Problem Set 2 - ECON 880

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In this exercise, we are interested in solving $Ax = b$, where

$$A = \begin{pmatrix} 54 & 14 & -11 & 2 \\ 14 & 50 & -4 & 29 \\ -11 & -4 & 55 & 22 \\ 2 & 29 & 22 & 95 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

using Gauss-Jacobi and Gauss-Siedel

1 Gauss-Jacobi

1. Consider the first equation from the first row of $Ax = b$:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

2. We can solve for x_1 in terms of $(x_2, \cdots, x_n)$ yielding $x_1 = a_{11}^{-1}(b_1 - a_{12}x_2 - \cdots - a_{1n}x_n)$
3. In general, if $a_{ii} \neq 0$, we can use the row of A to solve for x_i, finding:

$$\frac{1}{a_{ii}}\left(b_i - \sum_{i \neq j} a_{ij}x_j^k\right)$$

4. And son on.

2 Question 4

1. The linear convergence rate is defined by:

$$\lim_{n\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|} \leq \beta < 1$$

for some β.
We can rewrite the equation above:

$$|x_{k+1} - x^*| \leq \beta|x_k - x^*|$$

Also, We can write the above equation for x_k instead of x_{k+1}:

$$|x_k - x^*| \le \beta |x_{k-1} - x^*|$$

Write recursively:

$$|x_{k+1} - x^*| \le \beta |x_k - x^*| \cdots \le \beta^{n+1} |x_0 - x^*|$$

Hence, the above in equality is the necessary condition for the bisection method to be linearly convergence. In fact, we only can prove the necessary condition but sufficient condition.

2. The bisection method create the nested sequence as follow:

$$[a_n, b_n] \in [a_{n-1}, b_{n-1}] \cdots [a_0, b_0]$$

By the construction, we have:

$$a = a_0 \le a_1 \le a_2 \le \cdots \le a_n \cdots \le b_n \le \cdots \le b_2 \le b_1$$

3. $x_n = \frac{a_n + b_n}{2}$, also $a_n \le x^* \le b_n$ for all n. Implies:

$$|x_n - x^*| \le \frac{a_n + b_n}{2} - a_n = \frac{b_n - a_n}{2}$$

By induction:

$$|x_n - x^*| \le \frac{b_n - a_n}{2} = \frac{b_{n-1} - a_{n-1}}{2^2} = \frac{b_{n-2} - a_{n-2}}{2^3} = \frac{b_0 - a_0}{2^{n+1}}$$

let $\beta = \frac{1}{2}, |x_0 - x^*| = \frac{b_0 - a_0}{2}$, we can prove the necessary condition for the linearly convergence of the bisection method.

3 Question 5

The Newton method is given by:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

If $f(x_k) < 0$

1. Case 1: $x_k < x^*$
since $f'(x_k) \ne 0$

   - Case 1.1: $f'(x_k) > 0$ Hence, by the newton method recursive equation, we have:

     $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

     Since $f(x_k) < 0$ and $f'(x_k) > 0 \implies \frac{f(x_k)}{f'(x_k)} < 0 \implies -\frac{f(x_k)}{f'(x_k)} > 0$ Let $-\frac{f(x_k)}{f'(x_k)} = \epsilon$
     The newton equation can be written as: $x_{k+1} = x_k + \epsilon$ for some $\epsilon > 0$, so $x_{k+1} > x_k$ and $f'(x_k) > 0$ the function is strictly increasing, implies $f(x_{k+1}) > f(x_k)$, which is somehow lead us to closer to the root of $f(x)$.

- Case 1.2 $f'(x_k) > 0$ Using symmetric argument, now the Newton method will lead us to go to the left of x_k, now the Newton method may not give us useful information about the root of the function. If f does not have local maximum around x_k, then Newton method still can guide us to another root of f, the root in the left of x_k

2. Case 2: $x_k > x^*$

   - Case 2.1: $f'(x_k) > 0$
     In this case, f has local minimum around x_k, we obtain $\frac{f(x_k)}{f'(x_k)} < 0$ hence $x_{k+1} > x_k$, we go to the right of x_k, if the function does not have local maximum around x_k, Newton method can still guide us to the root on the right of x_k
   - Case 2,2: $f'(x_k) < 0$
     We obtain $\frac{f(x_k)}{f'(x_k)} > 0 \implies x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} < x_k$, The Newton Method wil guide us to the root of f.

3.1 Conclusion

If the function f does not have many Stationary point, the Newton Method will guide us to the root of the function.