School of Electrical Engineering and Robotics

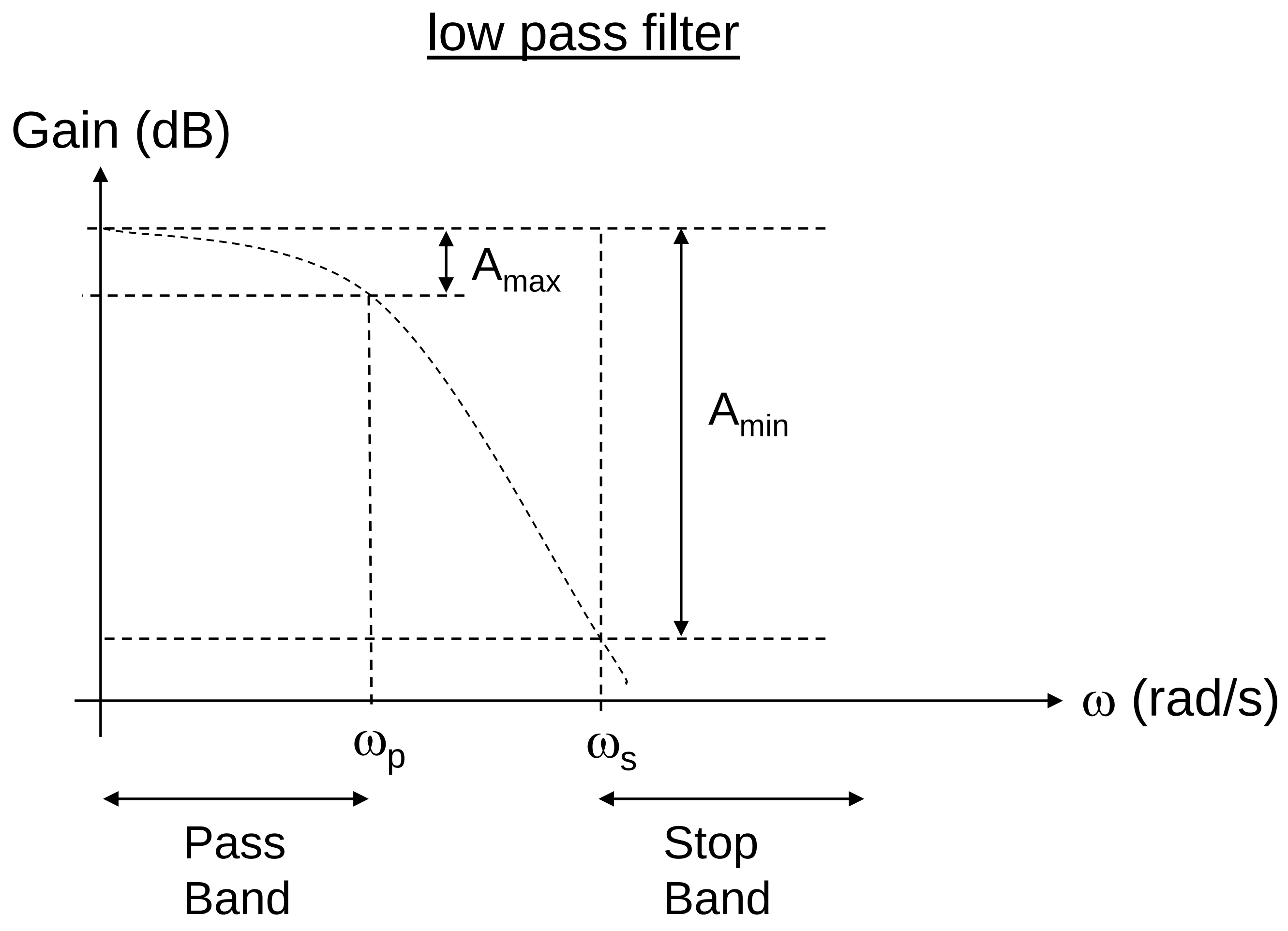
EGB348 Electronics

Filter Approximation

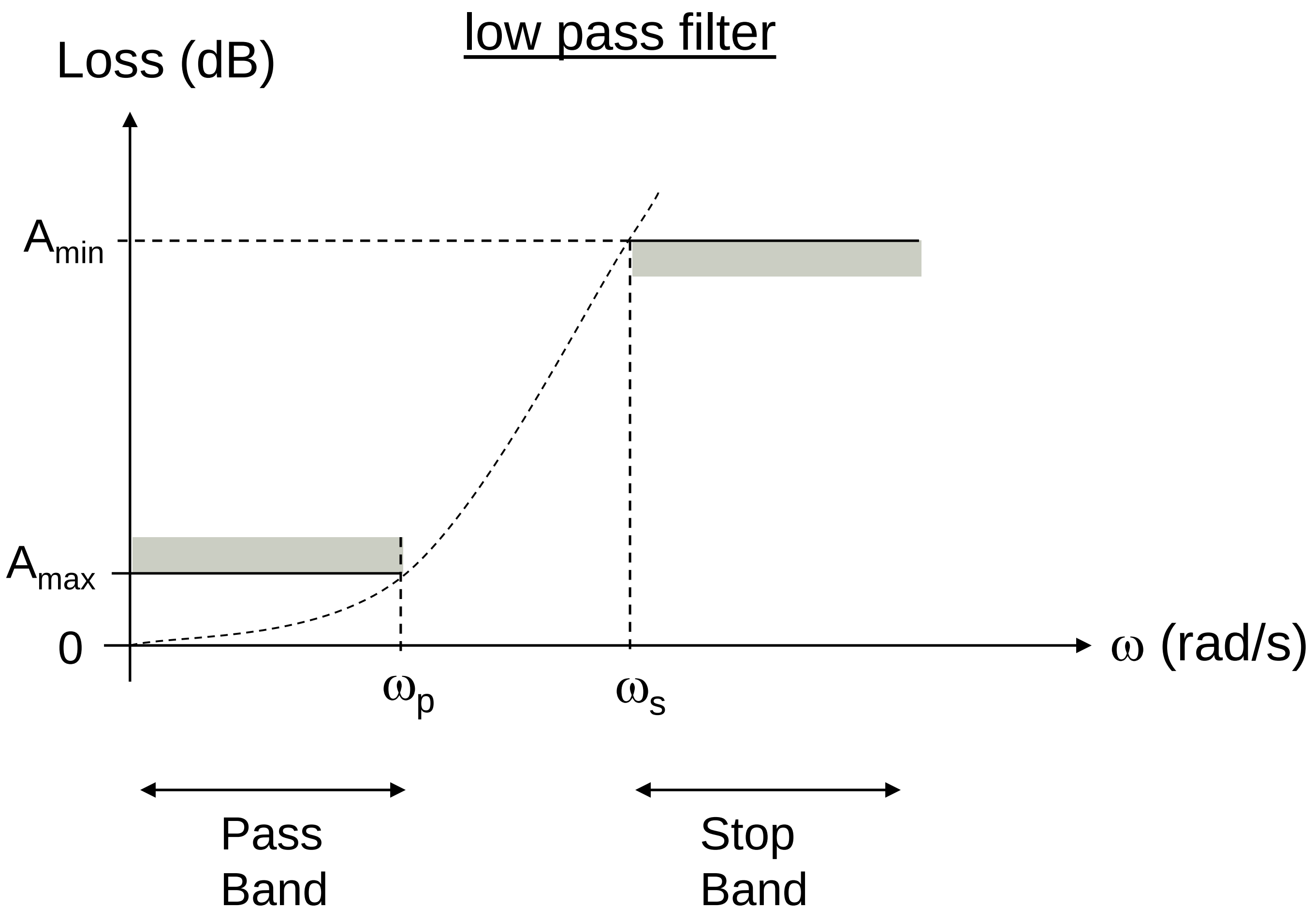
Jasmine Banks

Recommended Readings:

Filter Gain Characteristics



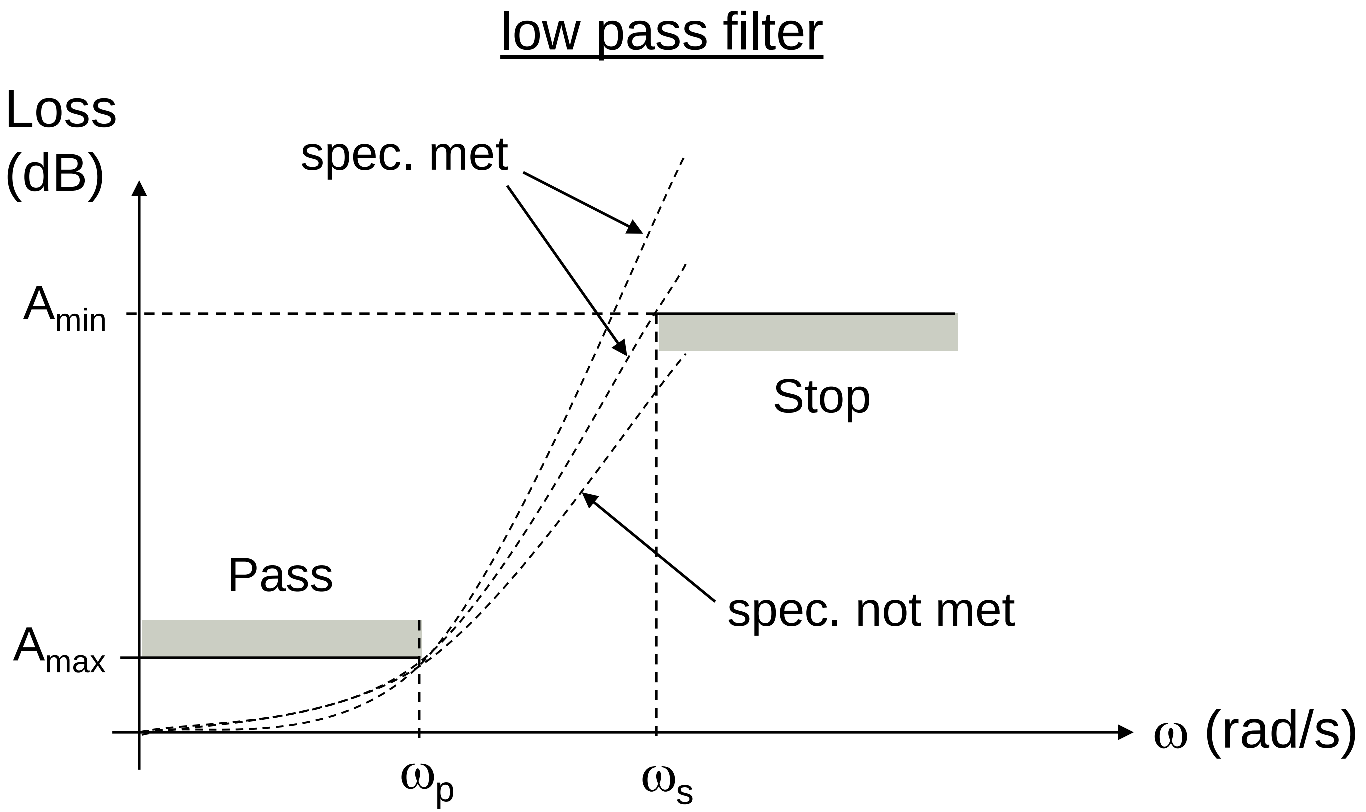
Filter Attenuation Characteristics



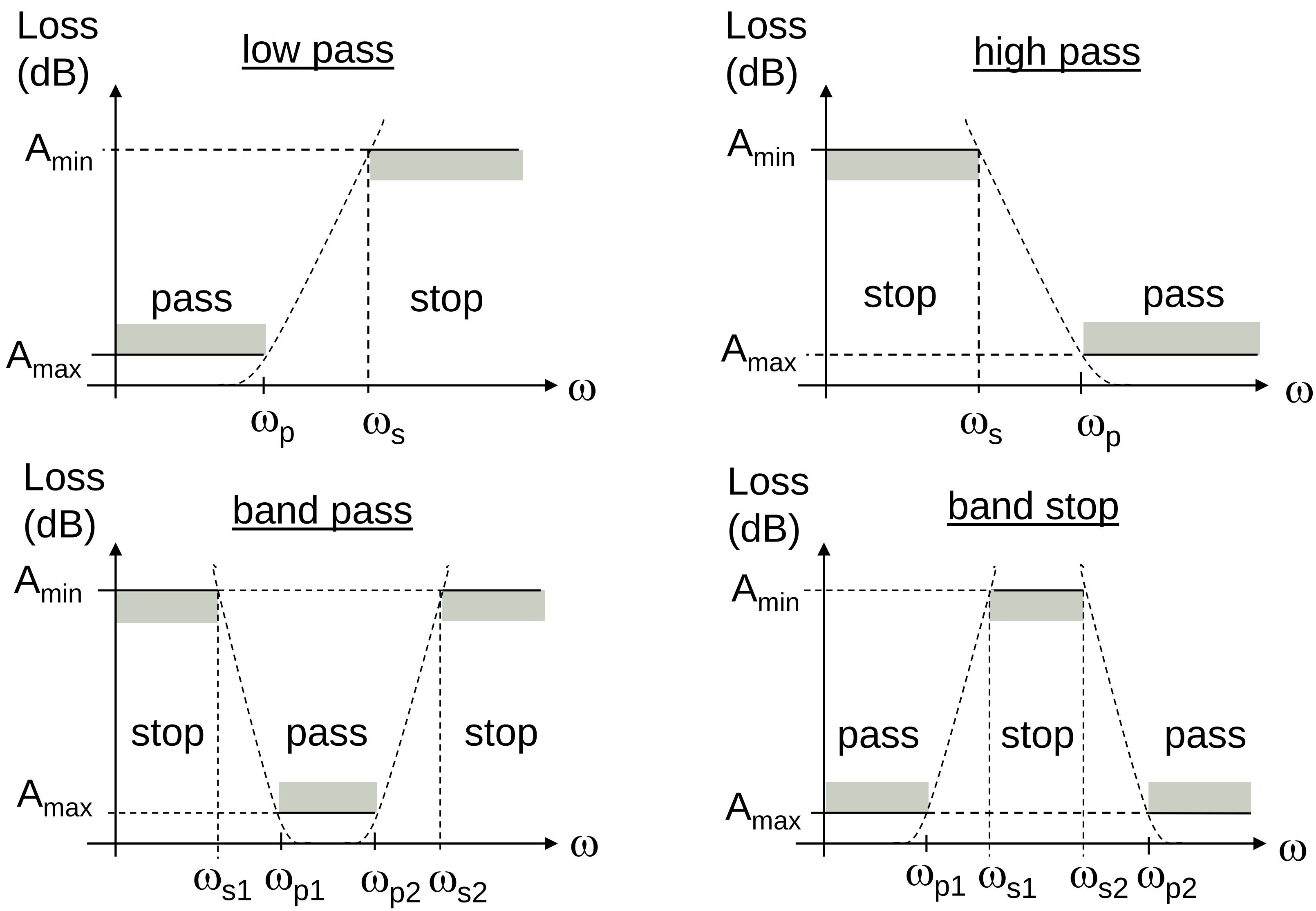
Filter Attenuation Characteristics

- Filter attenuation characteristic must stay outside the shaded region.
- A_{max} maximum attenuation that is allowed in the passband.
- A_{min} minimum attenuation that is required in the stopband.

Filter Attenuation Characteristics



Filter Attenuation Characteristics



Butterworth Filters

- Standard Butterworth function:

$$\left|T(j\Omega)\right|^2 = \frac{1}{1+\Omega^{2n}} \qquad \ldots(1)$$

$$\left|T(j\Omega)\right| = \frac{1}{\sqrt{1+\Omega^{2n}}}$$

- The Butterworth function is called maximally flat, because the first $2n-1$ derivatives of the denominator are zero at $\Omega = 0$.

Butterworth Filters

- "Adjustable" Butterworth function:

$$\Omega = \varepsilon^{\frac{1}{n}} \frac{\omega}{\omega_p}$$

$$\left|T(j\omega)\right|^2 = \frac{1}{1+\varepsilon^2\left(\omega/\omega_p\right)^{2n}}$$

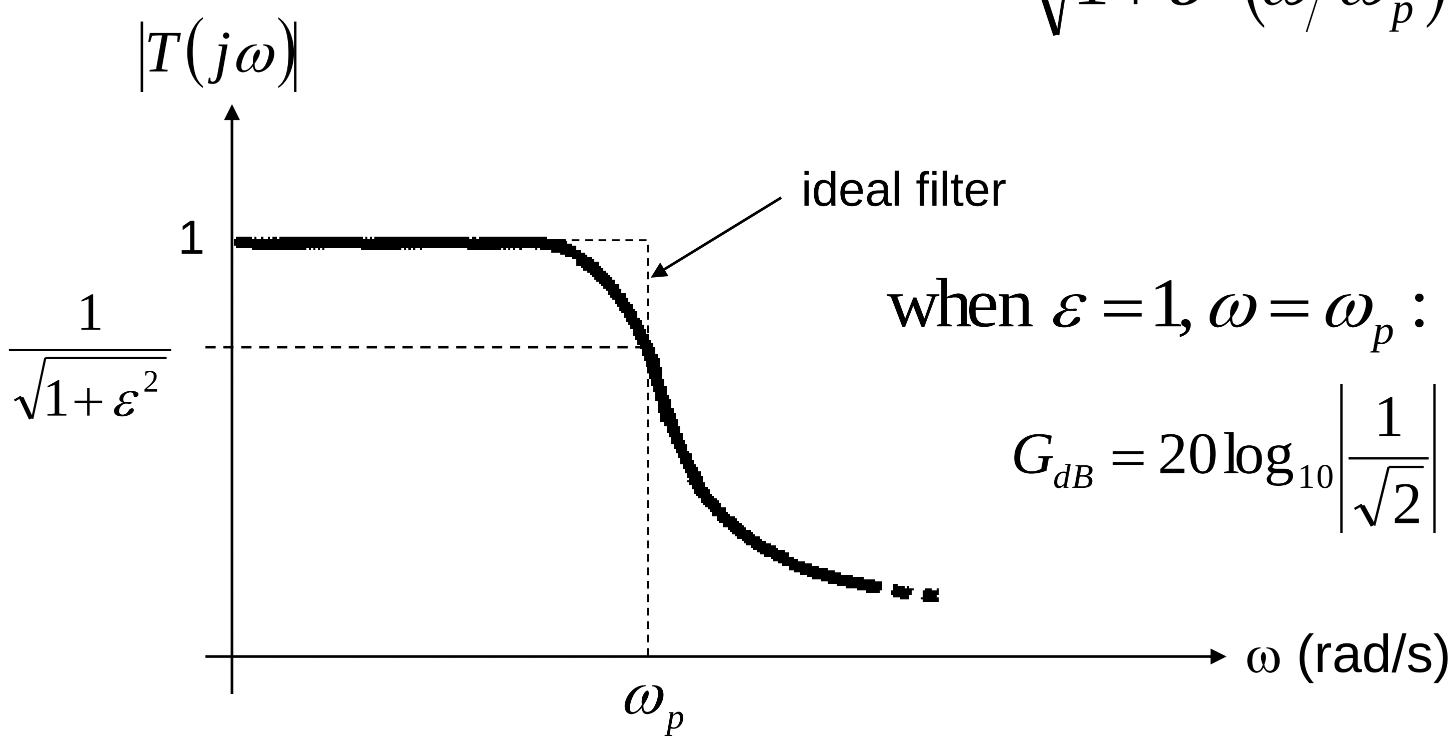
$$\left|T(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2\left(\omega/\omega_p\right)^{2n}}}$$

$\varepsilon =$ adjustment factor for max. passband attenuation

$\omega_p =$ cutoff frequency at edge of passband

Butterworth Filters

- Gain: $$|T(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2(\omega/\omega_p)^{2n}}}$$



when $\varepsilon = 1, \omega = \omega_p$:

$$G_{dB} = 20\log_{10}\left|\frac{1}{\sqrt{2}}\right| \approx -3\text{dB}$$

low pass filter

Butterworth Filters

- Consider the relationship:

$$\left|T(j\omega)\right|^2 = T(j\omega)T(j\omega)^* \\ = T(j\omega)T(-j\omega)$$

- Since $s = j\omega$:

$$\left|T(j\omega)\right|^2 = T(s)T(-s) \qquad \ldots(1)$$

Butterworth Pole Locations (n=1)

- Equation (1):

$$\left|\mathrm{T}(\mathrm{j}\Omega)\right|^2 = T(S)T(-S) \qquad \ldots(1)$$

- Standard Butterworth equation (2):

$$\left|\mathrm{T}(\mathrm{j}\Omega)\right|^2 = \frac{1}{1+\Omega^{2n}} \qquad \ldots(2)$$

- Equate (1) and (2), and substitute in $\Omega = S/j$:

$$T(S)T(-S) = \frac{1}{1+(-1)^n S^{2n}}$$

Butterworth Pole Locations (n=1)

