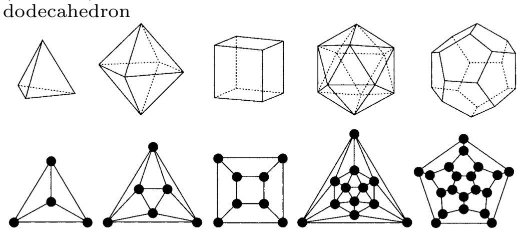
Combinatorics and Graph Theory Cheatsheet

Callum Burgher

1. Definitions

- Simple graph: No loops or multiple edges
- General graph: Multiple edges/loops allowed
- Vertex set: The set of vertices
- Edge set: The set of edges of a graph
- Adjacency: (vertex) If there is an edge joining them, (edge) if there is a common vertex
- Degree sequence: Sequence of vertex degrees from lowest (left) to highest (right)
- **Handshaking Lemma**: In any graph, the sum of all of the vertex degrees is an even number
- Corollary 1.2: In any graph, the number of vertices of odd degree is even
- Subgraph: If each vertex/edge belongs to the respective set
- Adjacency matrix: A, B, C, D along top and down left, then how many edges between the two
- Incidence matrix: Vertices down left, edges across top, 1 if connected, 0 if not
- Isomorphism: Two graphs G_1 and G_2 are **isomorhpic** if there is a one-to-one correspondence between the vertices of G_1 and those of G_2, such that the number of edges joining any two vertices of G_1 equals the number of edges joining the corresponding vertices of G_2
- Connected: If the graph cannot be expressed as a union of graphs
- Disconnected: If it is not connected
- Component: Any disconnected graph G can be expressed as the union of connected graphs, each of which is a component of G.
- Isolated vertex: A vertex of degree 0
- End vertex: A vertex of degree 1
- Subgraph: A graph H is a subgraph of G if each of its vertices belong to $V(G)$ and each of its edges belongs to $E(G)$.
- Null graphs: A graph whose edge-set is empty is a null graph.
- Complete graph: Each pair of distinct vertices are adjacent, K_n with $\frac{n(n-1)}{2}$ edges
- Cycle graphs: Each vertex has degree 2, C_n, n vertices
- Path graph: Removing an edge from C_n, P_n, n vertices
- Wheel: Obtained from C_{n-1} by joining each vertex to a new vertex, W_n, n vertices
- Regular graph: Each vertex has same degree
- Bipartite: Black to white, in form $G = G(A, B)$
- Complete bipartite: A biparite graph where each vertex in A is joined to every vertex in B by exactly one edge, in form $K_{r,s},\ r+s$ vertices, rs edges
- Cubic graphs: Regular degree 3, example is the Peterson graph (Pentagon with star in middle)
- Platonic graphs: Formed from the vertices and edges of the five regular (Platonic) solids – the tetrahedron, octahedron, cube, icosahedron and dodecahedron



- Cubes: Among the regular bipartite grpahs. The k-cube Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$ where each $a_i = 0$ or 1. Q_k has 2^k vertices and is regular of degree k

Digraphs

- Digraph: A direted graph
- Arc-family: The digraph equivalent of the edge-set
- Underlying graph: The graph obtained by removing the arrows from a digraph
- Isomorphic: If there is an isomorphism between their underlying graphs
- Weakly connected: If it cannot be expressed as the union of two digraphs
- Out-degree: outdeg(v) # of arcs vw
- In-degree: indeg(v) # of arcs wv
- **Handshaking Dilemma:** In any digraph the sum of all out-degrees = sum of all in-degrees
- Tournament: Any two vertices joined by exactly one arc

Infinite Graphs

- Infinite graph: Infinite set $V(G)$ of vertices and infinite set $E(G)$ of unordered pairs. If both are countably infinite, then G is a countable graph

  1. Locally finite if each vertex has finite degree
  2. Locally countable if each vertex has a countable degree

- **Thm 1.4**: Every connected locally countable infinite graph is a countable graph
- Corollary 1.5: Every locally finite infinite graph is a countable graph

2. Paths and Cycles

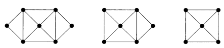
- Walk: Finite sequence of edges of the form $v_0v_1, v_1v_2, \ldots, v_{m-1}v_m$
- Length: # edges in a walk
- Trail: All distinct edges
- Path: A trail with distinct vertices except possibly $v_0 = v_m$
- Closed: $v_0 = v_m$
- Cycle: Closed with at least one edge
- **Thm 2.1:** A graph is bipartite $\iff$ every cycle has even length
- **Thm 2.2:** G simple graph on n vertices. If G has k components then the number m of edged of G satisfies $n - k \le m \le \frac{1}{2}(n-k)(n-k+1)$
- Corollary 2.3: Any simple graph with n vertices and more than $\frac{1}{2}(n-1)(n-2)$ edges is connected.
- Disconnecting set: In a connected graph is a set of edges whose deletion disconnects G.
- Cutset: A minimal disconnecting set
- Bridge: If a cutset only has one edge
- Edge-connectivity: If G is connected, then $\lambda(G)$ is the size of the smallest cutset in G
- K-edge-connected (k-e-c): If $\lambda(G) \ge k$
- **Thm 2.4:** A graph is k-e-c iff any two distinct vertices of G are joined by at least k paths, no two of which have any edges in common
- **Thm 2.5** A graph G with at least $k+1$ vertices is k-connected iff any two vertices of G are joined by at least k paths, no two of which have any other vertices in common.
- Separating set: In a connected graph G is a set of vertices whose deletion disconnects G
- Cut vertex: If a separating set contains only one vertex v, then v is a cut vertex
- Vertex connectivity: The size of the smallest separating set, $\kappa(G)$
- We also say G is k-connected if $\kappa(G) \ge k$
- Strongly connected (**Digraph**): if for any two vertices of D there is a directed path
- Orientable: If each edge of G can be directed so that the resulting digraph D is strongly connected; such a digraph is an **orientation** of G.
- **Thm 2.6:** A connected graph G is orientable iff each edge of G lies in at least 1 cycle
- **Infinite graphs**

  1. One way: $v_0 \to v_1 \to \cdots$
  2. Two way: $\cdots \to v_{-2} \to v_{-1} \to v_0 \to v_1 \to v_2 \cdots$

- **Thm 2.7 (König's Lemma):** Let G be a connected locally finite infinite graph. Then, for any vertex, v, of G, there exists a one way infinite path with initial vertex v

2.2 Eulerian Graphs

- Eulerian Graph: A connected graph is Eulerian if there exists a closed trail that includes every edge of G; such a trail is a Eulerian trail.
- A non-Eulerian graph G is semi-Eulerian if there exists a non closed trail that includes every edge of G. Below are Eulerian, S-Eulerian and N-Eulerian graphs respectively



- Lemma 2.8: If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.
- **Thm 2.9:** A connected graph G is Eulerian iff the degree of each vertex is even
- Corollary 2.10: A connected graph is Eulerian iff its set of edges can be split up into edge-disjoint cycles
- Corollary 2.11: A connected graph is semi-Eulerian iff it has exactly two vertices of odd degree.
- **Thm 2.12 (Fleury's Algorithm):** Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trail of G. Start at any vertex u an traverse the edges in an arbitrary manner, subject only to the following rules:

  1. Erase the edges as they are traversed, and if any isolated vertices result, erase them too
  2. At each stage, use a bridge only if there is no alternative **(don't burn your bridges too quick)**

  Digraphs

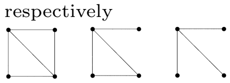
- A connected digraph D is Eulerian if there exists a closed directed trail that includes every arc of D.
- **Thm 2.13:** A strongly connected digraph is Eulerian iff, for every vertex v of D, $\text{outdeg}(v) = \text{indeg(v)}$
  Infinite graphs
- **Thm 2.14:** Let G be a countable connected graph which is Eulerian. Then

  1. G has no vertices of odd degree;
  2. For each finite subgraph H of G, the infinite graph K obtained by deleting from G the edges of H has at most two infinite components;
  3. if, in addition, each vertex of H has even degree, then K has exactly one infinite component.

- **Thm 2.15:** If G is a countable graph, then G is Eulerian iff the above conditions are satisfied

2.3 Hamiltonian (Di)graphs

- Hamiltonian cycle: If there exists a closed trail passing exactly once through each vertex of G: such a trail must be a cycle (except K_1). Such a cycle is a Hamiltonian cycle
- Hamiltonian graph: A graph with a Hamiltonian cycle
- Semi-Hamiltonian: A non- Hamiltonian graph is semi-Hamiltonian if there exists a path through every vertex. Below are H, S-H and N-H respectively



- **Thm 2.16:** If G is a simple graph and if $\deg(v)+\deg(w) \geq n$ for each pair of non-adjacent vertices v and w, then G is Hamiltonian.
- Corollary 2.17: If G is a simple graph with $n(\geq 3)$ vertices, and if $\deg(v) \geq \frac{n}{2}$ for each vertex v, then G is Hamiltonian.
- Hamiltonian **Digraph**: D is Hamiltonian if there is a directed cycle that includes every vertex of D.
- Semi- HD: A non-Hamiltonian digraph that contains a directed path through every vertex.
- **Thm 2.18:** Let D be a strongly connected digraph with n vertices. If $\text{outdeg}(v) \geq \frac{n}{2}$ and $\text{indeg}(v) \geq \frac{n}{2}$ for each vertex v, then D is Hamiltonian
- **Thm 2.19:**

  1. Every non-Hamiltonian tournament is semi-Hamiltonian (**Tournament:** Digraph where any two vertices are joined by exactly one arc)
  2. Every strongly connected tournament is Hamiltonian.

