Econ 780 HW Week 2

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Due:

Exercises for chapter 3

Exercise 1

- Let $d_0 = (1,1,1)^T, d_1 = (1,0,0)^T, d_2 = (0,1,0)^T, d_3 = (0,0,1)^T$.

  Let $z_1 = 1, z_2 = 1, z_3 = 1$
  Hence:
  $$d_0 = (1,1,1)^T = z_1(1,0,0)^T + z_2(0,1,0)^T + z_3(0,0,1)^T$$

  Since $z_1 = 1, z_2 = 1, z_3 = 1$ are nonnegative.Q.E.D

- No solution for Problem 2

  Suppose there exist $h = (x, y, z) \in R^3$
  such that:
  $$hd_0 = x + y + z < 0$$
  and
  $$hd_j \geq 0, j = 1, 2, 3$$
  $$hd_j \geq 0, j = 1, 2, 3 \implies x, y, z \geq 0 \implies x + y + z \geq 0$$
  A contradiction. Q.E.D

- $d_0 = (a, b, c)^T, a, b, c \geq 0$.
  let $z_1 = a, z_2 = b, z_3 = c$.
  Clearly $z_1, z_2, z_3 \geq 0$. and $d_0 = (a, b, c)^T = a(1,0,0)^T + b(0,1,0)^T + c(0,0,1)^T = z_1(1,0,0)^T + z_2(0,1,0)^T + z_3(0,0,1)^T = z_1d_1 + z_2d_2 + z_3d_3$. Q.E.D
  The solution is simply (a,b,c)

- No solution for problem 2

  Suppose there exist $h = (x, y, z) \in R^3$ such that:

$$hd_0 < 0$$

and

$$hd_j \geq 0, j = 1, 2, 3$$

$hd_0 < 0 \implies ax + by + cz < 0$
If a, b, and c are all positive, at least one of x, y, and z must be negative.
$hd_j \geq 0, j = 1, 2, 3 \implies x, y, z \geq 0$
This is essentially selecting the components because each d is a euclidian basis vector. So here we are saying that all of x, y, and z are positive. This is the contradiction. Since $a, b, c \geq 0 \implies ax + by + cz \geq 0$ a contraction.Q.E.D

- $d_0 = (-1, -1, -1)^T$
  Suppose there exist $z_1, z_2, z_3 \geq 0$ such that.

  $$(-1, -1, -1)^T = d_0 = z_1 d_1 + z_2 d_2 + z_3 d_3 = (z_1, z_2, z_3)^T$$

  Hence $z_1 = z_2 = z_3 = -1$ Here, we have found a solution, and in this solution, at least one component is negative (in fact all are). And because our matrix is invertible, there is only a single solution to the equation, namely this solution. Therefore, there cannot be a solution with all components positive.

  Prove there exist a solution for problem 2.
  Let $h = (1, 1, 1)$
  We have

  $$hd_0 = -3 < 0$$

  and

  $$hd_j = 1 \geq 0, j = 1, 2, 3$$

  Q.E.D

- $d_0 = (a, b, c)^T$, at least one coordinate is negative.
  WLOG, suppose $a < 0$.
  For problem 2, let $h = (1, 0, 0)$, we have

  $$hd_0 = a < 0$$

  and

  $$hd_1 = 1, hd_2 = hd_3 = 0$$

  Q.E.D.
  In general, put zero in the place that d_0 doesn't have negative coordinate and

1 in the place that d_0 have negative coordinate.

- For problem 1.
  Suppose there exist $z_1, z_2, z_3 \geq 0$ such that.

$$(a, b, c)^T = z_1 d_1 + z_2 d_2 + z_3 d_3 = (z_1, z_2, z_3)^T$$

Hence there at least one i such that $z_i < 0$.Q.E.D

Exercise 2.
Consider the augmented matrix.

