System Synthesis: Cart-Pendulum System

Lab 07 - EGH445 Modern Control - 2020

Electrical Engineering & Robotics (EER)
Queensland University of Technology

September 10, 2020

Important: Demonstration 1

Demonstration 1 requires you to **build** and **demonstrate** a **complete** and working **control system** for a continuous time **cart-pendulum model** in MATLAB/Simulink. Your demonstrations will be assessed **individually** and conducted during your usual **computer lab session**.

Preparation: You are required to create the system outlined in **Section 1** of this computer lab. Read the lab carefully and follow the instructions to complete the control system design.

Demonstration: You will be required to **run two scripts** created from this lab. These scripts are listed below and should automatically produce Figures 4 and 5 from this lab.

- `CP_ContrSys_Lin_a_MainFile_yourstudentnumber.m`
- `CP_ContrSys_Lin_b_MainFile_yourstudentnumber.m`

You will then be asked a **few questions** (orally) of increasing complexity to assess your understanding of the scripts, Simulink models and the concepts used in control system design.

Marking: The marks for the demonstration are allocated as follows: 35% for the scripts, 15% for the functions (correct expressions in fcn blocks and connections in Simulink model), and 50% for the questions.

⋆⋆ **Optional:** In addition to completing the tasks outlined in Section 1, consider building a new system outlined in **Section 2** of this computer lab. Create **two additional scripts** listed below to automatically produce equivalent figures to those shown in Figures 4 and 5 for the new system. Adequate demonstration of this system may receive an additional 10% for the assessment item. You cannot receive a total score for the demonstration greater than 100%.

- `CP_ContrSysDamp_Lin_a_MainFile_yourstudentnumber.m`
- `CP_ContrSysDamp_Lin_b_MainFile_yourstudentnumber.m`

Hint: *For more guidance, see Computer Labs 4 - 6. Consider attempting Lab 7 (this lab) before your in class session so you can use your lab session to refine your scripts and simulation models with help from the teaching team.*

Objectives

The objective of this Lab is to design and simulate in Matlab-Simulink a control system for the cart-pendulum system using position measurement only. The idea is to connect each required system component or synthesis the full control system (i.e. system model, controller and observer).

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1 Inverted pendulum on a cart

We use the cart-pendulum system as a benchmark to design a control system *using position measurements only.* The design is based on a linearised approximation, thus simulation of the nonlinear system is required to assess the performance of the closed loop, and the region of attraction ensured by the controller. Figure 1 shows the idealised model of the cart-pendulum system as used in previous labs. The system consists of a pendulum of mass m and length ℓ attached to a cart of mass M_c. The pendulum moves under the action of the gravity (g) and the cart moves on the horizontal direction and is actuated by the control force F. The state-space model can be written as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{x_4^2 \ell m \sin(x_2) - gm \sin(x_2)\cos(x_2) + F}{M_c + m - m\cos^2(x_2)} \\ \frac{-\ell m \sin(x_2)\cos(x_2) x_4^2 + g(M_c + m)\sin(x_2) - \cos(x_2) F}{\ell[M_c + m - m\cos^2(x_2)]} \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2}$$

where the states are

- x_1: the position of the cart,
- x_2: the angle of the pendulum,
- x_3: the velocity of the cart,
- x_4: the angular velocity of the pendulum.

We consider the values of the model parameters given in Table 1.

Table 1: Model parameters.

Parameter	value
m	0.15kg
M_c	0.4kg
ℓ	0.2m
g	9.81m/s^2

1.1 Problem formulation.

In this lab, we consider the control design problem for the cart-pendulum system using position measurement only. The control system is designed using tools for linear systems. We consider two equilibriums

$$\bar{x}_a = \begin{bmatrix} \bar{x}_{1a} \\ \bar{x}_{2a} \\ \bar{x}_{3a} \\ \bar{x}_{4a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \qquad \bar{x}_b = \begin{bmatrix} \bar{x}_{1b} \\ \bar{x}_{2b} \\ \bar{x}_{3b} \\ \bar{x}_{4b} \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

The objective is to use the controllers and observers designed in previous labs to stabilise the equilibrium points $\bar{x}_a$ and $\bar{x}_b$.

