Physics 137A - Quantum Mechanics - Summer 2019
Lecture Notes, Set 14

14.1 Spin in a Magnetic Field

14.1.1 Larmor Precession

A magnetic dipole $\vec{\mu}$ placed in a magnetic field $\vec{B}$ experiences a potential energy $U = -\vec{\mu} \cdot \vec{B}$. Classically, the dipole wants to align itself with the magnetic field. Consider a spin in a magnetic field (we will ignore the potential influence of orbital angular momentum for the rest of this discussion). The magnetic dipole vector operator is then given by

$$\hat{\vec{\mu}} \equiv \gamma \hat{\vec{S}}. \tag{14.1}$$

Let the magnetic field have magnitude B_0 and point in the $\check{n}$-direction. Then the Hamiltonian for the spin in the magnetic field is proportional to the $\check{n}$-component of spin,

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = -\gamma B_0 \hat{S}_n. \tag{14.2}$$

For a spin 1/2 particle the matrix representation of this Hamiltonian is

$$\hat{H} \doteq -\frac{\gamma B_0 \hbar}{2} \sigma_n = -\mu_B B_0 \sigma_n. \tag{14.3}$$

Since we know the eigenvalues of all of the components of the spin operator we know the eigenvalues of this Hamiltonian,

$$E_\pm = \mp \frac{\gamma B_0 \hbar}{2} = \mp \mu_B B_0, \tag{14.4}$$

where $E_\pm$ is the eigenvalue corresponding to the eigenket $|\check{n}, \pm\rangle$.

Without loss of generality, we may take $\vec{B}$ to be in the $\check{z}$-direction (that is, we let the magnetic field define what we call the z-direction for this setup). In this case we have

$$\hat{H} = -\gamma B_0 \hat{S}_z \doteq -\mu_B B_0 \sigma_z. \tag{14.5}$$

The eigenkets are just $|\uparrow\rangle$ and $|\downarrow\rangle$ and the eigenvalues are $-\gamma B_0 \hbar/2 = -\mu_B B_0$ and $+\gamma B_0 \hbar/2 = +\mu_B B_0$, respectively.

Suppose we start with an arbitrary spinor,

$$\chi(0) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{+i\varphi} \end{pmatrix}. \tag{14.6}$$

From Eq. 13.34 we know that the initial spin polarization vector is thus

$$\langle \vec{S} \rangle(0) = \frac{\hbar}{2} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}. \tag{14.7}$$

What happens as this spinor evolves in time? Well, the spinor is already nicely expanded in the eigenstates of the Hamiltonian and thus we just have to add on the relevant time-dependent phase factors $e^{-iE_\pm t/\hbar} = e^{\pm \gamma B_0 t/2}$ to the components!

$$\chi(t) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{+i\gamma B_0 t/2} \\ \sin\frac{\theta}{2}\, e^{+i\varphi} e^{-i\gamma B_0 t/2} \end{pmatrix}. \tag{14.8}$$

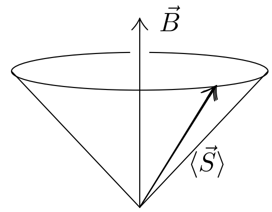

Figure 14.1: Larmor precession. The spin polarization vector $\langle\vec{S}\rangle$ precesses about the magnetic field vector $\vec{B}$ with the Larmor frequency γB_0.

INSERT PICTURE HERE

Figure 14.2: A $+\check{z}$-oriented Stern-Gerlach apparatus.

Next we want to compute the spin polarization vector. We *could* go through the full calculation but I much prefer tricking the answer into doing the work for us. In particular, let's pull an overall phase $e^{+i\gamma B_0 t/2}$ out of this spinor,

$$\chi(t) = e^{+i\gamma B_0 t/2}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\,e^{+i(\varphi-\gamma B_0 t)}\end{pmatrix}. \tag{14.9}$$

The overall phase factor has no physical consequence and we can see that the remaining spinor is exactly in the form of Eq. 13.34! Thus we find

$$\langle\vec{S}\rangle(t) = \frac{\hbar}{2}\begin{pmatrix}\sin\theta\cos(\varphi-\gamma B_0 t)\\ \sin\theta\sin(\varphi-\gamma B_0 t)\\ \cos\theta\end{pmatrix}. \tag{14.10}$$

The spin polarization vector is thus described by $\theta(t) = \theta$ and $\varphi(t) = \varphi - \gamma B_0 t$. The polar angle remains unchanged in time and the azimuthal angle is linear in time. That is, *the spin polarization vector precesses about the magnetic field.* This is called ***Larmor precession***. The frequency of precession is the ***Larmor frequency***,

$$\omega = \gamma B_0. \tag{14.11}$$

This precession is the ultimate foundation of ***nuclear magnetic resonance***.

14.1.2 The Stern-Gerlach Experiment

If we put a magnetic dipole in a *spatially-varying* magnetic field then it turns our we get a magnetic force! Consider a classical dipole $\vec{\mu}$ in a space-dependent magnetic field $\vec{B}$. Then, the classical force is

$$\vec{F} = -\vec{\nabla}U = \vec{\nabla}(\vec{\mu}\cdot\vec{B}). \tag{14.12}$$

Let's exploit this by setting up a particular magnetic field between the poles of a magnet as shown in Fig. 14.2. In Fig. 14.2 the $\check{z}$-direction is vertically up the page, in the direction of the magnetic field at the center, the $\check{x}$-direction points to the right, and the $\check{y}$-direction is into the page. The magnetic field in this case takes the form

$$\vec{B} = -\alpha x\check{x} + (B_0 + \alpha z)\check{z}, \tag{14.13}$$

where α depends on the angle of the south pole at the top. We call such a setup a ***Stern-Gerlach apparatus*** since this was the magnetic field configuration used by Otto Stern and Walther Gerlach to measure spin in 1922.[1] Since the magnetic field is mainly in the $\check{z}$-direction, we call this a ***$+\check{z}$-oriented Stern-Gerlach apparatus***. The classical force on a dipole in this field is thus given

[1] Gerlach, W.; Stern, O. (1922). "Der experimentelle Nachweis der Richtungsquantelung im Magnetfeld". Zeitschrift fr Physik. 9: 349352.

by

$$\vec{F} = -\alpha(\mu_x \check{x} - \mu_z \check{z}). \tag{14.14}$$

So what happens if we invoke quantum mechanics and let our particle have a spin. The Hamiltonian is then

$$\hat{H} = \frac{\hat{p}^2}{2m} - \gamma \hat{\vec{S}} \cdot \vec{B}. \tag{14.15}$$

Note that this Hamiltonian invokes both spatial *and* spin degrees of freedom. If $\alpha \ll 1$ then we can effectively ignore the x-component of the magnetic field. This is an *approximation* but it is justified by invoking Ehrenfest's theorem. We can expect that the Larmor precession about the magnetic field will cause the expectation value of the x-component of spin to oscillate at the (usually quite large) Larmor frequency. Since the x-component of force is dependent of the x-component of the angular momentum, the force essentially time-averages out to zero. This is equivalent to erasing the x-component of the magnetic field in the first place, which is why we make this approximation. Of course, this approximation is slightly unphysical since it violates Maxwell's equations. In 137B you can treat the bit we're ignoring here as a small perturbation in perturbation theory and see that the effect is indeed small. why could do the approximate.

