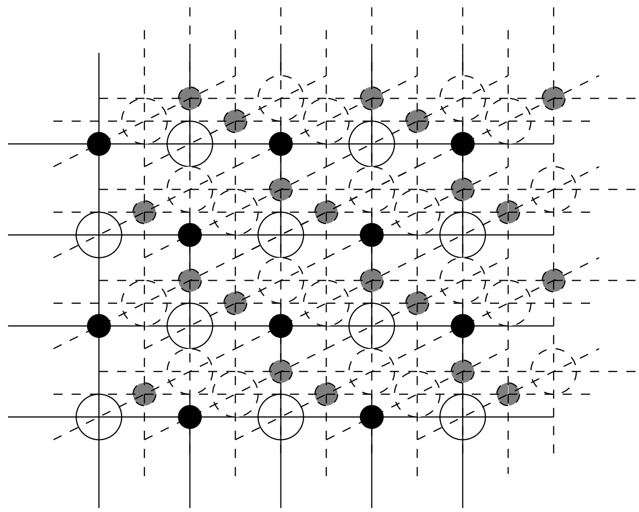

Figure 1.7: Structure of a crystal

directions along vectors of the same size. To understand its further symmetry, consider all those symmetries of the crystal that leave a given natrium atom fixed. Such a symmetry must permute the 6 chlorine atoms closest to the given natrium atom. These atoms form a regular octahedron, as depicted in Figure 1.8, and therefore we may say that the symmetry group of salt crystals is generated by that of the regular octahedron and three translations.

Exercise 1.1.7. This exercise concerns symmetries of one-dimensional figures. Considere therefore the real line $\mathbb{R}$ with the Euclidean metric, in which the distance between $x, y \in \mathbb{R}$ is $|x - y|$. A map g from $\mathbb{R}$ to itself is called an *isometry* if $|g(x) - g(y)| = |x - y|$ for all $x, y \in \mathbb{R}$.

1. Show that any isometry of $\mathbb{R}$ is either of the form $s_a : x \mapsto 2a - x$ (reflection in a) or of the form $t_a : x \mapsto x + a$ (translation over a).

2. Prove that $\{e\}$ and $\{e, s_0\}$, where e denotes the identity map on $\mathbb{R}$, are the only finite groups of isometries of $\mathbb{R}$.

To describe symmetries of *discrete* subsets of $\mathbb{R}$, we restrict our attention to groups G whose *translational part* $\{a \in \mathbb{R} \mid t_a \in G\}$ equals $\mathbb{Z}$.

3. Show that if such a group G contains s_0, then

$$G = \{t_a \mid a \in \mathbb{Z}\} \cup \{t_{\frac{a}{2}} \mid a \in \mathbb{Z}\}.$$

4. Conclude that there are only two ‘essentially different’ groups describing the symmetries of discrete subsets of $\mathbb{R}$ having translational symmetry.

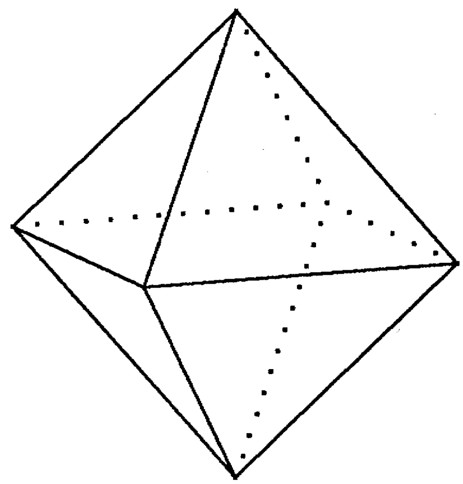

Figure 1.8: The octahedron.

Exercise 1.1.8. Analyse the symmetry groups of discrete subsets of the strip $\mathbb{R} \times [-1, 1]$ in a manner similar to that of Exercise 1.1.7. Compare the result with your answer to Exercise 1.1.3.

1.2 Basic notions

Formal definition of a group

Having seen some examples of groups, albeit from the narrow point of view of symmetry groups of figures in the plane or in three-dimensional space, we are ready for some abstract definitions that should reflect the experiences we had in those examples.

Definition 1.2.1. A *group* is a set G, together with an operation $\cdot$, which maps an ordered pair (g, h) of elements of G to another element $g \cdot h$ of G, satisfying the following axioms.

1. The operation is *associative*, i.e., for all $g, h, k \in G$ we have $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.
2. G contains an *identity element*, i.e., an element e that satisfies $e \cdot g = g \cdot e = g$ for all $g \in G$.
3. Each element of G has an *inverse*, i.e., for each $g \in G$ there is an $h \in G$ such that $g \cdot h = h \cdot g = e$. This element is denoted by g^{-1}.

The cardinality of G is called the *order* of the group, and often denoted by $|G|$; it may be infinite. If the operation is not only associative but commutative as well (meaning $g \cdot h = h \cdot g$ for all $g, h \in G$), then G is called an *Abelian* group.

Exercise 1.2.2. Prove that a group G cannot have more than one identity. Also, the notation g^{-1} for the inverse of g seems to indicate the uniqueness of that inverse; prove this.

Let us give some examples of groups.

Example 1.2.3.

1. All sets of transformations found in §1.1 form groups; in each case, the composition $\circ$ serves as operation $\cdot$. Some of them are Abelian; like the the chair, and some aren't, like those of the cube.

2. The real numbers $\mathbb{R}$ form a group with respect to addition $+$, the unit element being 0. They do *not* form a group with respect to ordinary multiplication, as 0 does not have an inverse. Leaving out 0 we do obtain an Abelian group $(\mathbb{R} \setminus \{0\}, \cdot)$, the unit element being 1.

3. Also $\mathbb{Q}$ and $\mathbb{Z}$ form additive groups. Again $\mathbb{Q} \setminus \{0\}$ is a multiplicative group. However, $\mathbb{Z}$ cannot be turned into a multiplicative group, the only invertible elements begin $\{\pm 1\}$.

4. Let X be a set. The bijections of X form a group. Composition is again the group operation. The group inverse coincides with the 'functional' inverse. The group is often denoted by $\mathrm{Sym}(X)$. If X is finite, say $|X| = n$, then $|\mathrm{Sym}(X)| = n!$. If two sets X and Y have the same cardinality, then the groups $\mathrm{Sym}(X)$ and $\mathrm{Sym}(Y)$ are essentially the same. For finite sets of cardinality n, the representative $\mathrm{Sym}(\{1, ..., n\})$ of this class of groups is denoted by $\mathrm{Sym}(n)$ or S_n.

   Its element are also called permutations. For $X = \{1, 2, 3\}$, we have

   $$\mathrm{Sym}(X) = \{e, (1, 2), (2, 3), (1, 3), (1, 2, 3), (3, 2, 1)\}.$$

   This notation is explained in Section 2.3.

5. Let V be a vector space. The set of all linear transformations of V, often denoted by $\mathrm{GL}(V)$, together with composition of maps, is a group. By fixing a basis of V, its element can be written as invertible matrices.

Let us make notation easier.

Definition 1.2.4. Usually, we leave out the operation $\cdot$ if it does not cause confusion. However, if the operation is written as $+$, it is never left out. For g in a group G we write g^n for the product

$$\underbrace{g \cdot g \cdot \ldots \cdot g}_{n}.$$

Our first lemma concerns the inverse of a product.

Lemma 1.2.5. *For two elements g, h in a group G we have $(gh)^{-1} = h^{-1}g^{-1}$, and $(g^n)^{-1} = (g^{-1})^n$. For the latter we also write g^{-n}.*

This is very natural, as Weyl points out in [10]:

> 'When you dress, it is not immaterial in which order you perform the operations; and when in dressing you start with the shirt and end up with the coat, then in undressing you observe the opposite order; first take off the coat and the shirt comes last.'

Exercise 1.2.6. Show that the notation for fractions which is usual for integers, does not work here: $\frac{h}{g}$ could stand for $h^{-1}g$ as well as for gh^{-1}, and the latter two may differ.

With the methane molecule in Figure 1.6, we say that some of the symmetries of the cube form a smaller group, namely the symmetry group of the tetrahedron. We found an example of a subgroup.

Definition 1.2.7. A *subgroup* of a group G is a subset H of G satisfying the following conditions.

1. H is non-empty.
2. H is closed under the operation, i.e., for all $g, h \in H$ we have $gh \in H$.
3. H is closed under taking inverses. This means that, for all $h \in H$, we have $h^{-1} \in H$.

Exercise 1.2.8. Prove that a subgroup of a group contains the group's identity. Prove that for a subset H of G to be a subgroup of G it is necessary and sufficient that H be non-empty and that $gh^{-1} \in H$ for all $g, h \in H$.

Exercise 1.2.9. Prove that for a subset H of a finite group G to be a subgroup, it is necessary and sufficient that H be non-empty and closed under the operation of G.

Generation

The following lemma is needed to formalize the concept of a subgroup generated by some elements.

Lemma 1.2.10. *The intersection of a family of subgroups of a group G is a subgroup of G.*

Definition 1.2.11. As a consequence of Lemma 1.2.10, for a subset S of G the intersection of all groups containing S is a subgroup. It is denoted by $\langle S \rangle$ and called *the subgroup generated by* S. Instead of $\langle \{g_1, \ldots, g_n\} \rangle$ we also write $\langle g_1, \ldots, g_n \rangle$. The elements $g_1, \ldots, g_n$ are called *generators* of this group.

Some groups are generated by a single element. For example, $\mathbb{Z}$ is generated by 1, and the symmetry group of the triquetrum depicted in Figure 1.9 is generated by a rotation around its center over $2\pi/3$.

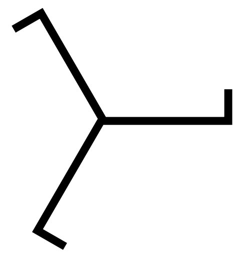

Figure 1.9: The triquetrum.

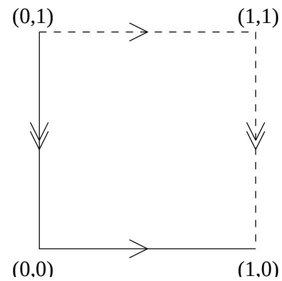

Figure 1.10: The flat torus.

Definition 1.2.12. If there exists an element $g \in G$ for which $G = \langle g \rangle$, then G is called a *cyclic* group.

Consider the integers modulo n, where n is a positive integer. In this set, which is denoted by $\mathbb{Z}/n\mathbb{Z}$ for reasons that will become clear later, two integers are identified whenever their difference is a multiple of n. This set forms a group, addition modulo n being the operation. The order of $\mathbb{Z}/n\mathbb{Z}$ is n, and it is a cyclic group. In fact, one could say that it is *the* cyclic group of this order, as all such groups are the same in a sense to be made precise later. Thus the symmetry group of the triquetrum (Figure 1.9) is 'the same' as $\mathbb{Z}/3\mathbb{Z}$.

Definition 1.2.13. Let g be an element of a group G. The order of $\langle g \rangle$ is called the *order* of g. It may be infinite.

Exercise 1.2.14. Prove that a cyclic group is Abelian.

Exercise 1.2.15. Consider the flat torus T. It can be seen as the square $S = [0,1] \times [0,1]$ with opposite edges glued together, such that the arrows in Figure 1.10 match. (Thus, the set is really in bijection with $[0,1) \times [0,1)$.) The torus has the following group structure: given two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ on the torus, their sum corresponds to the point r on S with the property that $r - (p_1 + q_1, p_2 + q_2)$, the latter term of which may not be in S, has integral entries.

Determine which elements of T have finite order, and which elements have infinite order.

Chapter 2

Permutation Groups

The realization of a group by means of symmetries (more generally, bijections of a set) is closely related to the structure of its subgroups. Let us first focus on the latter aspect.

2.1 Cosets and Lagrange

Recall the notion of subgroup from Definition 1.2.7.

Definition 2.1.1. For subsets S and T of a group G, we write ST for the set

$$\{st \mid s \in S, t \in T\}.$$

For $s \in G$ and $T \subset G$ we also write sT for $\{s\}T$ and Ts for $T\{s\}$.

Definition 2.1.2. For a subgroup H of G the sets of the form gH with $g \in G$ are called *left cosets* of H in G, and those of the form Hg are called *right cosets* of H in G. The set of all left cosets of H in G is denoted by G/H.

It would be logical to denote the set of right cosets of H in G by $H \setminus G$. Indeed, this is sometimes done, but this meaning of $\setminus$ should not be confused with the usual set-theoretic one. We shall only use G/H in these notes.

Example 2.1.3.

1. $G = \mathbb{Z}$, the additive group, and $H = \langle m \rangle = m\mathbb{Z}$, the subgroup generated by m, for some $m \in \mathbb{Z}$. The set of left cosets is $\{H, 1+H, 2+H, \ldots, m-1+H\}$. Since $x + H = H + x$, left cosets coincide with right cosets.

2. $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$, the symmetry group of an equilateral triangle (check). Take the subgroup $H = \langle (123) \rangle$. Then the set of left cosets is $\{H, (12)H\}$, where $(12)H = \{(12), (13), (23)\}$. Observe that $(12)H = H(12) = G \setminus H$, so left and right cosets coincide.

3. $G = \mathbb{R}^n$, the (additive) translation group of a real vector space and H is a linear subspace of G. Then the left cosets are the affine subspaces parallel to H.

4. A left coset need not be a right coset. Take for instance $G = S_3$ and H the cyclic subgroup $\langle(12)\rangle$. Then $(13)H = \{(13), (123)\}$ and $H(13) = \{(13), (132)\}$.

It is easy to verify that left cosets are equivalence classes of the equivalence relation

$$g \sim h \Leftrightarrow g^{-1}h \in H,$$

and similarly for right cosets. So we have the following lemma.

Lemma 2.1.4. *The left cosets of a subgroup H in the group G form a partition of the group G. For $g \in G$, the map $h \mapsto gh$ is a bijection between H and the left coset gH. A similar statement holds for right cosets.*

A similar statement holds for right cosets.

Exercise 2.1.5. Show that $(13)\langle(12)\rangle$ is not a right coset of $\langle(12)\rangle$ in S_3, and that $(12)\langle(123)\rangle$ is a right coset of $\langle(123)\rangle$ in S_3.

Definition 2.1.6. If the number of left cosets of a subgroup H in a group G is finite, this number is called the *index* of H in G and denoted by $[G : H]$.

Example 2.1.7. In the previous examples, we have $[\mathbb{Z} : \langle m\rangle] = m$ and $[S_3 : \langle(123)\rangle] = 2$.

Theorem 2.1.8 (Lagrange's Theorem). *If G is a finite group and H a subgroup, then $|H|$ divides $|G|$.*

Corollary 2.1.9. *If G is a finite group and $g \in G$, then the order of g divides that of G.*

Exercise 2.1.10. If H and K are subgroups of a finite group G, then $H \cap K$ is also a subgroup of G. Prove that $|HK| = \frac{|H||K|}{|H \cap K|}$.

2.2 Quotient groups and the homomorphism theorem

If two groups G and H and some of their elements are in the picture, we shall always make it clear to which of the two groups each element belongs. This enables us to leave out the operation symbol throughout, as the product of two elements of H is automatically understood to denote the product in H and not in G. Also, we shall use e for the unit element of both groups.

Next, we will see the functions between two groups that preserve the structure of groups.

Definition 2.2.1. Let G and H be groups. A *homomorphism* from G to H is a map $\phi : G \to H$ satisfying

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \text{ for all } g_1, g_2 \in G,$$

and $\phi(e_G) = e_H$. If, in addition, ϕ is a bijection, then ϕ is called an *isomorphism*. An *automorphism* of G is an isomorphism from G to itself. By $\mathrm{Aut}(G)$ we denote the set of all automorphisms of G.

If there exists an isomorphism from a group G to a group H, then G is called *isomorphic* to H. this fact is denoted by $G \cong H$, and it is clear that $\cong$ is an equivalence relation. We claimed earlier that the symmetry group of the triquetrum (Figure 1.9) is 'the same as' $\mathbb{Z}/3\mathbb{Z}$. The precise formulation is, that both groups are isomorphic.

Definition 2.2.2. For a group homomorphism $\phi : G \to H$ the subset

$$\{g \in G \mid \phi(g) = e\}$$

of G is called the *kernel* of ϕ, and denoted by $\ker(\phi)$. The set

$$\phi(G) = \{\phi(g) \mid g \in G\} \subseteq H$$

is called the *image* of ϕ, and denoted by $\mathrm{im}(\phi)$.

Exercise 2.2.3. Let $\phi : G \to H$ be a homomorphism. Then ϕ is injective if and only if $\ker \phi = \{e\}$.

Example 2.2.4. For $g \in G$, we define $\phi_g \colon G \to G$ by $\phi_g(h) = ghg^{-1}$. Then ϕ_g is an element of $\mathrm{Aut}(G)$, called *conjugation with* g. An automorphisms of G is called an *inner automorphism* if it is of the form ϕ_g form some g; it is called *outer* otherwise.

Exercise 2.2.5. Let ϕ be a group homomorphism from G to H. Prove that $\ker(\phi)$ and $\mathrm{im}(\phi)$ are subgroups of G and H, respectively. Show that $g \ker(\phi) g^{-1} \subseteq \ker(\phi)$ for all $g \in G$.

Exercise 2.2.6. Prove:

1. $\mathrm{Aut}(G)$ is a group with respect to composition as maps;
2. the set $\mathrm{Inn}(G) = \{\phi_g \mid g \in G\}$ is a subgroup of $\mathrm{Aut}(G)$. Here ϕ_g is denotes conjugation with g.

Example 2.2.7. For $G = \mathrm{GL}(V)$, the group of invertible linear transformations of a complex vector space V, the map $g \mapsto \det(g)$ is a homomorphism $G \to \mathbb{C}^*$. Its kernel is the group $\mathrm{SL}(V)$.

Example 2.2.8. The map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $x \mapsto x + m\mathbb{Z}$, is a homomorphism of additive groups. Its kernel is $m\mathbb{Z}$.

Example 2.2.9. There is an interesting homomorphism between $\mathrm{SL}(2, \mathbb{C})$ and the Lorentz group L, which is the group of all linear transformations of the vector space $\mathbb{R}^4$ that preserve the Lorentz metric

$$|x| := x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

To $x \in \mathbb{R}^4$ we associate a 2×2-matrix $\psi(x)$ as follows:

$$\psi(x) = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix},$$

so that $|x| = \det(\psi(x))$. Now, the map $\varphi : \mathrm{SL}(2, \mathbb{C}) \to L$ given by $\varphi(A)(x) = \psi^{-1}(A\psi(x)A^*)$ is a group homomorphism. Here $A^* = (\bar{a}_{ji})_{ij}$ if $A = (a_{ij})_{ij}$.

Indeed, ψ is a linear isomorphism from $\mathbb{R}^4$ onto

$$\psi(\mathbb{R}^4) = \left\{ \begin{pmatrix} x & y \\ z & u \end{pmatrix} | x, y, z, u \in \mathbb{C} \text{ and } \bar{x} = x, \bar{y} = z, \bar{u} = u \right\},$$

and one checks that this space is invariant under $M \mapsto AMA^*$ for any $A \in \mathfrak{G}L(2, \mathbb{C})$. Also, for $A \in \mathrm{SL}(2, \mathbb{C})$, $\varphi(A)$ preserves the metric due to the multiplicative properties of the determinant:

$$|\varphi(A)(x)| = \det(A\psi(x)A^*) = \det(A)\det(\psi(x))\det(A^*) = \det(\psi(x)) = |x|.$$

Exercise 2.2.10. Show that φ in Example 2.2.9 satisfies $\varphi(AB) = \varphi(A)\varphi(B)$, and thus finish the proof that φ is a homomorphism $\mathrm{SL}(2, \mathbb{C}) \to L$.

Exercise 2.2.11. Determine the subgroups of $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}_2/2\mathbb{Z})$. Is $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}_2/2\mathbb{Z})$ isomorphic to $\mathbb{Z}/4\mathbb{Z}$?

Exercise 2.2.12. Take the group $\mathbb{Z}/4\mathbb{Z}$, the subgroup $\langle(1234)\rangle$ of S_4 and the multiplicative subgroup generated by i in $\mathbb{C}$. Verify that they are all isomorphic.

As we have seen before, it is not always the case that left cosets and right cosets of a subgroup H are the same. When that happens, we say that H is normal.

Definition 2.2.13. A subgroup H of G is called *normal* if every left coset of H in G is a right coset of H in G, or, equivalently, if $gHg^{-1} \subset H$ for all $g \in G$. We denote it by $H \trianglelefteq G$.

If H is normal, then we can define an operation on the set G/H of all left cosets of H in H, as follows:

$$(gH) \cdot (kH) = (gk)H.$$

This operation turns G/H into a group, called the *quotient group of G by H*. The map $G \mapsto G/H, g \mapsto gH$ is a homomorphism $G \to G/H$.

Exercise 2.2.14. Verify that the operation on G/H is well-defined, i.e., does not depend on the choice of representatives g and k. Verify that G/H does indeed become a group with this operation. Note that one really needs normality of H.

Exercise 2.2.15. Prove that $\mathrm{Inn}(G)$ (cf. Example 2.2.4) is a normal subgroup of $\mathrm{Aut}(G)$ (cf. Definition 2.2.1).

Example 2.2.16.

1. The subgroup $\langle (12)(34), (13)(24) \rangle$ is normal in S_4.

2. Let G be the group of Euclidean motions in $\mathbb{R}^3$ (see Chapter 3). Write an element of G as
$$\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}$$
with $a \in \mathrm{SO}(3)$ and $v \in \mathbb{R}^3$. This presentation is convenient as the operation in G is now given by the multiplication of 4×4 matrices. Consider the subgroup N of all translations; thus, N consists of all matrices
$$\begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix}.$$
Then N is a normal subgroup of G.

Lemma 2.2.17. *The kernel of a group homomorphism $\phi : G \to H$ is a normal subgroup of G.*

See Exercise 2.2.5 for the proof.

Exercise 2.2.18. Prove that any subgroup of index 2 is normal.