Remark 1. *Notice that since we design a controller using the linearised model, we have to simulate the nonlinear model of the cart-pendulum system to assess the performance of the control system and study the stability of the desired equilibrium. Extensive simulation analysis should be conducted to study the region of convergency of the closed-loop nonlinear system. Alternatively, we can design the control system using nonlinear techniques, however these techniques are outside of the scope of our unit EGH445-Modern Control.*

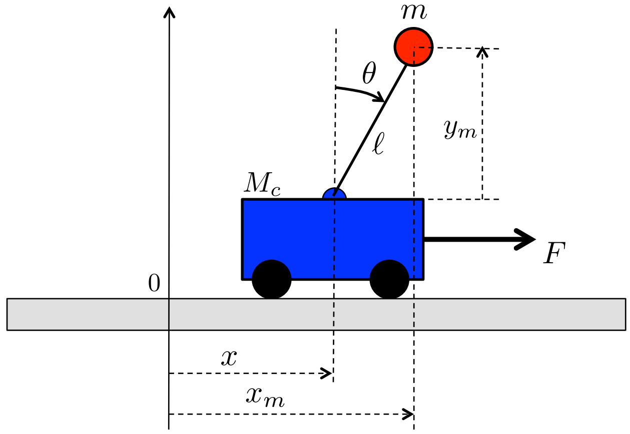

Figure 1: Cart-pendulum system.

1.2 Stabilisation of the desired equilibrium $\bar{x}_a$ using position measurements.

The task in this section is to design a control system to stabilise the desired equilibrium $\bar{x}_a$ using the linearised model under the assumption that the position of the cart and the angle of the pendulum are the only measurements available. Consider the controller and the observer designed in previous labs and use the states estimated by the observer to close the loop for the linearised model and the nonlinear models as shown in Figure 2 and Figure 3 respectively.

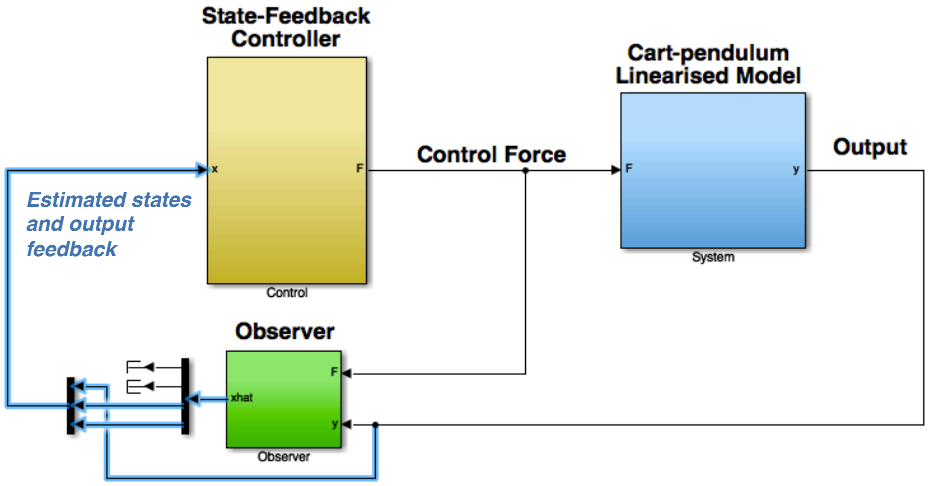

Figure 2: Linearised control system.

Write a script that performs the following tasks (build on scripts created in previous labs):

a) Define the parameters of the model.

b) Compute the matrices of the linearised model.

c) Determine if the linearised model is controllable and observable by computing the controllability and observability matrices and checking their rank. Hint: use the commands `ctrb` and `obsv`.

d) *Controller design.* Compute the controller $u = -K_a\,\tilde{x}_a$ such that the eigenvalues of the (linearised) closed loop are $\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = -5$ and $\lambda_4 = -6$.

e) *Observer design.* Compute the observer gain `La` such that the eigenvalues of the observer error dynamics are $\lambda_1 = -63$, $\lambda_2 = -64$, $\lambda_3 = -65$ and $\lambda_4 = -66$.

f) Simulate the linearised and nonlinear closed-loop systems:

   i) Linearised control system. Build a Simulink model of the linearised system in closed loop with the state-feedback controller and the observer designed in Lab 05 and Lab 06. Use the estimation of the system states $\hat{x}$ and the measurements to close the feedback loop. An example of the Simulink model is shown in Figure 2. Save your model as `CP_ContrSys_Lin_a_yourstudentnumber.slx`.

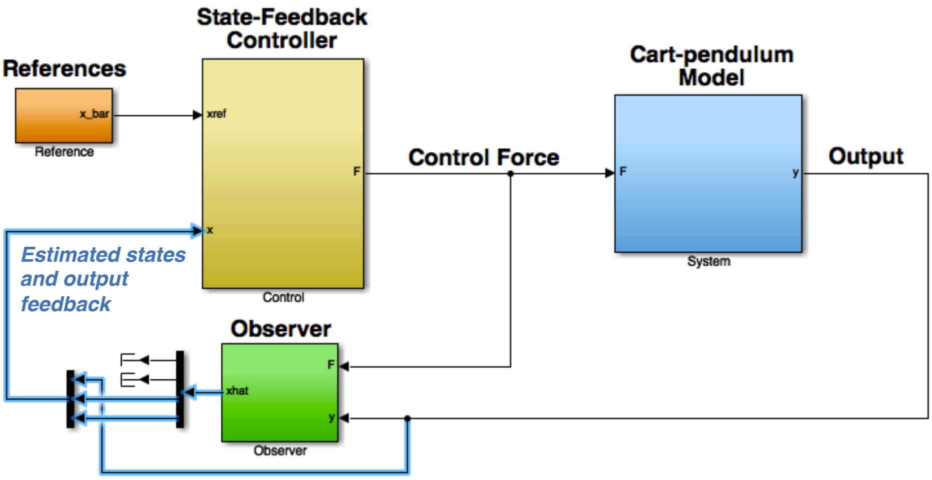

Figure 3: Nonlinear control system.

ii) *Nonlinear control system.* Build a Simulink model of the nonlinear system in closed loop with the state-feedback controller and the observer designed in Lab 05 and Lab 06. Use the estimation of the system states $\hat{x}$ and the measurements to close the feedback loop. An example of the Simulink model is shown in Figure 3. Save your model as `CP_ContrSys_NLin_yourstudentnumber.slx`. Contemplate the possibility of parametrising the controller and observer of the nonlinear control system for either the equilibrium $\bar{x}_a$ or the equilibrium $\bar{x}_b$.

iii) Export the states of the system, the estimation of the states, the control force and the simulation time from Simulink to Matlab.

iv) Simulate both the linearised and the nonlinear closed-loop models with the cart stating at 0.2m and the pendulum at 20deg. Set the initial conditions of the velocities to zero. That is $x(0) = \begin{bmatrix} 0.2 & 20\pi/180 & 0 & 0 \end{bmatrix}^\top$. Suggestion for the simulation: use the fixed-step solver `ode4` and select 0.02 as step time.

g) Plot the results of the simulations in a figure that shows the time histories of the position of the cart, the angle of the pendulum, the velocity of the cart, the angular velocity of the pendulum, the estimation of the states and the control force for both the linearised and nonlinear control systems. An example of the simulation results is shown in Figure 4.

h) Save the script as `CP_ContrSys_Lin_a_MainFile_yourstudentnumber.m`.

i) Run the script using different initial conditions for the simulation and analyse the results. The initial conditions should be defined in the script.

j) (Optional) Use the function `Cart_Pendulum_Animation.m` to create an animation of your control system.

Important: The plots and animation should be produced automatically when the script is executed without further user intervention.

