Econ 780 HW Chapter 4

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Due:

1 Exercise 4.4

$dX_t = \alpha X_t dt + \sigma_t dW_t$ when $\alpha \in R + \sigma$ is adapted
find $E[X_t]$
$\int X_t\, dt + \alpha \int X_t\, dt + \int \sigma_t\, dW_t$
$EX_t = \alpha \int E[X_t]\, dt$
$E[X_t] = \alpha \int_{s=0}^{s=t} m_s\, ds + m_s$
$M_t = \alpha \int_{s=0}^{s=t} m_s\, ds + m$
$M_t^* = \alpha M_t \rightarrow M_t = E^{\alpha t} + M_0$

2 Exercise 4.5

$dX_t = \alpha X_t dt + \sigma_t dW_t$ and $U_t \geq 0$ with $P = 1$ for all $k > 0$ show that X is submartingale
$E[X_t \mid F_S] \geq X_S$ with $S \leq t$
$X_t = \int_0^t \mu_t\, dt + \int_0^t \sigma_t\, dWt + X_0$
And for martingale: $dX_t = g_t dWt$
$E[X_t] = \int_0^t E(\mu_p)\, dp + X_0$
$= \int_0^t m_p\, dp + X_0$ where $m_s \geq 0$
thus $E[X_t] - E[X_s] = \int_0^t m_p\, dp - \int_0^s m_p\, dp$
$= \int_s^t m_p\, dp \geq 0$

Alternate proof:
$X_t = \int_0^t \mu_t\, dt + \int_0^t\, dWp + X_0$
$X_t = \int_0^t \mu_t\, dt + \int_0^t\, dWp + X_S$
$E[X_t] = \int_s^t E[\mu_p]\, dp + 0 + E[X_S]$
$E[X_t] = \int_s^t m_p\, dp + E[X_S]$ where $m_p \geq 0$
but $E[X_t \mid F_S] = \int_s^t m_p\, dp + E[X_S \mid F_S]$
$= \int_s^t m_p\, dp + X_S$ which is then deterministic due to the filter
thus $E[X_t \mid F_S] \geq X_S$

3 Exercise 4.6

Set $X_t = h(x_1, x_2, \cdots x_n)$ The Wieness is given by:
$W_1(t), \cdots, W_n(t)$ Apple Ito formula:

$$dX_t = \frac{1}{2}\left[\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t) dW_j(t)\right] + \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t)$$

Intergate both sides:

$$X_t = \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t) + \int_0^t \frac{1}{2}\left[\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t) dW_j(t)\right]$$

$$X_t = \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t) + \frac{1}{2}\int_0^t \left[\sum_{i \neq j}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t) dW_j(t)\right] + \frac{1}{2}\int_0^t \left[\sum_{i=j=1}^{n} \frac{\partial^2 h}{\partial {x_i}^2} dW_i(t)^2\right]$$

$$X_t = \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t) + \frac{1}{2}\int_0^t \left[\sum_{i \neq j}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t) dW_j(t)\right] + \frac{1}{2}\int_0^t \left[\sum_{i=j=1}^{n} \frac{\partial^2 h}{\partial {x_i}^2} ds\right]$$

Let the middle term = A, we have:

$$X_t = \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(s) + \frac{1}{2}\int_0^t \left[\sum_{i=j=1}^{n} \frac{\partial^2 h}{\partial {x_i}^2} ds\right] + A$$

We have:

$$dW_i dW_j = \rho_{i,j} t = 0$$

Since W_i and W_j are independent, $\rho i, j = 0$.
Hence $A = 0$

Take expectation both sides, conditional on the filtration F_W.

$$E(X_t \mid F_S) = E\left[\int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(s) \mid F_S\right] + \frac{1}{2} E \int_0^t \left[\sum_{i=j=1}^{n} \frac{\partial^2 h}{\partial {x_i}^2} ds \mid F_S\right]$$

Implies:

$$E(X_t \mid F_S) = E\left[\int_0^s \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i(s) \mid F_S\right] + \frac{1}{2} E \int_0^s \left[\sum_{i=j=1}^n \frac{\partial^2 h}{\partial {x_i}^2} ds \mid F_S\right]$$
$$+ E\left[\int_s^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i(s) \mid F_S\right] + \frac{1}{2} E \int_s^t \left[\sum_{i=j=1}^n \frac{\partial^2 h}{\partial {x_i}^2} ds \mid F_S\right]$$

Implies:

$$E(X_t \mid F_S) = X(S) + E\left[\int_s^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i(s) \mid F_S\right] + \frac{1}{2} E \int_s^t \left[\sum_{i=j=1}^n \frac{\partial^2 h}{\partial {x_i}^2} ds \mid F_S\right]$$

We have h is harmonic, so:

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} = 0 \implies \frac{1}{2} E \int_s^t \left[\sum_{i=j=1}^n \frac{\partial^2 h}{\partial {x_i}^2} ds \mid F_S\right] = 0$$

Also:

$$E\left[\int_s^t \sum_{i=1}^n \frac{\partial h}{\partial x_i} dW_i(s) \mid F_S\right] = 0$$

Since it is an expectation of a stochastic intergal.
Hence $E(X_t \mid F_S] = X(S)$.QED Similarly for h is a subharmonic function.