Three Magnetic Dipole Problem

Callebe R. Reis

July 25, 2023

Contents

1 Introduction

Magnetic systems often exhibit nonlinear responses to external magnetic fields, which means that, the relation between an applied magnetic field and the resulting magnetization is not linearly proportional, which can induce a variety of interesting phenomena, such as chaotic behavior, autoresonance, and turbulence, such as in earth's magneto tail [1–6].

This work focuses on investigating the dynamics of a system composed of three equidistantly spaced bar magnets fixed on a table, each free to rotate about its center.

In order to better analyze this system, we consider the length of the magnets to be small compared to the distance between them, allowing us to approximate the bar magnets for magnetic dipoles.

We studied the case where two magnets are much stronger than the third one, and, taking sufficient approximations, we were able to investigate the system as a single magnet in a homogeneous magnetic field whose direction oscillates in time.

This simplified system, can exhibit unexpected and chaotic behavior when, from an equilibrium initial state, a small increase in kinetic energy is applied to the small magnet through the oscillations of the magnetic field. This interesting behavior occurs at specific values of the physical parameters, such as the system's natural frequency and the frequency at which the magnetic field oscillates.

Comprehend the mechanism by which kinetic energy transfers among the magnetic field and the magnet was the motivation to this project. Having a good understanding of this phenomenon is crucial for gaining insights into the dynamics of magnetic systems and their potential applications.

One may suggest that, for a system composed of two magnetic dipoles there exists a nonlinear coupling, and the problem presents two different time scales depending on the magnitudes of the magnetic interactions, because they are of different nature [7]. As we'll see, such a phenomenon occurs for the single magnetic dipole in an oscillating magnetic field.

Previous works shown that the influence of small fluctuations in a system composed of two magnetic dipoles can lead to two different behaviors: the dipoles fluctuate around stable fixed points (with low amplitude fluctuations), or stochastic reversals occurs between stable fixed points (with strong fluctuations). Low energy fluctuations lead to disjoint basins of attraction near stable fixed points, while higher fluctuations connect basins (including an unstable fixed point) with stochastic reversals and Poisson-distributed waiting time [8]. In this work, we are interested in the second case, by each the system presents a rather unexpected behavior.

1.1 The triple dipole problem

A single magnetic dipole with magnetic moment $\boldsymbol{m}$, generates a magnetic field at a distance $\boldsymbol{r}$, accordingly to the expression:

$$\boldsymbol{B} = \frac{3\mu_0}{4\pi r^3}\left[(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \frac{1}{3}\boldsymbol{m}\right], \tag{1}$$

where, μ_0 is the magnetic permeability of the medium.

If subjected to an external magnetic field $\boldsymbol{B}_{ext}$, the magnetic dipole experiences a torque:

$$\boldsymbol{\tau} = \boldsymbol{m}\times\boldsymbol{B}_{ext}. \tag{2}$$

Now consider a system composed of three magnetic dipoles, each placed at a vertex of an equilateral triangle and equally spaced. The magnetic field generated by each of the dipoles is felt by the other two. By adding the torques at the dipole i, generated by the dipoles j and k, we find that, the resulting torque on dipole i is:

$$\begin{aligned}\boldsymbol{\tau}_i &= \boldsymbol{\tau}_{ij} + \boldsymbol{\tau}_{ik}\\ &= \boldsymbol{m}_i\times\boldsymbol{B}_{ji} + \boldsymbol{m_i}\times\boldsymbol{B}_{ki}\\ &= \boldsymbol{m}_i\times\boldsymbol{B}_{res}\end{aligned} \tag{3}$$

where, $\boldsymbol{B}_{res}$, represents the resulting magnetic field generated by the two dipoles j and k.

Considering the case where two dipoles have much greater moments of inertia than the third, and that the resulting magnetic field generated by the two heavy dipoles is approximately homogeneous, i.e,

$$\boldsymbol{B}_{res} = B(\cos(\theta_1), \sin(\theta_1), 0), \tag{4}$$

where θ_1 represents the angular displacement of the magnetic field.

If the smaller dipole has magnetic moment given by:

$$\boldsymbol{m}_1 = m_1(\cos(\theta), \sin(\theta), 0), \tag{5}$$

then, because $\boldsymbol{B}_{res}$ and $\boldsymbol{m_1}$ are in the same plane, the torque at the smaller dipole will have only components in the direction of $\hat{\boldsymbol{z}}$.