Theorem 2.2.19. *Let $\phi\colon G \to H$ be a group homomorphism. Then $G/\ker\phi \cong \mathrm{im}\,\phi$.*

Proof. Let $N := \ker\phi$. We claim that $\pi : G/N \to \mathrm{im}\,\phi, gN \mapsto \phi(g)$ is an isomorphism. Indeed, it is well-defined: if $g'N = gN$, then $g^{-1}g' \in N$ so that $\phi(g^{-1}g') = e$, hence $\phi(g') = \phi(g)$. Also, π is a homomorphism:
$$\pi((gN)(kN)) = \pi((gk)N) = \phi(gk) = \phi(g)\phi(k) = \pi(gN)\pi(kN).$$
Finally, if $\pi(gN) = e$, then $\phi(g) = e$, hence $gN = eN$; it follows that $\ker(\pi) = \{eN\}$, and Exercise 2.2.3 shows that π is injective. It is also surjective onto its image, hence it is an isomorphism. $\square$

Theorem 2.2.20 (First Isomorphism Theorem). *Let G be a group, and H, K normal subgroups of G such that $H \supseteq K$. Then K is a normal sugroup of H, H/K a normal subgroup of G/K, and $G/H \cong (G/K)/(H/K)$.*

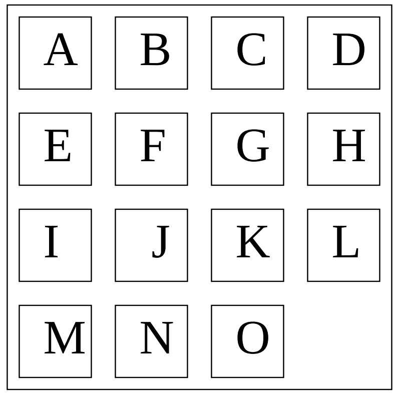

Figure 2.1: Initial state of Sam Loyd's 15-puzzle.

Proof. For any $g \in G$, we have $gKg^{-1} \subseteq K$, so *a fortiori* this holds for $g \in H$. Hence, K is normal in H. For any $g \in G$ and $h \in H$, we have $(gK)(hK)(gK)^{-1} = (ghg^{-1})K \in H/K$ as H is normal in G. Hence, H/K is normal in G/K.

Consider the map $\pi : G \to (G/K)/(H/K), g \mapsto (gK)(H/K)$. It is clearly a surjective homomorphism, and its kernel equals $\{g \in G \mid gK \in H/K\} = H$. By Theorem 2.2.19, $G/H \cong (G/K)/(H/K)$ as claimed. $\square$

Theorem 2.2.21 (Second Isomorphism Theorem). *Let G be a group, H a subgroup of G, and K a normal subgroup of G. Then HK is a subgroup of G, K a normal subgroup of HK, $H \cap K$ is a normal subgroup of H, and $HK/K \cong H/(H \cap K)$.*

Exercise 2.2.22. Prove the second Isomorphism Theorem.

2.3 Permutation groups

2.3.1 A Messed-Up Puzzle

Suppose that a friend borrows your 15-puzzle just after you solved it, so that it is in the initial state depicted in Figure 2.1. You leave the room to make some coffee and when you come back your friend returns the puzzle to you, sighing that it is too difficult for him, and asking you to show him how to solve it. At this time, the puzzle is in the state of Figure 2.2. Confidently you start moving around the plastic squares; for readers unfamiliar with the puzzle a possible sequence of moves is shown in Figure 2.3. As time passes, you start getting more and more nervous, for your friend is watching you and you don't seem to be able to transform the puzzle back into its initial state. What is wrong?

To answer this question, we shall look at some basic properties of permutations.

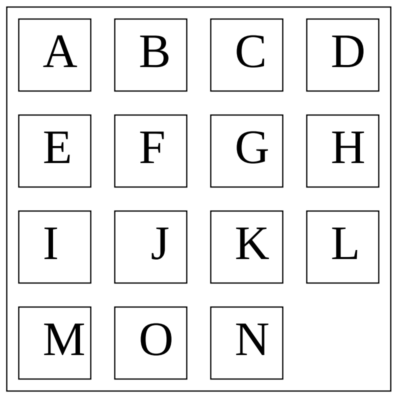

Figure 2.2: Messed up state of the 15-puzzle.

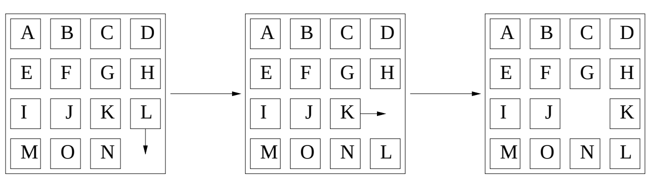

Figure 2.3: Possible moves of the 15-puzzle.

2.3.2 Permutations and their Signs

Recall the definition of a permutations and the group class S_n. Permutations can be represented in several ways, three of which we shall discuss here.

Two-row notation: for a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ one writes a $2 \times n$ matrix, in which both rows contain the numbers $1, \ldots, n$. The image of i is the number that appears below i in the matrix. An example is

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix} \in S_5.$$

Disjoint cycle notation: To explain this notation, consider the permutation π from the previous example. We have $\pi(1) = 5$, $\pi(5) = 3$ and $\pi(3)$ equals 1 again. The permutation that satisfies these equations and that fixes the other elements is denoted by

$$(1, 5, 3) = (5, 3, 1) = (3, 1, 5).$$

A permutation of this form is called a 3-*cycle*. But π does not fix 2 or 4; indeed, $\pi(2) = 4$ and $\pi(4) = 2$. We find that

$$\pi = (1, 5, 3)(2, 6) = (2, 6)(1, 5, 3),$$

i.e., π is the product of the 2-cycle $(2, 6)$ and the 3-cycle $(3, 5, 1)$. These are called *disjoint*, because each fixes the elements occurring in the other, and this fact makes them commute. Any permutation can be written as a product of disjoint cycles, and this factorization is unique up to changes in the order. In this notation, it is usual not to write down the 1-cycles. Leaving these out, it is not always clear to which S_n the permutation belongs, but this abuse of notation turns out to be convenient rather than confusing.

Permutation matrix notation: consider once again the permutation $\pi \in S_5$. With it we associate the 5×5-matrix $A_\pi = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 \text{ if } \pi(j) = i, \\ 0, \text{ otherwise.} \end{cases}$$

It can be checked that $A_{\pi\sigma} = A_\pi A_\sigma$, where we take the normal matrix product in the right-hand side. Hence S_n can be seen as a matrix group, multiplication being the ordinary matrix multiplication.

Diagram notation: in this notation, a permutation is represented by a diagram as follows: draw two rows of dots labeled $1, \ldots, n$, one under the other. Then draw a line from vertex i in the upper row to vertex $\pi(i)$ in the lower. In our example, we get the picture in Figure 2.4.

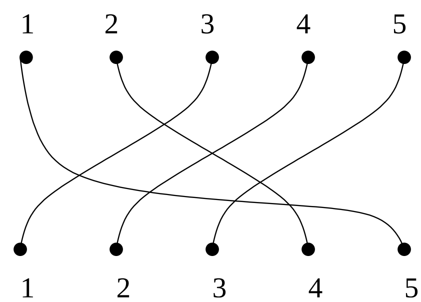

Figure 2.4: The diagram notation.

Exercise 2.3.1. Show that the 2-cycles in S_n generate that whole group. Can you give a smaller set of 2-cycles that generates the whole group? How many do you need?

Exercise 2.3.2. Determine the order of a k-cycle. Determine the order of $\pi = (2,6)(1,5,3)$. Give and prove a general formula for the order of a permutation, given in disjoint cycle notation, in terms of the lengths of the cycles.

Related to permutation matrices we have the following important definition.

Definition 2.3.3. For $\pi \in S_n$ we define the *sign* of π by

$$\operatorname{sgn} \pi := \det A_\pi.$$

The sign of a permutation π is either 1 or -1. In the former case, π is called an *even* permutation, in the latter an *odd* permutation.

Exercise 2.3.4. Show that $\operatorname{sgn} : S_n \to \{\pm 1\}$ is a group homomorphism.

The sign of a permutation can be interpreted in many ways. For example, in the diagram notation, the number of intersections of two lines is even if and only if the permutation is even.

Exercise 2.3.5. Prove that the sign of a k-cycle equals $(-1)^{k+1}$.

The even permutations in S_n form a subgroup, called the *alternating group* and denoted by A_n. It is the kernel of the homomorphism $\operatorname{sgn} : S_n \to \{\pm 1\}$ of Exercise 2.3.4.

2.3.3 No Return Possible

Let us return to Sam Loyd's 15-puzzle. It can be described in terms of states and moves. The Figures 2.1 and 2.2 depict *states* of the puzzle. Formally, a state can be described as a function

$$s : \{1, \dots, 16\} \to \{\mathrm{A}, \mathrm{B}, \dots, \mathrm{O}, *\}.$$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure 2.5: Numbering of the positions.

Here, $s(p)$ is the square occupying position p in the state s and $*$ stands for the empty square. The positions are numbered as in fig 2.5.

For $p \in \{1, \ldots, 16\}$, we denote by $M(p) \subseteq S_{16}$ the set of possible moves in a state s with $s(p) = *$. A move $m \in M(p)$ should be interpreted as moving the square from position j to position $m(j)$, for all $j = 1, \ldots, 16$. Hence, we have for example

$$\begin{aligned} M(1) &= \{(1,2), (1,5)\}, \\ M(2) &= \{(1,2), (2,3), (2,6)\}, \text{ and} \\ M(6) &= \{(2,6), (5,6), (6,7), (6,10)\}. \end{aligned}$$

Moves can be combined to form more general *transformations*. The set of all possible transformations starting in a state with the empty square in position p is denoted by $T(p)$, and it is defined as the set of all products (in S_{16}) of the form $m_k m_{k-1} \ldots m_1$, where

$$m_{j+1} \in M(m_j \circ m_{j-1} \circ \ldots \circ m_1(p))$$

for all $j \geq 0$. This reflects the condition that m_{j+1} should be a possible move, after applying the first j moves to s. In particular, in taking $k = 0$, we see that the identity is a possible transformation in any state. Also, if $t \in T(p)$ and $u \in T(t(p))$, then $u \circ t \in T(p)$. Finally, we have $t^{-1} \in T(t(p))$.

Moves and, more generally, transformations change the state of the puzzle. Let $s \in S$ with $s(p) = *$ and let $t \in T(p)$. Then t changes the state s to the state $t \cdot s$, which is defined by

$$(t \cdot s)(q) = s(t^{-1}(q)).$$

In words, the square at position q after applying t is the same as the square at position $t^{-1}(q)$ before that transformation.

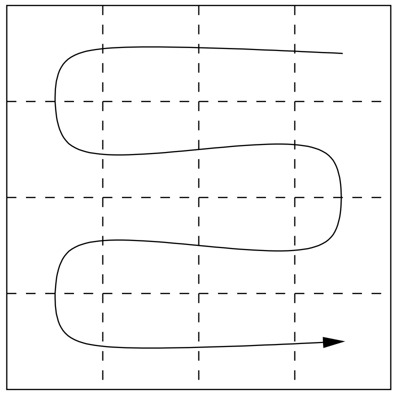

Figure 2.6: Ordering of the non-empty squares in each state.

Now that we captured Sam Loyd's 15-puzzle in a comfortable notation, we can formalize our question as follows: is there a $t_0 \in T(16)$ that transforms the messed up state s_1 of Figure 2.2 into the initial state s_0 of Figure 2.1, i.e., that satisfies

$$t_0 \cdot s_1 = s_0?$$

The answer is no, and we shall presently make clear why. In each state, order the non-empty squares according to the path in Figure 2.6. That is, the j-th *non-empty* square one encounters when following this path in a particular state s is called the j-th square in s, for $j = 1, \ldots, 15$. Now consider any position p, any $t \in T(p)$ and some $s \in S$ with $s(p) = *$. Then we define a permutation $\pi_t \in S_{15}$ by

$$\pi_t(i) = j \Leftrightarrow \text{ the } i\text{-th square in state } s \text{ is the } j\text{-th square in state } t \cdot s.$$

Note that this is independent of the choice of s. We have that

$$\pi_{u \circ t} = \pi_u \circ \pi_t \tag{2.1}$$

whenever the left-hand side makes sense, i.e., whenever $u \in T(t(p))$.

We see that the t_0 we would like to find equals $(14, 15)$, and then π_{t_0} also equals $(14, 15)$, considered as a permutation in S_{15}. This is an odd permutation. Now consider the move m depicted in figure 2.7. It interchanges the squares in position 6 and 10 in figure 2.5, so the move equals $(6, 10)$. The corresponding permutation π_m is the cycle $(10, 9, 8, 7, 6)$. A moment's thought leads to the conclusion that π_m is an odd cycle for *any* move m, hence an even permutation. As a consequence of this and equation (2.3.3), a sequence of such moves will also be mapped to an even permutation by π. Therefore, such a transformation cannot possibly be $(14, 15)$.

You conclude that your friend must have cheated while you were away making coffee, by lifting some of the squares out of the game and putting them back in a different way.

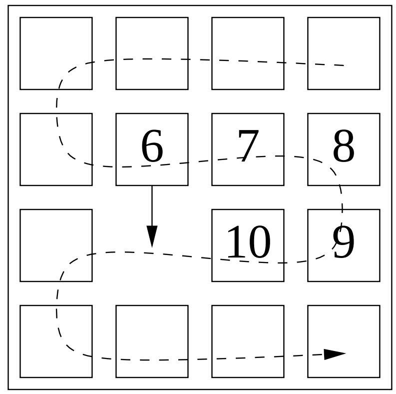

Figure 2.7: Moves correspond to odd cycles.

Exercise 2.3.6. Show that the set $\{\pi_t \mid t \in T(p) \text{ for some } p\}$ forms a subgroup of S_{15}.

Exercise 2.3.7. Prove that the set

$$\{t \in T(16) \mid t(16) = 16\}$$

is a subgroup of A_{16}. Use this to give another argument why your friend must have cheated.

2.4 Action of groups on sets

Although the transformations in the 15-puzzle do not quite form a group, their 'acting' on the puzzle states reflects an important phenomenon in group theory, namely that of an action.

Definition 2.4.1. Let G be a group and M a set. An *action* of G on M is a map $\alpha : G \times M \to M$ satisfying

1. $\alpha(e, m) = m$ for all $m \in M$, and
2. $\alpha(g, \alpha(h, m)) = \alpha(gh, m)$ for all $m \in M$ and $g, h \in G$.

If the action α is obvious from the context, we leave α out and write gm for $\alpha(g, m)$. Also, for subsets $S \subset G$ and $T \subset M$ we write ST for $\{\alpha(g, m) \mid g \in S, m \in T\}$. For $\{g\}T$ we write gT and for $S\{m\}$ we write Sm.

Remark 2.4.2. Given an action α, we have a homomorphism $G \to \mathrm{Sym}(M)$ given by $g \mapsto (m \mapsto \alpha(g, m))$. Conversely, given a homomorphism $\phi : G \to \mathrm{Sym}(M)$, we have an action $\alpha : G \times M \to M$ given by

$$\alpha(g, m) = \phi(g)m.$$

In other words, an action on M is nothing but a homomorphism $G \to \mathrm{Sym}(M)$.

Let us give some examples of a group acting on a set.

Example 2.4.3.

1. The group S_n acts on $\{1, \dots, n\}$ by $\pi \cdot i = \pi(i)$, and on the set of all two-element subsets of $\{1, \dots, n\}$ by $\pi(\{i, j\}) = \{\pi(i), \pi(j)\}$.
2. The group $\mathrm{GL}(3, \mathbb{R})$ acts on $\mathbb{R}^3$ by matrix-vector multiplication.
3. Consider the group of all motions of the plane, generated by all translations $(x, y) \mapsto (x + \alpha, y)$ for $\alpha \in \mathbb{R}$ in x-direction and all inflations $(x, y) \mapsto (x, \beta y)$ for $\beta \in \mathbb{R}^*$. This acts on the solutions of the differential equation
$$(\frac{\mathrm{d}}{\mathrm{d}x})^2(y) = -y.$$

Example 2.4.4. Consider $G = S_4$ and put $a = \{12, 34\}$, $b = \{13, 24\}$, $c = \{14, 23\}$. Here 12 stands for $\{1, 2\}$, etc. There is a natural action (as above) on the set of pairs, and similarly on the set $\{a, b, c\}$ of all partitions of $\{1, 2, 3, 4\}$ into two parts of size two. Thus, we find a homomorphism $S_4 \to \mathrm{Sym}(\{a, b, c\})$. It is surjective, and has kernel a group of order 4. (Write down its nontrivial elements!)

Example 2.4.5. The group $\mathrm{SL}(2, \mathbb{C})$ acts on the complex projective line $\mathbb{P}^1(\mathbb{C})$ as follows. Denote by $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1(\mathbb{C})$ be the map sending a pair of homogeneous coordinates to the corresponding point. The linear action of $\mathfrak{S}L(2, \mathbb{C})$ on $\mathbb{C}^2$ by left multiplication permutes the fibers of π, hence induces an action on $\mathbb{P}^1(\mathbb{C})$. On the affine part of $\mathbb{P}^1(\mathbb{C})$ where the second homogeneous coordinate is non-zero, we may normalize it to 1, and the action on the first coordinate x is given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax + b}{cx + d}.$$
Note, however, that this affine part is not invariant under the group.

Lemma 2.4.6. *Let G be a group. Then G acts on itself by conjugation, that is, the map $\alpha : G \times G \to G$ defined by $\alpha(h, g) = hgh^{-1}$ is an action.*

Proof. We have to check two things. First, that
$$\alpha(e, g) = ege^{-1} = g$$
for all $g \in G$, and secondly, that
$$\alpha(h, \alpha(k, g)) = h(kgk^{-1})h^{-1} = (hk)g(hk)^{-1} = \alpha(hk, g).$$
□

The following lemma allows us to partition a set M on which a group G acts into orbits.

Lemma 2.4.7. *Let G be a group acting on a set M. Then the sets $Gm = \{gm \mid g \in G\}$ for $m \in M$ partition the set M. The set Gm is called the* orbit *of m under G.*

Definition 2.4.8. An action of a group G on a set M is said to be *transitive* if it has only one orbit, or, stated differently, $Gm = M$ for any $m \in M$.

To define symmetry in a general context, we look at sets that are mapped to themselves by a given group.

Definition 2.4.9. Let G be a group acting on M. A subset $T \subseteq M$ is called *invariant* under $S \subseteq G$ if $ST \subseteq T$. In this case, the elements of S are called *symmetries* of T.

Definition 2.4.10. Let G be a group acting on M and let $m \in M$. Then the set

$$G_m := \{g \in G \mid gm = m\}$$

is called the *stabilizer* of m.

Lagrange's theorem has an important consequence for problems, in which one has to count the number of symmetries of a given set. It can be formulated as follows.

Theorem 2.4.11. *Let G be a group acting on a set M and let $m \in M$. Then G_m is a subgroup of G, and the map*

$$f : Gm \to G/G_m, \quad g \cdot m \to gG_m$$

is well-defined and bi-jective. As a consequence, if G is finite, then

$$|G| = |Gm||G_m|.$$

Remark 2.4.12. One must distinguish the elements of G from their actions on M. More precisely, the map sending $g \in G$ to the map $\alpha(g, \cdot) : M \to M$ is a homomorphism, but need not be an isomorphism.

We now want to make clear how the action of groups can help in classification and counting problems, and we shall do so by means of the following problem.

> Given is a finite set X with $|X| = v$. What is the number of non-isomorphic simple graphs with vertex set X?

This natural, but seemingly merely theoretical, question has strong analogue in counting all isomers with a given chemical formula, say C_4H_{10} (see [5]).

Of course, the question is formulated rather vaguely, and the first step in solving it is to formalize the meanings of the words *graph* and *non-isomorphic*. A simple graph is a function

$$f : \binom{X}{2} \to \{0, 1\}.$$

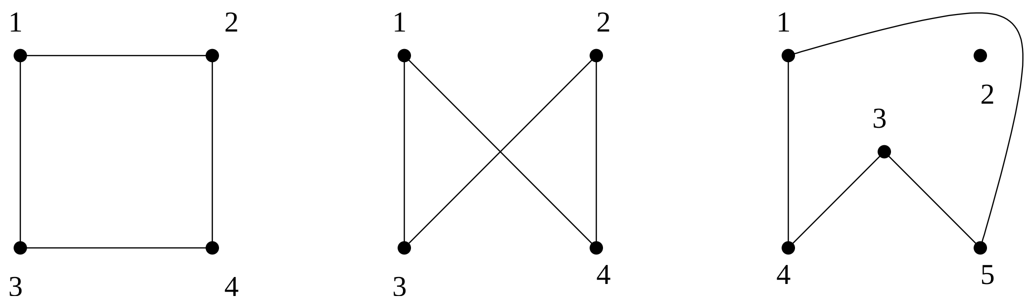

Figure 2.8: Three graphs, two of which are isomorphic.

Here $\binom{X}{2}$ stands for the set of all unordered pairs from X. If the function f takes the value 1 on such a pair $\{x, y\}$, this means that the pair is an *edge* in the graph. Often, graphs are associated with a picture like the ones in figure 2.8. The set of all graphs on X is denoted by $\text{Graphs}(X)$.

The first two graphs in figure 2.8 are essentially the same; to make them exactly the same one has to relabel the vertices. Such a relabeling is in fact a permutation of X. Formally, this leads to an action of $\text{Sym}(X)$ on $\text{Graphs}(X)$ in the following way. First of all $\text{Sym}(X)$ acts on $\binom{X}{2}$ by

$$\pi \cdot \{x, y\} = \{\pi(x), \pi(y)\}.$$

Furthermore, we define $\pi \cdot f$ by the commutativity of the following diagram.

$$\begin{tikzcd}
\binom{X}{2} \arrow[r, "f"] \arrow[d, "\pi\cdot"'] & \{0,1\} \arrow[d, "\mathrm{Id}"] \\
\binom{X}{2} \arrow[r, "\pi\cdot f"'] & \{0,1\}
\end{tikzcd}$$

This diagram tells us that $e \in \binom{X}{2}$ is an edge in the graph f if and only if $\pi \cdot e$ is an edge in the graph $\pi \cdot f$. In formula,

$$\pi \cdot f(x) = f(\pi^{-1}x).$$

Graphs that can be obtained from each other by the action of some $\pi \in \text{Sym}(X)$ are called *isomorphic*. Now our original question can be rephrased very elegantly as follows.

> How many orbits does the action of $\text{Sym}(X)$ have on $\text{Graphs}(X)$?

The following lemma answers this question in general.

Lemma 2.4.13 (Cauchy-Frobenius). *Let G be a finite group acting on a finite set M. Then the number of orbits of G on M equals*

$$\frac{1}{|G|} \sum_{g \in G} \mathit{fix}(g).$$

Here fix(g) denotes the number of $m \in M$ with $g \cdot m = m$.

Exercise 2.4.14. Prove Lemma 2.4.13.

To apply this lemma to counting graphs, we need to know for some $\pi \in \mathrm{Sym}(X)$, the number of $f \in \mathrm{Graphs}(X)$ that are fixed by π. The function f is fixed by π if and only if f is constant on the orbits of $\langle \pi \rangle$ in $\binom{X}{2}$. This reflects the condition that π should map edges to edges and non-edges to non-edges. Denoting the number of orbits of $\langle \pi \rangle$ on $\binom{X}{2}$ by $o(\pi)$, we find that the number of non-isomorphic graphs on X equals

$$\frac{1}{v!} \sum_{\pi \in \mathrm{Sym}(X)} 2^{o(\pi)},$$

where $v = |X|$.