Our approximate Hamiltonian for a $+\check{z}$-oriented Stern-Gerlach apparatus is

$$\hat{H} \approx \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} - \gamma(B_0 + \alpha\hat{z})\hat{S}_z. \tag{14.16}$$

The x- and y-spatial degrees of freedom are governed by free-particle Hamiltonians so we can put those aside for our purposes. Instead, we focus on the spin degrees of freedom and the z-spatial degree of freedom. If we go to a spinor representation in the usual $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis we have

$$\hat{H}_z = \frac{\hat{p}_z^2}{2m} - \mu_B(B_0 + \alpha\hat{z})\sigma_z = \begin{pmatrix} \frac{\hat{p}_z^2}{2m} - \mu_B B_0 - \mu_B\alpha\hat{z} & 0 \\ 0 & \frac{\hat{p}_z^2}{2m} + \mu_B B_0 + \mu_B\alpha\hat{z} \end{pmatrix}. \tag{14.17}$$

Since this is a diagonal matrix we can write the time-evolution operator,

$$\hat{U}(t) = e^{-i\hat{H}_z t/\hbar} = \begin{pmatrix} e^{-i(\hat{p}_z^2/2m - \mu_B B_0 - \mu_B\alpha\hat{z})t/\hbar} & 0 \\ 0 & e^{-i(\hat{p}_z^2/2m - \mu_B B_0 + \mu_B\alpha\hat{z})t/\hbar} \end{pmatrix}. \tag{14.18}$$

The most important piece for our discussion of the Stern-Gerlach experiment is the last term in the exponential, which we will isolate (the other terms are responsible for things like the Larmor precession and time-dependent phases). That is, let's focus on

$$\hat{U}_{SG}(t) = \begin{pmatrix} e^{i\mu_B\alpha\hat{z}t/\hbar} & 0 \\ 0 & e^{-i\mu_B\alpha\hat{z}t/\hbar} \end{pmatrix}. \tag{14.19}$$

Let's start with a state whose spatial and spin degrees of freedom are separable. Let the spatial bit be some Gaussian wave packet with $\langle p\rangle_z = 0$ and $\langle p\rangle_y > 0$ (so the particle can be interpreted as moving through the magnetic field). Let the spin bit be arbitrary, so

$$|\text{initial}\rangle = |\text{space}\rangle \otimes \big(a|\uparrow\rangle + b|\downarrow\rangle\big) \doteq \psi_0(\vec{x}) \begin{pmatrix} a \\ b \end{pmatrix}. \tag{14.20}$$

What happens when we act $\hat{U}_{SG}(t)$ on this? Well, the upper and lower components of our spinor get hit by different operators, so we get

$$\psi_0(\vec{x}) \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a\, e^{+i\mu_B\alpha\hat{z}t/\hbar}\psi_0(\vec{x}) \\ b\, e^{-i\mu_B\alpha\hat{z}t/\hbar}\psi_0(\vec{x}) \end{pmatrix} \tag{14.21}$$

We see that our time evolution made a separable state into an ***entangled state***, where the spin and space degrees of freedom have been entangled. What do the factors of $e^{\pm i\mu_B \alpha z t/\hbar}$ do to our wave function? You can check for yourself that it changes the expectation value of momentum by an amount $\pm\mu_B \alpha z t$! If we just look at the spin-up component of our state, we have gained a positive z-momentum and if we just look at the spin-down component of our state, we have gained a negative z-momentum. Our Gaussian wave packet starts to separate and move apart, with the spin-up bit moving in one direction and the spin-down bit moving in another direction. Thus, if we measure the position of our particle after it emerges from the beam we have effectively made a spin measurement! (Since the position measurement collapses our state and a spin-up particle, for example, has a much higher probability of being found with a positive z-position). This is how most measurement devices work! We correlate or entangle two different things - one thing that we want to measure, and one thing that is easy to read. For example, a balance scale correlates a mass measurement with a measurement of the position of an arrow.

The Stern-Gerlach apparatus is a spin measuring device! This is how we first learned of the quantized nature of spin. A beam of silver atoms (spin 1 particles) was shot through the device and instead of the beam fully splaying out vertically, it was found that only three discrete beams were produced. The deflections of the three beams precisely align with what would be expected for $S_z = +\hbar$, 0, and $-\hbar$.

Convert notes on how SG filters act like polarizers and can be used to prepare states to LaTeX for this subsection. See "Handwritten Notes".

14.1.3 Rotations

I still need to convert my written notes to LaTeX for this subsection. See "Handwritten Notes".

Here Ends Material for the Final - Summer 2019

14.2 Addition of Angular Momentum

Consider again an electron in hydrogen. We thoroughly discussed the electron's spatial states earlier. We formed a basis of spatial states or wave functions for an electron in hydrogen out of the simultaneous $\{\hat{H}, \hat{L}^2, \hat{L}_z\}$-eigenbasis. But as we just learned, the electron also has a spin angular momentum and thus the full state should contain information about both the spatial degrees of freedom *and* the spin degrees of freedom. We therefore need to learn how to handle systems with multiple angular momenta.

Let $\vec{L}$ and $\vec{S}$ be two classical angular momentum vectors and let $\hat{\vec{L}}$ and $\hat{\vec{S}}$ be the corresponding angular momentum operators. We define the ***total angular momentum*** (classical) vector and (quantum) operator as

$$\vec{J} = \vec{L} + \vec{S}, \qquad \hat{\vec{J}} \equiv \hat{\vec{L}} + \hat{\vec{S}}. \tag{14.22}$$

Note that I will introduce the formalism of angular momentum addition in the context of adding an orbital angular momentum ($\vec{L}$) and a spin angular momentum ($\vec{S}$). However, everything we do here will apply equally well for adding *any* two angular momenta! For example, the π^+ pion (introduced in Section 13.1.1) is a composite particle containing an up quark (u) and a down antiquark ($\bar{\mathrm{d}}$). The spin of the pion is the sum of the internal angular momenta of the component quarks, $\vec{S}_\pi = \vec{S}_{\mathrm{u}} + \vec{S}_{\bar{\mathrm{d}}}$. Another example of needing to add two spin angular momenta is in the hyperfine structure of hydrogen, where the Hamiltonian depends on the spin of the electron and the spin of the proton. We add two orbital angular momenta when we are considering the angular momentum states of helium. We can also add a combined angular momentum with a third angular momentum, such as when the nuclear spin interacts with the combined orbital- and spin-angular momenta of an electron in heavier atoms.

Going back to an electron in hydrogen, the spatial or orbital angular momentum degrees of freedom are contained in the orbital Hilbert space $\mathcal{H}_L$, which contains basis kets $\{|\ell, m_\ell\rangle\}$ and the spin degrees of freedom are contained in the spin Hilbert space $\mathcal{H}_S$, which contains basis kets $\{|s, m_s\rangle\}$ (even though the spin quantum number s is fixed for an electron we will carry it around with us, since s may be different or variable for other angular momentum addition situations). The Hilbert space containing the information about both of these angular momentum is the tensor product[2] of the individual spaces,

$$\mathcal{H}_J \equiv \mathcal{H}_L \otimes \mathcal{H}_S. \tag{14.23}$$

14.2.1 The Uncoupled Basis

We can form a basis for this space by combining the basis elements of the individual Hilbert spaces,

$$\text{Uncoupled basis states}: \qquad |\ell, s, m_\ell, m_s\rangle \equiv |\ell, m_\ell\rangle \otimes |s, m_s\rangle. \tag{14.24}$$

These are called ***uncoupled states*** (and the basis is called the ***uncoupled basis***) because we are separately specifying the orbital angular momentum information and the spin angular momentum information. That is, the uncoupled basis states are separable into their constituent angular momentum states. Since the orbital and spin angular momenta refer to entirely different degrees of freedom, any orbital angular momentum operator will commute with any spin angular momentum operator and thus the four operators $\{\hat{L}^2, \hat{S}^2, \hat{L}_z, \hat{S}_z\}$ form a complete set of commuting observables for the total angular momentum Hilbert space. The uncoupled basis is the simultaneous eigenbasis for this complete set,

$$\hat{L}^2|\ell, s, m_\ell, m_s\rangle = \hbar^2\ell(\ell+1)\,|\ell, s, m_\ell, m_s\rangle, \tag{14.25a}$$

$$\hat{S}^2|\ell, s, m_\ell, m_s\rangle = \hbar^2 s(s+1)\,|\ell, s, m_\ell, m_s\rangle, \tag{14.25b}$$

$$\hat{L}_z|\ell, s, m_\ell, m_s\rangle = \hbar m_\ell\,|\ell, s, m_\ell, m_s\rangle, \tag{14.25c}$$

$$\hat{S}_z|\ell, s, m_\ell, m_s\rangle = \hbar m_s\,|\ell, s, m_\ell, m_s\rangle. \tag{14.25d}$$

[2] Recall that the way we combine Hilbert spaces for different degrees of freedom is with the tensor product.

