

# Clebsch–Gordan coefficients

In physics, the **Clebsch–Gordan (CG) coefficients** are numbers that arise in angular momentum coupling in quantum mechanics. They appear as the expansion coefficients of total angular momentum eigenstates in an uncoupled tensor product basis. In more mathematical terms, the CG coefficients are used in representation theory, particularly of compact Lie groups, to perform the explicit direct sum decomposition of the tensor product of two irreducible representations (i.e., a reducible representation into irreducible representations, in cases where the numbers and types of irreducible components are already known abstractly). The name derives from the German mathematicians Alfred Clebsch and Paul Gordan, who encountered an equivalent problem in invariant theory.

From a vector calculus perspective, the CG coefficients associated with the SO(3) group can be defined simply in terms of integrals of products of spherical harmonics and their complex conjugates. The addition of spins in quantum-mechanical terms can be read directly from this approach as spherical harmonics are eigenfunctions of total angular momentum and projection thereof onto an axis, and the integrals correspond to the Hilbert space inner product.<sup>[1]</sup> From the formal definition of angular momentum, recursion relations for the Clebsch–Gordan coefficients can be found. There also exist complicated explicit formulas for their direct calculation.<sup>[2]</sup>

The formulas below use Dirac's bra–ket notation and the Condon–Shortley phase convention<sup>[3]</sup> is adopted.

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## Angular momentum operators

Angular momentum operators are self-adjoint operators  $\mathbf{j}_x$ ,  $\mathbf{j}_y$ , and  $\mathbf{j}_z$  that satisfy the commutation relations

$$[\mathbf{j}_k,\mathbf{j}_l]\equiv \mathbf{j}_k\mathbf{j}_l-\mathbf{j}_l\mathbf{j}_k=i\hbar\epsilon_{klm}\mathbf{j}_m\qquad k,l,m\in\{\mathbf{x},\mathbf{y},\mathbf{z}\},$$

where  $\epsilon_{klm}$  is the Levi-Civita symbol. Together the three operators define a *vector operator*, a rank one Cartesian tensor operator,

$$\mathbf{j}=(\mathbf{j}_x,\mathbf{j}_y,\mathbf{j}_z).$$

It also known as a spherical vector, since it is also a spherical tensor operator. It is only for rank one that spherical tensor operators coincide with the Cartesian tensor operators.

By developing this concept further, one can define another operator  $\mathbf{j}^2$  as the inner product of  $\mathbf{j}$  with itself:

$$\mathbf{j}^2=\mathbf{j}_x^2+\mathbf{j}_y^2+\mathbf{j}_z^2.$$

This is an example of a Casimir operator. It is diagonal and its eigenvalue characterizes the particular irreducible representation of the angular momentum algebra  $\mathfrak{so}(3)\cong\mathfrak{su}(2)$ . This is physically interpreted as the square of the total angular momentum of the states on which the representation acts.

One can also define *raising* ( $\mathbf{j}_+$ ) and *lowering* ( $\mathbf{j}_-$ ) operators, the so-called ladder operators,

$$\mathbf{j}_{\pm}=\mathbf{j}_x\pm i\mathbf{j}_y.$$

## Spherical basis for angular momentum eigenstates

It can be shown from the above definitions that  $\mathbf{j}^2$  commutes with  $j_x$ ,  $j_y$ , and  $j_z$ :

$$[\mathbf{j}^2, j_k] = 0 \quad k \in \{x, y, z\}.$$

When two Hermitian operators commute, a common set of eigenstates exists. Conventionally,  $\mathbf{j}^2$  and  $j_z$  are chosen. From the commutation relations, the possible eigenvalues can be found. These eigenstates are denoted  $|j\ m\rangle$  where  $j$  is the *angular momentum quantum number* and  $m$  is the *angular momentum projection* onto the z-axis.

They comprise the spherical basis, are complete, and satisfy the following eigenvalue equations,

$$\begin{aligned} \mathbf{j}^2 |j\ m\rangle &= \hbar^2 j(j+1) |j\ m\rangle, & j &\in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\} \\ j_z |j\ m\rangle &= \hbar m |j\ m\rangle, & m &\in \{-j, -j+1, \dots, j\}. \end{aligned}$$

The raising and lowering operators can be used to alter the value of  $m$ ,

$$j_{\pm} |j\ m\rangle = \hbar C_{\pm}(j, m) |j\ (m \pm 1)\rangle,$$

where the ladder coefficient is given by:

$$C_{\pm}(j, m) = \sqrt{j(j+1) - m(m \pm 1)} = \sqrt{(j \mp m)(j \pm m + 1)}. \quad (1)$$

In principle, one may also introduce a (possibly complex) phase factor in the definition of  $C_{\pm}(j, m)$ . The choice made in this article is in agreement with the Condon–Shortley phase convention. The angular momentum states are orthogonal (because their eigenvalues with respect to a Hermitian operator are distinct) and are assumed to be normalized,

$$\langle j\ m | j'\ m' \rangle = \delta_{j,j'} \delta_{m,m'}.$$

Here the italicized  $j$  and  $m$  denote integer or half-integer angular momentum quantum numbers of a particle or of a system. On the other hand, the roman  $j_x$ ,  $j_y$ ,  $j_z$ ,  $j_+$ ,  $j_-$ , and  $\mathbf{j}^2$  denote operators. The  $\delta$  symbols are Kronecker deltas.

## Tensor product space

We now consider systems with two physically different angular momenta  $j_1$  and  $j_2$ . Examples include the spin and the orbital angular momentum of a single electron, or the spins of two electrons, or the orbital angular momenta of two electrons. Mathematically, this means that the angular momentum operators act on a space  $\mathbf{V}_1$  of dimension  $2j_1 + 1$  and also on a space  $\mathbf{V}_2$  of dimension  $2j_2 + 1$ . We are then going to define a family of "total angular momentum" operators acting on the tensor product space  $\mathbf{V}_1 \otimes \mathbf{V}_2$ , which has dimension  $(2j_1 + 1)(2j_2 + 1)$ . The action of the total angular momentum operator on this space constitutes a representation of the  $\mathfrak{su}(2)$  Lie algebra, but a reducible one. The reduction of this reducible representation into irreducible pieces is the goal of Clebsch–Gordan theory.

Let  $V_1$  be the  $(2j_1 + 1)$ -dimensional vector space spanned by the states

$$|j_1\ m_1\rangle, \quad m_1 \in \{-j_1, -j_1 + 1, \dots, j_1\},$$

and  $V_2$  the  $(2j_2 + 1)$ -dimensional vector space spanned by the states

$$|j_2\ m_2\rangle, \quad m_2 \in \{-j_2, -j_2 + 1, \dots, j_2\}.$$

The tensor product of these spaces,  $V_3 \equiv V_1 \otimes V_2$ , has a  $(2j_1 + 1)(2j_2 + 1)$ -dimensional *uncoupled* basis

$$|j_1\ m_1\ j_2\ m_2\rangle \equiv |j_1\ m_1\rangle \otimes |j_2\ m_2\rangle, \quad m_1 \in \{-j_1, -j_1 + 1, \dots, j_1\}, \quad m_2 \in \{-j_2, -j_2 + 1, \dots, j_2\}.$$

Angular momentum operators are defined to act on states in  $V_3$  in the following manner:

$$(\mathbf{j} \otimes 1) |j_1\ m_1\ j_2\ m_2\rangle \equiv \mathbf{j} |j_1\ m_1\rangle \otimes |j_2\ m_2\rangle$$

and

$$(1 \otimes \mathbf{j}) |j_1\ m_1\ j_2\ m_2\rangle \equiv |j_1\ m_1\rangle \otimes \mathbf{j} |j_2\ m_2\rangle,$$

where  $1$  denotes the identity operator.