Exercise 2.4.15. Consider $X = \{1, 2, 3, 4, 5\}$ and the permutation $\pi = (1, 2)(3, 4, 5)$, and calculate the number $o(\pi)$. Try to find a general formula in terms of the cycle lengths of π and the number v. Observe that $o(\pi)$ only depends on the cycle type.

Example 2.4.16. For $v = 4$, we have $o(e) = 6$, $o((12)) = 4$, $o((123)) = 2$, $o((1, 2, 3, 4)) = 2$ and $o((12)(34)) = 4$; the conjugacy clases of each of these permutations have sizes $1, 6, 8, 6$ and 3, respectively. Hence, the number of non-isomorphic graphs on 4 vertices equals

$$(1 \cdot 2^6 + 6 \cdot 2^4 + 8 \cdot 2^2 + 6 \cdot 2^2 + 3 \cdot 2^4)/24 = 11.$$

Can you find all?

Chapter 3

Symmetry Groups in Euclidean Space

This chapter deals with groups of isometries of n-dimensional Euclidean space. We set up a framework of results that are valid for general n, and then we restrict our attention to $n = 2$ or 3. In these two cases, we classify the finite groups of isometries, and point out which of these are *crystallographic point groups*, i.e., preserve a 2- or 3-dimensional lattice, respectively. First, however, we consider an example that motivates the development of this theory.

3.1 Motivation

In low dimensions, there are few finite groups of isometries and this enables us, when given a two- or three-dimensional object, to determine its symmetry group simply by enumerating all axes of rotation and investigating if the object admits improper isometries such as inversion and reflections. As an example, suppose that we want to describe the symmetry group G of the CH_4 molecule depicted in Figure 3.1. The hydrogen atoms are arranged in a regular tetrahedron with the C-atom in the center.

As a start, let us compute $|G|$. Consider a fixed H-atom a. Its orbit Ga consists of all four H-atoms, as can be seen using the rotations of order 3 around axes through the C-atom and an H-atom. Applying Lagrange's theorem we find $|G| = 4|G_a|$. The stabilizer of a consists of symmetries of the remaining 4 atoms. Consider one of the other H atoms, say b. Its orbit under the stabilizer G_a has cardinality 3, and the stabilizer $(G_a)_b$ consists of the identity and the reflection in the plane through a, b and the C-atom, so that we find

$$|G| = |Ga||G_a| = |Ga||G_a b||(G_a)_b| = 4 \cdot 3 \cdot 2 = 24.$$

Only half of these isometries are orientation-preserving.

To gain more information on the group G, we try to find all rotation axes:

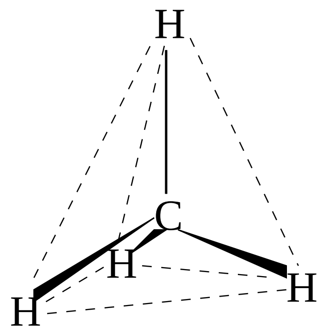

Figure 3.1: The CH_4 molecule.

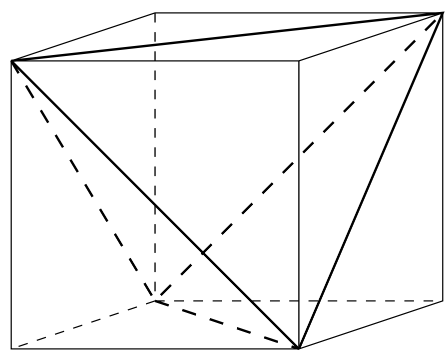

Figure 3.2: A regular tetrahedron inside a cube.

1. There are 4 three-fold axes, namely those through an H-atom and the center of the opposite side of the tetrahedron.

2. There are three two-fold axes, namely those through the centers of two perpendicular sides.

Later on, we shall see that this information determines the orientation preserving part of G uniquely; it is denoted by T. From the fact that G does not contain the inversion, but does contain other improper isometries, we may conclude that G is $T \cup i(W \setminus T)$, a group that can be described by Figure 3.2. Here we see a regular tetrahedron inside a cube. Half of the proper isometries of the cube are isometries of the tetrahedron, and the *improper* isometries of the tetrahedron may be obtained as follows: take any proper isometry of the cube which is *not* a isometry of the tetrahedron, and compose it with the inversion i. Note that $T \cup i(W\setminus)$ and W are isomorphic as abstract groups, but their realizations in terms of isometries are different in the sense that there is no isometry mapping one to the other. This point will also be clarified later.

Exercise 3.1.1. Consider the reflection in a plane through two H-atoms and the C-atom. How can it be written as the composition of the inversion and a proper isometry of the cube?

The finite symmetry group of the tetrahedron and the translations along each of three mutually orthogonal edges of the cube in Figure 3.2 generate the symmetry group of a crystal-like structure. For this reason, the group G is called a *crystallographic point group.*

3.2 Isometries of n-space

Consider the n-dimensional Euclidean space $\mathbb{R}^n$ with the distance function

$$d(x, y) = \|x - y\| = (x - y, x - y)^{1/2} = (\sum_i (x_i - y_i)^2)^{1/2}.$$

Definition 3.2.1. An *isometry* is a map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$d(g(x), g(y)) = d(x, y) \text{ for all } x, y \in \mathbb{R}^n.$$

The set of all isometries is a subgroup of $\mathrm{Sym}(\mathbb{R}^n)$, which we denote by $\mathrm{AO}(n, \mathbb{R})$.

The following definition and lemmas will explain the name of this group.

Definition 3.2.2. For a fixed $a \in \mathbb{R}^n$, the isometry $t_a : x \mapsto x + a$ is called the *translation over* a.

Lemma 3.2.3. *For any* $g \in \mathrm{AO}(n, \mathbb{R})$*, there exists a unique pair* (a, r) *where* $a \in \mathbb{R}^n$ *and* $r \in \mathrm{AO}(n, \mathbb{R})$ *fixes* 0*, such that* $g = t_a r$.

Proof. Set $a := g(0)$. Then $t_a^{-1} g(0) = 0$, so $t_a^{-1} g$ is an isometry fixing 0, and it is clear that $a = g(0)$ is the only value for which this is the case. □

Lemma 3.2.4. *Let* $r \in \mathrm{AO}(n, \mathbb{R})$ *be an isometry fixing* 0*. Then* r *is an orthogonal* $\mathbb{R}$*-linear map.*

Proof. Let $x, y \in \mathbb{R}^n$, and compute

$$(rx, ry) = (-d(rx, ry)^2 + d(rx, 0)^2 + d(ry, 0)^2)/2 = (-d(x, y)^2 + d(x, 0)^2 + d(y, 0)^2)/2 = (x, y),$$

where we use that r is an isometry fixing 0. Hence, r leaves the inner product invariant. This proves the second statement, if we take the first statement for granted. For any $z \in \mathbb{R}^n$, we have

$$(z, rx + ry) = (z, rx) + (z, ry) = (r^{-1} z, x + y) = (z, r(x + y)),$$

so that $rx + ry - r(x + y)$ is perpendicular to all of $\mathbb{R}^n$, hence zero. Similarly, one finds $r(\alpha x) = \alpha r x$ for $\alpha \in \mathbb{R}$. This proves the first statement. □

The group of all orthogonal linear maps $\mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\mathrm{O}(n, \mathbb{R})$. An isometry is thus an orthogonal linear map, followed by a translation; hence the name *affine orthogonal transformation*.

For a group G, we write

$$T(G) := \{a \in \mathbb{R}^n \mid t_a \in G\};$$

this is an additive subgroup of $\mathbb{R}^n$, which we shall identify with the group of all translations in G.

Using the linearity of isometries fixing 0, one can prove the following.

Proposition 3.2.5. *For a subgroup $G \subseteq \mathrm{AO}(n, \mathbb{R})$, the map $R : G \to \mathrm{O}(n, \mathbb{R})$ defined by*

$$g = t_a R(g) \text{ for some } a \in \mathbb{R}^n$$

is well-defined, and a group homomorphism. Its kernel is $T(G)$, and this set is invariant under $R(G)$.

Proof. Lemma 3.2.3 and 3.2.4 show that R is well-defined. Let $r_1, r_2 \in \mathrm{O}(n, \mathbb{R})$ and $a_1, a_2 \in \mathbb{R}^n$, and check that

$$t_{a_1} r_1 t_{a_2} r_2 = t_{a_1 + r_1 a_2} r_1 r_2.$$

This proves that R is a homomorphism. Its kernel is obviously $T(G)$. Let $r \in R(G)$ and $a_1 \in \mathbb{R}^n$ such that $t_{a_1} r_1 \in G$, and let $a_2 \in T(G)$. Then, for $x \in \mathbb{R}^n$,

$$\begin{aligned} t_{a_1} r_1 t_{a_2} r_1^{-1} t_{-a_1} x &= t_{a_1} r_1 t_{a_2} (r_1^{-1} x - r_1^{-1} a_1) \\ &= t_{a_1} (x - a_1 + r_1 a_2) \\ &= t_{r_1 a_2} x. \end{aligned}$$

This shows shows that $T(G)$ is invariant under $R(G)$. $\square$

We are interested in certain groups that act discretely.

Definition 3.2.6. A subset $V \subseteq \mathbb{R}^n$ is called *discrete* if for every point $p \in V$ there is an open neighbourhood $U \subseteq \mathbb{R}^n$ such that

$$U \cap V = \{p\}.$$

Definition 3.2.7. A discrete subgroup of $\mathbb{R}^n$ is called a *lattice*, and the dimension of its $\mathbb{R}$-linear span is called its *rank*.

Lemma 3.2.8. *Let $L \subseteq \mathbb{R}^n$ be a lattice. Then L is a closed set.*

Proof. As 0 is an isolated point, there is an $\epsilon > 0$ such that $\|v\| \geq \epsilon$ for all $v \in L \setminus \{0\}$. But then $\|v - w\| \geq \epsilon$ for all distinct $v, w \in L$. Hence, any sequence $(v_n)_n$ in L with $v_n - v_{n+1}$ converging to 0, is eventually constant. $\square$

Proposition 3.2.9. *Let $a_1, \ldots, a_k \in \mathbb{R}^n$ be linearly independent over $\mathbb{R}$. Then the set*

$$L = \sum_i \mathbb{Z} a_i$$

is a lattice. Conversely, every lattice is of this form.

Proof. To prove that an additive subgroup of $\mathbb{R}^n$ is a lattice, it suffices to prove that 0 is an isolated point. For L as in the lemma, let m be the minimum of the continuous function

$$x \to \| \sum_i x_i a_i \|$$

on the unit sphere $\{x \in \mathbb{R}^k \mid \|x\| = 1\}$. As the a_i are linearly independent, $m > 0$. For any non-zero $x \in \mathbb{Z}^k$, we find

$$\| \sum_i x_i a_i \| = \|x\| \| \sum_i \frac{x_i}{\|x\|} a_i \| \geq \|x\| m \geq m,$$

proving the first statement.

For the converse, we proceed by induction on the rank of the lattice. Clearly, a lattice of rank 0 is of the desired form. Now suppose that every lattice of rank $k-1$ is of that form, and let $L \subseteq \mathbb{R}^n$ be a lattice of rank k. First let us show that L is a closed subset of $\mathbb{R}^n$.

Being discrete and closed, L has only finitely many points in any compact set. In particular, one can choose a $v_1 \in L$ of minimal non-zero norm. Let $\pi : \mathbb{R}^n \to v_1^\perp$ denote the orthogonal projection along v_1. Then $\pi(L)$ is an additive subgroup of $v_1^\perp$. Let $v \in L$ be such that $\pi(v) \neq 0$. After subtracting an integer multiple of v_1 from v, we may assume that $\|v - \pi(v)\| \leq \|v_1\|/2$. Using $\|v\| \geq \|v_1\|$ and Pythagoras, we find $\|\pi(v)\| \geq \sqrt{3}/2 \|v_1\|$. This proves that $\pi(L)$ is discrete, hence a lattice. It has rank $k-1$, so there exist $v_2, \ldots, v_k \in L$ whose images under π are linearly independent, and such that

$$\pi(L) = \sum_{i=2}^k \mathbb{Z} \pi(v_i).$$

Now $v_1, \ldots, v_k$ are linearly independent. For $v \in L$ we can write

$$\pi(v) = \sum_{i=2}^k c_i \pi(v_i)$$

with $c_i \in \mathbb{Z}$. Then $v - \sum_{i=2}^k c_i v_k \in L$ is a scalar multiple of v_1, and by minimality of $\|v\|$ the scalar is an integer. Hence,

$$L = \sum_{i=1}^k \mathbb{Z} v_i.$$

$\square$

Definition 3.2.10. A subgroup $G \subseteq \mathrm{AO}(n, \mathbb{R})$ is said to act *discretely* if all its orbits in $\mathbb{R}^n$ are discrete.

If $G \subseteq \mathrm{AO}(n, \mathbb{R})$ is a group that acts discretely, then $T(G)$ does so, too. In general, this does not hold for $R(G)$, as the following example shows.

Example 3.2.11. Let $\alpha \in \mathbb{R}$ be an irrational multiple of π, and define $r \in \mathrm{O}(3, \mathbb{R})$ by

$$r = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider the group G generated by $g = t_{e_3} r$. From $g^n = t_{ne_3} r^n$ it is clear that G acts discretely, and that $R(G) = \{r^n \mid n \in \mathbb{Z}\}$. As r has infinite order, we have $T(G) = \{0\}$, and $R(G)$ does not act discretely: the $R(G)$-orbit of $(1, 0, 0)^T$ is an infinite set in the circle $\{x \mid x_3 = 0, \|x\| = 1\}$.

This phenomenon does not occur if we require $T(G)$ to have the maximal possible rank.

Theorem 3.2.12. *Let G be a subgroup of $\mathrm{O}(n, \mathbb{R})$ leaving invariant a lattice L of rank n. Then G is finite.*

Proof. Let $a_1, \ldots, a_n \in L$ be linearly independent. As L is discrete, it contains only finitely many vectors of norm $\|a_i\|$, for each i. Hence, the G-orbits Ga_i are finite. As the a_i span $\mathbb{R}^n$, the permutation representation of G on the finite set $X := \bigcup_{i=1}^{n} Ga_i$ is faithful. Hence, G is embedded into the finite group $\mathrm{Sym}(X)$. □

Definition 3.2.13. A subgroup $G \subseteq \mathrm{AO}(n, \mathbb{R})$ for which $T(G)$ is a lattice of rank n, is called a *crystallographic* group in n dimensions. A group $G \subseteq \mathrm{O}(n, \mathbb{R})$ leaving invariant a lattice of rank n, is called a *crystallographic point group* in n dimensions.

Corollary 3.2.14. *Let $G \subseteq \mathrm{AO}(n, \mathbb{R})$ be a crystallographic group. Then $R(G)$ is a crystallographic point group. Any crystallographic point group is finite.*

Proof. By Proposition 3.2.5, $T(G)$ is invariant under $R(G)$. Now apply Theorem 3.2.12. □

This corollary allows for the following approach to the classification of crystallographic groups in $\mathrm{O}(n, \mathbb{R})$: first classify all finite subgroups of $\mathrm{O}(n, \mathbb{R})$, then investigate for each of them whether it leaves invariant a lattice of rank n. Finally, for each pair (R, L) of a crystallographic point group R and a lattice L of rank n left invariant by R, there may be several non-equivalent groups G with $T(G) = L$ and $R(G) = R$. Among them is always the *semi-direct product* $L \rtimes R$: the subgroup of $\mathrm{AO}(n, \mathbb{R})$ generated by L and R.

In this setting, 'classification' means 'up to equivalence under $\mathrm{AO}(n, \mathbb{R})$', i.e., the groups G and hGh^{-1} for some $h \in \mathrm{AO}(n, \mathbb{R})$ are considered the same. This is a finer equivalence relation than isomorphism, as is shown in the following example.

Example 3.2.15. The groups C_2 and D_1 (see Section 3.3) are isomorphic as abstract groups, but not equivalent under $\mathrm{AO}(2,\mathbb{R})$. The same holds for the groups W and $T \cup i(W \setminus T)$ (see Section 3.4).

Exercise 3.2.16. Show that the symmetry group G of Figure 1.4 is not the semi-direct product of $R(G)$ and $T(G)$.

In the next sections, we shall carry out this classification in the cases $n = 2$ and 3, up to the point of determining the crystallographic point groups. The determinant of an orthogonal map turns out to be a useful tool there.

Definition 3.2.17. An isometry $r \in \mathrm{O}(n,\mathbb{R})$ is called *proper* or *orientation preserving* if $\det r = 1$. The set of all proper isometries in $\mathrm{O}(n,\mathbb{R})$ is denoted by $\mathrm{SO}(n,\mathbb{R})$. An isometry $g \in \mathrm{AO}(n,\mathbb{R})$ is called proper if $R(g)$ is proper. Otherwise, it is called *improper*.

Exercise 3.2.18. Prove that, in any subgroup $G \subseteq \mathrm{AO}(n,\mathbb{R})$ containing improper isometries, the proper ones form a subgroup of index 2.

Exercise 3.2.19. Consider the following map:

$$r_a : \quad v \mapsto v - 2(v,a)a,$$

where $a \in \mathbb{R}^n$ has norm 1 (that is, $||a|| = 1$). Prove that

1. it belongs to $O(n,\mathbb{R})$;
2. fixes each vector in the hyperplane $a^{\perp}$;
3. $r_a(a) = -a$;
4. $\det(r_a) = -1$;
5. r_a has order 2;

Such an element is called a *reflection*.

Exercise 3.2.20. Show that any finite subgroup $G \subseteq \mathrm{AO}(n,\mathbb{R})$ fixes a point in $\mathbb{R}^n$. Hint: consider the 'average' of an arbitrary orbit.

Exercise 3.2.21. Let G be a finite subgroup of the group $\mathrm{GL}(n,\mathbb{R})$ of all invertible $n \times n$-matrices. Show that G leaves invariant a positive definite symmetric bilinear form, and that it is hence conjugate to a subgroup of $\mathrm{O}(n,\mathbb{R})$. Hint: let $(.,.)$ denote the standard inner product, and define

$$\langle x, y\rangle := \frac{1}{|G|}\sum_{g\in G}(gx, gy)$$

for $x, y \in \mathbb{R}^n$. Show that $\langle .,.\rangle$ is symmetric and positive definite. Let A be the matrix with entries $a_{ij} = \langle e_i, e_j\rangle$, and $B \in \mathrm{GL}(n,\mathbb{R})$ such that $B^T B = A$. Show that $BGB^{-1} \subseteq \mathrm{O}(n,\mathbb{R})$.

3.3 The Finite Subgroups of $\mathrm{O}(2, \mathbb{R})$

In nature, as well as in art and many other areas, many plane figures with certain symmetries exist. One may think of flowers (seen from above), frieze patterns on picture frames, Escher's plane tilings and decorative wall papers. In order to describe these symmetries, we investigate the finite subgroups of $\mathrm{O}(2, \mathbb{R})$.

In two dimensions, we have two different types of isometries.

Lemma 3.3.1. *The linear map $r_\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix*

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

with respect to the standard basis is an element of $\mathrm{SO}(2, \mathbb{R})$*, called* rotation around 0 over ϕ*. Conversely, any element of this group equals some r_ϕ. Any improper isometry in* $\mathrm{O}(2, \mathbb{R})$ *is a reflection in a line, see Exercise 3.2.19.*

The proof of this lemma is straightforward. Let us describe two classes of finite subgroups of $\mathrm{O}(2, \mathbb{R})$.

1. The *cyclic group* of order n, denoted by C_n, and generated by $r_{2\pi/n}$.

2. The *dihedral group* of order $2n$, denoted by D_n, and generated by $r_{2\pi/n}$ and the reflection in a fixed line.

In fact, these are the only two classes, as we shall now prove. Let G be a finite subgroup of $\mathrm{O}(2, \mathbb{R})$. There are two possibilities:

1. $G \subseteq \mathrm{SO}(2, \mathbb{R})$. By Lemma 3.3.1, it consists of rotations r_ϕ. Let $\phi > 0$ be minimal such that $r_\phi \in G$. As G is finite, $\phi = 2\pi/n$ for some integer $n > 0$. If θ is any other angle such that $r_\theta \in G$, then $\theta = k\phi + \sigma$ for some integer k and angle $sigma \geq 0$ and $< \phi$. By minimality of ϕ, $\sigma = 0$. Hence G is generated by $r_{2\pi/n}$, and equal to C_n.

2. G contains a reflection g in a line l. Then $G \cap \mathrm{SO}(2, \mathbb{R})$ is a subgroup of index 2 in G, and by the first case generated by $r_{2\pi/n}$ for some integer $n > 0$. Now G is generated by the g and $r_{2\pi/n}$, and conjugate to D_n.

We have thus proved the following theorem.

Theorem 3.3.2. *Let G be a finite subgroup of* $\mathrm{O}(2, \mathbb{R})$*. Then G is conjugate to one of the groups C_n or D_n for some integer $n \geq 1$.*

Exercise 3.3.3. Show that $\mathrm{O}(n, \mathbb{R})$ is generated by reflections. Hint: show first that it is generated by elements that fix an $(n-2)$-dimensional subspace of $\mathbb{R}^n$ pointwise, and that $\mathrm{O}(2, \mathbb{R})$ is generated by reflections.

Using this classification, we can find which of the finite groups are crystallographic point groups.

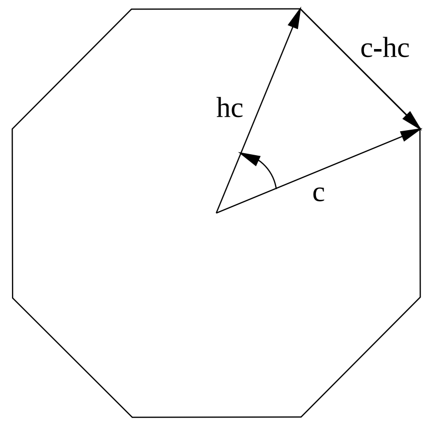

Figure 3.3: Why n cannot not be 8.

Theorem 3.3.4. *The following groups are crystallographic point groups:*

$$C_1, C_2, C_3, C_4, C_6,$$
$$D_1, D_2, D_3, D_4, D_6.$$

Moreover, any crystallographic point group G in two dimensions is conjugate to one of the above.

Proof. For the first part, one only needs to construct invariant lattices for D_4 and D_6 (see 3.3.5), as the other groups are contained in one of these see.

For the second statement, suppose that G is a crystallographic point group. According to Theorem 3.2.12, G must be finite. The lattice L is invariant under the subgroup K of $\mathrm{O}(2,\mathbb{R})$ generated by $G \cup \{i\}$, where i denotes the inversion. As the inversion commutes with all elements of G, K is finite (either equal to G or twice as large). Therefore, according to Theorem 3.3.2, K is conjugate to C_n or D_n for some n; as K contains the inversion, n must be even. So either $n \leq 6$, in which case G is of one of the types mentioned in the theorem, or $n \geq 8$. Suppose that the latter were the case and choose a vector c in the lattice of shortest possible length. Now $r_{2\pi/n}c \in L$ and $c - r_{2\pi/n}c \in L$. A little calculus shows that the latter vector is shorter than c because the rotation angle too small (for $n = 8$, this can be seen from Figure 3.3), and we arrive at a contradiction.