1.3 Stabilisation of the desired equilibrium $\bar{x}_b$ using position measurements.

The task in this section is to design a control system to stabilise the desired equilibrium $\bar{x}_b$ using the linearised model under the assumption that the position of the cart and the angle of the pendulum are the only measurements available. Consider the controller and the observer designed in previous labs and use the states estimated by the observer to close the loop for the linearised model and the nonlinear models. The Simulink block diagrams in this section are similar to the block diagrams in Section 1.2.

Write a script that performs the following tasks (build on scripts created in previous labs):

a) Define the parameters of the model.

b) Compute the matrices of the linearised model.

c) Determine if the linearised model is controllable and observable by computing the controllability and observability matrices and checking their rank. Hint: use the commands `ctrb` and `obsv`.

d) *Controller design.* Compute the controller $u = -K_b\,\tilde{x}_b$ such that the eigenvalues of the (linearised) closed loop are $\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = -5$ and $\lambda_4 = -6$.

e) *Observer design.* Compute the observer gain `Lb` such that the eigenvalues of the observer error dynamics are $\lambda_1 = -63$, $\lambda_2 = -64$, $\lambda_3 = -65$ and $\lambda_4 = -66$.

f) Simulate the linearised and nonlinear closed-loop systems:

   i) *Linearised control system.* Build a Simulink model of the linearised system in closed loop with the state-feedback controller and the observer designed in Lab 05 and Lab 06. Use the estimation of the system states $\hat{x}$ and the measurements to close the feedback loop. Save your model as `CP_ContrSys_Lin_b_yourstudentnumber.slx`.

   ii) *Nonlinear control system.* Build a Simulink model of the nonlinear system in closed loop with the state-feedback controller and the observer designed in Lab 05 and Lab 06. Use the estimation of the system states $\hat{x}$ and the measurements to close the feedback loop. Use the model `CP_ContrSys_NLin_yourstudentnumber.slx`.

   iii) Export the states of the system, the estimation of the states, the control force and the simulation time from Simulink to Matlab.

   iv) Simulate both the linearised and the nonlinear closed-loop models with the cart stating at 0.2m and the pendulum at 200deg. Set the initial conditions of the velocities to zero. That is $x(0) = \begin{bmatrix} 0.2 & 200\pi/180 & 0 & 0 \end{bmatrix}^\top$. Suggestion for the simulation: use the fixed-step solver `ode4` and select 0.02 as step time. Use the state of the system $\hat{x}$ to close the loop.

g) Plot the results of the simulations in a figure that shows the time histories of the position of the cart, the angle of the pendulum, the velocity of the cart, the angular velocity of the pendulum, the estimation of the states and the control force for both the linearised and nonlinear control systems. An example of the simulation results is shown in Figure 5.

h) Save the script as `CP_ContrSys_Lin_b_MainFile_yourstudentnumber.m`.

i) Run the script using different initial conditions for the simulation and analyse the results. The initial conditions should be defined in the script.

j) (Optional) Use the function `Cart_Pendulum_Animation.m` to create an animation of your control system.

Important: The plots and animation should be produced automatically when the script is executed without further user intervention.