- We have:

$$T(S)T(-S) = \frac{1}{1+(-1)^n S^{2n}}$$

- Let n=1. The above becomes:

$$T(S)T(-S) = \frac{1}{1-S^2}$$

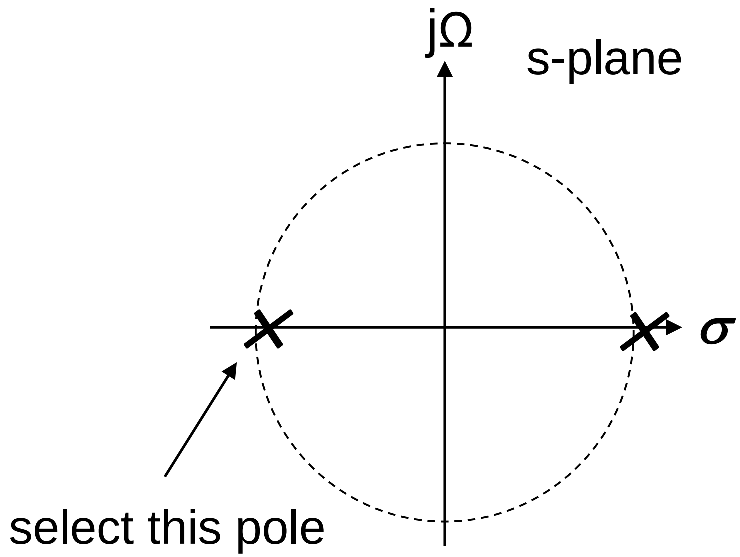
$$T(S)T(-S) = \underbrace{\frac{1}{1+S}}_{T(S)} \times \underbrace{\frac{1}{1-S}}_{T(-S)}$$

Butterworth Pole Locations (n=1)

- To find the poles, set the denominator = 0 and solve for S:

$$1 - S^2 = (1+S)(1-S) = 0$$

- Therefore the poles are located at S=±1.
- Plotting the poles in the s-plane:



- The pole in the right half s-plane (S=1) corresponds to an unstable system.
- Therefore we select T(S) with the pole in the left half plane

$$T(S) = \frac{1}{S+1}$$

Butterworth Pole Locations (n=2)

$$T(S)T(-S) = \frac{1}{1+(-1)^n S^{2n}}$$

- Let n=2. The above becomes:

$$|\mathrm{T}(\mathrm{j}\Omega)|^2 = T(S)T(-S) = \frac{1}{1+S^4}$$

$$T(S)T(-S) = \underbrace{\frac{1}{\left(S^2+\sqrt{2}S+1\right)}}_{T(S)} \times \underbrace{\frac{1}{\left(S^2-\sqrt{2}S+1\right)}}_{T(-S)}$$

Butterworth Pole Locations (n=2)

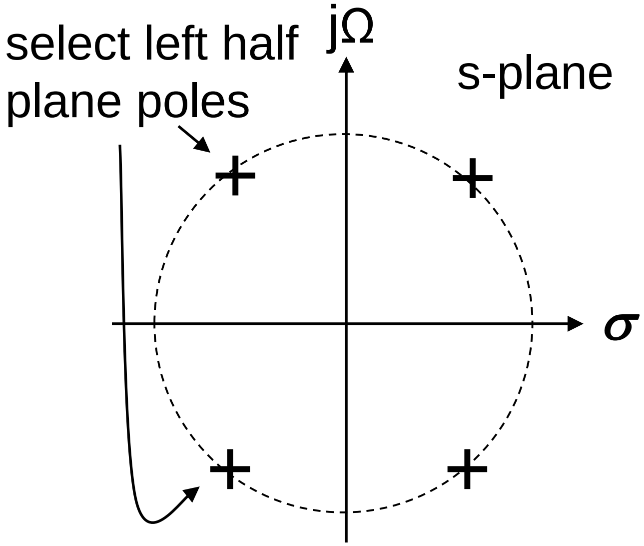
- To find the poles, set the denominator = 0 and solve for S:

$$S^4 + 1 = 0$$

- There are 4 solutions:

$$S = 0.707 + j0.707, \quad S = 0.707 - j0.707$$

$$S = -0.707 + j0.707, \quad S = -0.707 - j0.707$$



- We again select T(S) with the poles in the left half plane:

$$T(S) = \frac{1}{(S + 0.707 + j0.707)(S + 0.707 - j0.707)}$$

$$T(S) = \frac{1}{S^2 + \sqrt{2}S + 1}$$

Butterworth Pole Locations (n=3)

$$\left|\mathrm{T}(\mathrm{j}\Omega)\right|^2 = T(S)T(-S) = \frac{1}{1+(-1)^n S^{2n}}$$

- Let n=3. The above becomes:

$$\left|\mathrm{T}(\mathrm{j}\Omega)\right|^2 = T(S)T(-S) = \frac{1}{1-S^6}$$

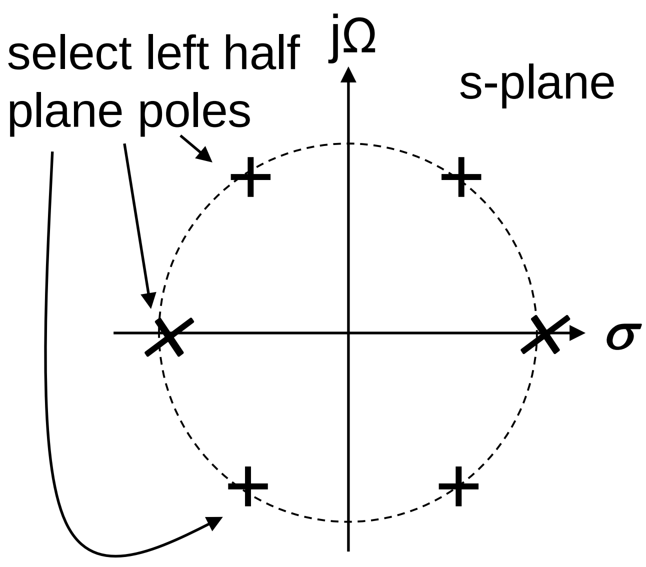
$$T(S)T(-S) = \underbrace{\frac{1}{(1+S)(S^2+S+1)}}_{T(S)} \times \underbrace{\frac{1}{(1-S)(S^2-S+1)}}_{T(-S)}$$

Butterworth Pole Locations (n=3)

- To find the poles, set the denominator = 0 and solve for S:

$$1 - S^6 = 0$$

- There are 6 solutions: $1\angle 0, 1\angle 60, 1\angle 120, 1\angle 180, 1\angle 240, 1\angle 300$



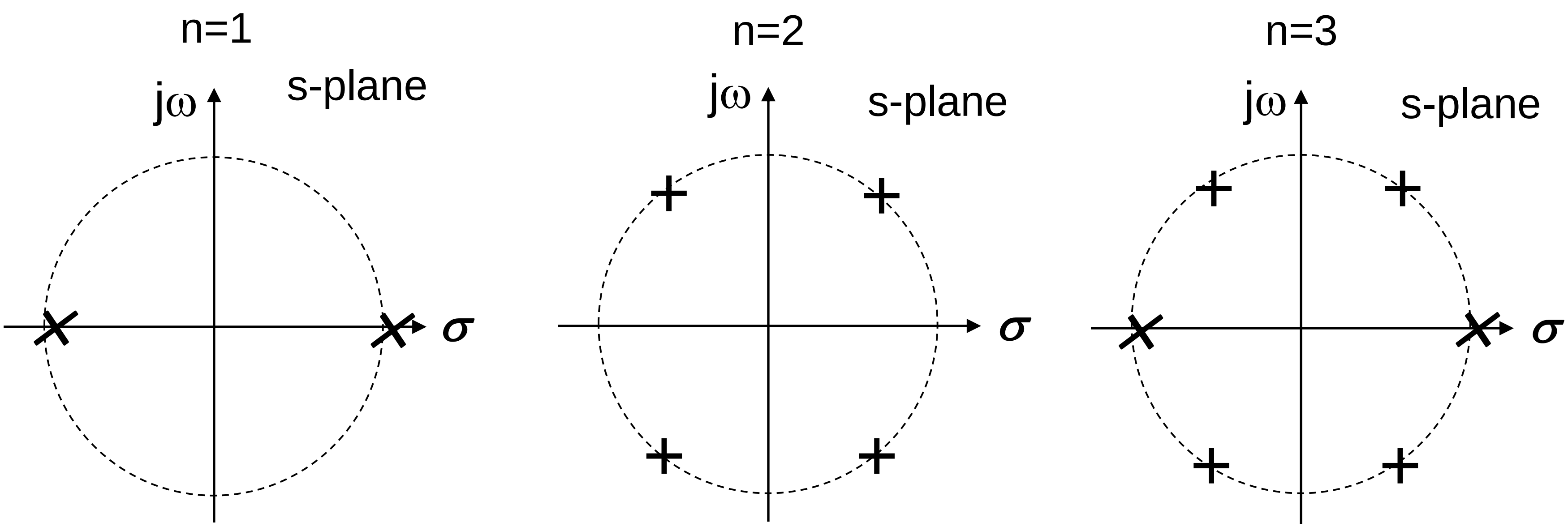
- We again select T(S) with the poles in the left half plane:

$$T(S) = \frac{1}{(S+1)\left(S + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(S + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)}$$

$$T(S) = \frac{1}{(S+1)(S^2 + S + 1)}$$

Butterworth Pole Locations

- For the standard Butterworth, the poles are always on the unit circle.
- there are never any poles on the jω axis.
- If n is odd there is always a pole at S=-1.
- Poles are separated by 180°/n.

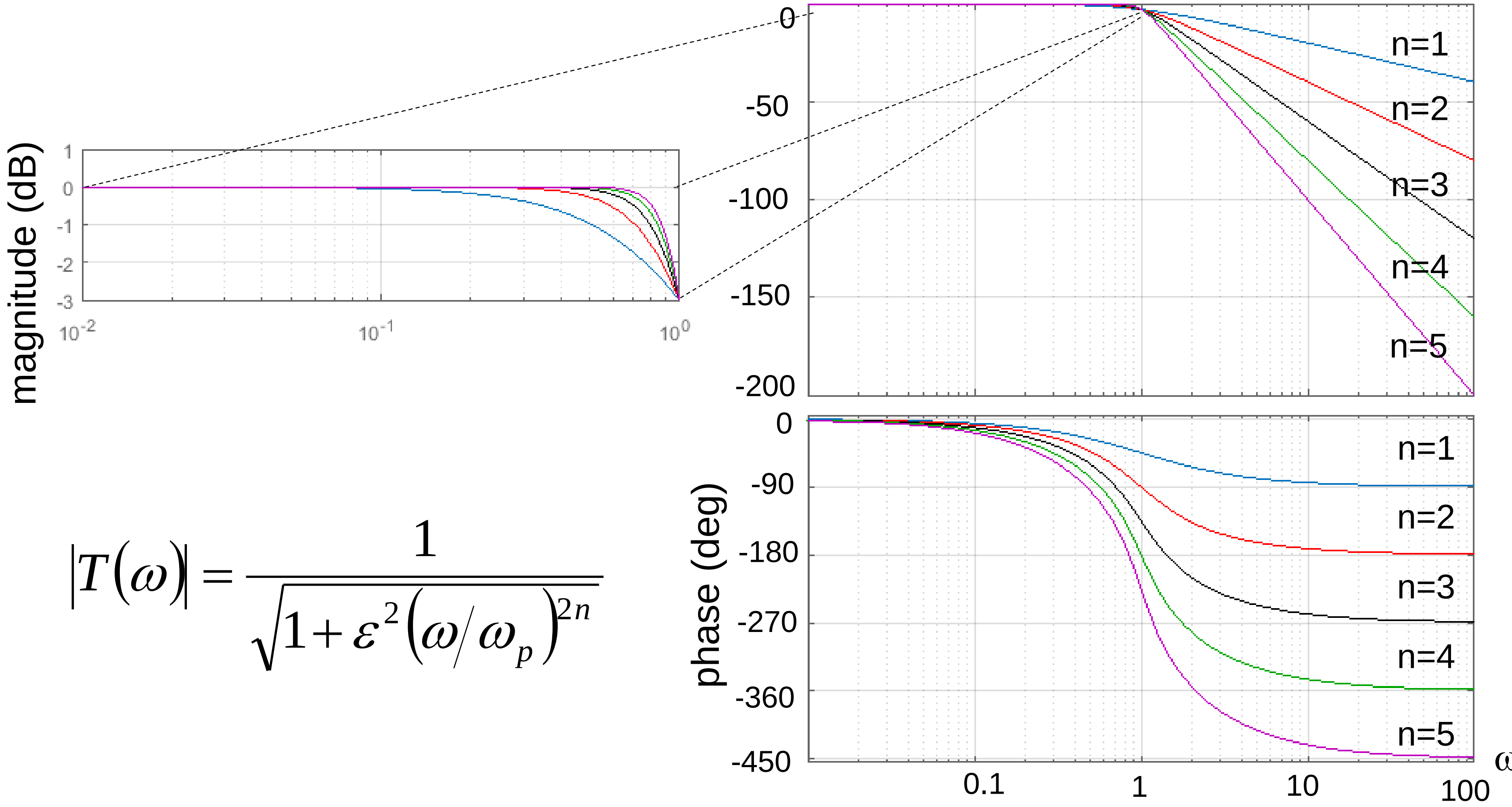


Butterworth Filters

- Butterworth functions:

n	Butterworth function
1	$(s+1)$
2	$(s^2+1.414s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$
5	$(s+1)(s^2+0.61803s+1)(s^2+1.61803s+1)$

Butterworth Filters

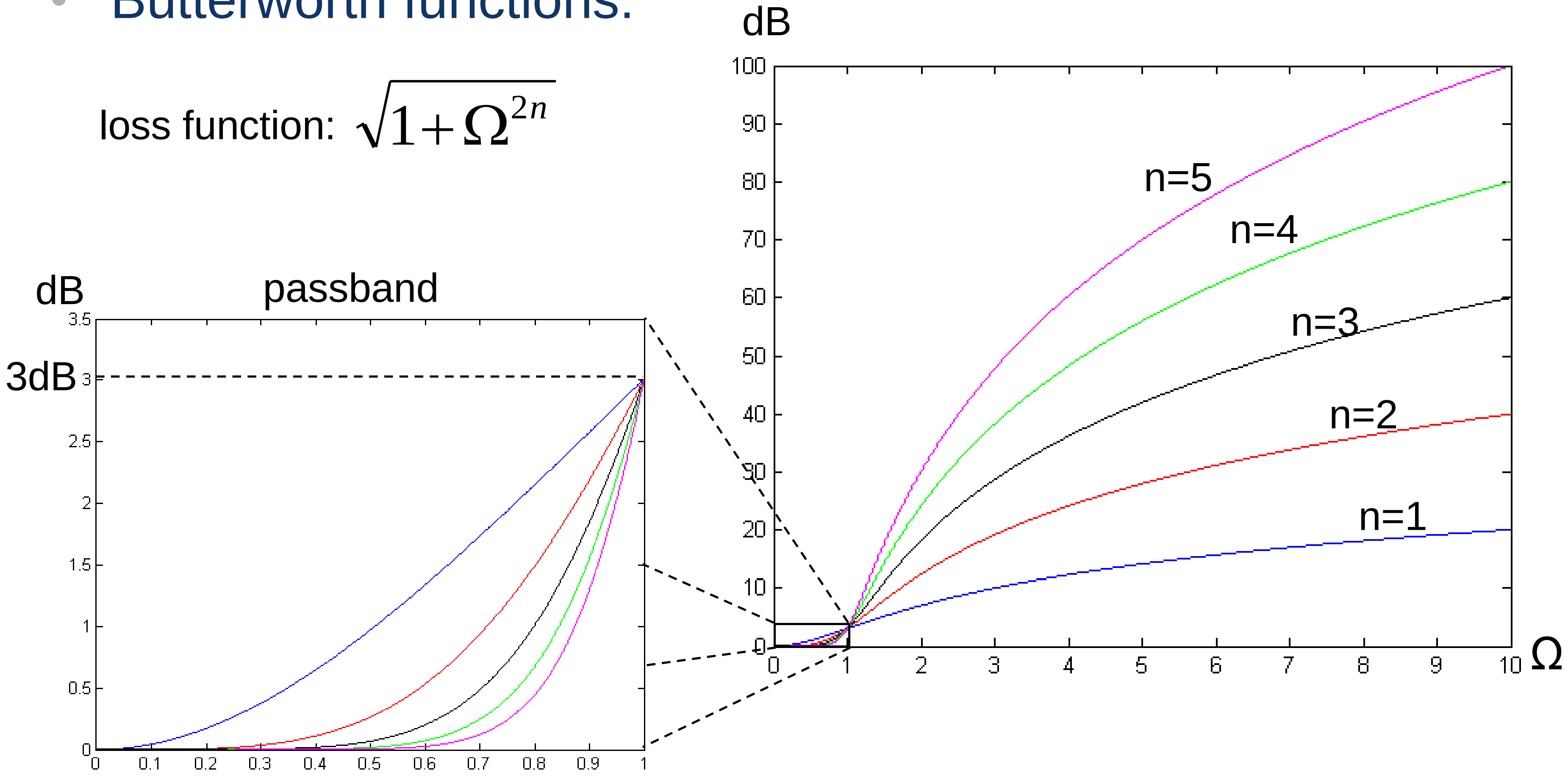


$$|T(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2\left(\omega/\omega_p\right)^{2n}}}$$

Butterworth Filters

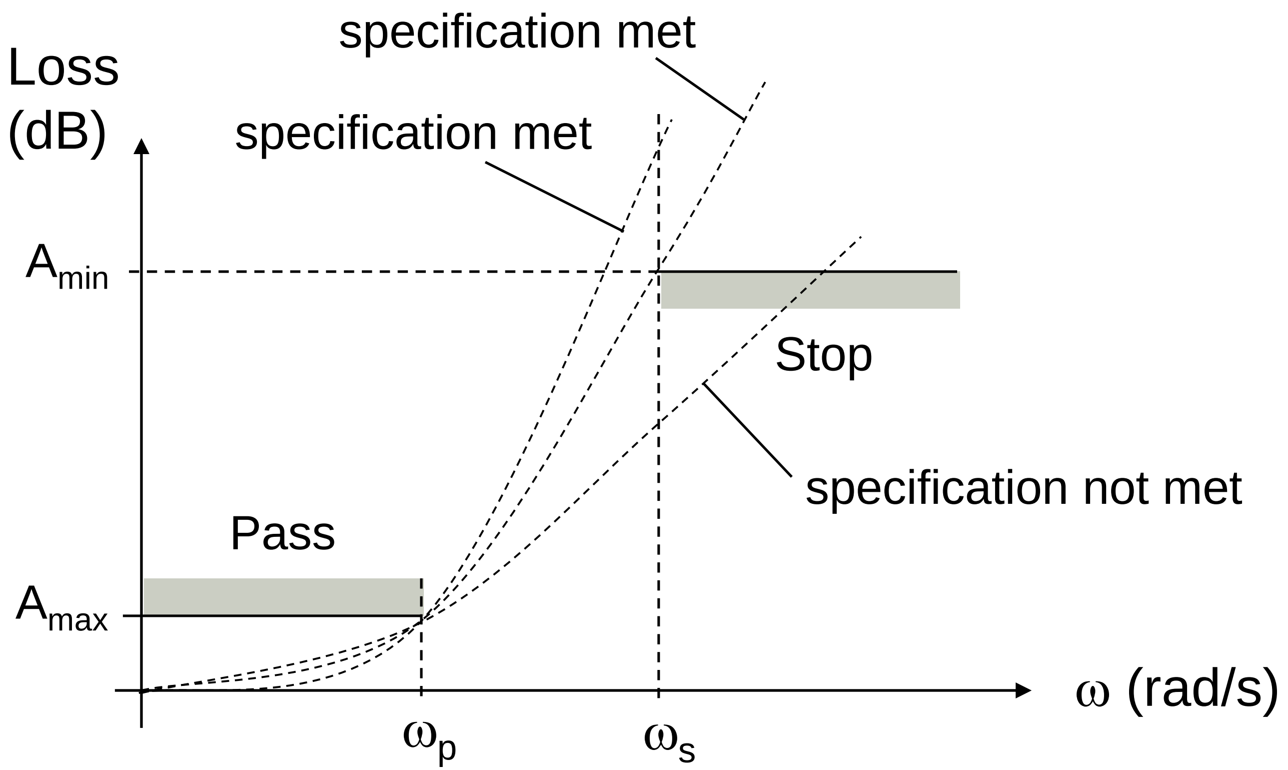
- Butterworth functions:

loss function: $\sqrt{1+\Omega^{2n}}$



Butterworth Filters

low pass filter



Butterworth Filters

- Attenuation or Loss in dB using the standard Butterworth function:

$$20\log_{10}\sqrt{1+\Omega^{2n}}$$

- We can adjust this to our specifications for loss at the edge of the passband and cutoff frequency by substituting:

$$\Omega = \varepsilon^{\frac{1}{n}} \frac{\omega}{\omega_p}$$

- Therefore, attenuation in dB becomes:

$$20\log_{10}\sqrt{1+\varepsilon^2\left(\omega/\omega_p\right)^{2n}}$$

Butterworth Filters

- Attenuation in dB: $20\log_{10}\sqrt{1+\varepsilon^2(\omega/\omega_p)^{2n}}$

- At the edge of the passband, $\omega=\omega_p$, and the loss = A_{max}:

$$A_{max} = 20\log_{10}\sqrt{1+\varepsilon^2(\omega_p/\omega_p)^{2n}}$$

$$A_{max} = 10\log_{10}\left(1+\varepsilon^2\right)$$

$$10^{0.1A_{max}} = 1+\varepsilon^2$$

$$\boxed{\varepsilon = \sqrt{10^{0.1A_{max}}-1}}$$

Butterworth Filters

- At the edge of the stopband, $\omega=\omega_s$, and the minimum loss is A_{min}:

$$A_{min} = 20\log_{10}\sqrt{1+\varepsilon^2\left(\omega_s/\omega_p\right)^{2n}}$$

$$A_{min} = 10\log_{10}\left(1+\varepsilon^2\left(\omega_s/\omega_p\right)^{2n}\right)$$

$$10^{0.1A_{min}} = 1+\varepsilon^2\left(\omega_s/\omega_p\right)^{2n}$$

$$\frac{10^{0.1A_{min}}-1}{\varepsilon^2} = \left(\omega_s/\omega_p\right)^{2n}$$

$$\log_{\omega_s/\omega_p}\left(\frac{10^{0.1A_{min}}-1}{\varepsilon^2}\right) = 2n$$