14.2.2 The Coupled Basis

It may seem intuitively obvious that if we add two angular momenta together we get something else which we can call an angular momentum (we do that classically all the time!) but it's good to check that this is indeed the case.

Example 14.1. *$\hat{\vec{J}}$ is an Angular Momentum*

To show that $\hat{\vec{J}}$ may be interpreted as an angular momentum we need to show that the components of $\hat{\vec{J}}$ obey the angular momentum algebra (the set of commutation relations Eq. 12.65). First, since both $\hat{\vec{L}}$ and $\hat{\vec{S}}$ are angular momenta we know that $[\hat{L}_a, \hat{L}_b] = \sum i\hbar\epsilon_{abc}\hat{L}_c$ and $[\hat{S}_a, \hat{S}_b] = \sum i\hbar\epsilon_{abc}\hat{S}_c$. We also know that all components of the orbital angular momentum commute with all components of the spin angular momentum, $[\hat{L}_a, \hat{S}_b] = 0$. Therefore,

$$[\hat{J}_a, \hat{J}_b] = [\hat{L}_a + \hat{S}_a, \hat{L}_b + \hat{S}_b] = [\hat{L}_a, \hat{L}_b] + \cancel{[\hat{L}_a, \hat{S}_b]} + \cancel{[\hat{S}_a, \hat{L}_b]} + [\hat{S}_a, \hat{S}_b]$$

$$= \sum_c i\hbar\epsilon_{abc}\hat{L}_c + \sum_c i\hbar\epsilon_{abc}\hat{S}_c = \sum_c i\hbar\epsilon_{abc}(\hat{L}_c + \hat{S}_c) = \sum_c i\hbar\epsilon_{abc}\hat{J}_c. \qquad \checkmark$$

This last expression is indeed the expected commutation relation for an angular momentum. ♦

In summary,

$$[\hat{L}_a, \hat{L}_b] = \sum_c i\hbar\epsilon_{abc}\hat{L}_c, \qquad [\hat{S}_a, \hat{S}_b] = \sum_c i\hbar\epsilon_{abc}\hat{S}_c, \qquad [\hat{L}_a, \hat{S}_b] = 0, \tag{14.26a}$$

$$[\hat{J}_a, \hat{J}_b] = \sum_c i\hbar\epsilon_{abc}\hat{J}_c. \tag{14.26b}$$

Since $\hat{\vec{J}}$ is an angular momentum we know from our earlier discussions of angular momentum that there are simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_z$. Moreover, we know the possibilities for the spectrum of these operators, namely that any eigenvalue of $\hat{J}^2$ must be of the form $\hbar^2 j(j+1)$ with j an integer or half integer and that any eigenvalue of $\hat{J}_z$ must be of the form $\hbar m_j$ with m_j ranging in integer steps from $-j$ to j. The question is which of the four operators that formed the "uncoupled" complete set of commuting observables is compatible with these operators. The operator $\hat{J}_z$ commutes with all four operators $\hat{L}^2$, $\hat{S}^2$, $\hat{L}_z$, and $\hat{S}_z$.[3] To determine the $\hat{J}^2$ commutators we first write the operator in terms of the angular momentum operators $\hat{\vec{L}}$ and $\hat{\vec{S}}$,

$$\hat{J}^2 = (\hat{\vec{L}} + \hat{\vec{S}})^2 = \hat{L}^2 + \hat{\vec{L}}\cdot\hat{\vec{S}} + \hat{\vec{S}}\cdot\hat{\vec{L}} + \hat{S}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{\vec{L}}\cdot\hat{\vec{S}}, \tag{14.27}$$

where we have used the fact that $\hat{\vec{L}}$ and $\hat{\vec{S}}$ commute. From this expression we can conclude that $\hat{J}^2$ commutes with $\hat{L}^2$ and $\hat{S}^2$ but does *not* commute with the components $\hat{L}_z$ or $\hat{S}_z$.[4] A complete set of commuting observables using the total angular momentum operators, then, is $\{\hat{L}^2, \hat{S}^2, \hat{J}^2, \hat{J}_z\}$. This complete set is used to form a new simultaneous eigenbasis, CSCO

$$\text{Coupled basis states}: \qquad |\ell, s; j, m_j\rangle. \tag{14.28}$$

$$\hat{L}^2|\ell, s; j, m_j\rangle = \hbar^2\ell(\ell+1)\,|\ell, s; j, m_j\rangle, \tag{14.29a}$$

$$\hat{S}^2|\ell, s; j, m_j\rangle = \hbar^2 s(s+1)\,|\ell, s; j, m_j\rangle, \tag{14.29b}$$

$$\hat{J}^2|\ell, s; j, m_j\rangle = \hbar^2 j(j+1)\,|\ell, s; j, m_j\rangle, \tag{14.29c}$$

$$\hat{J}_z|\ell, s; j, m_j\rangle = \hbar m_j\,|\ell, s; j, m_j\rangle. \tag{14.29d}$$

[3] Problem 13.2(a) of the Homework.
[4] Problem 13.2(b) of the Homework.

$|\ell, s\, m_\ell\, m_s\rangle = |\ell, m_\ell\rangle \otimes |s, m_s\rangle$

These are called ***coupled states*** (and the basis is called the ***coupled basis***) because we are “coupling” the two individual angular momenta together to create a total angular momentum. Note that, with the exception of the “stretch states” which we will discuss later, the coupled basis states are *not* separable into orbital and spin angular momentum states.

Rearranging Eq. 14.27 gives a very useful expression that will pop up repeatedly when you study the fine and hyperfine structures of hydrogen,

$$\hat{\vec{L}} \cdot \hat{\vec{S}} = \frac{1}{2}\left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2\right). \tag{14.30}$$

Question 14.1. *Consider the uncoupled basis state $|1, 2, 0, -1\rangle$. For which of the following operators is this state* not *an eigenstate? Determine the eigenvalues for the answers where the state* is *an eigenstate.*

(a) $\hat{L}^2$, (b) $\hat{S}^2$, (c) $\hat{J}^2$, (d) $\hat{L}_z$, (e) $\hat{S}_z$, (f) $\hat{J}_z$.

Question 14.2. *Consider the coupled basis state $|2, 1; 3, 0\rangle$. For which of the following operators is this state* not *an eigenstate? Determine the eigenvalues for the answers where the state* is *an eigenstate.*

(a) $\hat{L}^2$, (b) $\hat{S}^2$, (c) $\hat{J}^2$, (d) $\hat{L}_z$, (e) $\hat{S}_z$, (f) $\hat{J}_z$.

Question 14.3. *Determine the eigenvalues and eigenstates of $\hat{\vec{L}} \cdot \hat{\vec{S}}$.*

14.2.3 The Clebsch-Gordan Coefficients

To perform a change-of-basis from the uncoupled states to the coupled states we need to know how to expand the coupled states in terms of the uncoupled states. The most general expression we may consider is

$$|\ell, s; j, m_j\rangle = \sum_{\ell', s', m_\ell, m_s} c_{\ell', s', m_\ell, m_s} |\ell', s', m_\ell, m_s\rangle.$$

$\sum_{\ell'=0}^{\infty} \sum_{s'=0}^{\infty} \sum_{m=-\ell'}^{\ell'} \sum_{m_s=-s}^{s'}$

However, we may simplify this expansion considerably! For example, we can show that the only non-zero terms in the sum occur when $\ell = \ell'$ and $s = s'$.