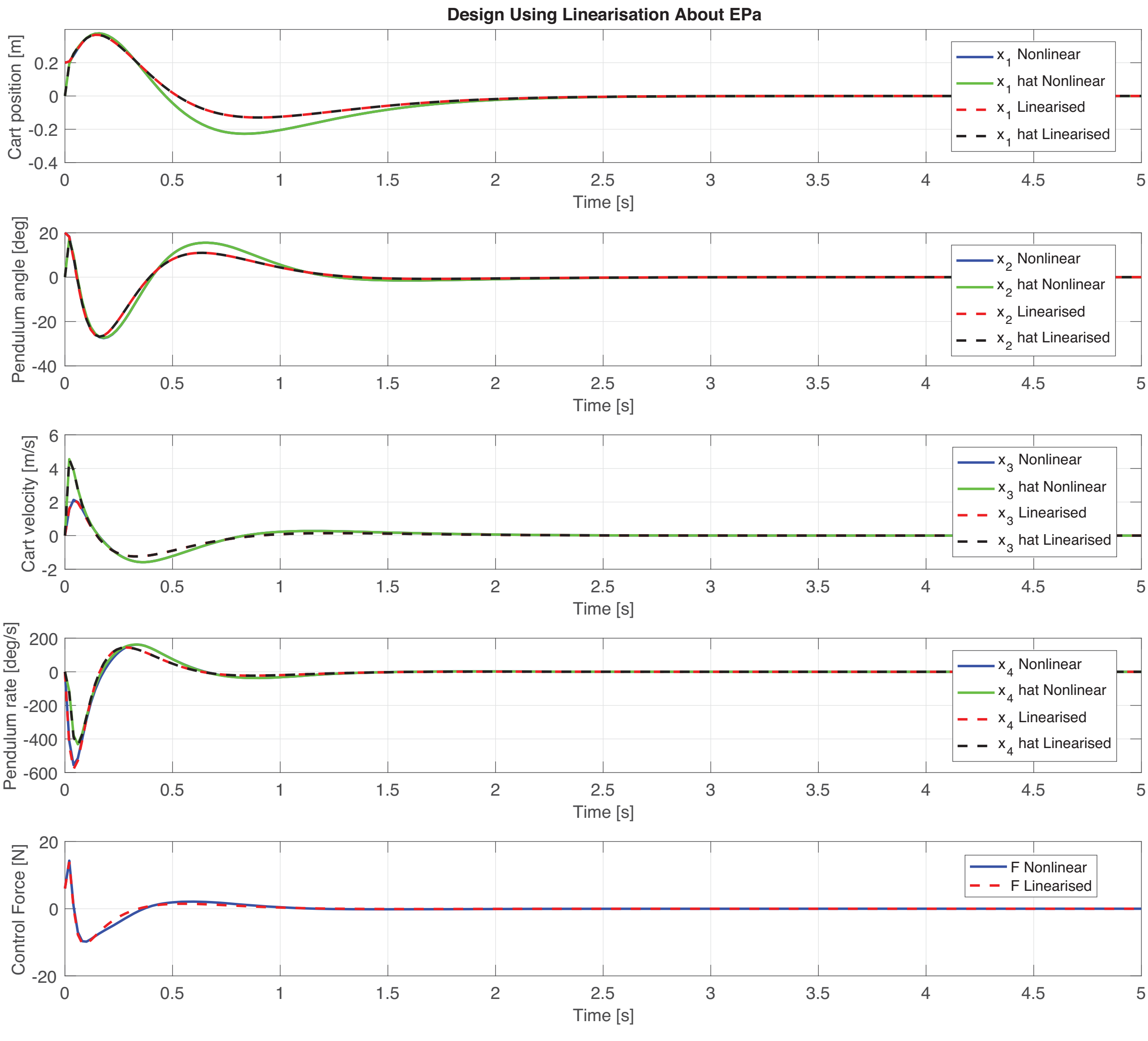

Figure 4: Time histories of the system states, the state estimations and control force.

