Three Magnetic Dipole Problem

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1 Introduction

Magnetic systems often exhibit nonlinear responses to external magnetic fields. This means, for example that the relation between an applied magnetic field and the resulting magnetization of a magnetizable material is not linearly proportional, which can induce a variety of interesting phenomena. Many examples can be mentioned, for example the as chaotic behavior of the system, autoresonance, and turbulence, such as in earth's magneto tail [1–6].

Previous works shown that the influence of small fluctuations in a system composed of two magnetic dipoles can lead to two different behaviors: the dipoles fluctuate around stable fixed points (with low amplitude fluctuations), or stochastic reversals occurs between stable fixed points (with strong fluctuations). Low energy fluctuations lead to disjoint basins of attraction near stable fixed points, while higher fluctuations connect basins (including an unstable fixed point) with stochastic reversals and Poisson-distributed waiting time [7,8].

One may suggest that, for a system composed of two magnetic dipoles, there exists a nonlinear coupling, and the problem presents two different time scales depending on the magnitudes of the magnetic interactions, because they are of different nature [9].

This work focuses on investigating the dynamics of a system composed of three equidistantly spaced bar magnets fixed on a table, each free to rotate about its center.

In order to better analyze this system, we consider the length of the magnets to be small compared to the distance between them, allowing us to approximate the bar magnets for magnetic dipoles.

We studied the case where two magnets are much stronger than the third one, and, taking sufficient approximations, we were able to investigate the system as a single magnet in a homogeneous magnetic field whose direction oscillates in time.

This simplified system can exhibit unexpected and chaotic behavior when, from an equilibrium initial state, a small perturbation that can increase kinetic energy of the system is applied to the small magnet through the oscillations of the external magnetic field. This interesting behavior occurs at specific values of the physical parameters, such as the system's natural frequency and the frequency at which the magnetic field oscillates.

Comprehending the mechanisms by which kinetic energy transfers among the magnetic field and the magnet was the original motivation to this project. Having a good understanding of this phenomenon is crucial for gaining insights into the dynamics of magnetic systems and their potential applications.

1.1 Theoretical model

A single magnetic dipole with magnetic moment $\boldsymbol{m}$, generates a magnetic field at a distance $\boldsymbol{r}$, accordingly to the expression:

$$\boldsymbol{B} = \frac{3\mu_0}{4\pi r^3}\left[(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \frac{1}{3}\boldsymbol{m}\right], \tag{1}$$

where, μ_0 is the magnetic permeability of the medium [10].

If subjected to an external magnetic field $\boldsymbol{B}_{ext}$, the magnetic dipole experiences a torque given by:

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}_{ext}. \tag{2}$$

Let us consider a system composed of three magnetic dipoles, each placed at a vertex of an equilateral triangle and equally spaced. The magnetic field generated by each of the dipoles is felt by the other two. By adding the torques at the dipole i, generated by the dipoles j and k, we find that, the resulting torque on dipole i is:

$$\begin{aligned}\boldsymbol{\tau}_i &= \boldsymbol{\tau}_{ij} + \boldsymbol{\tau}_{ik},\\ &= \boldsymbol{m}_i \times \boldsymbol{B}_{ji} + \boldsymbol{m_i} \times \boldsymbol{B}_{ki},\\ &= \boldsymbol{m}_i \times \boldsymbol{B}_{res},\end{aligned} \tag{3}$$

where, $\boldsymbol{B}_{res}$, represents the resulting magnetic field generated by the two dipoles j and k.

In this work, we analyze a simplified model where two dipoles have much greater moments of inertia than the third, and that the resulting magnetic field generated by the two heavy dipoles is approximately homogeneous, i.e,

$$\boldsymbol{B}_{res} = B(\cos(\theta_1), \sin(\theta_1), 0), \tag{4}$$

where θ_1 represents the angular displacement of the magnetic field.

If the smaller dipole has magnetic moment given by:

$$\boldsymbol{m}_1 = m_1(\cos(\theta), \sin(\theta), 0), \tag{5}$$

then, because $\boldsymbol{B}_{res}$ and $\boldsymbol{m_1}$ are in the same plane, the torque at the smaller dipole will have only components in the direction of $\hat{\boldsymbol{z}}$.

Therefore, in the above conditions, and accordingly to Newton 's Second Law of motion for spinning objects, the equation (3) gives rise to the equation of motion of a single dipole:

$$\tau_1 - I_1\ddot{\theta} = 0, \tag{6}$$

from where,

$$I\ddot{\theta} + m_1 B \sin(\theta - \theta_1) = 0. \tag{7}$$

s Now calling

$$\eta^2 = m_1 B/I, \tag{8}$$

then

$$\ddot{\theta} + \eta^2 \sin(\theta - \theta_1) = 0. \tag{9}$$

We remark that equation (9) is very similar to the pendulum equation, except for the θ_1 parameter.

