Chapter 2

1. Download and familiarize yourself with LaTeX and Inkscape.

2. A one period model is completely specified by its two prices processes B and S. Consider two versions of the one period model, (B', S') and (B'', S''), both arbitrage free.

   - Suppose $B_1' = B_1''$ and $S_t' = \alpha S_t''$ for $t = 0, 1$ for some positive constant α. What is the relationship between their martingale measures?
   - Suppose $B_1' = B_1''$, $S_t' = \alpha S_t''$ for $t = 1$, but $S_0' > \alpha S_0''$ for some positive constant α. What is the relationship between their martingale measures?
   - Suppose $B_1' < B_1''$ and $S_t' = \alpha S_t''$ for $t = 0, 1$ for some positive constant α. What is the relationship between their martingale measures?

3. Exercise 2.1.

4. Suppose we modified the one period model in the following way: $B_0 = b_0$ and $B_1 = b_1$ for some positive constants b_0 and b_1. S is still the same as in the book. Notice, the concepts of portfolios, value processes, and arbitrage still make sense in the modified model.

   - What should be the statement of Proposition 2.3?
   - What should be the statement of Definition 2.4?
   - What should be the formula for Q?

   In Class: As a student, you are usually just handed models and you take it for granted that they are the "right" ones. In the real world, you need to create the "right" model. There is a big gap between being able to solve problems in a model given to you and figuring what is the "right" model to use. This exercise helps us think a little more like in the real world.

   Create a one period trinomial model in the spirit of the binomial model. What should the concepts of portfolios, value processes, and arbitrage be?

   - What should Propositions 2.3, 2.5, 2.6 be?
   - How does the trinomial model differ from the binomial model in terms of the pricing of contingent claims?

5. Consider a two period ($T = 2$) binomial model with a stock S and a bond B. Assume $S_0 = B_0 = 1$ and $R = 0, d = \frac{2}{3}, u = \frac{3}{2}$.

   - Is the market arbitrage free?

- Introduce the contingent claim $X = \Phi(S_2)$ where $\Phi(\frac{9}{4}) = 1$ and Φ is zero for all other values of S_2. Write down a replicating portfolio strategy h for X. What is the date 0 arbitrage free price of X?

6. Exercise 2.3.

7. Familiarize yourself with induction.

   - Prove $1^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for all positive integers n.
   - Prove $4^n + 15n - 1$ is divisible by 9 for all positive integers n.

8. Exercise 2.4.

Chapter 3

1. Example of Farkas' Lemma: Suppose $\mathbb{R}^N = \mathbb{R}^3$, and there are 4 column vectors $d_0, d_1 = (1,0,0), d_2 = (0,1,0), d_3 = (0,0,1)$.

   - Suppose $d_0 = (1,1,1)$. Prove directly that there exists a solution to Problem 1 but not Problem 2.
   - Suppose $d_0 = (a,b,c)$ with $a,b,c \geq 0$. Prove directly that there exists a solution to Problem 1 but not Problem 2.
   - Suppose $d_0 = (-1,-1,-1)$. Prove directly that there exists a solution to Problem 2 but not Problem 1.
   - Suppose $d_0 = (a,b,c)$ such at least one coordinate is negative. Prove directly that there exists a solution to Problem 2 but not Problem 1.

2. Recall the matrix D from chapter 3. Introduce the matrix $\overline{D}$: The first column is the date 0 price of the assets while the rest of the matrix is D. Assume as in the book $P(\omega_j) > 0$ for all j.

   - Consider a $\overline{D} = \begin{bmatrix} 1 & 1 & 2 & 3 \end{bmatrix}$. Is there an arbitrage? If so construct an arbitrage portfolio. If not, find **all** martingale measures of the corresponding Z-market.
   - Consider $\overline{D} = \begin{bmatrix} 1 & 1.2 & 1.2 & 1.2 \\ 2 & 2.4 & 2.5 & 2.4 \end{bmatrix}$. Is there an arbitrage? If so construct an arbitrage portfolio. If not, find **all** martingale measures of the corresponding Z-market.

     Notice, this market has a risk-free asset with $R = 0.2$. Find a measure Q over the date 1 state space such that for both assets $i = 1, 2$, we have $S_0^i = \frac{1}{1+R}\mathbf{E}^Q S_1^i$. How do you reconcile the existence of Q with your answer to the first bullet point's question?

3. Consider the following trinomial economy:

$$\mathrm{S}(0) = 105, \quad \mathrm{B}(0) = 100 \quad \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} \quad \begin{array}{c} 120, \quad 105 \\ 105, \quad 105 \\ 100, \quad 105 \end{array}$$

- Is the market complete? Why or why not?

  Define $(AD_{\uparrow}, AD_{\rightarrow}, AD_{\downarrow})$ as the date 0 prices of the *Arrow-Debreu securities* (a type of contingent claim) that pay 1 if and only if the up state, middle state, and down state occur, respectively. We now investigate under what conditions the market consisting of the stock, bond, and all three Arrow-Debreu securities remains arbitrage free.