The <sup>[nb 1]</sup>**total angular momentum** operators are defined by the coproduct (or tensor product) of the two representations acting on  $V_1 \otimes V_2$ ,

$$\mathbf{J} \equiv \mathbf{j} \otimes 1 + 1 \otimes \mathbf{j}.$$

The total angular momentum operators can be shown to *satisfy the very same commutation relations*,

$$[J_k, J_l] = i\hbar \epsilon_{klm} J_m,$$

where  $k, l, m \in \{x, y, z\}$ . Indeed, the preceding construction is the standard method<sup>[4]</sup> for constructing an action of a Lie algebra on a tensor product representation.

Hence, a set of *coupled* eigenstates exist for the total angular momentum operator as well,

$$\begin{aligned}\mathbf{J}^2|[j_1 j_2] J M\rangle &= \hbar^2 J(J+1)|[j_1 j_2] J M\rangle \\ \mathbf{J}_z|[j_1 j_2] J M\rangle &= \hbar M|[j_1 j_2] J M\rangle\end{aligned}$$

for  $M \in \{-J, -J+1, \dots, J\}$ . Note that it is common to omit the  $[j_1 j_2]$  part.

The total angular momentum quantum number  $J$  must satisfy the triangular condition that

$$|j_1 - j_2| \leq J \leq j_1 + j_2,$$

such that the three nonnegative integer or half-integer values could correspond to the three sides of a triangle.<sup>[5]</sup>

The total number of total angular momentum eigenstates is necessarily equal to the dimension of  $V_3$ :

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1) .$$

As this computation suggests, the tensor product representation decomposes as the direct sum of one copy of each of the irreducible representations of dimension  $2\mathbf{J} + 1$ , where  $\mathbf{J}$  ranges from  $|j_1 - j_2|$  to  $j_1 + j_2$  in increments of 1.<sup>[6]</sup> As an example, consider the tensor product of the three-dimensional representation corresponding to  $\mathbf{j}_1 = \mathbf{1}$  with the two-dimensional representation with  $\mathbf{j}_2 = \mathbf{1}/2$ . The possible values of  $\mathbf{J}$  are then  $\mathbf{J} = \mathbf{1}/2$  and  $\mathbf{J} = \mathbf{3}/2$ . Thus, the six-dimensional tensor product representation decomposes as the direct sum of a two-dimensional representation and a four-dimensional representation.

The goal is now to describe the preceding decomposition explicitly, that is, to explicitly describe basis elements in the tensor product space for each of the component representations that arise.

The total angular momentum states form an orthonormal basis of  $V_3$ :

$$\langle J M | J' M' \rangle = \delta_{J,J'} \delta_{M,M'} .$$

These rules may be iterated to, e.g., combine  $n$  doublets ( $s=1/2$ ) to obtain the Clebsch–Gordan decomposition series, (Catalan's triangle),

$$\mathbf{2}^{\otimes n} = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \left( \frac{n+1-2k}{n+1} \binom{n+1}{k} \right) (\mathbf{n} + \mathbf{1} - \mathbf{2k}) ,$$

where  $\lfloor n/2 \rfloor$  is the integer floor function; and the number preceding the boldface irreducible representation dimensionality ( $2j+1$ ) label indicates multiplicity of that representation in the representation reduction.<sup>[7]</sup> For instance, from this formula, addition of three spin 1/2s yields a spin 3/2 and two spin 1/2s,  $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}$ .

## Formal definition of Clebsch–Gordan coefficients

The coupled states can be expanded via the completeness relation (resolution of identity) in the uncoupled basis

$$|J M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | J M\rangle \tag{2}$$

The expansion coefficients

$$\langle j_1 m_1 j_2 m_2 | J M\rangle$$

are the *Clebsch–Gordan coefficients*. Note that some authors write them in a different order such as  $\langle j_1 j_2; m_1 m_2 | J M\rangle$ . Another common notation is

$$\langle j_1 m_1 j_2 m_2 | J M\rangle = C_{j_1 m_1 j_2 m_2}^{JM}.$$

Applying the operators

$$\begin{aligned}\mathbf{J} &= \mathbf{j} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{j} \\ \mathbf{J}_z &= \mathbf{j}_z \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{j}_z\end{aligned}$$

to both sides of the defining equation shows that the Clebsch–Gordan coefficients can only be nonzero when

$$\begin{aligned}|j_1 - j_2| &\leq J \leq j_1 + j_2 \\ M &= m_1 + m_2\end{aligned} .$$

## Recursion relations

The recursion relations were discovered by physicist Giulio Racah from the Hebrew University of Jerusalem in 1941.

