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The Mathematics of Frets

Introduction

The mathematics of frets starts as early as the 16th century where Vicenzo Galilei (1520 – 1591), a theorist and musician, demonstrated that the frequency of an oscillating string is dependent on several factors: weight, tension and most importantly on the topic of frets, length. Not only did he establish scientifically how frets work, yet he changed the way musicians placed frets on their instruments by introducing a geometric method of fret placement known as the 'Rule of 18', thus replacing the trial-and-error method musicians used beforehand.

Main Body

Pythagoras

Pythagoras experimented with the idea of consonance using an instrument known as the Monocord, which consists of one or more movable bridges technically resembling the more modern fret. He used the instrument to observe the effect of changing the ratio of string segments (separated by a bridge) on the consonance of a note produced by a plucked string. He found that when the string segments had different lengths, they had different pitches and typically sounded dissonant. However, for some ratios, the 2 segments produced consonant sounds. He named the relationship between two notes an interval, which can be described using a ratio. For example, producing a fifth interval would have a ratio of 3:2 or 2:3 depending on whether the interval is ascending or descending.

The Relation Between Frequency and Length of a String

The fundamental frequency of an oscillating string (or the frequency of a fixed string's first harmonic) can be related to its length by the equation $f = \frac{v}{2L}$, where f is the frequency, v is the velocity of propagation and L is the (resonant) length of the string. Assuming the linear density of the string and tension remain constant (as $v = \sqrt{\frac{T}{\rho}}$, where T is tension and ρ is linear density), the relationship between the frequency and length of the string is inversely proportional. This is to say that as the length of the string decreases, the frequency it produces increases proportionally.

Tuning and Frets

Frets are used to discretize the continuous frequencies a vibrating string can produce. Assuming the guitar is tuned to 12 tone equal temperament (or 12-tet), the fret spacing between 2 adjacent notes must have a common ratio equal to the frequency ratio of a semitone (or $\sqrt[12]{2}$). This means the frequency of the note (and thus the length of the string) must change by a factor of this ratio as the adjacent fret is used. An exponential relationship can be clearly observed as each fret gets closer together further down the fretboard, which can be explained by the fact that the interval between n semitones is given by the exponential function $2^{\frac{n}{12}}$.

Placing Frets

Maths of Fret Placement

For accurate tuning, the placement of frets must be as accurate as possible so that the length of the string is shortened correctly. The ratio of the frequencies from the first fret to an open string should be $\sqrt[12]{2} : 1$, as the note the first fret produces is a semitone above the note produced by the open string. To produce the note a semitone above, the length of the string will have to be $\frac{1}{\sqrt[12]{2}}$ times the open string length. An equation can thus be derived for the position of a fret from the position of the previous fret by considering the previous length of the string, as shown in figure 1. The only issue with this method of fret placement is increased error due to the limited accuracy of measuring equipment. For example, it may be difficult to accurately measure any multiple of $\sqrt[12]{2}$ with a ruler, as a typical ruler has a best resolution of 1mm.

A geometric method, known as the 'rule of 18' says the position of a fret from its predecessor is the resting string length minus the previous fret length, divided by 18. The invention of the calculator later enabled a more accurate constant to divide by, which was 17.817. A mathematical formula for this method can be seen in figure 2.

The formula that relates string length and frequency can also be used to calculate the length of the shortened string, and consequently the position of a fret. Rewriting T as $4\rho L_o^2 f_o^2$, using constants for ρ, L_o (the length of the open string) and f_o (the frequency of the open string) appropriate to each string of an instrument, and substituting the value for T into the formula seen in figure 3 will produce a value for the length of the shortened string, L. $L_o - L$ can then be used to calculate the position of the fret, or the distance of the fret from the end of the string. This may be impractical when building an instrument, as knowledge of the frequency that each fret produces is required as well as knowledge of the linear density of each string on the instrument.

Figure 1

$$\left(1 - \frac{1}{\sqrt[12]{2}}\right) * L$$

(David Hornbeck, 2013)

Where L is the length of the string.

