

Econ 780    HW Week 2  
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Due:

### Exercises for chapter 3

#### Exercise 1

- Let  $d_0 = (1, 1, 1)^T, d_1 = (1, 0, 0)^T, d_2 = (0, 1, 0)^T, d_3 = (0, 0, 1)^T$ .

Let  $z_1 = 1, z_2 = 1, z_3 = 1$

Hence:

$$d_0 = (1, 1, 1)^T = z_1(1, 0, 0)^T + z_2(0, 1, 0)^T + z_3(0, 0, 1)^T$$

Since  $z_1 = 1, z_2 = 1, z_3 = 1$  are nonnegative. Q.E.D

- No solution for Problem 2

Suppose there exist  $h = (x, y, z) \in R^3$   
such that:

$$hd_0 = x + y + z < 0$$

and

$$hd_j \geq 0, j = 1, 2, 3$$

$$hd_j \geq 0, j = 1, 2, 3 \implies x, y, z \geq 0 \implies x + y + z \geq 0$$

A contradiction. Q.E.D

- $d_0 = (a, b, c)^T, a, b, c \geq 0$ .  
let  $z_1 = a, z_2 = b, z_3 = c$ .  
Clearly  $z_1, z_2, z_3 \geq 0$ . and  $d_0 = (a, b, c)^T = a(1, 0, 0)^T + b(0, 1, 0)^T + c(0, 0, 1)^T =$   
 $z_1(1, 0, 0)^T + z_2(0, 1, 0)^T + z_3(0, 0, 1)^T = z_1d_1 + z_2d_2 + z_3d_3$ . Q.E.D  
The solution is simply (a,b,c)

- No solution for problem 2

Suppose there exist  $h = (x, y, z) \in R^3$  such that:

$$hd_0 < 0$$

and

$$hd_j \geq 0, j = 1, 2, 3$$

$$hd_0 < 0 \implies ax + by + cz < 0$$

If a, b, and c are all positive, at least one of x, y, and z must be negative.

$$hd_j \geq 0, j = 1, 2, 3 \implies x, y, z \geq 0$$

This is essentially selecting the components because each d is a euclidian basis vector. So here we are saying that all of x, y, and z are positive. This is the contradiction. Since  $a, b, c \geq 0 \implies ax + by + cz \geq 0$  a contraction. Q.E.D

- $d_0 = (-1, -1, -1)^T$

Suppose there exist  $z_1, z_2, z_3 \geq 0$  such that.

$$(-1, -1, -1)^T = d_0 = z_1 d_1 + z_2 d_2 + z_3 d_3 = (z_1, z_2, z_3)^T$$

Hence  $z_1 = z_2 = z_3 = -1$  Here, we have found a solution, and in this solution, at least one component is negative (in fact all are). And because our matrix is invertible, there is only a single solution to the equation, namely this solution. Therefore, there cannot be a solution with all components positive.

Prove there exist a solution for problem 2.

Let  $h = (1, 1, 1)$

We have

$$hd_0 = -3 < 0$$

and

$$hd_j = 1 \geq 0, j = 1, 2, 3$$

Q.E.D

- $d_0 = (a, b, c)^T$ , at least one coordinate is negative.

WLOG, suppose  $a < 0$ .

For problem 2, let  $h = (1, 0, 0)$ , we have

$$hd_0 = a < 0$$

and

$$hd_1 = 1, hd_2 = hd_3 = 0$$

Q.E.D.

In general, put zero in the place that  $d_0$  doesn't have negative coordinate and

1 in the place that  $d_0$  have negative coordinate.

- For problem 1.

Suppose there exist  $z_1, z_2, z_3 \geq 0$  such that.

$$(a, b, c)^T = z_1 d_1 + z_2 d_2 + z_3 d_3 = (z_1, z_2, z_3)^T$$

Hence there at least one  $i$  such that  $z_i < 0$ . Q.E.D

Exercise 2.

Consider the augmented matrix.

$$\overline{D} = \begin{bmatrix} 1 & 1 & 2 & 3 \end{bmatrix}.$$

There is no arbitrage. since there only 1 matrix, so the only choice to make the value of the portfolio is zero in day zero is buy nothing, but it will turn out that the value of the portfolio at day 1 will be zero. So there will be no arbitrage opportunity.

let  $Q(w_1), Q(w_2), Q(w_3)$  be Martingale measure for the market.

We have the formula.

$$1 = \beta(Q(w_1) + 2Q(w_2) + 3Q(w_3))$$

and

$$Q(w_1) + Q(w_2) + Q(w_3) = 1$$

.

For some  $\beta$

For simplicity, I will use  $Q_1, Q_2, Q_3$  instead of  $Q(w_1), Q(w_2), Q(w_3)$

$$1 = \beta(Q(w_1) + 2Q(w_2) + 3Q(w_3)) \implies \frac{1}{\beta} = Q_1 + 2Q_2 + 3Q_3$$

Let write  $Q_1 = 1 - Q_2 - Q_3$  and substitute to the equation above, we obtain:

$$\frac{1}{\beta} = 1 - Q_2 - Q_3 + 2Q_2 + 3Q_3 = 1 + Q_2 + 2Q_3 \implies Q_2 = \frac{1}{\beta} - 1 - 2Q_3$$

again, substitute  $Q_2 = 1 - Q_1 - Q_3$  Obtain

$$\frac{1}{\beta} = Q_1 + 2(1 - Q_1 - Q_3) + 3Q_3 = 2 - Q_1 + Q_3 \implies Q_1 = 2 - \frac{1}{\beta} + Q_3$$

Now we will find the bounds for  $Q_2$  to be legitimate probability measure.

- $0 < Q_1 < 1 \implies 0 < 2 - \frac{1}{\beta} + Q_3 < 1 \implies \frac{1}{\beta} - 2 < Q_3 < \frac{1}{\beta} - 1$
  - $0 < Q_2 < 1 \implies 0 < \frac{1}{\beta} - 1 - 2Q_3 < 1 \implies \frac{\frac{1}{\beta} - 2}{2} < Q_3 < \frac{\frac{1}{\beta} - 1}{2}$
- Combine these two inequalities.

$$\frac{1 - 2\beta}{2\beta} < Q_3 < \frac{1 - \beta}{2\beta}$$

Hence the family of martingale measure can be writte as:

$$\left\{ 2 - \frac{1}{\beta} + Q_3, \frac{1}{\beta} - 1 - 2Q_3, Q_3 \mid \frac{1 - 2\beta}{2\beta} < Q_3 < \frac{1 - \beta}{2\beta} \right\}$$

- Consider  $\overline{D} = \begin{bmatrix} 1 & 1.2 & 1.2 & 1.2 \\ 2 & 2.4 & 2.5 & 2.4 \end{bmatrix}$ .

Let transfer the market to nomarlized market Z:

$$D^Z = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & \frac{2.5}{1.2} & 2 \end{bmatrix}.$$

Here asset 1 is risk-free asset with the interest rate i 0.

let  $\alpha_1, \alpha_2, \alpha_3$  strictly positive and  $\alpha_1 + \alpha_2 + \alpha_3 = 1$  the equation:

$2 = 2\alpha_1 + \frac{2.5}{1.2}\alpha_2 + 2(1 - \alpha_1 - \alpha_2) \implies \alpha_2 = 0$  a contradiction. hence the market is not arbitrage free.

In fact, in the original market. Let consider the portfolio  $h = (-2, 1)$ , hence  $V_0^h = -2 + 2 = 0$  and  $P(V_1^h \geq 0) = 1, P(V_1^h > 0) > 0$  Q.E.D