Therefore, in the above conditions, and accordingly to Newton 's Second Law of motion for spinning objects, the equation (3) gives rise to the equation of motion of a single dipole:

$$\begin{aligned}&\tau_1 - I_1\ddot{\theta} = 0\\ &I\ddot{\theta} + m_1 B\sin(\theta - \theta_1) = 0\end{aligned} \tag{6}$$

and calling

$$\eta^2 = m_1 B/I \tag{7}$$

then

$$\ddot{\theta} + \eta^2\sin(\theta - \theta_1) = 0. \tag{8}$$

Equation (8) is very similar to the pendulum equation, except for the θ_1 parameter.

1.2 Forced System

Considering a oscillating magnetic field, with frequency Ω, then:

$$\theta_1 = \varepsilon\sin(\Omega t) \tag{9}$$

and

$$\ddot{\theta} + \eta^2 \sin(\theta - \varepsilon \sin(\Omega t)) = 0. \tag{10}$$

Making the following change of variables:

$$\begin{aligned} \phi &= \theta - \varepsilon \sin(\Omega t), \\ \ddot{\phi} &= \ddot{\theta} + \varepsilon \Omega^2 \sin(\Omega t) \end{aligned} \tag{11}$$

resulting in,

$$\ddot{\phi} + \eta^2 \sin(\phi) = \Omega^2 \varepsilon \sin(\Omega t) \tag{12}$$

which is the equation of a forced pendulum.

(to be continued)

2 Numerical Methods

2.1 Runge-Kutta 4th order

In order to investigate the equation (8) numerically, we used the iterative method of Runge-Kutta of fourth order (RK4), which presents a low computational cost, and fourth order accuracy, i.e, at each iteration the error is $\mathcal{O}(h^4)$, where h is the integration step.

Before applying the method, we have transformed the second order differential equation of motion of dipole (8), into two first order differential equation by making the transformations:

$$\begin{aligned} x_1 &= \theta, \\ x_2 &= \dot{\theta}, \end{aligned} \tag{13}$$

leading to the system of first order differential equations below:

$$\begin{cases} \dot{x_1} &= x_2, \\ \dot{x_2} &= -\eta^2 \sin(x_1 - \varepsilon \sin(\Omega t)), \end{cases} \tag{14}$$

We used $h = 10^{-4}$ for simple graphs of the equation (8) and $h = 0.05$ for bifurcation diagrams,

2.2 Simpson's 1/3 rule and Fourier transform

After solving the differential equation numerically using the RK4 method, we used the Simpson's 1/3 rule, equation (16), to numerically integrate the Fourier transform for :

$$\frac{2}{T} \int_0^T f(t) e^{-i\xi t} dt \tag{15}$$

$$\int_a^b f(x)dx = \frac{1}{3} h \Bigg[f(a) + 4 \sum_{n=1}^{n/2} f(x_{2i-1}) + 2 \sum_{n=1}^{n/2-1} f(x_{2i}) + f(b) \Bigg] \tag{16}$$

2.3 Frequency Bifurcation Diagrams

For creating the bifurcation diagrams, figures 1 and 2, we vary the frequency of the magnetic field, Ω, from -3 to 3 using a step of 0.01, and for each frequency Ω we solved the equation (8) for $t \in [0, 500]$ using the RK4 method, then applied the Fourier transform and filtered out the frequencies with amplitude less than 20% of the maximum amplitude. Finally, for each Ω we plotted the frequencies obtained from the Fourier transform ω, creating a graph (Ω, ω) where the ω frequencies represent the frequency response of the system.

3 First Results

In this Section, we'll briefly discuss the dynamics of the equation (8), with mostly numerical results.

The dipole's dynamics was studied for small amplitude of oscillation of the magnetic field. We used two main values for the parameter ε, $\varepsilon = 0.05$ and $\varepsilon = 0.5$,

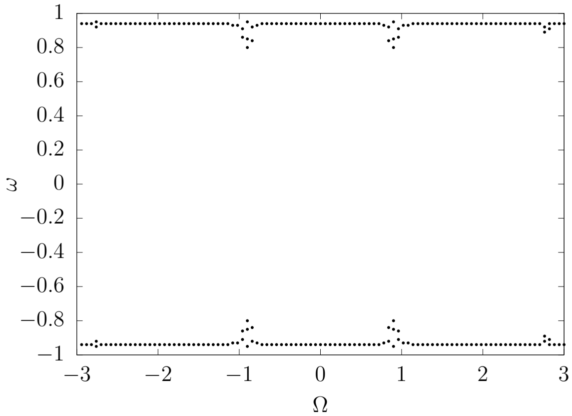


Figure 1: Bifurcation diagram of frequencies from the oscillating magnetic field Ω and the response frequencies on the dipole ω, for the case with $\varepsilon = 0.05$.