Applications:

- Weight: The number assigned to each edge
- Weighted graph: A connected graph in which a non-negative number is assigned to each edge.

The Shortest path problem:

Best shown in an example: See examples section

The Critical path problem

Same as above

The Chinese Postman problem

- Eulerian: The shortest path is the Eulerian circuit
- Semi-Eulerian:The minimum length of a solution is the total weight added up, plus the shortest path back from one vertex of odd degree to the other
- Non-Eulerian: TOO HARD

The Travelling salesman problem

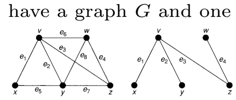
See examples for different questions asked

3. Trees

- Tree: A connected graph that contains no cycles. Trees are simple graph. Any two vertices are connected by exactly one edge
- **Thm 3.1:** Let T be a graph with n vertices. Then the following statements are equivalent:

  1. T is a tree
  2. T contains no cycles and has $n-1$ edges
  3. T is connected and has $n-1$ edges
  4. T is connected and each edge is a bridge
  5. Any two vertices of T are connected by exactly one path
  6. T contains no cycles, but the addition of any new edge creates exactly one cycle

- Spanning tree: Given any connected graph G, choose a cycle and remove one edge, the resulting graph remains connected, repeating this until there are no cycles and you have one
- Cylce rank: The number of edges removed to form a spanning tree, $\gamma(G) = m - n + 1 \in \mathbb{Z}^+ \cup \{0\}$
- Cutset rank: Number of edges in a spanning tree, $\xi(G) = n - 1$. Below we have a graph G and one of its spanning trees



- **Thm 3.2:** If T is any spanning tree of a connected graph G, then

  1. Each cutset of G has an edge in common with T
  2. Each cycle of G has an edge in common with the complement of T

3.2 Counting Trees

- **Thm 3.3 (Cayley):** There are n^{n-2} distinct labelled trees with n vertices.
- Corollary 3.4: The number of spanning trees of K_n is n^{n-2}
- **Thm 3.5:** Let G be a connected simple graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $M = (m_{ij})$ be the $n \times n$ matrix in which $m_{ii} = \deg(v_i), m_{ij} = -1$ if v_i and v_j are adjacent, and $m_{ij} = 0$ otherwise. Then the number of spanning trees of G is equal to thte cofactor of any element of M.

3.3 More Applications

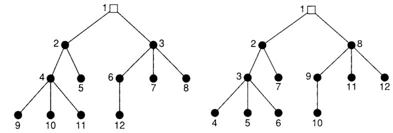
The minimum connector problem:

- **Thm 3.6 (Greedy's Algorithm):** Let G be a connected graph with n vertices. Then the following gives a solution of the minimum connector problem:

  1. Let e_1 be an edge of G of smallest weight
  2. Define $e_2, e_3, \ldots, e_{n-1}$ by choosing at each stage a new edge of smallest possible weight that forms no cycle with the previous edges e_i The required spanning tree is the subgraph T of G whose edges are $e_1, e_2, \ldots, e_{n-1}$

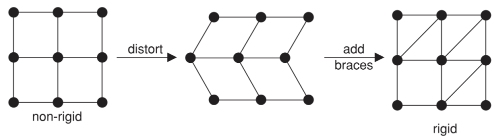
Searching Trees:

The two types are given below: Breadth-first and Depth-first



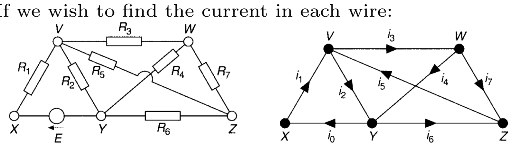
Bracing Rectangular Frameworks:

- **Thm 3.7:** A braced rectangular framework is rigid iff the corresponding bipartite graph is connected. (**bonus -** If the bipartite graph is a spanning tree, then the bracing is a minimum bracing)



Electrical Networks:

If we wish to find the current in each wire:



Assign an arbitrary direction to the current in each wire an apply Kirchoff's Laws:

1. The algebraic sum of the currents at each vertex is 0
2. The total voltage in each cycle is obtained by adding the products of the currents i_k and resistance R_k in that cycle

4. Planarity

4.1 Planar Graphs

- Planar graph: A graph that can be drawn in the plane without crossings
- **Thm 4.1:** K_5 and $K_{3,3}$ are non-planar.
- **Kuratowski's Thm:**
  1. Every subgraph of a planar graph is planar
  2. Every graph with a non-planar subgraph is non-planar
- Homeomorphic: Two graphs are said to be homeomorphic if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges
- **Thm 4.2 (Kuratowski):** A graph is planar iff it contains no subgraph homeomorphic to K_5 or $K_{3,3}$
- Contractible: A graph H is contractible to K_5 or $K_{3,3}$ if we can obtain them by successively contracting edges
- **Thm 4.3:** A graph is planar iff it contains no subgraph contractible to K_5 or $K_{3,3}$
- Every connected graph with 8 or fewer edges is planar
- **Infinite planar graphs (Thm 4.4:)** If G is a countable graph, every finite subgraph of which is planar, then G is planar.

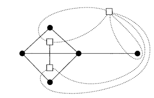
4.2 Euler's Formula

If G is planar, then any plane drawing of G divides the set of points of the plane not lying on G into regions, called faces.

- **Thm 4.5 (Euler):** Let G be a plane drawing of a planar graph, and let n, m and f denote respectively the number of vertices, edges and faces of G. Then
$$n - m + f = 2$$
- Corollary 4.6: Let G be a polyhedral graph. Then
$$n - m + f = 2$$
- Corollary 4.7: Let G be a plane graph with n vertices, m edges, f faces and k components. Then $n - m + f = k + 1$
- Corollary 4.8:
  1. If G is a simple connected planar graph with $n \geq 3$ vertices and m edges, then
  $$m \leq 3n - 6$$
  2. If, in addition, G has no triangles, then
  $$m \leq 2n - 4$$
- **Thm 4.10:** Every simple planar graph G contains a vertex of degree at most 5
- **Thm 4.11:** Let G be a simple graph with $n \geq 3$ vertices and m edges. Then the thickness $t(G)$ of G satisfies the inequalities
$$t(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil \quad \text{and} \quad t(G) \geq \left\lfloor \frac{m+3n-7}{3n-6} \right\rfloor$$

4.3 Dual Graphs

- Geometric Dual (G^*): Built from plane drawing of planar graph G as follows:
  1. Choose a point v^* in each face f of G. These are the vertices of G^*
  2. Corresponding to each edge e of G draw a line e^* that crosses **only** e and joins the vertices v^* in the faces f adjoining e, there are the edges of G^* Example below:

- Lemma 4.12: G connected plane graph, with usual n vertices, m edges and f faces. Its geometric dual G^* has n^* vertices, m^* edges and f^* faces. Then
$$n^* = f, \ m^* = m, \ f^* = n$$
- **Thm 4.13:** If G is a connected plane graph, then G^** is isomorphic to G.
- **Thm 4.14:** Let G be a planar graph and let G^* be a geometric dual of G. Then a set of edges in G forms a cycle in G iff the corresponding set of edges in G^* forms a cutset in G^*
- Corollary 4.15: A set of edges of G forms a cutset in G iff the corresponding set of edges of G^* forms a cycle in G^*
- **Thm 4.16:** If G^* is an abstract dual of G. Then G is an abstract dual of G^* (**Abstract dual:** if there is a 1-1 correspondence between the edges of G and those of G^*, with the property from Cor 4.15
- **Thm 4.17:** A graph is planar iff it has an abstract dual.

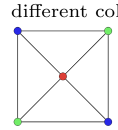
Graphs on other surfaces

- Genus: A surface is of genus g if it is topologically homeomorphic to a sphere with g handles.
- Sphere has genus 0, Doughnut (torus) has genus 1
- K_5 and $K_{3,3}$ are graphs of genus 1
- **Thm 4.18:** The genus of a graph does not exceed the crossing number. (crossing number- the smallest number of crossings, of two edges, that can occur when graph is drawn in the plane)
- **Thm 4.19:** Let G be a connected graph of genus g with n vertices, m edges and f faces. Then
$$n - m + f = 2 - 2g$$
- Corollary 4.20: The genus $g(G)$ of a simple graph G with $n \geq 4$ vertices and m edges satisfies the inequality
$$g(G) \geq \left\lceil \frac{(m-3n}{6} + 1 \right\rceil$$
- **Thm 4.21:** $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$

5. Colouring Graphs

5.1 Colouring Vertices

- k-colourable: If G is a graph with no loops, then G is k-colourable if we can assign one of k colours to each vertex so that adjacent vertices have different colours. W_5 is 3-colourable as seen below

- k-chromatic: If G is k-colourable but not $(k-1)-$colourable, alternatively we say the chromatic number of G is k, $\chi(G) = k$.
- **Thm 5.1:** If G is a simple graph with the largest vertex degree Δ, then G is $(\Delta + 1)-$colourable.
- **Brooke's Thm (5.2):** If G is a simple connected graph which is not a complete graph, and if the largest vertex-degree is $\Delta \geq 3$, then G is $\Delta-$colourable
- **Thm 5.3,4,5:** Every simple planar graph is 6, 5, 4-colourable

5.2 Chromatic Polynomials

- Chromatic function: G, a simple graph then $P_G(k)$ is the number of ways of colouring the vertices so that no two adjacent vertices have the same colour. P_G is the chromatic polynomial
- If G is a simple planar graph, then $P_G(4) > 0$
- **Thm 5.6:** Let G be a simple graph, and let $G - e$ and $G \backslash e$ be the graphs obtained by deleting and contracting and edge e. Then,
$$P_G(k) = P_{G-e}(k) - P_{G \backslash e}(k)$$
- Corollary 5.7: The chromatic function of a simple graph is a polynomial.