Figure 2

$$D_n = [(L - D_{n-1}) \div 17.817] + D_{n-1}$$

(Tom Morissey, 2016)

Where D_n denotes the length of fret n and L is the length of the open string.

Figure 3

$$L = \frac{\sqrt{\frac{T}{\rho}}}{2f}$$

(David Hornbeck, 2013)

Where f denotes frequency, T denotes the tension of the string, ρ is the linear (mass) density and L is the length of the string.

Strähle Construction

Daniel Strähle (1700 – 1746) invented a geometric method which was used to calculate fret positions for strings on fretted instruments. The method (shown below) determines the length a string needs to be to produce certain pitches using tempered tuning, without the need to calculate positions based on the frequency ratio of a semitone, a number that before the invention of the pocket calculator was difficult to compute. Even though this method is considered an approximation to 12 tone equal temperament, the method was still more accurate than other methods at the time of its creation, as most methods used approximations to $\sqrt[12]{2}$. For example, the 'Rule of 18' utilizes the ratio 18:17 as an approximation to $\sqrt[12]{2}$.

The method starts with the construction of a line segment QR joined to an apex O in the form of an isosceles triangle. The line segment QR is then divided into 12 parts, each labelled 1 through 12 and joined to O using a number of rays. The point P is then placed seven units above Q on the line QO and joined to R and then to M, a point the same distance away from P, P is from R. The intersections of the line MR and the rays from each segment of QR to O determines the lengths a string needs to be to produce the 12 pitches of an octave. The line MR can be pictured as the string of a fretted instrument and the points of intersection determine the positions of frets along the string.

Errors in Methods

Strähle's Construction

As each fret on an instrument tuned in 12-tet corresponds to a semitone, the length of the shortened string at each fret is said to be proportional to the "terms of the geometric progression r^n ($n = 0, 1, ..., 11$) where $r = 2^{1/2}$" (Schoenberg, I. J. 1976). The frets of an instrument in 12-tet can be assigned a position corresponding to each term, which Strähle's Construction attempts to approximate, with surprisingly small error. Strähle's Construction is thus an approximation to the exponential function r^n, which can be written as a linear fractional transformation given in the form $y = \frac{a+bx}{c+dx}$ where a, b, c and d are integer constants.

These are derived by considering a new line segment parallel to MR and intersecting QR at a new point D. As OQR and OQD are similar triangles, the length of QD can be determined as 24/7, and y must be ∞ if x is 24/7 due to division by 0. Since (0, 1) and (1, 2) are points satisfied by the construction, the constants can be determined as a = 10/7 and b = 24/7 and thus the equation in figure 4 is derived.

The relative error for each interval can then be calculated, the largest being the major sixth:

$$y = \frac{24 + 10(3/12)}{24 - 7(3/12)} = \frac{106}{89}$$

$$error = \frac{\frac{106}{89} - 2^{\frac{3}{12}}}{2^{\frac{3}{12}}} = 0.1517\%$$

This means Strähle's Construction is never more than a few cents from equal temperament and is therefore a good approximation.

Figure 4

$$y = \frac{24 + 10x}{24 - 7x}$$

(Schoenberg, I. J. 1976)

The Rule of 18

Most of the error produced by the 'Rule of 18' method is typically down to the limited accuracy of the tools used to measure distances between frets. This also leads to a cumulative error that grows for every fret, which can be clearly inferred by reviewing figure 2 – the position of the next fret relies on the position of the previous fret. In comparison to Strähle's Construction, it is a worse approximation, but it remained the main method of placing frets for hundreds of years, so it may be considered good enough.

Conclusion

Fret placement often becomes a compromise of practicality and accuracy, but there exists some methods (for example Strähle's Construction) that, to some extent, provides both. Even so, the 'Rule of 18' is still often used as an industry standard in the production of guitars despite the errors that arise in the practice of said method. In conclusion, the placement of frets on an instrument relies heavily on the concept of the musical interval as well as the mathematics of tempered tuning, which enabled mathematicians to produce various methods of fret placement that has shaped the way we construct instruments in the modern day.

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