School of Electrical Engineering and Robotics

EGB348 Electronics

Transfer Functions, Poles and Zeros

Jasmine Banks

Recommended Readings:

Systems

System:

- Physical (Electrical, Mechanical, …)
- Biological
- Organisational, ….

Entity whose behaviour is sought to be represented considering its interactions with others as inputs and outputs.

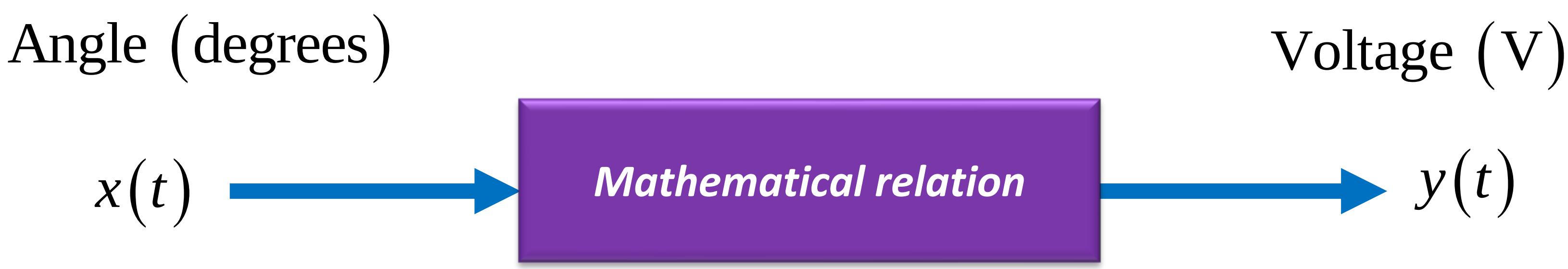
Usually (but not always) physical variables as functions of time.

Systems

- A mathematical model to relate system variables using known physical laws.

- For linear, time-invariant continuous time systems, the relationship between *x(t)* and *y(t)* is in general a linear constant coefficient differential equation. (Note: Not all systems are LTI)

Systems
Example – A transducer



$$y(t) = K\, x(t)$$

- In general the mathematical relationship is not so simple, and can involve a differential equation.
- It could make it easier if we can formulate the ratio between the output and the input, instead of having to formulate and solve the differential equation every time.

Laplace Transform

- Used to analyse and model linear systems
- Also defined for functions with discontinuities (eg, a switch)
- Time and frequency response characteristics can be easily determined
- Transient and steady-state responses determined simultaneously
- Can lead to intuitive understanding of linear systems and their applications.

Laplace Transform – Two Sided

The Two-Sided Laplace Transform is defined as

Note: $j = \sqrt{-1}$

$$F(s) = L\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)dt$$

where $s = \sigma + j\omega$ radians/second

and exists for when $\int_{-\infty}^{\infty} \left| e^{-\sigma t} f(t) \right| dt < \infty$

where σ rad/s is finite and real

Relationship with Fourier Transform (omitting scaling factor of $\frac{1}{\sqrt{2\pi}}$)

$$F(\omega) = F\{f(t)\} = L\{f(t)\}\Big|_{s=j\omega}$$

$$\Rightarrow F(s)\Big|_{s=j\omega} = \int_{-\infty}^{\infty} e^{j\omega t} f(t)dt$$

Laplace Transform – Causality

- Recall the two-sided Laplace transform

$$F(s) = L\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)dt$$

$$\text{where } s = \sigma + j\omega \text{ radians/se cond}$$

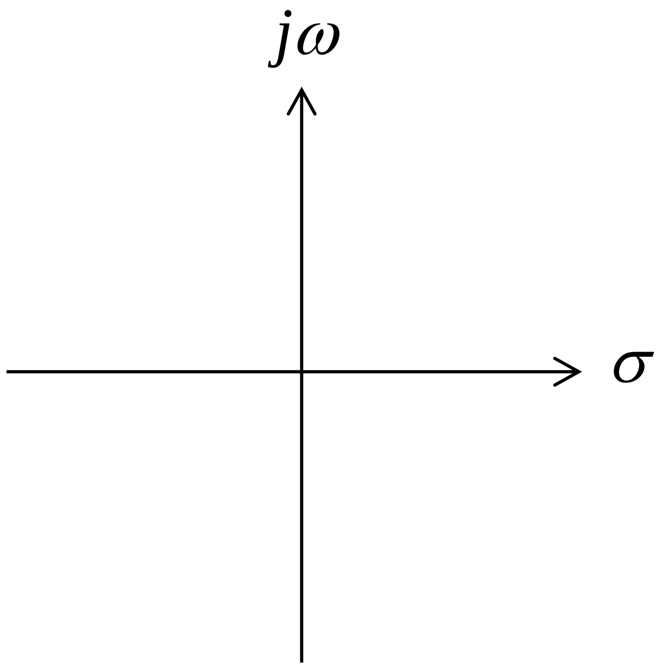
- Because time ranges from -∞ to +∞, the system is said to be **non-causal** (negative and positive time). Physical systems however only have positive time which exists from 0 to +∞ and are said to be **causal**. Therefore, for our purposes, the Laplace transform will be considered as

$$F(s) = L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)dt$$

- Physical systems in our everyday experience are causal.
- Non-causal systems are encountered in digital signals and systems.

Laplace Transform – Working Out

- Draw the s-plane and identify the axes.



- Find the Laplace Transform of:

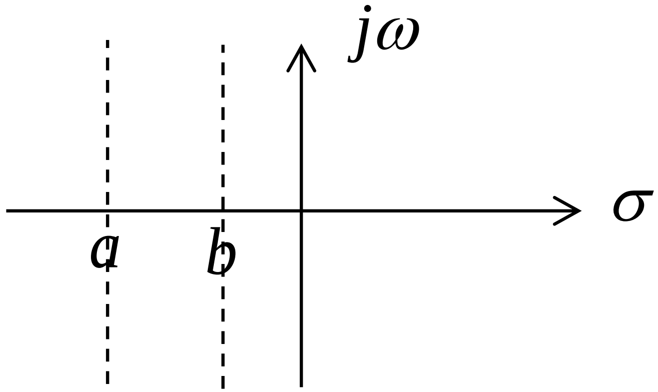
$$x(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where α is real-valued and positive.

$$X(s) = \int_0^\infty e^{-st} e^{-\alpha t} dt = \left[\frac{-e^{-st}}{(s+\alpha)} \right]_0^\infty$$

$$X(s) = \frac{1}{(s+\alpha)}$$

Laplace Transform – ROC

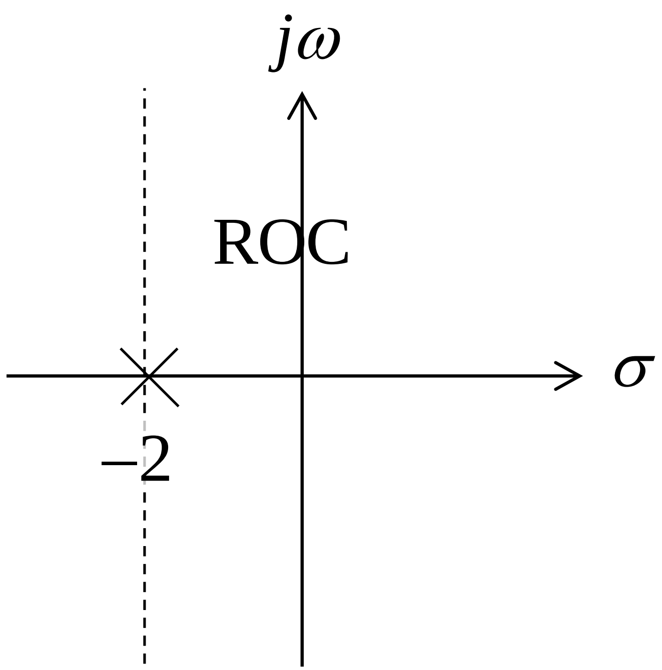


- The Laplace transform $F(s)$ typically exists for all complex numbers for a < Re{s} < b, where a and b are real constants. The values of s for which the Laplace transform exists is called the *region of convergence* (ROC).