5.3 Colouring Maps

- Map: A 3-connected plane graph
- k-colourable-(f) map: If its faces can be coloured with k colours so that no two faces with a boundary edge in common have the same colour.
- **Thm 5.8:** A map G is 2-colourable-(f) iff G is an Eulerian graph
- **Thm 5.9:** Let G be a plane graph without loops, and let G^* be a geometric dual of G. Then G is $k-$colourable-(v) iff G^* is $k-$colourable-(f)
- Corollary 5.10: The four colour theorem for maps is equivalent to the four colour theorem for planar graphs
- **Thm 5.11:** Let G be a cubic map. Then G is 3-colourable iff each face is bounded by an even number of edges
- **Thm 5.12:** In order to prove the four colour theorem, it is sufficient to prove that every cubic map is 4-colourable-(f)

5.4 The Four-Colour Theorem

- Unavoidable sets:

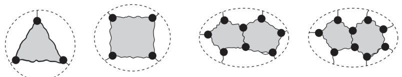
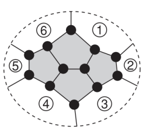


Figure 5.31

THEOREM 5.13 *The configurations in Fig. 5.31 form an unavoidable set.*

- Reducible: We say a configuration of faces is reducible if a 4-colouring of all the other faces can be extended (directly or after interchanging colours) to include the configuration
- **Birkhoff Diamond:** Another reducible configuration shown below.



- **Thm 5.14:** The Birkhoff diamond is reducible

5.5 Colouring Edges

- k-colourable-(e) if its edges can be coloured with k colours so that no two adjacent edges have the same colour
- Chromatic index: If G is k-colourable-(e) but not $(k-1)$-colourable-(e), we say that the chromatic index of G is k, and we write $\chi'(G) = k$
- **Thm 5.25 (Vizing):** If G is a simple graph with the largest vertex-degree Δ, then
$$\Delta \leq \chi'(G) \leq \Delta + 1$$
- **Thm 5.16:** $\chi'(K_n) = n$ if $n \geq 3$ is odd, and $\chi'(K_n) = n - 1$ if n is even
- **Thm 5.17:** The four-colour theorem is equivalent to the statement that $\chi'(G) = 3$ for each cubic map G
- **Thm 5.18 (König's):** If G is a bipartite graph with the largest vertex-degree Δ, then $\chi'(G) = \Delta$
- Corollary 5.19: $\chi'(K_{r,s}) = \max(r, s)$

6. Matching, Marriage and Menger's Thm

6.1 Hall's 'Marriage' Theorem

- **The Marriage Problem:** If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows.
- Complete matching: From V_1 to V_2 in a bipartite graph $G(V_1, V_2)$ is a one-to-one correspondence between the vertices in V_1 and some of the vertices in V_2, such that the corresponding vertices are joined. Then the marriage problem can be expressed as:

  If $G = G(V_1, v_2)$ is a bipartite graph, when does there exist a complete matching from V_1 to V_2 in G
- **Thm 6.1 (Hall):** A necessary and sufficient condition for a solution of the marriage problem is that each set of k girls collectively knows at least k boys, for $1 \leq k \leq m$
- Corollary 6.2: Let $G = G(V_1, V_2)$ be a bipartite graph, and for each subset A of V_1, let $\varphi(A)$ be the set of vertices of V_2 that are adjacent to at least one vertex of A. Then a complete matching from V_1 to V_2 exists iff $|A| \leq |\varphi(A)|$, for each subset A of V_1
- Transversal: If E is a non-empty finite set, and if $\mathcal{F} = (S_1, S_2, \ldots, S_m)$ is a family of non-empty subsets of E, then a **transversal** of $\mathcal{F}$ is a set of M distinct elements of E, one chosen from each set S_i; thus, $\{b_4, b_1, b_3, b_2\}$ is a transversal of the family
$$\mathcal{F} = (\{b_1, b_4, b_5\}, \{b_1\}, \{b_2, b_3, b_4\}, \{b_2, b_4\})$$
- Partial transversal: We call a transversal of a subfamily of $\mathcal{F}$ a **partial transversal** of $\mathcal{F}$. Note: Any subset of a partial transversal is a partial transversal
- **Thm 6.3:** Let E be a non-empty finite set, and let $\mathcal{F} = (S_1, S_2, \ldots, S_m)$ be a family of non-empty subsets of E. Then $\mathcal{F}$ has a transversal iff the union of any k of the subsets S_i contains at least k elements, for $1 \leq k \leq m$
- If E and $\mathcal{F}$ are as before, then $\mathcal{F}$ has a partial transversal of size t iff the union of any k of the subsets S_i contains at least $k + t - m$ elements

6.2 Menger's Theorem

- Edge-disjoint paths: The maximum number of paths from v to w in a graph G, no two of which have an edge in common
- Vertex-disjoint paths: The maximum number of paths from v to w, no two of which have a vertex in common
- $vw-$disconnecting set: A set E of edges of E such that each path from v to w includes an edge of E
- $vw-$separating set: A set of S vertices, other than v or w such that each path from v to w passes through a vertex of S
- **Thm 6.5:** The maximum number of edge disjoint paths connecting two distinct vertices v and w of a connected graph is equal to the minimum number of edges in a $vw-$disconnedting set
- **Thm 6.6 (Menger):** The maximum number of vertex-disjoint paths connecting two non-adjacent vertices v and w of a graph is equal to the minimum number of vertices in a $vw-$separating set
- Corollary 6.7: A graph G is $k-$edge-connected iff any two distinct vertices of G are connected by at least k edge-disjoint paths
- Corollary 6.8: A graph G with at least $k + 1$ vertices is $k-$connected iff any two disjoint vertices of G are connected by at least k vertex-disjoint paths
- **Thm 6.9:** The maximum number of arc-disjoint paths from a vertex V to a vertex w in a digraph is equal to the minimum number of arcs in a $vw-$disconnecting set
- **Thm 6.10** Menger's theorem implies Hall's Theorem

6.3 Network Flows

- Network: We define a network N to be a weighted digraph
- Capacity: Each arc a is assigned a non-negative real number $c(a)$
- The outdeg(x) of a vertex x is the sum of the capacities of the arcs of the form xz, indeg(x) is similarly defined
- **Handshaking dilemma for networks:** The sum of the out-degrees of the vertices of a network is equal to the sum of the in-degrees
- Source: A vertex with in-deg = 0
- Sink: A vertex with an out-deg=0
- Flow: A flow in a network is a function φ that assigns to each arc a a non-negative real number $\varphi(a)$, called the flow in a, in such a way that:

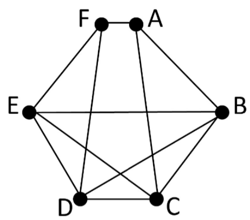
  1. for each arc $a, \varphi(a) \leq c(a)$
  2. the out-degree and in-degree of each vertex, other than v or w, are equal

- Zero flow: When the flow in every arc is 0, any other flow is a non-zero flow
- Saturated: An arc a for which $\varphi(a) = c(a)$, if not then unsaturated
- Cut: A set A of arcs such that each path from v to w includes an arc in A. A cut in a network is a $vw-$disconnecting set in the corresponding digraph D
- Capacity of a cut: The sum of the capacities of the arcs in the cut
- Minimum cuts: The cuts whose capacity is as small as possible
- **Thm 6.11 (Max-flow Min-cut Theorem):** In any network, the value of any maximum flow is equal to the capacity of any minimum cut
- Flow-augmenting paths: Paths that consist entirely of unsaturated arcs xz and zx carrying a non-zero flow

7. Examples

Normal 1st Questioners
(load of definition testers)

1. Consider the graph G depicted below.



(a) Is G Eulerian? Briefly say why or why not. [3 marks]
(b) Is G semi-Eulerian? Briefly say why or why not. [3 marks]
(c) Is G planar? Briefly say why or why not. [3 marks]
(d) Is G Hamiltonian? If yes find a Hamiltonian cycle, if not briefly say why not. [3 marks]
(e) What is the edge connectivity of G? (no justification needed) [2 marks]
(f) What is the vertex connectivity of G? (no justification needed) [2 marks]
(g) What is the chromatic number of G? (no justification needed) [2 marks]
(h) What is the chromatic index of G? (no justification needed) [2 marks]

[Total for this question: 20 marks]

1

(a) No, because there are vertices of odd degree (A and F).

(b) Yes, because there are exactly two vertices of odd degree (A and F).

(c) No, because it is contractible to K_5 (by contracting the edge AF).

(d) Yes, and $A - B - C - D - E - F - A$ forms a Hamiltonian cycle.