Applying the total angular momentum raising and lowering operators

$$\mathbf{J}_{\pm} = \mathbf{j}_{\pm} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{j}_{\pm}$$

to the left hand side of the defining equation gives

$$\begin{aligned}\mathbf{J}_{\pm} |[j_1 j_2] J M\rangle &= \hbar C_{\pm}(J, M) |[j_1 j_2] J (M \pm 1)\rangle \\ &= \hbar C_{\pm}(J, M) \sum_{m_1, m_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | J (M \pm 1)\rangle\end{aligned}$$

Applying the same operators to the right hand side gives

$$\begin{aligned} J_{\pm} \sum_{m_1, m_2} |j_1 m_1 j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | J M\rangle \\ = \hbar \sum_{m_1, m_2} \left( C_{\pm}(j_1, m_1) |j_1 (m_1 \pm 1) j_2 m_2\rangle + C_{\pm}(j_2, m_2) |j_1 m_1 j_2 (m_2 \pm 1)\rangle \right) \langle j_1 m_1 j_2 m_2 | J M\rangle \\ = \hbar \sum_{m_1, m_2} |j_1 m_1 j_2 m_2\rangle \left( C_{\pm}(j_1, m_1 \mp 1) \langle j_1 (m_1 \mp 1) j_2 m_2 | J M\rangle + C_{\pm}(j_2, m_2 \mp 1) \langle j_1 m_1 j_2 (m_2 \mp 1) | J M\rangle \right). \end{aligned}$$

where  $C_{\pm}$  was defined in [1](#). Combining these results gives recursion relations for the Clebsch–Gordan coefficients:

$$C_{\pm}(J, M) \langle j_1 m_1 j_2 m_2 | J (M \pm 1) \rangle = C_{\pm}(j_1, m_1 \mp 1) \langle j_1 (m_1 \mp 1) j_2 m_2 | J M \rangle + C_{\pm}(j_2, m_2 \mp 1) \langle j_1 m_1 j_2 (m_2 \mp 1) | J M \rangle.$$

Taking the upper sign with the condition that  $M = J$  gives initial recursion relation:

$$0 = C_{+}(j_1, m_1 - 1) \langle j_1 (m_1 - 1) j_2 m_2 | J J \rangle + C_{+}(j_2, m_2 - 1) \langle j_1 m_1 j_2 (m_2 - 1) | J J \rangle.$$

In the Condon–Shortley phase convention, one adds the constraint that

$$\langle j_1 m_1 j_2 m_2 | J M \rangle = (-1)^{m_1} \langle j_1 -m_1 j_2 m_2 | J M \rangle$$

(and is therefore also real).

The Clebsch–Gordan coefficients  $\langle j_1 m_1 j_2 m_2 | J M \rangle$  can then be found from these recursion relations. The normalization is fixed by the requirement that the sum of the squares, which equivalent to the requirement that the norm of the state  $|[j_1 j_2] J J \rangle$  must be one.

The lower sign in the recursion relation can be used to find all the Clebsch–Gordan coefficients with  $M = J - 1$ . Repeated use of that equation gives all coefficients.

This procedure to find the Clebsch–Gordan coefficients shows that they are all real in the Condon–Shortley phase convention.

