Problem Set 4 - ECON 880

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Gunawan, Minh Cao

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In this exercise, we are interested in solving the following minimization problem

$$\min_{x,y} f(x,y) = 100(y - x^2)^2 + (1 - x)^2$$

using both steepest descent and conjugate gradient method. We try different starting values $\tilde{x}^0 = (x^0, y^0)$ starting from $(0,0)$, $(-1,-1)$ and $(-1,2)$, and report $\tilde{x}^k = (x^k, y^k)$ for the first 5 iterations. Note that $(x,y) = (1,1)$ is a unique solution.

1 Steepest Descent

We use the following steps for the steepest descent method

Step 0. Choose $\tilde{x}^0$ (see above), $\delta = 10^{-5}$, $\epsilon = 10^{-10}$

Step 1. Compute $s^k = -\nabla f(\tilde{x}^k)$

Step 2. Solve $\lambda_k = \arg\min_\lambda f(\tilde{x}^k + \lambda s^k)$

Step 3. Set $\tilde{x}^{k+1} = \tilde{x}^k + \lambda_k s^k$

Step 4. If $||\tilde{x}^k - \tilde{x}^{k+1}|| < \epsilon(1 + ||\tilde{x}^k||)$, go to Step 5. Otherwise, go to Step 1.

Step 5. If $||\nabla f(\tilde{x}^k)|| < \delta(1 + f(x^k))$, stop and report success. Else, stop and report convergence to a nonoptimal point.

Table 1 shows the values of $\tilde{x}^k = (x^k, y^k)$ for $k = 1, 2, \ldots, 5$ for three different starting values. Steepest descent method leads to the correct solution using our specified tolerance levels at different numbers iteration.

$\tilde{x}^k = (x^k, y^k)$	x^k	y^k	x^k	y^k	x^k	y^k
$k = 0$	0	0	-1	-1	-1	2
$k = 1$	0.16133	0	0.25748	-0.37439	-1.348	1.8242
$k = 2$	0.16132	0.025915	0.0708	0.0003852	-1.337	1.8076
$k = 3$	0.2135	0.026592	0.48295	0.22121	-1.341	1.8049
$k = 4$	0.21326	0.045502	0.47934	0.22794	-1.3402	1.804
$k = 5$	0.2462	0.045405	0.49917	0.23846	-1.3398	1.8028
Convergence at	$k = 177,644$ (success)		$k = 1,409,717$ (success)		$k = 112,965$ (success)	
$\tilde{x}^{final}$	1	1	1	1	1	1

Table 1: Results for steepest descent method

2 Conjugate Gradient

We use the following steps for the conjugate gradient method

Step 0. Choose $\tilde{x}^0$ (see above), $\delta = 10^{-5}$, $\epsilon = 10^{-10}$, and $s^0 = -\nabla f(\tilde{x}^0)$

Step 1. Solve $\lambda_k = \arg\min_\lambda f(\tilde{x}^k + \lambda s^k)$

Step 2. Set $\tilde{x}^{k+1} = \tilde{x}^k + \lambda_k s^k$

Step 3. Compute the search direction $s^{k+1} = -\nabla f(\tilde{x}^{k+1}) + \frac{||\nabla f(\tilde{x}^{k+1})||^2}{||\nabla f(\tilde{x}^k)||^2} s^k$

Step 4. If $||\tilde{x}^k - \tilde{x}^{k+1}|| > \epsilon(1 + ||\tilde{x}^k||)$, go to Step 1. Otherwise, go to Step 5.

Step 5. If $||\nabla f(\tilde{x}^k)|| < \delta(1 + |f(x^k)|)$, stop and report success. Else, stop and report convergence to a nonoptimal point.

Table 2 shows the values of $\tilde{x}^k = (x^k, y^k)$ for $k = 1, 2, \ldots, 5$ for three different starting values. Conjugate gradient method does not lead to the correct solution $(1,1)$ at our specified tolerance levels. Although for starting values $(0,0)$ and $(-1,-1)$, the final values are close to the true solution, these are nonoptimal according to the specified δ.

$\tilde{x}^k = (x^k, y^k)$	x^k	y^k	x^k	y^k	x^k	y^k
$k = 0$	0	0	-1	-1	-1	2
$k = 1$	0.16133	0	0.25748	-0.37439	-1.348	1.8242
$k = 2$	0.29279	0.050517	0.11558	0.01084	-1.3367	1.8069
$k = 3$	0.42553	0.1432	0.31233	0.074648	-1.0855	1.1367
$k = 4$	0.5803	0.30357	0.48326	0.20686	-0.94572	0.82968
$k = 5$	0.84321	0.70196	0.71518	0.49616	-0.84507	0.63618
Convergence at	$k = 147$ (nonoptimal)		$k = 36$ (nonoptimal)		$k = 113$ (nonoptimal)	
$\tilde{x}^{final}$	1.00021	1.00037	1.00627	1.01202	1.43299	2.01758

Table 2: Results for conjugate gradient method