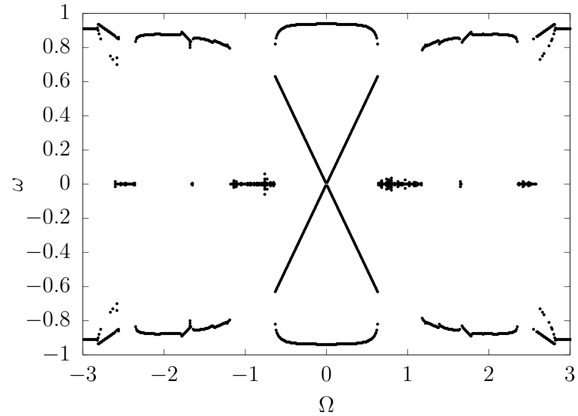


Figure 2: Bifurcation diagram of frequencies from the oscillating magnetic field Ω and the response frequencies on the dipole ω, for the case with $\varepsilon = 0.5$.

and the following initial conditions for both cases:

$$\begin{cases} \theta(0) & = 1 \\ \dot{\theta}(0) & = 0 \end{cases} \tag{17}$$

In the former case, $\varepsilon = 0.05$, we verify that the dipole's movement was periodic for all the values of the frequency of the magnetic field Ω in $[-3, 3]$, and as expected, the dipole behaved like a simple pendulum.

However, for the latter case, $\varepsilon = 0.5$, the dipole's movement was much more complex. For some subintervals in the frequency range in $[-3, 3]$, the movement was not periodic, figure 3b, and maybe not even limited.

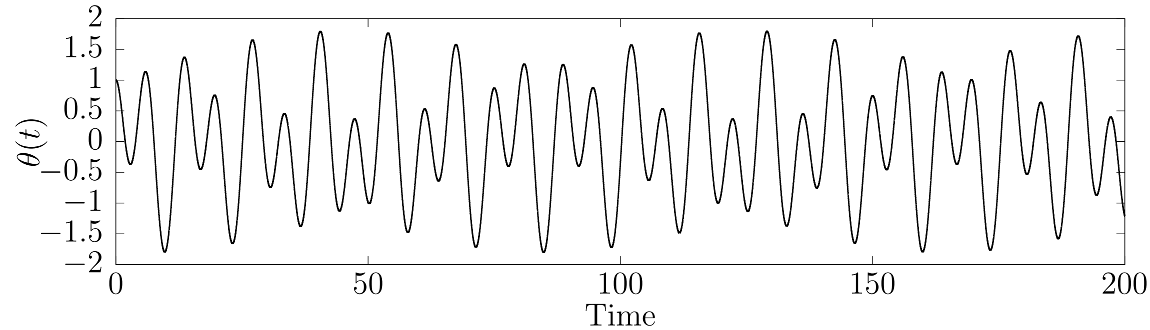
In the cases where the movement was oscillatory, the oscillation frequency was not comparable to the frequency of the magnetic field, and for the most time, the frequency of the movement was the natural frequency of the system.

4 Conclusion

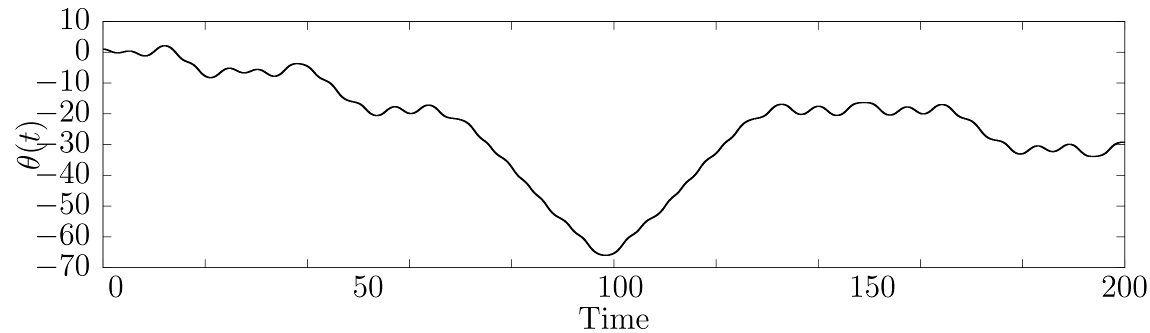
In this work we have investigated the dynamics of a dipole in an oscillating magnetic field, we found two main regimes, an oscillatory and a rotational one. In the oscillatory regime, the dipole behaves like a simple pendulum, where the frequency of oscillation is close related to it's natural frequency. However, in the rotational regime, the dipole shown to excitable for small oscillations of the magnetic field, at specific frequencies.