## Explicit expression

## Orthogonality relations

These are most clearly written down by introducing the alternative notation

$$\langle J M | j_1 m_1 j_2 m_2 \rangle \equiv \langle j_1 m_1 j_2 m_2 | J M \rangle$$

The first orthogonality relation is

$$\sum_{j_1+j_2} \sum_{M=-J}^J \langle j_1 m_1 j_2 m_2 | J M \rangle \langle J M | j_1 m'_1 j_2 m'_2 \rangle = \langle j_1 m_1 j_2 m_2 | j_1 m'_1 j_2 m'_2 \rangle = \delta_{m_1, m'_1} \delta_{m_2, m'_2}$$

(derived from the fact that  $1 \equiv \sum_X |X\rangle \langle X|$ ) and the second one is

$$\sum_{m_1, m_2} \langle J M | j_1 m_1 j_2 m_2 \rangle \langle j_1 m_1 j_2 m_2 | J' M' \rangle = \langle J M | J' M' \rangle = \delta_{J, J'} \delta_{M, M'}.$$

## Special cases

For  $J = 0$  the Clebsch–Gordan coefficients are given by

$$\langle j_1 m_1 j_2 m_2 | 0 0 \rangle = \delta_{j_1, j_2} \delta_{m_1, -m_2} \frac{(-1)^{j_1 - m_1}}{\sqrt{2j_1 + 1}}.$$

For  $J = j_1 + j_2$  and  $M = J$  we have

$$\langle j_1 j_1 j_2 j_2 | (j_1 + j_2) (j_1 + j_2) \rangle = 1.$$

For  $j_1 = j_2 = J/2$  and  $m_1 = -m_2$  we have

$$\langle j_1 m_1 j_1 (-m_1) | (2j_1) 0 \rangle = \frac{(2j_1)!^2}{(j_1 - m_1)!(j_1 + m_1)! \sqrt{(4j_1)!}}.$$

For  $j_1 = j_2 = m_1 = -m_2$  we have

$$\langle j_1 j_1 j_1 (-j_1) | J 0 \rangle = (2j_1)! \sqrt{\frac{2J+1}{(J+2j_1+1)!(2j_1-J)!}}.$$

For  $j_2 = 1, m_2 = 0$  we have

$$\begin{aligned}\langle j_1 m 1 0 | (j_1 + 1) m \rangle &= \sqrt{\frac{(j_1 - m + 1)(j_1 + m + 1)}{(2j_1 + 1)(j_1 + 1)}} \\ \langle j_1 m 1 0 | j_1 m \rangle &= \frac{m}{\sqrt{j_1(j_1 + 1)}} \\ \langle j_1 m 1 0 | (j_1 - 1) m \rangle &= -\sqrt{\frac{(j_1 - m)(j_1 + m)}{j_1(2j_1 + 1)}}\end{aligned}$$

For  $j_2 = 1/2$  we have

$$\begin{aligned}\left\langle j_1 \left(M - \frac{1}{2}\right) \frac{1}{2} \frac{1}{2} \left| \left(j_1 \pm \frac{1}{2}\right) M \right\rangle &= \pm \sqrt{\frac{1}{2} \left(1 \pm \frac{M}{j_1 + \frac{1}{2}}\right)} \\ \left\langle j_1 \left(M + \frac{1}{2}\right) \frac{1}{2} \left(-\frac{1}{2}\right) \left| \left(j_1 \pm \frac{1}{2}\right) M \right\rangle &= \sqrt{\frac{1}{2} \left(1 \mp \frac{M}{j_1 + \frac{1}{2}}\right)}\end{aligned}$$

## Symmetry properties

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$$\begin{aligned}\langle j_1 m_1 j_2 m_2 | J M \rangle &= (-1)^{j_1 + j_2 - J} \langle j_1 (-m_1) j_2 (-m_2) | J (-M) \rangle \\ &= (-1)^{j_1 + j_2 - J} \langle j_2 m_2 j_1 m_1 | J M \rangle \\ &= (-1)^{j_1 - m_1} \sqrt{\frac{2J + 1}{2j_2 + 1}} \langle j_1 m_1 J (-M) | j_2 (-m_2) \rangle \\ &= (-1)^{j_2 + m_2} \sqrt{\frac{2J + 1}{2j_1 + 1}} \langle J (-M) j_2 m_2 | j_1 (-m_1) \rangle \\ &= (-1)^{j_1 - m_1} \sqrt{\frac{2J + 1}{2j_2 + 1}} \langle J M j_1 (-m_1) | j_2 m_2 \rangle \\ &= (-1)^{j_2 + m_2} \sqrt{\frac{2J + 1}{2j_1 + 1}} \langle j_2 (-m_2) J M | j_1 m_1 \rangle\end{aligned}$$