- We will work with linear systems and input and output functions that have Laplace transforms and their ROCs can be determined. For ratios of polynomials in s, this is achieved through finding the roots of the denominator (poles) polynomials.

Laplace Transform – Working Out

- Draw the s-plane and identify ROC.



- Find ROC for the transform of:

$$x(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where $\alpha = 2$.

$$X(s) = \frac{1}{(s+2)} \qquad \text{for } s > -2$$

$$X(s) = \int_0^\infty \left| e^{-(s+2)t} \right| dt \; < \; \infty$$

Note: e^{at} if a is positive, will diverge

Laplace Transform – for <u>one-sided signals</u>

- Causal

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

where $s = \sigma + j\omega$ radians/second

- Convergent

$$\int_0^{\infty} \left| e^{-\sigma t} f(t) \right| dt < \infty$$

where σ rad/s is finite and real

- Equivalent to Fourier transform with $s = j\omega$ rad/s

$$F(\omega) = F(s)\Big|_{s=j\omega} = \int_0^{\infty} e^{j\omega t} f(t)dt$$

Laplace Transform Pairs

Description		$f(t)$ $(t \geq 0)$	$F(s)$
impulse		$\delta(t)$	1
step		$u_{-1}(t)$	$\frac{1}{s}$
ramp		t	$\frac{1}{s^2}$
exponential		e^{-at}	$\frac{1}{s+a}$
sine		$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
cosine		$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
damped ramp		te^{-at}	$\frac{1}{(s+a)^2}$
damped sine		$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
damped cosine		$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace Transform Theorems

1. $L[a\, f(t)] = a\, F(s)$ LINEARITY

2. $L[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$ SUPERPOSITION

3. $L[\, f(t-a)\, u_{-1}(t-a)] = e^{-as}F(s)$ TRANSLATION IN TIME

4. $L[t\, f(t)] = \frac{-d}{ds} F(s)$ COMPLEX DIFFERENTIATION

5. $L[e^{at}\, f(t)] = F(s-a)$ TRANSLATION IN S

6. $L\left[\frac{d^n f(t)}{dt^n}\right] =$ REAL DIFFERENTIATION

$s^nF(s) - s^{n-1}f(0) - s^{n-2}Df(0) \cdots - D^{n-1}f(0)$

7. $L\left[\int_0^{t_n} \cdots \int_0^{t_2} \int_0^{t_1} f(t)\, dt\, dt \cdots dt\right] = \frac{F(s)}{s^n}$ REAL INTEGRATION

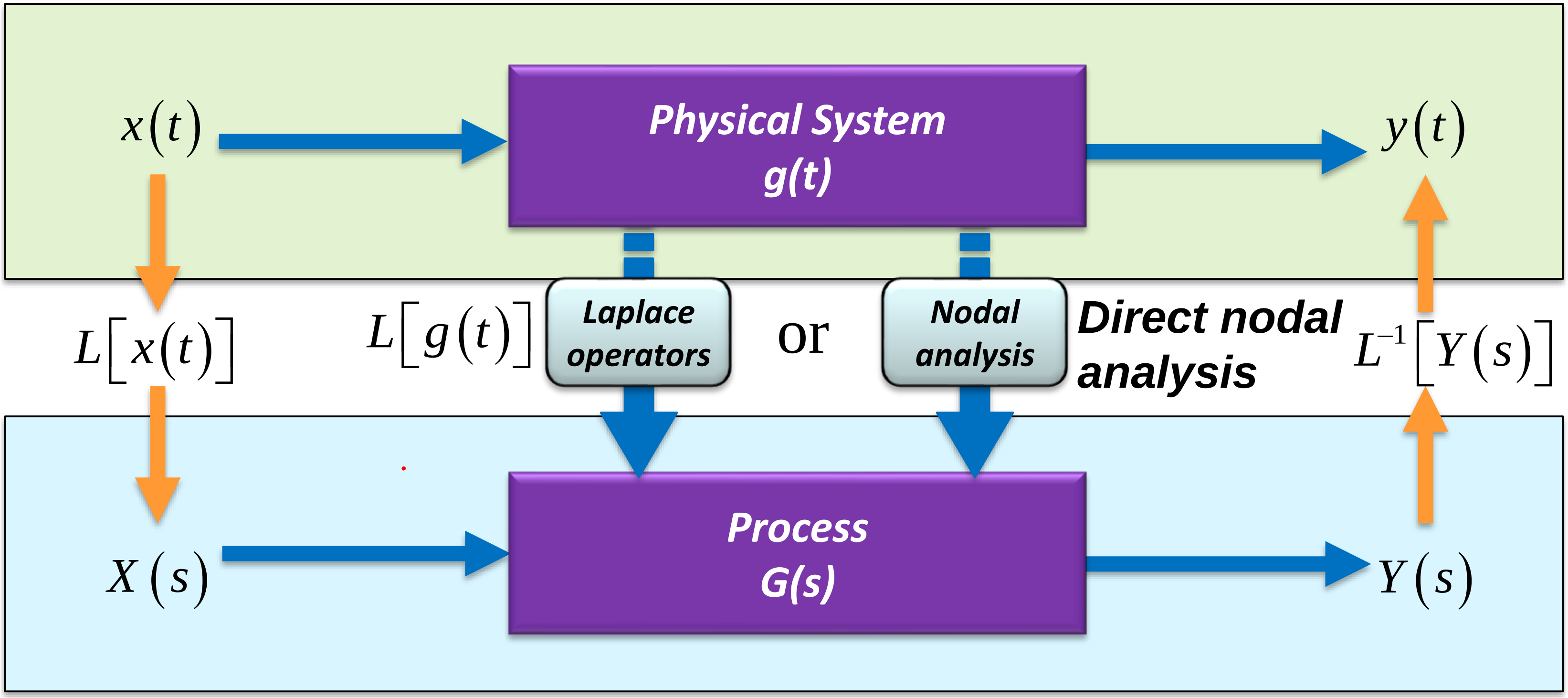
8. $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, F(s)$ FINAL VALUE

9. $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\, F(s)$ INITIAL VALUE

10. $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\, ds$ COMPLEX INTEGRATION

11. $L[f_1(t) * f_2(t)] = F_1(s).F_2(s)$ CONVOLUTION

The use of Laplace transforms and Transfer functions in system analysis



The relationship involving $g(t)$ is CONVOLUTION(harder) $\quad y(t) = \int_{-\infty}^{\infty} x(\tau) g(t-\tau) d\tau$

The relationship involving $G(s)$ is MULTIPLICATION(easier) $\quad Y(s) = G(s)\ X(s)$

Transfer Functions

- The *Transfer Function* of a circuit is normally given by:

$$G(s) = \frac{V_o(s)}{V_{in}(s)}$$

- Generally a transfer function is of the form of the ratio of a numerator and denominator polynomial in *s*.