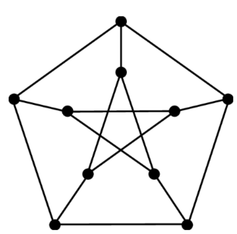
(e) 3

(f) 3

(g) 4

(h) 4

1. Consider the graph G depicted below:



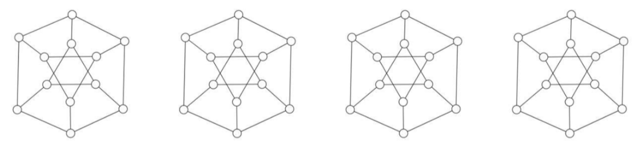
(a) Answer each yes/no question below, and briefly (with 1-2 sentences and/or a diagram) recount the reason for your answer. *Do not give a complete proof, just recall the main idea.*
  i. Is the graph G Hamiltonian? Give a reason for your answer. [2 marks]
  ii. Is the graph G semi-Hamiltonian? Give a reason for your answer. [2 marks]
  iii. Is the graph G bipartite? Give a reason for your answer. [2 marks]
  iv. Is the graph G Eulerian? Give a reason for your answer. [2 marks]
  v. Is the graph G planar? Give a reason for your answer. [2 marks]

(b) Provide each quantity below. *You do not need to give any justification.*
  i. What is the cycle rank $\gamma(\mathcal{P})$ of the graph G? [1 mark]
  ii. What is the cutset rank $\zeta(\mathcal{P})$ of the graph G? [1 mark]
  iii. What is the vertex connectivity $\kappa(\mathcal{P})$ of the graph G? [1 mark]
  iv. What is the trace of A^3, where A is the adjacency matrix of G? [1 mark]

(a) i. The Petersen graph is not Hamiltonian, we have seen the proof in lecture. It goes by contradiction: we assume it is Hamiltonian, so we start with a cycle of length 10, and we use the fact that there are no 2,3 or 4-cycles to produce the contradiction.
  ii. It is semi-Hamiltonian, as can be seen by visiting all vertices on the outer cycle, then moving to the inner cycle and traversing those.
  iii. It cannot be bipartite as it has a cycle of length 5.
  iv. It cannot be Eulerian as it has vertices of odd degree.
  v. It is not planar: if we contract all the edges which connect the inner 5-cycle to the outer 5-cycle, we obtain a K_5. We have shown as a corollary of Kuratowski's theorem that such a graph is not planar.

(b) i. 15+1-10 = 6
  ii. It is three. (We can easily see that it remains connected if we delete any two edges it remains connected, however if we delete the three edges around a vertex we disconnect that vertex).
  iii. This is also three (as above)
  iv. This is zero (since there are no triangles in the graph)

Q1. (1) (20 pts) Consider the connected graph G with 12 vertices and 18 edges, depicted below. It has been replicated 4 times in case you want to refer to a copy in your solutions.



(I) (II) (III) (IV)

(a) Is G Eulerian? (2pts) Briefly say why, or why not. (2pts)
(b) Is G bipartite? (2pts) Briefly say why, or why not. (2pts)
(c) Is G planar? (2pts) Briefly say why, or why not. (2pts)
(d) Indicate a Hamiltonian cycle in G on one of the figures. (2pts)
(e) What is the trace $\operatorname{tr} A^3$, where A is the adjacency matrix of G? (no justification required) (2pts)
(f) What is the cycle rank of G? (no justification required) (2pts)
(g) What is the edge connectivity of G? (no justification required) (2pts)

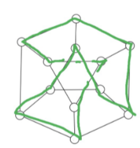
1

(a) It is not Eulerian, as there are vertices of odd degree.

(b) It is not bipartite, as there are cycles of odd length.

(c) It is planar, as can be seen by stretching one of the inner triangles to the exterior of the hexagon.

(d) Here is a Hamiltonian circuit:



(e) It has two triangles, and we have shown in the homework that the trace of A^3 on a simple graph is six times the number of triangles, hence the trace of A^3 is 12.

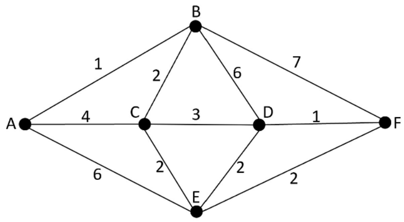
(f) Its cycle rank is $18 - 12 + 1 = 7$.

(g) Its edge connectivity is three. Clearly, deleting the three edges around any vertex disconnects the graph by isolating that vertex. However, we can see by inspection exploiting the symmetry of the graph that deleting any two edges will not disconnect it.

Dijkstra's, Guan's, Crit Path, T Salesman

2. This question concerns some of the optimization algorithms from the course.

   (a) In the graph below, the label on each edge represents the length of that edge.

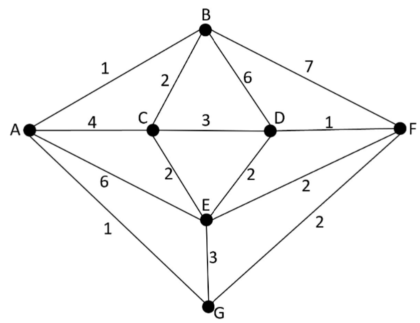


   i. Give a shortest path from vertex A to vertex F found using Dijkstra's algorithm.
   ii. What is the length of that path?
   iii. What are the permanent labels of each vertex A, B, C, D, E, and F?

   [8 marks]

   (b) The postman problem consists of finding a closed walk on a given weighted graph which includes each edge at least once and covers the least possible distance (where the label on each edge represents its length). What is the minimal length of such a walk for the graph in part (a)? No justification is needed. In particular, you don't need to present the walk in your solution. [6 marks]

   (c) The travelling salesman problem consists of finding a Hamiltonian cycle of minimal length on a given weighted graph. Find an optimal solution to the travelling salesman problem on the graph below and justify that it is optimal. [6 marks]



2

(a) (i) A shortest path is $A-B-C-D-F$. (Another is $A-B-C-E-F$.)

(ii) The shortest length is 7.

(iii) $l(A)=0$, $l(B)=1$, $l(C)=3$, $l(D)=6$, $l(E)=5$, $l(F)=7$.

(b) The minimal length of a solution to the postman problem in this graph is $36+7=43$ (where 36 is the sum of the weights of every edge and 7 is the length of the shortest path between the two odd-degree vertices, A and F).

(c) The graph has 7 vertices so a Hamiltonian cycle will have to have 7 edges. There are three edges of weight 1 and five edges of weight 2, so a lower bound for the total weight of a solution of the travelling salesman problem will be $1+1+1+2+2+2+2=11$. But by inspection we can see that the cycle $A-B-C-E-D-F-G-A$ has exactly that weight, so that must be an optimal solution.

For the first one, use the following: Start at A, set shortest path length to 0. Then see where you can reach from A. We see that we can reach B, C and E. Update the Previous Node column (PN) in the table. This shows which node you were at before getting to the current one. Then pick the one with the shortest length. In our case this is B. Confirm B's permanent label SP = 1. See where we can get to from B and update the table. We see we can get to C from A in 1 less going via B. This is now the shortest path to C. Update the table and then pick the next smallest number. Continue this until you have covered all vertices, where the final iteration of the table shows the permanent labels of each vertex.

N	SP	PN
A	0 ✓	
B	1	A
C	4	A
D		
E	6	A
F		

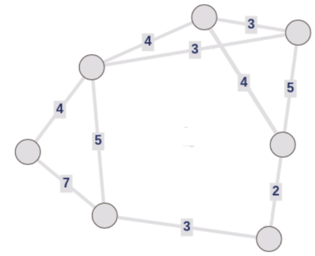
N	SP	PN
A	0	
B	1 ✓	A
C	3	B
D	7	B
E	6	A
F	8	B

N	SP	PN
A	0 ✓	
B	1 ✓	A
C	3 ✓	B
D	6	C
E	5	C
F	8	B

N	SP	PN
A	0 ✓	
B	1 ✓	A
C	3 ✓	B
D	6 ✓	C
E	5 ✓	C
F	7 ✓	E or D

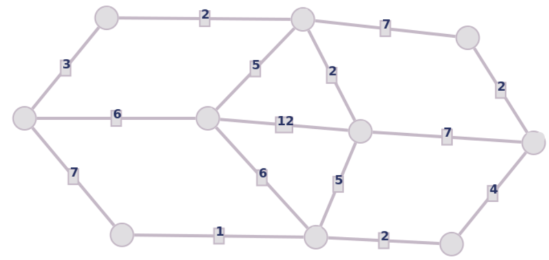
2. This question concerns various algorithms on graphs we have learned in the course.

   (a) Consider the "travelling salesman" problem on this weighted graph:



   i. Outline the list of steps one must take to obtain a lower bound for the solution to the travelling salesman problem. [3 marks]
   ii. Determine the lower bound according to your steps in this graph. [5 marks]
   iii. If you are able, produce an actual minimal solution to the travelling salesman problem. If you are not able, explain clearly what has gone wrong. [3 marks]

   (b) Now let us consider the special case of Kwan's solution to the "postman problem" in the case of the semi-Eulerian weighted graph below, for which the sum of the weights at all edges is 71.

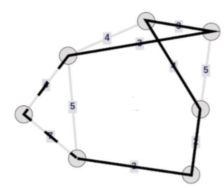


   i. What is the length of a solution to the postman problem on this graph? [6 marks]
   ii. Our postman feels it is dull to take the same route every day. Supposing the postman must begin at the post office at the leftmost vertex, and must always follow a fixed route for his delivery, how many choices does he have for his walk from the last delivery back to the post office which still minimize the length he walks? [3 marks]

Q2.

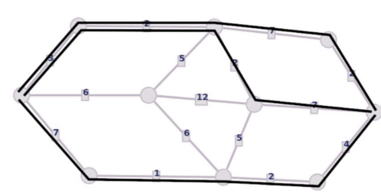
(a) i. We should find a vertex and delete it from the graph. We should then seek a spanning tree of minimum length for the remaining graph using the greedy algorithm. The minimum length of any Hamiltonian cycle must be greater than or equal to the weight of that minimal spanning tree, plus the weight of the smallest two edges emanating from the deleted graph. We are not guaranteed an actual solution, however as the tree we construct might not be a cycle.

ii. Let's elect to delete the vertex at the bottom left (it is convenient that it's degree two). Then we find a minimum spanning tree as depicted below with total length 14. Here the students may do something else and still get full marks). We get the tree depicted in dark black, and then we add back in the dotted edges.



iii. If the student has chosen the bottom left vertex as I have, then they will have produced a solution, and they only need to explain that it's indeed a cycle, and therefore a minimal Hamiltonian cycle (so they are mostly getting marks for having the good sense to choose the lower left vertex; if they understand the method they will know that this is a good way to ensure a solution). If not, they need to explain that it's not a tree (and in this case they get two of the three marks).