□

Exercise 3.3.5. Show that D_4 and D_6 are crystallographic point groups.

Note that we have only classified all crystallographic point groups, and not the crystallographic groups. There are 17 of the latter, and only 10 of the former; see [2].

3.4 The Finite Subgroups of $\mathrm{O}(3,\mathbb{R})$

In this section, we shall repeat the classification of §3.3, but now for three dimensions.

Lemma 3.4.1. *Any element $g \in \mathrm{SO}(3,\mathbb{R})$ fixes a line, i.e., with respect to some orthonormal basis g has the matrix*

$$\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix},$$

where M is the matrix of an element of $\mathrm{SO}(2,\mathbb{R})$.

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of g. As g leaves the norm invariant, all have norm 1. As g is proper, we have $\lambda_1\lambda_2\lambda_3 = 1$. Now either all are real, in which case at least one of them is 1, or we may assume that λ_1 is real and $\lambda_3 = \bar{\lambda_2}$. But then $\lambda_2\lambda_3 = |\lambda_2|^2 = 1$, so $\lambda_1 = 1$. □

An element of $\mathrm{SO}(3,\mathbb{R})$ is called a *rotation*, the line fixed by it its *axis* and the angle corresponding to M its *rotation angle.* We present some classes of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$.

1. Fix an axis l and consider the proper rotations around l over the angles $2k\pi/n$ for $k = 0, \ldots, n-1$. They form a group, denoted by C_n, due to its analogy to the class denoted by C_n in the two-dimensional setting.

2. Start with the group C_n of rotations around l and choose a second axis m perpendicular to l. Consider the group generated by C_n and the rotation around m over π. It is twice as large as C_n, and denoted by D'_n. Note that it contains only proper isometries. The proper symmetries of an n-prism form a group of type D'_n. Note that D'_1 is conjugate to C_2.

3. The five platonic solids (tetrahedron, cube, octahedron, dodecahedron and icosahedron) with vertices on the unit sphere yield examples of finite subgroups of $\mathrm{SO}(3,\mathbb{R})$. However, only three distinct classes arise:

   - T, the group of all proper rotations leaving a tetrahedron invariant.
   - W, the group of all proper rotations that are symmetries of a cube.
   - P, the group of all proper rotations leaving a dodecahedron invariant.

   The reason is that the octahedron and the cube are polar figures: it is easily seen that the proper symmetries of a cube are exactly those of the octahedron which has its vertices in the middles of the faces of the cube. A similar statement holds for the dodecahedron and the icosahedron.

Theorem 3.4.2. *Any finite subgroup of $\mathrm{SO}(3,\mathbb{R})$ is conjugate to one of the following.*

$$\begin{array}{ll} C_n & \textit{for } n \geq 1, \\ D'_n & \textit{for } n \geq 2, \\ T, W, P. & \end{array}$$

Group type	Order	Number of orbits on S	Their sizes	rotation axes
C_n	n	2	$1, 1$	$1 \times n$
D'_n	$2n$	3	$2, n, n$	$1 \times n, n \times 2$
T	12	3	$4, 4, 6$	$4 \times 3, 3 \times 2$
W	24	3	$8, 6, 12$	$4 \times 3, 3 \times 4, 6 \times 2$
P	60	3	$12, 20, 30$	$6 \times 5, 10 \times 3, 15 \times 2$

Table 3.1: The finite groups of proper rotations in space.

The following proof is due to Euler.

Proof. Let G be a finite subgroup of $\mathrm{SO}(3,\mathbb{R})$. Consider the set S defined by

$$S := \{p \in \mathbb{R}^3 \mid ||p|| = 1, \text{ there exists } g \in G \text{ such that } g \neq e \text{ and } g(p) = p\}.$$

Count the number N of pairs (g, p), with $e \neq g \in G$ and p on the unit sphere, that satisfy $g(p) = p$. On one hand, each $g \neq e$ fixes exactly two anti-podal points, hence this number is

$$N = 2(|G| - 1).$$

On the other hand, for each $p \in S$, the number of $g \neq e$ fixing p equals $|G_p| - 1$. Hence

$$N = \sum_{p \in S} (|G_p| - 1).$$

If $gp = p$ and $g \neq e$, and $h \in G$, then $(hgh^{-1})hp = hp$ and $hgh^{-1} \neq e$, so S is invariant under G. Partition S into orbits of G. The value of $|G_p|$ is constant on such an orbit o, namely $|G|/|o|$, due to Lagrange's theorem. Hence we find

$$2(|G| - 1) = \sum_o |o|(|G|/|o| - 1),$$

where the sum is taken over all orbits of G in S. Hence, dividing both sides by $|G|$, we get

$$2 - \frac{2}{|G|} = \sum_o (1 - \frac{|o|}{|G|}).$$

Hence, if we set $a_o := |G|/|o|$ for each orbit o on S, then we have

$$\sum_o (1 - 1/a_o) < 2.$$

As each $s \in S$ has non-trivial stabilizer, each orbit in S has size at most $|G|/2$; hence each a_o is at least 2. Now exercise 3.4.3 leads to Table 3.1. That is, it provides the numerical data for it; it remains to check that this data determines the group up to conjugation. □

Exercise 3.4.3. Consider the following function in GAP.

```
extend:=function(a,s)

#Pre-condition: a is a non-empty non-decreasing list of integers >=2,
#and s equals (sum i: (1-1/a[i])). This procedure ‘extends’ a in all
#possible ways, under the condition that this sum remains smaller than
#2.

local b,g,o;
        if s >= 2 then return false;
        else g:=2/(2-s); o:=List(a,x->g/x);
        Print("Orbit sizes:",o," |G|:",g,"\n");
                b:=a[Length(a)];
                while extend(Concatenation(a,[b]),s+(1-1/b)) do
                        b:=b+1;
                od;
                return true;
        fi;
end;
```

Explain why the function call `extend([2,2],1)` results in an endless loop. On the other hand, when called as `extend([2,3],1/2+2/3)`, the function does halt, and prints

```
Orbit sizes:[ 6/5, 4/5 ] |G|:12/5
Orbit sizes:[ 6, 4, 4 ] |G|:12
Orbit sizes:[ 12, 8, 6 ] |G|:24
Orbit sizes:[ 30, 20, 12 ] |G|:60
```

Compare this with Table 3.1.

Exercise 3.4.4. In Figure 3.4 we see a cube with vertices on the unit sphere. Its group of proper symmetries contains non-trivial rotation axes through the points drawn in that picture; they form the set S in the proof of Theorem 3.4.2. The points in the different orbits are given different shades of grey. Make similar pictures for the group types C_n, D'_n and T. Describe the symmetry axes for the icosahedron.

Now let us try to include improper rotations. Let G be a finite subgroup of $\mathrm{O}(3, \mathbb{R})$ containing improper rotations. Then $H := G \cap \mathrm{SO}(3, \mathbb{R})$ has index 2 in G. If G contains the inversion i, then we have a disjoint union

$$G = H \cup iH.$$

Note that this is isomorphic to $H \times C_2$ as an abstract group. Now assume that G does not contain i. Then the set

$$G' := H \cup i(G \setminus H)$$

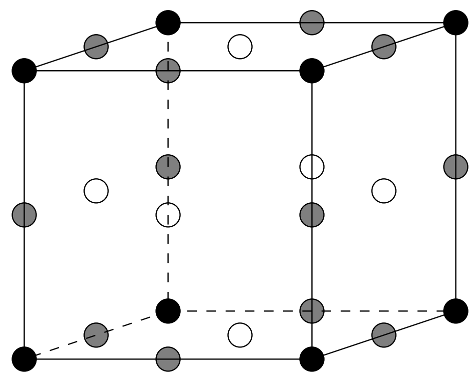

Figure 3.4: The orbits of W on the points on rotation axes.

is a subgroup of SO$(3, \mathbb{R})$ in which H has index 2, and one can recover G by

$$G = H \cup i(G' \setminus H).$$

Conversely, if $G', H \subseteq$ SO$(3, \mathbb{R})$ are finite subgroups, and H has index 2 in G', then this equation defines a finite subgroup of O$(3, \mathbb{R})$, which is isomorphic to G' as an abstract group, but not equivalent to it.

This enables us to classify all finite subgroups of O$(3, \mathbb{R})$.

Theorem 3.4.5. *Any finite subgroup of* O$(3, \mathbb{R})$ *is conjugate to one of the following:*

$$\begin{array}{ll} C_n, C_n \cup iC_n, C_n \cup i(C_{2n} \setminus C_n), & \textit{for } n \geq 1, \\ D'_n, D'_n \cup iD'_n, D_n \cup i(D'_{2n} \setminus D'_n), C_n \cup i(D'_n \setminus C_n), & \textit{for } n \geq 2, \\ T, W, P, T \cup iT, W \cup iW, P \cup iP, T \cup i(W \setminus T). & \end{array}$$

We do not give all details of the proof, but restrict ourselves to the following remarks.

1. Note that D'_1 and C_2 describe the same class; for this reason the second list starts with $n = 2$.

2. The last group in the list is made possible by the fact that a group of type W has a subgroup of type T of index 2. This can be seen as follows: consider the group of proper symmetries of a cube with center 0, and look at the lines between anti-podal vertices of the cube; they are axes of rotations of order 3 leaving the cube invariant; these proper rotations form the symmetry group of a regular tetrahedron with center 0 and points on those axes.

In applications, one wants to determine the (finite) symmetry group of a given object. The following procedure, based on Table 3.1 and the list in Theorem 3.4.5, can then be helpful.

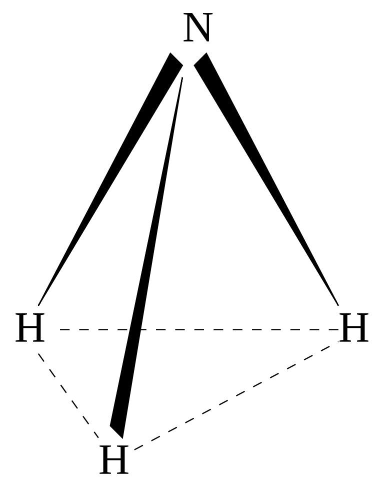

Figure 3.5: The NH_3- molecule.

1. List all rotation axes and their orders. They determine the subgroup H of G of all proper rotations.
2. Check whether G contains any improper rotations. If not, then $G = H$.
3. Suppose that G contains improper rotations. Check whether $i \in G$. If so, then $G = H \cup iH$.
4. Suppose that G contains improper rotations, but not the inversion. Then, according to the list of Theorem 3.4.5, there is only one possibility for G, unless H is of type C_n.
5. Suppose that $H = C_n$ and that G contains improper rotations, but not the inversion. Then if G is cyclic, it is of type $C_n \cup i(C_{2n} \setminus C_n)$. Otherwise, it is of type $C_n \cup i(D'_n \setminus C_n)$.

Exercise 3.4.6. Consider the molecule depicted in Figure 3.5. Identify the class of its symmetry group in the list of Theorem 3.4.5.

Exercise 3.4.7. Consider the full symmetry group of the tetrahedron, including its improper rotations. Which class in the list of Theorem 3.4.5 does it belong to?

Exercise 3.4.8. Determine the symmetry group of the CH_4 molecule depicted in Figure 3.1.

Exercise 3.4.9. Consider the ethane molecule in staggered configuration as depicted in Figure 3.6. What is the type of its symmetry group?

Exercise 3.4.10. Determine the symmetry group of the N_4P_4-molecule, depicted in Figure 3.7.

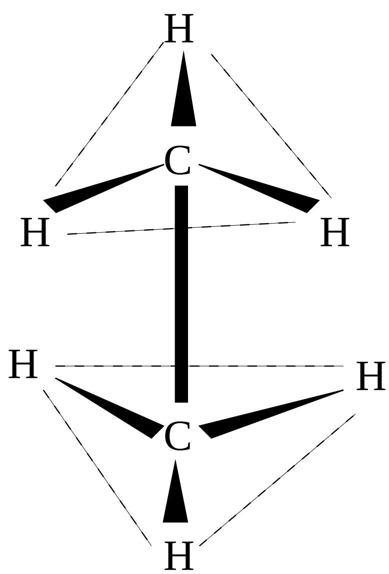

Figure 3.6: The ethane molecule in staggered configuration.

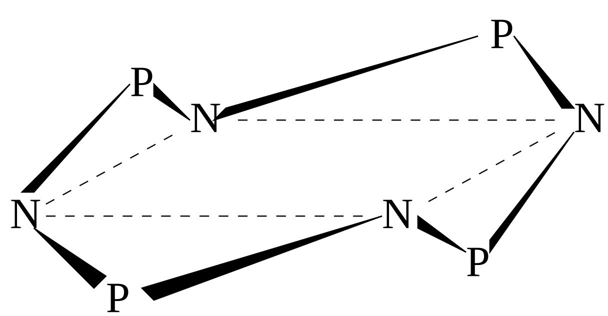

Figure 3.7: The ethane molecule in staggered configuration.

Most of the above examples are taken from [2].

Finally, the question arises which of the finite subgroups of $\mathrm{O}(3, \mathbb{R})$ are crystallographic point groups.

Theorem 3.4.11. *Let G be a crystallographic point group in 3 dimensions. Then G is conjugate to one of the following groups:*

$$\begin{array}{ll}
C_m, C_m \cup iC_m & m = 1, 2, 3, 4, 6 \\
D'_m, D'_m \cup iD'_m & m = 2, 3, 4, 6 \\
C_m \cup i(C_{2m} \setminus C_m) & m = 1, 2, 3 \\
D'_m \cup i(D'_{2m} \setminus D'_m) & m = 2, 3 \\
C_m \cup i(D'_m \setminus C_m) & m = 2, 3, 4, 6 \\
T, W, T \cup iT, W \cup iW, T \cup i(W \setminus T) &
\end{array}$$

Like in the two-dimensional case, a crystallographic point group may leave several lattices invariant. Indeed, there are 230 mutually non-equivalent crystallographic groups in 3 dimensions, while there are only 32 crystallographic point groups, see [2].

The proof runs like that of Theorem 3.3.4. For the sake of compatibility with other texts on crystallographic groups, especially texts on the applications in physics and chemistry, we include table 3.4. It is organized as follows: in the first column, one finds lattices classified up to their symmetry groups. The second column contains the groups leaving that lattice invariant (but no lattice with a smaller symmetry group), in our notation. The last row for each lattice is the full symmetry group of that lattice. There is only one exception to this rule: a trigonal lattice is really hexagonal. The third and fourth column contain two commonly used sets of symbols for them.

Type of Lattice	Our notation	Schoenflies	International
Triclinic	C_1	C_1	1
	$C_1 \cup iC_1$	$C_i = S_2$	$\bar{1}$
Monoclinic	C_2	C_2	2
	$C_1 \cup i(C_2 \setminus C_1)$	$C_s = C_{1h}$	m
	$C_2 \cup iC_2$	C_{2h}	2/m
Orthorhombic	D'_2	$D_2 = V$	2 2 2
	$C_2 \cup i(D'_2 \setminus C_2)$	C_{2v}	m m 2
	$D'_2 \cup iD'_2$	$D_{2h} = V_h$	m m m
Trigonal	C_3	C_3	3
	D'_3	D_3	3 2
	$C_3 \cup iC_3$	$C_{3i} = S_6$	$\bar{3}$
	$C_3 \cup i(D'_3 \setminus C_3)$	C_{3v}	3 m
	$D'_3 \cup iD'_3$	D_{3d}	$\bar{3}$ m
Tetragonal	C_4	C_4	4
	D'_4	D_4	4 2 2
	$C_4 \cup iC_4$	C_{4h}	4/m
	$C_2 \cup i(C_4 \setminus C_2)$	S_4	$\bar{4}$
	$C_4 \cup i(D'_4 \setminus C_4)$	C_{4v}	4 m m
	$D'_4 \cup i(D'_4 \setminus D'_2)$	$D_{2d} = V_d$	$\bar{4}$ 2 m
	$D'_4 \cup iD'_4$	D_{4h}	4/m m m
Hexagonal	C_6	C_6	6
	D'_6	D_6	6 2 2
	$C_6 \cup iC_6$	C_{6h}	6/m
	$C_3 \cup i(C_6 \setminus C_3)$	C_{3h}	$\bar{6}$
	$C_6 \cup i(D'_6 \setminus C_6)$	C_{6v}	6 m m
	$D_3 \cup i(D'_6 \setminus D'_3)$	D_{3h}	$\bar{6}$ m 2
	$D'_6 \cup iD'_6$	D_{6h}	6/m m m
Isometric	T	T	2 3
	W	O	4 3 2
	$T \cup iT$	T_h	m 3
	$T \cup i(W \setminus T)$	T_d	$\bar{4}$ 3 m
	$W \cup iW$	W_h	m 3 m

Table 3.2: A Dictionary of Crystallographic Group Names.

Chapter 4

Representation Theory

4.1 Linear representations of groups

An abstract group can be represented in various ways. In Chapter 2, we encountered permutation representations; in the present chapter we will start the theory of linear representations. In this chapter, G denotes a group and $\mathbb{K}$ a field of characteristic zero. All vector spaces will be over $\mathbb{K}$.

The notion

We introduce the concept of a linear representation of G.

Definition 4.1.1. A *(linear) representation* of a group G on a vector space V is a homomomorphism

$$\rho : G \to \mathrm{GL}(V).$$

If ρ is injective, it is called *faithful*.

After a choice of basis of V, we can view a representation as a homomorphism

$$G \to \mathrm{GL}(n, \mathbb{K}),$$

where $\mathrm{GL}(n, \mathbb{K})$ is the group of invertible matrices with entries in $\mathbb{K}$; such a map is called matrix representation of G.

If $\rho : G \to \mathrm{GL}(V)$ is a representation of G on V, then we often write gv instead of $\rho(g)v$, if no confusion is possible. Also, G is said to *act linearly* on V, and V is called a G*-module*. The *dimension* of the representation is by definition the dimension of V. Note that $(g, v) \mapsto \rho(g)v$ is indeed an action of G on V in the sense of Definition 2.4.1.

Exercise 4.1.2. To experience the use of matrix representations of groups, try to guess which orthogonal linear map is the product $\rho_x\rho_y\rho_z$. Here ρ_x stands for rotation over π about the x-axis, and similarly for y and z. To check your guess, represent the rotations by matrices and compute the product.

Example 4.1.3. Let G be the cyclic group C_n of order n with generator c. It has one-dimensional complex representations $r_l (l \in \mathbb{Z})$ defined by

$$r_l : C_n \to \mathrm{GL}(1, \mathbb{R}), \quad c^k \mapsto (e^{kl2\pi i/n}).$$

Note that r_l is faithful if and only if n and l are co-prime.

Example 4.1.4. Recall (2.3.2) the permutation matrix presentation of the symmetric group S_5 given in Chapter 2: to each permutation $\pi \in S_5$, we associate the 5×5 matrix M_π with $M_\pi e_i = e_{\pi(i)}$. This is a matrix representation of S_5.

The same thing works for S_3. The permutation (12) can be represented by the matrix

$$M_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and (123) can be represented by

$$M_{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We know that $(12)(123) = (23)$ and this can be seen from the matrices as well:

$$M_{(12)} M_{(123)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Exercise 4.1.5. Prove that the correspondence given above between S_3 and $GL(n, \mathbb{K})$ is a group homomorphism, that is, a matrix representation of S_3.

Example 4.1.6. Recall from 3.3 the dihedral group D_n of order $2n$. It contains the cyclic group C_n of order n with generator c as a subgroup of index 2 and has a reflection a (of order 2) with $aca = c^{-1}$. The group D_n has a two-dimensional real representation $r : D_n \to \mathrm{GL}(2, \mathbb{R})$ determined by

$$c^k \mapsto \begin{pmatrix} \cos(2k\pi/n) & -\sin(2k\pi/n) \\ \sin(2k\pi/n) & \cos(2k\pi/n) \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Example 4.1.7. An important example of a linear representation of a finite group G is the *regular representation*. Let $G = \{g_1, \ldots, g_n\}$ and take V to be the vector space of dimension n and basis $\{e_{g_1}, \ldots, e_{g_n}\}$. Define $\rho \colon G \to \mathrm{GL}(V)$ by $\rho(g_i)(e_{g_j}) = e_{g_i g_j}$, extended linearly to all of V.

Example 4.1.8. The above examples can be summarized in a more general construction. Let $\pi : G \to S_n$ be a permutation representation. Then π gives rise to a linear representation $M_\pi : G \to \mathrm{GL}(V)$, where V is a complex vector space with (formal) basis e_i $(i = 1, \ldots, n)$, in the following way:

$$M_\pi(g) e_j = e_{\pi(g)j} \quad (g \in G; j \in \{1, \ldots, n\}).$$

The regular representation of Example 4.1.7 is M_π with $\pi : G \to \mathrm{Sym}(G)$ the left regular permutation representation of G.

The matrix representation of Example 4.1.4 is M_π, where π is the natural permutation representation of S_n (actually, for $n = 5, 3$, respectively).

Example 4.1.9. An important class of constructions of new representations from known ones arises from function space constructions. Let $r : G \to \mathrm{GL}(V)$ be a linear representation of G on V. Consider the vector space $F(V)$, of all maps $V \to \mathbb{K}$ with pointwise addition $(f + g)x = f(x) + g(x)$ and pointwise scalar multiplication $(\lambda f)x = \lambda f(x)$ (where $\lambda \in \mathbb{K}$, $x \in V$, $f, f' : V \to \mathbb{K}$). Then r induces the following representation $r^* : G \to \mathrm{GL}(F(V))$:

$$r^*(g)(f)x = f(r(g)^{-1}x) \quad (x \in V, f \in F(V), g \in G).$$

As done before, we usually unwind the heavy notation by dropping r wherever possible, and write $(gf)(x) = f(g^{-1}x)$ instead.

Note that $F(V)$ has the dual space V^* of $\mathbb{K}$-linear functions as an invariant subspace.

Exercise 4.1.10. Let $\mathbb{R}[x]_{\leq 3}$ be the linear space of all polynomials in x of degree ≤ 3. Consider the $\mathbb{R}$-linear representation $\rho_a : \mathbb{R} \to \mathrm{GL}(\mathbb{R}[x]_{\leq 3})$ defined as $\rho_a p = p(x - a)$. Find a corresponding matrix representation.

Equivalence and Subrepresentations

Just as with groups, we are mainly interested in representations up to isomorphism.