Example 14.2. *$\ell' = \ell$ Based on a Physical Argument*

> There are a few ways to show that the only non-zero coefficients occur when $\ell' = \ell$. The first is a physical argument. If we are in the state $|\ell, s; j, m_j\rangle$ then we know that the probability of measuring L^2 and getting the result $\hbar^2 \ell(\ell + 1)$ is one and the probability of finding any other value is zero. By the last expression in Eq. 8.16 from Postulate 4, this means that the probability of finding $\hbar^2 \ell'(\ell' + 1)$ when $\ell \neq \ell'$ is
>
> $$P(\ell' \neq \ell) = 0 = \sum_{\ell' \neq \ell} \sum_{s', m_\ell, m_s} |c_{\ell', s', m_\ell, m_s}|^2 .$$
>
> The only way for the sum of magnitudes to be zero is if each of the magnitudes is zero so we conclude that $c_{\ell', s', m_\ell, m_s} = 0$ if $\ell \neq \ell'$. ♦

loop

Example 14.3. *$\ell' = \ell$ Based on the Hermiticity of $\hat{L}^2$*

A second way of showing this result is to use the fact that L^2 is an observable, which means $\hat{L}^2$ is Hermitian. By Fourier's trick, the coefficients are

$$c_{\ell',s',m_\ell,m_s} = \langle \ell', s', m_\ell, m_s | \ell, s; j, m_j \rangle.$$

From Section 7.4.1 we know that eigenkets belonging to different eigenvalues of a Hermitian operators must be orthogonal so this expression must evaluate to zero if $\ell \neq \ell'$. ♦

Example 14.4. *$\ell' = \ell$ Based on Matrix Elements*

As a last proof, consider the matrix element $\langle \ell' s' m_\ell m_s | \hat{L}^2 | \ell, s; j, m_j \rangle$. We may act the operator either on the ket to the right or on the bra to the left. Since both bra and ket in this expression are eigenstates of $\hat{L}^2$ and $\hat{L}^2$ is Hermitian we have

$$\begin{aligned} \text{Acting to the right:} \quad \langle \ell', s', m_\ell, m_s | \hat{L}^2 | \ell, s; j, m_j \rangle &= \langle \ell', s', m_\ell, m_s | \hbar^2 \ell(\ell+1) | \ell, s; j, m_j \rangle \\ \text{Acting to the left:} \quad \langle \ell', s', m_\ell, m_s | \hat{L}^2 | \ell, s; j, m_j \rangle &= \langle \ell', s', m_\ell, m_s | \hbar^2 \ell'(\ell'+1) | \ell, s; j, m_j \rangle. \end{aligned}$$

Setting these equal gives

$$\hbar^2 \ell(\ell+1) \langle \ell', s', m_\ell, m_s | \ell, s; j, m_j \rangle = \hbar^2 \ell'(\ell'+1) \langle \ell', s', m_\ell, m_s | \ell, s; j, m_j \rangle.$$

The braket here evaluates to the coefficient $c_{\ell' s' m_\ell m_s}$ from earlier and we are left with

$$\hbar^2 \Big(\ell(\ell+1) - \ell'(\ell'+1) \Big) c_{\ell' s' m_\ell m_s} = 0.$$

$\ell = \ell'$

If $\ell \neq \ell'$ we again find that $c_{\ell' s' m_\ell m_s} = 0$. ♦

We may therefore expand the coupled basis states as

$$|\ell, s; j, m_j\rangle = \sum_{m_\ell=-\ell}^{\ell} \sum_{m_s=-s}^{s} C^{\ell\, s\, j}_{m_\ell m_s m_j} |\ell, s, m_\ell, m_s\rangle. \tag{14.31}$$

The expansion coefficients $C^{\ell\, s\, j}_{m_\ell m_s m_j}$ are called the ***Clebsch-Gordan coefficients***.[5] We define them to be the expansion coefficients of the coupled basis kets in the uncoupled basis,

$$C^{\ell\, s\, j}_{m_\ell m_s m_j} \equiv \langle \ell, s, m_\ell, m_s | \ell, s; j, m_j \rangle \equiv \langle \ell, s, m_\ell, m_s | j, m_j \rangle, \tag{14.32}$$

where the ℓ and s values in the ket in the last expression can be suppressed since, as we saw above, they must be the same as the ℓ and s values in the bra. The Clebsch-Gordan coefficients are also the matrix elements of the change-of-basis matrix going from the uncoupled to the coupled basis. Since the coupled and uncoupled basis states are mainly defined as eigenstates there is some phase freedom in how they are defined. Once the phase of one of the uncoupled basis states is fixed, for example, the relative phases of all of the other uncoupled basis states is determined automatically through the raising and lowering operators. Similarly, we may (begin to) fix the phases of the coupled basis states by declaring that all of the Clebsch-Gordan coefficients are *real*, so that

$$\text{Phase Convention:} \qquad \langle \ell, s, m_\ell, m_s | j, m_j \rangle = \langle j, m_j | \ell, s, m_\ell, m_s \rangle. \tag{14.33}$$

[5] Personal anecdote time! The talk for my qualifying exam in graduate school was titled "The Asymptotics of the Clebsch-Gordon Coefficients and the 3j-Symbol," and dealt quite heavily with the Clebsch-Gordan coefficients. As this was one of the rites of passage for graduate school, I was understandably nervous. In particular, I wanted to make sure my presentation looked as flawless as possible. Since "Clebsch" has a whole lot of consonants at the end of it I double and triple checked that I was spelling the name correctly every time it popped up. So I'm standing in the room with my committee getting ready to begin with my title slide up when one of my committee members says "You know you misspelled that..." I objected, "No! I'm *sure* Clebsch is spelled correctly!" The devastating reply was "Yes, but 'Gordan' is spelled with an 'a'!"

Recall from earlier that $\hat{J}_z$ commuted with the complete set $\{\hat{L}^2, \hat{S}^2, \hat{L}_z, \hat{S}_z\}$. Since $\hat{J}_z = \hat{L}_z + \hat{S}_z$ the uncoupled basis kets from Eq. 14.24 are also eigenkets of $\hat{J}_z$,

$$\hat{J}_z|\ell, s, m_\ell, m_s\rangle = \hbar(m_\ell + m_s)\,|\ell, s, m_\ell, m_s\rangle. \tag{14.34}$$

Thus, by an argument similar to what was made in Example 14.4, the Clebsch-Gordan coefficient must be zero if $m_j \neq m_\ell + m_s$.

Expansion of Coupled State in Uncoupled Basis

$$|\ell, s; j, m_j\rangle = \sum_{m_\ell=-\ell}^{\ell} C^{\ell\, s\, j}_{m_\ell, m_s, m_j}\ |\ell, s, m_\ell, m_s\rangle, \tag{14.35a}$$

$$= \sum_{m_\ell=-\ell}^{\ell} \langle \ell, s, m_\ell, m_s | j, m_j\rangle\ |\ell, s, m_\ell, m_s\rangle, \tag{14.35b}$$

with $m_s \equiv m_j - m_\ell$.

Note that we may not get all $(2\ell+1)$ terms in the sum since we have the additional requirement that $|m_s| \leq s$!

Example 14.5. *Expansion of the Coupled State* $|2, 1; 3, 0\rangle$

To express the coupled state $|2, 1; 3, 0\rangle$ in terms of the uncoupled basis states we first find *which* uncoupled basis states can go into making this state. We know we must have $\ell = 2$ and $s = 1$. We also must have $m_\ell + m_s = m_j = 0$. Since $\ell = 2$ the possible (m_ℓ, m_s) pairs are

$|2,1,3,0\rangle$
$s\ \ell\ J\ m_j$

$m_\ell\ \ m_s$
~~2 -2~~
1 -1
0 0
-1 -1
~~2 2~~

$$(m_\ell, m_s) \in \Big\{(+2, -2),\ \ (+1, -1),\ \ (0, 0),\ \ (-1, +1),\ \ (-2, +2)\Big\}.$$

However, the first and last of these must be eliminated since $s = 1$ and thus $|m_s| \leq 1$. Therefore the expansion is

$$|2, 1; 3, 0\rangle = C^{2,1,3}_{1,-1,0}\,|2, 1, +1, -1\rangle + C^{2,1,3}_{0,0,0}\,|2, 1, 0, 0\rangle + C^{2,1,3}_{-1,1,0}\,|2, 1, -1, +1\rangle.$$