In general, the two large dipoles do not remain static, but oscillate with small amplitude:

$$\theta_1(t) = \varepsilon \sin(\Omega t). \tag{10}$$

Therefore, generating an oscillating magnetic field, with frequency Ω, which turns equation (9) into the equation (11).

$$\ddot{\theta} + \eta^2 \sin(\theta - \varepsilon \sin(\Omega t)) = 0. \tag{11}$$

This is the final equation that will be studied in this article.

2 Numerical Methods

2.1 Runge-Kutta 4th order

In order to investigate equation (9) numerically, we used the iterative method of Runge-Kutta of fourth order (RK4), which has a low computational cost, and fourth order accuracy, i.e, at each iteration the error is at most $\mathcal{O}(h^4)$, where h is the integration step.

Before applying the method, we have transformed the second order differential equation of motion of dipole (9), into two first order differential equation by making the transformations:

$$\begin{aligned} x_1 &= \theta, \\ x_2 &= \dot{\theta}, \end{aligned} \tag{12}$$

leading to the system of first order differential equations below:

$$\begin{cases} \dot{x_1} &= x_2, \\ \dot{x_2} &= -\eta^2 \sin(x_1 - \varepsilon \sin(\Omega t)), \end{cases} \tag{13}$$

We used $h = 10^{-4}$ for most simulations, equation (9) and $h = 0.05$ for the construction of the bifurcation diagrams.

2.2 Simpson's 1/3 rule and Fourier transform

After solving the differential equation numerically using the RK4 method, we used the Simpson's 1/3 rule, equation (15), to numerically integrate the Fourier transform, equation 14.

$$\mathcal{F}\{f(t)\} = \frac{2}{T}\int_0^T f(t)e^{-i\xi t}dt \tag{14}$$

$$\int_a^b f(x)dx = \frac{1}{3}h\Bigg[f(a) + 4\sum_{n=1}^{n/2} f(x_{2i-1}) + 2\sum_{n=1}^{n/2-1} f(x_{2i}) + f(b)\Bigg] \tag{15}$$

2.3 Bifurcation diagrams

For creating the bifurcation diagrams, figures 1 and 2, we vary the frequency of the magnetic field, Ω, from -3 to 3 using a step of 0.01, and for each frequency Ω we solved equation (9) for $t \in [0, 500]$ using the RK4 method, then applied the Fourier transform and filtered out the frequencies with amplitude less than 20% of the maximum amplitude. Finally, for each Ω we plotted the frequencies obtained from the Fourier transform, ω, creating a graph, (Ω, ω) where the ω frequencies represent the frequency response of the system.

3 Numerical results

In this Section, we'll discuss the dynamics of the equation (9), with mostly numerical results.

The dipole's dynamics was studied for small amplitude of oscillation of the magnetic field. We used two main values for the parameter ε, $\varepsilon = 0.05$ and $\varepsilon = 0.5$, and the following initial conditions for both cases:

$$\begin{cases} \theta(0) &= 1 \\ \dot{\theta}(0) &= 0 \end{cases} \tag{16}$$

In the former case, $\varepsilon = 0.05$, we verify that the dipole's movement was periodic for all

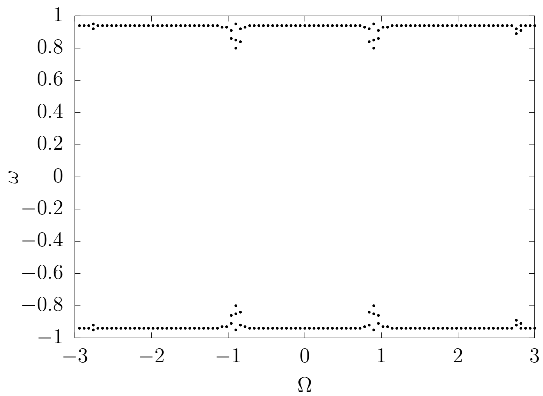

Figure 1: Bifurcation diagram of frequencies from the oscillating magnetic field Ω and the response frequencies on the dipole ω, for the case with $\varepsilon = 0.05$.

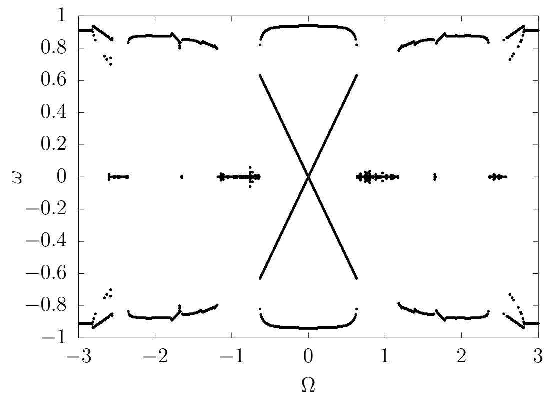

Figure 2: Bifurcation diagram of frequencies from the oscillating magnetic field Ω and the response frequencies on the dipole ω, for the case with $\varepsilon = 0.5$.

the values of the frequency of the magnetic field Ω in $[-3, 3]$, and as expected, the dipole behaved like a simple pendulum.

