Problem Set 5 - ECON 880

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Question 1

We have given $f(x) = (x^{1/2}+1)^{2/3}$, with a starting value $x_0 = 1$. The first derivative is given by

$$f'(x) = \frac{1}{3x^{1/2}(x^{1/2}+1)^{1/3}},$$

while the second one is given by

$$f''(x) = \frac{-1}{18x(x^{1/2}+1)^{4/3}} - \frac{1}{6x^{3/2}(x^{1/2}+1)^{1/3}}$$

1(a) Taylor Series Approximation

The first-order Taylor series approximation is given by:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) \\ &= 1.5874 + 0.2646(x-1) \end{aligned}$$

The second-order Taylor series approximation is given by:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{(x-x_0)^2}{2} f''(x_0) \\ &= 1.5874 + 0.2646(x-1) - 0.0772(x-1)^2 \end{aligned}$$

1(b) Padé Approximation

We want to compute Padé Approximation (1,1)

$$P_{M=1}^{N=1}(x) = \frac{a_0 + a_1(x-x_0)}{b_0 + b_1(x-x_0)},$$

where $b_0 = 1$ (normalized). The Padé coefficients are normally best found from an $(M+N)$-th order Taylor series expansion of $f(x)$:

$$\begin{aligned} T_2(x) &= \frac{a_0 + a_1(x-1)}{1 + b_1(x-1)} \\ 1.5874 + 0.2646(x-1) - 0.0772(x-1)^2 &= \frac{a_0 + a_1(x-1)}{1 + b_1(x-1)} \end{aligned}$$

Multiplying up the denominator of the RHS with the LHS gives the following equivalent set of coefficient relations:

$$\begin{aligned} a_0 &= 1.5874 && (1)\\ a_1 &= 1.5874b_1 + 0.2646 && (2)\\ 0 &= 0.2646b_1 - 0.0772 && (3) \end{aligned}$$

From equation 3, we obtain $b_1 = 0.2917$. Then, we plug it in into equation 2 to obtain $a_1 = 0.7276$. Finally, the Padé Approximation (1,1) for $f(x)$ is given by

$$P_{M=1}^{N=1}(x) = \frac{1.5874 + 0.7276(x-1)}{1 + 0.2917(x-1)},$$

Question 2

Consider the function

$$f(x) = e^{4x-2}$$

over $[0, 2]$ interval, and construct the following approximations to it:

(a) Chebyshev polynomials of degree 4; choose 5 points to use as nodes.

(b) Cubic spline over 5 equally spaced points in $[0, 2]$.

Then evaluate the approximations over 101 equally spaced points in $[0, 2]$ and plot them along with the true function.

2(a) Chebyshev Interpolation

We use the following algorithm[†] to construct the Chebyshev interpolation of order 4. The result is shown on Figure 1.

Step 1. Compute five Chebyshev interpolation nodes on $[0, 2]$ using the following formula

$$z_k = -\cos\left(\frac{2k-1}{2m}\pi\right), \quad k = 1, \dots, m$$

Step 2. Adjust the nodes to the $[0, 2]$ interval:

$$x_k = (z_k + 1)\left(\frac{2-0}{2}\right) + 0 = z_k + 1, \quad k = 1, \dots, m$$

Step 3. Evaluate f at the approximation nodes

$$y_k = f(x_k), \quad k = 1, \dots, m$$

Step 4. Compute Chebyshev coefficients $\{a_i\}_{i=0}^{n}$

$$a_i = \frac{\sum_{k=1}^{m} y_k T_i(z_k)}{\sum_{k=1}^{m} T_i(z_k)^2},$$

where $T_i(x) = 2xT_{i-1}(x) - T_{i-2}(x)$, with $T_0(x) = 1$ and $T_1(x) = x$, for $i = 2, \dots, n$

Step 5. The approximation for $f(x)$, $x \in [0, 2]$ is given by:

$$\hat{f}(x) = \sum_{i=0}^{n} a_i T_i(x-1).$$

[†]Kenneth L. Judd, 1998. "Numerical Methods in Economics," MIT Press Books, The MIT Press, p.223

2(a) Cubic Spline Interpolation

We use the following algorithm to construct the cubic spline interpolation. The result is shown on Figure 1.

Step 1. xxx

Step 2. xxx

Step 3. xxx

Step 4. xxx

Step 5. xxx

Comment: which method works better? Are higher order Chebyshev coefficients indeed smaller? Do approximations match the slope and curvature of the original function?

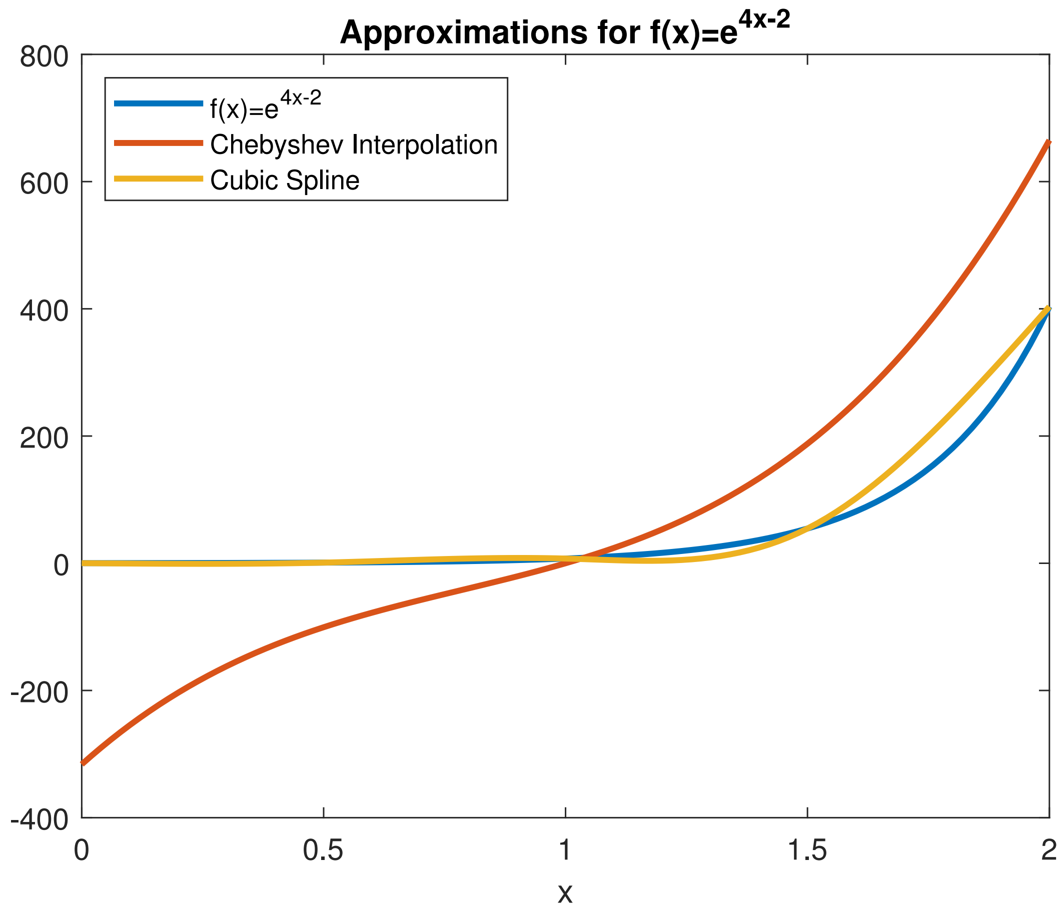

Figure 1: Approximations for $f(x) = e^{4x-2}$