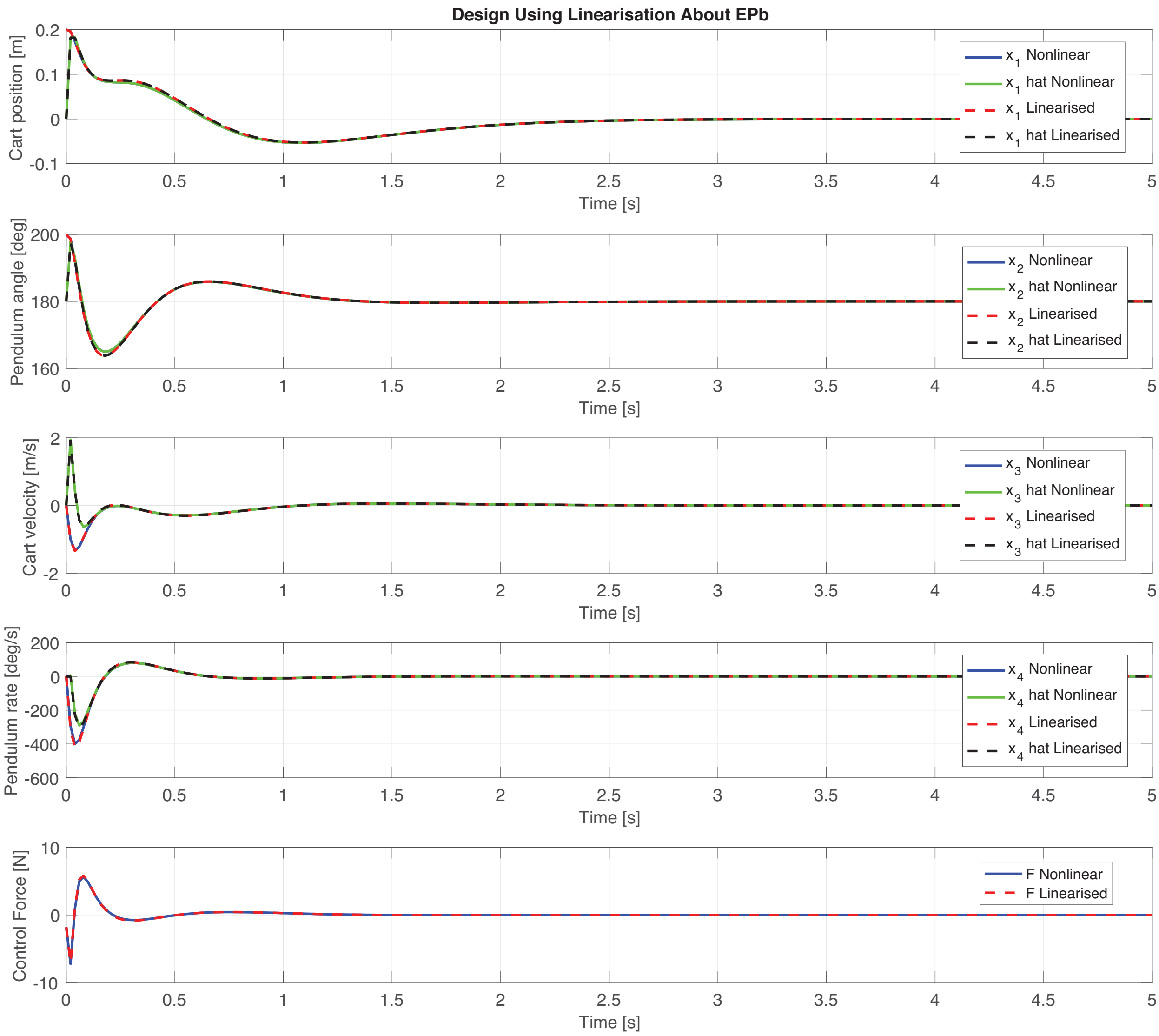

Figure 5: Time histories of the system states, the state estimations and control force.

2 [Optional] Model Reformulation

Over the past few computer labs we developed Matlab scripts and Simulink models for the cart-pendulum system utilising the equations of motion given by

$$(M_c + m)\,\ddot{q}_1 + m\,\ell\,\cos(q_2)\,\ddot{q}_2 - m\,\ell\,\sin(q_2)\,\dot{q}_2^2 \quad = \quad F, \tag{4}$$

$$m\,\ell\,\cos(q_2)\,\ddot{q}_1 + m\,\ell^2\,\ddot{q}_2 - m\,g\,\ell\,\sin(q_2) \quad = \quad 0, \tag{5}$$

where q_1 and q_2 are the position of the cart and the angle of the pendulum respectively, and g is the gravitational constant. However, if we simulate the non-linear model with no control force and initial conditions set to $\begin{bmatrix}0.2 & 20\pi/180 & 0 & 0\end{bmatrix}^\top$ the result is given by Figure 6. Can you notice anything odd about the results? *Hint:* look at the pendulum angle.

