Three Magnetic Dipole Problem

Callebe R. Reis

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1 Introduction

Magnetic systems often exhibit nonlinear responses to external magnetic fields, which means that, the correlation between an applied magnetic field and the resulting magnetization is not proportional, which can induce a variety of interesting phenomena, such as chaotic behavior, autoresonance, and turbulence in earth's magneto tail [1–6].

This work focuses on investigating the dynamics of a system composed of three equidistantly spaced bar magnets affixed to a table, each free to rotate about its center. In order to better analyze this system, we consider the length of the magnets to be small compared to the distance between them, allowing us to approximate the bar magnets for magnetic dipoles.

We studied the case where two magnets are much stronger than the third one, and, taking sufficient approximations, we were able to investigate the system as a single magnet in a homogeneous magnetic field that oscillates in direction.

From an initial equilibrium state, the simplified system can exhibit unexpected and chaotic behavior when a small increase in kinetic energy is applied to the small magnet through the oscillations of the magnetic field. This interesting behavior occurs at specific values of the physical parameters, such as the dipole moment and the frequency at which the field oscillates.

Understand the mechanism by which kinetic energy transfers among the magnetic field and the magnet was the motivation to this project. Having a good understanding of this phenomenon is crucial for gaining insights into the dynamics of magnetic systems and their potential applications.

One may suggest that, for a system composed of two magnetic dipoles there exists a nonlinear coupling, and the problem presents two different time scales depending on the magnitudes of the magnetic interactions, because they are of different nature [7]. As we'll see, such a phenomenon occurs for the single magnetic dipole in an oscillating magnetic field.

Previous works shown that the influence of small fluctuations in a system composed of two magnetic dipoles can lead to two different behaviors: the dipoles fluctuate around stable fixed points (with low amplitude fluctuations), or stochastic reversals occurs between stable fixed points (with strong fluctuations). Low energy fluctuations lead to disjoint basins of attraction near stable fixed points, while higher fluctuations connect basins (including an unstable fixed point) with stochastic reversals and Poisson-distributed waiting time [8]. In this work, we are interested in the second case, by each the system presents a rather unexpected behavior.

1.1 The triple dipole

A single magnetic dipole with magnetic moment $\boldsymbol{m}$, generates a magnetic field, at a distance $\boldsymbol{r}$, accordingly to the expression:

$$\boldsymbol{B} = \frac{3\mu_0}{4\pi r^3}\left[(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \frac{1}{3}\boldsymbol{m}\right], \tag{1}$$

where, μ_0 is the magnetic permeability of the medium.

When subjected to an external magnetic field $\boldsymbol{B}_{ext}$, it experiences a torque following the expression:

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}_{ext}. \tag{2}$$

Now consider a system composed of three magnetic dipoles, each placed at a vertex of an equilateral triangle and therefore equally spaced. The magnetic field generated by each of the dipoles is felt by the other two. Adding the torques at the dipole i, generated by the dipoles j and k, we find that the resulting torque is:

$$\begin{aligned}\boldsymbol{\tau}_i &= \boldsymbol{\tau}_{ij} + \boldsymbol{\tau}_{ik}\\ &= \boldsymbol{m}_i \times \boldsymbol{B}_{ji} + \boldsymbol{m_i} \times \boldsymbol{B}_{ki}\\ &= \boldsymbol{m}_i \times \boldsymbol{B}_{res}\end{aligned} \tag{3}$$

where, $\boldsymbol{B}_{res}$, represents the resulting magnetic field generated by the other two dipoles j and k.

Considering the case where two dipoles have much greater moments of inertia than the third, with magnetic moments given by:

$$\boldsymbol{m}_i = m_i(\cos(\theta), \sin(\theta), 0), \tag{4}$$

and that the resulting magnetic field generated by the two heavy dipoles is approximately homogeneous, i.e, under the following conditions:

$$\begin{aligned}I_1 &\ll I_2\\ I_1 &\ll I_3\\ \boldsymbol{B}_{res} &= B(\cos(\theta_1), \sin(\theta_1), 0)\end{aligned} \tag{5}$$

Note that $\boldsymbol{B}_{res}$ and $\boldsymbol{m_1}$ are in the same plane, and therefore the resulting torque $\boldsymbol{\tau}$ has components only in the $\hat{\boldsymbol{z}}$ direction.

Therefore, in the above conditions, and accordingly to Newton 's Second Law of motion for spinning objects, the equation (3) gives rise to the equation of motion of a single dipole:

$$\begin{aligned}&\tau_1 - I_1\ddot{\theta} = 0\\ &I\ddot{\theta} + m_1 B\sin(\theta - \theta_1) = 0\end{aligned} \tag{6}$$

Calling,

$$\omega^2 = m_1 B/I \tag{7}$$

then

$$\ddot{\theta} + \omega^2 \sin(\theta - \theta_1) = 0 \tag{8}$$

1.2 Forced System

The term θ_1 in the equation (8), represents the angular displacement of the magnetic field. If the magnetic field oscillates, with frequency Ω, as:

$$\theta_1 = \varepsilon \sin(\Omega t) \tag{9}$$

then (8) becomes:

$$\ddot{\theta} + \omega^2 \sin(\theta - \varepsilon\sin(\Omega t)) = 0 \tag{10}$$

Making the following change of variables:

$$\begin{aligned}\phi &= \theta - \varepsilon\sin(\Omega t),\\ \ddot{\phi} &= \ddot{\theta} + \varepsilon\Omega^2\sin(\Omega t)\end{aligned} \tag{11}$$

resulting in,

$$\ddot{\phi} + \omega^2 \sin(\phi) = \Omega^2 \varepsilon \sin(\Omega t) \tag{12}$$

which is the equation of a forced pendulum.

2 Numerical Methods

2.1 Runge-Kutta 4th order

In order to investigate the equation (8) numerically, we used the iterative method of Runge-Kutta of fourth order (RK4), which presents a low computational cost, and fourth order accuracy, i.e, at each iteration the error is $\mathcal{O}(h^4)$, where h is the integration step.

Before applying the method, we have transformed the second order differential equation of motion of dipole (8), into two first order differential equation by making the transformations:

$$\begin{aligned} x_1 &= \theta, \\ x_2 &= \dot{\theta}, \end{aligned} \tag{13}$$

leading to the system of first order differential equations below:

$$\begin{cases} \dot{x_1} &= x_2, \\ \dot{x_2} &= -\omega^2 \sin(x_1 - \varepsilon \sin(\Omega t)), \end{cases} \tag{14}$$

2.2 Fourier Transform

For the study of the frequencies present in the movement of the dipole, we first solved the differential equation numerically,

$$\int_a^b \tag{15}$$

List of Figures

References

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