Transfer Functions

- Example transfer functions:

$$G(s)=\frac{2}{s+2};\qquad \mathrm{G}(s)=\frac{1}{s^2+\sqrt{2}s+1};\qquad G(s)=\frac{s}{s+1}$$

- When realised using resistors, capacitors, inductors and active devices (op amps), numerator and denominator will be polynomials with real coefficients:

$$G(s)=\frac{a_n s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+a_0}{b_m s^m+b_{m-1}s^{m-1}+b_{m-2}s^{m-2}+\ldots+b_0}$$

where $a_{0..n}$ and $b_{0..m}$ are real coefficien ts

Poles and Zeros

- Can factor the numerator and denominator into a product of zeros and poles:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+a_i)}{\prod_{i=1}^{m}(s+b_i)}$$

zeros

poles

- For example:

$$G(s)=\frac{10}{(s+1)(s+2)}; \qquad G(s)=\frac{1}{\left(s+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)\left(s+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}\right)}; \qquad G(s)=\frac{2(s+3)}{(s+1)(s+2)}$$

Poles and Zeros

- The zeros are in the denominator.
- They are the values for which the transfer function becomes 0.
- For example:

$$G(s)=\frac{(s+1)}{(s+2)(s+3)}$$

- Zero is: $s=-1$

Poles and Zeros

- The poles are in the denominator.
- They are the values for which the transfer function becomes unbounded (infinite).
- For example:

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$

- Poles are: $s = -2; \quad s = -3$

Poles and Zeros

- More examples:

$$G(s)=\frac{10}{(s+1)(s+2)};\qquad G(s)=\frac{1}{\left(s+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)\left(s+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}\right)};\qquad G(s)=\frac{2(s+3)}{(s+1)(s+2)}$$

poles are:

-1

-2

poles are:

$-\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}$

$-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}$

poles are:

-1

-2

zero is:

-3

Poles and Zeros

- If there are complex poles and zeros then they must occur in complex conjugate pairs.
- For example:

$$G(s) = \frac{1}{s^2 + \sqrt{2}s + 1} = \frac{1}{\left(s + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)\left(s + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)}$$

- Poles and zeros in complex conjugate pairs can be represented as quadratic terms.

Poles and Zeros

- We could therefore think of the numerator and denominator of a transfer function as being able to be factored into first and second order terms:

$$G(s) = K \frac{\prod_{i=1}^{n}(s+a_i)}{\prod_{i=1}^{m}(s+b_i)} \times \frac{\prod_{i=1}^{p}(s^2+c_i s+d_i)}{\prod_{i=1}^{q}(s^2+e_i s+f_i)}$$

First order poles and zeros

quadratic poles and zeros

Poles and Zeros – Summary

G(s) can be expressed as:

$$G(s) = \frac{P(s)}{Q(s)} = \frac{a_w s^w + a_{w-1} s^{w-1} + \cdots + a_0 s^0}{s^n + b_{n-1} s^{n-1} + \cdots + b_0 s^0}$$

The highest power of s in the numerator, w, is the degree of P(s), and is the number of zeros of C(s)

The highest power of s in denominator, n, is the degree of Q(s) and is the number of poles of G(s)

The Characteristic Equation

Q(s) = 0

Roots of this equation are the location of the poles, or singularities of the system, and determine the character of the response.

Roots of P(s) are the locations of zeros.

When s = pole, G(s) = ∞

When s = zero, G(s) = 0

The s-plane

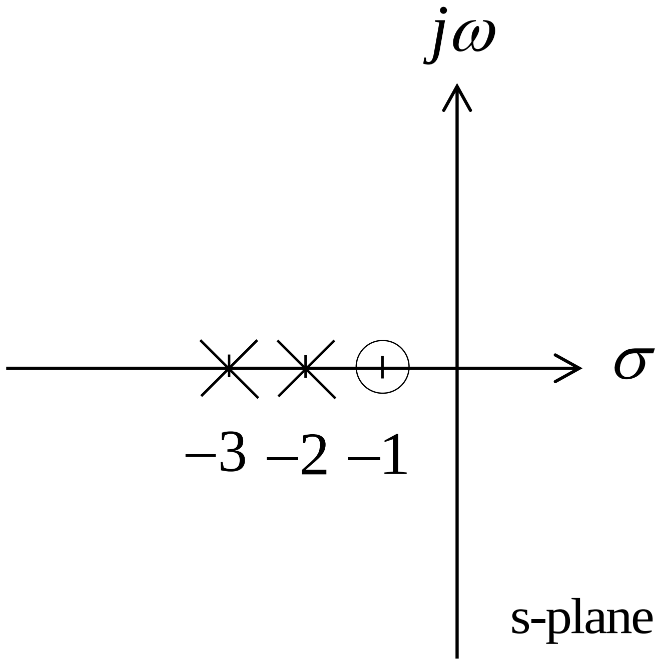
Roots of *P(s)* are zeros and plotted as "o" on the *s*-plane
Roots of *Q(s)* are poles and plotted as "x" on the *s*-plane

Poles and zeros for physical systems are always either real or complex conjugate pairs

Poles define the roots of the characteristic equation and determine the underlying characteristics of the time response

- For example:

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$

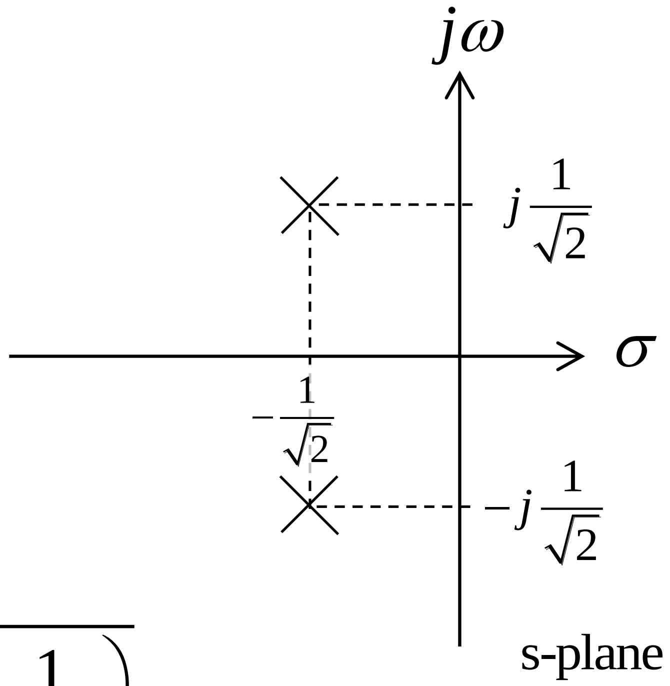


The s-plane

- Another example:

$$G(s)=\frac{1}{s^2+\sqrt{2}s+1}$$

$$G(s)=\frac{1}{\left(s+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)\left(s+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}\right)}$$



Factorisation

- Factorise the transfer function such that:

$$G(s) = \frac{P(s)}{(s+p_1)(s+p_2)(s+p_3)\ldots(s+p_n)}$$

$$G(s) = \frac{A_1}{(s+p_1)} + \frac{A_2}{(s+p_2)} + \frac{A_3}{(s+p_3)} + \ldots + \frac{A_n}{(s+p_n)}$$

- Then can find $g(t)$ by finding the inverse Laplace of each using tables.

Heaviside Partial Fraction Expansion

- Solving for $A_1,\ A_2,\ \dots\ A_n$:

$$G(s)=\frac{P(s)}{(s+p_1)(s+p_2)(s+p_3)\dots(s+p_n)}$$

$$G(s)=\frac{A_1}{(s+p_1)}+\frac{A_2}{(s+p_2)}+\frac{A_3}{(s+p_3)}+\dots+\frac{A_n}{(s+p_n)}$$

$$A_k=(s+p_1)G(s)\Big|_{s=-p_k}$$

Heaviside Partial Fraction Expansion – distinct roots example

- Example:

$$G(s)=\frac{4}{s(s+1)(s+2)}$$

$$G(s)=\frac{A}{s}+\frac{B}{(s+1)}+\frac{C}{(s+2)}$$

$$G(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

- Find $A,\ B,\ C$.