You should observe that the pendulum rotates continuously. This is not what we expect of the system, the pendulum should eventually stabilise and point down. Why does our model never stabilise? The answer, there is no damping component provided in the equations of motion above. Therefore we can remodel the system and provide a damping component by modelling some form of friction between the cart and the surface. The equations of motion must now include a damping coefficient $b = 1$ given by

$$(M_c + m)\,\ddot{q}_1 + m\,\ell\,\cos(q_2)\,\ddot{q}_2 - m\,\ell\,\sin(q_2)\,\dot{q}_2^2 + b\dot{q}_1 \quad = \quad F, \tag{6}$$

$$m\,\ell\,\cos(q_2)\,\ddot{q}_1 + m\,\ell^2\,\ddot{q}_2 - m\,g\,\ell\,\sin(q_2) \quad = \quad 0, \tag{7}$$

In control engineering, we often start with a simple system model and control design then evaluate its suitability to the real application. We then iteratively re-model the system and re-design the control system components to refine performance. As such, use the updated equations of motion to re-model and re-design your cart-pendulum control system. Using the steps outlined in **section 1** as a guide, perform the following:

1. Determine the non-linear state-space form (i.e. $\dot{\mathbf{x}} = \mathbf{M}$, where $\mathbf{M}$ is a matrix containing state variables, input values and constants only).

2. Determine the linear state-space model (i.e. $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$) by linearising the non-linear state-space model about equilibrium points $\bar{x}_a$ and $\bar{x}_b$.

3. Validate your new models via simulation with no input control force and appropriate initial conditions. Check the system eventually stabilises, similar to that shown in Figure 7 for the non-linear model.

4. Design a suitable state-feedback controller and observer for the new system model.

5. Simulate your complete control system design (using the linear and non-linear models) and compare your results for various initial conditions.

6. Save your designs in two new scripts:
   `CP_ContrSysDamp_Lin_a_MainFile_yourstudentnumber.m`
   `CP_ContrSysDamp_Lin_b_MainFile_yourstudentnumber.m`

   These scripts correspond to the equilibrium points $\bar{x}_a$ and $\bar{x}_b$ respectively for the cart-pendulm model. Similar to Section 1, these scripts will rely on Simulink models you must create in steps $(1) - (5)$:
   `CP_ContrSysDamp_NLin_yourstudentnumber.slx`
   `CP_ContrSysDamp_Lin_a_yourstudentnumber.slx`
   `CP_ContrSysDamp_Lin_b_yourstudentnumber.slx`