Butterworth Filters

Use the rule: $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$

$$\boxed{n = \frac{\log_{10}\left(\dfrac{10^{0.1A_{\min}} - 1}{\varepsilon^2}\right)}{2\log_{10}\left(\dfrac{\omega_s}{\omega_p}\right)}}$$

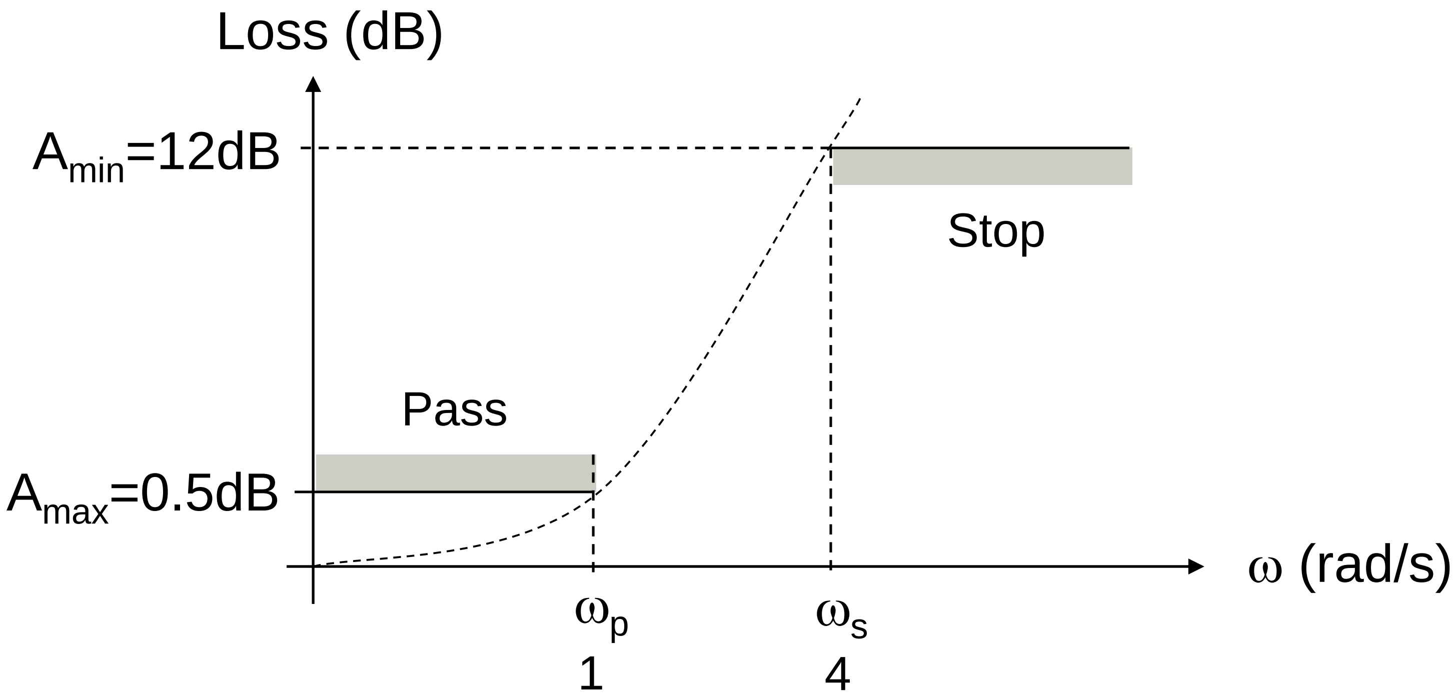
- Given A_{max}, A_{min}, ω_p and ω_s, the order of Butterworth filter required can be calculated.

Butterworth LP Filter

Example 3(a)

- Find the Butterworth approximation for a low pass filter whose requirements are characterised by:

A_{max} = 0.5dB, A_{min} = 12dB, ω_p = 1, ω_s = 4rad/s



Butterworth LP Filter

Example 3(a)

- First find ε:

$$\varepsilon = \sqrt{10^{0.1A_{\max}} - 1} = \sqrt{10^{0.05} - 1} = 0.35$$

- Now find order of filter required:

$$n = \frac{\log\left(\dfrac{10^{0.1A_{\min}} - 1}{\varepsilon^2}\right)}{2\log\left(\dfrac{\omega_s}{\omega_p}\right)} = \frac{\log\left(\dfrac{10^{1.2} - 1}{\varepsilon^2}\right)}{2\log\left(\dfrac{400}{100}\right)} = 1.73$$

- Therefore we will choose n = 2.

Butterworth LP Filter

Example 3(a)

- Second order Butterworth function:

$$T(S)=\frac{1}{S^2+\sqrt{2}S+1}$$

- Substitute $S=\left(\frac{\varepsilon^{1/n}}{\omega_p}\right)s=\left(\frac{0.35^{1/2}}{1}\right)s=Bs$:

$$T(s)=\frac{1}{B^2s^2+\sqrt{2}Bs+1}=\frac{1/B^2}{s^2+\left(\sqrt{2}/B\right)s+1/B^2}$$

$$T(s)=\frac{2.863}{s^2+2.393s+2.863}$$

Butterworth LP Filter

Example 3(a)

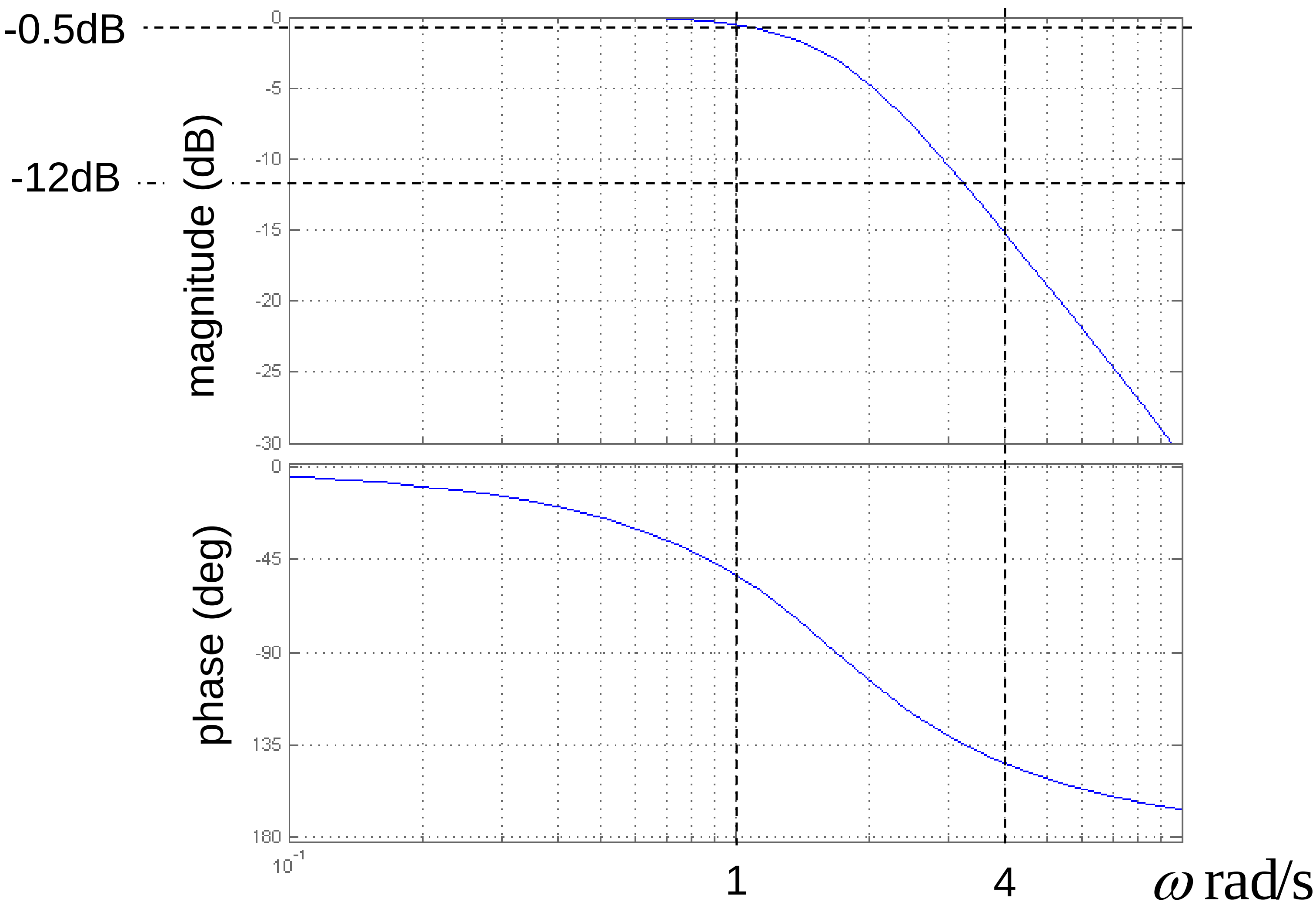
In Matlab:

```
Amax = 0.5;
Amin = 12;
wp = 1;
ws = 4;

epsilon = sqrt(10^(0.1*Amax) - 1);
n = log10((10^(Amin*0.1) - 1) / (10^(0.1*Amax) - 1)) /
(2*log10(ws/wp));
n = ceil(n);
B = epsilon^(1/n);

t1 = tf([0 0 1/(B*B)],[1 sqrt(2)/B 1/(B*B)]);
figure(1);
bodeplot(t1);
grid on;
```

Butterworth LP Filter



Butterworth LP Filter

Example 3(b)

- Design a prototype Sallen-Key circuit for this filter, with a gain of 10 in the passband, $\omega_p = 1\,\text{rad/s}$, and $C_1 = C_2 = 1\,\text{F}$.

$$T(s) = \frac{10 \times 2.863}{s^2 + 2.393\,s + 2.863}$$

- Circuit transfer function:

$$\frac{v_o(s)}{v_{in}(s)} = \frac{\dfrac{K}{R_1R_2C_1C_2}}{s^2 + s\left(\dfrac{1}{R_1C_2} + \dfrac{1}{R_2C_2} + \dfrac{1}{R_2C_1} - \dfrac{K}{R_2C_1}\right) + \dfrac{1}{R_1R_2C_1C_2}} \qquad K = 1 + \frac{R_4}{R_3}$$

Butterworth LP Filter

Example 3(b)

- Let $C_1 = C_2 = 1\,\mathrm{F}$

- Equating denominator coefficients we have:

$$K = 1 + \frac{R_4}{R_3} = 10 \qquad \ldots(1)$$

$$\frac{1}{R_1} + \frac{(2-K)}{R_2} = \sqrt{2} \qquad \ldots(2)$$

$$\frac{1}{R_1 R_2} = 1 \qquad \ldots(3)$$

Butterworth LP Filter

Example 3(b)

- Substituting (3) in (2):

$$R_2 + \frac{(2-K)}{R_2} = \sqrt{2}$$

$$R_2^2 - \sqrt{2}R_2 + (2-K) = 0$$

- Solution of a quadratic:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Butterworth LP Filter

Example 3(b)

- Therefore:

$$R_2 = \frac{\sqrt{2} \pm \sqrt{2 - 4(2 - K)}}{2}$$

- Since K=10:

$$R_2 = \frac{\sqrt{2} \pm \sqrt{2 - 4(2 - 10)}}{2}$$

$$R_2 = 3.6\Omega \quad \text{or} \quad -2.2\Omega$$ (use positive answer)

Butterworth LP Filter

Example 3(b)

- Use eqn(3) to calculate R_1:

$$R_1 = \frac{1}{R_2} = 0.276\Omega$$

- Now calculate R_3 and R_4.
- To reduce offset current effect, resistance seen by each input should be equal at DC:

$$R_3 \,//\, R_4 = R_1 + R_2$$

$$\frac{R_3 R_4}{R_3 + R_4} = R_1 + R_2$$

Butterworth LP Filter

Example 3(b)

- However, we know that: $$K = 1 + \frac{R_4}{R_3} = \frac{R_3 + R_4}{R_3}$$

- Therefore:

$$\frac{R_4}{K} = R_1 + R_2$$

$$R_4 = K\left(R_1 + R_2\right) = 10\left(3.62 + 0.276\right) = 38.98\Omega$$

and from re-arranging $K = 1 + \frac{R_4}{R_3}$:

$$R_3 = \frac{R_4}{K - 1} = \frac{38.98}{9} = 4.33\Omega$$

Butterworth LP Filter

Example 3(b)

- The component values now are:

$$C_1 = 1\text{F}$$
$$C_2 = 1\text{F}$$
$$R_1 = 0.276\Omega$$
$$R_2 = 3.6\Omega$$
$$R_3 = 4.33\Omega$$
$$R_4 = 38.98\Omega$$

- These values will now be scaled to new values:

$$C_1', C_2', R_1', R_2', R_3' \text{ and } R_4'$$

Butterworth LP Filter

Example 3(c)

- Choose a value for the capacitors, and scale the circuit so that the edge of the passband (f_p) is at 2kHz.

- Frequency scale factor:

$$K_f = \frac{\text{new cutoff freq.}}{\text{old cutoff freq.}} = \frac{2\pi 2\text{k}}{1} = 2\pi 2\text{k}$$

Butterworth LP Filter

<u>Example 3(c)</u>

- Choose C_1' and C_2', using rough rule:

$$\text{new value of } C = \frac{10}{f_c}\,\mu\text{F}$$

- Therefore:

$$C_1' = C_2' = \frac{10}{f_c}\,\mu\text{F} = \frac{10}{2\text{k}}\,\mu\text{F} = 0.005\,\mu\text{F} = 5\,\text{nF}$$

Butterworth LP Filter

Example 3(c)

- Work out magnitude scale factor.
- Since:

$$C_1' = \frac{C_1}{K_m K_f}$$

- The magnitude scale factor can be calculated:

$$K_m = \frac{C_1}{K_f C_1'} = \frac{1\text{F}}{(2\pi 2\text{k})5n\text{F}} = 15.9\text{k}$$

Butterworth LP Filter

Example 3(c)

- The new component values are as follows:

$$C_1' = \frac{C_1}{K_m K_f} = 5\text{nF}$$

$$C_2' = \frac{C_2}{K_m K_f} = 5\text{nF}$$

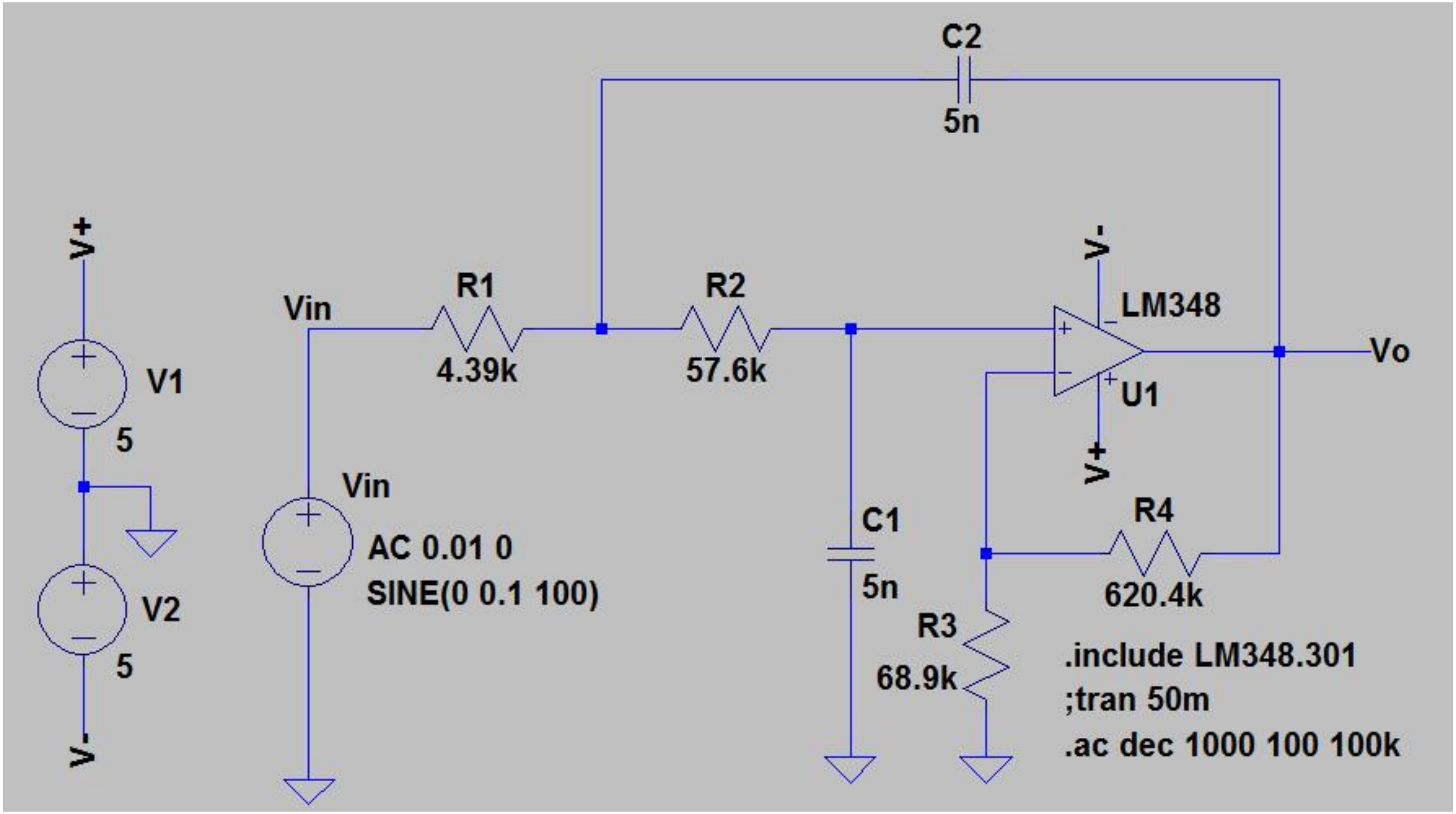
$$R_1' = K_m R_1 = 0.276 \times 15.9\text{k} = 4.39\text{k}\Omega$$

$$R_2' = K_m R_2 = 3.62 \times 15.9\text{k} = 57.6\text{k}\Omega$$

$$R_3' = K_m R_3 = 4.33 \times 15.9\text{k} = 68.9\text{k}\Omega$$

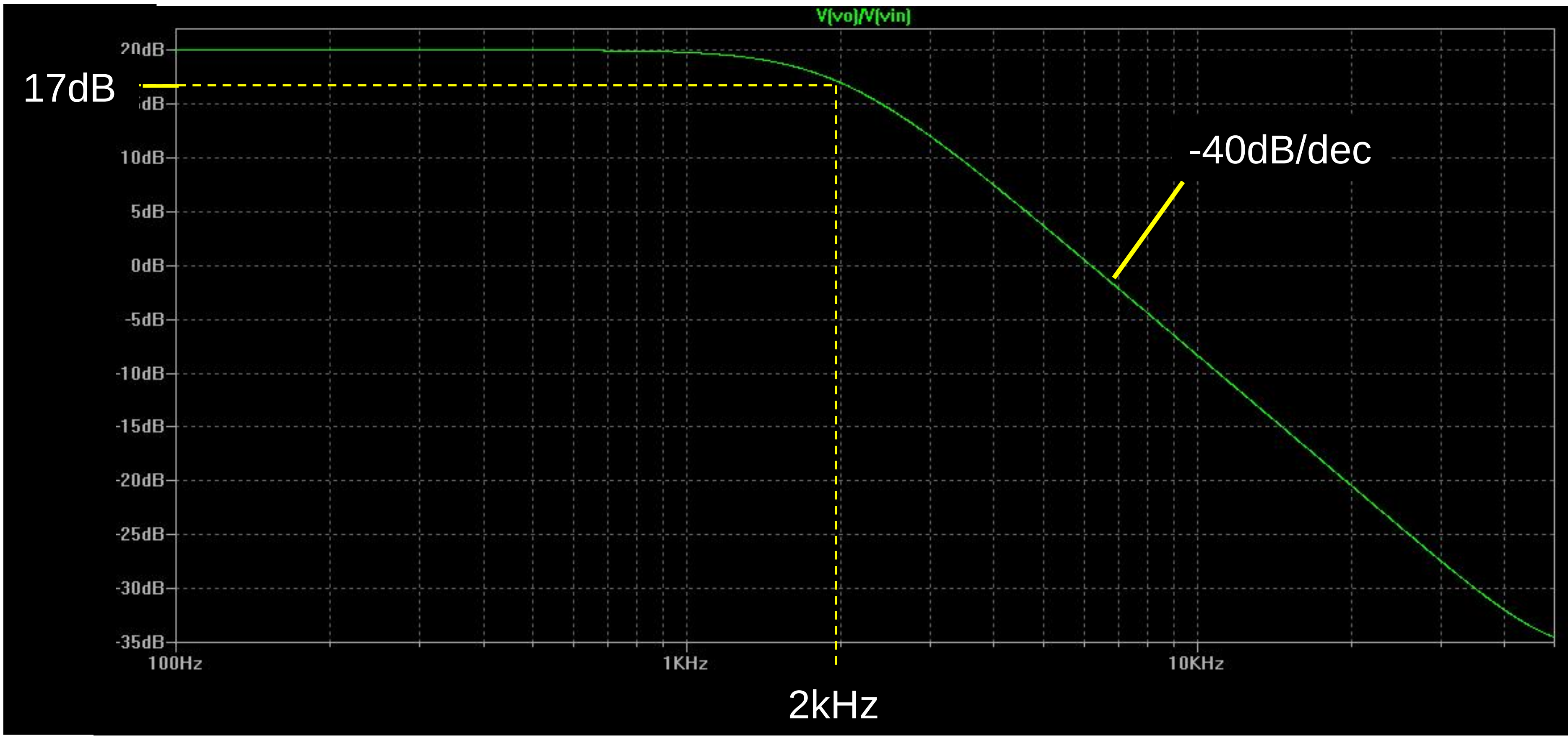
$$R_4' = K_m R_4 = 38.98 \times 15.9\text{k} = 620.4\text{k}\Omega$$