Heaviside Partial Fraction Expansion – distinct roots example

$$A = G(s)s\big|_{s=0} = \frac{4}{(0+1)(0+2)} = 2$$

$$B = G(s)(s+1)\big|_{s=-1} = \frac{4}{(-1)(-1+2)} = -4$$

$$C = G(s)(s+2)\big|_{s=-2} = \frac{4}{(-2)(-2+2)} = 2$$

$$G(s) = \frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

Heaviside Partial Fraction Expansion – distinct roots example

- Find the inverse Laplace transform of $G(s)$ in the previous example.

$$G(s)=\frac{2}{s}-\frac{4}{s+1}+\frac{2}{s+2}$$

$$g(t)=L^{-1}\left[G(s)\right]=L^{-1}\left[\frac{2}{s}-\frac{4}{s+1}+\frac{2}{s+2}\right]$$

$$g(t)=L^{-1}\left[\frac{2}{s}\right]-L^{-1}\left[\frac{4}{s+1}\right]+L^{-1}\left[\frac{2}{s+2}\right]$$

$$g(t)=2u(t)-4e^{-t}u(t)+2e^{-2t}u(t)$$

Heaviside Partial Fraction Expansion – multiple roots example

- Example:

$$G(s)=\frac{4(s+1)}{s(s+2)^2}$$

$$G(s)=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

- Find A, B, C.
- For terms such as B, involving lower order terms of the multiple root, the method involves multiplying $G(s)$ by the term for the multiple pole and then differentiating (k times for kth lower order)

Heaviside Partial Fraction Expansion – multiple roots example

$$A = G(s)s\Big|_{s=0} = \frac{4(0+1)}{(0+2)^2} = 1$$

$$C = G(s)(s+2)^2\Big|_{s=-2} = \frac{4(-2+1)}{(-2)} = 2$$

$$B = \frac{d}{ds}\left(G(s)(s+2)^2\right)\Big|_{s=-2} = \frac{d}{ds}\left(\frac{4(s+1)}{s}\right)\Bigg|_{s=-2} = \frac{4s-4(s+1)}{s^2}\Bigg|_{s=-2} = \frac{-4}{s^2}\Bigg|_{s=-2} = -1$$

$$G(s) = \frac{1}{s} - \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

Heaviside Partial Fraction Expansion – multiple roots example

- Find the inverse Laplace transform of $G(s)$ in the previous example.

$$G(s)=\frac{1}{s}-\frac{1}{s+2}+\frac{2}{(s+2)^2}$$

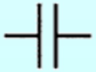
$$g(t)=L^{-1}\left[\frac{1}{s}-\frac{1}{s+2}+\frac{2}{(s+2)^2}\right]$$

$$g(t)=L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+2}\right]+L^{-1}\left[\frac{2}{(s+2)^2}\right]$$

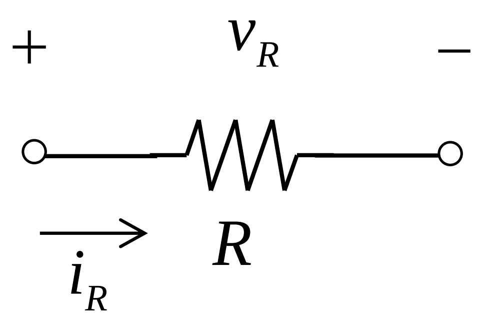
$$g(t)=u(t)+e^{-t}u(t)+te^{-2t}u(t)$$

Circuit Elements in the s domain

- Derive these from the time domain relationship using Laplace transform properties.

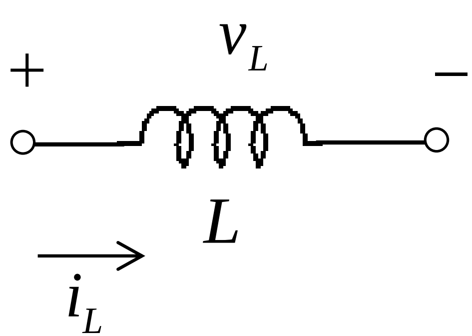
Element	Impedance (Z)	Admittance (Y)
	R	1/R
	sL	1/sL
	1/sC	sC

Circuit Elements in the s domain – Working out



$$v_R(t) = Ri_R(t); \quad \mathrm{V}_R(s) = RI_R(s); \quad \frac{\mathrm{V}_R(s)}{I_R(s)} = R$$

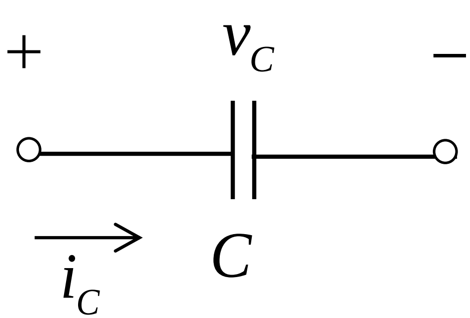
$$Z_R = R; \quad Y_R = \frac{1}{R}$$



$$v_L(t) = L\frac{di_L(t)}{dt}; \quad \mathrm{V}_L(s) = sLI_L(s) - Li_L(0^-);$$

with zero initial conditions:

$$\frac{V_L(s)}{I_L(s)} = sL; \quad Z_L = sL; \quad Y_L = \frac{1}{sL}$$



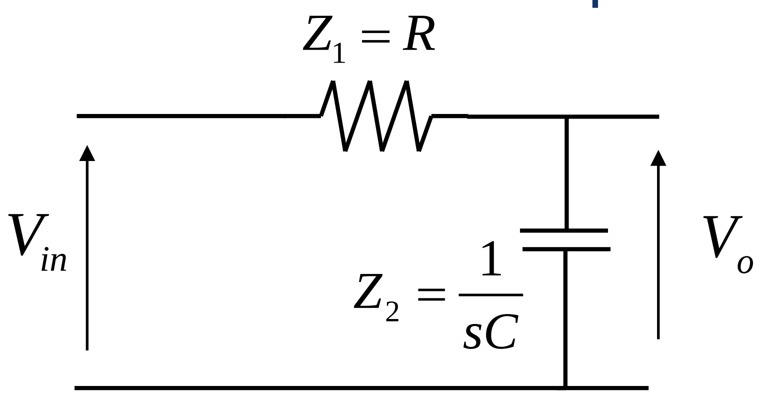
$$i_C(t) = C\frac{dv_C(t)}{dt}; \quad I_C(s) = C\left[sV_C(s) - v_C(0^-)\right]$$

with zero initial conditions:

$$\frac{V_C(s)}{I_C(s)} = \frac{1}{sC}; \quad Z_C = \frac{1}{sC}; \quad Y_C = sC$$

Circuit Analysis – RC Circuit

1. Find the relationship between the output and the input in the time domain and then use Laplace transforms to find the ratio of the output to the input in the s domain.



Assume zero initial condition.