Definition 4.1.11. Let r and s be representations of a group G on vector spaces V and W, respectively. A linear map $T : V \to W$ is said to *intertwine* r and s if $s(g)T = Tr(g)$ for all $g \in G$. In this case, T is called a *homomorphism of G-modules*. The space of all homomorphisms of G-modules is denoted by $\mathrm{Hom}_G(V, W)$.

The representations r and s are called *equivalent* if there exists a linear isomorphism $T : V \to W$ intertwining them. In this case, T is an *isomorphism of G-modules*.

It is straightforward to prove that this really defines an equivalence relation $\sim$ on representations.

Exercise 4.1.12. Consider the linear representations r_l of C_n defined in Example 4.1.3. For which pairs (l, m) are the representations r_l and r_m equivalent?

Exercise 4.1.13. Consider the linear representations ρ_a of Exercise 4.1.10. Show that $\rho_a \sim \rho_b$ if and only if $ab \neq 0$.

In Example 4.1.9, we saw that V^* is a invariant under the action of G on $F(V)$. This gives rise to the following definition.

Definition 4.1.14. Let r be a representation of a group G on V. A subspace W is *invariant* if $r(g)W \subset W$ for all $g \in G$. In this case, $r|_W : G \to \mathrm{GL}(W)$ given by $r|_W(g) = r(g)|_W$, is called a *subrepresentation* of r, and W is called a *sub-G-module* of V.

If G is clear from the context, we leave it out and call W a submodule of V.

Example 4.1.15. Suppose that $T : V \to W$ is a G-module homomorphism of G on V and W. Then $\ker T$ is a G-invariant subspace of V and $\operatorname{im} T$ an invariant subspace of W.

Example 4.1.16. Consider the two-dimensional matrix representation $\rho\colon \mathbb{R} \to \mathrm{GL}(\mathbb{R}^2)$ defined by

$$M_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then it is not hard to see that the 1 dimensional subspace spanned by $(1, 0)$ is an invariant subspace.

Example 4.1.17. Let V be a finite-dimensional vector space over $\mathbb{K}$, and let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group g. Let $P(V)$ be the set of all polynomial functions on V, i.e., the set of all elements of $F(V)$ (see Example 4.1.9) that can be written as a $\mathbb{K}$-linear combination of products of elements of V^*. The subspace $P(V)$ of $F(V)$ is invariant under G, and so are its homogeneous components $P(V)_d$, defined by

$$P(V)_d = \{f \in P(V) \mid f(\lambda x) = \lambda^d f(x) \text{ for all } x \in V \text{ and } \lambda \in \mathbb{K}\}.$$

We find that $P(V)_0$ is the one-dimensional space of constant functions, and $P(V)_1 = V^*$ has dimension $n := \dim V$. Choosing a basis $x_1, \ldots, x_n$ of V^*, we find that $P(V)_d$ is precisely the $\mathbb{K}$-linear span of monomials $x_1^{a_1} \cdots x_n^{a_n}$ where $(a_1, \ldots, a_n) \in \mathbb{N}^n$ satisfies $a_1 + \ldots + a_n = d$. This implies $\dim P(V)_d = \binom{n+d-1}{d}$; prove this!

To make all this more explicit, suppose that $n = 2$, and consider an element $A \in \mathrm{GL}(V)$ whose matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the basis e_1, e_2 dual to x_1, x_2.

We compute the matrix of A on $P(V)_1$ with respect to x_1, x_2. We have

$$A^{-1} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so

$$\begin{array}{rcccl} Ax_1(e_1) & = & x_1(A^{-1}e_1) & = & (ad-bc)^{-1}x_1(de_1 - ce_2) = (ad-bc)^{-1}d \\ Ax_1(e_2) & = & x_1(A^{-1}e_2) & = & (ad-bc)^{-1}x_1(-be_1 + ae_2) = -(ad-bc)^{-1}b \\ Ax_2(e_1) & = & x_2(A^{-1}e_1) & = & (ad-bc)^{-1}x_2(de_1 - ce_2) = -(ad-bc)^{-1}c \\ Ax_2(e_2) & = & x_2(A^{-1}e_2) & = & (ad-bc)^{-1}x_2(-be_1 + ae_2) = (ad-bc)^{-1}a \end{array}$$

from which we conclude $Ax_1 = (ad - bc)^{-1}(dx_1 - bx_2)$ and $Ax_2 = (ad - bc)^{-1}(-cx_1 + ax_2)$. Hence A has matrix

$$(ad-bc)^{-1}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

on the basis x_1, x_2 of $P(V)_1$. Note that this is the tranposed inverse of A.

Finally, let us compute the matrix of A on $P(V)_2$ with respect to the basis x_1^2, x_1x_2, x_2^2. Now

$$\begin{array}{ccccc}
Ax_1^2 & = & (Ax_1)^2 & = & (ad-bc)^{-2}(dx_1 - bx_2)^2 \\
 & & & = & (ad-bc)^{-2}(d^2x_1^2 - 2bdx_1x_2 + b^2x_2^2) \\
A(x_1x_2) & = & (Ax_1)(Ax_2) & = & (ad-bc)^{-2}(dx_1 - bx_2)(-cx_1 + ax_2) \\
 & & & = & (ad-bc)^{-2}(-cdx_1^2 + (ad+bc)x_1x_2 - abx_2^2) \\
Ax_2^2 & = & (Ax_2)^2 & = & (ad-bc)^{-2}(-cx_1 + ax_2)^2 \\
 & & & = & (ad-bc)^{-2}(c^2x_1^2 - 2acx_1x_2 + a^2x_2^2),
\end{array}$$

so the matrix sought for is

$$(ad-bc)^{-2}\begin{pmatrix} d^2 & -cd & c^2 \\ -2bd & ad+bc & -2ac \\ b^2 & -ab & a^2 \end{pmatrix}.$$

In conclusion, we have found a matrix representation of $\mathrm{GL}(2, \mathbb{K})$ on $\mathbb{K}^3$ sending A to the 3×3 matrix just computed. Can you determine its kernel?

Exercise 4.1.18. Find the invariant subspaces of the matrix representation $\rho\colon \mathbb{R} \to \mathrm{GL}(\mathbb{R}^2)$ defined by

$$M_t = \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}.$$

Lemma 4.1.19. *Suppose that G is finite, and that ρ is a representation of G on a vector space V. Let U be a sub-G-module of V. Then there exists a sub-G-module W of V such that $V = U \oplus W$.*

Proof. Let W' be any vector space complement of U in V, and let π' be the projection of V onto U with kernel W'. Define a new map

$$\pi := \frac{1}{|G|}\sum_{g \in G} \rho(g)\pi'\rho(g^{-1})$$

on V. We claim that π is a projection commuting with all $\rho(h),\ h \in G$. To see that it does indeed commute with $\rho(h)$, consider

$$\pi\rho(h) = \frac{1}{|G|}\sum_{g \in G} \rho(g)\pi'\rho(g^{-1}h);$$

by the dummy transformation $g \mapsto hg$ this equals

$$\begin{aligned}
&\frac{1}{|G|}\sum_{g\in G}\rho(hg)\pi'\rho((hg)^{-1}h)\\
&=\frac{1}{|G|}\sum_{g\in G}\rho(h)\rho(g)\pi'\rho(g^{-1}h^{-1}h)\\
&=\rho(h)\frac{1}{|G|}\sum_{g\in G}\rho(g)\pi'\rho(g^{-1})\\
&=\rho(h)\pi,
\end{aligned}$$

as claimed. To verify that π is a projection, we first note that $\pi\pi' = \pi'$; indeed, both are zero on W' and the identity on U. Compute

$$\begin{aligned}
\pi^2 &= \pi\frac{1}{|G|}\sum_{g\in G}\rho(g)\pi'\rho(g^{-1})\\
&=\frac{1}{|G|}\sum_{g\in G}\rho(g)\pi\pi'\rho(g^{-1})\\
&=\frac{1}{|G|}\sum_{g\in G}\rho(g)\pi'\rho(g^{-1})\\
&=\pi,
\end{aligned}$$

where the second equality follows from the fact that π commutes with each $\rho(g)$. This concludes the proof that π is a projection. This implies that $V = \mathrm{im}(\pi)\oplus\ker(\pi)$, where both $\mathrm{im}(\pi)$ and $\ker(\pi)$ are sub-G-modules of V by Exercise 4.1.15. Now note that $\mathrm{im}(\pi) = \mathrm{im}(\pi') = U$, so that we may take $W = \ker(\pi)$ to conclude the proof of the lemma. □

Remark 4.1.20. Let ρ be a linear representation of G on a finite-dimensional complex vector space V. Suppose that V is endowed with a Hermitian inner product $(.,.)$ satisfying

$$(\rho(g)(v), \rho(g)(w)) = (v, w) \quad \forall v, w \in V,\ g \in G.$$

Now if U is a G-invariant subspace of V, then the orthogonal complement $U^\perp$ of U in V is an invariant subspace of V complementary to U.

It is important note that, if G is finite, one can always build an invariant Hermitian inner product $(.|.)$ from an arbitrary one $(.,.)$ as was done in Exercise 3.2.21 for positive definite inner products:

$$(v|w) = \sum_{g\in G}(\rho(g)(v), \rho(g)(w)).$$

Thus, the above argument gives an alternative proof of the lemma in the case where V is a finite-dimensional complex vector space.

Exercise 4.1.21. Show that in Exercises 4.1.10 and 4.1.18, there are invariant subspaces that have no invariant complements.

Definition 4.1.22. A G-module V is called *reducible* if it has sub-G-modules other than 0 and V; it is called *irreducible* otherwise. If V is the direct sum of irreducible sub-G-modules, then V is called *completely reducible*.

This terminology is also used for the corresponding representations.

Exercise 4.1.23. Consider the cyclic group C_n of order n with generator c. It has a two-dimensional real representation defined by

$$r : C_n \to \mathrm{GL}(2, \mathbb{R}), c^k \mapsto \begin{pmatrix} \cos(2k\pi/n) & -\sin(2k\pi/n) \\ \sin(2k\pi/n) & \cos(2k\pi/n) \end{pmatrix}.$$

Show that it is irreducible. If we replace $\mathbb{R}$ by $\mathbb{C}$, we obtain a complex representation. Show that this representation is completely reducible.

Exercise 4.1.24. Show that any irreducible representation of an Abelian group G over an algebraically closed field is one-dimensional.

For finite groups, irreducible representations are the building blocks of all representations.

Theorem 4.1.25. *For a finite group G, every finite-dimensional G-module is completely reducible.*

Proof. Proceed by induction. Suppose that the stament holds for all representations of dimension smaller than n, and let V be an n-dimensional representation. If V is irreducible, then we are done. Otherwise there exists a sub-G-module U with $0 \subsetneq U \subsetneq V$. By Lemma 4.1.19, U has a invariant complementary sub-G-module W. As both $\dim U$ and $\dim W$ are smaller than n, we may apply the induction hypothesis, and find that U is the direct sum of irreducible sub-G-modules $U_1, \ldots, U_k$ of U and W is the direct sum of irreducible sub-G-modules $W_1, \ldots, W_l$. But then

$$V = U_1 \oplus \ldots \oplus U_k \oplus W_1 \oplus \ldots \oplus W_l.$$

□

Remark 4.1.26. The representation of $\mathbb{R}$ in Example 4.1.16 is reducible, but it is not the direct sum of irreducibles. Exercise 4.1.21 also gives examples of this phenomenon. This shows that we need at least some condition on G (like finiteness) for Theorem 4.1.25 to hold.

Exercise 4.1.27. Let p be a prime, and consider the cyclic group C_p with generator c. The map $C_p \to \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ defined by

$$c^k \mapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

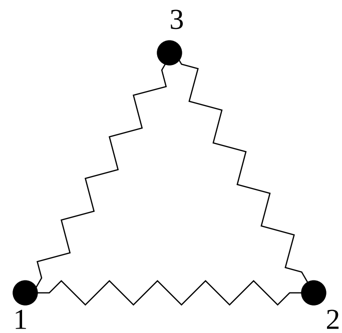

Figure 4.1: A system with 3 point masses and springs.

is a matrix representation of C_p over $\mathbb{Z}/p\mathbb{Z}$. The one-dimensional subspace spanned by $(1,0)^T$ is invariant, but it has no invariant complement.

In general, representation theory of a group G over a field of characteristic p tends to be much harder if p divides $|G|$ than otherwise.

An important result about irreducible representations known as Schur's Lemma.

Lemma 4.1.28 (Schur's Lemma). *Suppose that $\mathbb{K}$ is algebraically closed, and let V be an irreducible finite-dimensional G-module over $\mathbb{K}$. Then any homomorphism of G-modules $V \to V$ is a scalar.*

In other words, a matrix commuting with all matrices of an irreducible matrix representation is a multiple of the identity matrix.

Proof. Let $T : V \to V$ be a homomorphism of G-modules, and let $\lambda \in \mathbb{K}$ be an eigenvalue of T. Then $T - \lambda I$ is also a homomorphism of G-modules. By Example 4.1.15, its kernel is a sub-G-module of V, and it is non-zero as λ is an eigenvalue of T. As V is irreducible, $\ker(T - \lambda I)$ equals V. □

Exercise 4.1.29. Compute the set of all 2×2 matrices commuting with C_n in one of the 2-dimensional complex representations given in Example 4.1.3. Conclude that the representation is not irreducible.

4.2 Decomposing Displacements

Consider the system of Figure 4.1. It depicts three point masses (labelled with $P = \{1,2,3\}$) at positions $(-\frac{1}{2}, -\frac{1}{2}\sqrt{3}), (-\frac{1}{2}, \frac{1}{2}\sqrt{3}), (0,1)$ in two-dimensional space $E = \mathbb{R}^2$, and these masses are connected with springs of equal lengths. The system can be thought of as vibrating (in two dimensions) about its equilibrium state, in which the point masses form an equilateral triangle.

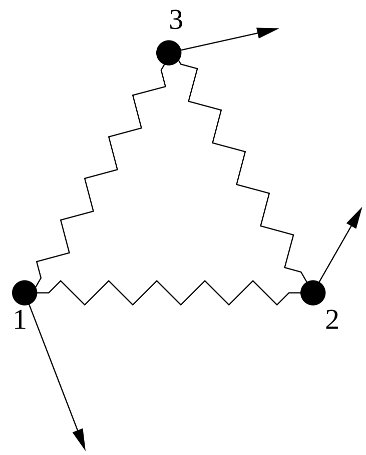

Figure 4.2: A displacement.

A *displacement* of the system is a function $f : P \to \mathbb{R}^2$ which attaches to each point mass a velocity vector. Displacements can be depicted as in Figure 4.2. The set of all displacements is an $\mathbb{R}$-linear space, which we denote by Γ. Its dimension is clearly $3 \times 2 = 6$, i.e., there are 6 degrees of freedom.

We would like to decompose Γ into subspaces in a manner that is consistent with the symmetry group D_3 that acts on the system. To be precise, we have a permutation representation $\pi : D_3 \to \mathrm{Sym}(P)$. The map is in fact an isomorphism, and we shall identify D_3 with $\mathrm{Sym}(P) = S_3$. For example, the permutation (12) corresponds to reflection in the line bisecting the edge 12 and meeting point 3.

The group D_3 also acts linearly on E. The corresponding linear representation $D_3 \to \mathrm{GL}(E)$ is given by

$$(12) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } (123) \mapsto \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

The space Γ becomes a D_3-module by setting

$$(gf)(i) = gf(g^{-1}i),\ \ i \in P, f \in \Gamma, g \in G.$$

Our goal can now be formalized as follows: try to write Γ as a direct sum of irreducible submodules. Theorem 4.1.25 tells us that this is possible.

Here, we shall try to solve this problem 'by hand', without using too much representation theory. In this way, the reader will appreciate the elegant character arguments which are to be given in Example 5.2.23.

As a first step, we can write each displacement in a unique way as a purely 'radial' displacement and a purely 'tangental' displacement (see Figure 4.4), and the spaces Γ_{rad} and Γ_{tan} of radial and tangental displacements, respectively, are three-dimensional submodules of Γ.

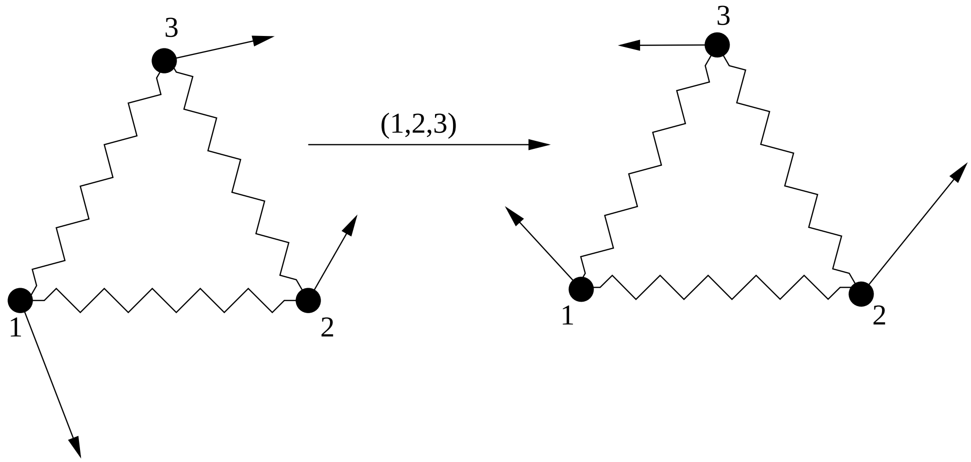

Figure 4.3: The action of (123) on a displacement.

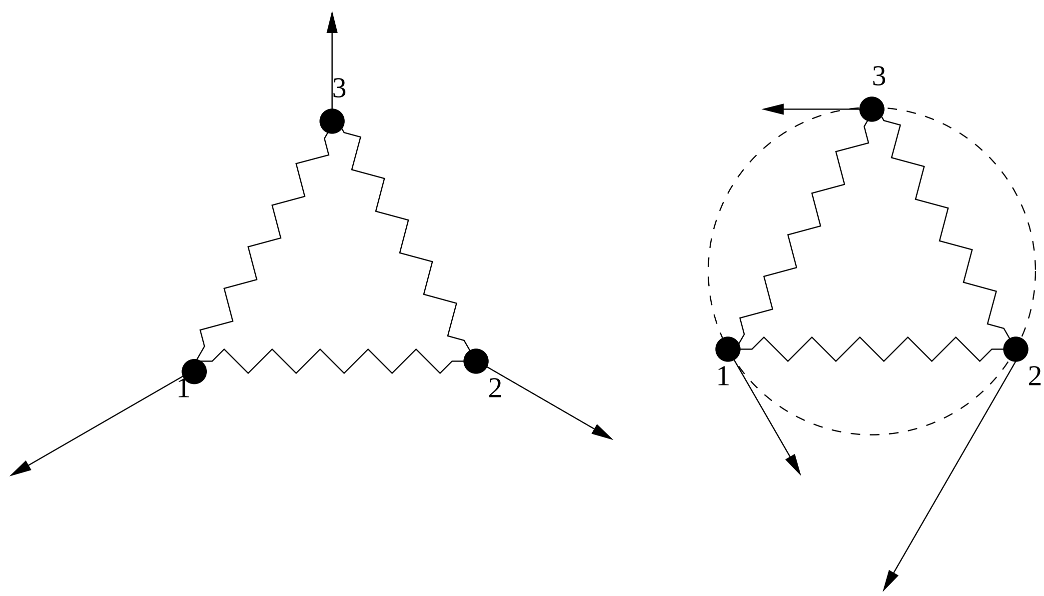

Figure 4.4: A radial (left) and a tangental (right) displacement.

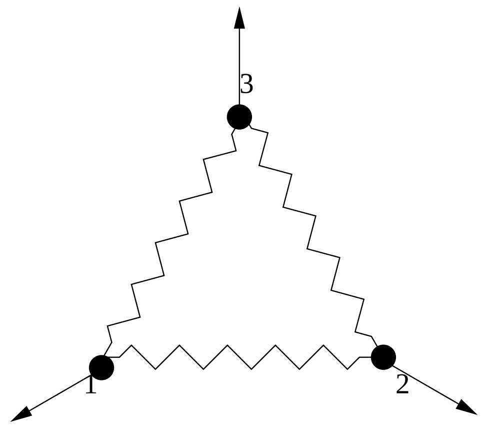

Figure 4.5: A special radial displacement.

The submodule Γ_{rad} contains a one-dimensional submodule Γ^1_{rad}, consisting of displacements with the 'same' velocity vector in all three directions, such as the one depicted in Figure 4.5. To be precise: this space consists of all radial displacements f for which $(1i)(f(1)) = f(i)$.

The space Γ^1_{rad} must have an invariant complement in Γ_{rad}, and looking at the pictures we find that the space Γ^2_{rad} of radial displacements spanned by the two in Figure 4.6 is also invariant. Note that this space consists of all radial displacements for which $(13)f(1) + (12)f(1) + f(1) = 0$.

One may wonder if the two-dimensional module Γ^2_{rad} can be decomposed further, but this is not the case, as we shall see in Example 5.2.23. The representation Γ_{tan} can be decomposed in a similar fashion. However, instead of going into details, we will first treat more representation theory, and come back to this application in Example 5.2.23.

Notes

This section is based on a similar (but more difficult) example in [9]. If your interest is aroused by this example, you are encouraged to read the pages in that book concerned with the displacemements of a tetrahedral molecule.

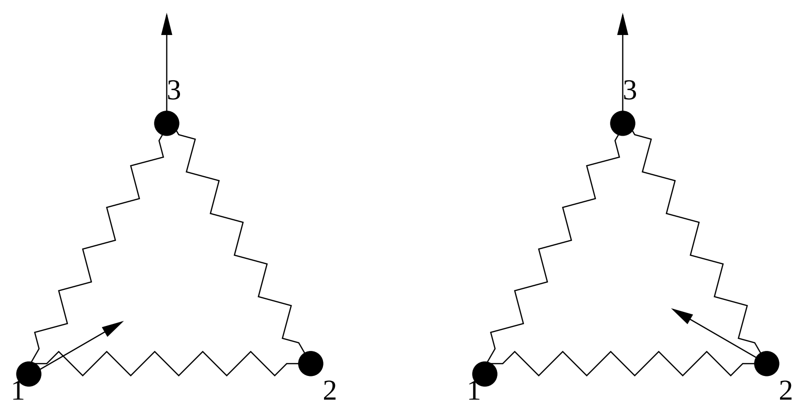

Figure 4.6: Two more special radial displacements.

Chapter 5

Character Tables

In this chapter, G denotes a finite group and $\mathbb{K}$ an algebraically closed field of characteristic 0, e.g. $\mathbb{C}$.

5.1 Characters

An extremely useful notion regarding matrices is the *trace*. The *trace* of a matrix $A = (a_{ij})_{n\times n}$ is $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$.