A convenient way to derive these relations is by converting the Clebsch–Gordan coefficients to [Wigner 3-j symbols](#) using [3](#). The symmetry properties of Wigner 3-j symbols are much simpler.

## Rules for phase factors

Care is needed when simplifying phase factors: a quantum number may be a half-integer rather than an integer, therefore  $(-1)^{2k}$  is not necessarily 1 for a given quantum number  $k$  unless it can be proven to be an integer. Instead, it is replaced by the following weaker rule:

$$(-1)^{4k} = 1$$

for any angular-momentum-like quantum number  $k$ .

Nonetheless, a combination of  $j_i$  and  $m_i$  is always an integer, so the stronger rule applies for these combinations:

$$(-1)^{2(j_i - m_i)} = 1$$

This identity also holds if the sign of either  $j_i$  or  $m_i$  or both is reversed.

It is useful to observe that any phase factor for a given  $(j_i, m_i)$  pair can be reduced to the canonical form:

$$(-1)^{aj_i + b(j_i - m_i)}$$

where  $a \in \{0, 1, 2, 3\}$  and  $b \in \{0, 1\}$  (other conventions are possible too). Converting phase factors into this form makes it easy to tell whether two phase factors are equivalent. (Note that this form is only *locally* canonical: it fails to take into account the rules that govern combinations of  $(j_i, m_i)$  pairs such as the one described in the next paragraph.)

An additional rule holds for combinations of  $j_1, j_2$ , and  $j_3$  that are related by a Clebsch–Gordan coefficient or Wigner 3-j symbol:

$$(-1)^{2(j_1 + j_2 + j_3)} = 1$$

This identity also holds if the sign of any  $j_i$  is reversed, or if any of them are substituted with an  $m_i$  instead.

## Relation to Wigner 3-j symbols

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Clebsch–Gordan coefficients are related to [Wigner 3-j symbols](#) which have more convenient symmetry relations.

$$\begin{aligned}\langle j_1 m_1 j_2 m_2 | J M \rangle &= (-1)^{-j_1+j_2-M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \\ &= (-1)^{2j_2} (-1)^{J-M} \sqrt{2J+1} \begin{pmatrix} j_1 & J & j_2 \\ m_1 & -M & m_2 \end{pmatrix}\end{aligned}$$

The factor  $(-1)^{2j_2}$  is due to the Condon–Shortley constraint that  $\langle j_1 j_1 j_2 (J-j_1) | J J \rangle > 0$ , while  $(-1)^{J-M}$  is due to the time-reversed nature of  $|J M\rangle$ .

## Relation to Wigner D-matrices

$$\begin{aligned}\int_0^{2\pi} d\alpha \int_0^\pi \sin\beta d\beta \int_0^{2\pi} d\gamma D_{M,K}^J(\alpha, \beta, \gamma)^* D_{m_1,k_1}^{j_1}(\alpha, \beta, \gamma) D_{m_2,k_2}^{j_2}(\alpha, \beta, \gamma) \\ = \frac{8\pi^2}{2J+1} \langle j_1 m_1 j_2 m_2 | J M \rangle \langle j_1 k_1 j_2 k_2 | J K \rangle\end{aligned}$$

## Relation to spherical harmonics

In the case where integers are involved, the coefficients can be related to [integrals of spherical harmonics](#):

$$\int_{4\pi} Y_{\ell_1}^{m_1*}(\Omega) Y_{\ell_2}^{m_2*}(\Omega) Y_L^M(\Omega) d\Omega = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}} \langle \ell_1 0 \ell_2 0 | L 0 \rangle \langle \ell_1 m_1 \ell_2 m_2 | L M \rangle$$