Node equation at V_O :

$$\frac{v_{in}(t) - v_O(t)}{R} = C\frac{d}{dt}v_o(t)$$

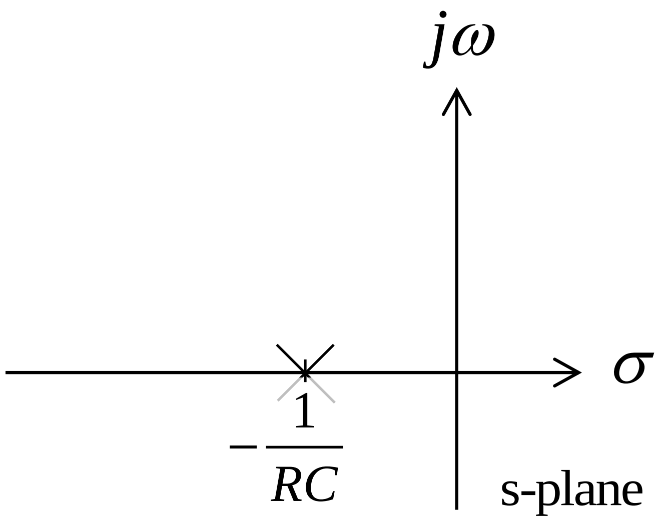
$$v_O(t) + RC\frac{d}{dt}v_o(t) = v_{in}(t)$$

Take Laplace transforms:

$$V_O(s) + RCV_o(s) = V_{in}(s)$$

$$\frac{V_O(s)}{V_{in}(s)} = \frac{1}{1 + sCR}$$

Circuit Analysis – RC Circuit



$$G(s) = \frac{V_O(s)}{V_{in}(s)} = \frac{1}{1+sCR} = \frac{\frac{1}{CR}}{s+\frac{1}{CR}}$$

Find the impulse response:

$$v_{in}(t) = \delta(t) \qquad V_{in}(s) = 1$$

$$V_O(s) = G(s)V_{in}(s) = \frac{\frac{1}{CR}}{s+\frac{1}{CR}} \times 1 = \frac{\frac{1}{CR}}{s+\frac{1}{CR}}$$

$$g(t) = v_O(t) = L^{-1}\left[V_O(s)\right] = \frac{1}{CR} e^{-\frac{t}{CR}} u(t)$$

Circuit Analysis – RC Circuit

$$G(s) = \frac{V_O(s)}{V_{in}(s)} = \frac{\frac{1}{CR}}{s + \frac{1}{CR}}$$

Find the step response: $v_{in}(t) = u(t)$ $V_{in}(s) = \frac{1}{s}$

$$V_O(s) = G(s)V_{in}(s) = \frac{\frac{1}{CR}}{s\left(s + \frac{1}{CR}\right)}$$

$$v_O(t) = L^{-1}\left[V_O(s)\right] = \left(1 - e^{-\frac{t}{CR}}\right)u(t)$$

Circuit Analysis in the s domain

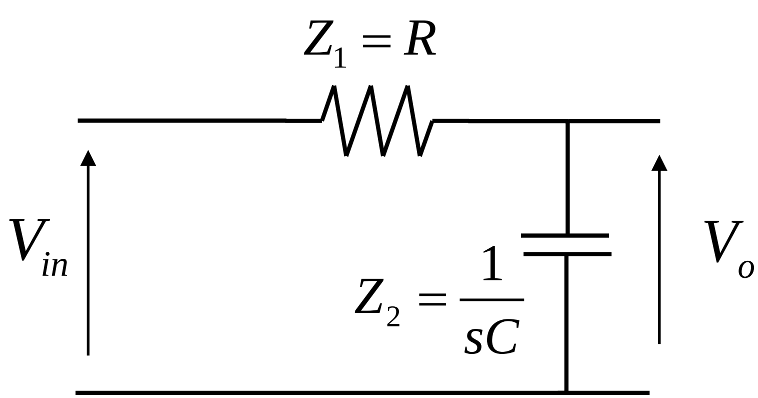
- Mesh analysis or Node analysis or any of the circuit theorems such as Thévenin or Norton can be applied after converting to the s domain. Usually the input and output node voltages are of interest (in a 2 port network)

- Systematic method for nodal analysis
  1. # equations = # unknown node voltages
  2. Equation written for each node other than the reference ground node (assumed 0).
  3. For each node, the equation is given by:

$$\text{Algebraic sum of currents entering the node} = 0$$

Circuit Analysis – RC Circuit

2. Find the transfer function for the RC circuit shown using systematic nodal analysis. .



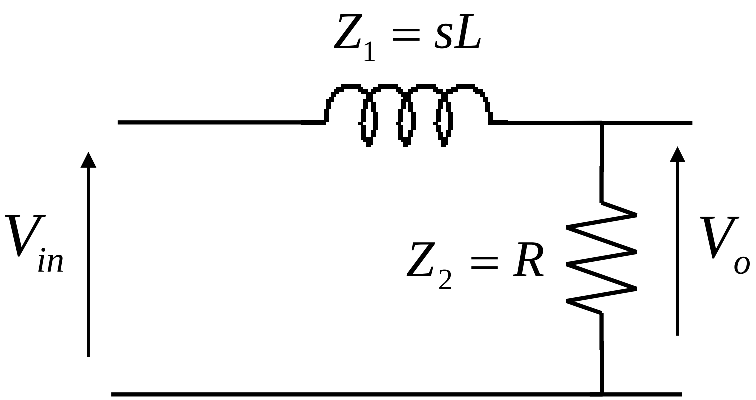
$$V_O(s)\left(\frac{1}{R} + sC\right) - \frac{V_{in}(s)}{R} = 0$$

$$V_O(s)\left(\frac{1+sCR}{R}\right) = \frac{V_{in}(s)}{R}$$

$$\frac{V_O(s)}{V_{in}(s)} = \frac{1}{1+sCR} = \frac{\frac{1}{CR}}{s + \frac{1}{CR}}$$

Circuit Analysis – RL Circuit

- Find the transfer function for the general RL circuit shown using systematic nodal analysis.



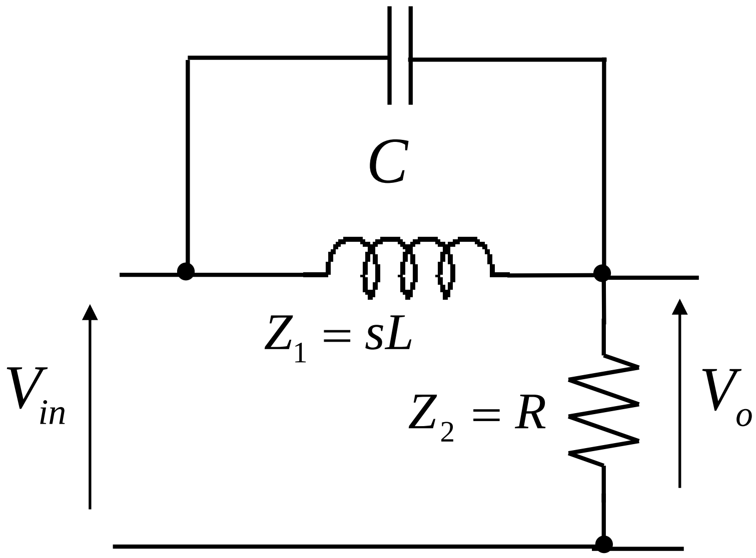
$$V_O(s)\left(\frac{1}{sL}+\frac{1}{R}\right)-\frac{V_{in}(s)}{sL}=0$$

$$V_O(s)\left(\frac{R+sL}{sLR}\right)=\frac{V_{in}(s)}{sL}$$

$$\frac{V_O(s)}{V_{in}(s)}=\frac{R}{R+sL}=\frac{\frac{R}{L}}{s+\frac{R}{L}}$$

Circuit Analysis – Parallel RLC Circuit

- Find the transfer function for the general parallel RLC circuit shown using systematic nodal analysis.



$$V_O(s)\left(\frac{1}{sL}+\frac{1}{R}+sC\right)-V_{in}(s)\left(\frac{1}{sL}+sC\right)=0$$

