# Econ 780 HW Chapter 3

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Due:

## Exercises for chapter 3

1. Recall the matrix $D$ from chapter 3. Introduce the matrix $\overline{D}$: The first column is the date 0 price of the assets while the rest of the matrix is $D$. Assume as in the book $P(\omega_j) > 0$ for all $j$. Consider the following market:

$$\overline{D} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & 3 & 3 \end{bmatrix},$$

As in the book, denote the two assets by $S^1$ and $S^2$.

- Is there a risk-free rate for this market?
  - We found that the initial prices could be generated by

$$(1/6 - 3/6z_3, 1/3 - 3/3z_3, z_3)$$

We require that all three be positive, which happens when $z_3$ is between 0 and 1/3. This means the sum of the numbers is between 1/2 and 1/3 and $R$ must be between 1 and 2.
Minh Comment. I think there is no risk free rate on that market, because if there is a "mix" can yield equal value for the three state of the world. so the last state will be 3(x+y), which is never can equal 2x+3y in state 2!?

- Fixing $S^1$ as numeraire, what are all the martingale measures?
  - Not sure on this one.
Minh solution The numeraire market given by:

$$\overline{D} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1.5 & 1 \end{bmatrix},$$

Let $q_1, q_2, q_3 > 0, q_1 + q_2 + q_3 = 1$ be a martingale measure. we have the system of equation:

$$1 = q_1 + q_2 + q + 3$$

$$1 = 1.5q_2 + q_3$$

solving we have the family of martingale measure:

$$\{q_1, 2q_1, 1-3q_1 \mid 0 < q_1 < \frac{1}{3},\}$$

- Describe another asset $S^3$ such that the market $(S^1, S^2, S^3)$ is complete *but not arbitrage free.*
  - An additional asset could be added to make the following matrix:

$$\overline{D} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & 3 & 3 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- Introduce the asset $X$ where $X_0 = 1$ and $X_1 = [3\ 0\ \frac{8}{3}]$ and consider the market $(S^1, S^2, X)$. Fixing $S^1$ as numeraire, is there a martingale measure? If so what is it? If not, find an arbitrage portfolio.
  - The new matrix is

$$D = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 3 \\ 3 & 0 & 8/3 \end{bmatrix}$$

    Which is non-singular. Thus, any way to get $S_0$ from linear combinations of these columns is the only linear combination that works. We calculated that linear combination, and we got negative values for the first two vectors, which means there is arbitrage. We just need to find an arbitrage portfolio. Any holding vector whose components add to zero is free because the initial prices are all 1. So we need to find a holding vector whose components add to 1 and give positive or zero dot products with the columns of $D$ (with at least one being positive). We could do that with $h = (1, -.5, -.5)$

2. Consider the following market

$$\overline{D} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 8 & 4 & 2 \end{bmatrix}$$

As in the book, denote the two assets by $S^1$ and $S^2$.

a. Create the normalized market by choosing $S^1$ as numeraire. Find all martingale measures.

Express you answer in the following way: Let $q_1$ denote the probability of $\omega_1$ under an arbitrary martingale measure. Characterize the set of martingale measures by stating what values $q_1$ can take, and, for each such $q_1$, what are the corresponding probabilities for $\omega_2$ and $\omega_3$.

For example, you could write: "*the set of martingale measures is* $\{(q_1, \frac{1}{2} + q_1, \frac{1}{2} - 2q_1) \mid q_1 \in (0, \frac{1}{4})\}$." (This is the wrong answer of course).

- the set of martingale measures is $\{(q_1, 1 - 3q_1, 2q_1) \mid q_1 \in (0, \frac{1}{3})\}$

b. True of False. If we choose $S^2$ instead of $S^1$ as numeraire, we still generate the same set of martingale measures. (No need to show work.)

- True. It's just scaling the vectors, so our linear combination numbers would be different, but we scale those anyway to make the probability measure. This means the answer to this problem would be the same if we didn't scale at all or choose a numeraire.

c. A put option $PutK$ on $S^2$ with strike price $K$ maturing at date 1 is a contingent claim with date 1 payoff $PutK_1 = \max\{K - S^2, 0\}$. What is $Put5_1(\omega_i)$ for $i = 1, 2, 3$?