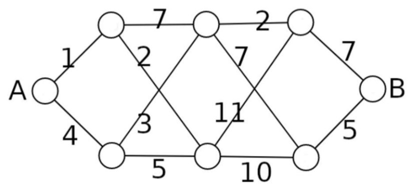
(b) i. The students must apply Kwan's algorithm noting that the graph given is semi-Eulerian with start and endpoints on the left and right (this is certainly a visual hint). They should find the shortest path between the start and end to be of length 14, which together with the length 71 of any semi-Hamiltonian cycle gives a minimal length circuit of 85. The students may give any one of the three depicted solutions.



ii. There are three minimal solutions, each depicted in the diagram above, and hence there are three possible return routes once the route is finished.

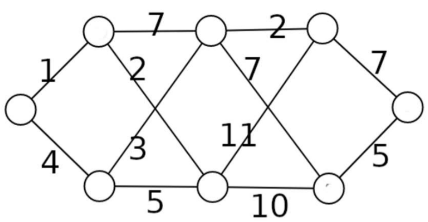
Q2. (2) (20pts) This question concerns some of the optimization algorithms from the course.

(a) On the graph below, indicate the minimal length path from vertex A to vertex B using the algorithm from the textbook (4pts).



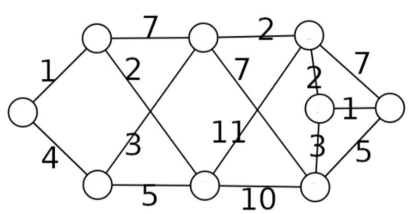
(b) On the graph above, correctly label each vertex with its permanent label according to the "Shortest path problem" algorithm from Section 8 of the chapter "Paths and Cycles" in Wilson's texbook. (4pts)

(c) On the graph below, draw in a minimal length spanning tree using the greedy algorithm (4pts):



(d) In the graph above, how many minimal length spanning trees are there (2pts)?

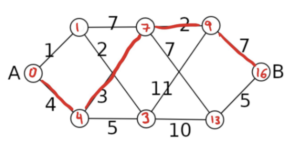
(e) Determine a lower bound for a solution of the travelling salesman problem for the following graph (4pts):



(f) Does your construction yield a solution of the salesman problem with that lower bound? Why or why not? (2pts)

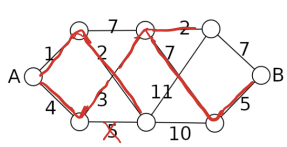
2

(a) See:



(b) See numbers in circles in graph above.

(c) See:



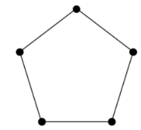
(d) There are two solutions, because at the point when we added the edge of length 7 we had two choices. Note that the third 7 would have created a cycle.

(e) We see that a lower bound to the travelling salesman problem would be obtained by adding the two edges of length 1 and 2 emanating from the new vertex to the spanning tree which had weight 24, obtaining a lower bound of 27.

(f) No it is not realized because the result of adding those two edges is not a cycle (for either choice).

Chromatic Polynomials + Trees etc

3. (a) Compute the chromatic polynomial of the cycle graph C_5. Show your work. You may use your knowledge of basic chromatic polynomials such as those of trees and complete graphs. You don't need to fully expand or fully factor your answer. **[4 marks]**

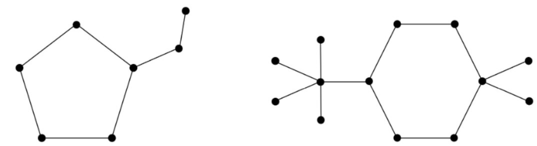


(b) Prove that the chromatic polynomial of the cycle graph C_n, for $n \geq 3$ is given by

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1).$$

[6 marks]

(c) What are the chromatic numbers $\chi(G)$ and $\chi(H)$ of the two graphs below? **[4 marks]**



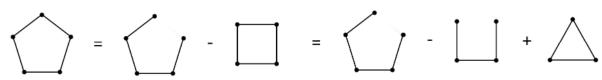
G H

(d) The graphs G and H above are examples of graphs containing exactly one cycle. What is the chromatic number of a graph with n vertices that contains exactly one cycle, and that cycle is of length ℓ? Justify your answer. **[6 marks]**

[Total for this question: 20 marks]

3

1. We use the deletion-contraction formula to begin computing the chromatic polynomial $P_{C_5}(k)$:



We can then continue the computation by using the known formulas for the chromatic polynomials of trees and of the complete graph K_3:

$$\begin{aligned} P_{C_5}(k) &= \dots \\ &= k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2) \\ &= k^5 - 5k^4 + 10k^3 - 10k^2 + 4k \end{aligned}$$

2. We prove this by induction on n.

The base case is for $n = 3$. Since C_3 is the complete graph K_3 we know its chromatic polynomial:

$$P_{C_3}(k) = P_{K_3}(k) = k(k-1)(k-2).$$

On the other hand,

$$(k-1)^3 + (-1)^3(k-1) = (k^3 - 3k^2 + 3k - 1) - (k-1) = k^3 - 3k^2 + 2k = k(k-1)(k-2).$$

Now we assume by induction that

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$

and aim to show that

$$P_{C_{n+1}}(k) = (k-1)^{n+1} + (-1)^{n+1}(k-1).$$

We will use the deletion-contraction formula: when we delete an edge of C_{n+1} we obtain a line graph with $(n+1)$ vertices. In particular this is a tree on $(n+1)$ vertices and hence its chromatic polynomial is

$$k(k-1)^n.$$

When we contract an edge of C_{n+1} we obtain C_n, for which we can use the formula from the induction hypothesis. Therefore:

$$\begin{aligned} P_{C_{n+1}}(k) &= k(k-1)^n - \left((k-1)^n + (-1)^n(k-1)\right) \\ &= (k-1)(k-1)^n - (-1)^n(k-1) \\ &= (k-1)^{n+1} + (-1)^{n+1}(k-1), \end{aligned}$$

as we wanted to show.

3. $\chi(G) = 3$ and $\chi(H) = 2$.

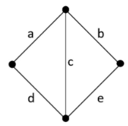
4. Let G be a graph with n vertices that contains exactly one cycle, and that cycle is of length ℓ. Then

$$\chi(G) = \begin{cases} 2, & \text{if } \ell \text{ even} \\ 3, & \text{if } \ell \text{ odd} \end{cases}$$

To see why, let C_ℓ be the subgraph of G consisting of only that cycle. The subgraph C_ℓ can be vertex-coloured with either two or three colours, depending on whether ℓ is even or odd respectively (if ℓ is even we alternate vertex colours, but if ℓ is odd we need an extra colour in order to colour the final vertex when going around the cycle). As for the trees coming out of the various vertices of C_ℓ, we can always colour them using only two colours.

4. Note: parts (a)-(d) are not related to part (e).

(a) Find all spanning trees of the following graph H: [4 marks]



(b) Up to isomorphism, how many distinct spanning trees are there? What are they? [3 marks]

(c) Compute the cycle rank of H and find a fundamental set of cycles. [3 marks]

(d) Compute the cutset rank of H and find the fundamental set of cutsets associated with the spanning tree $T = \{a, d, e\}$. (Write each cutset as a set of edges.) [4 marks]

(e) Let G be a planar simple connected graph that is self-dual, that is, the dual G^* is isomorphic to the original graph G. Show that G must contain at least three triangular faces. [6 marks]

[Total for this question: 20 marks]

4

(a) There are eight spanning trees of H:

$\{a, b, c\}$, $\{a, b, d\}$,$\{a, b, e\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, d\}$,$\{b, d, e\}$,$\{c, d, e\}$.

(b) Up to isomorphism there are two distinct spanning trees of H: one is the line graph with four vertices and three edges (such as $\{b, a, d\}$), and the other is a star with one central vertex of degree three connecting to three vertices of degree one (such as $\{a, b, c\}$).

(c) The cycle rank of H is $m - n + 1 = 5 - 4 + 1 = 2$. One example of a fundamental set of cycles is the set consisting of the two triangles acd and cbe.

(d) The cutset rank of H is $n - 1 = 4 - 1 = 3$. The fundamental set of cutsets associated with the spanning tree $T = \{a, d, e\}$ consists of the cutsets $\{a, b, c\}$, $\{b, c, d\}$, and $\{b, e\}$.

(e) Let G be a planar simple connected graph with n vertices, m edges, and f faces. Then the dual graph G^* has f vertices, m edges, and n vertices. Thus, if we assume that G is isomorphic to G^* then we have $n = f$. In particular, this means that for a self-dual graph, the Euler formula for planar graphs $n - m + f = 2$ becomes $m = 2f - 2$.

Counting the edges around each face we have on the one hand:

$$\sum_{\text{faces}} (\text{number of edges around each face}) = 2m = 4f - 4,$$

where in the second equality we used the Euler formula for self-dual graphs derived above.