However, for the latter case, $\varepsilon = 0.5$, the dipole's movement was much more complex. For some frequencies of oscillation of the external magnetic field, Ω, the dynamics of magnetic dipole begin to differ from a regular pendulum, and the movement alternates between rotational and oscillatory, figure 3b. In this case, the motion it is not periodic, and therefore the Fourier transform cannot give accurate results, resulting in regions of frequencies near zero as shown the bifurcation diagram, figure 2.

4 Conclusion

In this work we have investigated the dynamics of a dipole in an external oscillating magnetic field. We found that, for the most frequencies in the frequency range, $[-3, 3]$, the dynamics of the dipole is very similar to a simple pendulum, figures 3a, 3c. Nonetheless, for some specific subranges of the frequency range, a small perturbation of the external magnetic field can cause the dipole oscillate at a higher energy, figure 3b, and in a non-periodic manner, differing from a regular pendulum.

In order to investigate the system fully equipped with three magnets, we need to develop new methodologies, given that the movement in this case can present a dynamic even more complex than the one we studied here.

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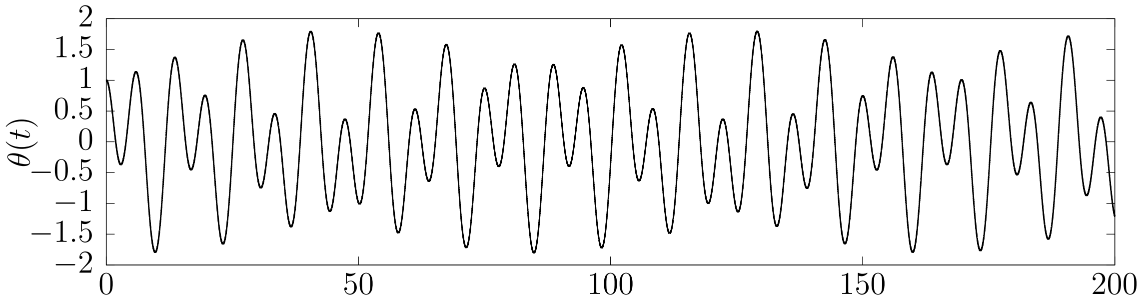
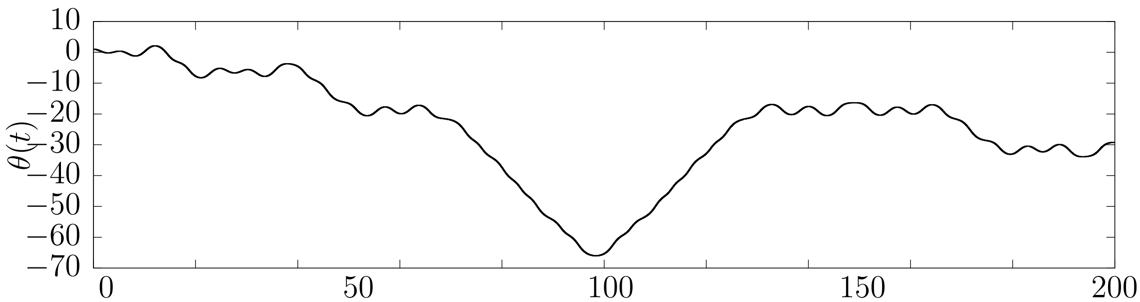
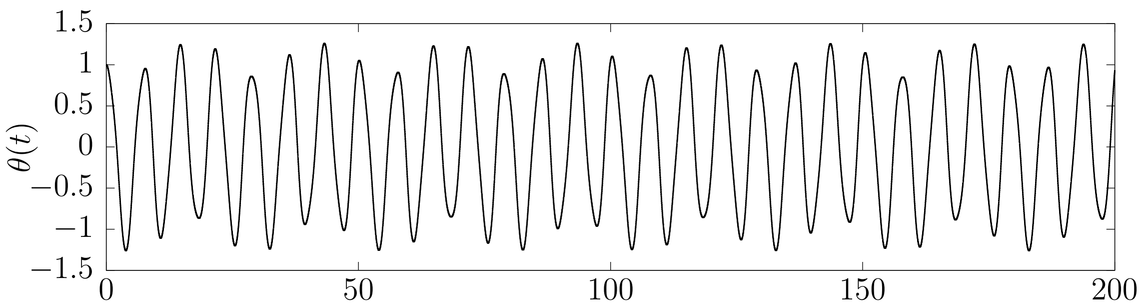


(a) Movement of the dipole with frequency of the external magnetic field being 0.5.



(b) Movement of the dipole with frequency of the external magnetic field being 0.75.



(c) Movement of the dipole with frequency of the external magnetic field being 2.0.

Figure 3: Movement of the dipole for different frequencies of the magnetic field. Using $\varepsilon = 0.5$