Butterworth LP Filter



Butterworth LP Filter

Frequency response in LTSpice:



Butterworth LP Filter

Example 3(c)

- Transfer function:

$$\frac{V_o}{V_{in}} = \frac{\dfrac{K}{R_1R_2C_1C_2}}{s^2 + s\left(\dfrac{1}{R_1C_2} + \dfrac{1}{R_2C_2} + \dfrac{1}{R_2C_1} - \dfrac{K}{R_2C_1}\right) + \dfrac{1}{R_1R_2C_1C_2}}$$

$$\frac{V_o}{V_{in}} = \frac{1.58\times10^9}{s^2 + 17.8\times10^3 s + 1.58\times10^8}$$

- To display Bode plot in Matlab:

```
t1 = tf([1.58e9],[1 17.8e3 1.58e8]);
bode (t1);
grid on;
```

Butterworth LP Filter

Example 3(c)

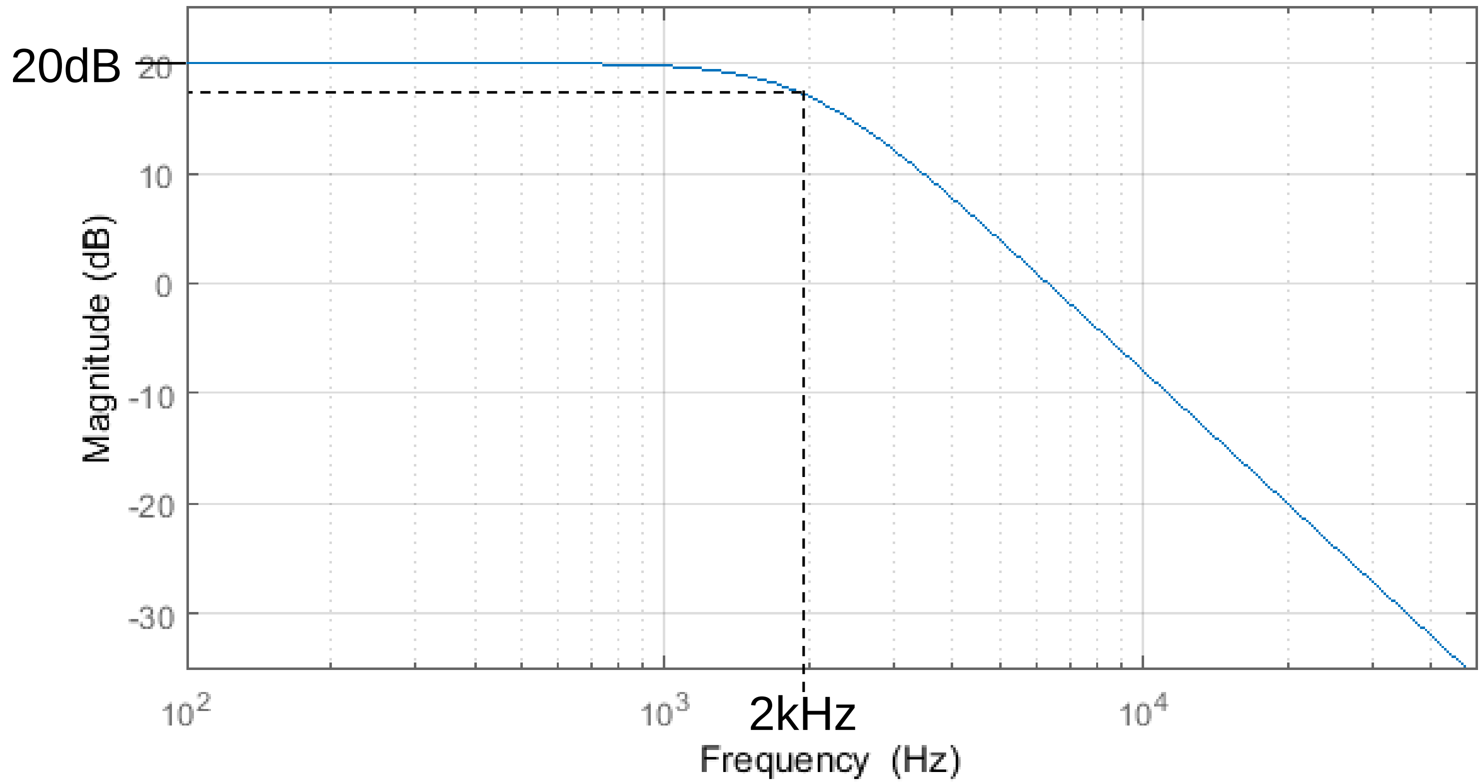
- To display Bode plot in Matlab:

```
K1 = 1 + R4A/R3A;
num = K1/(R1A*R2A*C1A*C2A);
den1 = 1/(R1A*C2A) + 1/(R2A*C2A) + (1-K1)/(R2A*C1A);
den2 = 1/(R1A*R2A*C1A*C2A);

t1 = tf([0 0 num],[1 den1 den2]);
figure(2);
h = bodeplot(t1);
setoptions(h,'FreqUnits','Hz');
grid on;
```

Butterworth LP Filter

Example 3(c)



Chebyshev Filters

- Butterworth approximation is maximally flat at DC.
- The approximation to a flat passband gets progressively poorer as ω approaches ω_p.

- Chebyshev uses an equiripple characteristic in the passband.

- Usually requires a lower order than Butterworth for same stopband attenuation.

Chebyshev Filters

- Chebyshev filter:

$$|\mathrm{T}(\mathrm{j}\Omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(\Omega)}}$$

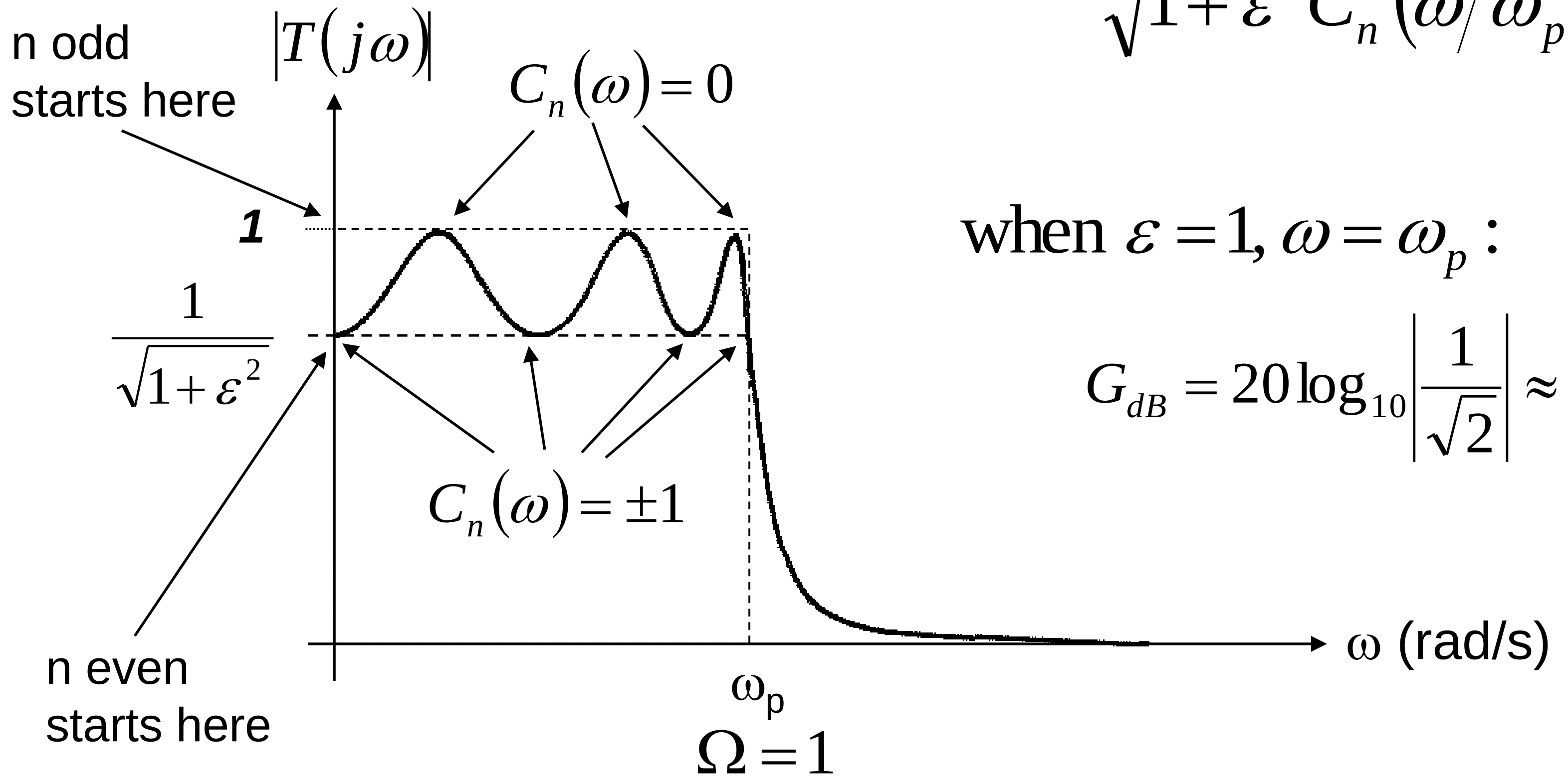
where $C_n(\Omega)$ is Chebyshev polynomial of the first kind of degree n.

- Ω is the standardised frequency:

$$\Omega = \frac{\omega}{\omega_p}$$

Chebyshev Filters

- Gain: $$|T(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(\omega/\omega_p)}}$$



when $\varepsilon = 1, \omega = \omega_p$:

$$G_{dB} = 20\log_{10}\left|\frac{1}{\sqrt{2}}\right| \approx -3\text{dB}$$

<u>low pass filter</u>

Chebyshev Filters

- The function $C_n(\Omega)$ needs to oscillate over ±1.
- One function that will oscillate over ±1 is the sinusoid:

$$C(x) = \cos(nx)$$

where n is an integer.

- To limit to $|x| \le \pm 1$, we introduce:

$$x = \cos^{-1}(\Omega)$$

- Therefore we have:

$$C_n(\Omega) = \cos\left(n\cos^{-1}(\Omega)\right) \qquad \text{for } |\Omega| \le 1$$

Chebyshev Filters

- Chebyshev function:

$$C_n(\Omega) = \cos\left(n\cos^{-1}(\Omega)\right) \qquad \text{for } |\Omega| \le 1$$

- For Ω > 1, assume: $\cos^{-1}\Omega = jz$ ***…(1)***
- Therefore we have: $\Omega = \cos jz = \frac{1}{2}\left(e^{j(jz)} + e^{-j(jz)}\right) = \cosh z$

$$z = \cosh^{-1}\Omega$$

- Substituting in (1): $\cos^{-1}\Omega = j\cosh^{-1}\Omega$
- The Chebyshev function becomes:

$$C_n(\Omega) = \cos\left(nj\cosh^{-1}(\Omega)\right) = \cosh\left(n\cosh^{-1}(\Omega)\right) \quad \text{for } |\Omega| > 1$$

Chebyshev Filters

- Chebyshev functions:

$$C_n(\Omega) = \begin{cases} \cos\left(n\cos^{-1}(\Omega)\right) & \text{for } |\Omega| \le 1 \\ \cosh\left(n\cosh^{-1}(\Omega)\right) & \text{for } |\Omega| > 1 \end{cases}$$

Chebyshev Filters

- Hyperbolic functions: sinh, cosh, tanh

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)\ ; \qquad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)\ ; \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

- Trigonometric functions:

$$\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)\ ; \qquad \cos x = \frac{1}{2j}\left(e^{jx} + e^{-jx}\right)\ ; \qquad \tan x = \frac{\sin x}{\cos x}$$

Chebyshev Filters

- This function can be expressed as a polynomial.
- Consider:

$$C_{n+1}(\Omega)+C_{n-1}(\Omega)=\cos\left[(n+1)\cos^{-1}(\Omega)\right]+\cos\left[(n-1)\cos^{-1}(\Omega)\right]$$

- Using the identity:

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$

- We get:

$$\begin{aligned}C_{n+1}(\Omega)+C_{n-1}(\Omega)&=2\cos\left(n\cos^{-1}(\Omega)\right)\cos\left(\cos^{-1}(\Omega)\right)\\&=2\cos\left(n\cos^{-1}(\Omega)\right)\Omega\\&=2\Omega C_n(\Omega)\end{aligned}$$

Chebyshev Filters

- Therefore we obtain a recursive relationship that allows us to determine each polynomial C_n.

$$C_{n+1}(\Omega) = 2\Omega C_n(\Omega) - C_{n-1}(\Omega)$$

- Since: $C_0(\Omega) = \cos(0) = 1$

$$C_1(\Omega) = \cos\left(\cos^{-1}(\Omega)\right) = \Omega$$

- Polynomials can be calculated for successive n:

$$C_2(\Omega) = 2\Omega(\Omega) - 1 = 2\Omega^2 - 1$$
$$C_3(\Omega) = 2\Omega\left(2\Omega^2 - 1\right) - \Omega = 4\Omega^3 - 3\Omega$$
$$C_4(\Omega) = 2\Omega\left(4\Omega^2 - 3\Omega\right) - \left(2\Omega^2 - 1\right) = 8\Omega^4 - 8\Omega^2 + 1$$
$$C_5(\Omega) = 2\Omega\left(8\Omega^4 - 8\Omega^2 + 1\right) - \left(4\Omega^3 - 3\Omega\right) = 16\Omega^5 - 20\Omega^3 + 5\Omega$$

Chebyshev Filters

- Chebyshev functions:

$$\Omega = \omega / \omega_p$$ standardised frequency

$$C_n(\Omega) = \begin{cases} \cos\left(n \cos^{-1}(\Omega)\right) & \text{for } |\Omega| \le 1 \\ \cosh\left(n \cosh^{-1}(\Omega)\right) & \text{for } |\Omega| > 1 \end{cases}$$

n	Chebychev Polynomial
0	$C_0(\Omega) = 1$
1	$C_1(\Omega) = \Omega$
2	$C_2(\Omega) = 2\Omega^2 - 1$
3	$C_3(\Omega) = 4\Omega^3 - 3\Omega$
4	$C_4(\Omega) = 8\Omega^4 - 8\Omega^2 + 1$
5	$C_5(\Omega) = 16\Omega^5 - 20\Omega^3 + 5\Omega$
n	$C_{n+1}(\Omega) = 2\Omega C_n(\Omega) - C_{n+1}(\Omega)$

Chebyshev Approximation

- The Chebyshev functions:

$$C_n(\Omega) = \begin{cases} \cos\left(n\cos^{-1}(\Omega)\right) & \text{for } |\Omega| \le 1 \\ \cosh\left(n\cosh^{-1}(\Omega)\right) & \text{for } |\Omega| > 1 \end{cases}$$

are equivalent to the Chebyshev polynomial for each n.

Chebyshev Filters

- Hyperbolic functions: sinh, cosh, tanh

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \; ; \qquad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) \; ; \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

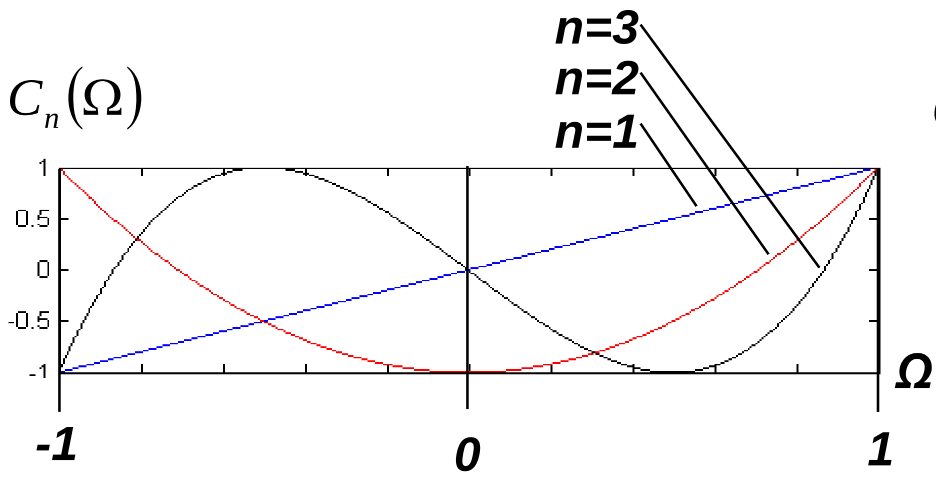
- Trigonometric functions:

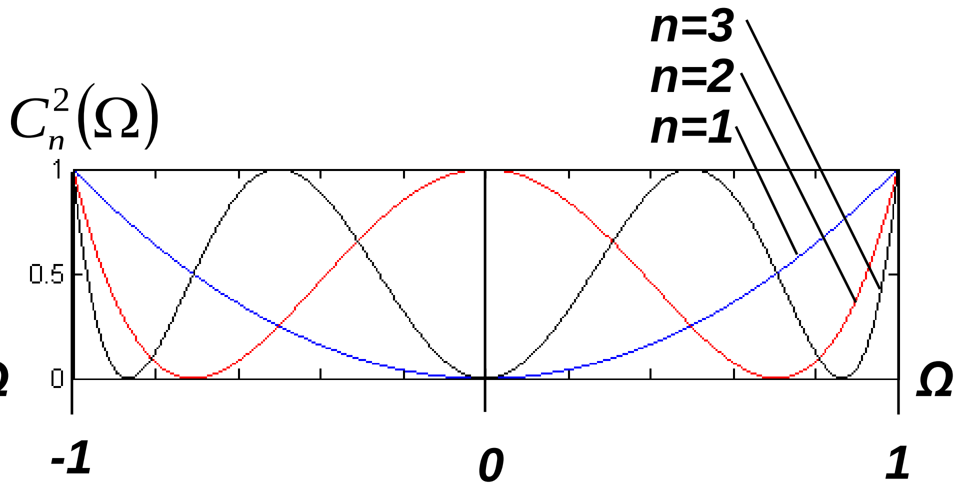
$$\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) \; ; \qquad \cos x = \frac{1}{2j}\left(e^{jx} + e^{-jx}\right) \; ; \qquad \tan x = \frac{\sin x}{\cos x}$$