On the other hand, if we assume towards a contradiction that G has three or fewer triangular faces, then those three faces must have at least three edges and all other $(f - 3)$ faces must have at least four edges, so:

$$\sum_{\text{faces}} (\text{number of edges around each face}) \geq 3 + 3 + 3 + 4(f - 3) = 4f - 3.$$

Putting those two equations together we reach a contradiction:

$$4f - 4 \geq 4f - 3 \iff -1 \geq 0.$$

3. This question concerns results about counting labelled trees and labelled forests.

(a) Recall the enumeration of labelled trees via so-called Prüfer sequences, presented as Cayley's theorem in the textbook.

i. Consider the sequence $(1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2)$ consisting of the sequence 1,2 repeated 7 times. Draw the shape of the corresponding tree. You again do not need to label it, but make sure the shape of the tree is clear, in particular the number of vertices and edges. [2 marks]

ii. Write a Prüfer sequence whose associated labelled tree consists of exactly 8 vertices of degree 3, and 10 vertices of degree one (and no other vertices besides those). [3 marks]

iii. How many such labelled trees with the above properties are there, counted up to isomorphism of labelled trees? [5 marks]

iv. Will all such *labelled* trees have isomorphic underlying *unlabelled tree*? Give a reason for your answer. [2 marks]

(b) Recall that a forest is a possibly disconnected graph all of whose connected components are trees. How many labelled forests are there with exactly 3 connected components and 7 vertices, giving your final answer as an integer? [8 marks]

Q3.

(a) They should depict a tree which has two adjacent central vertices, from each of which 7 additional vertices extrude.

(b) They may provide any sequence of length 16 in which only eight numbers appear, each exactly twice. These will be the 8 vertices of degree 3 in the resulting tree, and the 10 which were omitted will be the vertices of degree one.

(c) As reasoned above, there are $\binom{18}{8}$ ways to decide which 8 will be degree three, and then having chosen those there are $16!/2^8$ possible arrangements. So all in all we have $\binom{18}{8} \cdot 16!/2^8$ such trees.

(d) Certainly not. If we delete all the 10 vertices of degree 1, we will obtain a tree with vertices of degree 1, 2 or 3. Any such tree at all such with k vertices of degree two and $2l$ vertices of degree 1, such that $k + 2l = 10$ can arise in this way. So there are many possible shapes for the original tree. The students may give other explanations.

In Q3(a) assume that G is simple and connected.

Q3. (a) Let G be a graph with $p \geq 11$ vertices. Prove that G and $\overline{G}$ (the complement of G) cannot be both planar. **[5 marks]**

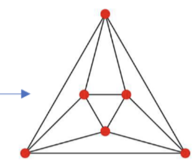
(b) Prove that a bipartite graph G with an odd number of vertices cannot have a Hamiltonian cycle. **[5 marks]**

(c) (i) Draw a plane embedding of the octahedron so that all edges are straight lines.

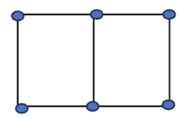
(ii) Let G be the plane embedding from previous part (i). Draw a geometric dual of G ~~such that all edges are straight lines.~~

(iii) What is the chromatic number of the dual? Give an example of such colouring to justify your answer. **[5 marks]**

(we didn't talk at all about edges being straight lines, so here is the answer to (c) (i) so you can do parts (ii) and (iii))



(d) Compute the chromatic polynomial of the graph



3

(a) Let G be a simple connected planar graph with $p \geq 11$ vertices and m edges. It is a corollary of Euler's formula that

$$m \leq 3p - 6.$$

Assume towards a contradiction that $\overline{G}$ is also planar. The graph $\overline{G}$ has $\frac{p(p-1)}{2} - m$ edges, and thus

$$\frac{p(p-1)}{2} - m \leq 3p - 6.$$

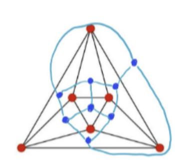
Adding the two inequalities above yields

$$\frac{p(p-1)}{2} \leq 6p - 12 \iff p^2 - 13p + 24 \leq 0 \iff 2.2280 \leq p \leq 10.772,$$

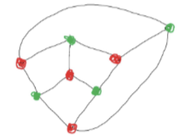
which contradicts the assumption that $p \geq 11$.

(b) Let G be a bipartite graph with an odd number of vertices. We know that a graph is bipartite if and only if every cycle of G has even length. In particular, there is no cycle that contains every vertex of G (exactly once), and thus there exists no Hamiltonian cycle of G.

(c) Part (i) is given in the mock exam. Part (ii) is

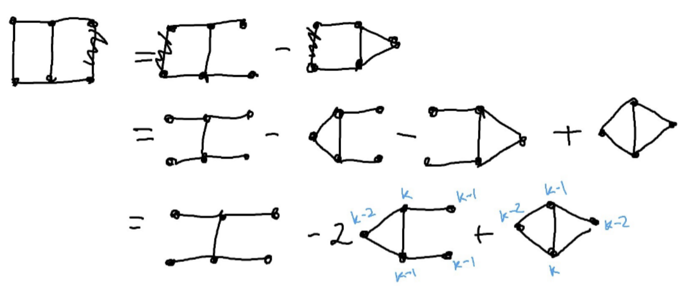


Part (iii): the chromatic number of the dual is two:



(d) We begin by using the contraction-deletion formula: The first graph in the last equality is a tree on 6 vertices, and for the other two we can simply calculate the number of possible colourings by looking at each vertex:

$$P_G(k) = k(k-1)^5 - 2k(k-1)^3(k-2) + k(k-1)(k-2)^2.$$



Q4. (c) Let $G = (V, E)$ be a bipartite graph such that $V = V_1 \cup V_2$ is the partition of vertices. Prove that $\sum_{u \in V_1} \deg u = \sum_{v \in V_2} \deg v$. [Hint: You may use the double counting rule.] **[3 marks]**

(d) There are some men and 15 women in the room. Each man shook hands with exactly 6 women, and each woman shook hands with exactly 8 men. How many men are in the room?[Hint: You might use the result from part (c).] **[4 marks]**

(a) Let T be a tree of order $p \geq 2$. Prove that T is bipartite. [Hint: You may use any results from the course if they are stated clearly.] [5 marks]

(by order they mean number of vertices)

(b) For each of the following four properties give an example of a graph G with $p \geq 4$ such that

- **G is non-planar, hamiltonian and eulerian**
- **G is planar, non-hamiltonian, non-eulerian**
- **G is bipartite, non-eulerian and hamiltonian**
- **G is bipartite, eulerian and non-hamiltonian.**

(p means the number of vertices)

You must justify your answer. [8 marks]

4

(a) Maybe it's overkill, but we can prove this by induction on the number of edges of G, let us call it m. The base case is $m = 0$: in that case the sum of the degrees of the vertices on each of the V_i's is 0, so the equality we wanted to prove boils down to $0 = 0$ which is true.

The induction hypothesis is that if G has m edges then

$$\sum_{u \in V_1} \deg(u) = \sum_{v \in V_2} \deg(v). \tag{1}$$

Now for the induction step let us assume that G has $m + 1$ edges. Let G' be the graph that we obtain by removing an edge e of G; this graph has m edges and therefore (1) holds for it. Now we add the edge e back to G' to obtain G. Because G is bipartite this edge connects some vertex $u \in V_1$ so some vertex $v \in V_2$. Therefore, putting e back in adds 1 to either side of (1) and thus we have the result that we wanted to prove about the graph G.

(b) This is an application of the formula proved above: the set V_1 consists of the k men and the set V_2 consists of the 15 women. We draw an edge between a vertex in V_1 and a vertex in V_2 if those two people shook hands. If each man shook hands with exactly six women, we have that

$$\sum_{u \in V_1} \deg(u) = 6k.$$

Similarly, each woman shook hands with exactly eight men means that

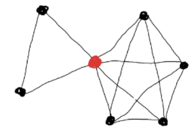
$$\sum_{v \in V_2} \deg(v) = 15 \times 8 = 120.$$

Thus, by (1) above, we have $6k = 120$ and thus $k = 20$: there were twenty men.

(c) Let T be a tree with $p \geq 2$ vertices. We want to show that T is bipartite. The quickest way is to combine the following two results: "a graph is bipartite if and only if it has no cycles of odd length" and "trees have no cycles" (and hence in particular no cycles of odd length). But let's prove it more constructively:

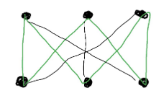
We are going to build two subsets A and B of vertices of T such that every edge of T connects a vertex in A to a vertex in B. Pick some vertex v_0 of T, this vertex will be in A. Recall that in a tree there exists only one path connecting one vertex to another (otherwise, concatenating those two paths would yield a cycle). Now take any vertex in T that is an even number of edges away from v_0 and place it also in A, and take any vertex in T that is an odd number of edges away from v_0 and place it in B. Let's check that indeed T is bipartite, that is, that no two vertices of A are connected by an edge and no two vertices of B are connected by an edge.

If two vertices v_1 and v_2 in A were connected by an edge e, then concatenating the path from v_0 to v_1 with the edge e with the path from v_2 to v_0 would yield a cycle, which cannot exist because trees don't have cycles. A similar argument holds for B.

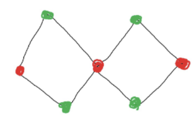


The graph above is non-planar (contains K_5 as a subgraph), non-hamiltonian (any cycle that contains both vertices of the K_5 subgraph and vertices of the triangle subgraph must go through the red vertex twice), and eulerian (every vertex has even degree).

The graph $K_{3,3}$ drawn below is bipartite (by definition), non-eulerian (there exist vertices of odd degree), and hamiltonian (see cycle in green).



The graph below is bipartite (the two subsets of vertices are colored green and red), eulerian (every vertex has even degree) and non-hamiltonian (any cycle that contains vertices from both diamonds must go through the middle red vertex twice).