- Use the stock and bond price dynamics to find two constraints on the values of these AD securities. Use these constraints to write both $AD_{\uparrow}$ and $AD_{\rightarrow}$ solely as a function of $AD_{\downarrow}$.
- Identify the range of values of $AD_{\downarrow}$ that ensure the market is arbitrage free.
- Introduce the call option with strike $K = 105$. Identify the range of arbitrage-free prices.
- Introduce the call option with strike $K = 100$. Identify the range of arbitrage-free prices. This answer is qualitatively different than the pervious one. Why?

In Class: Consider the following market:

$$\overline{D} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1/2 & 1 & 1 & 1/2 \end{bmatrix},$$

As in the book, denote the two assets by S^1 and S^2.

- Is there a risk-free rate?
- Is the market arbitrage free?

In Class: If (S^1, S^2) contains an arbitrage, is it possible for (S^1, S^2, S^3) to be arbitrage free for some S^3?

In Class: If (S^1, S^2) is arbitrage free and complete, is it possible for (S^1, S^2, S^3) to be arbitrage free but incomplete for some S^3?

In Class: Suppose there exist S^i, S^j such that $S_0^i = S_0^j$, $S_1^i(\omega_k) \geq S_1^j(\omega_k)$ for all k and there exists some k^* such that $S_1^i(\omega_{k^*}) > S_1^j(\omega_{k^*})$. Prove the market is not arbitrage free.

In Class: Consider three hypothetical Kalshi contracts: 1. Next month fed will raise rates. 2. Next month unemployment will rise. 3. Next month fed will raise rates and unemployment will fall.

- Suppose right now, the prices for these contracts are x, y, and z. How would you go about telling if there is arbitrage?

4. Recall the matrix D from chapter 3. Introduce the matrix $\overline{D}$: The first column is the date 0 price of the assets while the rest of the matrix is D. Assume as in the book $P(\omega_j) > 0$ for all j. Consider the following market:

$$\overline{D} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & 3 & 3 \end{bmatrix},$$

As in the book, denote the two assets by S^1 and S^2.

- Is there a risk-free rate for this market?
- Fixing S^1 as numeraire, what are all the martingale measures?
- Describe another asset S^3 such that the market (S^1, S^2, S^3) is complete *but not arbitrage free.*
- Introduce the asset X where $X_0 = 1$ and $X_1 = [3\ 0\ \frac{8}{3}]$ and consider the market (S^1, S^2, X). Fixing S^1 as numeraire, is there a martingale measure? If so what is it? If not, find an arbitrage portfolio.

5. Consider the following market

$$\overline{D} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 8 & 4 & 2 \end{bmatrix}$$

As in the book, denote the two assets by S^1 and S^2.

a. Create the normalized market by choosing S^1 as numeraire. Find all martingale measures.

   Express you answer in the following way: Let q_1 denote the probability of ω_1 under an arbitrary martingale measure. Characterize the set of martingale measures by stating what values q_1 can take, and, for each such q_1, what are the corresponding probabilities for ω_2 and ω_3.

   For example, you could write: "*the set of martingale measures is* $\{(q_1, \frac{1}{2} + q_1, \frac{1}{2} - 2q_1) \mid q_1 \in (0, \frac{1}{4})\}$." (This is the wrong answer of course).

b. True of False. If we choose S^2 instead of S^1 as numeraire, we still generate the same set of martingale measures. (No need to show work.)

c. A put option $PutK$ on S^2 with strike price K maturing at date 1 is a contingent claim with date 1 payoff $PutK_1 = \max\{K - S^2, 0\}$. What is $Put5_1(\omega_i)$ for $i = 1, 2, 3$?

d. Suppose the date 0 price of $Put5$ is $Put5_0 = \frac{7}{8}$. Is the market $(S^1, S^2, Put5)$ arbitrage free?

e. Continuing to assume the date 0 price of $Put5$ is $\frac{7}{8}$. What is the arbitrage free date 0 price of $Put4$?

6. Exercise 3.2.

7. Exercise 3.3.

Chapter 4

1. Exercise 4.1.

2. Exercise 4.2.

3. Let W be Brownian motion.
   - Let p be any integer ≥ 2. Provide a formula for $\int_0^t W(s)^{p-1} dW(s)$ that does not involve a stochastic integral.
   - Use induction to show that $\mathbf{E}W(t)^p = t^{\frac{p}{2}}(p-1)!!$ for all even p. (Definition: For an odd number k, the expression $k!!$ means $1 \cdot 3 \cdot \ldots \cdot k$. For example, $7!! = 1 \cdot 3 \cdot 5 \cdot 7$.)
   - What is $\mathbf{E}W(t)^p$ for all odd p?

4. Exercise 4.4.

5. Exercise 4.5.

6. Exercise 4.6.

7. Exercise 4.7.

8. Exercise 4.8.

9. Let $Z_t = te^{W_t}$ where W_t is Brownian motion. Find dZ_t.

10. Let $D_t = (X_t + Y_t)^2$ where X_t and Y_t are two independent Brownian motions. Compute $\mathbf{E}D_{10}$.

11. Let $dX_t = X_t dW_t$ where W_t is Brownian motion. Define $Z_t = X_t^\gamma$ where γ is a positive constant. Find dZ_t. Simplify the expression so that the SDE is a function of Z_t instead of X_t.