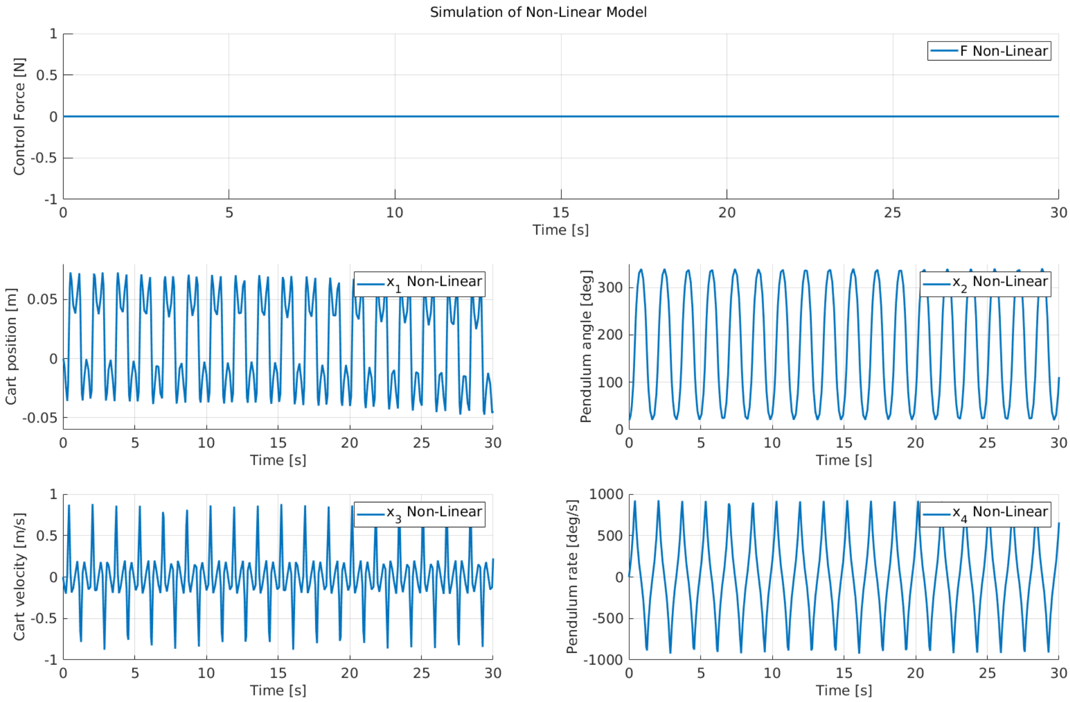

Figure 6: Time histories of the system states with zero control force and initial conditions of $\begin{bmatrix} 0 & 20\pi/180 & 0 & 0 \end{bmatrix}^\top$ for the non-linear model without damping.

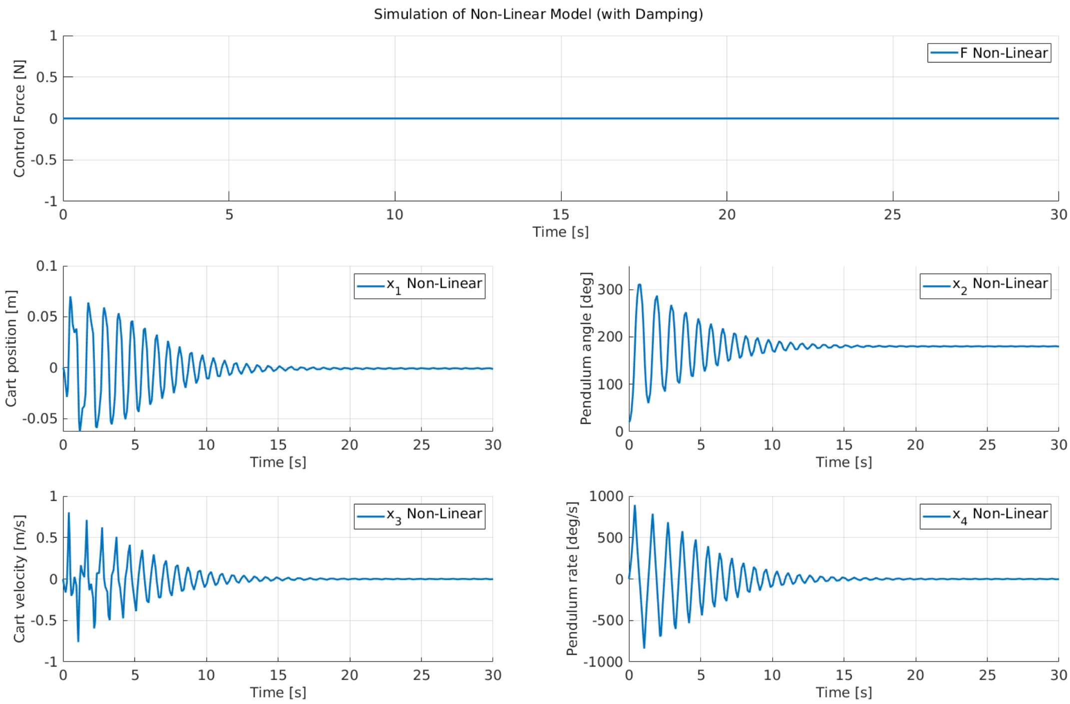

Figure 7: Time histories of the system states with zero control force and initial conditions of $\begin{bmatrix} 0 & 20\pi/180 & 0 & 0 \end{bmatrix}^\top$ for the non-linear model with damping.