To get the actual coefficients we use the table of Clebsch-Gordan coefficients (Fig. 14.3). Section 14.2.4 explains how to read the table. We find

$$C^{2,1,3}_{1,-1,0} = \sqrt{\frac{1}{5}}, \qquad C^{2,1,3}_{0,0,0} = \sqrt{\frac{3}{5}}, \qquad C^{2,1,3}_{-1,1,0} = \sqrt{\frac{1}{5}},$$

$$|2, 1; 3, 0\rangle = \sqrt{\frac{1}{5}}\,|2, 1, +1, -1\rangle + \sqrt{\frac{3}{5}}\,|2, 1, 0, 0\rangle + \sqrt{\frac{1}{5}}\,|2, 1, -1, +1\rangle. \qquad \blacklozenge$$

Once we know how to expand the coupled basis states in terms of the uncoupled basis states we know how to do a change-of-basis in the other direction as well! In particular, we take the uncoupled basis states and insert a resolution of the identity in terms of coupled basis states,

$$|\ell, s, m_\ell, m_s\rangle = \sum_j \sum_{m_j=-j}^{j} |\ell, s; j, m_j\rangle\langle \ell, s, j, m_j|\ell, s, m_\ell, m_s\rangle, \tag{14.36}$$

where we have used the fact that we must have the ℓ and s quantum numbers agree in the expansion. As discussed earlier we must have $m_j = m_\ell + m_s$ so the sum over m_j reduces to a single term. Next we analyze the sum over j. At this point j only has to satisfy the general rules for angular momentum, so the sum over j includes the integers *and* the half-integers. However, we *also* know that m_j must range in integer steps from $-j$ to j. Thus if m_j is an integer then so is j and if m_j is a half-integer then so is j. The m_ℓ and m_s values are similarly linked to the ℓ and s values.

If $\ell + s$ is an integer then j is an integer and if $\ell + s$ is a half-integer then j is a half-integer.

triangle relationship

If we add two classical vectors $\vec{L}$ and $\vec{S}$ to make a third vector $\vec{J} = \vec{L} + \vec{S}$ of lengths L, S, and J, respectively, then we know that $|L - S| \leq J \leq L + S$, where the inequalities are saturated when $\vec{L}$ and $\vec{S}$ are either aligned or anti-aligned. It turns out that a similar inequality holds for the relevant quantum numbers,[6]

$$|\ell - s| \leq j \leq \ell + s. \tag{14.37}$$

Expansion of Uncoupled State in Coupled Basis

$$|\ell, s, m_\ell, m_s\rangle = \sum_{j=|\ell-s|}^{\ell+s} C^{\ell\, s\, j}_{m_\ell, m_s, m_j}\ |\ell, s; j, m_j\rangle, \tag{14.38a}$$

interge

$m_j = m_\ell + m_s$

$\hookrightarrow$ j should be interge

$$= \sum_{j=|\ell-s|}^{\ell+s} \langle \ell, s, m_\ell, m_s | j, m_j\rangle\ |\ell, s; j, m_j\rangle, \tag{14.38b}$$

with $m_j \equiv m_\ell + m_s$.

In the above equations the sum takes integer steps (so that if $\ell = \frac{3}{2}$ and $s = 2$ the sum runs over the values $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$). We have also written the braket $\langle \ell, s, j, m_j | \ell, s, m_\ell, m_s\rangle$ from Eq. 14.36 as $\langle \ell, s, m_\ell, m_s | j, m_j\rangle$ in Eq. 14.38b by using the reality of the Clebsch-Gordan coefficients (which allows us to flip the bra and the ket as in Eq. 14.33) and the implicit understanding that the ket has quantum numbers ℓ and s (as in Eq. 14.32).

Example 14.6. *Expansion of the Uncoupled State* $\left|\frac{3}{2}, 1, \frac{1}{2}, 0\right\rangle$

To express the uncoupled state $\left|\frac{3}{2}, 1, \frac{1}{2}, 0\right\rangle$ in terms of the coupled basis states we first find *which* coupled basis states can go into making this state. We know we must have $\ell = \frac{3}{2}$ and $s = 1$. We also know that we must have $m_j = m_\ell + m_s = \frac{1}{2} + 0 = \frac{1}{2}$. Since $\ell = \frac{3}{2}$ and $s = 1$ the possible values of j are are

$$j \in \left\{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}\right\}$$

Therefore the expansion is

$$\left|\tfrac{3}{2}, 1, \tfrac{1}{2}, 0\right\rangle = C^{\frac{3}{2},1,\frac{1}{2}}_{\frac{1}{2},0,\frac{1}{2}}\ \left|\tfrac{3}{2}, 1; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + C^{\frac{3}{2},1,\frac{3}{2}}_{\frac{1}{2},0,\frac{1}{2}}\ \left|\tfrac{3}{2}, 1; \tfrac{3}{2}, \tfrac{1}{2}\right\rangle + C^{\frac{3}{2},1,\frac{5}{2}}_{\frac{1}{2},0,\frac{1}{2}}\ \left|\tfrac{3}{2}, 1; \tfrac{5}{2}, \tfrac{1}{2}\right\rangle.$$

To get the actual coefficients we again use the table of Clebsch-Gordan coefficients (Fig. 14.3) and find

$$C^{\frac{3}{2},1,\frac{1}{2}}_{\frac{1}{2},0,\frac{1}{2}} = -\sqrt{\frac{1}{3}}, \qquad C^{\frac{3}{2},1,\frac{3}{2}}_{\frac{1}{2},0,\frac{1}{2}} = \sqrt{\frac{1}{15}}, \qquad C^{\frac{3}{2},1,\frac{5}{2}}_{\frac{1}{2},0,\frac{1}{2}} = \sqrt{\frac{3}{5}},$$

$$\left|\tfrac{3}{2}, 1, \tfrac{1}{2}, 0\right\rangle = -\sqrt{\frac{1}{3}}\ \left|\tfrac{3}{2}, 1; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\frac{1}{15}}\ \left|\tfrac{3}{2}, 1; \tfrac{3}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\ \left|\tfrac{3}{2}, 1; \tfrac{5}{2}, \tfrac{1}{2}\right\rangle.$$

♦

Question 14.4. *If we have a coupled state with $j = 0$, what must we require of ℓ and s?*

(a) $\ell = 0$, (b) $s = 0$, (c) $\ell = 0$ and $s = 0$, (d) Merely $\ell = s$.

---

[6] This can be proved or shown with various degrees of hand-waving. A simple counting of dimensions shows that, in order for the number of states in the coupled and uncoupled bases to agree, we must have j ranging from $|\ell - s|$ to $\ell + s$ in integer steps. The rigorous proof gets into a subfield of group theory called ***representation theory***.

Question 14.5. *Which uncoupled states occur in the decomposition of the coupled state* $\left|\frac{1}{2}, \frac{3}{2}; 1, -1\right\rangle$*?*

Question 14.6. *Which coupled states occur in the decomposition of the uncoupled state* $|2, 1, 0, 0\rangle$*?*

Question 14.7. *The state* $\left|\frac{3}{2}, 1, \frac{1}{2}, 0\right\rangle$ *from Example 14.6 is prepared, after which the magnitude of the total angular momentum is measured. What is the probability of measuring* $j = \frac{1}{2}$*?*

(a)	$-\frac{1}{\sqrt{3}} = -0.333$,	(b)	0.000,	(c)	$\frac{1}{15} = 0.067$,	(d)	$\frac{1}{3} = 0.333$,
(e)	$\frac{1}{\sqrt{15}} = 0.258$,	(f)	$\frac{3}{5} = 0.600$,	(g)	$\frac{1}{\sqrt{3}} = 0.577$,	(h)	$\sqrt{\frac{3}{5}} = 0.774$.