Miscellaneous + End Proofs

5. For each question below, you will be expected to write original proofs consistent with the writing standards for the course. As with the hand-in assignments, partial credit will be awarded for significant progress towards a solution.

(a) Show that, for any colouring of the *edges* of the complete graph K_{17} by colours Red, Green, or Blue, there exists some triangle $K_3 \subset K_{17}$ all of whose edges are coloured with the same colour. [10 marks]

(b) Show that for any number n, there exists some integer m such that in any colouring of the graph K_m from a palette of n distinct colours there exists a triangle $K_3 \subset K_m$ all of whose edges are coloured with the same colour. [10 marks]

Q5.

(a) Consider some vertex of G, and look at all its red neighbours. Supposing there are at least six of them, then we know that no edge between any of the neighbors can be red; hence they are either blue or green, and then the 6 people at a party problem implies there is a monochrome triangle. A similar argument applies if our chosen vertex has at least six green or six blue neighbors. By pigeonhole principle one of these must apply, since we have 16 neighbours and only 3 colors. Hence we are done.

(b) We proceed by induction. The base case is when $n = 2$ and we may take $m = 6$, as this is the six people at a party question. For the induction step, fix some number n of colours, and we may assume that there exists m' with the property that any coloring of $K_{m'}$ with n colors contains a monocrhomatic triangle. Now we consider $n + 1$ colours. Let $m = (n + 1) \cdot (m' - 1) + 2$. Then, arguing as above we see that if some vertex v has $(m' - 1)$ edges of some fixed colour, then by induction there is either a monochromatic triangle among their neighbours, or a triangle of that colour connecting two such edges. However, also arguing as above, the pigeonhole principle will tell us there must be at least one color with $(m' - 1)$ edges adjacent to v, and we are done.

Q5. **(d) Determine whether K_4 is a subgraph of the complete bipartite graph $K_{4,4}$. If it is, then exhibit it. If it is not, then explain why not. [10 marks]**

(20pts) Recall that a simple graph is triangle-free if it contains no subgraph isomorphic to the triangle K_3. In the "six people at a party" problem, we have proved that any triangle-free graph with at least six vertices contains a null graph N_3. In your homework you have proved that any triangle-free graph with at least 9 vertices contains a null graph N_4.

(a) (10pts) Prove that a triangle-free simple graph G with at least 14 vertices must contain as a subgraph the null graph N_5.

5

(a) The graph K_4 is not a subgraph of of the complete bipartite graph $K_{4,4}$. Indeed, K_4 contains a triangle, and so if K_4 were a subgraph of $K_{4,4}$ then $K_{4,4}$ would contain that triangle, but bipartite graphs do not have cycles of odd length.

(b) Chose some vertex v in G. If v has at least five neighbors, then none of those can be connected to one another since that would form a triangle, hence the five neighbours of v define an N_5 in G. If v has at most four neighbours, then it means there are at least nine vertices which are not connected to v. By the homework problem recalled in the text of the question, the subgraph with those nine vertices contains an N_4. This N_4 together with v define the required N_5.

Other

1.10: Show that there are exactly $n^{n(n-1)/2}$ labelled simple graphs on n vertices.

Proof: We proceed by induction

- Base Case: $n = 2$

  We start with $n = 2$ since there are no simple graphs with less than 2 vertices that can be labelled differently. using this we obtain: $2^{n(n-1)/2} = 2^{2(2-1)/2} = 2^1 = 2$. The base case holds.

- Inductive Hypothesis: $n^{n(n-1)/2}$ holds for $n = k$, i.e., $k^{k(k-1)/2}$ is the number of labelled simple graphs on k vertices

- Inductive step: We now consider the case where $n = k+1$. We want to show that $(k+1)^{(k+1)(k+1-1)/2} = (k+1)^{(k+1)(k)/2}$ for a simple graph on $k+1$ vertices.

  Using the IH, we assumed that there are$k^{k(k-1)/2}$ simple labelled graphs with k vertices. We can then add a vertex to each of these graphs where we then have the option to choose which of the k vertices in the original graph to connect this one to, each one generates a distinct graph. By doing this, we have 2^k new graphs for each of the $k^{k(k-1)/2}$ graphs. So we have
  $$\begin{aligned}\left[2^{k(k-1)/2}\right]\left[2^k\right] &= 2^{k(k-1)/2\ +\ k}\\ &= 2^{\frac{k^2-k+2k}{2}}\\ &= 2^{\frac{k^2+k}{2}}\\ &= 2^{\frac{k(k+1)}{2}} \quad \text{as required.}\end{aligned}$$

  Thus, by the principle of mathematical induction...

1.15: If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree

Proof: For a contradiction, assume that a graph G does not contain two or more vertices of the same degree, i.e., one vertex has a different degree. Let v_0 be some vertex in G such that $\deg(v_0) = 0$. For a graph that does not contain any vertices w with the same degree as, we require a vertex, v_{n-1}, such that $\deg(v_{n-1}) = n-1$. However, given that the graph G is simple, v_{n-1} would be adjacent to all vertices in G, including v_0, which is a contradiction since $\deg(v_0) = 0$. Therefore our assumption is false and the required result holds.

2.7 (i): Show that, if G is a connected graph with minimum degree k, then $\lambda(G) \leq k$

Proof: Assume we have a connected graph G with minimum degree k such that $\lambda(G) > k$. This implies, by theorem 2.4, that any two distinct vertices of G are joined by at least $k+1$ distinct paths. Let v be a vertex in G with minimum degree such that $\deg(v) = k$. We require that we can get to some other arbitrary vertex in G by at least $k+1$ paths. However, since $\deg(v) = k$, we only have k distinct "ways out" of v. This implies the graph must be at most k-edge-connected and therefore, $\lambda(G) \leq k$.

Hand-ins

H1 Question 1 [5pts] Prove that for every simple graph G with nine vertices, either G contains a subgraph isomorphic to the complete graph K_4, or else G^c contains a subgraph isomorphic to the complete graph K_3.

Solution: We begin by noting that it cannot be the case that every vertex of G has exactly five neighbours. This is because, by the Handshaking Lemma, the sum of all the vertex-degrees is an even number but $9 \times 5 = 45$ is not even. Thus, at least one of the following two situations must occur:

(A) There exists a vertex with less than or equal to four neighbours.

(B) There exists a vertex with greater than or equal to six neighbours.

We will now see that each of these situations implies the desired conclusion.
In case (A), let v denote a vertex with at most four neighbours. We let S denote the set of non-neighbours to v, which is necessarily of size at least four. If any pair $\{x, y\}$ of elements of S are also not connected to one another, then $\{v, x, y\}$ gives a triangle in G^c. On the other hand if all elements of S are connected, then they form a complete graph on at least 4 vertices, which therefore contains a K_4 in G, and we are done.
In case (B), let v denote a vertex with six or more neighbours, and denote this set of neighbours (and any edges that exists between them in G) by T. By the six-people-at-a-party result, either T^c contains a triangle (in which case we are done), or else T contains a triangle, $\{x, y, z\}$, in which case $\{v, x, y, z\}$ determines a K_4 in G.

H2 Question 1 [5pts]

(a) Show that if a simple graph G with $\geqslant 3$ vertices is 2-connected then it can be built up by starting with a cycle G_0 and then repeating the following steps until getting the whole graph:

  (i) pick two vertices u_k, v_k in the graph G_k built so far;

  (ii) add a path P_k between u_k and v_k, of any length, that doesn't have any edges or vertices in common with G_k – this creates G_{k+1}.

(b) ***CANCELLED.***

Note 1: You can use Theorem 2.5 from the textbook without proving it.

Note 2: Part (a) became an if-then (instead of an if-and-only-if) and part (b) is cancelled.

Lemma A: In a 2-connected graph with $\geqslant 3$ vertices, any two edges lie on a common cycle. ***(You can use this without proving it.)***

Solution:

The process described in the question is called an ***open ear*** decomposition. We want to show that if a graph is 2-connected then it has an open ear decomposition, i.e., it can be built iteratively from a cycle by adding on paths connecting distinct vertices on the previous iteration.

Let G is 2-connected. Then:

(1) we can start the process (i.e., there is indeed a cycle in G to be used as G_0),

(2) that at each step we can proceed (i.e., there is path P_k as described), and

(3) that the process terminates at some point.

Ok, so here we go:

(1) Pick two vertices of G. Then by Menger's Theorem (Theorem 2.5 in the textbook), they are joined by at least two vertex-disjont paths. The union of two such paths makes up a cycle, call it G_0.

(2) Assume that we are at the step of having G_k. If $G_k = G$ then we have concluded the process. Otherwise, there exist edges of G not already in G_k, pick one such edge e_k. Pick also one edge f_k in G_k. By Lemma A, there exists a cycle in G that contains both e_k and f_k, let us call that cycle C_k. If we remove from C_k all the edges it has in common with G_k, we are left with a collection of paths (and some isolated vertices), each with two endpoints on G_k. Let P_k be the path in that collection that contains the edge e_k, and set G_{k+1} to be the union of G_k and P_k.

(3) The graph G has a finite number of vertices and edges, so eventually we will use up all of them and we will have $G_k = G$, where the process terminates.

H3 Question 1 [10 pts] A planar graph G is ***outerplanar*** if it can be drawn on the plane in such a way that all the vertices lie on the exterior boundary.

(a) Show that K_4 and $K_{2,3}$ are not outerplanar. (5pts)

(b) Deduce that if G is an outerplanar graph, then G contains no subgraph that is a subdivision[1] of K_4 or $K_{2,3}$.(5pts)

Remark: For simplicity, you may use the following version of Kuratowski's theorem, which we also discussed in lecture: A graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.