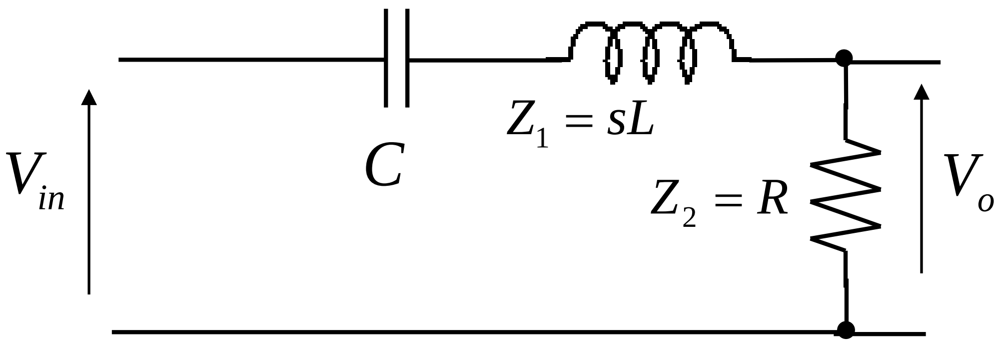
$$V_O(s)\left(\frac{R+sL+s^2CLR}{sLR}\right)=V_{in}(s)\left(\frac{1+s^2CL}{sL}\right)$$

$$\frac{V_O(s)}{V_{in}(s)}=\frac{R+s^2CLR}{R+sL+s^2CLR}$$

$$\frac{V_O(s)}{V_{in}(s)}=\frac{s^2+\frac{1}{CL}}{s^2+\frac{1}{CR}s+\frac{1}{CL}}$$

Circuit Analysis – Series RLC Circuit

- Find the transfer function for the general series RLC circuit shown using systematic nodal analysis.



Easier to use voltage divider:

$$V_O(s) = \frac{R}{\left(sL + \frac{1}{sC} + R\right)} V_{in}(s)$$

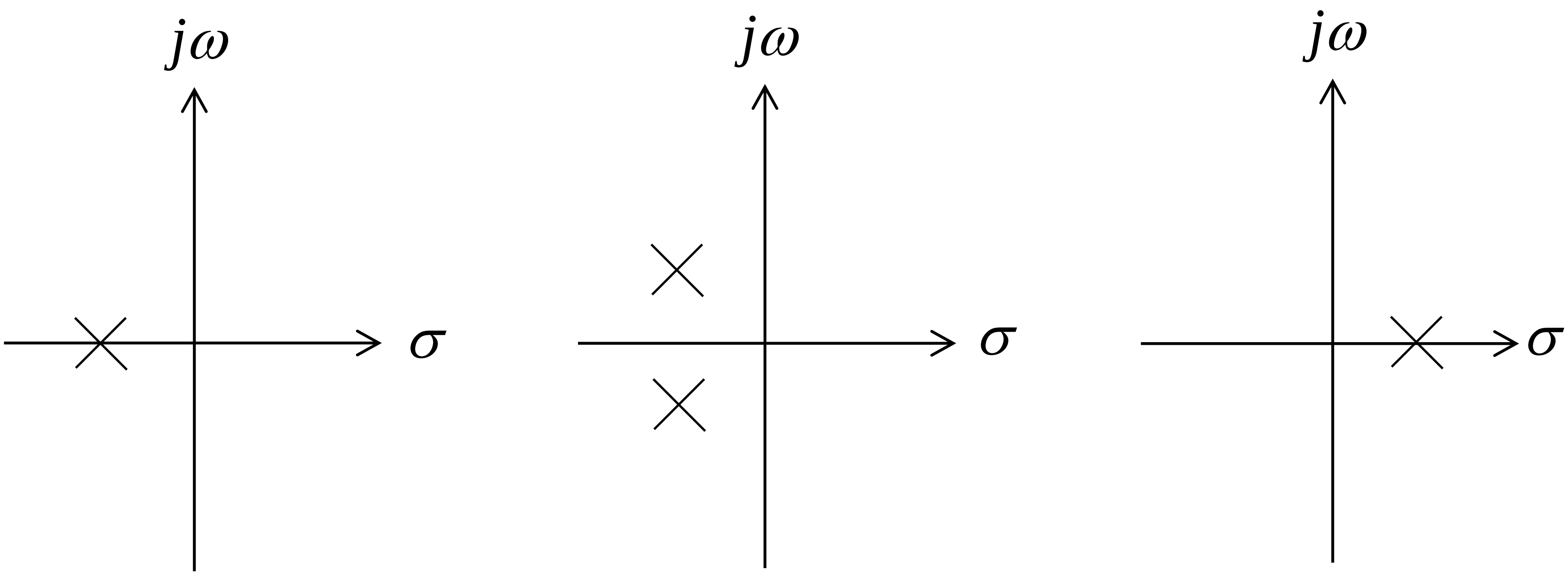
$$\frac{V_O(s)}{V_{in}(s)} = \left(\frac{sCR}{s^2LC + 1 + sCR}\right)$$

$$\frac{V_O(s)}{V_{in}(s)} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{CL}}$$

Transfer function and stability

- How do we know if a system is stable?

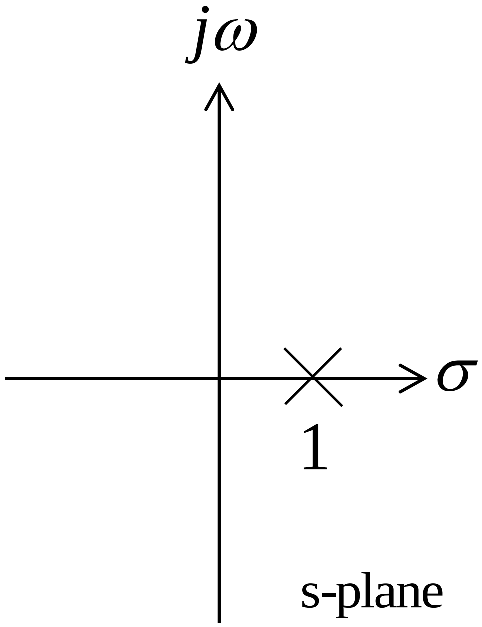
  Poles are on the left half of the s-plane.



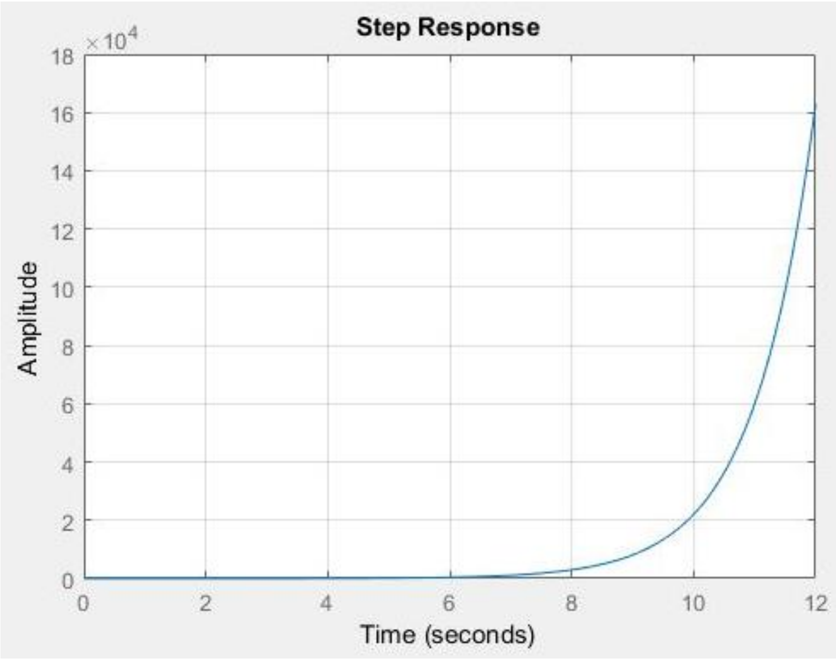
Stable? YES   Stable? YES   Stable? NO

Transfer function and stability

- Example:



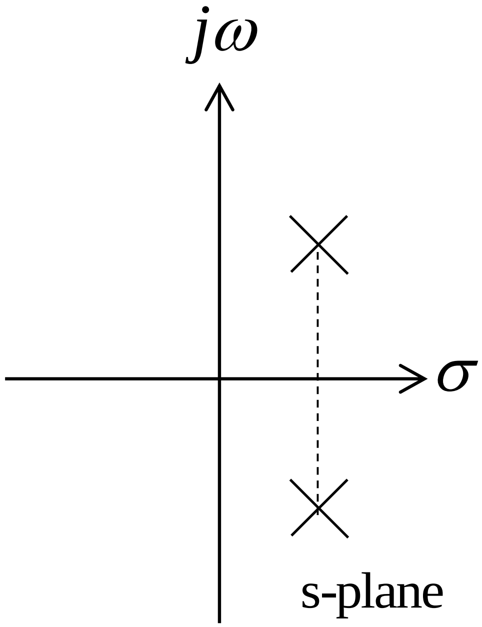
$$G(s) = \frac{1}{s-1}$$

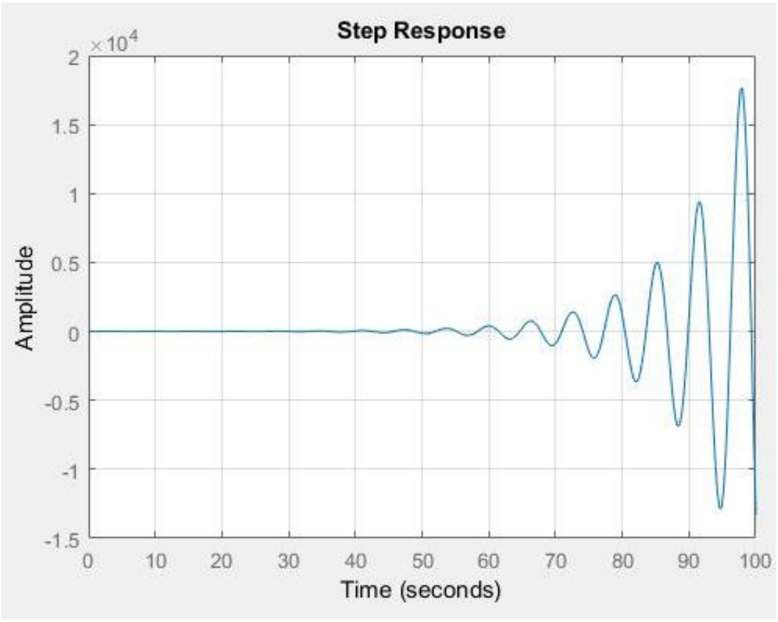


Output goes to infinity

Transfer function and stability

- Example:

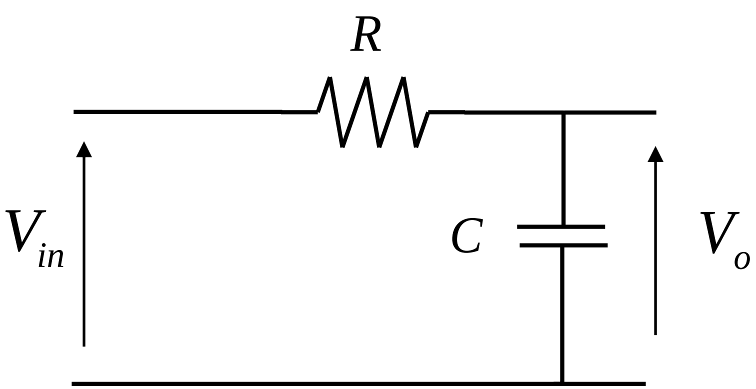




$$G(s) = \frac{1}{s^2 - 0.2s + 1}$$

Output oscillates

Time Response – First Order RC Circuit



Transfer function:

$$\frac{V_O(s)}{V_{in}(s)} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \qquad G(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}} \quad \text{where} \quad \tau = RC$$

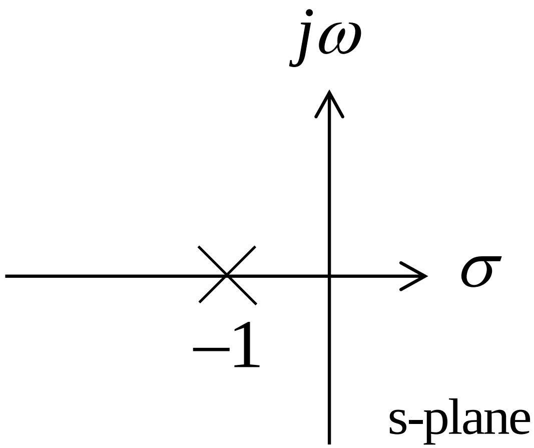
$$\frac{V_O(s)}{V_{in}(s)} = \frac{a}{s+a} \quad (\text{form used in tables})$$

Time Response – First Order RC Circuit

Prototype components:

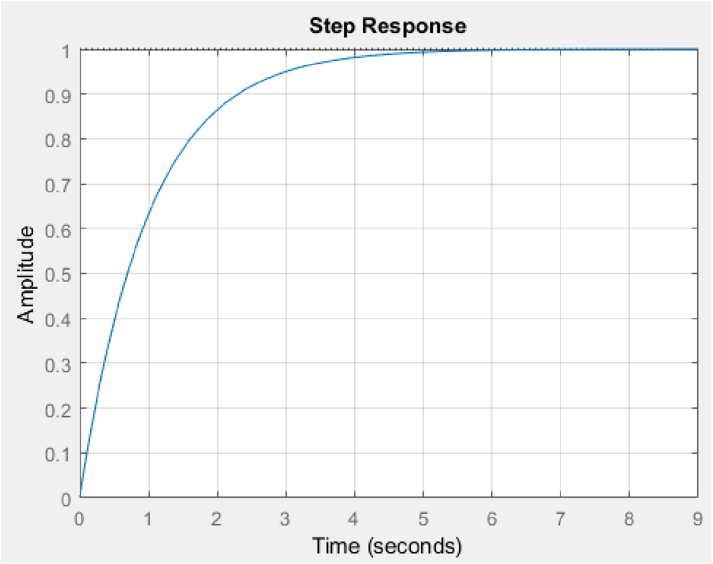
$$C = 1\ \mathrm{F}$$

$$R = 1\ \Omega$$



Transfer function:

$$\frac{V_O(s)}{V_{in}(s)} = \frac{1}{s+1}$$



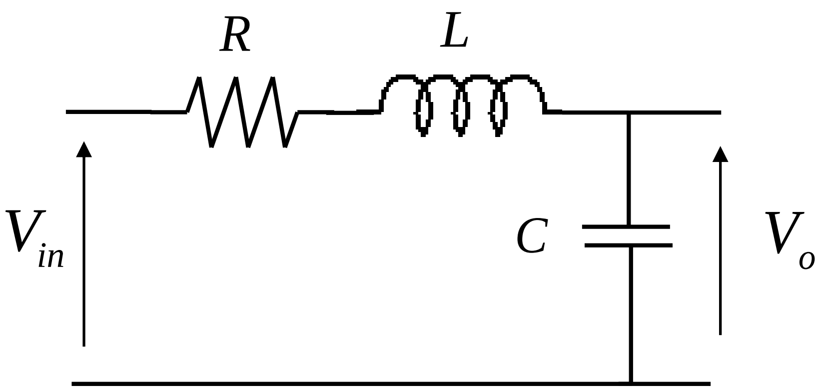
Unit step response:

$$v_o(t) = \left(1 - e^{-at}\right)u(t)$$

At $t = \tau$:

$$v_o(t) = 1 - e^{-t} = 0.632$$

Time Response – Second Order RLC Circuit



Transfer function:

$$V_O(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} V_{in}(s)$$

$$\frac{V_O(s)}{V_{in}(s)} = \frac{\frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