Chebyshev Approximation

- ***Chebyshev functions:*** for $|\Omega| \le 1$

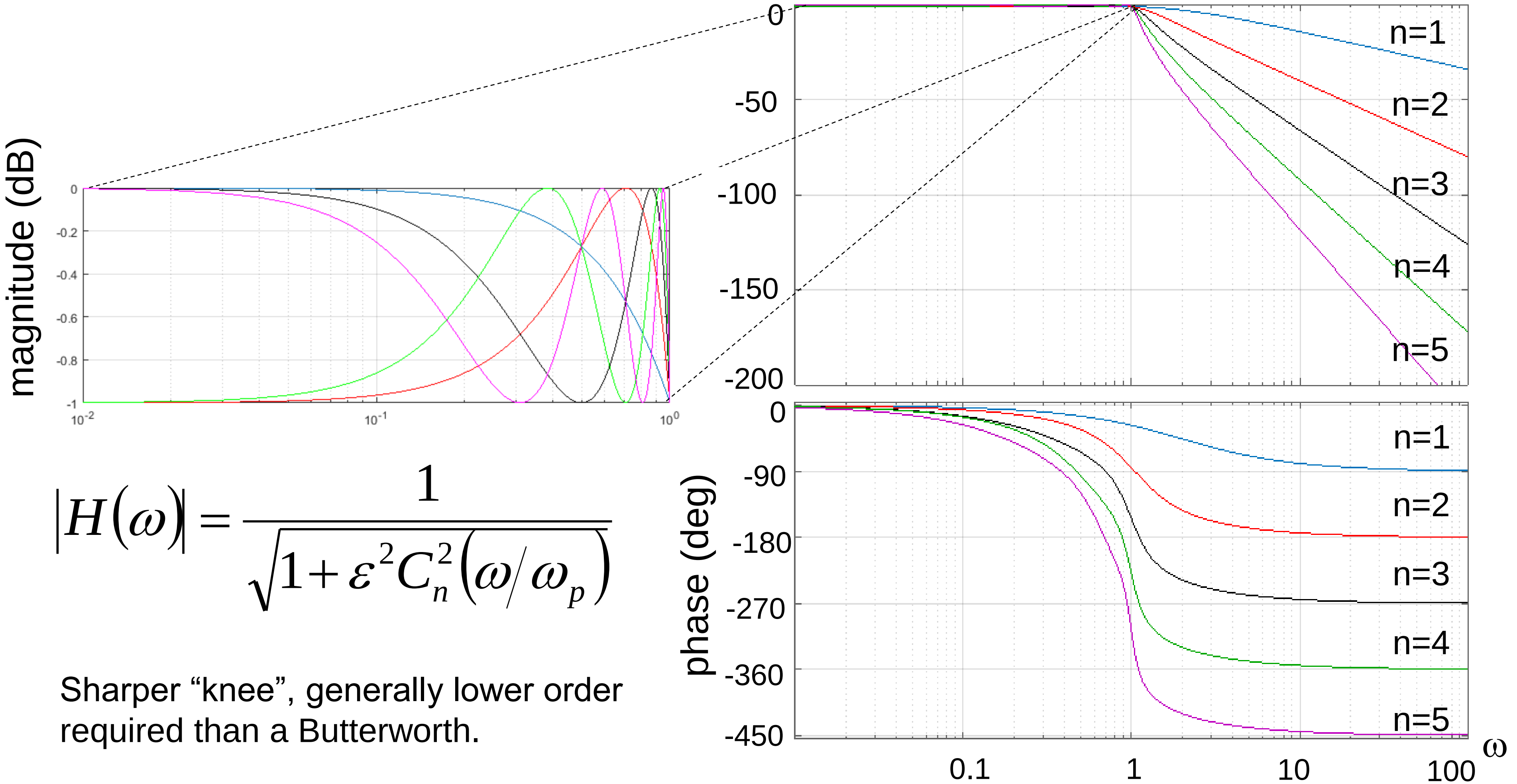
$$C_1(\Omega) = \Omega$$
$$C_2(\Omega) = 2\Omega^2 - 1$$
$$C_3(\Omega) = 4\Omega^3 - 3\Omega$$





$C_n(1) = 1$ for all n

Chebyshev Filters



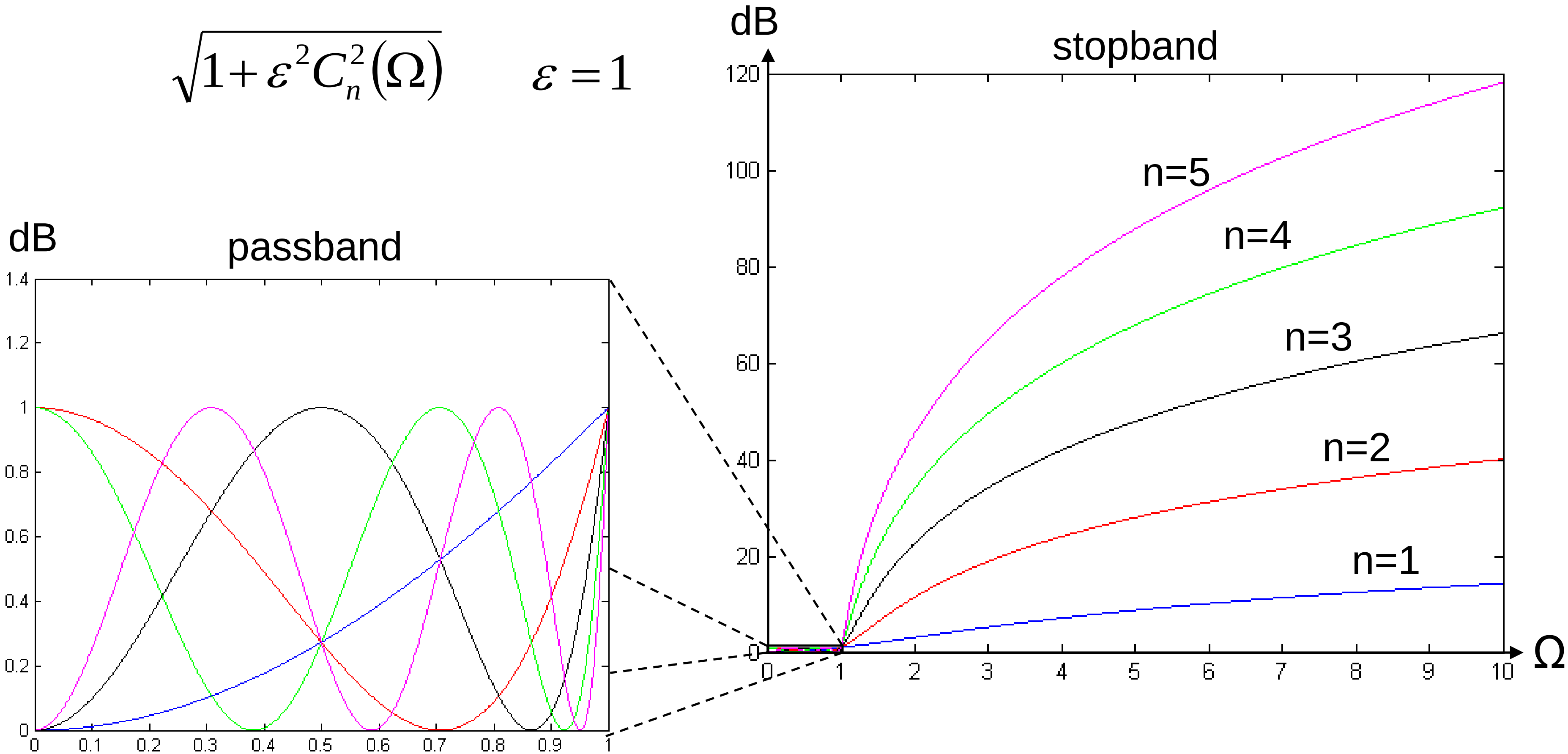
$$\left|H(\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2\left(\omega/\omega_p\right)}}$$

Sharper "knee", generally lower order required than a Butterworth.

Chebyshev Filters

Plot of loss function:

$$\sqrt{1+\varepsilon^2 C_n^2(\Omega)} \qquad \varepsilon = 1$$



Chebyshev Pole Locations

- We need to find the roots of the denominator of T(S).

$$\left|T_n(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\Omega)} = T(S)T(-S)$$

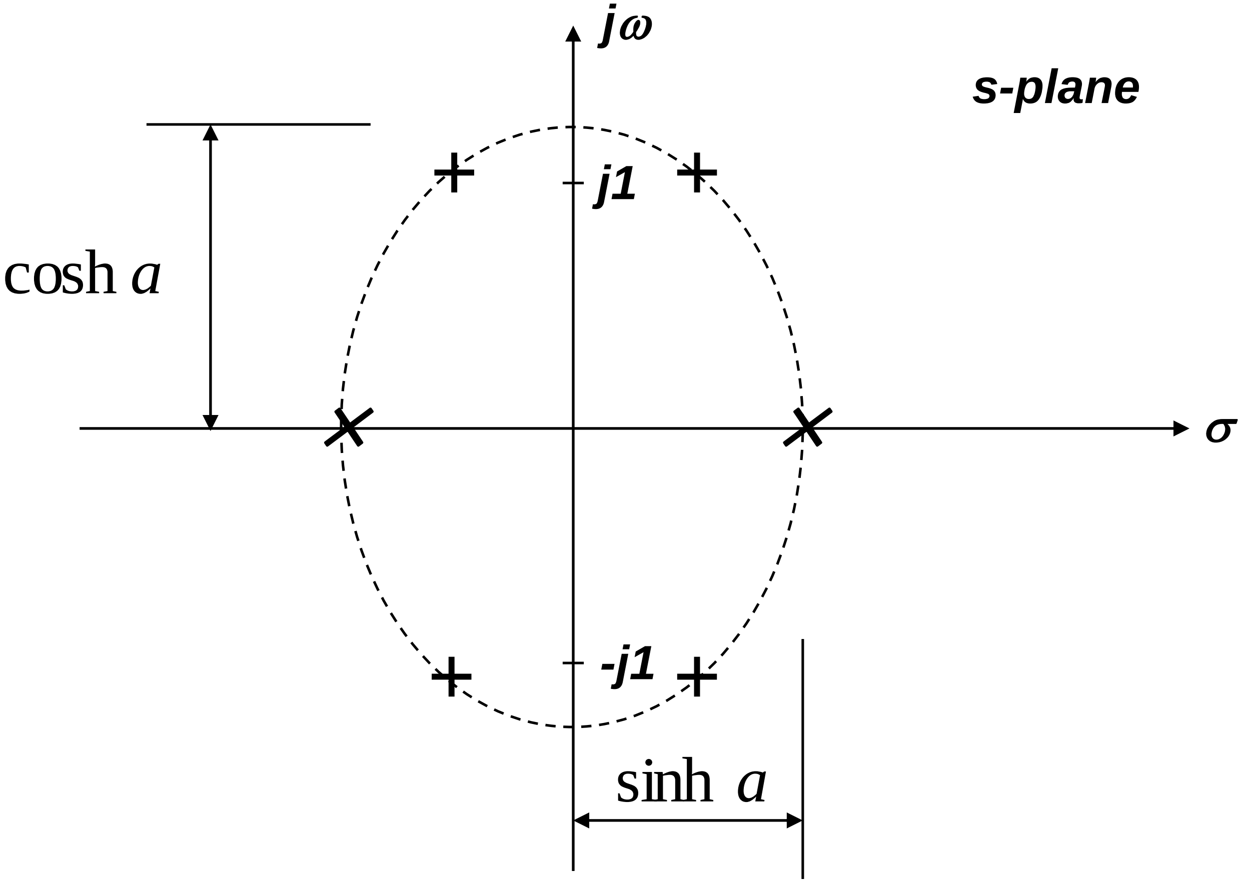
- The roots are: $s_k = \sigma_k \pm j\omega_k \qquad k = 1,2,\ldots,n$

where

$$\sigma_k = \pm \sinh(a)\sin\left(\frac{2k-1}{2n}\pi\right)$$

$$\omega_k = \cosh(a)\cos\left(\frac{2k-1}{2n}\pi\right) \qquad a = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}$$

Chebyshev Pole Locations



Chebyshev Pole Locations

- As with the Butterworth approximation only the left half plane poles are associated with T(S).
- Furthermore:

$$\left(\frac{\sigma_k}{\sinh a}\right)^2 + \left(\frac{\omega_k}{\cosh a}\right)^2 = 1$$

- This means that the roots of the Chebyshev approximation lie on an ellipse in the s-plane.

Chebyshev Filters

- Tables are available giving pole locations for Chebyshev filters for values of A_{max} and n.

$$A_{max} = 0.25\,\mathrm{dB}$$

n	Denominator of T(s)	Numerator, K
1	$(s + 4.10811)$	4.10811
2	$(s^2 + 1.79668\,s + 2.11403)$	2.05405
3	$(s^2 + 0.76722\,s + 1.33863)(s + 0.76722)$	1.02702
4	$(s^2 + 0.42504\,s + 1.16195)(s^2 + 1.02613\,s + 0.45485)$	0.51352
5	$(s^2 + 0.27005\,s + 1.09543)(s^2 + 0.70700\,s + 0.53642)(s + 0.43695)$	0.25676

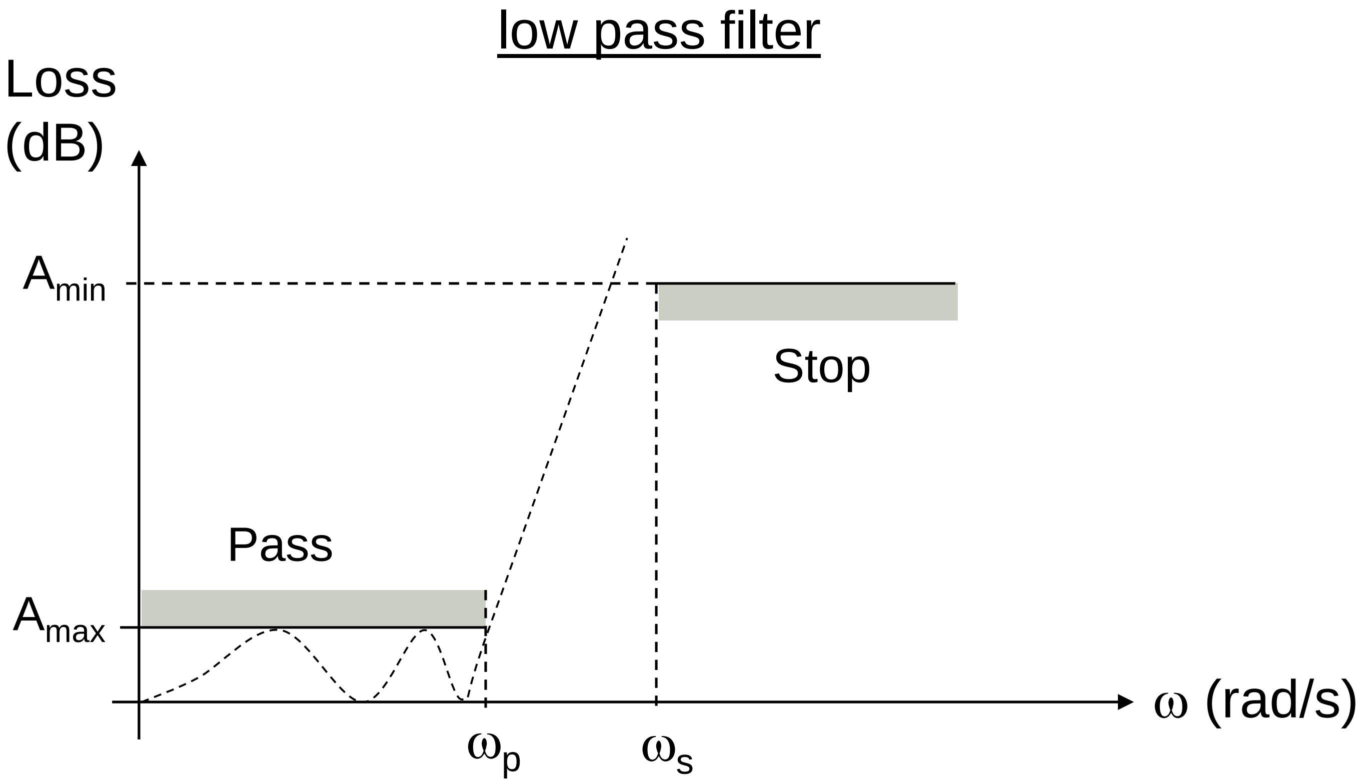
$$A_{\max} = 0.5\text{dB}$$

n	Denominator of T(s)	Numerator, K
1	$(s+2.86278)$	2.86278
2	$(s^2+1.42562\,s+1.51620)$	1.43138
3	$(s^2+0.62646\,s+1.14245)(s+0.62646)$	0.71570
4	$(s^2+0.35071\,s+1.06352)(s^2+0.84668\,s+0.356412)$	0.35785
5	$(s^2+0.22393\,s+1.03578)(s^2+0.58625\,s+0.47677)(s+0.362332)$	0.17892

$$A_{\max} = 1.0\text{dB}$$

n	Denominator of T(s)	Numerator, K
1	$(s+1.96523)$	1.96523
2	$(s^2+1.09773\,s+1.10251)$	0.98261
3	$(s^2+0.49417\,s+0.99420)(s+0.49417)$	0.49130
4	$(s^2+0.27907\,s+0.98650)(s^2+0.67374\,s+0.27940)$	0.24565
5	$(s^2+0.17892\,s+0.98831)(s^2+0.46841\,s+0.42930)(s+0.28949)$	0.12283

Chebyshev Filters



Chebyshev Filters

- Attenuation or Loss in dB for the Chebyshev filter:

$$20\log_{10}\sqrt{1+\varepsilon^2 C_n^2(\Omega)}$$

- We can adjust this to our cutoff frequency by substituting:

$$\Omega = \frac{\omega}{\omega_p}$$

- Therefore, attenuation in dB becomes:

$$20\log_{10}\sqrt{1+\varepsilon^2 C_n^2\left(\omega/\omega_p\right)}$$

Chebyshev Filters

- Attenuation in dB: $20\log_{10}\sqrt{1+\varepsilon^2 C_n^2(\omega/\omega_p)}$

- At the edge of the passband, ω=ω_p, and the loss = A_{max}:

$$A_{max} = 20\log_{10}\sqrt{1+\varepsilon^2 C_n^2(\omega_p/\omega_p)}$$

- Since $C_n(1)=1$ for all n:

$$A_{max} = 10\log_{10}(1+\varepsilon^2)$$

$$\boxed{\varepsilon = \sqrt{10^{0.1A_{max}}-1}}$$

Chebyshev Filters

- At the edge of the stopband, the minimum loss is A_{min}.

$$A_{\min} = 20\log_{10}\sqrt{1+\varepsilon^2 C_n^2\left(\omega_s/\omega_p\right)}$$

$$A_{\min} = 10\log_{10}\left(1+\varepsilon^2 C_n^2\left(\omega_s/\omega_p\right)\right)$$

$$10^{0.1A_{\min}} = 1+\varepsilon^2 C_n^2\left(\omega_s/\omega_p\right)$$

$$\sqrt{\frac{10^{0.1A_{\min}}-1}{\varepsilon^2}} = C_n\left(\omega_s/\omega_p\right)$$

Chebyshev Filters

$$\sqrt{\frac{10^{0.1\mathrm{A}_{\min}}-1}{\varepsilon^2}}=C_n\left(\omega_s/\omega_p\right)$$

- Since $C_n(\Omega)$ = cosh (n $\cosh^{-1}\Omega$), for $|\Omega|>1$:

$$\sqrt{\frac{10^{0.1\mathrm{A}_{\min}}-1}{\varepsilon^2}}=\cosh\left(n\cosh^{-1}\left(\omega_s/\omega_p\right)\right)$$

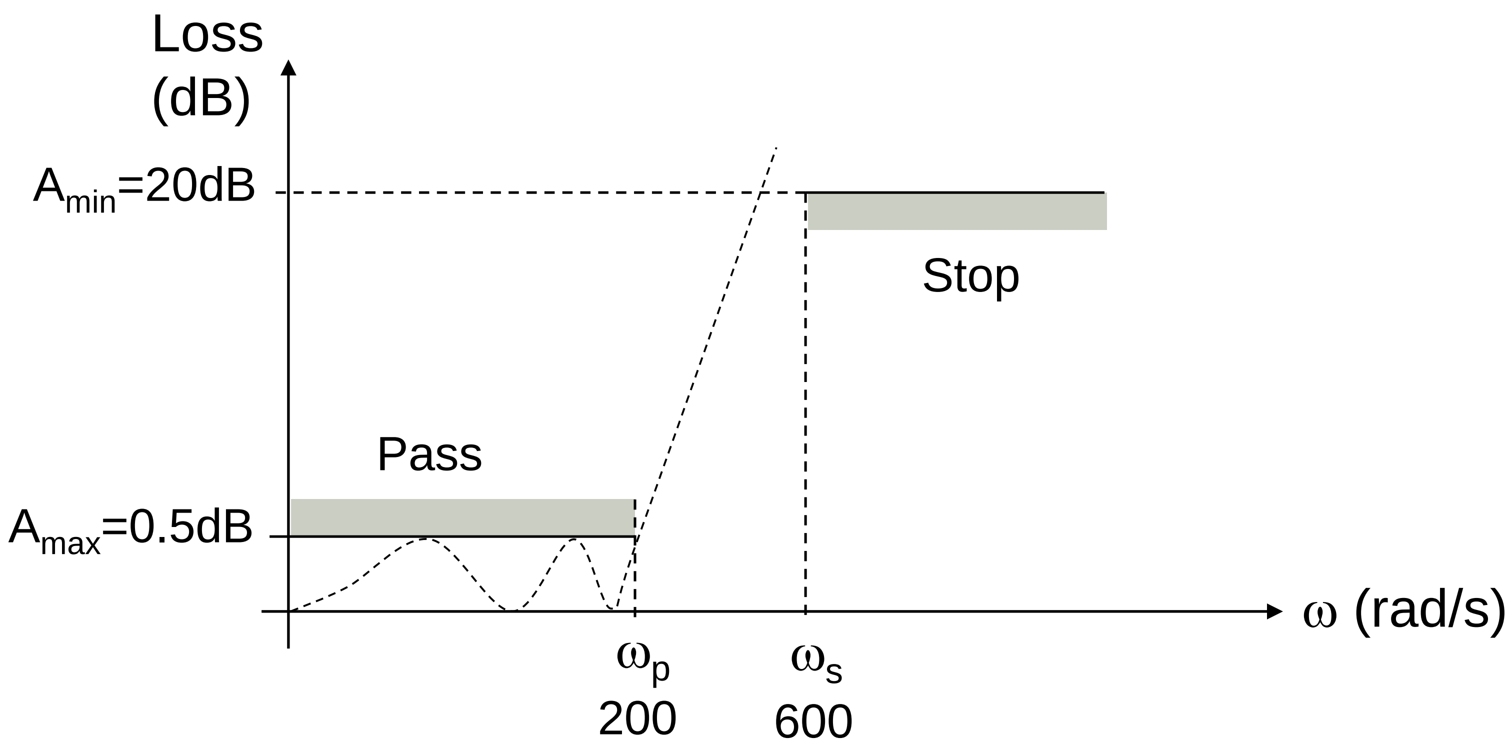
$$\boxed{n=\frac{\cosh^{-1}\sqrt{\frac{10^{0.1A_{\min}}-1}{\varepsilon^2}}}{\cosh^{-1}\left(\omega_s/\omega_p\right)}}$$

Chebyshev LP Filter

Example 4

- Find the Chebyshev approximation for a low pass filter whose requirements are:

  A_{max} = 0.5dB, A_{min} = 20dB, ω_p = 200, ω_s = 600rad/s



Chebyshev LP Filter

Example 4

- First find ε:

$$\varepsilon = \sqrt{10^{0.1A_{\max}} - 1} = \sqrt{10^{0.05} - 1} = 0.35$$

- Now find order of filter required:

$$n = \frac{\cosh^{-1}\sqrt{\frac{10^{0.1A_{\min}} - 1}{\varepsilon^2}}}{\cosh^{-1}\left(\omega_s / \omega_p\right)} = \frac{\cosh^{-1}\sqrt{\frac{10^{2.0} - 1}{\varepsilon^2}}}{\cosh^{-1}\left(600/200\right)} = 2.3$$

- Therefore we will choose n = 3.