(a) We first show that K_4 is not outerplanar, the proof for $K_{2,3}$ is analogous.

  Suppose by contradiction that K_4 is indeed outerplanar. Add a single vertex v to the outside face and connect it to all four vertices of K_4. The resulting graph is a K_5, by construction. However, the resulting graph is also planar, because with all original vertices on the exterior boundary, I can draw the new edges without crossing any pre-existing edge. But K_5 is not a planar graph, which gives us our contradiction.

  For $K_{2,3}$ we do the same thing, but instead connecting the new vertex v to just the 3 relevant vertices of $K_{2,3}$ in order to get $K_{3,3}$.

  Hence K_4 and $K_{2,3}$ are not outerplanar.

(b) We first argue that a subgraph of an outerplanar graph is outerplanar: indeed, a subgraph can be obtained by

  (1) possibly deleting vertices of G along with any edges connected with them, and then

  (2) possibly deleting some more edges.

  As neither of these actions changes the outerplanarity of the graph, we can conclude that a subgraph of an outerplanar graph is outerplanar.

  Second, we argue that a subdivision of a graph that is not outerplanar is not outerplanar. We do this by showing that if H_0 is a subdivision of H and H_0 is outerplanar, then H must have been outerplanar as well. Indeed, if H_0 is a subdivision of H then in particular we can recover H from H_0 by smoothing out the edges that were subdivided: if uv was replaced by uw and wv (with the vertex w added), then contract one of the edges uw or uv – this "undoes" the original subdivision. But this smoothing out procedure does not change the outerplanarity of the graph, so H is outerplanar as well.

  Therefore, if G is an outerplanar graph, then any subgraph of it has to be outerplanar as well, and in particular, that means a subgraph of G cannot be a subdivision of K_4 and $K_{2,3}$, which would not be outerplanar, by the paragraph above and part (a).

[1] A subdivision of a graph H is a graph that is obtained from H by a sequence of edge subdivision operations. An edge subdivision operation for an edge uv deletes that edge from the graph and replaces it by two edges, uw and wv, along with a new vertex w. We mentioned this in lecture.

H4 Question 1 [5 pts]

Recall that $\chi(G)$ denotes the chromatic number of a simple graph G. Prove by induction on n that if G is a simple graph with n vertices and $\overline{G}$ is its complement, then
$$\chi(G)+\chi(\overline{G}) \leqslant n+1.$$

Solution

We will prove by induction on n (the number of vertices of G) that $\chi(G)+\chi(\overline{G}) \leqslant n+1$.

Base case: If G has $n=1$ vertex (and therefore no edges, as it is a simple graph), then $\chi(G)=\chi(\overline{G})=1$, and so indeed
$$\chi(G)+\chi(\overline{G})=1+1.$$

Induction hypothesis: If G is a simple graph on n vertices then $\chi(G)+\chi(\overline{G}) \leqslant n+1$.

Induction step: Let G be a graph on $n+1$ vertices. Pick a vertex v in G. Then $G-v$ is a graph on n vertices and so using the induction hypothesis, we have
$$\chi(G-v)+\chi(\overline{G}-v) \leqslant n+1. \tag{1}$$

We now prove an auxiliary claim:

Claim: If v is a vertex of G, then $\chi(G) \leqslant \chi(G-v)+1$, and if equality happens then $\deg_G(v) \geqslant \chi(G-v)$.

Proof of Claim: Consider a colouring of $G-v$ that uses $\chi(G-v)$ colours. Then, in order to produce a colouring of G we just need to figure out how to colour the vertex v. Depending on the colours of the vertices adjacent to v, we can either colour v with one of the already existing $\chi(G-v)$ colours (and in that case $\chi(G)=\chi(G-v)$), or we may need an extra colour for the vertex v (in which case $\chi(G)=\chi(G-v)+1$). So we have proved that
$$\chi(G) \leqslant \chi(G-v)+1.$$

Furthermore, if we are in the case of needing an extra colour for the vertex v (that is, $\chi(G)=\chi(G-v)+1$) then v must be connected to at least $\chi(G-v)$ vertices (and those vertices need to all be coloured in all distinct colours), so:
$$\text{If } \chi(G)=\chi(G-v)+1 \text{ then } \deg_G(v) \geqslant \chi(G-v).$$

Now we continue our proof by induction. By the Claim proved above, we also have
$$\chi(G) \leqslant \chi(G-v)+1 \qquad \text{and} \qquad \chi(\overline{G}) \leqslant \chi(\overline{G}-v)+1. \tag{2}$$

Putting these three inequalities together we get
$$\chi(G)+\chi(\overline{G}) \leqslant \chi(G-v)+\chi(\overline{G}-v)+2 \leqslant n+3. \tag{3}$$

But what we want to show is that
$$\chi(G)+\chi(\overline{G}) \leqslant n+2,$$
so we need to show that we can't have equalities in (3). But in order for that to happen, we would need to also have equalities in (1) and (2). By the Claim proved above, this would mean that
$$\deg_G(v) \geqslant \chi(G-v) \qquad \text{and} \qquad \deg_{\overline{G}}(v) \geqslant \chi(\overline{G}-v).$$

We know that the degrees of a vertex v in G and in $\overline{G}$ must add up to the number of vertices of G minus 1, so
$$n=\deg_G(v)+\deg_{\overline{G}}(v) \geqslant \chi(G-v)+\chi(\overline{G}-v)=n+1,$$
where the last equality is because we are assuming that (1) is an equality. Since we concluded that if (3) is an equality then $n \geqslant n+1$, which is a contradiction, it must be that (3) is always a strict inequality. Because all these numbers are integers, that can be rewritten as
$$\chi(G)+\chi(\overline{G}) \leqslant n+2$$
as desired.

Alternate Proof: We will note quickly that as in the book we do not consider the null graph with 0 vertices to be simple. Let's begin, we will proceed by induction on the number of vertices. Firstly, the base case
$$\begin{aligned} n=1 &\implies \chi(G)=1 \\ &\implies \chi(\overline{G})=1 \\ &\implies 1+1 \leq 1+1. \end{aligned}$$

Now, we will state our **inductive hypothesis:** Let G be a simple graph with n vertices, then
$$\chi(G)+\chi(\overline{G}) \leq n+1.$$

Our inductive step is follows. Let G now be a simple graph with $\boldsymbol{n+1}$ vertices, and without loss of generality fix a vertex v of G and let k be its degree in G, so that the degree of v in $\overline{G}$ is $n-k$. Now let us look at the graph $G-v$, if we have $\chi(G-v)>k$ then adding v back to $G-v$ does not increase the chromatic number, this is because we can assign a colour to v using an existing colour different from the colours of each of its k neighbours. If it is not the case that $\chi(G-v)>k$, (i.e., $\chi(G-v) \leqslant k$) then the chromatic number will be increased by at most one (as we need to pick another colour to give to v that has not already been chosen). Similarly if we look at the graph $\overline{G}-v$, if we have $\chi(\overline{G}-v)>n-k$ then adding v back to $\overline{G}$ does not increase the chromatic number, again due to the fact that we can assign one of the existing colours that differs from each of v's $n-k$ neighbours. Else we increase the chromatic number of $\overline{G}$ by at most one because we need exactly one extra colour.

Thus, if either $\chi(G-v)>k$ or $\chi(\overline{G}-v)>n-k$ holds, then, using the inductive hypothesis, we obtain the result:
$$\chi(G)+\chi(\overline{G}) \leq \chi(G-v)+\chi(\overline{G}-v)+1 \leq n+2.$$

Otherwise we have $\chi(G-v) \leqslant k$ and $\chi(\overline{G}-v) \leq n-k$, which implies
$$\chi(G)+\chi(\overline{G}) \leq \chi(G-v)+\chi(\overline{G}-v)+2 \leq k+n-k+2=n+2.$$

Which concludes the proof. Since the base case holds, and the inductive hypothesis (G on n vertices) implies the statement holds for G on $n+1$ vertices. By the principle of mathematical induction, we conclude that $\chi(G)+\chi(\overline{G}) \leqslant n+1$ holds for any simple graph G on n vertices.

Type of Graph	Vertices	Edges	Chromatic Number	Chromatic Polynomial	Chromatic Index	Deg of Vertices
Null (N_n)	n	none	1	λ^n	0	0
Complete (K_n)	n	$\frac{n(n-1)}{2}$	n	$\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$	$n/n-1$ (odd/even)	$n-1$
Cycle (C_n)	n	n	2 if n even 3 if n odd	$(\lambda-1)^n+(-1)^n(\lambda-1)$	2/3 (even/ odd)	2
Path (P_n)	n	$n-1$	2	$\lambda(\lambda-1)^{n-1}$	2	1 (end vertices) 2 (others)
Wheel (W_n)	n	$2(n-1)$	3 if n odd 4 if n even	$\lambda\big[(\lambda-2)^{n-1}-(-1)^n(\lambda-2)\big]$	$n-1$	$n-1$ (center vertex) 3 (outer vertices)
Cubic	n	$3n/2$	$\leqslant 4$	Varies	3	3
Platonic	Varies	Varies	Varies	Varies	Varies	Regular
Bipartite	n	m	2	Varies	Δ (max degree)	Varies
Complete Bipartite ($K_{r,s}$)	$r+s$	rs	2 if $r,s>1$ 1 if $r=1$ or $s=1$	$\lambda(\lambda-1)^{r+s-1}$	Δ	r (for s vertices) s (for r vertices)
Cubes (Q_k)	2^k	$k\cdot 2^{k-1}$	2	Varies	k	k