

# **X100/701**

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QUALIFICATIONS  
2006

FRIDAY, 19 MAY  
1.00 PM – 4.00 PM

MATHEMATICS  
ADVANCED HIGHER

**Read carefully**

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**



**Answer all the questions.**

1. Calculate the inverse of the matrix  $\begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$ .

For what value of  $x$  is this matrix singular?

4

2. Differentiate, simplifying your answers:

(a)  $2 \tan^{-1} \sqrt{1+x}$ , where  $x > -1$ ;

3

(b)  $\frac{1+\ln x}{3x}$ , where  $x > 0$ .

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3. Express the complex number  $z = -i + \frac{1}{1-i}$  in the form  $z = x + iy$ , stating the values of  $x$  and  $y$ .

3

Find the modulus and argument of  $z$  and plot  $z$  and  $\bar{z}$  on an Argand diagram.

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4. Given  $xy - x = 4$ , use implicit differentiation to obtain  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .

2

Hence obtain  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ .

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5. Obtain algebraically the fixed point of the iterative scheme given by

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n^2} \right), \quad n = 0, 1, 2, \dots .$$

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6. Find  $\int \frac{12x^3 - 6x}{x^4 - x^2 + 1} dx$ .

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7. For all natural numbers  $n$ , prove whether the following results are true or false.

(a)  $n^3 - n$  is always divisible by 6.

(b)  $n^3 + n + 5$  is always prime.

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8. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0,$$

given that when  $x = 0$ ,  $y = 0$  and  $\frac{dy}{dx} = 2$ .

6

9. Use Gaussian elimination to obtain solutions of the equations

$$\begin{aligned}2x - y + 2z &= 1 \\x + y - 2z &= 2 \\x - 2y + 4z &= -1.\end{aligned}$$

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10. The amount  $x$  micrograms of an impurity removed per kg of a substance by a chemical process depends on the temperature  $T$  °C as follows:

$$x = T^3 - 90T^2 + 2400T, \quad 10 \leq T \leq 60.$$

At what temperature in the given range should the process be carried out to remove as much impurity per kg as possible?

4

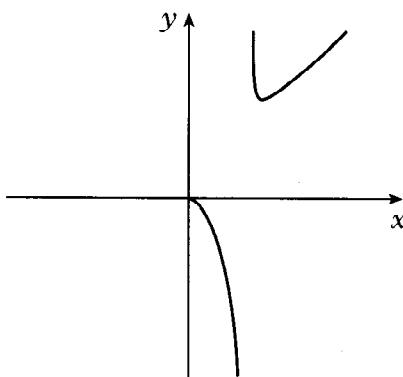
11. Show that  $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$ , where  $0 < \theta < \frac{\pi}{2}$ .

1

By expressing  $y = \cot^{-1} x$  as  $x = \cot y$ , obtain  $\frac{dy}{dx}$  in terms of  $x$ .

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- 12.



The diagram shows part of the graph of a function  $f$  which satisfies the following conditions:

- (i)  $f$  is an even function;
- (ii) two of the asymptotes of the graph  $y = f(x)$  are  $y = x$  and  $x = 1$ .

Copy the diagram and complete the graph. Write down equations for the other two asymptotes.

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13. The square matrices  $A$  and  $B$  are such that  $AB = BA$ . Prove by induction that  $A^n B = B A^n$  for all integers  $n \geq 1$ .

5

[Turn over for Questions 14 to 17 on Page four]

Marks

14. (a) Determine whether  $f(x) = x^2 \sin x$  is odd, even or neither. Justify your answer. 3
- (b) Use integration by parts to find  $\int x^2 \sin x \, dx$ . 4
- (c) Hence find the area bounded by  $y = x^2 \sin x$ , the lines  $x = -\frac{\pi}{4}$ ,  $x = \frac{\pi}{4}$  and the  $x$ -axis. 3
15. Obtain an equation for the plane passing through the point  $P(1, 1, 0)$  which is perpendicular to the line  $L$  given by  

$$\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1}. \quad 3$$
- Find the coordinates of the point  $Q$  where the plane and  $L$  intersect. 4
- Hence, or otherwise, obtain the shortest distance from  $P$  to  $L$  and explain why this is the shortest distance. 2, 1
16. The first three terms of a geometric sequence are  

$$\frac{x(x+1)}{(x-2)}, \frac{x(x+1)^2}{(x-2)^2} \text{ and } \frac{x(x+1)^3}{(x-2)^3}, \text{ where } x < 2.$$
- (a) Obtain expressions for the common ratio and the  $n$ th term of the sequence. 3
- (b) Find an expression for the sum of the first  $n$  terms of the sequence. 3
- (c) Obtain the range of values of  $x$  for which the sequence has a sum to infinity and find an expression for the sum to infinity. 4
17. (a) Show that  $\int \sin^2 x \cos^2 x \, dx = \int \cos^2 x \, dx - \int \cos^4 x \, dx$ . 1
- (b) By writing  $\cos^4 x = \cos x \cos^3 x$  and using integration by parts, show that  

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{1}{4} + 3 \int_0^{\pi/4} \sin^2 x \cos^2 x \, dx. \quad 3$$
- (c) Show that  $\int_0^{\pi/4} \cos^2 x \, dx = \frac{\pi+2}{8}$ . 3
- (d) Hence, using the above results, show that  

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{3\pi+8}{32}. \quad 3$$

[END OF QUESTION PAPER]