(Note that Butterworth would have required n=4)

Chebyshev LP Filter

Example 4

- From tables, for A_{max}=0.5dB, and n=3:

$$T(s)=\frac{0.71570}{(s^2+0.62646s+1.14245)(s+0.62646)}$$

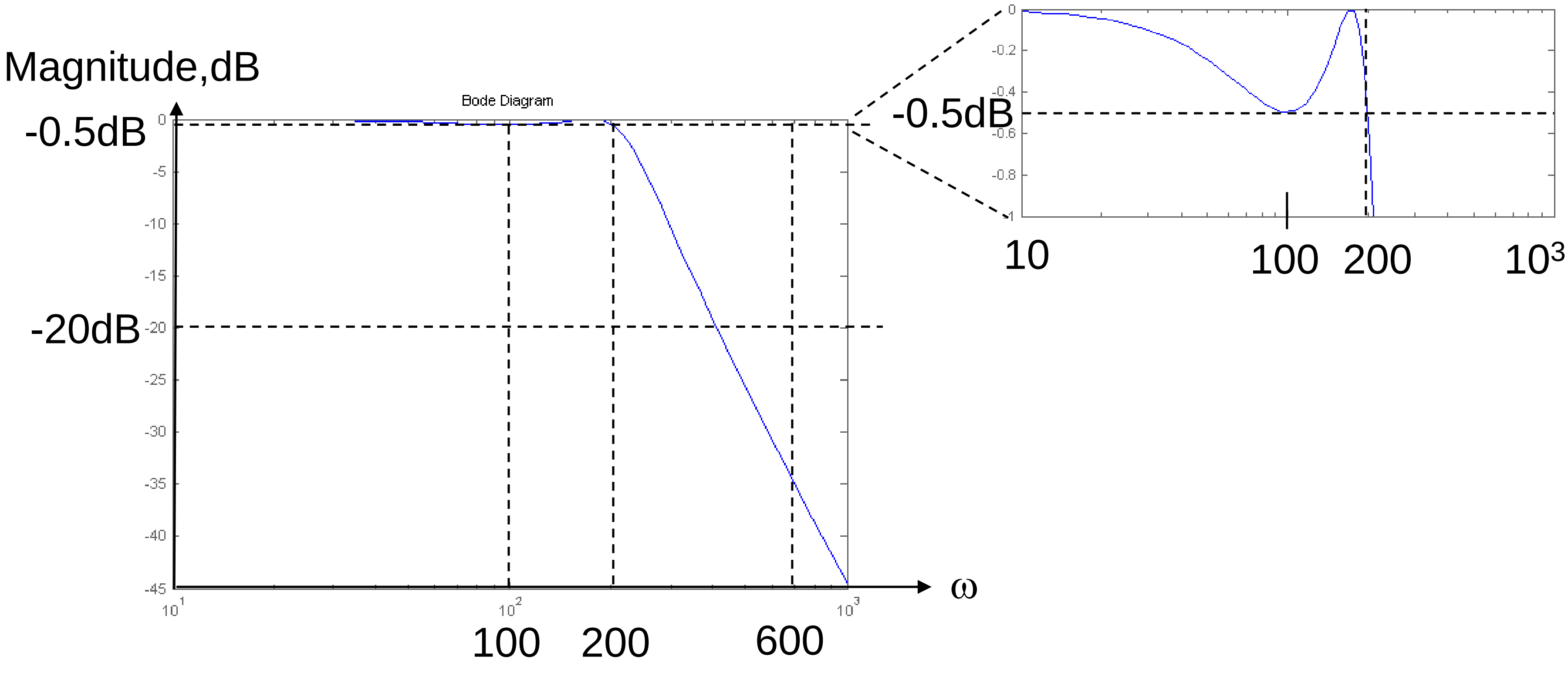
- Substitute $\frac{s}{\omega_p}=\frac{s}{200}$:

$$T(s)=\frac{5725600}{(s^2+125.3s+45698)(s+125.3)}$$

Chebyshev LP Filter

Example 4

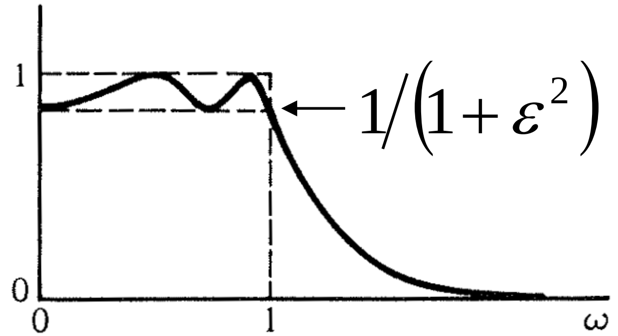
$$T(s)=\frac{5725600}{\left(s^2+125.3s+45698\right)\left(s+125.3\right)}$$



Inverse Chebyshev Approximation

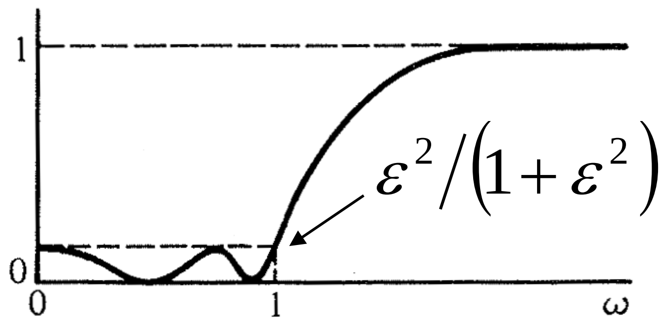
- Given the Chebyshev magnitude response:

$$\left|T_n(j\omega)\right|^2 = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$



- We subtract this function from 1, resulting in:

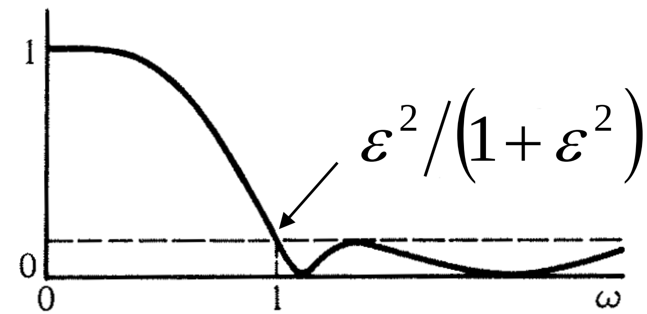
$$1-\left|T_n(j\omega)\right|^2 = \frac{\varepsilon^2 C_n^2(\omega)}{1+\varepsilon^2 C_n^2(\omega)}$$



Inverse Chebyshev Approximation

- Finally we invert frequency by replacing ω with 1/ω, giving the inverse Chebyshev response:

$$\left|T(j\omega)\right|^2 = \frac{\varepsilon^2 C_n^2(1/\omega)}{1+\varepsilon^2 C_n^2(1/\omega)}$$



- Equiripple in the stop band and maximally flat in the passband.

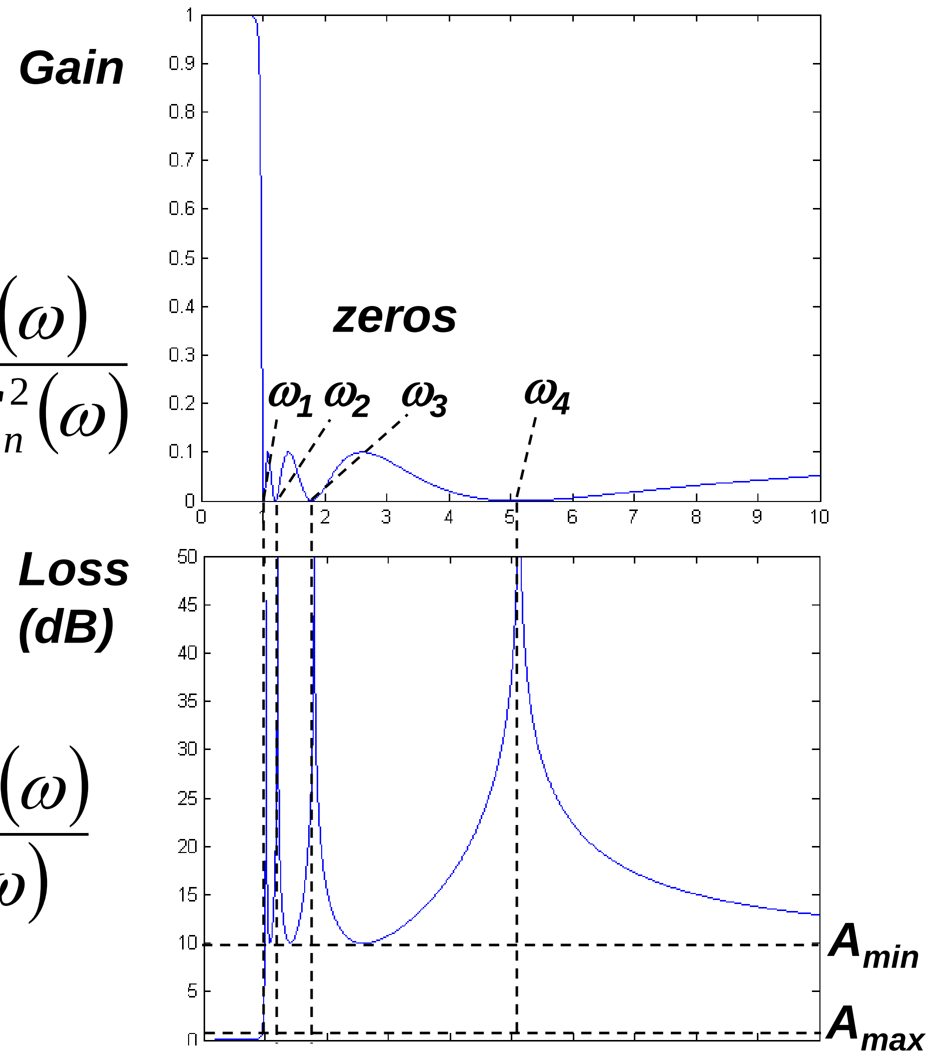
Inverse Chebyshev Approximation

- Typical plot of gain and attenuation functions:

$$|T(j\omega)|^2 = \frac{\varepsilon^2 C_n^2(\omega)}{1+\varepsilon^2 C_n^2(\omega)}$$

- zeros in gain function are poles in loss function

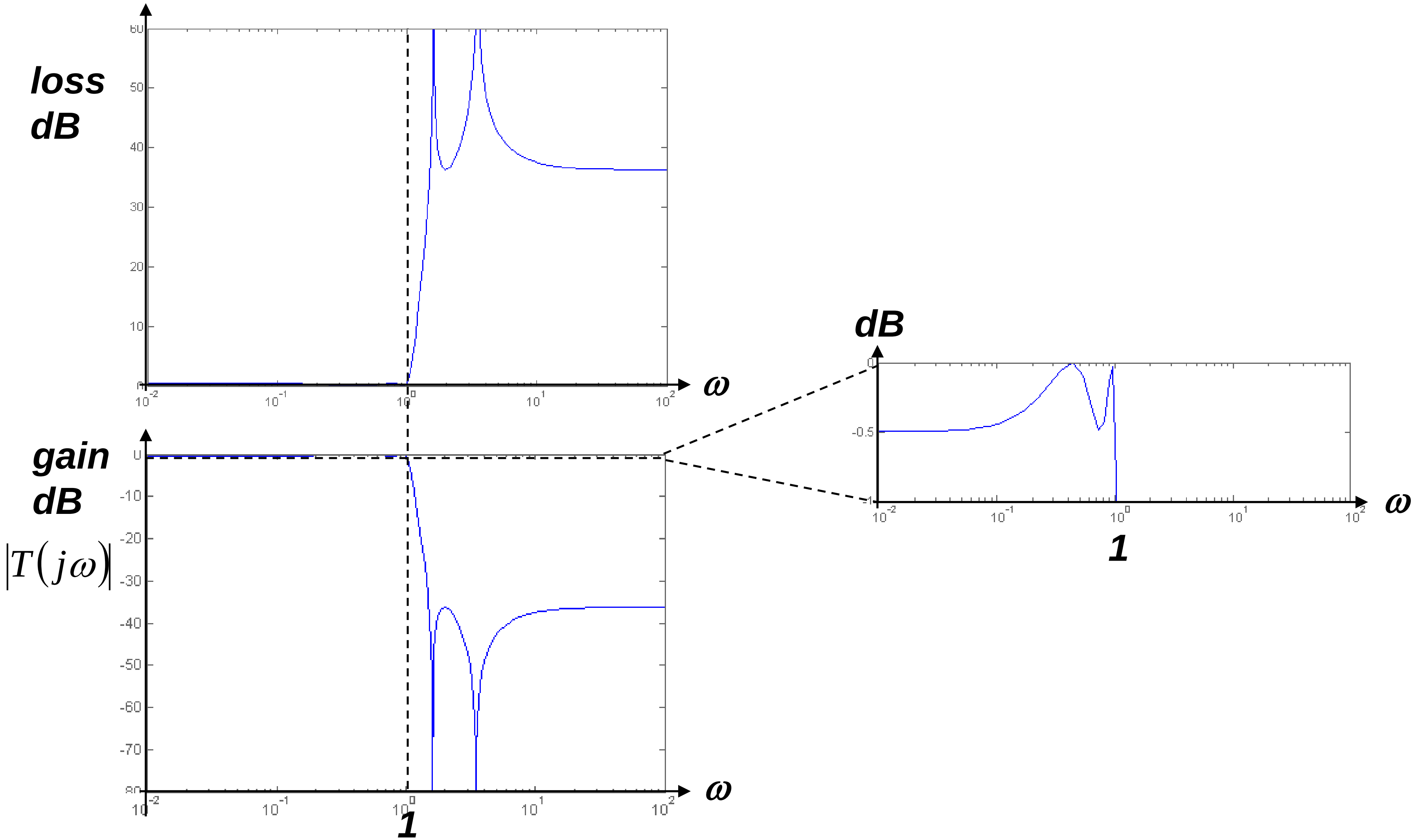
$$10\log \frac{1+\varepsilon^2 C_n^2(\omega)}{\varepsilon^2 C_n^2(\omega)}$$



Elliptic (Cauer) Approximation

- Have a flatter stopband attenuation than the Butterworth or Chebyshev
- Elliptic approximation is a rational function of poles and zeros.
- For a given filter requirement, generally the elliptic approximation will result in a lower order than a Butterworth or Chebyshev.

Elliptic (Cauer) Approximation



Elliptic (Cauer) Approximation

- Designers of elliptic filters can use extensive tables, or computerised algorithms.
- Tables are normalised for ω_s/ω_p, and give the poles, zeros and A_{min}.
- A different table is needed for each A_{max}.
- A portion of such tables is shown.

$$A_{max} = 0.5\text{dB}$$

$$\omega_s / \omega_p = 1.5$$

n	Numerator K	Numerator and Denominator of T(s)		A_{min}
2	0.38540	N	$\left(s^2 + 3.92705\right)$	8.3
		D	$\left(s^2 + 1.03153\,s + 1.60319\right)$	
3	0.31410	N	$\left(s^2 + 2.80601\right)$	21.9
		D	$\left(s^2 + 0.45286\,s + 1.14917\right)\left(s + 0.766952\right)$	
4	0.015397	N	$\left(s^2 + 2.53555\right)\left(s^2 + 12.09931\right)$	36.3
		D	$\left(s^2 + 0.25496\,s + 1.06044\right)\left(s^2 + 0.92001\,s + 0.47183\right)$	
5	0.019197	N	$\left(s^2 + 2.42551\right)\left(s^2 + 5.43764\right)$	50.6
		D	$\left(s^2 + 0.16346\,s + 1.03189\right)\left(s^2 + 0.57023\,s + 0.57601\right)\left(s + 0.42597\right)$	

$$A_{max} = 0.5\text{dB}$$

$$\omega_s / \omega_p = 2.0$$

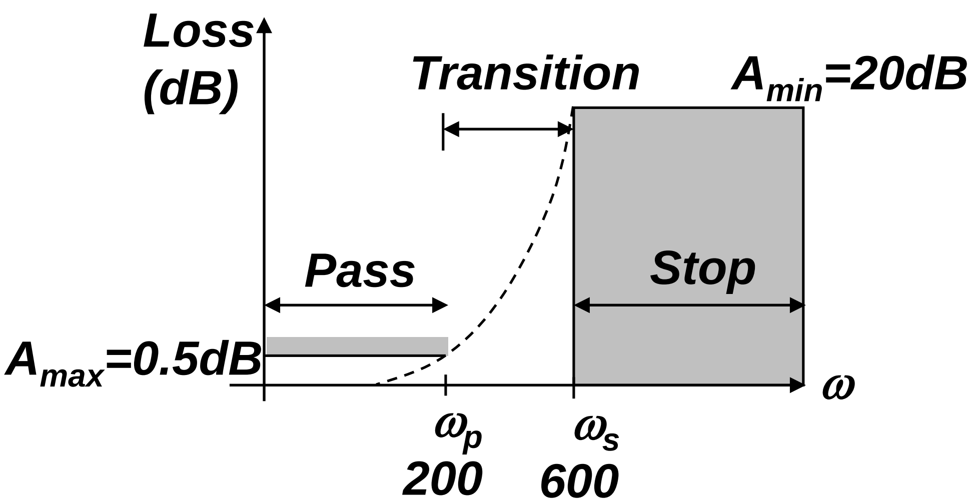
n	Numerator K	Numerator and Denominator of T(s)		A_{min}
2	0.20133	N	$\left(s^2 + 7.4641\right)$	13.9
		D	$\left(s^2 + 1.24504\,s + 1.59179\right)$	
3	0.15424	N	$\left(s^2 + 5.15321\right)$	31.2
		D	$\left(s^2 + 0.53787\,s + 1.14849\right)\left(s + 0.69212\right)$	
4	0.0036987	N	$\left(s^2 + 4.59326\right)\left(s^2 + 24.22720\right)$	48.6
		D	$\left(s^2 + 0.30116\,s + 1.06258\right)\left(s^2 + 0.88456\,s + 0.41032\right)$	
5	0.0046205	N	$\left(s^2 + 4.36495\right)\left(s^2 + 10.56773\right)$	66.1
		D	$\left(s^2 + 0.19255\,s + 1.03402\right)\left(s^2 + 0.58054\,s + 0.52500\right)\left(s + 0.392612\right)$	

Elliptic Example

Example

- ***Find the Elliptic approximation for a low pass filter whose requirements are characterised by:***

  A_{max} = 0.5dB, A_{min} = 20dB, ω_p = 200, ω_s = 600rad/s



Elliptic Example

- ***First find ω_s/ω_p:*** $\dfrac{\omega_s}{\omega_p} = \dfrac{600}{200} = 3$
- ***From tables for ω_s/ω_p =3, A_{max}=0.5dB and A_{min}=20dB:***