14.2.4 The Table of Clebsch-Gordan Coefficients

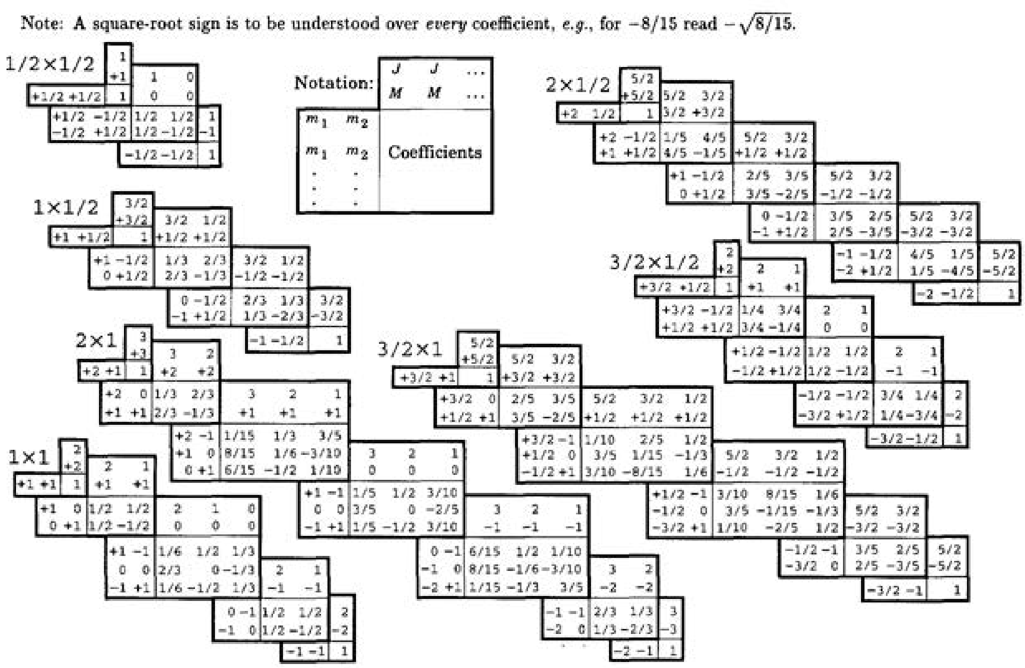

Figure 14.3: The Table of Clebsch-Gordan Coefficients.

It is important to know how to read the Clebsch-Gordan coefficients off of the table presented in Griffiths (and, most notably, compiled by the Particle Data Group, `http://pdg.lbl.gov/2002/clebrpp.pdf`). For example, let's say we are adding two angular momenta with $\ell = 1$ and $s = 1$. First we go to the "$\ell \times s$" block of tables. This block contains a number of sub-tables, with each sub-table characterized by a particular value of m_j. The two left-most columns in each sub-table give the two values m_ℓ and m_s. The two top-most rows in each sub-table give the two values j and m_j. Note that $m_\ell + m_s$ always equals this value of m_j for each row! This is a reflection of the fact that the Clebsch-Gordan coefficients are all zero unless $m_\ell + m_s = m_j$. The other entries give numbers which are related to the values of the coefficients. A square root sign is understood for every entry; the minus sign, if present, goes *outside* the square root.

Example 14.7. *The Clebsch-Gordan Coefficient* $\langle 1,1,-1,+1\,|\,1,0\rangle$.

> To find the coefficient $C^{1,1,1}_{-1,1,0} = \langle 1,1,-1,1\,|\,1,0\rangle$ ($\ell = 1$, $s = 1$, $j = 1$, $m_\ell = -1$, $m_s = +1$, $m_j = 0$), we first go to the $\ell = 1$, $s = 1$ block of tables. This block has five sub-tables. We are interested in the $m_j = 0$ subtable, which is in the middle. We choose the second column, labeled by $j = 1$, and the third row, labeled by $m_\ell = -1$, $m_s = +1$. The entry reads "-1/2" so the Clebsch-Gordan coefficient is identified as
>
> $$C^{1,1,1}_{-1,1,0} = \langle 1,1,-1,+1\,|\,1,0\rangle = -\frac{1}{\sqrt{2}}. \qquad \blacklozenge$$

An important thing to note is that in the table, the quantum number ℓ is always greater than or equal to the quantum number s. If we are looking at a problem where this is not true, we merely need to flip the order of ℓ and s (and m_ℓ and m_s).

Example 14.8. *The Clebsch-Gordan Coefficient* $\langle 1,\frac{3}{2},+1,+\frac{1}{2}\,|\,\frac{5}{2},\frac{3}{2}\rangle$.

> The coefficient $\langle 1,\frac{3}{2},+1,+\frac{1}{2}\,|\,\frac{5}{2},\frac{3}{2}\rangle$ has $\ell = 1 < s = 3/2$ so we have to flip our ℓ and s values to continue:
>
> $$\langle 1,\tfrac{3}{2},+1,+\tfrac{1}{2}\,|\,\tfrac{5}{2},\tfrac{3}{2}\rangle = \langle \tfrac{3}{2},1,+\tfrac{1}{2},+1\,|\,\tfrac{5}{2},\tfrac{3}{2}\rangle.$$
>
> To read this we go to the $\ell = 3/2$, $s = 1$ block of tables. We go to the second (of six) subtables, characterized by $m_j = +3/2$. The second row ($m_\ell = +1/2$, $m_s = +1$) and first column ($j = 5/2$, $m_j = 3/2$) reads "3/5" so
>
> $$\langle 1,\tfrac{3}{2},+1,+\tfrac{1}{2}\,|\,\tfrac{5}{2},\tfrac{3}{2}\rangle = \sqrt{\frac{3}{5}}. \qquad \blacklozenge$$

We can also use the table to easily read off how to decompose a basis ket in one basis into a linear combination of basis kets in the other basis. For example, if we want to express a ket $|\ell,s,m_\ell,m_s\rangle$ in the basis $|\ell,s;j,m_j\rangle$ we find the table $\ell\times s$, then the row corresponding to m_ℓ, m_s. Then we read horizontally and get the coefficients times the kets given by the columns.

Example 14.9. *Expanding* $|\frac{3}{2},1,+\frac{1}{2},0\rangle$ *in the Coupled Basis.*

> In Griffiths, in the $3/2\times 1$ table a horizontal row is shaded. This is the row with $m_\ell = +1/2$ and $m_s = 0$. Thus,
>
> $$\left|\tfrac{3}{2},1,+\tfrac{1}{2},0\right\rangle = +\sqrt{\frac{3}{5}}\left|\tfrac{3}{2},1;\tfrac{5}{2},\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{15}}\left|\tfrac{3}{2},1;\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\tfrac{3}{2},1;\tfrac{1}{2},\tfrac{1}{2}\right\rangle. \qquad \blacklozenge$$

Similarly, if we want to express a ket $|\ell,s;j,m_j\rangle$ in the basis $|\ell,s,m_\ell,m_s\rangle$, we find the table $\ell\times s$, then the column corresponding to j,m_j. Then we read vertically and get the coefficients times the kets given by the row.

Example 14.10. *Expanding* $|2,1;3,0\rangle$ *in the Uncoupled Basis.*

> In Griffiths, in 2×1 table a vertical row is shaded. This is the row with $j = 3$ and $m_j = 0$. Thus,
>
> $$|2,1;3,0\rangle = +\frac{1}{\sqrt{5}}\,|2,1,+1,-1\rangle + \sqrt{\frac{3}{5}}\,|2,1,0,0\rangle + \frac{1}{\sqrt{5}}\,|2,1,-1,+1\rangle. \qquad \blacklozenge$$

Question 14.8. *What is the Clebsch-Gordan coefficient* $\langle 2,1,-2,0\,|\,2,-2\rangle$

(a) $\frac{1}{\sqrt{3}}$, (b) $\sqrt{\frac{2}{3}}$, (c) $\sqrt{-\frac{2}{3}}$, (d) $-\sqrt{\frac{2}{3}}$, (e) 0.