In order to investigate the system fully equipped with three magnets, we need to develop new methodologies, given that the movement in this case can present a dynamic even more complex than the one we studied here.

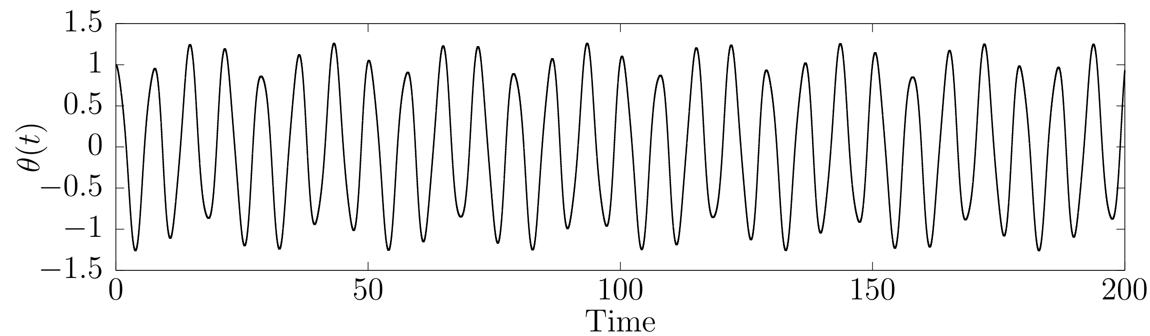
List of Figures



(a) Movement of the dipole with frequency of the external magnetic field being 0.5.



(b) Movement of the dipole with frequency of the external magnetic field being 0.75.



(c) Movement of the dipole with frequency of the external magnetic field being 2.0.

Figure 3: Movement of the dipole for different frequencies of the magnetic field. Using $\varepsilon = 0.5$

References

[1] E. M. Clements, Raja Das, Manh-Huong Phan, Ling Li, Veerle Keppens, David Mandrus, Michael Osofsky, and Hariharan Srikanth. Magnetic field dependence of nonlinear magnetic response and tricritical point in the monoaxial chiral helimagnet $\mathrm{cr}_{1/3}\mathrm{Nbs}_2$. *Phys. Rev. B*, 97:214438, Jun 2018.

[2] V. L. Bratman, G. G. Denisov, N. S. Ginzburg, and M. I. Petelin. Fel's with bragg reflection resonators: Cyclotron autoresonance masers versus ubitrons. *IEEE Journal of Quantum Electronics*, 19, 1983.

[3] A. Loeb and L. Friedland. Autoresonance laser accelerator. *Physical Review A*, 33, 1986.

[4] M. et al. Veranda. Helically self-organized pinches: dynamical regimes and magnetic chaos healing. *Nucl. Fusion 60 016007*, 2019.

[5] Steven L. Brunton, Bingni W. Brunton, Joshua L. Proctor, Eurika Kaiser, and J. Nathan Kutz. Chaos as an intermittently forced linear system. *Nature Communications*, 8, 2017.

[6] Tom Chang. Self-organized criticality, multi-fractal spectra, sporadic localized reconnections and intermittent turbulence in the magnetotail. *Physics of Plasmas*, 6, 1999.

[7] D. Laroze, P. Vargas, C. Cortes, and G. Gutierrez. Dynamics of two interacting dipoles. *Journal of Magnetism and Magnetic Materials*, 320(8):1440–1448, 2008.

[8] Nicolas Plihon, Sophie Miralles, Mickael Bourgoin, and Jean-Fran çois Pinton. Stochastic reversal dynamics of two interacting magnetic dipoles: A simple model experiment. *Phys. Rev. E*, 94:012224, Jul 2016.