12. Let $Z_t = \exp(X_t + Y_t - t)$ where X_t and Y_t are two independent Brownian motions. Compute $\mathbf{E}Z_{10}$.

Chapter 5

1. Exercise 5.1.

2. Exercise 5.6.

3. Exercise 5.7.

4. Exercise 5.8.

5. Let W be Brownian motion.

   - Compute $\mathbf{E}e^{at+bW(t)}$ where a and b are constants.
   - Solve the following boundary value problem in the domain $[0, T] \times \mathbb{R}$,

   $$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0 \\ F(T,x) = A + Bx \end{aligned}$$

   where r, σ, A, B are all constants. Simplify your expression as much as possible.

6. Exercise 5.9.

7. Exercise 5.10.

8. Exercise 5.11.

9. Exercise 5.12.

10. Exercise 5.15.

11. Exercise 5.16.

Chapter 6

1. Consider a market with the following assets:

$$\begin{aligned} dB_t &= 0dt \\ dS_t &= S_t dW_t \end{aligned}$$

with $B_0 = S_0 = 1$.

Introduce the following self-financing portfolio strategy $h_t = (h_t^B, h_t^S)$: At each date, the strategy puts half of the value of the portfolio in B and S each. Derive the stochastic differential equation for h_t^S.

(Hint: Assume h_t^S is an Ito process, satisfying $dh_t^S = \mu_t^S dt + \sigma_t^S dW_t$ for some adapted processes μ_t^S and σ_t^S. Then solve for μ_t^S and σ_t^S.)

In Class: Let's derive the stochastic differential equation for h_t^B.

Chapter 7

1. Exercise 7.2.
2. Exercise 7.4.
3. Exercise 7.5.

   In Class: Exercise 7.7.
4. Exercise 7.6.
5. Exercise 7.9.
6. Consider the following Black-Scholes model over the time interval $[0, T]$:

$$\begin{aligned} dB(t) &= rB(t)dt \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW^P(t) \end{aligned}$$

   where $\sigma > 0, r, \alpha$ are constants and W^P is Brownian motion under the objective measure P. A forward contract on $S(T)^2$ is created at date t with delivery date T. Find the forward price by first writing it as an expectation and then simplifying the expression down to a function of $S(t)$.

7. Consider the following generalization of the Black-Scholes model over the time interval $[0, T]$:

$$\begin{aligned} dB(t) &= rB(t)dt \\ dS(t) &= \alpha S(t)dt + \sigma_1 S(t)dW_1^P(t) + \sigma_2 S(t)dW_2^P(t) \end{aligned}$$

   where W_1^P and W_2^P are *correlated* Brownian motions under the objective measure P with constant correlation $\rho > 0$.

   - Rewrite the SDE for S so that it is being driven by a single Brownian motion under P.

- Use the Black-Scholes formula to provide a similar formula for the date t price of the T-claim $\max\{S(T)^\beta - K, 0\}$ where β and $K > 0$ are constants.

Chapter 8

1. Exercise 8.3.

Chapter 10

1. 10.1.
2. 10.2.
3. 10.3.
4. Consider the following Black-Scholes model over the time interval $[0, T]$:

$$
\begin{aligned}
dB(t) &= rB(t)dt \\
dS(t) &= \alpha S(t)dt + \sigma S(t)dW^P(t)
\end{aligned}
$$

   where $B(0) = S(0) = 1$ and $\sigma > 0, r, \alpha$ are constants and W^P is Brownian motion under the objective measure P. Introduce the following T-claim:

$$
X = \Phi(S(T)) = \begin{cases} K & \text{if } S(T) \leq A \\ K + A - S(T) & \text{if } S(T) \in [A, K + A] \\ \frac{K+A}{2} - \frac{S(T)}{2} & \text{if } S(T) \geq K + A \end{cases}
$$

   - Create a constant replicating portfolio consisting solely of B, S and European call options.
   - Let $c(t, s, K)$ and $\Delta(t, s, K)$ denote the date t price and delta, respectively, of a European call on $S(T)$ maturing at date T with strike K given $S(t) = s$.

     Consider the portfolio consisting of a single unit of X at time t, given $S(t) = s$ and $B(t) = e^{rt}B(0)$. The holder of the portfolio wants to readjust so that

     1. Δ of the portfolio becomes zero,
     2. the total value of the portfolio stays the same,
     3. the portfolio continues to feature exactly 1 unit of X.

     Readjustment can only be done by buying/shorting units of the underlying assets, B and S. What is the value of the portfolio? How many units of B and S should the readjusted portfolio feature? Express your answers in terms of c, Δ, t, and the parameters of the model.

5. 10.6.

6. **Project**: Today, you want to create a call option that gives the holder the option to exchange some Tesla stock for some Apple stock on May 7th. Design such an option that would attract a decent amount of investors and figure out a fair price to charge. *There are no right answers. But there are good answers and bad answers.*