Question 14.9. *Expand the coupled state* $\left|\frac{1}{2},\frac{1}{2};0,0\right\rangle$ *in terms of the uncoupled states.*

14.3 Examples of Adding Angular Momenta

Before getting into a few specific examples, physicists and mathematicians both have their own ways of expressing the addition of angular momentum rules. For a single angular momentum $\vec{L}$, the collection of quantum states forms what is known as a ***representation*** of the rotation group.[7] Under rotations, states of a given ℓ will become a linear combination of other states with the *same* value of ℓ (rotations don't affect the length of the angular momentum vector). When we add an $\ell = \frac{1}{2}$ angular momentum to an $s = \frac{1}{2}$ angular momentum, we find that the possible results for the total angular momentum quantum number are $j = 0$ and $j = 1$. Under rotations, the $j = 0$ state is unaffected and the $j = 1$ states get mixed together. We express this as $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1}$. A physicist's translation of this may be "combining a spin-1/2 angular momentum with a spin-1/2 angular momentum gives states with total angular momentum of either 0 or 1". Rather than referring to the quantum number ℓ, mathematicians may refer to the angular momenta by their dimension. Since m_ℓ ranges in integer steps from $-\ell$ to ℓ, the dimension of an angular momentum ℓ is $(2\ell+1)$. For $\ell = s = 1/2$, $j = 0$, and $j = 1$, the dimensions are 2, 1, and 3, respectively. Thus the mathematician's way of expressing the addition of two spin-1/2 particles would be $2 \otimes 2 = 1 \oplus 3$. In general:

$$\text{In terms of Spin:} \qquad \boldsymbol{\ell} \otimes \mathbf{s} = \bigoplus_{j=|\ell-s|}^{\ell+s} \mathbf{j} = |\boldsymbol{\ell} - \mathbf{s}| \oplus \cdots \oplus (\boldsymbol{\ell} + \mathbf{s}), \tag{14.39a}$$

$$\text{In terms of Dimension:} \qquad (2\ell+1) \otimes (2s+1) = \bigoplus_{j=|\ell-s|}^{\ell+s} (2j+1). \tag{14.39b}$$

Note that we do in fact have

$$(2\ell+1)(2s+1) = \sum_{j=|\ell-s|}^{\ell+s} (2j+1),$$

which means we have the same number of uncoupled basis states as we have coupled basis states, which is what we require!

14.3.1 Adding One Angular Momentum to a Spin-0 Particle

When $s = 0$ (or $\ell = 0$) the rules for adding the two angular momentum degrees of freedom becomes trivial since there isn't any freedom for the second angular momentum! The only $|s, m_s\rangle$ state available is the state $|0, 0\rangle$. Thus by our rules we must have $m_j = m_\ell + 0$ and $|\ell - 0| \le j \le \ell + 0$, which means $(j, m_j) = (\ell, m_\ell)$,

$$|\ell, 0, m_\ell, 0\rangle = |\ell, 0; j = \ell, m_j = m_\ell\rangle. \tag{14.40}$$

Unsurprisingly, if we add a spin-0 particle (or a zero-angular momentum contribution) to our system, the total angular momentum is identical to the first angular momentum.

$$\text{In terms of Spin:} \qquad \boldsymbol{\ell} \otimes \mathbf{0} = \boldsymbol{\ell}, \tag{14.41a}$$

$$\text{In terms of Dimension:} \qquad (2\ell+1) \otimes 1 = (2\ell+1). \tag{14.41b}$$

14.3.2 Adding Two Spin-1/2 Particles

Now consider adding two spin-1/2 particles. Rather than using $\vec{L}$ and $\vec{S}$ we will use $\vec{S}_1$ and $\vec{S}_2$ to talk about the two spins, since we know for a fact that they are spins. (Only the symbols we are using have changed, none of the actual math or theory). In particular, we have

$$\hat{\vec{J}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2, \tag{14.42}$$

[7] As we have seen, angular momentum is intimately tied to rotations. Rotational symmetry of a system implies that angular momentum is conserved by Noether's Theorem, and angular momentum "generates" rotations.

with quantum numbers $s_1 = s_2 = 1/2$. The possible values of the total angular momentum quantum number j range from $|s_1 - s_2| = 0$ to $s_1 + s_2 = 1$, so

$$\text{In terms of Spin:} \quad \mathbf{\tfrac{1}{2}} \otimes \mathbf{\tfrac{1}{2}} = \mathbf{0} \oplus \mathbf{1}, \tag{14.43a}$$

$$\text{In terms of Dimension:} \quad 2 \otimes 2 = 1 \oplus 3. \tag{14.43b}$$

14.3.2.1 Blastoff!... I mean... The Uncoupled Basis!

First let's talk about the uncoupled basis, made up of states $|s_1, s_2, m_1, m_2\rangle = \left|\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right\rangle$. Recall that when talking about the states of a spin-1/2 particle, rather than excessively writing "1/2", in the ket we just used arrows as our labels. We will do the same thing here for our uncoupled basis states,

$$\begin{aligned} |{\uparrow\uparrow}\rangle &\equiv \left|\frac{1}{2}, \frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right\rangle; \qquad & |{\uparrow\downarrow}\rangle &\equiv \left|\frac{1}{2}, \frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right\rangle; \\ |{\downarrow\uparrow}\rangle &\equiv \left|\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right\rangle; \qquad & |{\downarrow\downarrow}\rangle &\equiv \left|\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle. \end{aligned}$$

We can also use this notation to talk about other separable states!

Example 14.11. *A Combined Spin State*

> ***Question:*** *In a two-spin system, the x-component of the first spin is measured to be $+\hbar/2$ and the y-component of the second spin is measured to be $-\hbar/2$. What is the state in the uncoupled basis?*
>
> To find the spin state, we just "smoosh together" the individual spin states $|{\rightarrow}\rangle$ (for the first spin) and $|\odot\rangle$ (for the second spin),
>
> $$|{\rightarrow}, \odot\rangle = |{\rightarrow}\rangle \otimes |\odot\rangle.$$
>
> To find this in the uncoupled basis we just plug in what $|{\rightarrow}\rangle$ and $|\odot\rangle$ are in terms of the S_z-eigenbasis from Eqs. 13.26a and 13.27b,
>
> $$\begin{aligned} |{\rightarrow}, \odot\rangle &= \left(\frac{1}{\sqrt{2}}\Big(|{\uparrow}\rangle + |{\downarrow}\rangle\Big)\right) \otimes \left(\frac{1}{\sqrt{2}}\Big(|{\uparrow}\rangle - i|{\downarrow}\rangle\Big)\right) \\ &= \frac{1}{2}\Big(|{\uparrow\uparrow}\rangle - i|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle - i|{\downarrow\downarrow}\rangle\Big). \end{aligned}$$ ♦

We can "plot" our four uncoupled basis states in a state diagram as shown in Fig. 14.4 which shows the quantum number m_1 for $\hat{S}_{1,z}$ on the horizontal axis, the quantum number m_2 for $\hat{S}_{2,z}$ on the vertical axis. Allowed states are plotted as bullet points on the diagram. Note that we can get from one state to another state in our diagram by using the raising and lowering operators $\hat{S}_{1,\pm}$ and $\hat{S}_{2,\pm}$.

Question 14.10. *What is $\hat{S}_{1,-}$ acting on the state $|{\uparrow\downarrow}\rangle$?*

14.3.2.2 The Coupled Basis

Now let's construct the coupled basis! We know we have four coupled basis states: one state with $j = 0$ ($m_j = 0$) called the ***singlet state*** and three states with $j = 1$ ($m_j = -1, 0, 1$) called the ***triplet states***. Using the table of Clebsch-Gordan coefficients we can read off the coupled states in

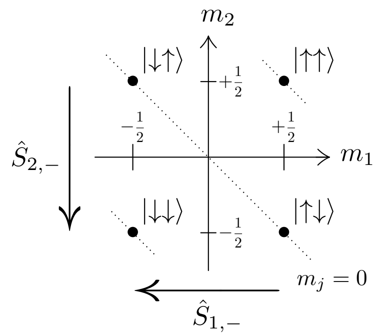

Figure 14.4: Allowed angular momentum states in the addition of two spin-1/2 particles in the uncoupled basis $\left|\frac{1}{2}, \frac{1}{2}, m_1, m_2\right\rangle$. The dotted diagonal lines running from the upper left to the lower right represent states with the same value of $m_j = m_1 + m_2$.