Standard form:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{1}{LC}} \qquad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

ω_n = natural frequency

ζ = damping ratio

Time Response – Second Order RLC Circuit

Poles:

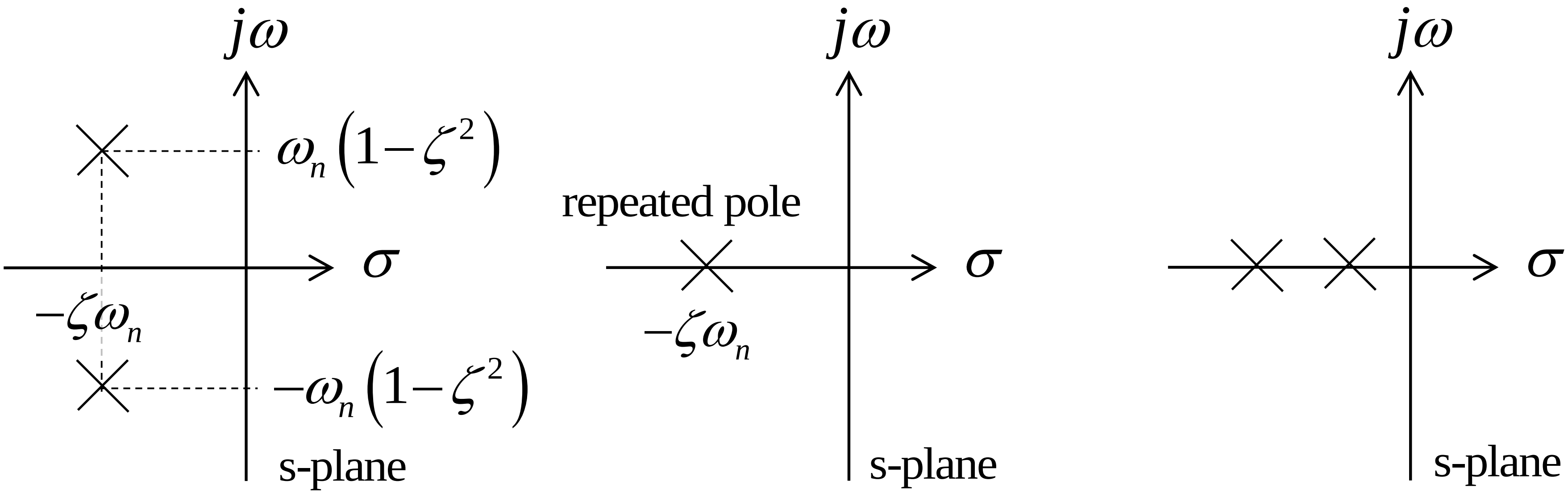
$$s=\frac{-2\zeta\omega_n \pm \sqrt{\left(2\zeta\omega_n\right)^2-4\times 1\times \omega_n^2}}{2\times 1}$$

$$s=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$s=\begin{cases} -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} & 0<\zeta<1 \quad \text{underdamped} \\ -\zeta\omega_n,\ \ -\zeta\omega_n & \zeta=1 \quad \text{critically damped} \\ -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2} & \zeta>1 \quad \text{overdamped} \end{cases}$$

51

Time Response – Second Order RLC Circuit



underdamped critically damped overdamped

Time Response – Second Order RLC Circuit

Prototype components:

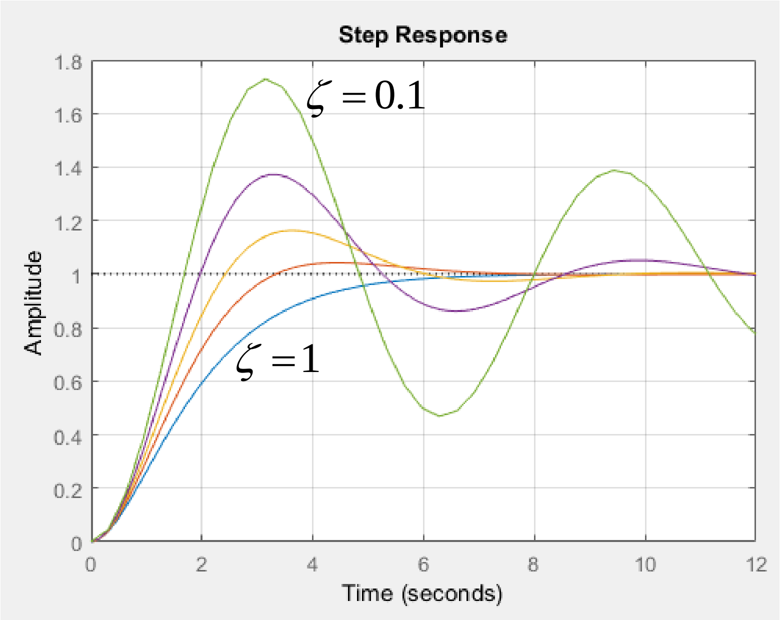
$$C = 1\ \text{F}$$

$$L = 1\ \text{H}$$

$$R = 2,\ \sqrt{2},\ 1.0,\ 0.6,\ 0.2\ \Omega$$

damping ratio

$$\zeta = \underbrace{1}_{\text{critically damped}},\ \underbrace{\frac{1}{\sqrt{2}},\ 0.5,\ 0.3,\ 0.1}_{\text{underdamped}}$$



Time Response – Second Order RLC Circuit

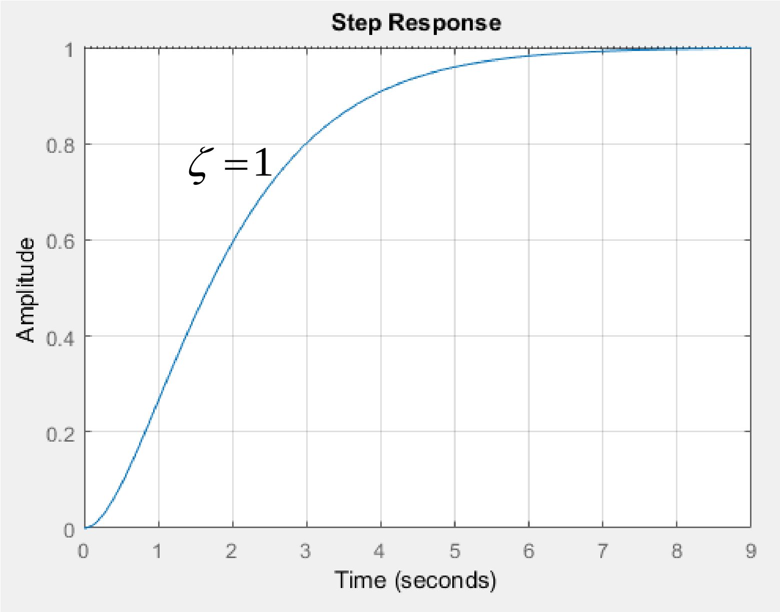
Prototype components:

$$C = 1\ \text{F}$$
$$L = 1\ \text{H}$$
$$R = 2\ \Omega$$

damping ratio

$$\zeta = 1$$

critically damped



Transfer function:

$$\frac{V_O(s)}{V_{in}(s)} = \frac{1}{s^2 + 2s + 1}$$

Time Response – Second Order RLC Circuit

Prototype components:

$$C = 1\ \mathrm{F}$$

$$L = 1\ \mathrm{H}$$

$$R = 2,\ 4,\ 10,\ 20\ \Omega$$

damping ratio

$$\zeta = \underbrace{1}_{\text{critically damped}},\ \underbrace{2,\ 5,\ 10}_{\text{overdamped}}$$