Exercise 5.1.1. Prove that for two matrices A and B, one has $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Exercise 5.1.2. Show that in the case of a permutation representation $\pi\colon G \to S_n$, the number $\mathrm{tr}(M_\pi(g))$ (see Example 4.1.8) is equal to the number of fixed points of the permutation $\pi(g)$.

Definition 5.1.3. Let r be a representation of a group G on a finite-dimensional vector space V. The *character* of r is the function χ^r defined on G by

$$\chi^r(g) = \mathrm{tr}(r(g)) \quad (g \in G).$$

If r is irreducible, then the character χ^r is also called irreducible.

Thus, the character of r on g is the sum of the eigenvalues of $r(g)$ counted with multiplicities.

Remark 5.1.4. If G has finite order n, and $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional representation, then $\rho(g)^n = I$ for all $g \in G$. Hence, over the algebraic closure of $\mathbb{K}$, each $\rho(g)$ is diagonalizable, and all its eigenvalues are n-th roots of unity. If $\mathbb{K} = \mathbb{C}$, then all eigenvalues have absolute value 1.

We next give some properties of complex characters.

Lemma 5.1.5. *For any complex representation $r\colon G \to \mathrm{GL}(V)$ of a finite group, we have:*

1. $\chi^r(e) = \mathrm{tr}(I) = \dim V$;
2. $\chi^r(g^{-1}) = \overline{\chi^r(g)}$;
3. $\chi^r(ghg^{-1}) = \chi^r(h)$ *for all* $g, h \in G$.

Proof. The first part is easy. As for the second part, we have seen that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $r(g)$ have absolute value 1. Therefore:

$$\chi^r(g^{-1}) = \mathrm{tr}(r(g^{-1})) = \mathrm{tr}(r(g)^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\mathrm{tr}(r(g))} = \overline{\chi^r(g)}.$$

The last part follows directly from Exercise 5.1.1. □

The last part states that characters are constant on conjugacy classes of G.

Example 5.1.6. Take $G = \mathrm{GL}(2, \mathbb{K})$. The trace of an arbitrary element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $a+d$. This is the character of the identity, which in this case is a representation (often called the *natural representation*).

But, as we have seen in Example 4.1.17, there are also linear actions on $P(V)_1$ and on $P(V)_2$. We read off the corresponding characters as traces of the matrices found in 4.1.17. They are $(a+d)/(ad-bc)$ on $P(V)_1$, and $(a^2+ad+bc+d^2)/(ad-bc)^2$ on $P(V)_2$.

Lemma 5.1.7. *Let* $V_1, \ldots, V_s$ *be representations of* G *with characters respectively,* $\chi_1, \ldots, \chi_s$*. Then the character of the representation* $V_1 \oplus \cdots \oplus V_s$ *is the sum of characters* $\chi_1 + \cdots + \chi_s$*.*

5.2 Orthogonality of irreducible characters

The following theorem is very important in the representation theory of finite groups.

Theorem 5.2.1. *Let* G *be a finite group. The irreducible complex characters of* G *form an orthonormal basis for the space of class functions on* G*, with respect to the Hermitian inner product* $(\cdot \mid \cdot)$*.*

We need a definition to make clear what this theorem states.

Definition 5.2.2. A *class function* on a group G is a function $f : G \to \mathbb{C}$ that is constant on conjugacy classes, or, equivalently, that is fixed by the linear action of G on $\mathbb{C}^G$ defined by

$$(g \cdot f)(h) := f(ghg^{-1}).$$

The space of all class functions is denoted by $\mathcal{H}$. The Hermitian inner product is given by

$$(\chi \mid \psi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}$$

for $\chi, \psi \in \mathcal{H}$.

We have noticed earlier that characters are class functions. We divide the proof of Theorem 5.2.1 into two parts: first, we will show the orthonormality of the irreducible characters, and then their being a full basis of the space of class functions. For our first purpose, we introduce the tensor product of two representations.

Definition 5.2.3. Let V and W be vector spaces over $\mathbb{K}$. Then $V \otimes W$, called the *tensor product of* U *and* V, is the quotient of the free vector space over $\mathbb{K}$ with basis $\{v \otimes w \mid v \in V,\ w \in W\}$ by its subspace spanned by all elements of the following forms (for $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{K}$):

$$\begin{aligned}&(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w,\\ &v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2,\\ &(\lambda v) \otimes w - \lambda(v \otimes w),\ \text{and}\\ &v \otimes (\lambda w) - \lambda(v \otimes w).\end{aligned}$$

Exercise 5.2.4. Show that if $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases of V and W, respectively, then

$$\{v_i \otimes w_j \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$$

is a basis of $V \otimes W$.

It is not hard to see that if U, V, and W are vector spaces over $\mathbb{K}$, then the spaces $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are canonically isomorphic. We will identify them, leave out parentheses, and write $T^d(V)$ for the d-fold tensor product of V with itself. The direct sum of all $T^d(V)$ for $d \in \mathbb{N}$ (including 0, for which $T^d(V)$ is isomorphic to $\mathbb{K}$) forms an associative algebra with respect to the operator $\otimes$; this algebra is called the *tensor algebra on* V and denoted by $T(V)$.

Definition 5.2.5. Let V be a vector space over $\mathbb{K}$ and let d be a natural number. Then the quotient of $T^d(V)$ by the subspace spanned by the set

$$\{v_1 \otimes v_2 \otimes \ldots \otimes v_d - v_{\pi(1)} \otimes v_{\pi(2)} \otimes \ldots \otimes v_{\pi(d)} \mid \pi \in S_d\}$$

is denoted by $S^d(V)$, and called the *d-th symmetric power* of V.

Just like $T(V)$, the direct sum of all $S^d(V)$ for $d \in \mathbb{N}$ forms an associative algebra, called the *symmetric algebra on* V and denoted by $S(V)$. This algebra is commutative, and to stress this fact we usually leave out the operator $\otimes$ from expressions in $S(V)$.

Exercise 5.2.6. Show that if $v_1, \ldots, v_n$ is a basis of V, then the set

$$\{v_1^{m_1} \cdots v_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{N},\ m_1 + \ldots + m_n = d\}$$

is a basis for $S^d(V)$. Prove that $S^d(V^*)$ is canonically isomorphic to $P(V)_d$ (see Example 4.1.17).

Definition 5.2.7. Let $r : G \to \mathrm{GL}(V)$ and $s : G \to \mathrm{GL}(W)$ be linear representations of G. Then we define a linear representation $r \otimes s : G \to \mathrm{GL}(V \otimes W)$ of G by

$$(r \otimes s)(g)(v \otimes w) := (r(g)v) \otimes (s(g)w),$$

extended linearly to all of $V \otimes W$. This representation is denoted by $r \otimes s$ and called the *tensor product of* r *and* s.

Lemma 5.2.8. *Let $G, r, V, s,$ and W be as in Definition 5.2.7. Then the character of $r \otimes s$ satisfies $\chi^{r \otimes s} = \chi^r \chi^s$.*

Proof. Fix bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ of V and W, respectively. Consider, for a fixed $g \in G$, the matrix $A = (a_{ij,kl})$ of $(r \otimes s)(g)$ with respect to the basis $v_i \otimes w_j$ of $V \otimes W$; We have

$$(r \otimes s)(g)(v_k \otimes w_l) = \sum_{i,j} a_{ij,kl} v_i \otimes w_j. \tag{5.1}$$

On the other hand, by the definition of the tensor representation, the left-hand side equals

$$(r(g)v_k) \otimes (s(g)w_l) = (\sum_i r_{ik} v_i) \otimes (\sum_j s_{jl} w_j) = \sum_{i,j} r_{ik} s_{jl} v_i \otimes w_j. \tag{5.2}$$

Comparing the right-hand sides of equations (5.1) and (5.2) we obtain

$$a_{ij,kl} = r_{ik} s_{jl},$$

and therefore

$$\chi^{s \otimes r}(g) = \sum_{i,j} a_{ij,ij} = \sum_{i,j} r_{ii} s_{jj} = \chi^r(g) \chi^s(g),$$

concluding the proof. $\square$

Exercise 5.2.9. Let V be a G-module, and consider the linear isomorphism $\sigma : V \otimes V \to V \otimes V$ determined by $\sigma(x \otimes y) = y \otimes x$ for $x, y \in V$. Show that σ has precisely two eigenspaces, with eigenvalues 1 and -1, and that both spaces are G-invariant.

Another important ingredient in our proof of Theorem 5.2.1 is the so-called dual representation.

Definition 5.2.10. Let $r : G \to \mathrm{GL}(V)$ be a representation of the group G. Then we write V^* for the dual of V (in the notation of Example 4.1.17, this is $P(V)_1$), and r^* for the corresponding representation, given by

$$(r^*(g)f)(v) := f(r(g)^{-1}v).$$

Exercise 5.2.11. Show that $\chi^{r^*} = \overline{\chi^r}$.

One final lemma before proceeding with the proof of Theorem 5.2.1.

Lemma 5.2.12. *Let r, V, s, W, G be as in Definition 5.2.7. Define a representation t of G in* $\mathrm{Hom}(V, W)$*, the space of all $\mathbb{C}$-linear maps from V to W by*

$$t(g)H := s(g)Hr(g)^{-1}, \quad H \in \mathrm{Hom}(V, W).$$

This representation is equivalent to $s \otimes r^*$.

Proof. Define $T : W \otimes V^* \to \mathrm{Hom}(V, W)$ by

$$T(w \otimes f)(v) = f(v) \cdot w, \quad v \in V, f \in V^*, w \in W,$$

and extend it linearly. It is a matter of elementary linear algebra to see that T is a linear isomorphism. We claim that it intertwines the representations t and $s \otimes r^*$. The best way to see this is to do the computation yourself, so we leave it as an exercise for the reader. □

Exercise 5.2.13. Prove that the map T defined in the proof of Lemma 5.2.12 is really an intertwining map.

If we choose a basis $(v_j)_j$ of V with dual basis $(v^j)_j$ of V^*, and a basis $(w_i)_i$ of W, then the map T sends $w_i \otimes v^j$ to the linear map $V \to W$ having the matrix E_{ij} (the matrix with 1 on position (i, j), and zeroes elsewhere) with respect to the bases $(v_j)_j$ and $(w_i)_i$.

Having completed all preparations, of which the tensor product is a very useful tool in its own right, we proceed with the first part of the proof of this section's main theorem.

Proof of Theorem 5.2.1, first part. Let r, V, s, W, G be as in Definition 5.2.7 and t as in Lemma 5.2.12, and moreover, assume that both representations r and s are irreducible. We wish to compute the inner product $(\chi^s \mid \chi^r)$. It can be written in a particularly nice form, using tensor products.

Define the linear map $P : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ by

$$P = \frac{1}{|G|} \sum_{g \in G} t(g).$$

The trace of P equals

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} \chi^t(g) &= \frac{1}{|G|} \sum_{g \in G} \chi^s(g) \chi^{r^*}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi^s(g) \overline{\chi^r(g)}) \\ &= (\chi^s \mid \chi^r). \end{aligned}$$

Here we used Lemmas 5.2.12, 5.2.8 and Exercise 5.2.11.

We claim that P is a projection from $\mathrm{Hom}(V,W)$ onto the space $\mathrm{Hom}_G(V,W)$. To show that it is indeed a projection, let $H \in \mathrm{Hom}(V,W)$, and compute

$$\begin{aligned}
P^2H &= \frac{1}{|G|}\sum_g \frac{1}{|G|}\sum_h s(g)s(h)Hr(h^{-1})r(g^{-1}) \\
&= \frac{1}{|G|}\sum_g \frac{1}{|G|}\sum_h s(gh)Hr((gh)^{-1}) \\
&= \frac{1}{|G|}\sum_g \frac{1}{|G|}\sum_h s(h)Hr(h^{-1}) \\
&= \frac{1}{|G|}\sum_g PH \\
&= PH,
\end{aligned}$$

where in third equality the dummy h is replaced by $g^{-1}h$. This proves that P is a projection. To see that $\mathrm{im}(P)$ is contained in $\mathrm{Hom}_G(V,W)$, let $g_0 \in G$, $H \in \mathrm{Hom}(V,W)$, and $v \in V$, and compute

$$\begin{aligned}
(PH)(g_0v) &= \frac{1}{|G|}\sum_g s(g)Hr(g^{-1}g_0)v \\
&= \frac{1}{|G|}\sum_g s(g_0g)Hr(g^{-1}g_0^{-1}g_0)v \\
&= s(g_0)\frac{1}{|G|}\sum_g s(g)Hr(g^{-1})v \\
&= g_0(PHv),
\end{aligned}$$

where in the second equality, the dummy g is replaced by g_0g. Finally, if $H \in \mathrm{Hom}_G(V,W)$, then

$$\begin{aligned}
PH &= \frac{1}{|G|}\sum_g s(g)Hr(g^{-1}) \\
&= \frac{1}{|G|}\sum_g s(g)s(g^{-1})H \\
&= H,
\end{aligned}$$

so that $\mathrm{im}(P) = \mathrm{Hom}_G(V,W)$. This proves the claim. We now distinguish two cases.

First, suppose that r and s are not equivalent. Then, by Schur's lemma, the space $\mathrm{Hom}_G(V,W)$ is zero-dimensional, so that P and, *a fortiori*, its trace are 0. In this case $(\chi^s \mid \chi^r) = 0$.

Alternatively, suppose that r and s are equivalent. As we only wish to compute the inner product of their characters, we may assume that they are equal: $W = V$ and $s = r$. In this case, the image of P is the linear span of the identity map I on V. Taking any basis of $\mathrm{Hom}(V,V)$ that starts with I, we see that the matrix of P with respect to that basis has only non-zero elements on the first row. The only contribution to the trace of P is therefore the element in the upper left corner. Since $PI = I$, that element is 1. Hence $(\chi^r \mid \chi^r) = 1$. □

This proves that the characters are an orthonormal system in the space of class functions on G.

Corollary 5.2.14. *The number of non-isomorphic irreducible representations of G is finite.*

Proof. The dimension of $\mathcal{H}$ is the number of conjugacy classes of G. The irreducible characters form an orthonormal, whence independent, set in this finite-dimensional space by the above. □

Lemma 5.2.15. *Let r be a complex representation of G on a space V, and write*

$$V = V_1 \oplus \cdots \oplus V_t,$$

where the V_i are irreducible submodules. Let s be an irreducible representation of G on a space W. Then the number of i for which V_i is isomorphic to W is equal to $(\chi^r|\chi^s)$. In particular, this number does not depend on the decomposition in irreducibles.

Proof. If χ_i is the character of V_i then the character χ^r of V can be written as $\chi^r = \chi_1 + \cdots + \chi_t$. Hence

$$(\chi^r|\chi^s) = (\chi_1|\chi^s) + \cdots + (\chi_t|\chi^s),$$

and by the above $(\chi_i|\chi^s)$ is one if $V_i \cong W$, and zero otherwise. □

Corollary 5.2.16. *Two representations of a finite group are equivalent if and only if they have the same character.*

Recall that the *regular* representation $\rho\colon G \to \mathrm{GL}(V)$ of the group G is defined by $\rho(g)(e_h) = e_{gh}$, with V the vector space of dimension $|G|$ and basis $\{e_g \mid g \in G\}$.

Proposition 5.2.17. *The character of the regular representation ρ of G satisfies $\rho(g) = \delta_{g,e}|G|$ $(g \in G)$.*

Proof. Since $eg = g$, the transformation $\rho(e)$ fixes each element g of G. Therefore, $\mathrm{tr}(\rho(e)) = n = |G|$. If $g \neq e$, then $e_h \neq e_{gh}$ for each $h \in G$. This implies that $\mathrm{tr}(\rho(g)) = 0$. □

Lemma 5.2.18. *Every irreducible representation W of G is a sub-representation of the regular representation with multiplicity* $\dim W$.

Proof. From Proposition 5.2.17 we have

$$(\chi^\rho \mid \chi^s) = \frac{1}{|G|}\sum_{h\in G}\chi^\rho(h)\overline{\chi^s(h)} = \frac{1}{|G|}|G|\overline{\chi^s(e)} = \dim(W),$$

and the result follows from Lemma 5.2.15. $\square$

Corollary 5.2.19. *Let $r_1, r_2, \ldots, r_s$ be the irreducible representations of G. Then*

$$|G| = \sum n_i^2$$

where n_i is the dimension of the representation r_i.

Proof. We saw in the lemma that the regular representation ρ can be written as a sum of all irreducible representations r_i of G. Moreover, each of these representations appears n_i times in the decomposition. So

$$\rho = \overbrace{r_1 \oplus \ldots \oplus r_1}^{n_1 \text{ times}} \oplus \ldots \oplus \overbrace{r_s \oplus \cdots \oplus r_s}^{n_s \text{ times}}.$$

Therefore $|G| = \chi^\rho(e) = \sum_{i=1}^s n_i \chi^{r_i}(e) = \sum_{i=1}^s n_i^2$. $\square$

Suppose that we manage to construct some non-isomorphic irreducible representations of G. Then Corollary 5.2.19 can be used to decide whether we have already find all.

In order to finish the proof of Theorem 5.2.1, we still need to show that the irreducible characters span $\mathcal{H}$.

Lemma 5.2.20. *Suppose that $r : G \to \mathrm{GL}(V)$ is an irreducible n-dimensional complex representation of G, and let $f \in \mathcal{H}$. Then the linear map*

$$H := \frac{1}{|G|}\sum_{g\in G} f(g)r(g)$$

equals

$$\frac{1}{n}(\overline{f} \mid \chi^r)I.$$

Proof. First of all, note that H verifies $r(g)H = Hr(g)$ for all $g \in G$. Hence, as r is irreducible, H must be multiplication by a scalar λI. Its trace equals $n\lambda$. On the other hand, it equals

$$\frac{1}{|G|}\sum_{g\in G} f(g)\chi^r(g) = (\overline{f} \mid \chi^r).$$

Consequently,

$$\lambda = (\overline{f} \mid \chi^r)/n.$$

$\square$

Proof of Theorem 5.2.1, second part. In order to prove that the irreducible characters form a basis of $\mathcal{H}$, it suffices to prove that the orthoplement of their linear span in $\mathcal{H}$ is zero. Hence, let $f \in \mathcal{H}$ be orthogonal to all irreducible characters. According to Lemma 5.2.20, the linear map

$$H = \frac{1}{|G|} \sum_{g \in G} f(g) r(g)$$

is zero for all irreducible representations r. But, writing an arbitrary representation as a direct sum of irreducible ones, we see that this linear map is zero for *any* representation r. Let us apply this knowledge to the regular representation. Compute

$$He_h = \frac{1}{|G|} \sum_{g \in G} f(g) r(g) e_h = \frac{1}{|G|} \sum_{g \in G} f(g) e_{gh}.$$

Thus, for H to be zero, f must be identically zero. This concludes the proof. □

Theorem 5.2.21. *The number of irreducible characters equals the number of conjugacy classes of G.*

Proof. The dimension of $\mathcal{H}$ equals the number of conjugacy classes of G, and the irreducible characters form a basis of this linear space. □

Half of the following proposition is Exercise 4.1.24, and can be done without character theory.

Proposition 5.2.22. *The group G is Abelian if and only if all of its irreducible representations are one dimensional.*

Proof. Denote by s the number of irreducible characters of G and denote their dimensions, as before by n_i.

If G is Abelian, all conjugacy classes consist of a single element of G. Then by the theorem above, there are $|G|$ irreducible representations of G. Hence by Corollary 5.2.19, all n_i's are equal to 1. This settles one implication.

As for the converse, suppose that $n_i = 1$ for all i. Then $|G| = s$ and so there are $|G|$ distinct conjugacy classes of G. Therefore, G is Abelian. □

Example 5.2.23. Recall from Section 4.2 the D_3-module Γ of displacements. Let us employ character theory to find the decomposition of Γ into irreducible modules. First of all, note that D_3 has three conjugacy classes, with representatives $(1), (12)$ and (123). Hence D_3 has three distinct irreducible characters by Theorem 5.2.21. Let $V = \mathbb{R}^3$, and let $s : D_3 \to \mathrm{GL}(V)$ be the linear representation corresponding to the permutation representation of D_3 on $\{1, 2, 3\}$ (see Example 4.1.8). Then

$$\chi^s((1)) = 3, \ \chi^s((1,2)) = 1, \ \text{and} \ \chi^s((1,2,3)) = 0.$$

We have $(\chi^s|\chi^s) = (9 + 3 * 1)/6 = 2$, so that V is reducible. Indeed, the line spanned by $(1, 1, 1) \in V$ is invariant. Let r_1 be the representation of D_3 on this line, and let r_2 be the representation of D_3 on an invariant complement. Then

$$\chi^{r_1}((1)) = 1,\ \chi^{r_1}((1,2)) = 1,\ \text{and}\ \chi^{r_1}((1,2,3)) = 1$$

and

$$\chi^{r_2}((1)) = 2,\ \chi^{r_2}((1,2)) = 0,\ \text{and}\ \chi^{r_2}((1,2,3)) = -1.$$

Both χ^{r_1} and χ^{r_2} have squared norm 1, so they are irreducible. The representation of D_3 on E is equivalent to r_2, as can be read of from the traces of the matrices on page 4.2.

An element of Γ can be extended in a unique way to a linear map $V \to E$, so that we may identify Γ with the space $\mathrm{Hom}_K(V, E)$ of K-linear maps from V to E. By Lemma 5.2.12, $\mathrm{Hom}_K(V, E)$ is isomorphic to $V^* \otimes E$; let $r : D_3 \to V^* \otimes E$ be the corresponding representation. Lemma 5.2.8 shows that $\chi^r = \overline{\chi^s}\chi^{r_2}$. Hence,

$$\chi^r((1)) = 6,\ \chi^r((1,2)) = 0,\ \text{and}\ \chi^r((1,2,3)) = 0.$$

First, we can decompose $\chi^r = (\chi^{r_1} + \chi^{r_2})\chi^{r_2} = \chi^{r_1}\chi^{r_2} + (\chi^{r_2})^2 = \chi^{r_2} + (\chi^{r_2})^2$. The character $(\chi^{r_2})^2$ has squared norm 3, so it is reducible. We have $(\chi^{r_1}|(\chi^{r_2})^2) = 1$ and also $(\chi^{r_2}|(\chi^{r_2})^2) = 1$, so that $(\chi^{r_2})^2 - \chi^{r_1} - \chi^{r_2}$ is the character of a third irreducible representation, say r_3, of dimension 1. Its character equals

$$\chi^{r_3}((1)) = 1,\ \chi^{r_3}((12)) = -1,\ \text{and}\ \chi^{(r_3)}(123) = 1,$$

that is, $\chi^{r_3} = \mathrm{sgn}$. By Theorem 5.2.21, the characters r_1, sgn, r_2 exhaust all irreducible characters of D_3. We conclude that the character χ^r of D_3 on Γ decomposes into $\chi^{r_1} + 2\chi^{r_2} + \chi^{r_3}$.

