Problem Set 2 - ECON 880

Spring 2022 - University of Kansas

Gunawan, Minh Cao

February 17, 2022

In this exercise, we are interested in solving $Ax = b$, where

$$A = \begin{pmatrix} 54 & 14 & -11 & 2 \\ 14 & 50 & -4 & 29 \\ -11 & -4 & 55 & 22 \\ 2 & 29 & 22 & 95 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

using Gauss-Jacobi and Gauss-Siedel

1 Gauss-Jacobi

1. Consider the first equation from the first row of $Ax = b$:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

2. We can solve for x_1 in terms of $(x_2, \cdots, x_n)$ yielding $x_1 = a_{11}^{-1}(b_1 - a_{12}x_2 - \cdots - a_{1n}x_n)$
3. In general, if $a_{ii} \neq 0$, we can use the row of A to solve for x_i, finding:

$$\frac{1}{a_{ii}}\left(b_i - \sum_{i \neq j} a_{ij}x_j^k\right)$$

4. And son on.

2 Question 4

1. The linear convergence rate is defined by:

$$\lim_{n \to \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|} \leq \beta < 1$$

for some β.
We can rewrite the equation above:

$$|x_{k+1} - x^*| \leq \beta |x_k - x^*|$$

Also, We can write the above equation for x_k instead of x_{k+1}:

$$|x_k - x^*| \leq \beta |x_{k-1} - x^*|$$

Write recursively:

$$|x_{k+1} - x^*| \leq \beta |x_k - x^*| \cdots \leq \beta^{n+1} |x_0 - x^*|$$

Hence, the above in equality is the necessary condition for the bisection method to be linearly convergence. In fact, we only can prove the necessary condition but sufficient condition.

2. The bisection method create the nested sequence as follow:

$$[a_n, b_n] \in [a_{n-1}, b_{n-1}] \cdots [a_0, b_0]$$

By the construction, we have:

$$a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$$

3. $x_n = \frac{a_n + b_n}{2}$, also $a_n \leq x^* \leq b_n$ for all n. Implies:

$$|x_n - x^*| \leq \frac{a_n + b_n}{2} - a_n = \frac{b_n - a_n}{2}$$

By induction:

$$|x_n - x^*| \leq \frac{b_n - a_n}{2} = \frac{b_{n-1} - a_{n-1}}{2^2} = \frac{b_{n-2} - a_{n-2}}{2^3} = \frac{b_0 - a_0}{2^{n+1}}$$

let $\beta = \frac{1}{2}, |x_0 - x^*| = \frac{b_0 - a_0}{2}$, we can prove the necessary condition for the linearly convergence of the bisection method.

3 Question 5

The Newton method is given by:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

If $f(x_k) < 0$

1. Case 1: $x_k < x^*$
since $f'(x_k) \neq 0$

2. Case 1.1: $f'(x_k) > 0$ Hence, by the newton method recursive equation, we have:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Since $f(x_k) < 0$ and $f'(x_k) > 0 \implies \frac{f(x_k)}{f'(x_k)} < 0 \implies -\frac{f(x_k)}{f'(x_k)} > 0$ Let $-\frac{f(x_k)}{f'(x_k)} = \epsilon$
The newton equation can be written as: $x_{k+1} = x_k + \epsilon$ for some $\epsilon > 0$, so $x_{k+1} > x_k$ and $f'(x_k) > 0$ the function is strictly increasing, implies $f(x_{k_1}) > f(x_k)$, which is somehow lead us to closer to the root of $f(x)$.

Symestric argument for other cases:

3.1 Notice

In the argument above, we rely on the fact that the function f is piecewise monotonic, but in some case, f may behave so that Newton method cannot guide us the the true answer.

ADD PICTURE HERE: