Problem Set 3 - ECON 880

Spring 2022 - University of Kansas

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Problem 1

In this exercise, we are interested in solving $Ax = b$, where

$$A = \begin{pmatrix} 54 & 14 & -11 & 2 \\ 14 & 50 & -4 & 29 \\ -11 & -4 & 55 & 22 \\ 2 & 29 & 22 & 95 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

using Gauss-Jacobi and Gauss-Seidel method. Both methods yield the same result

$$x = \begin{pmatrix} 0.0189 \\ 0.0168 \\ 0.0234 \\ -0.0004 \end{pmatrix}.$$

Gauss-Jacobi method required 0.0207 seconds with 45 iterations until convergence. The residual is given by

$$10^{-11} \times \begin{pmatrix} 0.0361 \\ -0.1211 \\ -0.0922 \\ 0.2351 \end{pmatrix}$$

Gauss-Seidel method required 0.0193 seconds with 23 iterations until convergence. The residual is given by

$$10^{-12} \times \begin{pmatrix} 0.1013 \\ -0.1849 \\ -0.1201 \\ -0.0002 \end{pmatrix}$$

Problem 2

In this exercise, we are interested in solving $Bq = r$ using extrapolation, where

$$B = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.6 & 1 & 0.1 \\ 0.2 & 0.4 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix}.$$

Following Ken Judd's definition[†], we first define $G = I - B$, and run the following iteration

$$q^{k+1} = \omega G q^k + \omega r + (1-\omega) q^k,$$

where we pick $\omega = 1.05$, tolerance level 10^{-13}, and initial value $q_0 = (0,0,0)'$. The extrapolation converged after $k = 97$ iterations, with the residual

$$Bq - r = 10^{-12} \times \begin{pmatrix} 0.1670 \\ -0.2371 \\ 0.1279 \end{pmatrix}.$$

The solution to the linear equation system is

$$q = \begin{pmatrix} 1.6716 \\ 5.8651 \\ 1.3196 \end{pmatrix}$$

Problem 3

We want to solve the following functions

1. $\sin(2\pi x) - 2x = 0$
2. $\sin(2\pi x) - x = 0$
3. $\sin(2\pi x) - 0.5x = 0$

using 1) Bisection, 2) Newton method, 3) Secant method, and 4) fixed-point iteration. We want to evaluate for what value of initial guess $x_0 \in [-2, 2]$ these methods converge. We proceed by first plotting all the three functions on Figure 1. From these graphs, we see that within the interval $[-2, 2]$, function 1 and 2 have both two roots, while function 3 has seven roots.

3(a) Bisection

In order for Bisection to work, we need to pick two values x_{low} and x_{high} so that $f(x_{low}) \cdot f(x_{high}) < 0$. The range of admissible values for x for the three functions above is summarized in Table 1

3(b) Newton Method

1 Question 4

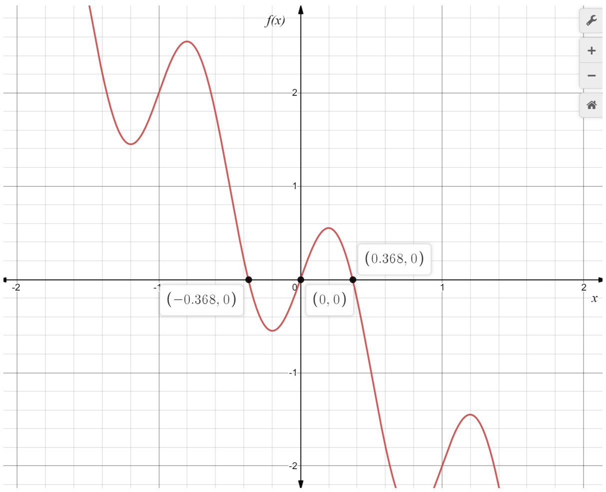
1. The linear convergence rate is defined by:

$$\lim_{n\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|} \le \beta < 1$$

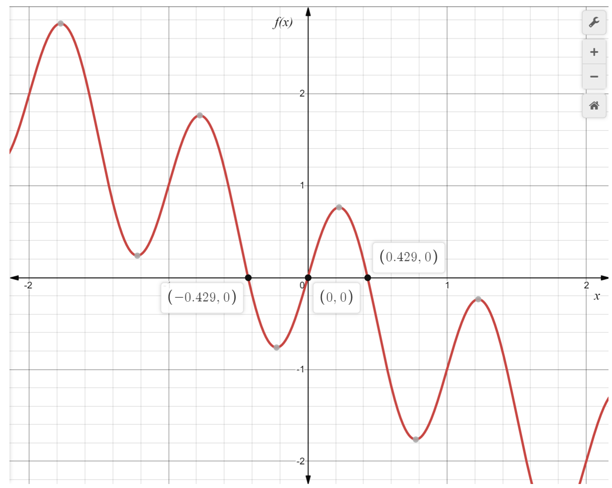
   for some β.
   We can rewrite the equation above:

$$|x_{k+1} - x^*| \le \beta |x_k - x^*|$$

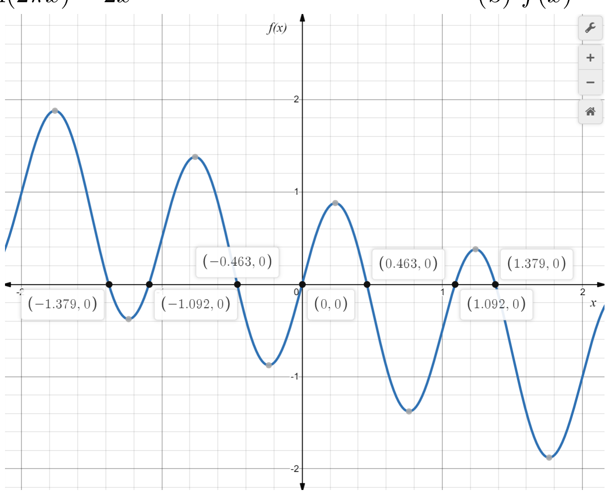
[†]Kenneth L. Judd, 1998. "Numerical Methods in Economics," MIT Press Books, The MIT Press, p.78-79



(a) $f(x) = \sin(2\pi x) - 2x$



(b) $f(x) = \sin(2\pi x) - x$



(c) $f(x) = \sin(2\pi x) - 0.5x$

Figure 1: Function plots for Problem 3

Function	Root	Admissible Range	
		x_{low}	x_{high}
$\sin(2\pi x) - 2x = 0$	$x_1 = -0.368$	$[-2, -0.368)$	$(-0.368, 0)$
	$x_2 = 0$	$(-0.368, 0)$	$(0, 0.368)$
	$x_3 = 0.368$	$(0, 0.368)$	$(0.368, 2]$
$\sin(2\pi x) - x = 0$	$x_1 = -0.429$	$[-2, -0.429)$	$(-0.429, 0)$
	$x_2 = 0$	$(-0.429, 0)$	$(0, 0.429)$
	$x_3 = 0.429$	$(0, 0.429)$	$(0.429, 2]$
$\sin(2\pi x) - 0.5x = 0$	$x_1 = -1.379$	$[-2, -1.379)$	$(-1.379, -1.092)$
	$x_2 = -1.092$	$(-1.379, -1.092)$	$(-1.092, -0.463)$
	$x_3 = -0.463$	$(-1.092, -0.463)$	$(-0.463, 0)$
	$x_4 = 0$	$(-0.463, 0)$	$(0, 0.463)$
	$x_5 = 0.463$	$(0, 0.463)$	$(0.463, 1.092)$
	$x_6 = 1.092$	$(0.463, 1.092)$	$(1.092, 1.379)$
	$x_7 = 1.379$	$(1.092, 1.379)$	$(1.379, 2]$

Table 1: Polynomial evaluation costs

Also, We can write the above equation for x_k instead of x_{k+1}:

$$|x_k - x^*| \leq \beta |x_{k-1} - x^*|$$

Write recursively:

$$|x_{k+1} - x^*| \leq \beta |x_k - x^*| \cdots \leq \beta^{n+1} |x_0 - x^*|$$

Hence, the above in equality is the necessary condition for the bisection method to be linearly convergence. In fact, we only can prove the necessary condition but sufficient condition.

2. The bisection method create the nested sequence as follow:

$$[a_n, b_n] \in [a_{n-1}, b_{n-1}] \cdots [a_0, b_0]$$

By the construction, we have:

$$a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$$

3. $x_n = \frac{a_n + b_n}{2}$, also $a_n \leq x^* \leq b_n$ for all n. Implies:

$$|x_n - x^*| \leq \frac{a_n + b_n}{2} - a_n = \frac{b_n - a_n}{2}$$

By induction:

$$|x_n - x^*| \leq \frac{b_n - a_n}{2} = \frac{b_{n-1} - a_{n-1}}{2^2} = \frac{b_{n-2} - a_{n-2}}{2^3} = \frac{b_0 - a_0}{2^{n+1}}$$

let $\beta = \frac{1}{2}$,$|x_0 - x^*| = \frac{b_0 - a_0}{2}$, we can prove the necessary condition for the linearly convergence of the bisection method.

2 Question 5

The Newton method is given by:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

If $f(x_k) < 0$

1. Case 1: $x_k < x^*$
   since $f'(x_k) \neq 0$

2. Case 1.1: $f'(x_k) > 0$ Hence, by the newton method recursive equation, we have:

   $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

   Since $f(x_k) < 0$ and $f'(x_k) > 0 \implies \frac{f(x_k)}{f'(x_k)} < 0 \implies -\frac{f(x_k)}{f'(x_k)} > 0$ Let $-\frac{f(x_k)}{f'(x_k)} = \epsilon$
   The newton equation can be written as: $x_{k+1} = x_k + \epsilon$ for some $\epsilon > 0$, so $x_{k+1} > x_k$ and $f'(x_k) > 0$ the function is strictly increasing, implies $f(x_{k_1}) > f(x_k)$, which is somehow lead us to closer to the root of $f(x)$.

   Symestric argument for other cases:

   ### 2.1 Notice

   In the argument above, we rely on the fact that the function f is piecewise monotonic, but in some case, f may behave so that Newton method cannot guide us the the true answer.

   ADD PICTURE HERE: