Three Magnetic Dipole Problem

Callebe R. Reis

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1 Introduction

Magnetic systems often exhibit nonlinear responses to external magnetic fields, which means that, the correlation between an applied magnetic field and the resulting magnetization is not proportional, which can induce a variety of interesting phenomena, such as chaotic behavior, autoressonance, and turbulance in earth's magnetotail [1–6].

This work focuses on investigating the dynamics of a system composed of three equidistantly spaced bar magnets affixed to a table, each free to rotate about its center. In order to better analyze this system, we consider the length of the magnets to be small compared to the distance between them, allowing us to approximate the bar magnets for magnetic dipoles.

We studied the case where two magnets are much stronger than the third one, and, taking sufficient approximations, we were able to investigate the system as a single magnet in a homogeneous magnetic field that oscillates in direction.

From an initial equilibrium state, the simplified system can exhibit unexpected and chaotic behavior when a small increase in kinetic energy is applied to the small magnet through the oscillations of the magnetic field. This interesting behavior occurs at specific values of the physical parameters, such as the dipole moment and the frequency at which the field oscillates.

Understand the mechanism by which kinetic energy transfers among the magnetic field and the magnet was the motivation to this project. Having a good understanding of this phenomenon is crucial for gaining insights into the dynamics of magnetic systems and their potential applications.

One may suggest that, for a system composed of two magnetic dipoles there exists a nonlinear coupling, and the problem presents two different time scales depending on the magnitudes of the magnetic interactions, because they are of different nature [7]. As we'll see, such a phenomenon occurs for the single magnetic dipole in an oscillating magnetic field.

Previous works shown that the influence of small fluctuations in a system composed of two magnetic dipoles can lead to two different behaviors: the dipoles fluctuate around stable fixed points (with low amplitude fluctuations), or stochastic reversals occours between stable fixed points (with strong fluctuations). Low energy fluctuations lead to disjoint basins of attraction near stable fixed points, while higher fluctuations connect basins (including an unstable fixed point) with stochastic reversals and Poisson-distributed waiting time [8]. In this work, we are interested in the second case, by each the system presents a rather unexpected behavior.

1.1 The triple dipole

A single magnetic dipole with magnetic moment $\boldsymbol{m}$, generates a magnetic field, at a distance $\boldsymbol{r}$, accordindly to the expression:

$$\boldsymbol{B} = \frac{3\mu_0}{4\pi r^3}\left[(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \frac{1}{3}\boldsymbol{m}\right], \qquad (1)$$

where, μ_0 is the magnetic permeability of the medium.

When subjected to an external magnetic field $\boldsymbol{B}_{ext}$, it experiences a torque following the expression:

$$\boldsymbol{\tau} = \boldsymbol{m}\times\boldsymbol{B}_{ext}. \qquad (2)$$

Now consider a system composed of three magnetic dipoles, each placed at a vertex of an equilateral triangle and therefore equally spaced. The magnetic field generated by each of the dipoles is felt by the other two, and at each dipole, and adding the torques at the dipole i, generated by the dipoles j and k, we find that the resulting torque is:

$$\begin{aligned}\boldsymbol{\tau}_i &= \boldsymbol{\tau}_{ij} + \boldsymbol{\tau}_{ik}\\ &= \boldsymbol{m}_i\times\boldsymbol{B}_{ji} + \boldsymbol{m_i}\times\boldsymbol{B}_{ki}\\ &= \boldsymbol{m}_i\times\boldsymbol{B}_{res}\end{aligned} \qquad (3)$$

where, $\boldsymbol{B}_{res}$, represents the resulting magnetic field generated by the other two dipoles j and k.

Considering the special case where there are two strong dipoles, and one weak one, with magnetic moment given by:

$$\boldsymbol{m_1} = m_1(\cos(\theta),\sin(\theta),0), \qquad (4)$$

and that the resulting magnetic field generated by the two strong dipoles is approximately homogeneous, i.e, under the follwing conditions:

$$\begin{aligned}m_1 &\ll m_2\\ m_1 &\ll m_3\\ \boldsymbol{B}_{res} &= B(\cos(\theta_1),\sin(\theta_1),0)\end{aligned} \qquad (5)$$

Note that $\boldsymbol{B}_{res}$ and $\boldsymbol{m_1}$ are in the same plane, and therefore the resulting torque $\boldsymbol{\tau}$ has components only in the $\hat{\boldsymbol{z}}$ direction.

Therefore, in the above conditions, and acoordindly to Newton 's Second Law of motion for spinning objects, the equation (3) gives rise to the equation of motion of a single dipole:

$$\begin{aligned}&\tau_1 - I_1\ddot{\theta} = 0\\ &I\ddot{\theta} + m_1 B\sin(\theta-\theta_1) = 0\end{aligned} \qquad (6)$$

Calling,

$$\omega^2 = m_1 B/I \qquad (7)$$

then

$$\ddot{\theta} + \omega^2\sin(\theta-\theta_1) = 0 \qquad (8)$$

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References

[1] E. M. Clements, Raja Das, Manh-Huong Phan, Ling Li, Veerle Keppens, David Mandrus, Michael Osofsky, and Hariharan Srikanth. Magnetic field dependence of nonlinear magnetic response and tricritical point in the monoaxial chiral helimagnet $cr_{1/3}Nbs_2$. *Phys. Rev. B*, 97:214438, Jun 2018.

[2] V. L. Bratman, G. G. Denisov, N. S. Ginzburg, and M. I. Petelin. Fel's with bragg reflection resonators: Cyclotron autoresonance masers versus ubitrons. *IEEE Journal of Quantum Electronics*, 19, 1983.

[3] A. Loeb and L. Friedland. Autoresonance laser accelerator. *Physical Review A*, 33, 1986.

[4] M. et al. Veranda. Helically self-organized pinches: dynamical regimes and magnetic chaos healing. 2019.

[5] Steven L. Brunton, Bingni W. Brunton, Joshua L. Proctor, Eurika Kaiser, and J. Nathan Kutz. Chaos as an intermittently forced linear system. *Nature Communications*, 8, 2017.

[6] Tom Chang. Self-organized criticality, multi-fractal spectra, sporadic localized reconnections and intermittent turbulence in the magnetotail. *Physics of Plasmas*, 6, 1999.

[7] D. Laroze, P. Vargas, C. Cortes, and G. Gutierrez. Dynamics of two interacting dipoles. *Journal of Magnetism and Magnetic Materials*, 320(8):1440–1448, 2008.

[8] Nicolas Plihon, Sophie Miralles, Mickael Bourgoin, and Jean-Fran çois Pinton. Stochastic reversal dynamics of two interacting magnetic dipoles: A simple model experiment. *Phys. Rev. E*, 94:012224, Jul 2016.