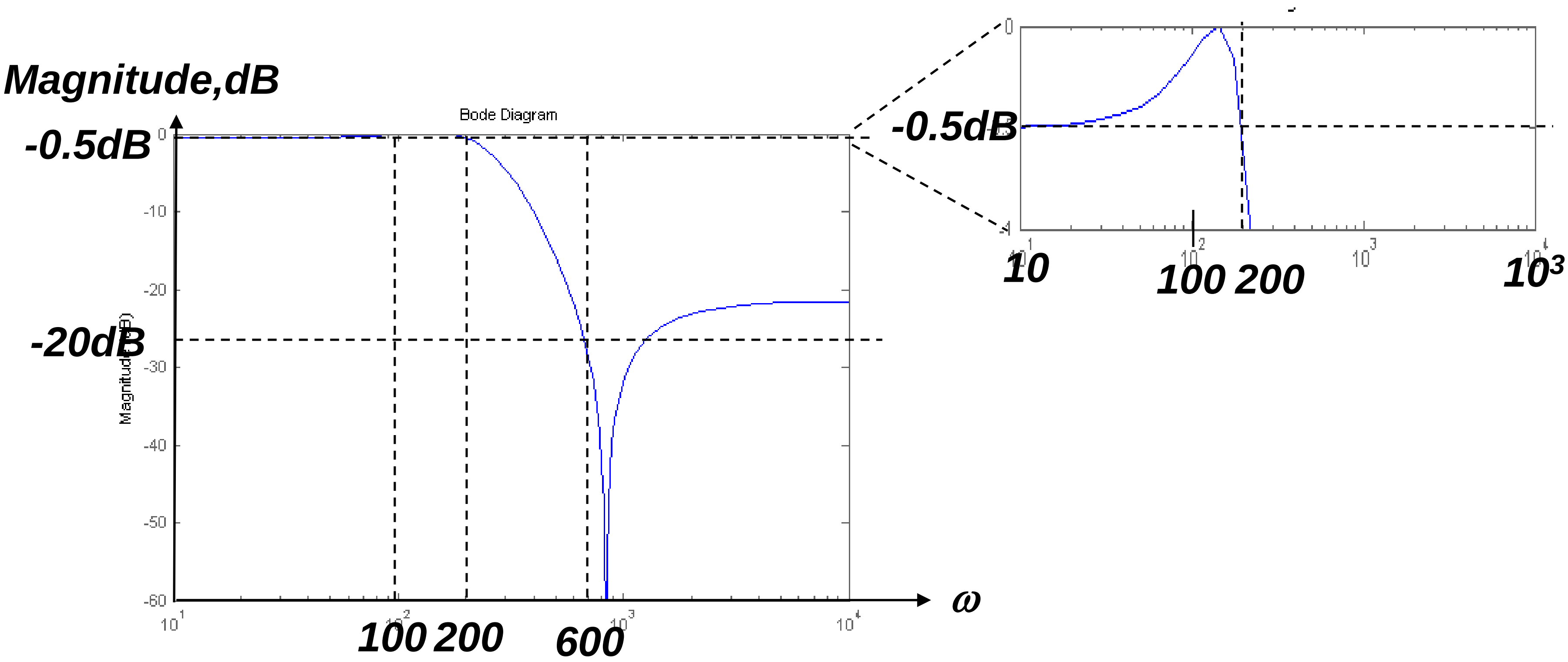
$$T(s) = \frac{0.083974\left(s^2 + 17.48528\right)}{s^2 + 1.35715s + 1.55532}$$

- ***Substitute*** $\dfrac{s}{\omega_p} = \dfrac{s}{200}$ ***:***

$$T(s) = \frac{0.083974\left(s^2 + 699411.2\right)}{s^2 + 271.43s + 62212.8}$$

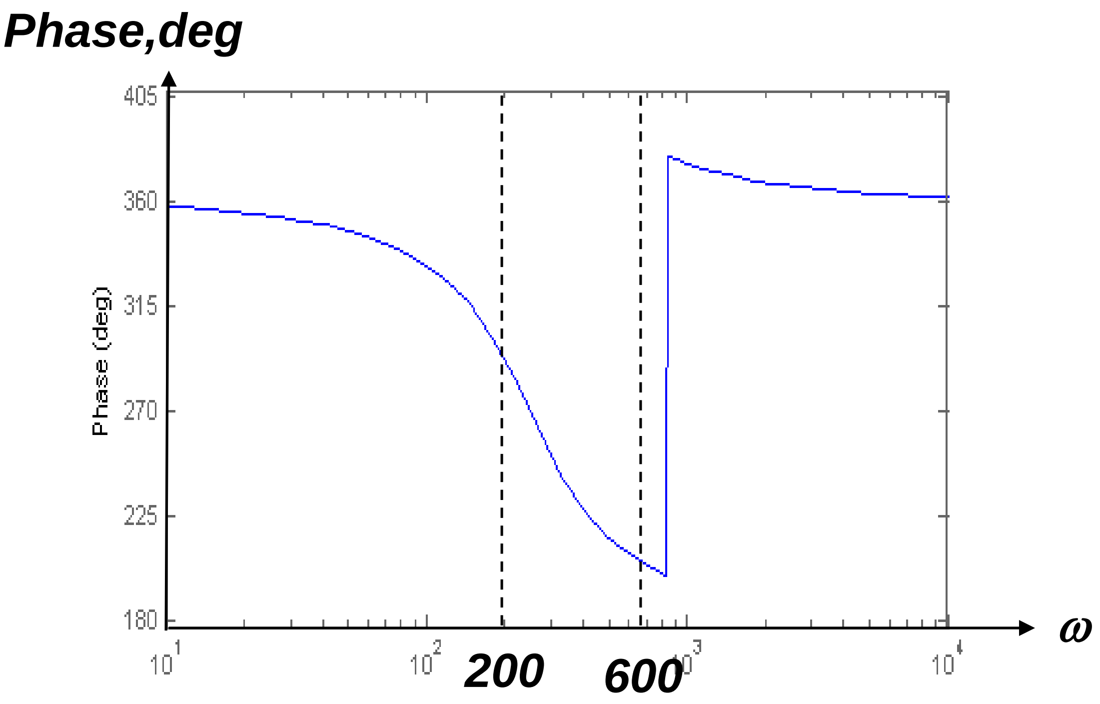
Elliptic Example

$$T(s)=\frac{0.083974\left(s^{2}+699411.2\right)}{s^{2}+271.43s+62212.8}$$



Elliptic Example

$$T(s) = \frac{0.083974\left(s^2 + 699411.2\right)}{s^2 + 271.43s + 62212.8}$$



Frequency Transformations

- Last few sections dealt with design of low pass filters.
- These approximations can be adapted to high pass, band pass, and band stop filters.

High Pass

- High pass filter can be transformed to a low pass with the transformation:

$$\boxed{S = \frac{\omega_{p(hp)}}{s}}$$

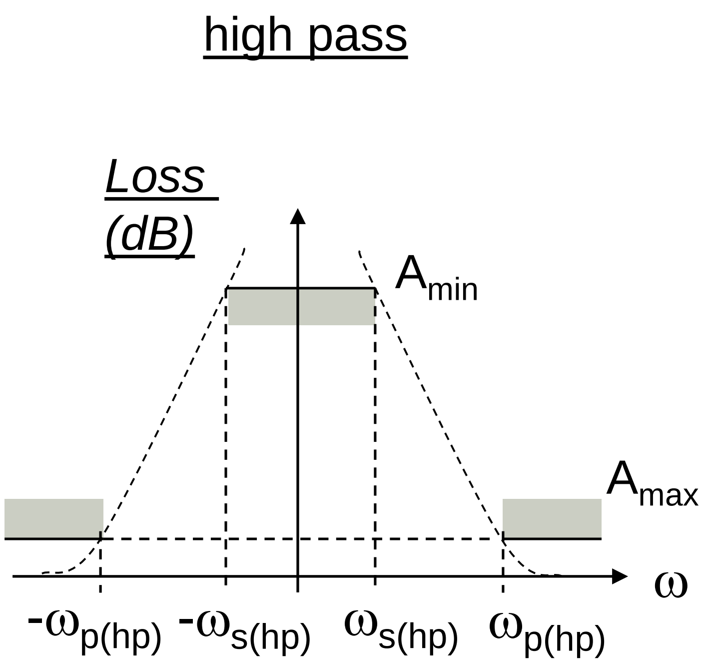
- Since $S = j\Omega$ and $s = j\omega$:

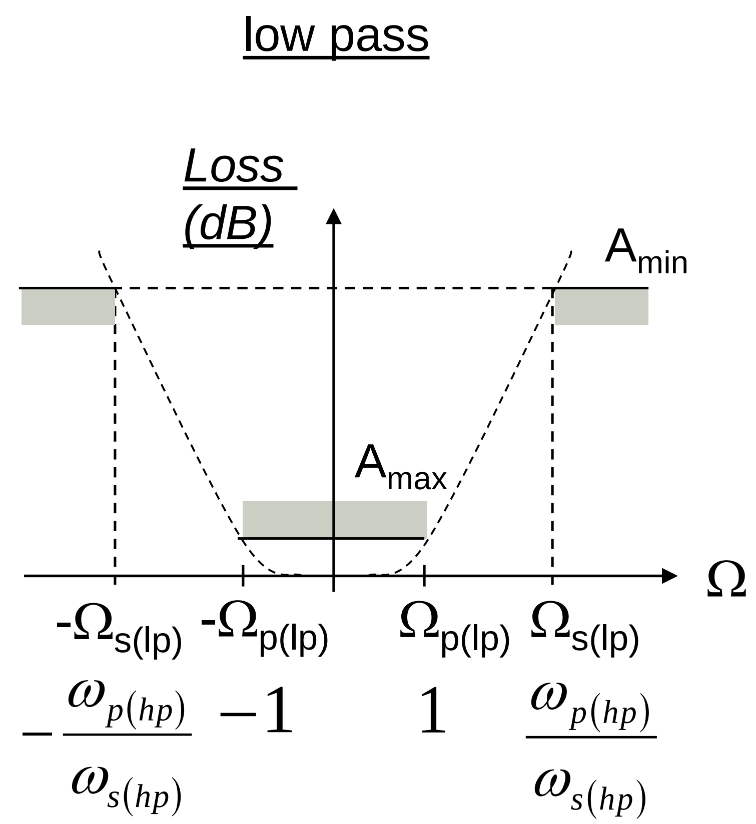
$$\Omega = -\frac{\omega_{p(hp)}}{\omega}$$

- LP filter requirements:

$$\boxed{\begin{aligned} &\Omega_{p(lp)} = 1 \\ &\Omega_{s(lp)} = \omega_{p(hp)} / \omega_{s(hp)} \end{aligned}}$$

High Pass

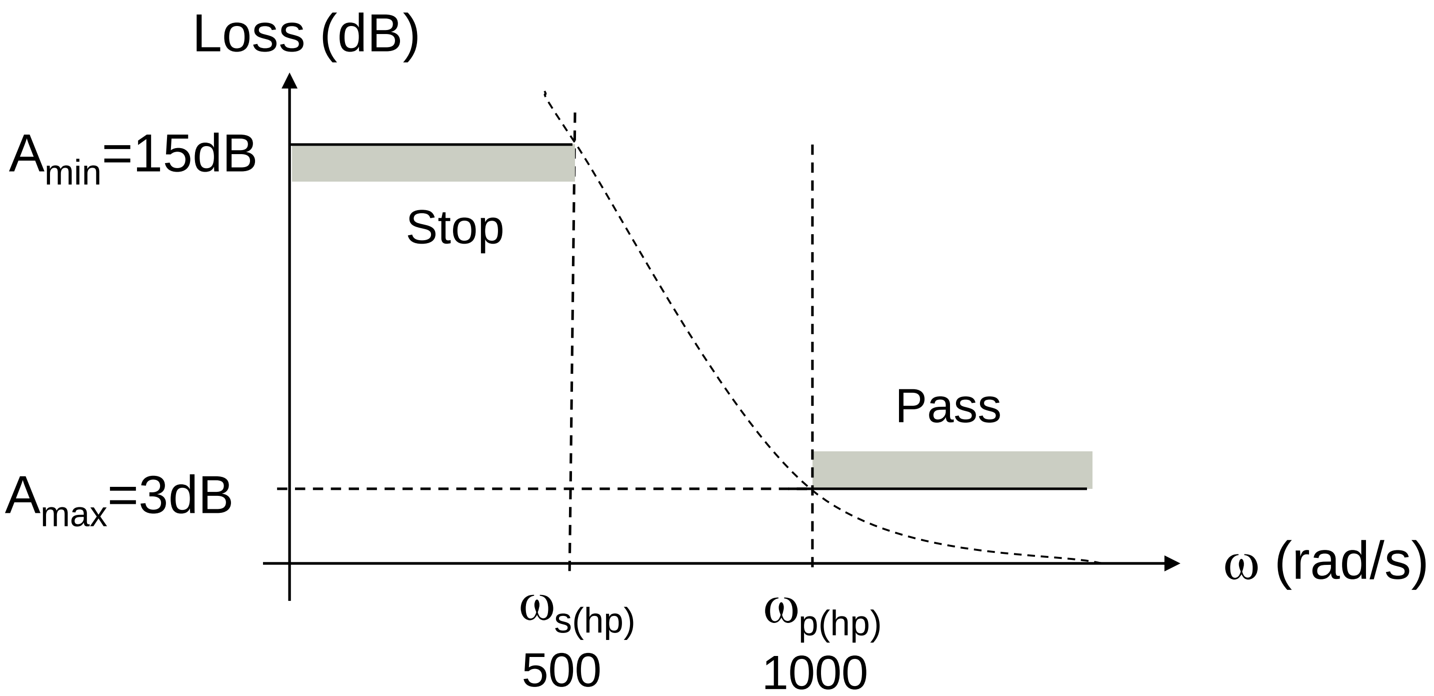

high pass
Loss (dB)
A_min
A_max
ω
-ω_p(hp)
-ω_s(hp)
ω_s(hp)
ω_p(hp)


low pass
Loss (dB)
A_min
A_max
Ω
-Ω_s(lp)
-Ω_p(lp)
Ω_p(lp)
Ω_s(lp)
$-\frac{\omega_{p(hp)}}{\omega_{s(hp)}}$
-1
1
$\frac{\omega_{p(hp)}}{\omega_{s(hp)}}$

High Pass – Example

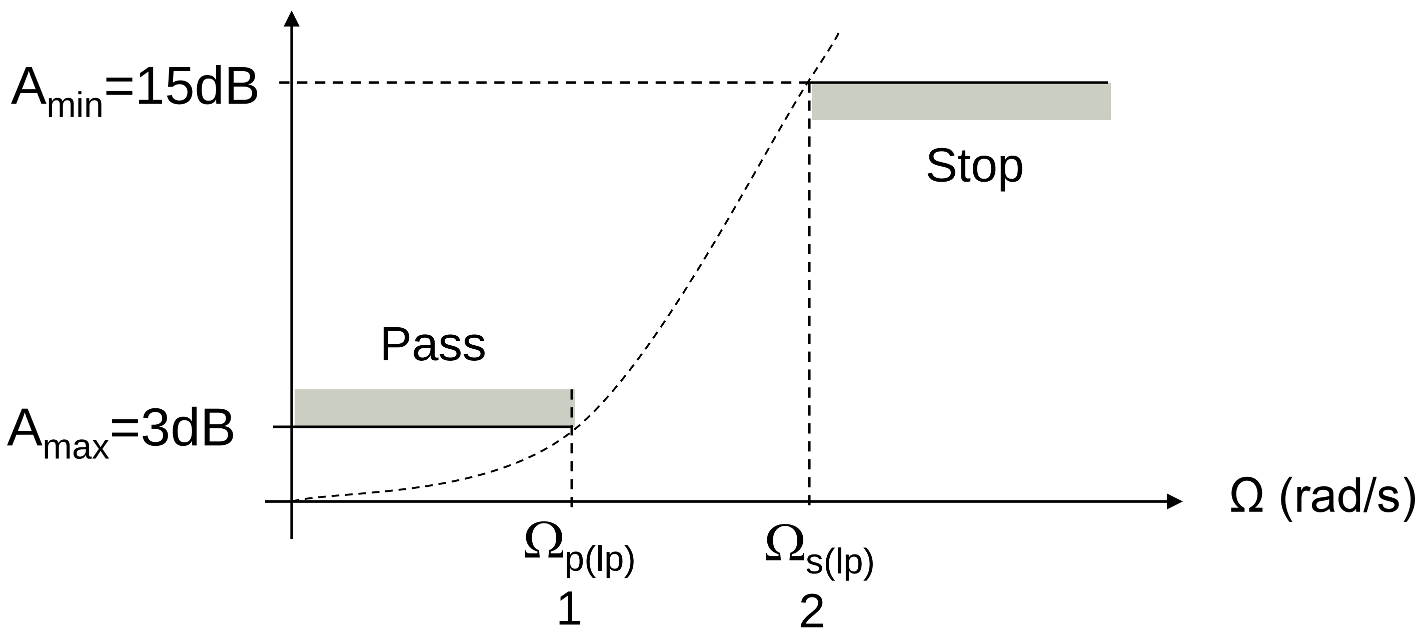
- Find a Butterworth approximation for the following high pass filter requirements:

  A_{min} = 15dB, A_{max} = 3dB, $\omega_{p(hp)}$=1000 rad/s, $\omega_{s(hp)}$=500 rad/s



High Pass – Example

- The equivalent LP filter requirements are:



$$\Omega_{p(lp)} = \frac{\omega_{p(hp)}}{\omega_{p(hp)}} = 1\,\text{rad/s} \qquad \Omega_{s(lp)} = \frac{\omega_{p(hp)}}{\omega_{s(hp)}} = \frac{1000}{500} = 2\,\text{rad/s}$$

High Pass – Example

- As previously, first find ε:

$$\varepsilon = \sqrt{10^{0.1A_{\max}} - 1} = \sqrt{10^{0.3} - 1} = 1$$

(since A_{max} = 3dB)

- Now find order of filter required:

$$n = \frac{\log\left(\dfrac{10^{0.1A_{\min}} - 1}{\varepsilon^2}\right)}{2\log\left(\dfrac{\Omega_{s(lp)}}{\Omega_{p(lp)}}\right)} = \frac{\log\left(\dfrac{10^{1.5} - 1}{\varepsilon^2}\right)}{2\log\left(\dfrac{2}{1}\right)} = 2.47$$

- Therefore we will choose n = 3.

High Pass – Example

- Third order Butterworth function:

$$T(S) = \frac{1}{(S^2 + S + 1)(S + 1)}$$

- Now substitute:

$$S = \left(\varepsilon^{1/n} / \Omega_p\right)s = \left(1^{1/2}/1\right)s = s$$

$$T(s) = \frac{1}{(s^2 + s + 1)(s + 1)}$$

High Pass – Example

- To convert LP function to HP, substitute:

$$S = \frac{\omega_{p(hp)}}{s} = \frac{1000}{s}$$

$$T(s) = \frac{1}{\left(\left(\frac{1000}{s}\right)^2 + \frac{1000}{s} + 1\right)\left(\frac{1000}{s} + 1\right)}$$

$$T(s) = \frac{s^3}{\left(s^2 + 1000s + 10^6\right)\left(s + 1000\right)}$$

High Pass – Example

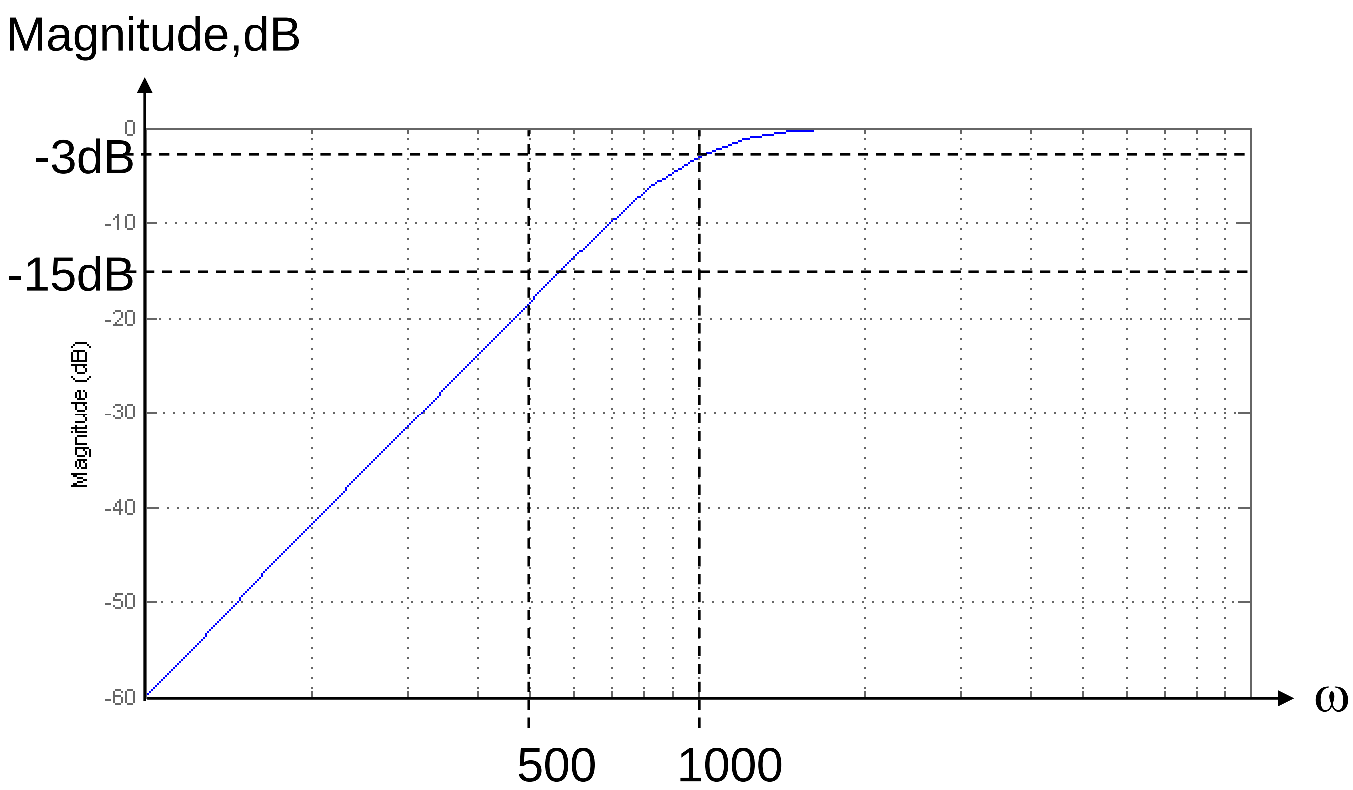
$$T(s) = \frac{s^3}{\left(s^2 + 1000s + 10^6\right)\left(s + 1000\right)}$$