It follows from this and orthonormality of the spherical harmonics that CG coefficients are in fact the expansion coefficients of a product of two spherical harmonics in terms of a single spherical harmonic:

$$Y_{\ell_1}^{m_1}(\Omega) Y_{\ell_2}^{m_2}(\Omega) = \sum_{L,M} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}} \langle \ell_1 0 \ell_2 0 | L 0 \rangle \langle \ell_1 m_1 \ell_2 m_2 | L M \rangle Y_L^M(\Omega)$$

## Other Properties

$$\sum_m (-1)^{j-m} \langle j m j (-m) | J 0 \rangle = \delta_{J,0} \sqrt{2j+1}$$

## SU(*n*) Clebsch–Gordan coefficients

For arbitrary groups and their representations, Clebsch–Gordan coefficients are not known in general. However, algorithms to produce Clebsch–Gordan coefficients for the [special unitary group](#) are known.<sup>[8][9]</sup> In particular, [SU\(3\) Clebsch-Gordan coefficients](#) have been computed and tabulated because of their utility in characterizing hadronic decays, where a flavor-SU(3) symmetry exists that relates the up, down, and strange quarks.<sup>[10][11][12]</sup> A [web interface for tabulating SU\(N\) Clebsch–Gordan coefficients](#) (<http://homepages.physik.uni-muenchen.de/~vondelft/Papers/ClebschGordan/>) is readily available.

## See also

- [3-j symbol](#)
- [6-j symbol](#)
- [9-j symbol](#)
- [Racah W-coefficient](#)
- [Spherical harmonics](#)
- [Spherical basis](#)
- [Tensor products of representations](#)
- [Associated Legendre polynomials](#)
- [Angular momentum](#)
- [Angular momentum coupling](#)
- [Total angular momentum quantum number](#)
- [Azimuthal quantum number](#)
- [Table of Clebsch–Gordan coefficients](#)
- [Wigner D-matrix](#)
- [Wigner–Eckart theorem](#)
- [Angular momentum diagrams \(quantum mechanics\)](#)
- [Clebsch–Gordan coefficient for SU\(3\)](#)
- [Littlewood-Richardson coefficient](#)

## Remarks

- The word "total" is often overloaded to mean several different things. In this article, "total angular momentum" refers to a generic sum of two angular momentum operators **j**<sub>1</sub> and **j**<sub>2</sub>. It is not to be confused with the other common use of the term "total angular momentum" that refers specifically to the sum of [orbital angular momentum](#) and [spin](#).

## Notes

- [Greiner & Müller 1994](#)
- [Edmonds 1957](#)
- [Condon & Shortley 1970](#)
- [Hall 2015](#) Section 4.3.2
- [Merzbacher 1998](#)
- [Hall 2015](#) Appendix C
- [Zachos, C K \(1992\). "Altering the Symmetry of Wavefunctions in Quantum Algebras and Supersymmetry". \*Modern Physics Letters\*. \*\*A7\*\* \(18\): 1595–1600. arXiv:\[hep-th/9203027\]\(https://arxiv.org/abs/hep-th/9203027\) \(<https://arxiv.org/abs/hep-th/9203027>\). Bibcode:1992MPLA....7.1595Z \(<https://ui.adsabs.harvard.edu/abs/1992MPLA....7.1595Z>\). doi:10.1142/S0217732392001270 \(<https://doi.org/10.1142%2FS0217732392001270>\).](#)
- [Alex et al. 2011](#)
- [Kaplan & Resnikoff 1967](#)
- [de Swart 1963](#)
- [Kaeding 1995](#)

## References

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- Clebsch–Gordan, 3-j and 6-j Coefficient Web Calculator (<http://www.volya.net/index.php?id=vc>)
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- Web interface for tabulating SU(N) Clebsch–Gordan coefficients (<http://homepages.physik.uni-muenchen.de/~vondelft/Papers/ClebschGordan/>)

## Further reading

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