5.3 Character tables

Let $r_1, r_2, \ldots, r_s$ be the distinct irreducible representations of G and let $\chi_1, \chi_2, \ldots, \chi_s$ be their characters. Let $C_1, C_2, \ldots, C_s$ be the distinct conjugacy classes of G, with representatives $g_1, \ldots, g_s$, respectively.

Construct a table with rows labelled by $\chi_1, \chi_2, \ldots, \chi_s$ and columns labelled by $g_1, g_2, \ldots, g_s$; under each g_i we write the size of C_i. The (i, j)-entry of the table is $\chi_i(g_j)$. It is common to take $C_1 = \{e\} = \{g_1\}$, so that the first column contains the dimensions of the respective characters, and also to take χ_1 to be the trivial representation, so that the first row contains only ones.

This table is called the *character table* of G.

Example 5.3.1. Let us check the character table of S_3. First notice that there are three conjugacy classes: $C_1 = e$, $C_2 = \{(12), (13), (23)\}$ consisting of all transpositions and $C_3 = \{(123), (132)\}$ consisting of all 3-cycles. By Theorem 5.2.21, we know that there are three irreducible representations.

By convention, χ_1 is the trivial character. For the second one we take the *sign* representation. Recall this is the one that assigns to each cycle its sign

(2-cycles are odd, 3-cycles are even). Since it is a 1-dimensional representation, it is equal to its character.

Finally for the third representation, we proceed as follows. The permutation representation π of S_3 on $\mathbb{C}^3$ has an invariant subspace U spanned by $(1,1,1)$. Its orthogonal complement is also an invariant subspace, say V; it consists of all $(z_1, z_2, z_3) \in \mathbb{C}^3$ such that $z_1 + z_2 + z_3 = 0$. The latter representation has dimension 2 and is irreducible; it is called the *standard* representation of S_3. To find its character, we observe that $\chi^V = \chi^\pi - \chi^U$. But U is just the trivial representation and χ^π is equal to 3 on C_1, 1 on C_2 and 0 on C_3 (why? see Exercise 5.1.2)

We have found the complete character table.

S_3	(1)	(12)	(123)
	1	3	2
trivial U	1	1	1
sign	1	-1	1
standard V	2	0	-1

Proposition 5.3.2. *The standard representation of S_n is irreducible for all $n > 1$.*

Proof. Let χ be the character of the n-dimensional linear representation corresponding to the permutation representation of S_n on $\{1, \ldots, n\}$ (see Example 4.1.8), and let χ_1 be the trivial character of S_n. We wish to show that $\chi - \chi_1$ is irreducible; to this effect, it suffices to prove that $(\chi|\chi) = 2$. Denoting by $\mathrm{fix}(\sigma)$ the set of fixed points of $\sigma \in S_n$ on $\{1, \ldots, n\}$, we have $\chi(\sigma) = |\,\mathrm{fix}(\sigma)|$. Hence, we find

$$(\chi|\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} |\,\mathrm{fix}(\sigma)|^2.$$

To evaluate the right-hand side of this equation we count the number of permutations having precisely k fixed points ($0 \leq k \leq n$). This number is clearly equal to

$$\binom{n}{k} F(n-k),$$

where $F(m)$ denotes the number of fixed-point-free permutations in S_m. A standard inclusion-exclusion argument shows that

$$F(m) = \sum_{i=0}^{m} (-1)^i \frac{m!}{i!}.$$

Combining the above considerations we find

$$\begin{aligned}(\chi|\chi) &= \frac{1}{n!}\sum_{k=0}^{n} k^2 \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \frac{(n-k)!}{i!} \\ &= \sum_{k=0}^{n} k^2 \frac{1}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} \\ &=: M_n.\end{aligned}$$

It is readily seen that $M_2 = 2$. To prove that $M_n = 2$ for all $n \geq 2$, we note that for $n \geq 3$ we have:

$$M_n - M_{n-1} = \sum_{i=0}^{n} (n-i)^2 (-1)^i \frac{1}{(n-i)!i!},$$

and it is not hard to prove that the right-hand side is zero for $n \geq 3$. □

Example 5.3.3. Consider the cyclic group C_n with generator c, and let $\zeta \in \mathbb{C}$ be a primitive n-th root of unity. Define, for $j = 0, \ldots, n-1$, the map $\chi_j : C_n \to \mathbb{C}$ by $\chi_j(c^k) := \zeta^{jk}$. Then the χ_j are characters of C_n, and irreducible as they are one-dimensional. By Theorem 5.2.21 there are no other characters, so that the character table of C_n is as follows.

C_n	e	c	c^2	$\ldots$	c^{n-1}
	1	1	1	$\ldots$	1
χ_1	1	1	1	$\ldots$	1
χ_2	1	ζ	ζ^2	$\ldots$	ζ^{n-1}
χ_3	1	ζ^2	ζ^4	$\ldots$	$\zeta^{2(n-1)}$
.	.	.	.	$\ldots$	.
.	.	.	.	$\ldots$	.
.	.	.	.	$\ldots$	.
χ_n	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\ldots$	$\zeta^{(n-1)^2}$

Example 5.3.4. To compute the character table of S_4, we first count the number of conjugacy classes. They are the identity, the 2-cycles, the 3-cycles, the 4-cycles and the product of two distinct 2-cycles; five in total. Hence, by Theorem 5.2.21, S_4 has five distinct irreducible characters. It has three of them in common with S_3 (or, indeed, with any symmetric group; see Proposition 5.3.2), namely the trivial character, the sign and the standard character. They are listed in the following partial character table.

S_4	(1)	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
χ_1 (trivial)	1	1	1	1	1
χ_2 (sign)	1	-1	1	-1	1
χ_4 (standard)	3	1	0	-1	-1

By Lemma 5.2.19, the squares of the dimensions of the irreducible characters sum up to $|S_4| = 24$, and we only have $1 + 1 + 9 = 11$ so far. We conclude that the two remaining irreducible characters have dimensions 2 and 3. The latter is the product $\chi_4 := \chi_2\chi_4$, which is irreducible as it has norm 1. The remaining 2-dimensional character χ_3 can be found using the orthogonality relations among characters. We then find the following character table of S_4.

S_4	(1)	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
$\chi_5 = \chi_2\chi_4$	3	-1	0	1	-1

Exercise 5.3.5. Compute the number of elements in each conjugacy class of S_4.

Exercise 5.3.6. Consider the permutation representation of S_4 on partitions of $\{1, 2, 3, 4\}$ into two sets of size 2; that is, the action of S_4 on the set

$$\{\{A, B\} \mid |A| = |B| = 2,\ A \cup B = \{1, 2, 3, 4\},\ A \cap B = \emptyset\}.$$

Show that the character of the corresponding linear representation is $\chi_1 + \chi_3$; this explains the character χ_3.

A more or less explicit description of the irreducible representations of S_n is known for general n. It involves the beautiful combinatorics of Young tableaux, a good textbook on which is Fulton's book [4].

Exercise 5.3.7. Construct the character table of D_4.

Exercise 5.3.8. Compute the character table of the Klein group $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Example 5.3.9. Consider the subgroup T of the orthogonal group $SO_3(\mathbb{R})$ consisting of the proper rotations of the tetrahedron. We wish to compute its character table. The conjugacy classes are those of $(1), (234), (243)$ and $(13)(24)$ (make a picture that corresponds to this numbering), and their cardinalities are 1, 4, 4 and 3, respectively. Thus, there must be 4 irreducible characters, of representations with dimensions $m_1 \leq m_2 \leq m_3 \leq m_4$. The sum of their squares should equal $|T| = 12$. Their turns out to be only one solution, namely $m_1 = m_2 = m_3 = 1$ and $m_4 = 3$. Hence, apart from the trivial character, there should be two more characters of degree 1. Let r be one of them. As r is nothing but a homomorphism from T to $\mathbb{C}^*$, the complex number $r((234))$ should have multiplicative order 3; hence it is either $\omega = e^{2\pi/3}$ or ω^2. The image of (243) is fixed in both cases, and so is the image of (13)(24), as χ^r should be orthogonal to the trivial representation. This explains the first three lines in Table 5.1. The last line can be obtained from the orthonormality conditions.

T	(1)	(234)	(243)	(13)(24)
	1	4	4	3
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1
χ_4	3	0	0	-1

Table 5.1: The character table of the tetrahedral group

Exercise 5.3.10. Find a representation corresponding to the last line of Table 5.1.

Exercise 5.3.11. Let G be a group and let H be a normal subgroup of G. Let r be a representation of G with the property that $H \subseteq \ker r$. Show that r induces a representation of G/H. Conversely, any representation of G/H can be lifted to one on G. Apply this principle to the group T, which has the Klein 4-group as a normal subgroup.

Exercise 5.3.12. Prove the column orthogonality of the character table: for $g, h \in G$, and $\chi_1, \ldots, \chi_s$ a complete set of irreducible characters of G,

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} = |C_G(h)|\delta_{g,h}.$$

This formula is also useful for the construction of character tables.

Exercise 5.3.13. Let χ be the character of a complex linear representation of G on V. Show that the character of G on the symmetric square $S^2(V)$ of V equals

$$g \mapsto \frac{\chi(g)^2 + \chi(g^2)}{2}.$$

What is the character of G on $S^3(V)$? Use these results to find new characters of A_5 from the irreducible 4-dimensional character occurring in the standard permutation representation of degree 5.

Exercise 5.3.14. Consider the alternating group A_5 on 5 letters.

1. Write down representatives for each of its conjugacy classes.

2. Determine the sizes of the corresponding conjugacy classes.

3. Show that A_5 has only one irreducible character of degree 1. (Hint: if there were one more, then there would have to be a normal subgroup N such that A_5/N is abelian.)

4. The rotation group of the icosahedron is A_5. This gives an irreducible representation ρ of A_5 of degree 3. Write down its character.

5. Verify that the composition of ρ with conjugation by $(1, 2)$ gives a non-equivalent representation and determine its character.

6. Apply the symmetric square formula of Exercise 5.3.13 to ρ to find a 6-dimensional character. Show that this character splits into the trivial character and an irreducible 5-dimensional character.

7. Conclude from the orthogonality of the character table that there is one more irreducible character. Compute this character, and compare it with the character of the restriction of the standard representation of S_5 to A_5.

5.4 Application: symmetry of vibrations

There exist many spectroscopic techniques in chemistry that can be used to measure symmetry properties of molecules of a given substance. In this section, we describe two of them: infrared (IR) spectroscopy and Raman spectroscopy. In the former method, the substance is exposed to radiation of various frequencies, and the intensity of the light leaving the substance is measured. If the radiation excites the molecules from the ground state into vibration, then it is absorbed, giving rise to a (downward) peak in the intensity (plotted as a function of the frequency). As we have seen in Section 4.2, there are several essentially different vibrations, corresponding to irreducible submodules of the representation of the symmetry group of the molecule on the space of all displacements. As with Raman spectroscopy—another technique whose physical details we do not discuss here—only actual vibrations may contribute a peak to the intensity spectrum, i.e., the (infinitesimal) isometries of three-dimensional space do not contribute. However, not even all irreducible modules in the module of vibrations can be observed by IR spectroscopy; due to technical properties of this technique, only those irreducibles occurring also in the natural 3-dimensional representation do actually contribute a peak in the spectrum. This distinguishes IR from Raman spectroscopy, which allows to observe all irreducibles that occur in the symmetric square of the aforementioned natural representation.

Hence, to predict the number of peaks using either of the above methods, for a given molecule with n molecules, we proceed as follows.

1. Determine the symmetry group G of the molecule at hand, and compute or look up its character table.

2. Determine the character χ_{dis} of G on the $3n$-dimensional module of displacements of the molecule, as well as its character χ_{iso} on the 6-dimensional submodule of (infinitesimal) isometries of $\mathbb{R}^3$. Then

$$\chi_{\text{vib}} := \chi_{\text{dis}} - \chi_{\text{iso}}$$

is the character of G on vibrations of the molecule.

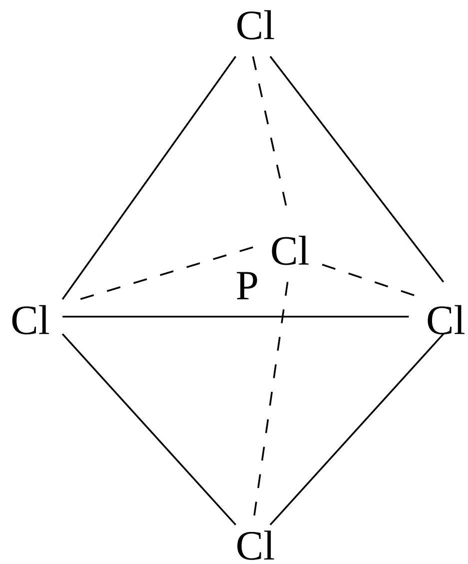

Figure 5.1: The PCl_5 molecule

3. Compute the character χ_1 of G on $\mathbb{R}^3$, and the character χ_2 of G on the symmetric square of $\mathbb{R}^3$.

4. Then the number of peaks in IR spectroscopy is the number of irreducible characters in χ_{vib} (counted with multiplicity) that also occur in χ_1, and and the number of peaks in Raman spectroscopy is the number of irreducible characters in χ_{vib} (counted with multiplicity) that also occur in χ_2.

Example 5.4.1. We want to predict the number of peaks in IR and Raman spectroscopy when applied to the molecule PCl_5 of Figure 5.1. In these chemical applications, it is common to use the Schoenflies notation (see Table 3.2), and we will adhere to this tradition in the present example.

1. The symmetry group of this molecule is D_{3h}, whose character table is as follows (E is the identity, C_i stands for a rotation of order i, σ_h and σ_v are reflections in a horizontal and vertical plane, respectively, and S_3 is the composition of a C_3 and a σ_h).

D_{3h}	E	C_3	$3C_2$	σ_h	S_3	σ_v
	1	2	3	1	2	3
A_1'	1	1	1	1	1	1
A_2'	1	1	-1	1	1	-1
E'	2	-1	0	2	-1	0
A_1''	1	1	1	-1	-1	-1
A_2''	1	1	-1	-1	-1	1
E''	2	-1	0	-2	1	0

2. Let χ_1 denote the character of D_{3h} in its natural representation on $\mathbb{R}^3$, and let χ_3 denote the character of D_{3h} corresponding to its permutation representation on the 6 molecules of PCl_5 (see Example 4.1.8). Their values are easily calculated to be as follows.

D_{3h}	E	C_3	$3C_2$	σ_h	S_3	σ_v
χ_1	3	0	-1	1	-2	1
χ_3	6	3	2	4	1	4

As in Example 5.2.23, the character of the module of displacements equals the dual of χ_3 times χ_1. Having only real values, χ_3 is self-dual, so that we find the following values for χ_{dis}.

D_{3h}	E	C_3	$3C_2$	σ_h	S_3	σ_v
χ_{dis}	18	0	-2	4	-2	4

This character can be decomposed into irreducible ones as follows:

$$\chi_{\text{dis}} = 2A_1' + A_2' + 4E' + 3A_2'' + 2E''$$

The character χ_{iso} is the sum of the characters of D_{3h} on translations and on rotations. The character of D_{3h} on translations equals $\chi_3 = E' + A_2''$, and the character χ_4 on rotations turns out to be as follows.

D_{3h}	E	C_3	$3C_2$	σ_h	S_3	σ_v
χ_4	3	0	-1	-1	2	-1

We find that $\chi_4 = A_2' + E''$. We conclude that

$$\chi_{\text{vib}} = 2A_1' + 3E' + 2A_2'' + E''.$$

3. We have already computed χ_1. Using Exercise 5.3.13 we find the following values for its symmetric square χ_2.

D_{3h}	E	C_3	$3C_2$	σ_h	S_3	σ_v
χ_2	6	0	2	2	2	2

We find that $\chi_2 = 2A_1' + E' + E''$, while $\chi_1 = E' + A_2''$.

4. From the above, we conclude that only vibrations with characters E' or A_2'' are observable with IR spectroscopy; the number of such irreducible characters in χ_{vib} is 5, so that is the predicted number of peaks when using this technique. Using Raman spectroscopy, only vibrations with characters A_1', E' and E'' can be observed; this leads to 6 peaks.

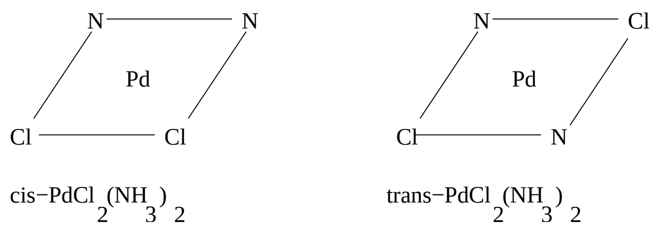

Figure 5.2: Two configurations of $PdCl_2(NH_3)_2$

The following exercises shows how spectroscopic techniques may or may not be used to recognize molecules.

Exercise 5.4.2. The square planar coordination compound $PdCl_2(NH_3)_2$ can be prepared with two different configurations; see Figure 5.2. Is it possible to discern the cis- and the trans-configurations using information concerning $Pd - Cl$ stretch vibrations in IR spectroscopy?

Exercise 5.4.3. Is it possible to deduce the symmetry of $Ni(CO)_4$ (tetrahedron or square plane) from the IR and/or Raman spectra of the CO-stretches?

5.5 Notes

Most of the content of this chapter can be found in Serre's book [8]. The last section is entirely based on Theo Beelen's lectures in the course for which these notes were prepared.

Chapter 6

Some compact Lie groups and their representations

In the previous two chapters we got acquainted with the representation theory of *finite* groups. In this chapter we will treat some of the representation theory of *compact Lie groups*, without going into the details of defining a Lie group and showing, for example, the existence of an invariant measure. A good reference, on which in fact most of this chapter is based, is [1]. Some general, less detailed remarks can be found in [8].

6.1 Some examples of Lie groups

One can think of a Lie group G as a smooth surface embedded in some Euclidean space. Moreover, G carries the structure of a group in such a way that the multiplication map $(g,h) \mapsto g \cdot h, G \times G \to G$ and the inverse map $g \mapsto g^{-1}, G \to G$ are smooth. Examples, some of which we already saw, are:

1. $\mathrm{GL}(n,\mathbb{R})$,
2. $\mathrm{SL}(n,\mathbb{R})$,
3. $\mathrm{O}(n,\mathbb{R})$,
4. $\mathrm{SO}(n,\mathbb{R})$,
5. $\mathrm{U}(n) := \{g \in M_n(\mathbb{C}) \mid g^* g = I\}$, where g^* stands for the conjugate transpose of g, i.e., $g^* = (\overline{g}_{ji})_{ij}$ if $g = (g_{ij})_{ij}$.
6. $\mathrm{SU}(n) := \{g \in \mathrm{U}(n) \mid \det(g) = 1\}$.

The latter two groups are called the *unitary group* and the *special unitary group*. A more intrinsic definition is the following: let $(.,.)$ be a Hermitian inner product on a finite-dimensional complex vector space V. Then the corresponding unitary group is the group of all linear maps on V with $(gv, gw) = (v, w)$ for all $g \in G$ and $v, w \in V$. Note that each of the real (complex) matrix groups above is the zero set of some smooth map F from $M_m(\mathbb{R})$ ($M_m(\mathbb{C})$) to some Euclidean space $\mathbb{R}^d$ such that the rank of the Jacobian of F at every point of the zero set of F is d. If, in this situation, the zero set of F is a group with respect to matrix multiplication—as it is in the above cases—then it is automatically a Lie group. Such Lie groups are called *linear* because they are given by a linear representation. In the present chapter, this class of Lie groups suffices for our needs.

Exercise 6.1.1. How can $\mathrm{GL}(n, \mathbb{R})$ be constructed from $M_{n+1}(\mathbb{R})$ via the above construction? What about $(\mathbb{R}, +)$?

Exercise 6.1.2. Show that $\mathrm{O}(n, \mathbb{R}) \cong \mathrm{SO}(n, \mathbb{R}) \rtimes (\mathbb{Z}/2\mathbb{Z})$. Here $\rtimes$ means that the subgroup at the left is a normal subgroup and the subgroup at the right is a complementary subgroup in the sense that their product is the whole group and their intersection is the trivial group. Such a product is called *semi-direct.* If n is odd, then it is a direct product.

Exercise 6.1.3. Show that, in $\mathrm{SU}(2)$, any element is conjugate to a diagonal matrix of the form

$$e(t) := \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \text{ for certain } t \in \mathbb{R}.$$

The theory of Lie groups is a blend of group theory and topology (surfaces, continuity, differentiability, etc.), as might be expected. For example, one can ask whether a given Lie group is (path-wise) connected, or whether it is compact. A typical lemma from the theory is the statement that the connected component of the identity element is a normal subgroup.

Exercise 6.1.4. Show that $\mathrm{O}(n, \mathbb{R})$ is not connected.

Proposition 6.1.5. *The Lie group* $\mathrm{SO}(n, \mathbb{R})$ *is connected.*

Proof. We will connect an arbitrary matrix $g = (a_{ij})_{ij}$ in $\mathrm{SO}(n, \mathbb{R})$ to the identity via a continuous and piecewise smooth path in $\mathrm{SO}(n, \mathbb{R})$. The first step is to connect it to a diagonal matrix; to this end, suppose first that there exists a position (i, j) such that $i > j$ and $a_{ij} \neq 0$. Then one can choose this position such that, in addition, $a_{i'j'} = 0$ for all (i', j') with $i' > j'$ and either $j' < j$ or $j' = j$ and $i' > i$. Schematically, g has the following form:

$$g = \begin{pmatrix} T & * & * & * & * \\ 0 & a_{jj} & * & a_{ji} & * \\ 0 & * & * & * & * \\ 0 & a_{ij} & * & a_{ii} & * \\ 0 & 0 & * & * & * \end{pmatrix},$$

where T is a $(j-1) \times (j-1)$ upper triangular matrix. Now let $r(t),\ t \in \mathbb{R}$ be the real $n \times n$ matrix given by

$$r(t)_{i'j'} = \begin{cases} \cos(t), \text{ if } i' = j' = j \text{ or } i' = j' = i, \\ \sin(t), \text{ if } i' = i \text{ and } j' = j, \\ -\sin(t), \text{ if } i' = j \text{ and } j' = i, \\ 1, \text{ if } i \neq i' = j' \neq j, \text{ and} \\ 0 \text{ otherwise.} \end{cases}$$

Then $r(t)$ has the following block form:

$$r(t) = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & -\sin(t) & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & \sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix};$$

note that $r(t) \in \mathrm{SO}(n, \mathbb{R})$ for all $t \in \mathbb{R}$. Now we have

$$c(t) := r(t)g = \begin{pmatrix} T & * & * & * & * \\ 0 & a_{jj}\cos(t) - a_{ij}\sin(t) & * & a_{ji}\cos(t) - a_{ii}\sin(t) & * \\ 0 & * & * & * & * \\ 0 & a_{jj}\sin(t) + a_{ij}\cos(t) & * & a_{ji}\sin(t) + a_{ii}\cos(t) & * \\ 0 & 0 & * & * & * \end{pmatrix}$$

Now $\mathbb{R}(\sin(t), \cos(t))$ runs through all 1-dimensional subspaces of $\mathbb{R}^2$ as t runs from 0 to π, so that there exists a $t_0 \in [0, \pi)$ for which $(\sin(t_0), \cos(t_0))$ is perpendicular to (a_{jj}, a_{ij}) in the standard inner product; we find that $c(t_0)_{ij} = 0$, in addition to $c(t_0)_{i'j'} = 0$ for all (i', j') with $i' > j'$ and either $j' < j$ or $j' = j$ and $i' > i$. Moreover, the path $c(t),\ t \in [0, t_0]$ lies in $\mathrm{SO}(n, \mathbb{R})$ and connects g to $c(t_0)$.