- To plot in Matlab:

```
t1 = tf([1 0 0 0],[1 2000 2e6 1e9]);
bode (t1);
grid on;
```

High Pass – Example

$$T(s) = \frac{s^3}{\left(s^2 + 1000\,s + 10^6\right)\left(s + 1000\right)}$$



Band Pass

- Band pass filter can be transformed to a low pass with the transformation:

$$S = \frac{s^2 + \omega_o^2}{\mathrm{BW}s}$$

where

$$\mathrm{BW} = \omega_{p2} - \omega_{p1} = \frac{\omega_0}{Q}$$

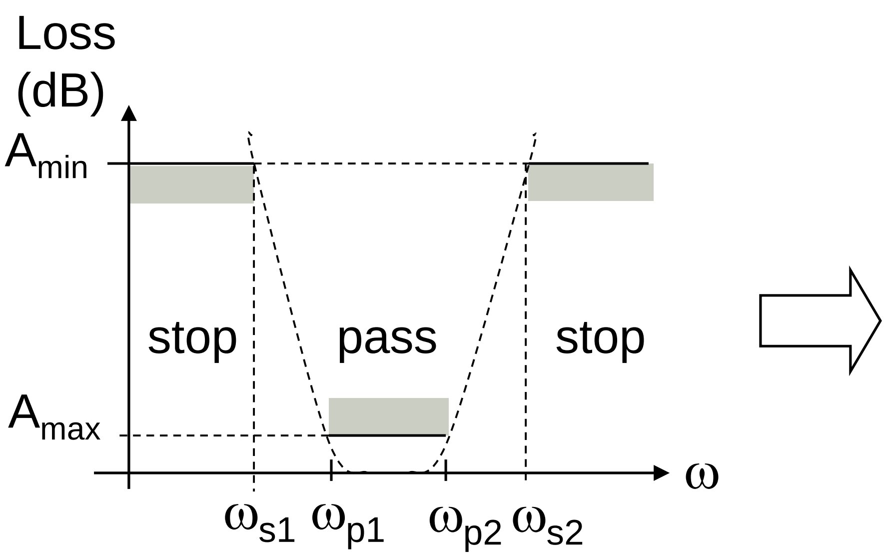
$$\omega_o = \sqrt{\omega_{p1}\omega_{p2}}$$

- Since $S = j\Omega$ and $s = j\omega$:

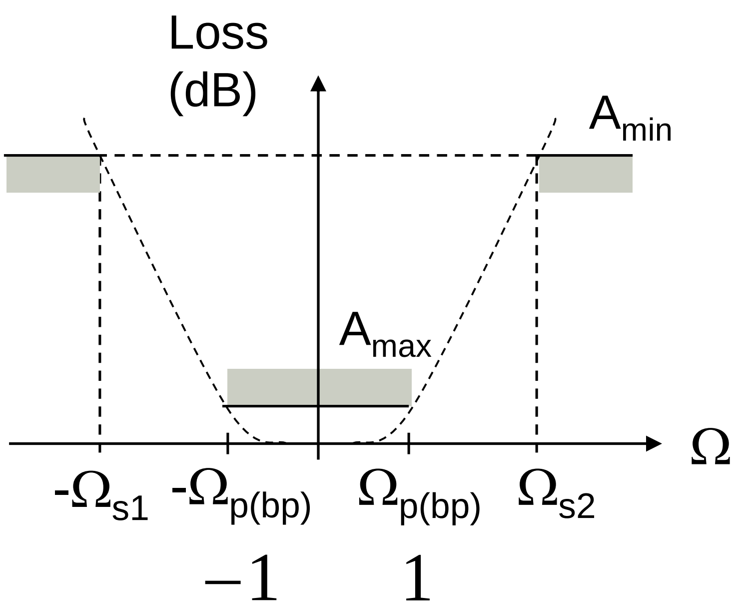
$$\Omega = \frac{\omega^2 - \omega_o^2}{(\omega_{p2} - \omega_{p1})\omega}$$

Band Pass

band pass



low pass



$$\Omega_{s(bp)} = \min\left(\left|\Omega_{s1}\right|, \left|\Omega_{s2}\right|\right)$$

Band Pass

- Transform BP cutoff frequencies to LP:

$$\Omega_{s1} = \frac{\omega_{s1}^2 - \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})\omega_{s1}}$$

$$\Omega_{p2} = \frac{\omega_{p2}^2 - \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})\omega_{p2}} = 1$$

$$\Omega_{p1} = \frac{\omega_{p1}^2 - \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})\omega_{p1}} = -1$$

$$\Omega_{s2} = \frac{\omega_{s2}^2 - \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})\omega_{s2}}$$

- Choose steeper specification, ie, whichever of Ω_{s1} or Ω_{s2} has the smaller absolute value.

Band Pass

- If the band pass filter is symmetrical:

$$\omega_0^2 = \omega_{p1}\omega_{p2} = \omega_{s1}\omega_{s2}$$

- The equations simplify to:

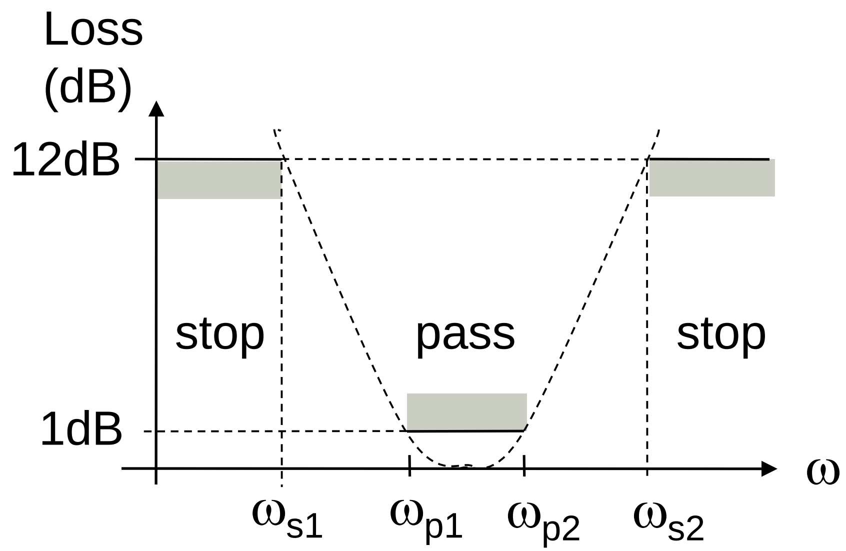
$$\Omega_{s1} = \frac{\omega_{s1} - \omega_{s2}}{\omega_{p2} - \omega_{p1}} \qquad \Omega_{p2} = 1$$

$$\Omega_{p1} = -1 \qquad \Omega_{s2} \to \frac{\omega_{s2} - \omega_{s1}}{\omega_{p2} - \omega_{p1}} = -\Omega_{s1}$$

Band Pass – Example

- Find a Butterworth approximation for the following band pass filter requirements:

  A_{min} = 12dB, A_{max} = 1dB, ω_{s1}=500 rad/s, ω_{p1}=1000 rad/s, ω_{p1}=2000 rad/s, ω_{s1}=3500 rad/s



Band Pass – Example

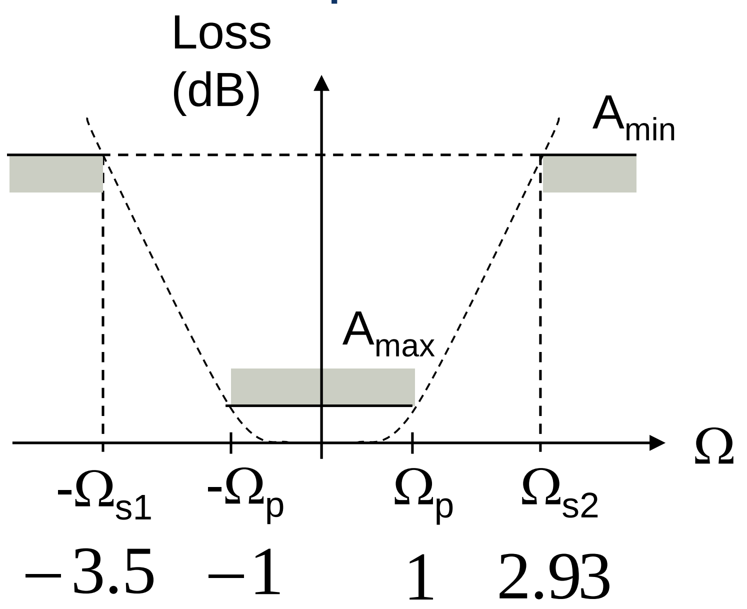
- Transform BP frequencies to LP:

$$\Omega_{s1} = \frac{\omega_{s1}^2 - \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})\omega_{s1}} = \frac{500^2 - 1000 \times 2000}{(2000 - 1000)500} = -3.5$$

$$\Omega_{s2} = \frac{\omega_{s2}^2 - \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})\omega_{s2}} = \frac{3500^2 - 1000 \times 2000}{(2000 - 1000)3500} = 2.93$$

Band Pass – Example

- The equivalent LP filter requirements are:



- Take steeper requirement:

$$\Omega_{p(bp)} = 1\,\text{rad/s} \qquad \Omega_{s(bp)} = 2.93\,\text{rad/s}$$

Band Pass – Example

- As previously, first find ε:

$$\varepsilon = \sqrt{10^{0.1A_{\max}} - 1} = \sqrt{10^{0.1} - 1} = 0.5088$$

- Now find order of filter required:

$$n = \frac{\log_{10}\left(\dfrac{10^{0.1A_{\min}} - 1}{\varepsilon^2}\right)}{2\log_{10}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\dfrac{10^{1.2} - 1}{\varepsilon^2}\right)}{2\log_{10}\left(\dfrac{2.93}{1}\right)} = 1.88$$

- Therefore we will choose n = 2.

Band Pass – Example

- Second order Butterworth function:

$$T(S) = \frac{1}{S^2 + \sqrt{2}S + 1}$$

- Now substitute:

$$S = \left(\varepsilon^{1/n} / \Omega_p\right)s = \left(0.5088^{1/2} / 1\right)s = 0.7133s = Bs$$

$$T(s) = \frac{1}{B^2 s^2 + \sqrt{2}Bs + 1} = \frac{1/B^2}{s^2 + \left(\sqrt{2}/B\right)s + 1/B^2}$$

$$T(s) = \frac{1.965}{s^2 + 1.983s + 1.965}$$

Band Pass – Example

- To convert LP function to BP, substitute:

$$s = \frac{s^2 + \omega_o^2}{(\omega_{p2} - \omega_{p1})s} = \frac{s^2 + 1000 \times 2000}{(2000 - 1000)s} = \frac{s^2 + 2 \times 10^6}{1000s}$$

- Therefore:

$$T(s) = \frac{1.965}{\left(\frac{s^2 + 2 \times 10^6}{10^3 s}\right)^2 + 1.983\left(\frac{s^2 + 2 \times 10^6}{10^3 s}\right) + 1.965}$$

Band Pass – Example

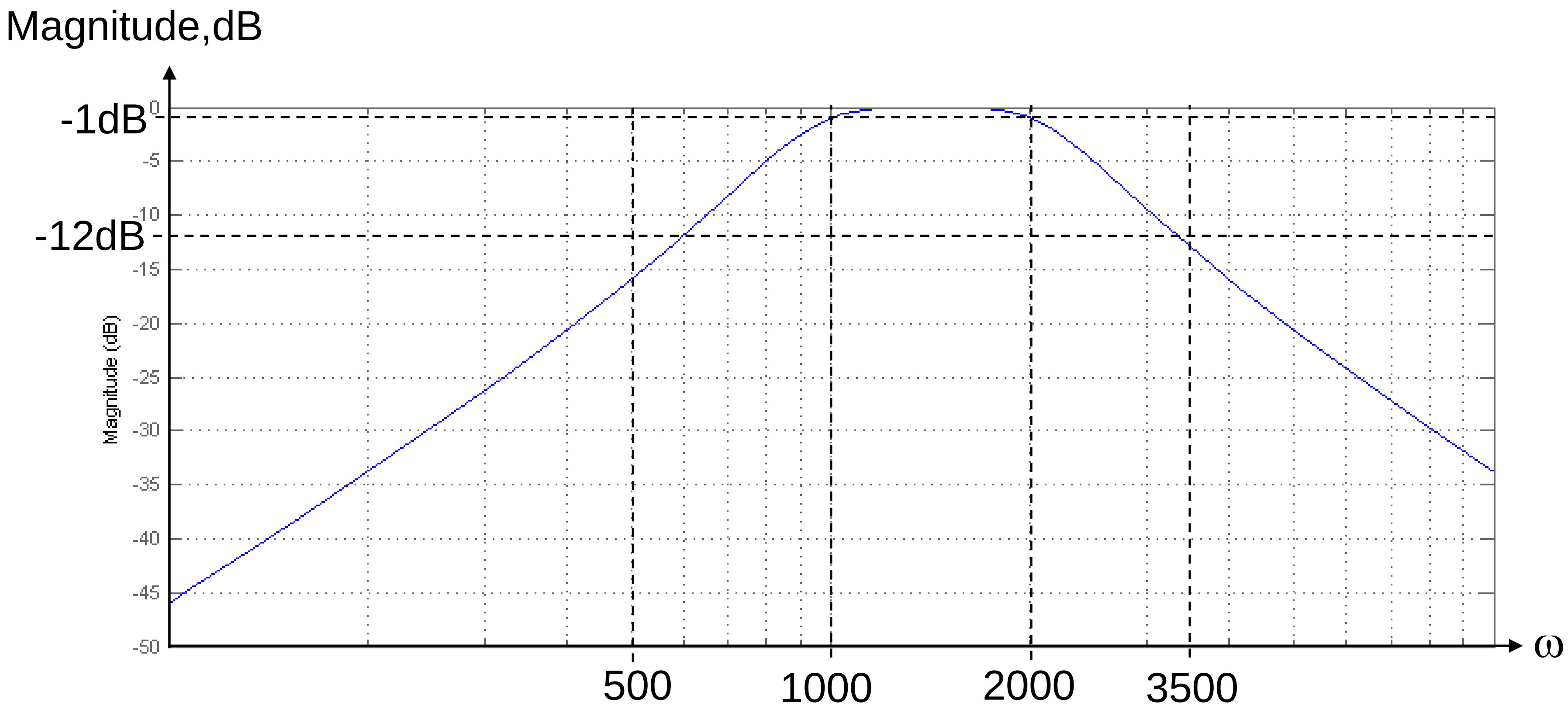
$$T(s) = \frac{1.965\times10^{6}\, s^{2}}{s^{4} + 1.983\times10^{3} s^{3} + 5.965\times10^{6} s^{2} + 3.965\times10^{9} s + 4\times10^{12}}$$

- To plot in Matlab:

```
t1 = tf([0 0 1.965e6 0 0 ],[1 1983 5.965e6 3.965e9 4e12]);
bode (t1);
grid on;
```

Band Pass – Example

$$T(s) = \frac{1.965\times10^{6}\, s^{2}}{s^{4} + 1.983\times10^{3} s^{3} + 5.965\times10^{6} s^{2} + 3.965\times10^{9} s + 4\times10^{12}}$$



Band Stop

- Band stop filter can be transformed to a low pass with the transformation:

where

$$S = \frac{BWs}{s^2 + \omega_o^2}$$

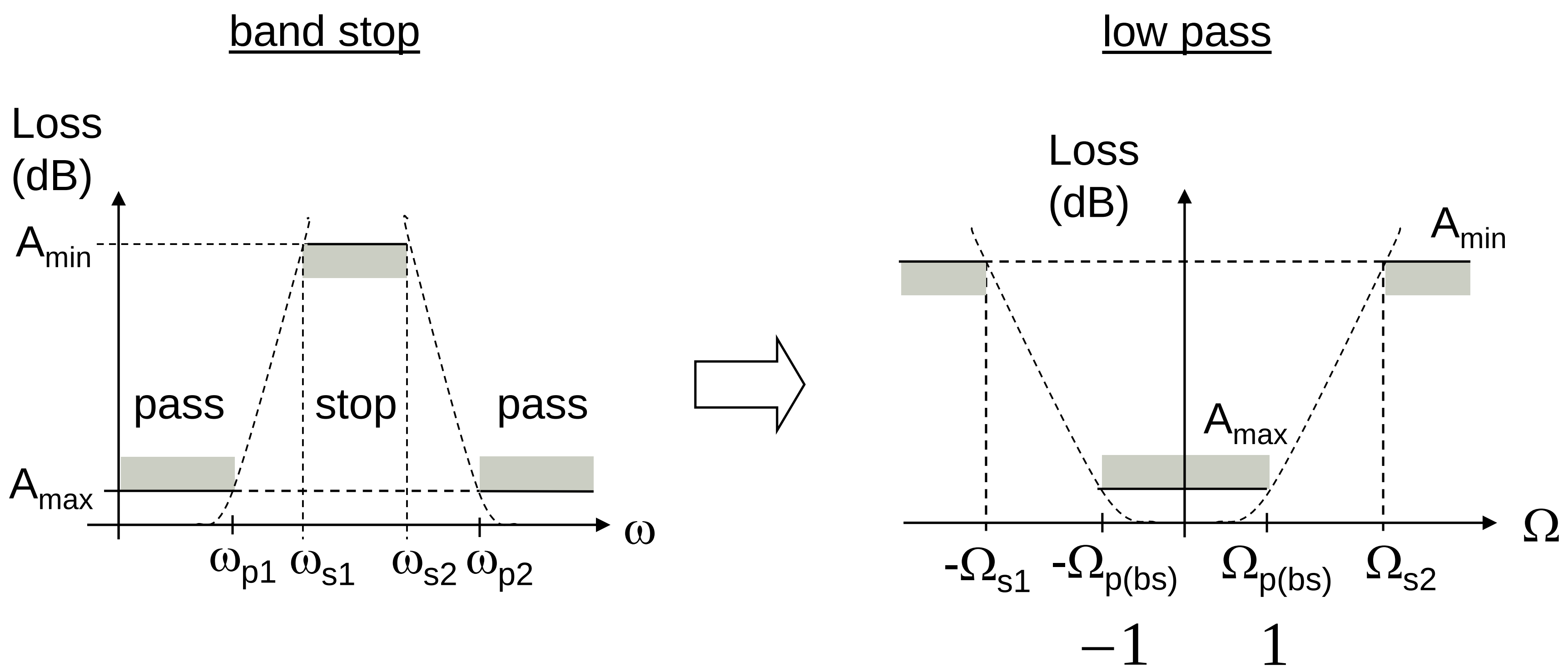
$$BW = \omega_{p2} - \omega_{p1} = \frac{\omega_0}{Q}$$

$$\omega_o = \sqrt{\omega_{p1}\omega_{p2}}$$

- Since $S = j\Omega$ and $s = j\omega$:

$$\Omega = \frac{(\omega_{p2} - \omega_{p1})\omega}{\omega_o^2 - \omega^2}$$

Band Stop



$$\Omega_{s(bs)} = \min\left(\left|\Omega_{s1}\right|, \left|\Omega_{s2}\right|\right)$$

Band Stop

- Transform BP cutoff frequencies to LP:

$$\Omega_{p1} = \frac{(\omega_{p2} - \omega_{p1})\omega_{p1}}{\omega_{p1}\omega_{p2} - \omega_{p1}^2} = 1 \qquad \Omega_{s2} = \frac{(\omega_{p2} - \omega_{p1})\omega_{s2}}{\omega_{p1}\omega_{p2} - \omega_{s2}^2}$$

$$\Omega_{s1} = \frac{(\omega_{p2} - \omega_{p1})\omega_{s1}}{\omega_{p1}\omega_{p2} - \omega_{s1}^2} \qquad \Omega_{p2} = \frac{(\omega_{p2} - \omega_{p1})\omega_{p2}}{\omega_{p1}\omega_{p2} - \omega_{p2}^2} = -1$$

- Choose steeper specification, ie, whichever of Ω_{s1} or Ω_{s2} has the smaller absolute value.

Band Stop

- If the band stop filter is symmetrical:

$$\omega_0^2 = \omega_{p1}\omega_{p2} = \omega_{s1}\omega_{s2}$$

- The equations simplify to:

$$\Omega_{p1} = 1$$

$$\Omega_{s2} = \frac{\omega_{p2} - \omega_{p1}}{\omega_{s1} - \omega_{s2}} = \text{-}\Omega_{s1}$$

$$\Omega_{s1} = \frac{\omega_{p2} - \omega_{p1}}{\omega_{s2} - \omega_{s1}}$$

$$\Omega_{p2} = -1$$