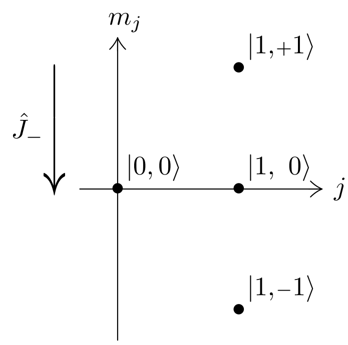

Figure 14.5: Allowed angular momentum states in the addition of two spin-1/2 particles in the coupled basis $\left|\frac{1}{2}, \frac{1}{2}; j, m_j\right\rangle$. The dotted horizontal lines connect states with the same value of $m_j = m_1 + m_2$. Each vertical column of dots represents one "ladder" with a fixed value of j.

terms of the uncoupled basis states,

Singlet state:
$$|0,0\rangle = \frac{1}{\sqrt{2}}\Big(|{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle\Big), \tag{14.44a}$$

Triplet states:
$$|1,+1\rangle = |{\uparrow\uparrow}\rangle, \tag{14.44b}$$
$$|1,\ 0\ \rangle = \frac{1}{\sqrt{2}}\Big(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle\Big), \tag{14.44c}$$
$$|1,-1\rangle = |{\downarrow\downarrow}\rangle. \tag{14.44d}$$

Convert notes to LaTeX for the following topics. See "Handwritten Notes".

- Show how to generate the table of Clebsch-Gordan coefficients
- Matrix Representations - Change of Basis Matrix, J^2, L_z, etc.
- Example: Spin-spin hyperfine coupling leads to 21cm line.

14.3.3 Adding an l=1 to an s=1 Angular Momentum

The last case we will consider is the addition of an angular momentum of $\ell = 1$ with another angular momentum of $s = 1$,

$$\text{In terms of Spin:} \quad \mathbf{1} \otimes \mathbf{1} = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}, \tag{14.45a}$$
$$\text{In terms of Dimension:} \quad 3 \otimes 3 = 1 \oplus 3 \oplus 5. \tag{14.45b}$$

Convert notes to LaTeX for the following topics. See “Handwritten Notes”.

- Discuss how to generate the table of Clebsch-Gordan coefficients
- Classical analogue - two vectors give a scalar, a vector, and a symmetric, traceless 2x2 matrix

14.4 Answers

Answer 14.1. *This state is in the uncoupled basis so it is by definition an eigenstate of $\hat{L}^2$, $\hat{S}^2$, $\hat{L}_z$, and $\hat{S}_z$ with eigenvalues $2\hbar^2$, $6\hbar^2$, 0, and $-\hbar$, respectively. We can also have $\hat{J}_z = \hat{L}_z + \hat{S}_z$ so $\hat{J}_z|1,2,0,-1\rangle = (0)|1,2,0,-1\rangle + (-\hbar)|1,2,0,-1\rangle = -\hbar|1,2,0,-1\rangle$. Thus this state* is *an eigenstate of $\hat{J}_z$ with eigenvalue $-\hbar$. The operator $\hat{J}^2$ does* not *commute with the complete set and we can* not *claim that this is an eigenstate. The correct answer is Choice **C**. [*Note: The fact that $\hat{J}^2$ doesn't commute with the other operators is actually not enough to claim that the state isn't an eigenstate. For us to solidify our statement we can explicitly expand $\hat{J}^2$ out in terms of the orbital and spin operators (which will involve the ladder operators) and check that this is not in fact an eigenstate.*]*

Answer 14.2. *This state is in the coupled basis so it is by definition an eigenstate of $\hat{L}^2$, $\hat{S}^2$, $\hat{J}^2$, and $\hat{J}_z$ with eigenvalues $6\hbar^2$, $2\hbar^2$, $12\hbar^2$, and 0, respectively. The operators $\hat{L}_z$ and $\hat{S}_z$ do* not *commute with the complete set and we can* not *claim that this is an eigenstate. The correct answers are Choices **D** and **E**.*

Answer 14.3. *The coupled basis states are eigenstates of $\hat{J}^2$, $\hat{L}^2$, and $\hat{S}^2$ and so will be eigenstates of $\hat{\vec{L}} \cdot \hat{\vec{S}}$. The eigenvalue for state $|\ell, s; j, m_j\rangle$ is $\frac{\hbar^2}{2}(j(j+1) - \ell(\ell+1) - s(s+1))$.*

Answer 14.4. *Since we must have $|\ell - s| \leq 0 \leq \ell + s$ we require that $\ell = s$, Choice **D** (of which Choice C is one special case, but not the* only *case).*

Answer 14.5. *We must have $\ell = \frac{1}{2}$, $s = \frac{3}{2}$, and $m_\ell + m_s = m_j = -1$. Since $|m_\ell| \leq \ell$ and $|m_s| \leq s$, there are only two possible uncoupled states in the decomposition: $\left|\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right\rangle$ and $\left|\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$.*

Answer 14.6. *We must have $\ell = 2$, $s = 1$, $m_j = m_\ell + m_s = 0$, and $|2-1| \leq j \leq (2+1)$. Thus, we have coupled states $|2,1;1,0\rangle$, $|2,1;2,0\rangle$, and $|2,1;3,0\rangle$ in the decomposition.*

Answer 14.7. *First we use Example 14.6 to expand $\left|\frac{3}{2}, 1, \frac{1}{2}, 0\right\rangle$ in the coupled basis,*

$$\left|\tfrac{3}{2}, 1, \tfrac{1}{2}, 0\right\rangle = -\sqrt{\frac{1}{3}}\left|\tfrac{3}{2}, 1; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\frac{1}{15}}\left|\tfrac{3}{2}, 1; \tfrac{3}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\left|\tfrac{3}{2}, 1; \tfrac{5}{2}, \tfrac{1}{2}\right\rangle.$$

Now it's easy to find the probability of measuring $j = 1/2$ since we are in an eigenbasis for which $\hat{J}^2$ is part of the CSCO. We just identify the terms that contain basis states with $j = 1/2$ and sum the magnitude-squared of their coefficients! That is, we can use, by Postulate 4

$$\begin{aligned} P(j = \tfrac{1}{2}) &= \left|\left|\hat{P}_{j=1/2}\left|\tfrac{3}{2}, 1, \tfrac{1}{2}, 0\right\rangle\right|\right|^2 \\ &= \left|\left|-\sqrt{\frac{1}{3}}\left|\tfrac{3}{2}, 1; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \cancel{\sqrt{\frac{1}{15}}\left|\tfrac{3}{2}, 1; \tfrac{3}{2}, \tfrac{1}{2}\right\rangle} + \cancel{\sqrt{\frac{3}{5}}\left|\tfrac{3}{2}, 1; \tfrac{5}{2}, \tfrac{1}{2}\right\rangle}\right|\right|^2 \\ &= \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}. \end{aligned}$$

*Thus the correct answer is Choice **D**.*

Answer 14.8. *The relevant entry in $m_j = -2$ sub-table of the 2×1 block is "-2/3" so the correct Clebsch-Gordan cofficient is $-\sqrt{2/3}$ (with the negative sign* outside *the square root), Choice **D**.*

Answer 14.9. *Using the second row of the second of three sub-tables in the $1/2 \times 1/2$ block we find*

$$\left|\tfrac{1}{2}, \tfrac{1}{2}; 0, 0\right\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}, \tfrac{1}{2}, +\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle.$$

Answer 14.10. *The lowering operator $\hat{S}_{1,-}$ only acts on the first spin, so*

$$\hat{S}_{1,-}|\uparrow\downarrow\rangle = \left(\hat{S}_-|\uparrow\rangle\right) \otimes |\downarrow\rangle = \left(\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}|\downarrow\rangle\right) \otimes |\downarrow\rangle = \hbar|\downarrow\downarrow\rangle.$$