Repeating the above construction, we find a path connecting g to an upper triangular matrix h. Being an element of $\mathrm{O}(n, \mathbb{R})$, h is in fact diagonal and has only entries ± 1 on the diagonal. Moreover, as $\det(h) = 1$, the number of -1s is even, and they can be partitioned into pairs. Each such pair can be smoothly transformed to 1s by multiplication from the left with an appropriate $r(t),\ t \in [0, \pi]$ as above. □

This proposition implies that $\mathrm{SO}(n, \mathbb{R})$ is the connected component of the identity in $\mathrm{O}(n, \mathbb{R})$. The following exercise will be used later on, for the representation theory of $\mathrm{SO}(3, \mathbb{R})$.

Exercise 6.1.6. Show that $\mathrm{SU}(2)$ is connected. Hint: find a continuous path from any given element of $\mathrm{SU}(2)$ to some $e(t)$ as in Exercise 6.1.3.

6.2 Representation theory of compact Lie groups

Compact Lie groups (i.e., with the property that their embedding in Euclidean space is closed and bounded) are particularly nice, because much of the representation theory of finite groups works just as well for these. The main tool is the so-called *Haar measure*, which enables us to take 'the average over a group' just as we did in finite group theory. Let us state the existence and uniqueness of this measure in a theorem.

Theorem 6.2.1. *Let G be a compact real Lie group, and denote by $C(G)$ the linear space of all continuous functions $G \to \mathbb{R}$ on G. There is a unique map $f \mapsto \int_G f(g)\mathrm{d}g, C(G) \to \mathbb{R}$ which is*

1. linear*, i.e., for all $e, f \in C(G)$ and $\alpha \in \mathbb{R}$, we have*

$$\int_G (e+f)(g)\mathrm{d}g = \int_G f(g)\mathrm{d}g + \int_G e(g)\mathrm{d}g \text{ and } \int_G (\alpha f)(g)\mathrm{d}g = \alpha \int_G f(g)\mathrm{d}g,$$

2. monotonous*, i.e., if $e(g) \leq f(g)$ for all $g \in G$, then also*

$$\int_G e(g)\mathrm{d}g \leq \int_G f(g)\mathrm{d}g,$$

3. left-invariant*, i.e., for the action of G induced on $C(G)$ by the left multiplication, we have for all $h \in G$ and $f \in C(G)$ that*

$$\int_G h \cdot f(g)\mathrm{d}g = \int_G f(g)\mathrm{d}g,$$

where $h \cdot f$ is the function $g \mapsto f(h^{-1}g)$ $(h \in G)$.

4. *and* normalized*, i.e.,*

$$\int_G 1\mathrm{d}g = 1.$$

*This linear map is called the invariant (*Haar*)-integral.*

Exercise 6.2.2. A finite group is a compact Lie group. What is the invariant integral?

In order to find an invariant Haar integral on the compact Lie group $T = \{e(t) \mid t \in \mathbb{R}\}$, where $e(t)$ is defined as in Exercise 6.1.3, we search for invariant *measures* on $e(t)$, that is, differential forms invariant under left multiplication. In general, given a representation $r : G \to \mathrm{GL}(V)$, we can make invariant measures on G by taking certain entries of the matrix $r(g)^{-1}\mathrm{d}r(g)$. For the group T and the natural representation, this is $e(t)^{-1}\mathrm{d}e(t)$. The $1,1$ entry of this matrix is $e^{-it}\mathrm{d}e^{it} = i\mathrm{d}t$. Since $\int_0^{2\pi} i\mathrm{d}t = 2\pi i$, the invariant Haar-integral, viewed as a map assigning a real number to a function $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$, is

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t.$$

A similar but more laborious computation yields that the invariant Haar measure for $SU_2(\mathbb{C})$ is

$$\frac{1}{2\pi^2}\sin^2\theta\sin\psi \mathrm{d}\theta\mathrm{d}\psi\mathrm{d}\phi,$$

where

$$\begin{pmatrix} \cos\theta - i\sin\theta\cos\psi & -\sin\theta\sin\psi e^{i\phi} \\ \sin\theta\sin\psi e^{-i\phi} & \cos\theta + i\sin\theta\cos\psi \end{pmatrix}$$

is a general element of $SU_2(\mathbb{C})$.

The result of the following exercise will be used in proving that certain irreducible representations of $\mathrm{SU}_2(\mathbb{C})$ exhaust all irreducible representations.

Exercise 6.2.3. Show that the map sending a class function $f : \mathrm{SU}(2) \to \mathbb{C}$ to $(f \circ e) : \mathbb{R} \to \mathbb{C}$, where e is the one-parameter subgroup of $\mathrm{SU}(2)$ defined in Exercise 6.1.3, is a linear bijection between the space $\mathcal{H}$ of continuous class functions on $\mathrm{SU}(2)$ and the linear space of all *even*, 2π-periodic continuous functions $\mathbb{R} \to \mathbb{C}$. Moreover, the norm corresponding to the invariant inner product (6.1) below corresponds, under this linear bijection, to the normalized L_2 inner product on the space of continuous even 2π-periodic functions on $\mathbb{R}$.

One could define representations of compact Lie groups in Hilbert spaces, but here we shall restrict our attention to finite-dimensional representations. Also, we shall only consider complex representations.

Definition 6.2.4. A finite-dimensional complex representation of the Lie group G is a smooth homomorphism $G \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space.

In fact, one can prove that continuous homomorphisms from a Lie group G are automatically smooth.

For finite-dimensional complex representations of compact Lie groups, almost all theorems that we proved in the preceding chapters for finite groups hold, be it that at some places a sum must be replaced by an integral. To give an idea how things work, one can try the following exercise.

Exercise 6.2.5. Let $r : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of the compact Lie group G, and let $W \subset V$ be an invariant subspace. Show that W has an invariant complement in V.

We will study the representations of the groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3, \mathbb{R})$, so it is good to verify that these are indeed compact Lie groups.

Exercise 6.2.6. In this exercise we prove that $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{O}(n, \mathbb{R})$ and $\mathrm{SO}(n, \mathbb{R})$ are compact Lie groups. To this end, endow the space $M_n(\mathbb{C})$ with the Hermitian inner product given by

$$((a_{ij})_{ij}, (b_{ij})_{ij}) := \sum_{i,j} a_{ij}\overline{b_{ij}}.$$

1. Show that $(a, b) = \mathrm{tr}(ab^*)$ for all $a, b \in M_n(\mathbb{C})$.

2. Show that $\mathrm{U}(n)$ is bounded with respect to the norm corresponding to this inner product.
3. Show that $\mathrm{U}(n)$ is the zero set of a smooth map $F : M_n(\mathbb{C}) \to \mathbb{R}^{n^2}$, whose Jacobian has rank n^2 at each element of $U(n)$.

Thus, $\mathrm{U}(n)$ is a Lie group by the construction of the beginning of this chapter, and compact as it is a closed and bounded set in $M_n(\mathbb{C})$. Find similar arguments for the other groups above. Show, on the contrary, that the group $\mathrm{O}(n, \mathbb{C})$ of complex matrices g satisfying $g^T g = I$is not compact for $n \geq 2$.

A final remark concerns the compact analogon of the inner product on the space of complex-valued functions on a finite group: in the case of a compact Lie group, one should consider the space of Hilbert space of continuous class functions with Hermitian inner product defined by

$$(f|h) := \int_G f(g)\overline{h(g)}dg. \tag{6.1}$$

Here, the irreducible characters form an orthonormal Hilbert basis in the space of continuous class functions.

6.3 The irreducible representations of $\mathrm{SU}(2)$

This section is based on section II.5 of [1].

Let $r : \mathrm{SU}(2) \to \mathrm{GL}(V)$ be the standard representation of $\mathrm{SU}(2)$ on $V = \mathbb{C}^2$, the action being given by matrix-column multiplication (the elements of V are understood to be columns). As we have seen before, this action induces linear actions on $P(V)_d$, the space of homogeneous polynomials of degree d defined on V, by letting

$$(gP)(\xi) = P(g^{-1}\xi) \text{ for } \xi \in V, P \in P(V)_d, g \in \mathrm{SU}(2).$$

Let $r_d : \mathrm{SU}(2) \to \mathrm{GL}(P(V)_d)$ be the corresponding representation and χ_d its character.

We are ready to state the main theorem of this section.

Theorem 6.3.1. *The representations r_d are irreducible for all $d = 1, 2, \ldots$.*

We will prove this theorem using the result of the following exercise.

Exercise 6.3.2. Let $r : G \to \mathrm{GL}(V)$ be a representation of a group G. For r to be irreducible, it suffices that any linear map $A : V \to V$ that commutes with all $r(g), g \in G$, is a scalar.

Proof of theorem 6.3.1. Let $A \in \mathrm{End}(P(V)_d)$ be a linear map that commutes with all $r_d(g)$ for $g \in \mathrm{SU}(2)$.

The space $P(V)_d$ is spanned by the polynomials $P_k : \xi \mapsto \xi_1^k \xi_2^{d-k}$ for $k = 0, \ldots, d$, and is therefore $(d+1)$-dimensional.

Consider the diagonal subgroup of SU(2). It consists of all matrices of the form

$$g_a := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

Note that the element $a \in \mathbb{C}$ must necessarily have norm 1. We compute

$$\begin{aligned}(g_a P_k)(\xi) &= P_k(g_{\frac{1}{a}}\xi) \\ &= (\xi_1/a)^k (a\xi_2)^{d-k} \\ &= a^{d-2k} P_k(\xi).\end{aligned}$$

Hence we conclude that P_k is an eigenfunction of $r_d(g_a)$ with eigenvalue a^{d-2k}. Choose a_0 such that none of these eigenvalues coincide. If A commutes with $r_d(g_{a_0})$, then the eigenspaces of the latter map are invariant under A. Hence, as these eigenspaces are all one-dimensional, there are complex numbers c_k for $k = 0, \dots, d$ such that

$$AP_k = c_k P_k.$$

It remains to show that all c_k are equal. For this purpose, consider the real rotations s_t defined by

$$s_t := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and compute

$$\begin{aligned}(s_t P_n)(\xi) &= P_n(s_{-t}\xi) \\ &= (\xi_1 \cos t + \xi_2 \sin t)^n \\ &= \sum_k \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot P_k(\xi)\end{aligned}$$

Applying A to the result, we get

$$A(s_t P_n) = \sum_k \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot c_k P_k.$$

First applying A and then $r_d(s_t)$, and using the fact that the two commute, we find that also

$$A(s_t P_n) = \sum_k \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot c_n P_k.$$

Taking t such that $\cos t \sin t \neq 0$, it follows from the fact that the P_k are linearly independent that $c_n = c_k$ for all $k = 1, \dots, n$, and hence that $A = c_n I$. □

We wish to prove that these are all irreducible characters. To this end we shall use some Fourier theory, and the result of Exercise 6.2.3.

Theorem 6.3.3. *The representations r_d are, up to equivalence, all irreducible representations of the group* SU(2).

Proof. In view of Exercise 6.2.3, we can identify $\mathcal{H}$ with the space of even 2π-periodic complex-valued functions on $\mathbb{R}$. The matrix of $r_d(e(t))$ with respect to the basis $P_0, \ldots, P_d$ is diagonal, and its trace is

$$\kappa_d(t) = \sum_{k=0}^{d} e^{i(d-2k)t}.$$

It is readily verified that $\kappa_0(t) = 1$ (this corresponds to the trivial representation of the group), and that $\kappa_1(t) = 2\cos t$. Indeed, one has $\kappa_{d+2}(t) = \kappa_d(t) + 2\cos((d+2)t)$. Hence, the functions κ_d for $d = 0, 1, 2, \ldots$ generate the same space as the functions $\cos dt$. From Fourier theory we know that this subspace is dense in the Hilbert space of continuous even 2π-periodic functions. This implies, using the isomorphism from Exercise 6.2.3, that the characters χ_d span a dense subspace in the Hilbert space of continuous class functions on SU(2). Any irreducible character different from all r_d would have inner product 0 with all r_d, and hence be zero. This concludes the proof that the r_d exhaust the set of irreducible characters of SU(2). □

6.4 The irreducible representations of $\mathrm{SO}(3, \mathbb{R})$

We will obtain the irreducible representations of $\mathrm{SO}(3,\mathbb{R})$ from those of SU(2) by means of the following lemma, which appeared earlier as exercise 5.3.11.

Lemma 6.4.1. *Let G and H be groups, $\phi : G \to H$ a surjective homomorphism. Then the (irreducible) representations of H are in bijective correspondence with the (irreducible) representations r of G with* $\ker r \supseteq \ker \phi$.

Thus, we wish to find a surjective homomorphism of SU(2) onto SO(3) and to determine its kernel. Recall the homomorphism $\phi : \mathrm{SL}(2,\mathbb{C}) \to L$ of Example 2.2.9. Here, L is the group of Lorentz-norm preserving linear maps from M to itself, where M is the $\mathbb{R}$-linear span of the following four complex 2×2-matrices.

$$e_0 = I, e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It can be checked that in this way

$$\det(x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

So L is the group

$$\{B \in \mathrm{GL}(M) \mid \det(Bx) = \det(x), \text{ for all } x \in M\}.$$

The homomorphism ϕ was defined by

$$\phi(A)(x) = AxA^*, \text{ for all } x \in M, A \in \mathrm{SL}(2,\mathbb{C}).$$

From now on, ϕ will denote the restriction of this homomorphism to SU(2). We claim that the image is the group

$$G := \{B \in L \mid Be_0 = e_0 \text{ and } \det B = 1\},$$

which is, by its action on the invariant 3-space $e_0^\perp$, easily seen to be isomorphic to $\mathrm{SO}(3,\mathbb{R})$.

First of all $\phi(\mathrm{SU}(2)) \subseteq G$. Indeed, if $A \in \mathrm{SU}(2)$, then $A^* = A^{-1}$. Therefore, $\phi(A)(e_0) = e_0$, since e_o is just the 2×2 identity matrix. Also, the function $\det \circ \phi$ is a continuous function $\mathrm{SU}(2) \to \mathbb{R}$, which can apparently take only values in $\{-1, 1\}$. As SU(2) is connected, and $\det(\phi(I)) = 1$, the determinant of any element in the image $\phi(\mathrm{SU}(2))$ is 1.

Now let us prove that $G \subseteq \phi(\mathrm{SU}(2))$. We follow the approach given in [3]. First we prove the following two lemmas.

Lemma 6.4.2. *Let F be a subgroup of* $\mathrm{SO}(3,\mathbb{R})$ *with the following two properties.*

1. *F acts transitively on the unit sphere.*
2. *There is an axis such that F contains all rotations around that axis.*

Then $F = \mathrm{SO}(3,\mathbb{R})$.

Exercise 6.4.3. Prove this lemma.

Lemma 6.4.4. *For every non-zero $x \in e_0^\perp$ there is an $A \in \mathrm{SU}(2)$ such that $\phi(A)x = ce_3$ for some $c \in \mathbb{R}$ and $c > 0$. Here $e_0^\perp$ denotes the orthogonal complement of e_0 with respect to the Lorentz inner product, i.e.,*

$$e_0^\perp = \{x_1 e_1 + x_2 e_2 + x_3 e_3 \mid x_i \in \mathbb{R}\} \subseteq M.$$

Proof. Any such x is in fact a Hermitian matrix with trace 0. As such it has two real eigenvalues c, $-c$, where $c > 0$, (note that x is non-zero). Let two (necessarily orthogonal) eigencolumns of norm 1 corresponding to c and $-c$ be the first and second column of a matrix B, respectively. Then the usual coordinate transformation rule gives

$$x = B^{-1}(cH)B.$$

We have that $B \in \mathrm{U}(2)$. Now take $A = dB$ where $d^2 = 1/(\det B)$; then A is still unitary, since $|d| = 1$. Furthermore, we have

$$AxA^{-1} = (dB)x(dB)^{-1} = dd^{-1}BxB^{-1} = cH,$$

and $\det A = d^2 \det B = 1$. □

Now let us continue with the proof that $F := \phi(\mathrm{SU}(2))$ coincides with G. The group F acts on the unit sphere S in $e_0^\perp$ defined by $x_1^2 + x_2^2 + x_3^2$, and by Lemma 6.4.4 this action is transitive, since any vector in the sphere is mapped to e_3 by some group element. Hence, to apply Lemma 6.4.2 to F, we need only check that F contains all rotations around some axis.

We leave this as an exercise to the reader:

Exercise 6.4.5. Show that $\phi(e(t))$ is the rotation around $\langle e_3 \rangle$ through an angle of $2t$. Here $e : \mathbb{R} \to \mathrm{SU}(2)$ is the one-parameter group defined earlier.

We conclude that $\phi : \mathrm{SU}(2) \to G$ is a surjective homomorphism. Its kernel contains $\{I, -I\}$, as is easily checked, and the following exercise shows that this is the whole kernel.

Exercise 6.4.6. Show that $\ker \phi = \{I, -I\}$. Hint: the elements of this kernel must commute with all of M, hence in particular with e_1, e_2, and e_3.

Finally we can apply lemma 6.4.1: the irreducible representations of SO(3) correspond to those of SU(2) for which $-I$ is mapped to the identity transformation. More precisely, we wish to know for which d we have $r_d(-I) = I$. For this to be the case, it is necessary and sufficient that

$$((-I)P)(\xi) = P(\xi) \text{ for all } \xi \in V, P \in P(V)_d.$$

This holds if and only if d is even. Hence the irreducible representations are

$$W_d := P(V)_{2d}.$$

Note that the dimension of W_d is $2d + 1$.

In the following example, we will see another interpretation of the irreducible representations of $\mathrm{SO}(3, \mathbb{R})$.

Example 6.4.7. Consider $P(V)_d$ the space of homogeneous polynomials of degree d defined on the vector space V which will be of dimension three here. As we have seen before, the group $\mathrm{GL}(3, \mathbb{R})$ acts on this space $P(V)_d$ by $(A \cdot p)(x) = p(A^{-1}x)$, for all $A \in \mathrm{GL}(3, \mathbb{R})$, $p \in P(V)_d$ and $x \in \mathbb{R}^3$. So does the subgroup SO(3).

Take the subspace U of $P(V)_2$ generated by the polynomial $x_1^2 + x_2^2 + x_3^2$. This is an invariant subspace for the action of SO(3). Hence $P(V)_2$ is not irreducible. Actually the spaces $P(V)_d$ are not irreducible for all $d \geq 2$.

To find SO(3)- invariant subspaces of $P(V)_d$, we need to define the *Laplace* operator Δ on $\mathbb{R}^3$: $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. The space of harmonic polynomials $\mathfrak{h}_d$ is the vector space:

$$\mathfrak{h}_d = \{p \in P(V)_d \mid \Delta p = 0\}.$$

As was noticed in Example 4.1.17, the dimension of $P(V)_d$ is $\dim P(V)_d = \binom{d+2}{d}$.

We now want to prove that the dimension of $\mathfrak{h}_d$ is $2d + 1$. A homogeneous polynomial p of degree d can be written as a sum of homogeneous polynomials p_k in x_2 and x_3 with power of x_1's as coefficients:

$$\begin{aligned} p(x_1, x_2, x_3) &= p_0(x_2, x_3) + x_1 p_1(x_2, x_3) + \frac{x_1^2}{2} p_2(x_2, x_3) + \cdots + \frac{x_1^d}{d!} p_d(x_2, x_3) \\ &= \sum_{k=0}^{d} \frac{x_1^k}{k!} p_k(x_2, x_3) \end{aligned}$$

where the p_k's have degree $d-k$. If one applies the Laplace operator to p (you see why we took factorials):

$$\Delta(p) = \sum_{k=0}^{d-2} \frac{x_1^k}{k!} p_{k+2}(x_2, x_3) + \sum_{k=0}^{d} \frac{x_1^k}{k!} \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) p_k(x_2, x_3)$$

Hence an element $p \in P(V)_d$ is in $\mathfrak{h}_d$ if and only if for all $k = 0, \ldots, d-2$:

$$p_{k+2}(x_2, x_3) = -\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) p_k(x_2, x_3)$$

This formula implies that p in $\mathfrak{h}_d$ depends only on the first two homogeneous polynomials p_0 and p_1. Therefore we conclude that the dimension of $\mathfrak{h}_d$ is precisely the number of homogeneous polynomials in two variables of degree d and $d-1$. That is, $\dim \mathfrak{h}_d = 2d+1$.

Of course, the degree of $\mathfrak{h}_d$ reminds us of the irreducible representations W_d of SO(3) we met before. Indeed, we have the following.

Theorem 6.4.8. *The space $\mathfrak{h}_d$ of harmonic polynomials of degree d is an irreducible representation of* SO(3)*.*

To see that what we just stated is true, we should first check that $\mathfrak{h}_d$ is a representation of SO(3). We state here the following fact without proof.

Fact: The action of the Laplace operator on $P(V)_d$ commutes with the action of SO(3).

From this we conclude that $\mathfrak{h}_d$ is a representation of SO(3).

To prove that $\mathfrak{h}_d$ is irreducible, we claim that $\mathfrak{h}_d = W_d$. Set p a harmonic polynomial in $\mathfrak{h}_d$, $p(x_1, x_2, x_3) = (x_2 + ix_3)^d$. Let $R(t)$ be the matrix of SO(3) defined by

$$R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

Apply $R(t)$ to p. That gives us $R(t)p = \exp(-idt)p$. So p generates an invariant subspace W of $\mathfrak{h}_d$ under the action of $R(t)$.

Any element in SO(3) is conjugate to $R(t)$. So in order to find the character of W_d on $R(t)$ it suffices to compute it on $R(t)$. That is just the value of the character χ_{2d} of r_d (the irreducible representation of SU(2)) at $e^{t/2}$, i.e., $\sum_{k=0}^{2d} e^{i(d-k)t}$.

By the decomposition of $\mathfrak{h}_d$ in irreducible W_s's, we know that its character is a linear combination of $\exp(ist)$. So what we saw above about the action of $R(t)$ on W leads us to conclude that $\mathfrak{h}_d = W_d$.

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