$\overline{D} = \begin{bmatrix} 1 & 1 & 2 & 3 \end{bmatrix}.$

There is no arbitrage. since there only 1 matrix, so the only choice to make the value of the profolio is zero in day zero is buy nothing, but it will turnout that the value of the porlio at day 1 will be zero. So there will be no arbitrage opportunity.

let $Q(w_1), Q(w_2), Q(w_3)$be Martingale measure for the market.
We have the formular.

$$1 = \beta(Q(w_1 + 2Q(w2) + 3Q(w_3))$$

and

$$Q(w_1) + Q(w_2) + Q(w_3) = 1$$

.
For some β
For simplicity, I will use Q_1, Q_2, Q_3 instead of $Q(w_1), Q(w_2), Q(w_3)$

$$1 = \beta(Q(w_1 + 2Q(w2) + 3Q(w_3)) \implies \frac{1}{\beta} = Q_1 + 2Q_2 + 3Q_3$$

Let write $Q_1 = 1 - Q_2 - Q_3$ and substitiue to the equation above, we obtain:

$$\frac{1}{\beta} = 1 - Q_2 - Q_3 + 2Q_2 + 3Q_3 = 1 + Q_2 + 2Q_3 \implies Q_2 = \frac{1}{\beta} - 1 - 2Q_3$$

again, substitute $Q_2 = 1 - Q_1 - Q_3$ Obtain

$$\frac{1}{\beta} = Q_1 + 2(1 - Q_1 - Q_3) + 3Q_3 = 2 - Q_1 + Q_3 \implies Q_1 = 2 - \frac{1}{\beta} + Q_3$$

.

Now we will find the bounds for Q_2 to be legitimate probability measure.

- $0 < Q_1 < 1 \implies 0 < 2 - \frac{1}{\beta} + Q_3 < 1 \implies \frac{1}{\beta} - 2 < Q_3 < \frac{1}{\beta} - 1$
- $0 < Q_2 < 1 \implies 0 < \frac{1}{\beta} - 1 - 2Q_3 < 1 \implies \frac{\frac{1}{\beta}-2}{2} < Q_3 < \frac{\frac{1}{\beta}-1}{2}$
Combine these two inequalities.

$$\frac{1-2\beta}{2\beta} < Q_3 < \frac{1-\beta}{2\beta}$$

Hence the family of martingale measure can be writte as:

$$\{2 - \frac{1}{\beta} + Q_3, \frac{1}{\beta} - 1 - 2Q_3, Q_3 \mid \frac{1-2\beta}{2\beta} < Q_3 < \frac{1-\beta}{2\beta}\}$$

- Consider $\overline{D} = \begin{bmatrix} 1 & 1.2 & 1.2 & 1.2 \\ 2 & 2.4 & 2.5 & 2.4 \end{bmatrix}$.

Consider the arbitrage portfolio:

$h = (-2, 1)$, hence ${V_0}^h = -2 + 2 = 0$ and $P({V_1}^h \geq 0) = 1, P({V_1}^h > 0) > 0$ Q.E.D

Let $q_1, q_2, q_3 \geq 0, q_1 + q_2 + q_3 = 1$ be a measure that satisfies:

$${S_0}^i = \frac{1}{1+R} E^Q {S_1}^i$$

for $i = 1, 2$
Clearly, True for $i = 1$
For $i = 2$:

$${S_0}^i = \frac{1}{1+R} E^Q {S_1}^i \implies 2.4 = 2.4q_1 + 2.5q_2 + 2.4q_3$$
$$\implies 2.4 = 2.4(q_1 + q_3) + 2.4q_2 \implies 2.4 = 2.4(1 - q_2) + 2.5Q_2$$

$$\implies 2.4 = 2.4 - 2.4q_2 + 2.5q_2 \implies q_2 = 0$$

In this part, the measure that satisfies ${S_0}^i = \frac{1}{1+R} E^Q {S_1}^i$ for $i = 1, 2$ is not the "martingale measure". since we have to put zero weight in q_2. Compare to the first bullet point, we can have an infinitely many martingale measure that satisfies the condition.