- $Put5_1(\omega_i) = (0, 1, 3)$

d. Suppose the date 0 price of $Put5$ is $Put5_0 = \frac{7}{8}$. Is the market $(S^1, S^2, Put5)$ arbitrage free?

- i THINK THERE IS SOMETHING WRONG WITH THIS ANSWER!!!
- No it is not. To prove this, we try to create 7/8 with the dot product of our martingale measures above and $(0, 1, 3)$. This gives us $7/8 = 1 + 3q_1$, but $q_1$ must be positive, so we have a contradiction.
  Minh solution:

$$\overline{D} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 1 \\ \frac{7}{8} & 0 & 0.5 & 1.5 \end{bmatrix}$$

  To find the martingale measure, we need to solve:

$$q_1 + q_2 + q_3 = 1$$

$$4q_1 + 2q_2 + q_3 = 2$$

$$0.5q_2 + 1.5q_3 = \frac{7}{8}$$

  Obtain: $\{0.25, 0.25, 0.5\}$ is the unique martingale measure.

e. Continuing to assume the date 0 price of $Put5$ is $\frac{7}{8}$. What is the arbitrage free date 0 price of $Put4$?

- The replicating holding vector is $(-4/3, 1/3, 4/3)$, and $hS_0$ is 0.5.
- Minh solution The arbitrage free price of Put4 Should be
  $V_0{}^P ut4 = 0.25 * 0 + 0.25 * 0 + 0.5 * 2 = 1$

3. Exercise 3.2.

- If $X$ is replicable, then:

$$\exists h; hS_1 = X$$

- Now, using the definition of the value of a portfolio:

$$V_0^h = hS_0$$

- If there is no arbitrage, then there is a column vector, $z$ such that $S_0$ can be formed by $z$'s scaling of $S_1$'s columns. Thus,

$$\exists z; S_0 = S_1 z$$

- We can use this equation with the second bullet to get:

$$V_0^h = hS_0 = hS_1 z$$

- If we want to turn z into a vector that scales to 1 (to make it a probability measure), we would want to divide its components by the sum of all the components, i.e. to normalize it. So to do this, we can multiply and divide by the sum, and we can call the new normalized z, q:

$$V_0^h = hS_0 = hS_1 z \frac{\Sigma z_i}{\Sigma z_i} = \Sigma z_i * hS_1 q$$

- But we know that $\Sigma z_i$ has a name, and it is $\beta$, which is also $1/1+R$. Thus,

$$V_0^h = hS_0 = hS_1 z \frac{\Sigma z_i}{\Sigma z_i} = \Sigma z_i * hS_1 q = \frac{1}{1+R} hS_1 q$$

- Now we use our existence in the very first step, which is our replicating portfolio. Thus, we get (and removing the intermediate steps):

$$V_0^h = \frac{1}{1+R} hS_1 q = \frac{1}{1+R} Xq$$

- And lastly, we say that $Xq$ is the expectation of $X$ under $S$:

$$V_0^h = \frac{1}{1+R} Xq = \frac{1}{1+R} \mathbb{E}^Q[X]$$

4. Exercise 3.3.

   - We want to prove that the arbitrage free price of an claim $X$ is given by:

$$\Pi(0, X) = \mathbb{E}^P[\Lambda(\omega) X(\omega)]$$

- We start with the definition of $\Lambda(\omega)$:

$$\Lambda(\omega) = \frac{1}{1+R}L(\omega) = \frac{1}{1+R}\frac{q_i}{p_i}$$

- We now substitute this into the first equation:

$$\Pi(0, X) = \mathbb{E}^P[\frac{1}{1+R}\frac{q_i}{p_i}X(\omega)] = \frac{1}{1+R}\mathbb{E}^P[\frac{q_i}{p_i}X(\omega)]$$

- We now expand the expectation:

$$\Pi(0, X) = \mathbb{E}^P[\Lambda(\omega)X(\omega)] = \frac{1}{1+R}\mathbb{E}^Q[X]$$

Which was our goal.